Matrix integrals, Toda symmetries, Virasoro constraints and orthogonal polynomials

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Symmetries of the infinite Toda lattice. The symmetries for the infinite Toda lattice,

$$\frac{\partial L}{\partial t_n} = [\frac{1}{2}(L^n)_s, L], \quad n = 1, 2, \ldots, \tag{0.1}$$

viewed as isospectral deformations of bi-infinite tridiagonal matrices L, are time-dependent vector fields transversal to the Toda hierarchy; bracketing a symmetry with a Toda vector field yields another vector field in the hierarchy. As is well known, the Toda hierarchy is intimately related to the Lie algebra splitting of $gl(\infty)$,

$$gl(\infty) = \mathcal{D}_s \oplus \mathcal{D}_b \ni A = A_s + A_b, \tag{0.2}$$

into the algebras of skew-symmetric A_s and lower triangular (including the diagonal) matrices A_b (Borel matrices). We show that this splitting plays a prominent role also in the construction of the Toda symmetries and their action on $\tau-$functions; it also plays a crucial role in obtaining the Virasoro constraints for matrix integrals and it ties up elegantly with the theory of orthogonal polynomials .

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Define matrices δ and ε, with $[\delta, \varepsilon] = 1$, acting on characters $\chi(z) = (\chi_n(z))_{n\in\mathbf{Z}} = (z^n)_{n\in\mathbf{Z}}$ as

$$\delta\chi = z\chi \text{ and } \varepsilon\chi = \frac{\partial}{\partial z}\chi \ . \tag{0.3}$$

This enables us to define a wave operator S, a wave vector Ψ,

$$L = S\delta S^{-1} \quad \text{and} \quad \Psi = S \exp^{\frac{1}{2}\sum_1^\infty t_i z^i} \chi(z), \tag{0.4}$$

and an operator M, reminiscent of Orlov and Schulman's M-operator for the KP-equation, such that

$$L\Psi = z\Psi \quad \text{and} \quad M\Psi = \frac{\partial}{\partial z}\Psi, \tag{0.5}$$

thus leading to identities of the form:

$$M^\beta L^\alpha \Psi = z^\alpha (\frac{\partial}{\partial z})^\beta \Psi. \tag{0.6}$$

The vector Ψ and the matrices S, L and M evolve in a way, which is compatible with the algebra splitting above,

$$\frac{\partial \Psi}{\partial t_n} = \frac{1}{2}(L^n)_s \Psi \quad \text{and} \quad \frac{\partial S}{\partial t_n} = -\frac{1}{2}(L^n)_b S, \tag{0.7}$$

$$\frac{\partial L}{\partial t_n} = [\frac{1}{2}(L^n)_s, L] \quad \text{and} \quad \frac{\partial M}{\partial t_n} = [\frac{1}{2}(L^n)_s, M], \tag{0.8}$$

and the wave vector Ψ has the following representation in terms of a vector[1] of τ-functions $\tau = (\tau_n)_{n\in Z}$:

$$\Psi(t,z) = e^{\frac{1}{2}\Sigma t_i z^i} \Big(z^n \frac{e^{-\eta}\tau_n(t)}{\sqrt{\tau_n(t)\tau_{n+1}(t)}} \Big)_{n\in\mathbf{Z}} \equiv (z^n \Psi_n)_{n\in\mathbf{Z}}. \tag{0.9}$$

The wave vector defines a t-dependent flag

$$\ldots \supset W_{k-1}^t \supset W_k^t \supset W_{k+1}^t \supset \ldots$$

[1] with $\eta = \sum_1^\infty \frac{z^{-i}}{i} \frac{\partial}{\partial t_i}$; note the 1/2 appearing in Ψ

of nested linear spaces, spanned by functions of z,

$$W_k^t \equiv \mathrm{span}\{z^k\Psi_k, z^{k+1}\Psi_{k+1}, \ldots\} \tag{0.10}$$

Formula (0.6) motivates us to give the following definition of symmetry vector fields (symmetries), acting on the manifold of wave functions Ψ and inducing a Lax pair on the manifold of L-operators [2]:

$$\mathbf{Y}_{z^\alpha(\frac{\partial}{\partial z})^\beta}\Psi = -(M^\beta L^\alpha)_b\Psi \quad \text{and} \quad \mathbf{Y}_{z^\alpha(\frac{\partial}{\partial z})^\beta}L = [-(M^\beta L^\alpha)_b, L]. \tag{0.11}$$

It turns out that *only the vector fields*

$$\mathbf{Y}_{\ell,n} := \mathbf{Y}_{z^{n+\ell}(\frac{\partial}{\partial z})^n} = -(M^n L^{n+\ell})_b, \text{for } n = 0, \ell \ \in \mathbf{Z} \ \text{ and } \ n = 1, \ell \geq -1, \tag{0.12}$$

conserve the tridiagonal nature of the matrices L. The expressions (0.12), for $n = 1$, $\ell < -1$ have no geometrical meaning, as the corresponding vector fields move you out of the space of tridiagonal matrices. This phenomenon is totally analogous to the KdV case (or pth Gel'fand-Dickey), where a certain algebra of symmetries, a representation of the sub-algebra [3] $\mathrm{Diff}(S^1)^+ \subset \mathrm{Diff}(S^1)$ of holomorphic vector fields on the circle, maintains the differential nature of the 2nd order operator $\frac{\partial^2}{\partial x^2} + q(x)$ (or pth order differential operator).

