International Mathematical Forum, Vol. 9, 2014, no. 18, 871 - 890
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/imf.2014.4488

Large and Moderate Deviation Principles for Kernel Distribution Estimator

Yousri Slaoui

Université de Poitiers
Laboratoire de Mathématiques et Application
Futuroscope Chasseneuil, France

Copyright © 2014 Yousri Slaoui. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

In this paper we prove large and moderate deviations principles for the kernel estimator of a distribution function introduced by Nadaraya [1964. Some new estimates for distribution functions. Theory Probab. Appl. 9, 497-500]. We provide results both for the pointwise and the uniform deviations.

Mathematics Subject Classifiation: 62E20, 60F10

Keywords: Distribution estimation; Large and Moderate deviations principles

1 Introduction

Let $X_1, \ldots, X_n$ be independent, identically distributed of random variables, and let f and F denote respectively the probability density of X_1 and the distribution function of X_1. Nadaraya (1964) introduce a kernel K (that is, a function satisfying $\int_{\mathbb{R}} K(x)dx = 1$), a function $\mathcal{K}$ (that is, a function defined by $\mathcal{K}(z) = \int_{-\infty}^{z} K(u)\, du$), and a bandwidth (h_n) (that is, a sequence of positive real numbers that goes to zero). The estimator proposed by Nadaraya (1964) to estimate the distribution function F at the point x is given by

$$F_n(x) = \frac{1}{n}\sum_{k=1}^{n} \mathcal{K}\left(\frac{x - X_k}{h_n}\right). \tag{1}$$

Some theoretical properties of the estimator F_n have been investigated (see among many others, Nadaraya (1964), Reiss (1981), and Hill (1985)). Reiss (1981) and Falk (1983) showed that the kernel distribution estimator (1) have an asymptotically better performance than empirical distribution function, which does not take into account the smoothness of F.

Recently, large and moderate deviations results have been proved for the well-known nonrecursive kernel density estimator introduced by Rosenblatt (1956) (see also Parzen, 1962). The large deviations principle has been studied by Louani (1998) and Worms (2001). Gao (2003) and Mokkadem et al. (2005) extend these results and provide moderate deviations principles. The purpose of this paper is to establish large and moderate deviations principles for the nonrecursive distribution estimator (1).

Let us first recall that a $\mathbb{R}^m$-valued sequence $(Z_n)_{n\geq 1}$ satisfies a large deviations principle (LDP) with speed (ν_n) and good rate function I if :

1. (ν_n) is a positive sequence such that $\lim_{n\to\infty}\nu_n = \infty$;
2. $I : \mathbb{R}^m \to [0,\infty]$ has compact level sets;
3. for every borel set $B \subset \mathbb{R}^m$,

$$\begin{aligned} -\inf_{x\in\overset{\circ}{B}} I(x) &\leq \liminf_{n\to\infty} \nu_n^{-1}\log\mathbb{P}[Z_n\in B] \\ &\leq \limsup_{n\to\infty} \nu_n^{-1}\log\mathbb{P}[Z_n\in B] \leq -\inf_{x\in\overline{B}} I(x), \end{aligned}$$

where $\overset{\circ}{B}$ and $\overline{B}$ denote the interior and the closure of B respectively. Moreover, let (v_n) be a nonrandom sequence that goes to infinity; if $(v_n Z_n)$ satisfies a LDP, then (Z_n) is said to satisfy a moderate deviations principle (MDP).

The first aim of this paper is to establish pointwise LDP for the kernel distribution estimator (1).

We show that using the bandwiths defined as $h_n = h(n)$ for all n, where h is a regularly varing function with exponent $(-a)$, $a \in]0,1[$. We prove that the sequence $(F_n(x) - F(x))$ satisfies a LDP with speed (n) and the rate function defined as follows:

$$\begin{cases} if \quad F(x)\neq 0, \quad I_x : t \to F(x)\, I\left(1+\frac{t}{F(x)}\right) \\ if \quad F(x) = 0, \quad I_x(0) = 0 \quad and \quad I_x(t) = +\infty \quad for \quad t\neq 0. \end{cases} \tag{2}$$

where

$$I(t) = \sup_{u\in\mathbb{R}}\{ut - \psi(u)\}$$
$$\psi(u) = \exp(u) - 1.$$

Our second aim is to provide pointwise MDP for the distribution estimator defined by (1). For any positive sequence (v_n) satisfying

$$\lim_{n\to\infty} v_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{v_n^2}{n} = 0$$

and general bandwidths (h_n), we prove that the sequence

$$v_n \left(F_n \left(x\right) - F\left(x\right)\right)$$

satisfies a LDP of speed (n/v_n^2) and rate function $J_x(.)$ defined by

$$\begin{cases} if \quad f(x) \neq 0, & J_x : t \to \frac{t^2}{2F(x)} \\ if \quad f(x) = 0, & J_x(0) = 0 \quad and \quad J_x(t) = +\infty \quad for \quad t \neq 0. \end{cases} \tag{3}$$

Finally, we give a uniform version of the previous results. More precisely, let U be a subset of $\mathbb{R}$; we establish large and moderate deviations principles for the sequence $(\sup_{x\in U} |F_n(x) - F(x)|)$.

2 Assumptions and main results

We define the following class of regularly varying sequences.

Definition 1. Let $\gamma \in \mathbb{R}$ and $(v_n)_{n\geq 1}$ be a nonrandom positive sequence. We say that $(v_n) \in \mathcal{GS}(\gamma)$ if

$$\lim_{n\to+\infty} n \left[1 - \frac{v_{n-1}}{v_n}\right] = \gamma. \tag{4}$$

Condition (4) was introduced by Galambos and Seneta (1973) to define regularly varying sequences (see also Bojanic and Seneta, 1973). Typical sequences in $\mathcal{GS}(\gamma)$ are, for $b \in \mathbb{R}$, $n^\gamma (\log n)^b$, $n^\gamma (\log\log n)^b$, and so on.

2.1 Pointwise LDP for the Nadaraya's distribution estimator

To establish pointwise LDP for F_n, we need the following assumptions.

(L1) $K : \mathbb{R} \to \mathbb{R}$ is a bounded and integrable function satisfying $\int_{\mathbb{R}} K(z)\,dz = 1$, and $\int_{\mathbb{R}} zK(z)\,dz = 0$.

(L2) $(h_n) \in \mathcal{GS}(-a)$ with $a \in]0, 1[$.

The following Theorem gives the pointwise LDP for F_n in this case.

Theorem 1 (Pointwise LDP for Nadaraya's distribution estimator).
Let Assumptions $(L1)$ and $(L2)$ hold and assume that F is continuous at x. Then, the sequence $(F_n(x) - F(x))$ satisfies a LDP with speed (n) and rate function defined by (2).

2.2 Pointwise MDP for the Nadaraya's distribution estimator

Let (v_n) be a positive sequence; we assume that

(M1) $K : \mathbb{R} \to \mathbb{R}$ is a continuous, bounded function satisfying $\int_{\mathbb{R}} K(z)\, dz = 1$, and, $\int_{\mathbb{R}} zK(z)\, dz = 0$ and $\int_{\mathbb{R}} z^2 |K(z)|dz < \infty$.