According to a non-commutative Lie algebra splitting theorem, due to ([A-S-V]), stated in section 2 and adapted to the Toda lattice, we have a Lie algebra anti-homomorphism:

(0.13)

$$w_2 \equiv = \{z^{n+\ell}(\frac{\partial}{\partial z})^n, n = 0, \ell \in \mathbf{Z}, \text{or } n = 1, \ell \geq -1\}$$

$$\rightarrow \{\text{tangent vector fields on the } \Psi\text{-manifold}\} :$$

$$z^{n+\ell}(\frac{\partial}{\partial z})^n \mapsto \mathbf{Y}_{z^{n+\ell}(\frac{\partial}{\partial z})^n}, \qquad \text{acting on } \Psi \text{ as in (0.11)},$$

to wit,

$$[\mathbf{Y}_{\ell,0}, \mathbf{Y}_{m,1}] = \ell\mathbf{Y}_{m+\ell,0} \quad \text{and} \quad [\mathbf{Y}_{\ell,1}, \mathbf{Y}_{m,1}] = (\ell - m)\mathbf{Y}_{m+\ell,1}. \tag{0.14}$$

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[2] sometimes $\mathbf{Y}_{M^\beta L^\alpha}$ will stand for $\mathbf{Y}_{z^\alpha(\frac{\partial}{\partial z})^\beta}$

[3] $\mathrm{Diff}(S^1)^+ := \mathrm{span}\{z^{k+1}\frac{\partial}{\partial z}, \ k \geq -1\}$

Transferring symmetries from the wave vector to the τ-function . An important part of the paper (section 3) is devoted to understanding how, in the general Toda context, the symmetries $\mathbf{Y}_{\ell,n}$ acting on the manifold of wave vectors Ψ induce vector fields $\mathcal{L}_{\ell,n}$ on the manifold of τ-vectors $\tau=(\tau_j)_{j\in\mathbf{Z}}$, for $n=0$ or 1; this new result is contained in Theorem 3.2:

(0.15)
$$\mathbf{Y}_{\ell,n}\log\Psi=(e^{-\eta}-1)\mathcal{L}_{\ell,n}\log\tau+\frac{1}{2}\Big(\mathcal{L}_{\ell,n}\log(\frac{\tau}{\tau_\delta})\Big). \qquad (\textit{Fundamental relation})$$

for $n=0,\ \ell\in\mathbf{Z}$ and $n=1,\ \ell\geq-1$

where " $\mathbf{Y}_{\ell,n}\log$" and "$\mathcal{L}_{\ell,n}\log$" act as logarithmic derivatives[4], where $\mathcal{L}_{\ell,n}f=(\mathcal{L}^j_{\ell,n}f_j)_{j\in\mathbf{Z}}$, and [5]

(0.16) $$\mathcal{L}^j_{\ell,1}=J^{(2)}_\ell+(2j-\ell-1)J^{(1)}_\ell+(j^2-j)J^{(0)}_\ell,\quad \mathcal{L}^j_{\ell,0}=2J^{(1)}_\ell+2jJ^{(0)}_\ell.$$

Note the validity of the relation not only for infinite matrices, but also for semi-infinite matrices. Also note the robustness of formula (0.15): it has been shown to be valid in the KP-case (continuous) and the 2-dimensional Toda lattice (discrete) by [A-S-V]. We give here an independent proof of this relation, although it could probably have been derived from the [A-S-V]-vertex operator identity for the two-dimensional Toda lattice. The rest of the paper will be devoted to an application of the fundamental relation.

Orthogonal polynomials, skew-symmetric matrices and Virasoro constraints. Consider now in section 4 an orthonormal polynomial basis $(p_n(t,z))_{n\geq0}$ of $\mathcal{H}^+\equiv\{1,z,z^2,\dots\}$ with regard to the weight $\rho_0(z)e^{\sum_0^\infty t_iz^i}dz=e^{-V_0+\sum_0^\infty t_iz^i}$ on the interval $[a,b]$, $-\infty\leq a<b\leq\infty$, satisfying:
(0.17)
$$-\frac{\rho_0'}{\rho_0}=V_0'=\frac{\sum_0^\infty b_iz^i}{\sum_0^\infty a_iz^i}=:\frac{h_0}{f_0}\quad\text{and}\quad f_0(a)\rho(a)a^k=f_0(b)\rho(b)b^k=0\ (k=0,1,\dots).$$

The polynomials $p_n(t,z)$ are t-deformations of $p_n(0,z)$, through the exponential in the weight. Then the semi-infinite vector Ψ and semi-infinite matrices L and M,

[4]for instance $\mathbf{Y}_{\ell,n}\log\Psi_j:=\frac{-\left(((M^nL^{n+\ell})_b\Psi\right)_j}{\Psi_j}$ and $\Big(\mathcal{L}_{\ell,n}\log(\frac{\tau}{\tau_\delta})\Big)_j\equiv\frac{\mathcal{L}_{\ell,n}\tau_j}{\tau_j}-\frac{\mathcal{L}_{\ell,n}\tau_{j+1}}{\tau_{j+1}}$

[5]set $J^{(0)}_n=\delta_{n,0}$, $J^{(1)}_n=\frac{\partial}{\partial t_n}+\frac{1}{2}(-n)t_{-n}$, and $J^{(2)}_n=\Sigma_{i+j=n}:J^{(1)}_iJ^{(1)}_j:$

defined by

$$(0.18)\qquad \Psi(t,z)\equiv \exp^{\frac{1}{2}\sum_1^\infty t_i z^i}\left(p_n(t,z)\right)_{n\geq 0},\quad z\Psi=L\Psi\quad\text{and}\quad \frac{\partial}{\partial z}\Psi=M\Psi$$

are solutions of the Toda differential equations (0.7) and (0.8). Moreover, $\Psi(t,z)$ can be represented by (0.9) with $\tau_0=1$ and

$$(0.19)\qquad \tau_n=\frac{1}{\Omega_n n!}\int_{\mathcal{M}_n(a,b)} dZ\, e^{-Tr\,V_0(Z)+\sum_1^\infty t_i\,Tr\,Z^i},\quad n\geq 1;$$

here the integration is taken over a subspace $\mathcal{M}_n(a,b)$ of the space of $n\times n$ Hermitean matrices Z, with eigenvalues $\in[a,b]$.

We prove in Theorem 4.2 that in terms of the matrices L and M, defined in (0.18) and in terms of the anti-commutator $\{A,B\}:=\frac{1}{2}(AB+BA)$, the semi-infinite matrices[6]

(0.20)

$$V_m:=\{Q,L^{m+1}\}=QL^{m+1}+\frac{m+1}{2}L^m f_0(L),\ \text{ with }\ V_{-1}=Q:=Mf_0(L)+\frac{(f_0\rho_0)'}{2\rho_0}(L).$$

are skew-symmetric for $m\geq -1$ and form a representation of the Lie algebra of holomorphic vector fields $\mathrm{Diff}(S^1)^+$, i.e. they satisfy

$$(0.21)\qquad [V_m,V_n]=(n-m)\sum_{i\geq 0}a_i V_{m+n+i},\quad m,n\geq -1.$$

Thus, in terms of the splitting (0.2), we have for orthogonal polynomials the following identities:

$$(0.22)\qquad (V_m)_b=0,\quad \text{for all}\quad m\geq -1,$$

leading to the *vanishing* of a whole algebra of symmetry vector fields $\mathbf{Y}_{V_m}$ on the locus of wave functions Ψ, defined in (0.18); then using the fundamental relation (0.15) to transfer the vanishing statement to the τ-functions τ_n, we find the Virasoro-type constraints for the τ_n, $n\geq 0$ and for $m=-1,0,1,2,\ldots$:

$$(0.23)\qquad \sum_{i\geq 0}\left(a_i(J^{(2)}_{i+m}+2n\,J^{(1)}_{i+m}+n^2\,J^{(0)}_{i+m})-b_i(J^{(1)}_{i+m+1}+n\,J^{(0)}_{i+m+1})\right)\tau_n=0,$$

[6]the matrices V_m are not to be confused with the potential V_0, appearing in the weight.

in terms of the coefficients a_i and b_i of f_0 and h_0 (see (0.18)). In his fundamental paper [W], Witten had observed, as an incidental fact, that in the case of Hermite polynomials $V_{-1} = M - L$ is a skew-symmetric matrix. In this paper we show that skew-symmetry plays a crucial role; in fact the *Virasoro constraints* (0.23) are tantamount to *the skew-symmetry of the semi-infinite matrices* (0.20).

They can also be obtained, with uninspired tears, upon substituting

$$Z \mapsto Z + \varepsilon f_0(Z) Z^{k+1} \tag{0.24}$$

in the integrand of (0.19); then the linear terms in ε in the integral (0.19) must vanish and yield the same Virasoro-type constraints (0.23) for each of the integrals τ_n as is carried out in the appendix.

At the same time, the methods above solve the "*string equation*", which is : for given f_0, find two semi-infinite matrices, a symmetric L and a skew-symmetric Q, satisfying

$$[L, Q] = f_0(L). \tag{0.25}$$

For the **classical orthogonal polynomials**, as explained in section 6, the matrices L and Q, matrix realizations of the operators z and $\sqrt{\frac{f_0}{\rho_0}} \frac{\partial}{\partial z} \sqrt{\rho_0 f_0}$ respectively, acting on the space of polynomials, are both tridiagonal, with L symmetric and Q skew-symmetric. In addition, the matrices L and Q stabilize the flag, defined in (0.10), in the following sense:

$$zW_k \subset W_{k-1} \quad \text{and} \quad T_{-1} W_k \subset W_{k-1}. \tag{0.28}$$

This result is related to a classical Theorem of Bochner; see [C].

The results in this paper have been lectured on at CIMPA (1991), Como, Utrecht (1992) and Cortona (1993); see the lecture notes [vM]. Grinevich, Orlov and Schulman made a laconic remark in a 1993 paper [GOS, p. 298] about defining symmetries for the Toda lattice. We thank A. Grünbaum, L. Haine, V. Kac, A. Magnus, A. Morozov, T. Shiota, Cr. Tracy and E. Witten for conversations, regarding various aspects of this work. We also thank S. D'Addato-Muës for unscrambling an often messy manuscript.

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1 The Toda lattice revisited

On $\mathbf{Z}$ we define the function χ (character)

$$\chi : \mathbf{Z} \times \mathbf{C} \to \mathbf{C} : (n, z) \mapsto \chi_n(z) = z^n$$

and the matrix-operators δ and ε

(1.1)

$$\delta = \left(\begin{array}{ccc|cccc} \ddots & & & & & & \\ & 0 & 1 & & & & \\ & & 0 & 1 & & & \\ \hline & & & 0 & 1 & & \\ & & & & 0 & 1 & \\ & & & & & 0 & \\ & & & & & & \ddots \end{array}\right) \quad \text{and } \varepsilon = \left(\begin{array}{ccc|cccc} \ddots & & & & & & \\ -2 & 0 & & & & & \\ & -1 & 0 & & & & \\ \hline & & 0 & 0 & & & \\ & & & 1 & 0 & & \\ & & & & 2 & 0 & \\ & & & & & & \ddots \end{array}\right)$$

acting on χ as

$$\delta\chi = z\chi \text{ and } \varepsilon\chi = \frac{\partial}{\partial z}\chi; \tag{1.2}$$

they satisfy

$$[\delta, \varepsilon] = 1.$$

Proposition 1.1. *The infinite matrix*[7]

$$\tilde{L} = \sum_{j \leq 1} a_j \delta^j \in \mathcal{D}_{-\infty,1}, \qquad a_j(t) = \ diag(a_j(n,t))_{n \in \mathbf{Z}}, a_1 = 1, \tag{1.3}$$