(M2) $(h_n) \in \mathcal{GS}(-a)$ with $a \in]0, 1[$.

(M3) F is bounded, twice differentiable, and $F^{(2)}(x)$ is bounded.

(M4) $\lim_{n\to\infty} v_n = \infty$ and $\lim_{n\to\infty} \frac{v_n^2}{n} = 0$.

The following Theorem gives the pointwise MDP for F_n.

Theorem 2 (Pointwise MDP for the kernel distribution estimator (1)).
Let Assumptions $(M1) - (M4)$ hold and assume that F is continuous at x. Then, the sequence $(F_n(x) - F(x))$ satisfies a MDP with speed (n/v_n^2) and rate function J_x defined in (3).

2.3 Uniform LDP and MDP for the Nadaraya's distribution estimator

To establish uniform large deviations principles for the distribution estimator defined by (1) on a bounded set, we need the following assumptions:

(U1) $i)$ $\int_{\mathbb{R}} zK(z)\, dz = 0$ and $\int_{\mathbb{R}} z^2|K(z)|dz < \infty$.
$ii)$ $\mathcal{K}$ is Hölder continuous.

(U2) F is bounded, twice differentiable, and, $\sup_{x\in\mathbb{R}} |F^{(2)}(x)| < \infty$.

(U3) $\lim_{n\to\infty} \frac{v_n^2 \log v_n}{n} = 0$.

Set $U \subseteq \mathbb{R}$; in order to state in a compact form the uniform large and moderate deviations principles for the distribution estimator defined by (1) on U, we set:

$$\begin{aligned}
g_U(\delta) &= \begin{cases} \|F\|_{U,\infty} I\left(1 + \frac{\delta}{\|F\|_{U,\infty}}\right) & \text{when } v_n \equiv 1 \quad , \ (L1) \text{ and } (L2) \text{ hold} \\ \frac{\delta^2}{2\|F\|_{U,\infty}} & \text{when } v_n \to \infty \quad , \ (M1) - (M4) \text{ hold} \end{cases} \\
\tilde{g}_U(\delta) &= \min\{g_U(\delta), g_U(-\delta)\}
\end{aligned}$$

where $\|F\|_{U,\infty} = \sup_{x\in U} |F(x)|$.

Remark 1. The functions $g_U(.)$ and $\tilde{g}_U(.)$ are non-negative, continuous, increasing on $]0,+\infty[$ and decreasing on $]-\infty,0[$, with a unique global minimum in 0 ($\tilde{g}_U(0) = g_U(0) = 0$). They are thus good rate functions (and $g_U(.)$ is strictly convex).

Theorem 3 below states uniform LDP on U in the case U is bounded, and Theorem 4 in the case U is unbounded.

Theorem 3 (Uniform deviations on a bounded set for the kernel distribution estimator (1)). Let $(U1)-(U3)$ hold. Then for any bounded subset U of $\mathbb{R}$ and for all $\delta > 0$,

$$\lim_{n\to\infty} n^{-1}v_n^2 \log \mathbb{P}\left[\sup_{x\in U} v_n |F_n(x) - F(x)| \geq \delta\right] = -\tilde{g}_U(\delta) \tag{5}$$

To establish uniform large deviations principles for the distribution estimator (1) on an unbounded set, we need the following additionnal assumptions:

(U4) *i)* There exists $\beta > 0$ such that $\int_{\mathbb{R}} \|x\|^{\beta} f(x)\,dx < \infty$.
ii) F is uniformly continuous.

(U5) There exists $\tau > 0$ such that $z \mapsto \|z\|^{\tau}\mathcal{K}(z)$ is a bounded function.

(U6) *i)* There exists $\zeta > 0$ such that $\int_{\mathbb{R}} \|z\|^{\zeta} |K(z)|\,dz < \infty$
ii) There exists $\eta > 0$ such that $z \mapsto \|z\|^{\eta} F(z)$ is a bounded function.

Theorem 4 (Uniform deviations on an unbounded set for the estimator defined by (1)). Let $(U1)-(U6)$ hold. Then for any subset U of $\mathbb{R}$ and for all $\delta > 0$,

$$\begin{aligned} -\tilde{g}_U(\delta) &\leq \liminf_{n\to\infty} n^{-1}v_n^2 \log \mathbb{P}\left[\sup_{x\in U} v_n |F_n(x) - F(x)| \geq \delta\right] \\ &\leq \limsup_{n\to\infty} n^{-1}v_n^2 \log \mathbb{P}\left[\sup_{x\in U} v_n |F_n(x) - F(x)| \geq \delta\right] \leq -\frac{\beta}{\beta+1}\tilde{g}_U(\delta) \end{aligned}$$

The following corollary is a straightforward consequence of Theorem 4.

Corollary 1. Under the assumptions of Theorem 4, if $\int_{\mathbb{R}} \|x\|^{\xi} F(x)\,dx < \infty$ for all ξ in $\mathbb{R}$, then for any subset U of $\mathbb{R}$,

$$\lim_{n\to\infty} n^{-1}v_n^2 \log \mathbb{P}\left[\sup_{x\in U} v_n |F_n(x) - F(x)| \geq \delta\right] = -\tilde{g}_U(\delta) \tag{6}$$

Comment. Since the sequence $(\sup_{x\in U} |F_n(x) - F(x)|)$ is positive and since $\tilde{g}_U$ is continuous on $[0,+\infty[$, increasing and goes to infinity as $\delta \to \infty$, the application of Lemma 5 in Worms (2001) allows to deduce from (5) or (6) that $\sup_{x\in U} |F_n(x) - F(x)|$ satisfies a LDP with speed (n) and good rate function $\tilde{g}_U$ on $\mathbb{R}+$.

3 Proofs

Throught this section we use the following notation:

$$Y_{k,n} = \mathcal{K}\left(\frac{x - X_k}{h_n}\right) \tag{7}$$

Noting that, in view of (1), we have

$$F_n(x) - \mathbb{E}[F_n(x)] = \frac{1}{n}\sum_{k=1}^{n}(Y_{k,n} - \mathbb{E}[Y_{k,n}])$$

Let (Ψ_n) and (B_n) be the sequences defined as

$$\begin{aligned}\Psi_n(x) &= \frac{1}{n}\sum_{k=1}^{n}(Y_{k,n} - \mathbb{E}[Y_{k,n}])\\ B_n(x) &= \mathbb{E}[F_n(x)] - F(x)\end{aligned}$$

We have:

$$F_n(x) - F(x) = \Psi_n(x) + B_n(x) \tag{8}$$

Theorems 1, 2, 3 and 4 are consequences of (8) and the following propositions.

Proposition 1 (Pointwise LDP and MDP for (Ψ_n)).

1. Under the assumptions $(L1)$ and $(L2)$, the sequence $(F_n(x) - \mathbb{E}(F_n(x)))$ satisfies a LDP with speed (n) and rate function I_x.

2. Under the assumptions $(M1) - (M4)$, the sequence $(v_n\Psi_n(x))$ satisfies a LDP with speed (n/v_n^2) and rate function J_x.

Proposition 2 (Uniform LDP and MDP for (Ψ_n)).

1. Let $(U1) - (U3)$ hold. Then for any bounded subset U of $\mathbb{R}$ and for all $\delta > 0$,

$$\lim_{n\to\infty} n^{-1}v_n^2 \log \mathbb{P}\left[\sup_{x\in U} v_n|\Psi_n(x)| \geq \delta\right] = -\tilde{g}_U(\delta)$$

2. Let $(U1) - (U6)$ hold. Then for any subset U of $\mathbb{R}$ and for all $\delta > 0$,

$$\begin{aligned}-\tilde{g}_U(\delta) &\leq \liminf_{n\to\infty} n^{-1}v_n^2 \log \mathbb{P}\left[\sup_{x\in U} v_n|\Psi_n(x)| \geq \delta\right]\\ &\leq \limsup_{n\to\infty} n^{-1}v_n^2 \log \mathbb{P}\left[\sup_{x\in U} v_n|\Psi_n(x)| \geq \delta\right] \leq -\frac{\xi}{\xi + d}\tilde{g}_U(\delta)\end{aligned}$$

Proposition 3 (Pointwise and uniform convergence rate of (B_n)).
Let Assumptions $(M1)-(M3)$ hold.

1. If f' is continuous at x. We have

   If $a \leq 1/3$, then
$$B_n(x) = O\left(h_n^2\right).$$
   If $a > 1/3$, then
$$B_n(x) = o\left(\sqrt{n^{-1}h_n}\right).$$

2. If $(U2)$ holds, then:

   If $a \leq 1/3$, then
$$\sup_{x\in\mathbb{R}} |B_n(x)| = O\left(h_n^2\right).$$
   If $a > 1/3$, then
$$\sup_{x\in\mathbb{R}} |B_n(x)| = o\left(\sqrt{n^{-1}h_n}\right).$$

Set $x \in \mathbb{R}$; since the assumptions of Theorems 1 guarantee that $\lim_{n\to\infty} B_n(x) = 0$, Theorem 1 is a straightforward consequence of the application of Part 1 (respectively of Part 2) of Proposition 1. Moreover, under the assumptions of Theorem 2, we have by application of Propostion 3, $\lim_{n\to\infty} v_n B_n(x) = 0$; Theorem 2 thus straightfully follows from the application of Part 3 of Proposition 1. Finaly, Theorem 3 and 4 follows from Proposition 2 and the second part of Proposition 3.
We now state a preliminary lemma, which will be used in the proof of Proposition 1. For any $u \in \mathbb{R}$, Set

$$\begin{aligned}
\Lambda_{n,x}(u) &= n^{-1}v_n^2 \log \mathbb{E}\left[\exp\left(u\frac{n}{v_n}\Psi_n(x)\right)\right] \\
\Lambda_x^L(u) &= F(x)\left(\psi(u) - u\right), \\
\Lambda_x^M(u) &= \frac{u^2}{2}F(x)
\end{aligned}$$

Lemma 1. [Convergence of $\Lambda_{n,x}$]

1. (Pointwise convergence)
   If F is continuous at x, then for all $u \in \mathbb{R}$
$$\lim_{n\to\infty} \Lambda_{n,x}(u) = \Lambda_x(u) \tag{9}$$
   where
$$\Lambda_x(u) = \begin{cases} \Lambda_x^L(u) & \text{when} \quad v_n \equiv 1 \\ \Lambda_x^M(u) & \text{when} \quad v_n \to \infty \end{cases}$$

2. (Uniform convergence)
   If F is uniformly continuous, then the convergence (9) holds uniformly in $x \in U$.

Our proofs are now organized as follows: Lemma 1 is proved in Section 3.1, Proposition 1 in Section 3.4 and Proposition 2 in Section 3.3.

3.1 Proof of Lemma 1.

Set $u \in \mathbb{R}$, $u_n = u/v_n$ and $a_n = n$. We have:

$$\begin{aligned}
\Lambda_{n,x}(u) &= \frac{v_n^2}{a_n}\log\mathbb{E}\left[\exp\left(u_n a_n \Psi_n(x)\right)\right] \\
&= \frac{v_n^2}{a_n}\log\mathbb{E}\left[\exp\left(u_n\sum_{k=1}^{n}\left(Y_{k,n}-\mathbb{E}\left[Y_{k,n}\right]\right)\right)\right] \\
&= \frac{v_n^2}{a_n}\sum_{k=1}^{n}\log\mathbb{E}\left[\exp\left(u_nY_{k,n}\right)\right]-uv_n\mathbb{E}\left[Y_{1,n}\right]
\end{aligned}$$

By Taylor expansion, there exists $c_{k,n}$ between 1 and $\mathbb{E}\left[\exp\left(u_nY_{k,n}\right)\right]$ such that

$$\log\mathbb{E}\left[\exp\left(u_nY_{k,n}\right)\right]=\mathbb{E}\left[\exp\left(u_nY_{k,n}\right)-1\right]-\frac{1}{2c_{k,n}^2}\left(\mathbb{E}\left[\exp\left(u_nY_{k,n}\right)-1\right]\right)^2$$

and $\Lambda_{n,x}$ can be rewriten as

$$\begin{aligned}
\Lambda_{n,x}(u) &= \frac{v_n^2}{a_n}\sum_{k=1}^{n}\mathbb{E}\left[\exp\left(u_nY_{k,n}\right)-1\right]-\frac{v_n^2}{2a_n}\sum_{k=1}^{n}\frac{1}{c_{k,n}^2}\left(\mathbb{E}\left[\exp\left(u_nY_{k,n}\right)-1\right]\right)^2 \\
&\quad -uv_n\mathbb{E}\left[Y_{1,n}\right] \qquad (10)
\end{aligned}$$

First case: $v_n \to \infty$. A Taylor's expansion implies the existence of $c'_{k,n}$ between 0 and $u_nY_{k,n}$ such that

$$\mathbb{E}\left[\exp\left(u_nY_{k,n}\right)-1\right]=u_n\mathbb{E}\left[Y_{k,n}\right]+\frac{1}{2}u_n^2\mathbb{E}\left[Y_{k,n}^2\right]+\frac{1}{6}u_n^3\mathbb{E}\left[Y_{k,n}^3e^{c'_{k,n}}\right]$$

Therefore,

$$\begin{aligned}
\Lambda_{n,x}(u) &= \frac{1}{2}u^2a_n\sum_{k=1}^{n}\mathbb{E}\left[Y_{k,n}^2\right]+\frac{1}{6}u^2\frac{u_n}{a_n}\sum_{k=1}^{n}\mathbb{E}\left[Y_{k,n}^3e^{c'_{k,n}}\right] \\
&\quad -\frac{v_n^2}{2a_n}\sum_{k=1}^{n}\frac{1}{c_{k,n}^2}\left(\mathbb{E}\left[\exp\left(u_nY_{k,n}\right)-1\right]\right)^2 \\
&= \frac{1}{2}u^2F(x)+R_{n,x}^{(1)}(u)+R_{n,x}^{(2)}(u) \qquad (11)
\end{aligned}$$

with

$$\begin{aligned} R_{n,x}^{(1)}(u) &= u^2 \int_{\mathbb{R}} K(z)\,\mathcal{K}(-z)\,[F(x+zh_n)-F(x)]\,dz \\ R_{n,x}^{(2)}(u) &= \frac{1}{6}\frac{u^3}{v_n}\frac{1}{a_n}\sum_{k=1}^{n}\mathbb{E}\left[Y_k^3 e^{c'_{k,n}}\right] - \frac{v_n^2}{2a_n}\sum_{k=1}^{n}\frac{1}{c_{k,n}^2}\left(\mathbb{E}\left[\exp(u_n Y_{k,n})-1\right]\right)^2 \end{aligned}$$