[7] $\mathcal{D}_{k,\ell}$ ($k < \ell \in \mathbf{Z}$) denotes the set of band matrices with zeros outside the strip (k, ℓ). The symbols $(\ )_+, (\ )_-, (\ )_0$ denote the projection of a matrix onto $\mathcal{D}_{0,\infty}$, $\mathcal{D}_{-\infty,-1}$ and $\mathcal{D}_{00}$ respectively.

subjected to the deformation equations

$$\frac{\partial \tilde{L}}{\partial t_n} = [(\tilde{L}^n)_+, \tilde{L}] = [-(\tilde{L}^n)_-, \tilde{L}], \qquad n = 1, 2, \dots \ . \tag{1.4}$$

has, for generic initial conditions, a representation in terms of τ*-functions* τ_n

$$\tilde{L} = \tilde{S}\delta\tilde{S}^{-1} = \Big(\dots, (\frac{\gamma_n}{\gamma_{n-1}})^2, \frac{\partial}{\partial t_1}\log\gamma_n^2, 1, 0, \dots\Big)_{n\in\mathbf{Z}}, \tag{1.5}$$

where[8]

$$\gamma_n = \sqrt{\frac{\tau_{n+1}}{\tau_n}}, \quad \tilde{S} = \frac{\tau(t-[\delta^{-1}])}{\tau(t)} = \frac{\sum_{n=0}^{\infty} p_n(-\tilde{\partial})\tau(t)\delta^{-n}}{\tau(t)} \tag{1.6}$$

The wave operator $\tilde{S}$ *and the wave vector*[9]

$$\tilde{\Psi} := e^{\sum_1^\infty t_i z^i}\tilde{S}\chi(z) = \Big(z^n e^{\Sigma}\frac{e^{-\eta}\tau_n}{\tau_n}\Big)_{n\in\mathbf{Z}} =: (z^n\tilde{\Psi}_n)_{n\in\mathbf{Z}} \tag{1.7}$$

satisfy

$$\tilde{L}\tilde{\Psi} = z\tilde{\Psi}, \quad \frac{\partial\tilde{\Psi}}{\partial t_n} = (\tilde{L}^n)_+\tilde{\Psi}, \quad \frac{\partial\tilde{S}}{\partial t_n} = -(\tilde{L}^n)_- S. \tag{1.8}$$

Proof: The proof of this statement can be deduced from the work of Ueno-Takasaki [U-T]; we consider only a few points: from equation (1.6), it follows that:

$$\tilde{S} = I - A\delta^{-1} - B\delta^{-2} - \dots, \quad \text{and} \quad \tilde{S}^{-1} = I + A\delta^{-1} + B\delta^{-2} + A\delta^{-1}A\delta^{-1} + \dots \ . \tag{1.9}$$

with

$$A = \frac{\partial}{\partial t_1}\log\tau, \quad \text{and} \quad B = -\frac{p_2(-\tilde{\partial})\tau(t)}{\tau(t)}.$$

Then, calling $p_k\tau := p_k(-\tilde{\partial})\tau$, we have

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[8] The p_k's are the elementary Schur polynomials $e^{\sum_1^\infty t_i z^i} = \sum_0^\infty p_k(t)z^k$ and $p_k(-\tilde{\partial}) := p_k(-\frac{\partial}{\partial t_1}, -\frac{1}{2}\frac{\partial}{\partial t_2}, -\frac{1}{3}\frac{\partial}{\partial t_3}, \dots)$. Also $[\alpha] := (\alpha, \frac{\alpha^2}{2}, \frac{\alpha^3}{3}, \dots)$.

[9] set $\eta = \sum_1^\infty \frac{z^{-i}}{i}\frac{\partial}{\partial t_i}$ and $\Sigma = \sum_1^\infty t_j z^j$

(1.10)

$$\begin{aligned}\tilde{L} = \tilde{S}\delta\tilde{S}^{-1} &= \delta + (A_{n+1} - A_n)_{n\in\mathbf{Z}}\delta^0 + (B_{n+1} - B_n + A_nA_{n+1} - A_n^2)_{n\in\mathbf{Z}}\delta^{-1} + \dots \\ &= \delta + \Big(\frac{\partial}{\partial t_1}\log\frac{\tau_{n+1}}{\tau_n}\Big)_{n\in\mathbf{Z}}\delta^0 \\ &\quad + \Big(-\frac{p_2\tau_{n+1}}{\tau_{n+1}} + \frac{p_2\tau_n}{\tau_n} + \frac{p_1\tau_n}{\tau_n}\frac{p_1\tau_{n+1}}{\tau_{n+1}} - (\frac{p_1\tau_n}{\tau_n})^2\Big)_{n\in\mathbf{Z}}\delta^{-1} + \dots,\end{aligned}$$

yielding the representation (1.5), except for the δ^{-1}-term, which we discuss next.