Since F is continuous, we have $\lim_{n\to\infty}|F(x+zh_n)-F(x)|=0$, and thus, by the dominated convergence theorem, $(M1)$ implies that

$$\lim_{n\to\infty}\int_{\mathbb{R}} K(z)\,\mathcal{K}(-z)\,|F(x+zh_n)-F(x)|\,dz = 0,$$

it follows that $\lim_{n\to\infty}\left|R_{n,x}^{(1)}(u)\right| = 0$.
Moreover, in view of (7), we have $|Y_{k,n}|\le\|\mathcal{K}\|_\infty$, then

$$\begin{aligned} c'_{k,n} &\le |u_n Y_{k,n}| \\ &\le |u_n|\,\|\mathcal{K}\|_\infty \end{aligned} \tag{12}$$

Noting that $\mathbb{E}|Y_{k,n}|^3 \le 3\|F\|_\infty \int_{\mathbb{R}}|K(z)|\,|\mathcal{K}^2(z)|\,dz$. Hence, it follows from (12), there exists a positive constant c_1 such that, for n large enough,

$$\left|\frac{u^3}{v_n}\frac{1}{a_n}\sum_{k=1}^{n}\mathbb{E}\left[Y_k^3 e^{c'_{k,n}}\right]\right| \le c_1 e^{|u_n|\|\mathcal{K}\|_\infty}\frac{u^3}{v_n}\|F\|_\infty\int_{\mathbb{R}}|K(z)|\,\left|\mathcal{K}^2(z)\right|dz \tag{13}$$

which goes to 0 as $n\to\infty$ since $v_n\to\infty$.
In the same way, there exists a positive constant c_2 such that, for n large enough,

$$\begin{aligned} &\left|\frac{v_n^2}{2a_n}\sum_{k=1}^{n}\frac{1}{c_{k,n}^2}\left(\mathbb{E}\left[\exp(u_n Y_{k,n})-1\right]\right)^2\right| \\ &\quad\le \frac{v_n^2}{2a_n}\sum_{k=1}^{n}\left(\mathbb{E}\left[\exp(u_n Y_{k,n})-1\right]\right)^2 \\ &\quad\le c_2\frac{u^2}{2}h_n\|f\|_\infty^2\exp\left(|u_n|\,\|K\|_\infty\right)\left(\int_{\mathbb{R}}|\mathcal{K}(-z)|\,dz\right)^2 \end{aligned} \tag{14}$$

The combination of (13) and (14) ensures that $\lim_{n\to\infty}\left|R_{n,x}^{(2)}(u)\right| = 0$. Then, we obtain from (11), $\lim_{n\to\infty}\Lambda_{n,x}(u)=\Lambda_x^M(u)$.

Second case: $(v_n) \equiv 1$. It follows from (10), that

$$\begin{aligned}\Lambda_{n,x}(u) &= \frac{1}{a_n}\sum_{k=1}^{n}\mathbb{E}\left[\exp\left(uY_{k,n}\right)-1\right]-\frac{1}{2a_n}\sum_{k=1}^{n}\frac{1}{c_{k,n}^2}\left(\mathbb{E}\left[\exp\left(uY_{k,n}\right)-1\right]\right)^2\\ &\quad -u\mathbb{E}\left[Y_{1,n}\right]\end{aligned}$$

Moreover, using integration by parts, we get

$$\Lambda_{n,x}(u) = uF(x)\int_{\mathbb{R}}K(z)\left(\exp\left(u\mathcal{K}(-z)\right)-1\right)dz - R_{n,x}^{(3)}(u)+R_{n,x}^{(4)}(u) \quad (15)$$

with

$$\begin{aligned}R_{n,x}^{(3)}(u) &= \frac{1}{2a_n}\sum_{k=1}^{n}\frac{1}{c_{k,n}^2}\left(\mathbb{E}\left[\exp\left(uY_{k,n}\right)-1\right]\right)^2\\ R_{n,x}^{(4)}(u) &= u\int_{\mathbb{R}}K(z)\left(\exp\left(u\mathcal{K}(-z)\right)-1\right)\left[F(x+zh_n)-F(x)\right]dz.\end{aligned}$$

It follows from (14), that $\lim_{n\to\infty}\left|R_{n,x}^{(3)}(u)\right| = 0$.
Since $\left|e^t-1\right| \leq |t|\,e^{|t|}$, we have

$$\left|R_{n,x}^{(4)}(u)\right| \leq u^2 e^{|u|\|\mathcal{K}\|_\infty}\int_{\mathbb{R}}|K(z)|\,|\mathcal{K}(-z)|\,|F(x+zh_n)-F(x)|\,dz.$$

Then, the dominated convergence theorem ensures that $\lim_{n\to\infty} R_{n,x}^{(4)}(u) = 0$.
In the case F is uniformly continuous, set $\varepsilon > 0$ and let $M > 0$ such that $2\|F\|_\infty \int_{\|z\|\leq M}|K(z)|\,|\mathcal{K}(-z)|\,dz \leq \varepsilon/2$. We need to prove that for n sufficiently large

$$\sup_{x\in\mathbb{R}}\int_{\|z\|\leq M}|K(z)|\,|\mathcal{K}(-z)|\,|F(x+zh_n)-F(x)|\,dz \leq \varepsilon/2$$

which is a straightforward consequence of the uniform continuity of F.

Then, it follows from (15), that

$$\begin{aligned}\lim_{n\to\infty}\Lambda_{n,x}(u) &= uF(x)\int_{\mathbb{R}}K(z)\left(\exp\left(u\mathcal{K}(-z)\right)-1\right)dz\\ &= F(x)\left(\exp(u)-1-u\right)\\ &= \Lambda_x^L(u)\end{aligned}$$

and thus Lemma 1 is proved.

3.2 Proof of Proposition 1

To prove Proposition 1, we apply Lemma 1 and the following result (see Puhalskii, 1994).

Lemma 2. Let (Z_n) be a sequence of real random variables, (ν_n) a positive sequence satisfying $\lim_{n\to\infty} \nu_n = +\infty$, and suppose that there exists some convex non-negative function Γ defined on $\mathbb{R}$ such that

$$\forall u \in \mathbb{R}, \lim_{n\to\infty} \frac{1}{\nu_n} \log \mathbb{E}\left[\exp\left(u\nu_n Z_n\right)\right] = \Gamma\left(u\right).$$

If the Legendre function Γ^* of Γ is a strictly convex function, then the sequence (Z_n) satisfies a LDP of speed (ν_n) and good rate fonction Γ^*.