To the Toda problem is associated a flag of nested planes $\tilde{W}_{n+1} \subset \tilde{W}_n \in Gr_n$,

$$\begin{aligned}\tilde{W}_n &\equiv \operatorname{span}\{z^n\tilde{\Psi}_n, z^{n+1}\tilde{\Psi}_{n+1}, \dots\} \\ &= \operatorname{span} z^n\{\tilde{\Psi}_n, \frac{\partial}{\partial t_1}\tilde{\Psi}_n, (\frac{\partial}{\partial t_1})^2\tilde{\Psi}_n, \dots\}.\end{aligned}$$

The inclusion $\tilde{W}_{n+1} \subset \tilde{W}_n$ implies, by noting $\tilde{\Psi}_k = 1 + O(z^{-1})$, that

$$z\tilde{\Psi}_{n+1} = \frac{\partial}{\partial t_1}\tilde{\Psi}_n - \alpha\tilde{\Psi}_n \quad \text{for some} \quad \alpha = \alpha(t). \tag{1.11}$$

Then $\alpha(t) = \frac{\partial}{\partial t_1}\log\tau_{n+1}/\tau_n$ and putting this expression in (1.11) yields[10]

$$\{\tau_n(t - [z^{-1}]), \tau_{n+1}(t)\} + z\Big(\tau_n(t - [z^{-1}])\tau_{n+1}(t) - \tau_{n+1}(t - [z^{-1}])\tau_n(t)\Big) = 0.$$

Expanding this expression in powers of z^{-1} and dividing the coefficient of z^{-1} by $\tau_n\tau_{n+1}$ yield

$$-\frac{p_2\tau_{n+1}}{\tau_{n+1}} + \frac{p_2\tau_n}{\tau_n} + \frac{p_1\tau_n}{\tau_n}\frac{p_1\tau_{n+1}}{\tau_{n+1}} - \frac{\frac{\partial^2\tau_n}{\partial t_1^2}}{\tau_n} = 0.$$

Combining this relation with the customary Hirota bilinear relations, the simplest one being:

$$-\frac{1}{2}\frac{\partial^2}{\partial t_1^2}\tau_n \circ \tau_n + \tau_{n-1}\tau_{n+1} = 0, \quad \text{i.e.} \quad \frac{\partial^2}{\partial t_1^2}\log\tau_n = \frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2},$$

we find

$$-\frac{p_2\tau_{n+1}}{\tau_{n+1}} + \frac{p_2\tau_n}{\tau_n} + \frac{p_1\tau_n}{\tau_n}\frac{p_1\tau_{n+1}}{\tau_{n+1}} - (\frac{p_1\tau_n}{\tau_n})^2 = \frac{\partial^2}{\partial t_1^2}\log\tau_n = \frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2},$$

[10] $\{f, g\} = \frac{\partial f}{\partial t_1}g - f\frac{\partial g}{\partial t_1}$

and thus the representation (1.5) of $\tilde{L}$, ending the proof of Proposition 1.1.

Henceforth, we assume $\tilde{L}$ as in proposition 1.1, but in addition *tridiagonal*: $\tilde{L} = \sum_{-1\leq j\leq 1} a_j\delta^j$; this submanifold is invariant under the vector field (1.4); indeed, more generally if $\tilde{L} = \sum_{j\leq 1} a_j\delta^j$ is a $N+1$ band matrix, i.e. $a_j = 0$ for $j \leq -N \leq 0$, then $\tilde{L}$ remains a $N+1$ band matrix under the Toda vector fields. Moreover consider the Lie algebra decomposition, alluded to in (0.2), of $gl(\infty) = \mathcal{D}_s \oplus \mathcal{D}_b \ni A = (A)_s + (A)_b$ in skew-symmetric plus lower Borel part (lower triangular, including the diagonal).

Theorem 1.2. *Considering the submanifold of tridiagonal matrices $\tilde{L}$ of proposition 1.1 and remembering the form of the diagonal matrix $\gamma = (\gamma_n)_{n\in\mathbf{Z}}$, with $\gamma_n = \sqrt{\frac{\tau_{n+1}}{\tau_n}}$, we define a new wave operator S and wave vector Ψ :*

$$S := \gamma^{-1}\tilde{S} \quad \text{and} \quad \Psi := S\chi(z)e^{\Sigma/2}; \tag{1.12}$$

also define

$$L := S\delta S^{-1} \quad \text{and} \quad M := S(\varepsilon + \frac{1}{2}\sum_1^\infty kt_k\delta^{k-1})S^{-1}. \tag{1.13}$$

Then the tridiagonal matrix

$$L = \Big(\dots, 0, \frac{\gamma_n}{\gamma_{n-1}}, \frac{\partial}{\partial t_1}\log\gamma_n^2, \frac{\gamma_{n+1}}{\gamma_n}, 0, \dots\Big)_{n\in\mathbf{Z}} \tag{1.14}$$

is symmetric and

$$\Psi = \gamma^{-1}\tilde{\Psi} = e^{\Sigma/2}\gamma^{-1}\chi\frac{e^{-\eta}\tau}{\tau} =: (z^n\Psi_n)_{n\in\mathbf{Z}} \text{ with } \Psi_n := e^{\Sigma/2}\frac{e^{-\eta}\tau_n}{\sqrt{\tau_n\tau_{n+1}}}. \tag{1.15}$$

The new quantities satisfy:

$$\frac{\partial\log\gamma}{\partial t_n} = \frac{1}{2}(L^n)_0\ , \quad \frac{\partial S}{\partial t_n} = -\frac{1}{2}(L^n)_b S \quad \text{and} \quad \frac{\partial\Psi}{\partial t_n} = \frac{1}{2}(L^n)_s\Psi, \tag{1.16}$$

$$L\Psi = z\Psi \quad \text{and} \quad M\Psi = \frac{\partial}{\partial z}\Psi, \quad \text{with} \quad [L, M] = 1, \tag{1.17}$$

and

$$\frac{\partial L}{\partial t_n} = [\frac{1}{2}(L^n)_s, L], \quad \frac{\partial M}{\partial t_n} = [\frac{1}{2}(L^n)_s, M]. \tag{1.18}$$