In our framework, when $v_n \equiv 1$, we take $Z_n = F_n(x) - \mathbb{E}(F_n(x))$, $\nu_n = n$ and $\Gamma = \Lambda_x^L$. In this case, the Legendre transform of $\Gamma = \Lambda_x^L$ is the rate function $I_x : t \to F(x) I\left(1 + \frac{t}{F(x)}\right)$, since ψ is strictly convex, then its Cramer transform I is a good rate function on $\mathbb{R}$ (see Dembo and Zeitouni, 1998). Otherwise, when, $v_n \to \infty$, we take $Z_n = v_n\left(F_n(x) - \mathbb{E}(F_n(x))\right)$, $\nu_n = n/v_n^2$ and $\Gamma = \Lambda_x^M$; Γ^* is then the quadratic rate function J_x defined in (3) and thus Proposition 1 follows.

3.3 Proof of Proposition 2

In order to prove Proposition 2, we first establish some lemmas.

Lemma 3. Let $\phi : \mathbb{R}^+ \to \mathbb{R}$ be the function defined for $\delta > 0$ as

$$\phi(\delta) = \begin{cases} (\psi')^{-1}\left(1 + \frac{\delta}{\|F\|_{U,\infty}}\right) & \text{when } v_n \equiv 1 \quad , \quad (L1) \text{ and } (L2) \text{ hold} \\ \frac{\delta}{\|F\|_{U,\infty}} & \text{when } v_n \to \infty \quad , \quad (M1) - (M4) \text{ hold} \end{cases}$$

1. $\sup_{u\in\mathbb{R}} \{u\delta - \sup_{x\in U} \Lambda_x(u)\}$ equals $g_U(\delta)$ and is achieved for $u = \phi(\delta) > 0$.
2. $\sup_{u\in\mathbb{R}} \{-u\delta - \sup_{x\in U} \Lambda_x(u)\}$ equals $g_U(\delta)$ and is achieved for $u = \phi(-\delta) < 0$.

Proof of Lemma 3 . We just prove the first part, the proof of the second part one being similar.

- First case $v_n \to 1$. Since $e^t \geq 1 + t$, for all t, we have $\psi(u) \geq u$ and therefore,

$$\begin{aligned} u\delta - \sup_{x\in U} \Lambda_x(u) &= u\delta - \|F\|_{U,\infty}\left(\psi(u) - u\right) \\ &= \|F\|_{U,\infty}\left[u\left(1 + \frac{\delta}{\|F\|_{U,\infty}}\right) - \psi(u)\right] \end{aligned}$$

The function $u \mapsto u\delta - \sup_{x\in U} \Lambda_x(u)$ has second derivative $-\|F\|_{U,\infty}\psi''(u) < 0$ and thus it has a unique maximum achieved for

$$u_0 = (\psi')^{-1}\left(1 + \frac{\delta}{\|F\|_{U,\infty}}\right)$$

Now, since ψ' is increasing and since $\psi'(0) = 1$, we deduce that $u_0 > 0$.

- Second case $v_n \to \infty$. In this case, we have

$$u\delta - \sup_{x\in U} \Lambda_x(u) = u\delta - \frac{u^2}{2}\|F\|_{U,\infty}.$$

The function $u \mapsto u\delta - \sup_{x\in U} \Lambda_x(u)$ has second derivative $-\|F\|_{U,\infty} < 0$ and thus it has a unique maximum achieved for

$$u_0 = \frac{\delta}{\|F\|_{U,\infty}} > 0$$

Lemma 4.

- In the case when $(v_n) \equiv 1$, let $(L1)$ and $(L2)$ hold;
- In the case when $v_n \to \infty$, let $(M1) - (M4)$ hold.
  Then for any $\delta > 0$,

$$\begin{aligned}
\lim_{n\to\infty} \frac{v_n^2}{n} \log \sup_{x\in U} \mathbb{P}\left[v_n \Psi_n(x) \geq \delta\right] &= -g_U(\delta) \\
\lim_{n\to\infty} \frac{v_n^2}{n} \log \sup_{x\in U} \mathbb{P}\left[v_n \Psi_n(x) \leq -\delta\right] &= -g_U(-\delta) \\
\lim_{n\to\infty} \frac{v_n^2}{n} \log \sup_{x\in U} \mathbb{P}\left[v_n |\Psi_n(x)| \leq -\delta\right] &= -\tilde{g}_U(-\delta)
\end{aligned}$$

Proof of Lemma 4. The proof of Lemma 4 is similar to the proof of Lemma 4 in Mokkadem et al. (2006).

Lemma 5. Let Assumptions $(U1) - (U3)$ hold and assume that either $(v_n) \equiv 1$ or $(U4)$ holds.

1. If U is a bounded set, then for any $\delta > 0$, we have

$$\lim_{n\to\infty} \frac{v_n^2}{n} \log \mathbb{P}\left[\sup_{x\in U} v_n |\Psi_n(x)|\right] \leq -\tilde{g}_U(\delta)$$

2. If U is an unbounded set, then, for any $b > 0$ and $\delta > 0$,

$$\limsup_{n\to\infty} \frac{v_n^2}{n} \log \mathbb{P}\left[\sup_{x\in U, \|x\|\leq w_n} v_n |\Psi_n(x)|\right] \leq b - \tilde{g}_U(\delta)$$

where $w_n = \exp\left(b\frac{n}{v_n^2}\right)$.

Proof of Lemma 5. Set $\rho \in]0, \delta[$, let β denote the Hölder order of $\mathcal{K}$, and $\|\mathcal{K}\|_H$ its corresponding Hölder norm. Set $w_n = \exp\left(b\frac{n}{v_n^2}\right)$ and

$$R_n = \left(\frac{\rho}{2\|\mathcal{K}\|_H v_n h_n^{-\beta}}\right)^{\frac{1}{\beta}}$$

We begin with the proof of the second part of Lemma 5. There exist $N'(n)$ points of $\mathbb{R}$, $y_1^{(n)}, y_2^{(n)}, \ldots, y_{N'(n)}^{(n)}$ such that the ball $\{x \in \mathbb{R}; \|x\| \leq w_n\}$ can covered by the $N'(n)$ balls $B_i^{(n)} = \left\{x \in \mathbb{R}; \|x - y_i^{(n)}\| \leq R_n\right\}$ and such that $N'(n) \leq 2\left(\frac{2w_n}{R_n}\right)$. Considering only the $N(n)$ balls that intersect $\{x \in U; \|x\| \leq w_n\}$, we can write

$$\{x \in U; \|x\| \leq w_n\} \subset \cup_{i=1}^{N(n)} B_i^{(n)}.$$

For each $i \in \{1, \ldots, N(n)\}$, set $x_i^{(n)} \in B_i^{(n)} \cap U$. We then have:

$$\begin{aligned}
\mathbb{P}\left[\sup_{x\in U, \|x\|\leq w_n} v_n |\Psi_n(x)| \geq \delta\right] &\leq \sum_{i=1}^{N(n)} \mathbb{P}\left[\sup_{x\in B_i^{(n)}} v_n |\Psi_n(x)| \geq \delta\right] \\
&\leq N(n) \max_{1\leq i\leq N(n)} \mathbb{P}\left[\sup_{x\in B_i^{(n)}} v_n |\Psi_n(x)| \geq \delta\right].
\end{aligned}$$

Now, for any $i \in \{1, \ldots, N(n)\}$ and any $x \in B_i^{(n)}$,

$$\begin{aligned}
v_n |\Psi_n(x)| &\leq v_n \left|\Psi_n\left(x_i^{(n)}\right)\right| \\
&\quad + \frac{v_n}{n}\sum_{k=1}^{n} \left|\mathcal{K}\left(\frac{x - X_k}{h_n}\right) - \mathcal{K}\left(\frac{x_i^{(n)} - X_k}{h_n}\right)\right| \\
&\quad + \frac{v_n}{n}\sum_{k=1}^{n} \mathbb{E}\left|\mathcal{K}\left(\frac{x - X_k}{h_n}\right) - \mathcal{K}\left(\frac{x_i^{(n)} - X_k}{h_n}\right)\right| \\
&\leq v_n \left|\Psi_n\left(x_i^{(n)}\right)\right| + 2\frac{v_n}{n}\|\mathcal{K}\|_H \sum_{k=1}^{n}\left(\frac{\|x - x_i^{(n)}\|}{h_n}\right)^{\beta} \\
&\leq v_n \left|\left(x_i^{(n)}\right)\right| + 2v_n\|\mathcal{K}\|_H h_n^{-\beta} R_n^{\beta} \\
&\leq v_n \left|\Psi_n\left(x_i^{(n)}\right)\right| + \rho
\end{aligned}$$

Hence, we deduce that

$$\begin{aligned}
\mathbb{P}\left[\sup_{x\in U, \|x\|\leq w_n} v_n |\Psi_n(x)| \geq \delta\right] &\leq N(n) \max_{1\leq i\leq N(n)} \mathbb{P}\left[v_n \left|\Psi_n\left(x_i^{(n)}\right)\right| \geq \delta - \rho\right] \\
&\leq N(n) \sup_{x\in U} \mathbb{P}\left[v_n \left|\Psi_n\left(x_i^{(n)}\right)\right| \geq \delta - \rho\right]
\end{aligned}$$

Further, by definition of $N(n)$ and w_n, we have

$$\log N(n) \leq \log N'(n) \leq b\frac{n}{v_n^2} + 2\log 2 - \log R_n$$

and

$$\frac{v_n^2}{n}\log R_n = \frac{1}{\beta}\frac{v_n^2}{n}\left[\log\rho - \log\left(2\|\mathcal{K}\|_H\right) - \log v_n + \beta\log h_n\right].$$

Then, in view of $(U3)$, we have

$$\limsup_{n\to\infty}\frac{v_n^2}{n}\log N(n) \leq b \tag{16}$$

The application of Lemma 4 then yields

$$\begin{aligned}\limsup_{n\to\infty}\frac{v_n^2}{n}\log\mathbb{P}\left[\sup_{x\in U,\|x\|\leq w_n} v_n\left|\Psi_n(x)\right| \geq \delta\right] &\leq \limsup_{n\to\infty}\frac{v_n^2}{n}\log N(n) - \tilde{g}_U(\delta-\rho)\\ &\leq b - \tilde{g}_U(\delta-\rho).\end{aligned}$$

Since the inequality holds for any $\rho \in]0,\delta[$, part 2 of Lemma 5 thus follows from the continuity of $\tilde{g}_U$.

Let us now consider part 1 of Lemma 5. This part is proved by following the same steps as for part 2, except that the number $N(n)$ of balls covering U is at most the integer part of (Δ/R_n), where Δ denotes the diameter of $\overline{U}$. Relation (16) then becomes

$$\limsup_{n\to\infty}\frac{v_n^2}{n}\log R_n \leq 0$$

and Lemma 5 is proved.

Lemma 6. Let $(U1)\,i)$, $(M2)$ and $(U6)\,i)$ hold. Assume that either $(v_n)\equiv 1$ or $(U3)$ and $(U6)\,ii)$ hold. Moreover assume that F is continuous. For any $b>0$ if we set $w_n = \exp\left(b\frac{n}{v_n^2}\right)$ then, for any $\rho>0$, we have, for n large enough,

$$\sup_{x\in U,\|x\|\geq w_n}\frac{v_n}{n}\sum_{k=1}^{n}\left|\mathbb{E}\left[\mathcal{K}\left(\frac{x-X_k}{h_n}\right)\right]\right| \leq \rho$$

Proof of Lemma 6. We have

$$\frac{v_n}{n}\sum_{k=1}^{n}\mathbb{E}\left[\mathcal{K}\left(\frac{x-X_k}{h_n}\right)\right] = v_n\int_{\mathbb{R}} K(z)\,F(x+zh_n)\,dz. \tag{17}$$

Set $\rho > 0$. In the case $(v_n) \equiv 1$, we set M such that $\|F\|_\infty \int_{\|z\|>M} |K(z)|\, dz \le \rho/2$; it follows that

$$\frac{v_n}{n}\sum_{k=1}^{n}\left|\mathbb{E}\left[\mathcal{K}\left(\frac{x-X_k}{h_n}\right)\right]\right|$$
$$\le \frac{\rho}{2} + F(x)\int_{\|z\|\le M}|K(z)|\, dz + \int_{\|z\|>M}|K(z)|\,|F(x+zh_n)-F(x)|\, dz.$$

Lemma 6 then follows from the fact that F fulfills $(U6)\, ii)$. As matter of fact, this conditions implies that $\lim_{\|x\|\to\infty, x\in\overline{U}} F(x) = 0$ and that the third term in the right-hand-side of the previous inequality goes to 0 as $n\to\infty$ (by the dominated convergence).
Let us now assume that $\lim_{n\to\infty} v_n = \infty$; relation (17) can be rewritten as

$$\begin{aligned}\frac{v_n}{n}\sum_{k=1}^{n}\mathbb{E}\left[\mathcal{K}\left(\frac{x-X_k}{h_n}\right)\right] &= v_n\int_{\|z\|\le w_n/2} K(z)\, F(x+zh_n)\, dz \\ &\quad + v_n\int_{\|z\|\ge w_n/2} K(z)\, F(x+zh_n)\, dz.\end{aligned}$$

First, since $\|x\| \ge w_n$ and $\|z\| \le w_n/2$, we have

$$\begin{aligned}\|x+zh_n\| &\ge w_n(1-h_n/2) \\ &\ge w_n/2 \quad \text{for } n \text{ large enough.}\end{aligned}$$

Moreover, in view of assumptions $(U3)$, for all $\xi > 0$,

$$\lim_{n\to\infty}\frac{v_n}{w_n^{\xi}} = \lim_{n\to\infty}\exp\left\{-b\xi\frac{n}{v_n^2}\left(1-\frac{1}{b\xi}\frac{v_n^2\log v_n}{n}\right)\right\} = 0. \tag{18}$$