Proof of Theorem 1.2: For a given initial condition $\gamma'(0)$, the system of partial differential equations in γ'

$$\frac{\partial}{\partial t_k}\log\gamma' = \frac{1}{2}(\tilde{L}^k)_0; \tag{1.19}$$

has, by Frobenius theorem, a unique solution, since

$$\begin{aligned}\left(\frac{\partial}{\partial t_k}\tilde{L}^n - \frac{\partial}{\partial t_n}\tilde{L}^k\right)_0 &= -[(\tilde{L}^k)_-, \tilde{L}^n]_0 + [\tilde{L}^k, (\tilde{L}^n)_+]_0 \\ &= -[(\tilde{L}^k)_-, (\tilde{L}^n)_+]_0 + [(\tilde{L}^k)_-, (\tilde{L}^n)_+]_0 = 0,\end{aligned} \tag{1.20}$$

using $[A_+, B_+]_0 = 0$ and $[A_-, B_-]_0 = 0$ for arbitrary matrices A and B. Given this solution $\gamma'(t)$, define $S' := \gamma'^{-1}\tilde{S}$ and $L' := S'\delta S'^{-1} = \gamma'^{-1}\tilde{L}\gamma'$, and using $\partial\tilde{S}/\partial t_n = -(\tilde{L}^n)_-\tilde{S}$, compute

$$\begin{aligned}\frac{\partial S'}{\partial t_k} = \frac{\partial(\gamma'^{-1}\tilde{S})}{\partial t_k} &= \frac{\partial\gamma'^{-1}}{\partial t_k}\tilde{S} + \gamma'^{-1}\frac{\partial\tilde{S}}{\partial t_k} \\ &= -\gamma'^{-1}(\frac{\partial\gamma'}{\partial t_k})\gamma'^{-1}\tilde{S} + \gamma'^{-1}\frac{\partial\tilde{S}}{\partial t_k} \\ &= -\gamma'^{-1}\frac{1}{2}(\tilde{L}^k)_0\tilde{S} - \gamma'^{-1}(\tilde{L}^k)_-\tilde{S} \\ &= -\frac{1}{2}(L'^k)_0 S' - (L'^k)_- S' \quad \text{since } L' = \gamma'^{-1}\tilde{L}\gamma' \text{ and } S' := \gamma'^{-1}\tilde{S} \\ &= -\left((L'^k)_- + \frac{1}{2}(L'^k)_0\right)S' = -\frac{1}{2}(L'^k)_{b'}S',\end{aligned}$$

where

$$A_{b'} := 2A_- + A_0 \quad \text{and} \quad A_{s'} := A - A_{b'}.$$

It follows at once that

$$\frac{\partial L'}{\partial t_n} = [-\frac{1}{2}(L'^n)_{b'}, L'] = [\frac{1}{2}(L'^n)_{s'}, L'] \tag{1.21}$$

Observe now that the manifold of symmetric tridiagonal matrices A is invariant under the vector fields

$$\frac{\partial A}{\partial t_n} = [-\frac{1}{2}(A^n)_{b'}, A] = [\frac{1}{2}(A^n)_{s'}, A], \tag{1.22}$$

since for A symmetric, the operations $()_{b'}$ and $()_{s'}$ coincide with the decomposition (0.2):

$$A_{b'} = A_b \text{ and } A_{s'} = A_s.$$

Now according to formula (1.5), picking $\gamma'(0) := \gamma(0) = \sqrt{\frac{\tau_{n+1}(0)}{\tau_n(0)}}$ as initial condition for the system of pde's (1.19), makes $L'(0) = \gamma'(0)^{-1}\tilde{L}(0)\gamma(0)'$ symmetric. Since L' was shown to evolve according to (1.21) or (1.22), and since its initial condition is symmetric, the matrix L' remains symmetric in t. Since $L'(t) := \gamma'^{-1}(t)\tilde{L}(t)\gamma'(t)$ is symmetric and since, by definition, $L(t) := \gamma^{-1}(t)\tilde{L}(t)\gamma(t)$ is also symmetric, we have $L'(t) = L(t)$, and thus $\gamma'(t) = c\gamma(t)$ for some constant c; but c must be 1, since $\gamma'(0) = \gamma(0)$. This proves (1.14), (1.16) and the first halfs of (1.17) and (1.18).

Besides multiplication of Ψ by z, which is represented by the matrix L, we also consider differentiation $\partial/\partial z$ of Ψ, which we represent by a matrix M:

$$\begin{aligned} \frac{\partial \Psi}{\partial z} &= e^{\Sigma/2}S\frac{\partial}{\partial z}\chi + \frac{1}{2}(\sum_1^\infty k t_k z^{k-1})e^{\Sigma/2}S\chi \\ &=: \left(P + \frac{1}{2}\sum_1^\infty k t_k L^{k-1}\right)\Psi =: M\Psi \end{aligned} \tag{1.23}$$

with

$$[L, M] = 1 \text{ and } P := S\varepsilon S^{-1} \in \mathcal{D}_{-\infty,-1}.$$

Finally compute

$$\begin{aligned} \frac{\partial M}{\partial t_n} &= \frac{\partial S}{\partial t_n}(S^{-1}S)(\varepsilon + \frac{1}{2}\Sigma k t_k \delta^{k-1})S^{-1} \\ &\quad + \frac{1}{2}nL^{n-1} - S(\varepsilon + \frac{1}{2}\Sigma k t_k \delta^{k-1})S^{-1}\frac{\partial S}{\partial t_n}S^{-1} \\ &= -\frac{1}{2}(L^n)_b M + [\frac{1}{2}L^n, M] + \frac{1}{2}M(L^n)_b \\ &= [-\frac{1}{2}(L^n)_b + \frac{1}{2}L^n, M] \\ &= [\frac{1}{2}(L^n)_s, M], \end{aligned}$$

ending the proof of theorem 1.2.
Remark 1.2.1: Theorem 1.2 remains valid for semi-infinite matrices L; the proof would only require minor modifications.