Set $M_f = \sup_{x\in\mathbb{R}}\|x\|^{\eta}F(x)$. Assumption $(U6)\, ii)$ and equation (18) implie that, for n sufficiently large,

$$\begin{aligned}&\sup_{\|x\|\ge w_n} v_n\int_{\|z\|\le w_n/2}|K(z)\, F(x+zh_n)|\, dz \\ &\le M_f\sup_{\|x\|\ge w_n} v_n\int_{\|z\|\le w_n/2}|K(z)|\,\|x+zh_n\|^{-\eta}dz \\ &\le 2^{\eta}M_f\frac{v_n}{w_n^{\eta}}\int_{\mathbb{R}}|K(z)|\, dz \\ &\le \frac{\rho}{2}.\end{aligned}$$

Moreover, in view of $(U3)$, $(U6)\,i)$ and (18), for n sufficiently large,

$$\begin{aligned}\sup_{\|x\|\geq w_n} v_n \int_{\|z\|>w_n/2} &\left|K(z)\,F(x+zh_n)\right| dz \\ &\leq 2^{\zeta} M_f \frac{v_n}{w_n^{\zeta}} \int_{\|z\|>w_n/2} \|z\|^{\zeta} \left|K(z)\right| dz \\ &\leq \frac{\rho}{2}.\end{aligned}$$

This concludes the proof of Lemma 6. Since K is a bounded function that vanishes at infinity, we have $\lim_{\|x\|\to\infty} |\Psi_n(x)| = 0$ for every $n \geq 1$. Moreover, since K is assumed to be continuous, Ψ_n is continuous, and this ensures the existence of a random variable s_n such that

$$|\Psi_n(s_n)| = \sup_{x\in U} |\Psi_n(x)|.$$

Lemma 7.
Let Assumptions $(U1)-(U3)$, $(U4)\,ii)$ and $(U5)$ hold. Suppose either $(v_n) \equiv 1$ or $(H6)$ hold. For any $b > 0$, set $w_n = \exp\left(b\frac{n}{v_n^2}\right)$; for any $\delta > 0$, we have

$$\limsup_{n\to\infty} \frac{v_n^2}{n} \log \mathbb{P}\left[\|s_n\| \geq w_n \quad \text{and} \quad |\Psi_n(s_n)| \geq \delta\right] \quad \leq \quad -b\beta \tag{19}$$

Proof of Lemma 7. We first note that $s_n \in \overline{U}$ and therefore

$$\begin{aligned}&\|s_n\| \geq w_n \quad \text{and} \quad v_n\left|\Psi_n(s_n)\right| \geq \delta \\ &\quad \Rightarrow \|s_n\| \geq w_n \quad \text{and} \quad \frac{v_n}{n}\left|\sum_{k=1}^{n}\mathcal{K}\left(\frac{s_n-X_k}{h_n}\right)\right| + \frac{v_n}{n}\mathbb{E}\left|\sum_{k=1}^{n}\mathcal{K}\left(\frac{s_n-X_k}{h_n}\right)\right| \geq \delta \\ &\quad \Rightarrow \|s_n\| \geq w_n \quad \text{and} \quad \frac{v_n}{n}\sum_{k=1}^{n}\left|\mathcal{K}\left(\frac{s_n-X_k}{h_n}\right)\right| \geq \delta \\ &\qquad - \sup_{\|x\|\geq w_n, x\in\overline{U}} \frac{v_n}{n}\sum_{k=1}^{n}\mathbb{E}\left|\mathcal{K}\left(\frac{s_n-X_k}{h_n}\right)\right|\end{aligned}$$

Set $\rho \in\,]0,\delta[$; the application of Lemma 6 ensures that, for n large enough,

$$\begin{aligned}&\|s_n\| \geq w_n \quad \text{and} \quad v_n\left|\Psi_n(s_n)\right| \geq \delta \\ &\quad \Rightarrow \|s_n\| \geq w_n \quad \text{and} \quad \frac{v_n}{n}\left|\sum_{k=1}^{n}\mathcal{K}\left(\frac{s_n-X_k}{h_n}\right)\right| \geq \delta-\rho.\end{aligned}$$

Set $\kappa = \sup_{x\in\mathbb{R}} \|x\|^{\gamma} |\mathcal{K}(x)|$ (see Assumption $(U5)$). We obtain, for n sufficiently large,

$$
\begin{aligned}
&\|s_n\| \geq w_n \quad \text{and} \quad v_n |\Psi_n(s_n)| \geq \delta \\
&\quad \Rightarrow \|s_n\| \geq w_n \quad \text{and} \quad \exists k \in \{1, \dots, n\} \quad \text{such that} \quad v_n \left|\mathcal{K}\left(\frac{s_n - X_k}{h_n}\right)\right| \geq \delta - \rho \\
&\quad \Rightarrow \|s_n\| \geq w_n \quad \text{and} \quad \exists k \in \{1, \dots, n\} \quad \text{such that} \quad \kappa h_n^{\gamma} \geq v_n^{-1} \|s_n - X_k\|^{\gamma} (\delta - \rho) \\
&\quad \Rightarrow \|s_n\| \geq w_n \quad \text{and} \quad \exists k \in \{1, \dots, n\} \quad \text{such that} \quad \left|\|s_n\| - \|X_k\|\right| \leq \left[\frac{\kappa v_n h_n^{\gamma}}{\delta - \rho}\right]^{\frac{1}{\gamma}} \\
&\quad \Rightarrow \|s_n\| \geq w_n \quad \text{and} \quad \exists k \in \{1, \dots, n\} \quad \text{such that} \quad \|X_k\| \leq \|s_n\| - \left[\frac{\kappa v_n h_n^{\gamma}}{\delta - \rho}\right]^{\frac{1}{\gamma}} \\
&\quad \Rightarrow \|s_n\| \geq w_n \quad \text{and} \quad \exists k \in \{1, \dots, n\} \quad \text{such that} \quad \|X_k\| \leq w_n (1 - u_{n,k}) \quad \text{with} \\
&\qquad u_{n,k} = w_n^{-1} v_n^{\frac{1}{\gamma}} h_n \left(\frac{\kappa}{\delta - \rho}\right)^{\frac{1}{\gamma}}.
\end{aligned}
$$

Moreover, we can write $u_{n,k}$ as

$$
u_{n,k} = \exp\left(-b\frac{n}{v_n^2}\left[1 - \frac{v_n^2 \log v_n}{n}\frac{1}{b\gamma} - \frac{v_n^2 \log(h_n)}{n}\frac{1}{b}\right]\right)\left(\frac{\kappa}{\delta - \rho}\right)^{\frac{1}{\gamma}}
$$

and assumption $(U3)$ ensure that $\lim_{n\to\infty} u_{n,k} = 0$, it then follows that $1 - u_{n,k} > 0$ for n sufficiently large; therefore we can deduce that (see Assumption $(U4)\,i)$):