2 Symmetries of the Toda lattice and the w_2-algebra

Symmetries are t-dependent vector fields on the manifold of wave functions Ψ, which commute with and are transversal to the Toda vector fields, without affecting the t-variables. We shall need the following Lie algebra splitting lemmas, dealing with operators and their eigenfunctions, due to [A-S-V].

Lemma 2.1. *Let $\mathcal{D}$ be a Lie algebra with a vector space decomposition $\mathcal{D} = \mathcal{D}_+ \oplus \mathcal{D}_-$ into two Lie subalgebras $\mathcal{D}_+$ and $\mathcal{D}_-$; let V be a representation space of $\mathcal{D}$, and let $\mathcal{M} \subset V$ be a submanifold preserved under the vector fields defined by the action of $\mathcal{D}_-$, i.e.,*

$$\mathcal{D}_- \cdot x \subset T_x\mathcal{M}, \quad \forall x \in \mathcal{M}.$$

For any function $p: \mathcal{M} \to \mathcal{D}$, let $\mathbf{Y}_p$ be the vector field on $\mathcal{M}$ defined by

$$\mathbf{Y}_p(x) := -p(x)_- \cdot x, \quad x \in \mathcal{M}.$$

(a) Consider a set $\mathcal{A}$ of functions $p: \mathcal{M} \to \mathcal{D}$ such that

$$\mathbf{Y}_q p = [-q_-, p], \quad \forall p, q \in \mathcal{A},$$

holds. Then $\mathbf{Y}: p \mapsto \mathbf{Y}_p$ gives a Lie algebra homomorphism of the Lie algebra generated by $\mathcal{A}$ to the Lie algebra $\mathcal{X}(\mathcal{M})$ of vector fields on $\mathcal{M}$:

$$[\mathbf{Y}_{p_1}, \mathbf{Y}_{p_2}] = \mathbf{Y}_{[p_1,p_2]}, \quad \forall p_1, p_2 \in \mathcal{A},$$

and hence we can assume without loss of generality that $\mathcal{A}$ itself is a Lie algebra.
(b) Suppose for a subset $\mathcal{B} \subset \mathcal{A}$ of functions

$$\mathbf{Z}_q(x) := q(x)_+ \cdot x \in T_x\mathcal{M}, \quad \forall x \in \mathcal{M}, q \in \mathcal{B},$$

and hence defines another vector field $\mathbf{Z}_q \in \mathcal{X}(\mathcal{M})$ when $q \in \mathcal{B}$, and such that

$$\mathbf{Z}_q p = [q_+, p], \quad \forall p \in \mathcal{A}, q \in \mathcal{B},$$

holds. Then

$$[\mathbf{Y}_p, \mathbf{Z}_q] = 0, \quad \forall p \in \mathcal{A}, q \in \mathcal{B}.$$

Remark 2.1.1: A special case of this which applies to many integrable systems is: $V = \mathcal{D}'$, a Lie algebra containing $\mathcal{D}$, and $\mathcal{D}$ acts on $\mathcal{D}'$ by Lie bracket, i.e., $\mathbf{Y}_p(x) = [-p(x)_-, x]$, etc.

Proof: To sketch the proof, let $p_1,\ p_2,\ p \in \mathcal{A}$ and $q \in \mathcal{B}$; then the commutators have the following form:

$$[\mathbf{Y}_{p_1}, \mathbf{Y}_{p_2}](x) = Z_1(x) \ \text{ and } \ [\mathbf{Z}_q, \mathbf{Y}_p](x) = Z_2 x,$$

where, using $\mathcal{D}_-$ and $\mathcal{D}_+$ are Lie subalgebras,

$$\begin{aligned} Z_1 &:= (\mathbf{Y}_{p_1}(p_2))_- - (\mathbf{Y}_{p_2}(p_1))_- + [p_{1-}, p_{2-}] \\ &= [-p_{1-}, p_2]_- - [-p_{2-}, p_1]_- + [p_{1-}, p_{2-}] \\ &= (-[p_{1-}, p_2] - [p_1, p_{2-}] + [p_{1-}, p_{2-}] - [p_{1+}, p_{2+}])_- = -[p_1, p_2]_-. \end{aligned}$$

and

$$\begin{aligned} Z_2 &:= (\mathbf{Z}_q(p))_- + (\mathbf{Y}_p(q))_+ - [q_+, p_-] \\ &= [q_+, p]_- + [-p_-, q]_+ - [q_+, p_-] \\ &= [q_+, p_-]_- + [-p_-, q_+]_+ - [q_+, p_-] = [q_+, p_-] - [q_+, p_-] = 0, \end{aligned}$$

ending the proof of the lemma.

In the setup of the lemma, if we are given a Lie algebra (anti)homomorphism $\phi : g \to \mathcal{A}$, we denote $\mathbf{Y}_{\phi(x)}$ by $\mathbf{Y}_x$ and $\mathbf{Z}_{\phi(x)}$ by $\mathbf{Z}_x$ if there is no fear of confusion.