$$
\begin{aligned}
\mathbb{P}\left[\|s_n\| \geq w_n \quad \text{and} \quad v_n |\Psi_n(s_n)| \geq \delta\right] &\leq \sum_{i=1}^{n} \mathbb{P}\left[\|X_k\|^{\beta} \geq w_n^{\beta} (1 - u_{n,k})^{\beta}\right] \\
&\leq \sum_{i=1}^{n} \mathbb{E}\left(\|X_k\|^{\beta}\right) w_n^{-\beta} (1 - u_{n,k})^{-\beta} \\
&\leq n\mathbb{E}\left(\|X_1\|^{\beta}\right) w_n^{-\beta} \max_{1\leq k\leq n} (1 - u_{n,k})^{-\beta}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\frac{v_n^2}{n} \log \mathbb{P}\left[\|s_n\| \geq w_n \quad \text{and} \quad v_n |\Psi_n(s_n)| \geq \delta\right] \\
&\qquad \leq \frac{v_n^2}{n}\left[\log n + \log \mathbb{E}\left(\|X_1\|^{\beta}\right) - b\beta n v_n^2 - \beta \log \max_{1\leq k\leq n} (1 - u_{n,k})\right],
\end{aligned}
$$

and, thanks to assumptions $(U3)$, it follows that

$$
\limsup_{n\to\infty} \frac{v_n^2}{n} \log \mathbb{P}\left[\|s_n\| \geq w_n \quad \text{and} \quad v_n |\Psi_n(s_n)| \geq \delta\right] \quad \leq \quad -b\beta,
$$

which concludes the proof of Lemma 7.

3.4 Proof of Proposition 2

Let us at first note that the lower bound

$$\liminf_{n\to\infty} \frac{v_n^2}{n} \log \mathbb{P}\left[\sup_{x\in U} v_n \left|\Psi_n(x)\right| \geq \delta\right] \geq -\tilde{g}_U(\delta) \tag{20}$$

follows from the application of Proposition 1 at a point $x_0 \in \overline{U}$ such that $F(x_0) = \|F\|_{U,\infty}$.
In the case U is bounded, Proposition 2 is thus a straightforward consequence of (20) and the first part of Lemma 5. Let us now consider the case U is unbounded.
Set $\delta > 0$ and, for any $b > 0$ set $w_n = \exp\left(b\frac{n}{v_n^2}\right)$. Since, by definition of s_n,

$$\begin{aligned}
&\mathbb{P}\left[\sup_{x\in U} v_n \left|\Psi_n(x)\right| \geq \delta\right] \\
&\quad \leq \mathbb{P}\left[\sup_{x\in U, \|x\|\leq w_n} v_n \left|\Psi_n(x)\right| \geq \delta\right] + \mathbb{P}\left[\|s_n\| \geq w_n \text{ and } v_n \left|\Psi_n(x)\right| \geq \delta\right],
\end{aligned}$$

it follows from Lemmas 5 and 7 that

$$\limsup_{n\to\infty} \frac{v_n^2}{n} \log \mathbb{P}\left[\sup_{x\in U} v_n \left|\Psi_n(x)\right| \geq \delta\right] \leq \max\left\{-b\beta; b - \tilde{g}_U(\delta)\right\}$$

and consequently

$$\limsup_{n\to\infty} \frac{v_n^2}{n} \log \mathbb{P}\left[\sup_{x\in U} v_n \left|\Psi_n(x)\right| \geq \delta\right] \leq \inf_{b>0} \max\left\{-b\beta; b - \tilde{g}_U(\delta)\right\}.$$

Since the infimum in the right-hand-side of the previous bound is achieved for $b = \tilde{g}_U(\delta)/(\beta+1)$ and equals $-\beta\tilde{g}_U(\delta)/(\beta+1)$, we obtain the upper bound

$$\limsup_{n\to\infty} \frac{v_n^2}{n} \log \mathbb{P}\left[\sup_{x\in U} v_n \left|\Psi_n(x)\right| \geq \delta\right] \leq -\frac{\beta}{\beta+1}\tilde{g}_U(\delta)$$

which concludes the proof of Proposition 2.

3.5 Proof of Proposition 3

It follows from (1), that

$$\begin{aligned}
\mathbb{E}\left[F_n(x)\right] &= \int_{\mathbb{R}} \mathcal{K}\left(\frac{x-y}{h_n}\right) f(y)\,dy \\
&= \int_{\mathbb{R}} K(z)\, F(x + zh_n)\,dz \\
&= F(x) + \frac{1}{2} h_n^2 F^{(2)}(x) \int_{\mathbb{R}} z^2 K(z)\,dz + \eta(x) \qquad (21)
\end{aligned}$$

with

$$\eta(x) = \int_{\mathbb{R}} \left[F(x + zh_n) - F(x) - zh_n F'(x) - \frac{1}{2} z^2 h_n^2 F^{(2)}(x) \right] K(z)\, dz$$

Since F is continuous, we have $\lim_{n\to\infty} \left| F(x + zh_n) - F(x) - zh_n F'(x) - \frac{1}{2} z^2 h_n^2 F^{(2)}(x) \right| = 0$, and thus by the dominated convergence theorem, we have $\lim_{n\to\infty} \eta(x) = 0$, and thus Part 1 of Proposition 3 is completed. Since $\sup_{x\in\mathbb{R}} \|F^{(2)}(x)\| < +\infty$, Part 2 follows.

References

A. Azzalini, A note on the estimation of a distribution function and quantiles by a kernel method, *Biometrika*, **68** (1981), 326–328.

R. Bojanic and E. Seneta, A unified theory of regularly varying sequences, *Math. Z*, **134** (1973), 91–106.

A. Dembo and O. Zeitouni, Large deviations techniques and applications. *Springer, Application of mathematics, New-York*, 1998

M. Falk, Relative efficiency and deficiency of kernel type estimator of smooth distribution functions, *Stat. Neerl*, **37** (1983), 73–83.

J. Galambos and E. Seneta, Regularly varying sequences, *Amer. Math. Soc*, **41** (1973), 110–116.

F. Gao, Moderate deviations and large deviations for kernel density estimators, *J. Theoret. Probab*, **16** (2003), 401–418.

P.D. Hill, Kernel estimation of a distribution function, *Commun. Stat-Theor. M*, **14** (1985), 605–620.

D. Louani, Large deviations limit theorems for the kernel density estimator, *Scand. J. Statist*, **25** (1998), 243-253.

A. Mokkadem & M. Pelletier and J. Worms, Large and moderate deviations principles for kernel estimation of a multivariate density and its partial derivatives, *Austral. J. Statist*, **4** (2005), 489–502.

A. Mokkadem, M. Pelletier and B. Thiam, Large and moderate deviations principles for recursive kernel estimation of a multivariate density and its partial derivatives, *Serdica Math. J*, **32** (2006), 323–354.

E.A. Nadaraya, Some new estimates for distribution functions. *Theory Prob. Appl*, **9** (1964), 497–500.

E. Parzen, On estimate of a probability density and mode, *Ann. Math. Statist*, **33** (1962), 1065–1076.

A.A. Puhalskii, The method of stochastic exponentials for large deviations, *Stochastic Process. Appl*, **54** (1994), 45–70.

R.D. Reiss, Nonparametric estimation of smooth distribution function, *Scand. J. Statist*, **8** (1981), 116–119.

M. Rosenblatt, Remarks on some nonparametric estimate of a density function, *Ann. Math. Statist*, **27** (1956), 832–837.

J. Worms, Moderate and large deviations of some dependent variables, Part II: Some kernel estimators, *Math. Methods Statist*, **10** (2001), 161–193.

Received: May 1, 2014