Theorem 2.2. *Let L represent an infinite symmetric tridiagonal matrix, flowing according to the Toda vector fields. There is a Lie algebra anti-homomorphism*

$$w_2^+ = \{z^{n+\ell}(\frac{\partial}{\partial z})^n, n = 0, \ell \in \mathbf{Z} \text{ or } n = 1, \ell \geq -1\} \to \left\{ \begin{array}{l} \text{vector fields on the} \\ \text{manifold of wave functions} \end{array} \right\}$$

$$z^{n+\ell}(\frac{\partial}{\partial z})^n \longrightarrow \mathbf{Y}_{\ell,n}\Psi = -(M^n L^{n+\ell})_b \Psi$$

satisfying

$$[\mathbf{Y}_{\ell,0}, \mathbf{Y}_{m,0}] = \frac{\ell}{2}\delta_{\ell+m}, \quad [\mathbf{Y}_{\ell,0}, \mathbf{Y}_{m,1}] = \ell \mathbf{Y}_{m+\ell,0} \text{ and } [\mathbf{Y}_{\ell,1}, \mathbf{Y}_{m,1}] = (\ell - m)\mathbf{Y}_{m+\ell,1}.$$

They commute with the Toda vector fields:

$$[\mathbf{Y}_{\ell,n}, \frac{\partial}{\partial t_k}] = 0.$$

Note the vector fields $\mathbf{Y}_{\ell,n}$ *induce vector fields on* S *and* $L = S\delta S^{-1}$

$$\mathbf{Y}_{\ell,n}(S) = -(M^n L^{n+\ell})_b S \text{ and } \mathbf{Y}_{\ell,n}(L) = [-(M^n L^{n+\ell})_b, L].$$

Proof of Theorem 2.2: Taking into account the notation of 1.13 and in view of Lemma 2.1 and the remark 2.1.1, set

$$\mathcal{D} := gl(\infty), \quad \mathcal{D}_+ := \mathcal{D}_s, \quad \mathcal{D}_- := \mathcal{D}_b \quad \text{and} \quad \mathcal{D}' := \mathcal{D} \times \mathcal{D}$$

on which $\mathcal{D}$ acts via diagonal embedding $\mathcal{D} \hookrightarrow \mathcal{D}' : p \mapsto (p,p)$.

$$\begin{aligned}
V &:= \mathcal{D} \\
\mathcal{M} &:= \text{respectively, the space of wave operators} S, \text{ of wave functions } \Psi \\
&\quad\;\; \text{or of pairs } (L,M) = (S\delta S^{-1}, S(\varepsilon + \frac{1}{2}\sum_1^\infty k t_k \delta^{k-1})S^{-1}), \text{ with an} \\
&\quad\;\; \text{infinite symmetric tridiagonal matrix } L \\
\mathcal{A} &:= \text{span} \begin{cases} M^n L^{n+\ell}, & n = 0, \ell \in \mathbf{Z} \\ \text{or} & n = 1, \ell \geq -1 \end{cases} \\
\mathcal{B} &:= \text{span}\,\{L^\alpha, \alpha \in \mathbf{Z}\}
\end{aligned}$$

and

$$g := w_2^+$$

with the antihomomorphism $\phi : g \to \mathcal{A}$ given by

$$\phi(z^\alpha \partial_z^\beta) := M^\beta L^\alpha.$$

Then the vector fields take the form:

$$\mathbf{Y}_p \Psi = -p_b \Psi, \quad \mathbf{Y}_p S = -p_b S, \quad p \in \mathcal{A}$$

$$\mathbf{Y}_p(L,M) = ([-p_b, L], [-p_b, M]), \quad p \in \mathcal{A}$$

$$\begin{aligned}
\mathbf{Z}_{L^n/2}\Psi &= \tfrac{1}{2}(L^n)_s \Psi, & \mathbf{Z}_{L^n/2} S &= -\tfrac{1}{2}(L^n)_b S \\
&= \tfrac{\partial}{\partial t_n}\Psi & &= \tfrac{\partial}{\partial t_n} S
\end{aligned}$$

by Theorem 1.2,

$$\mathbf{Z}_{L^n/2}(L,M) = ([\frac{1}{2}(L^n)_s, L], [\frac{1}{2}(L^n)_s, M]) = \frac{\partial}{\partial t_n}(L,M) \qquad \text{by (1.18).}$$

Note that the vector fields

$$\mathbf{Y}_{m,0} \equiv \mathbf{Y}_{L^m} \text{ all } m \in \mathbf{Z} \text{ and } \mathbf{Y}_{\ell,1} = \mathbf{Y}_{ML^{\ell+1}}, \text{ all } \ell \in \mathbf{Z}, \geq -1 \tag{2.1}$$

are tangent to $\mathcal{M}$. Indeed for $m < 0$, the vector field reads

$$\frac{\partial}{\partial s_m}\Psi = \mathbf{Y}_{m,0}\Psi = -(L^m)_b\Psi = -L^m\Psi = -z^m\Psi \quad \text{for} \quad m < 0. \tag{2.2}$$

The solution to this equation with initial condition $\Psi^{(0)}$ is given by

$$\Psi = e^{-s_m z^m}\Psi^{(0)}(t,z)$$

i.e., every component of the vector $\Psi^{(0)}$ is multiplied by the same exponential factor, and so is each τ-function:

$$\tau_k(t) = \tau_k^0(t)e^{-ms_m t_{-m}}.$$

Since the entries of the tridiagonal matrix only depend on ratios of τ-functions, this exponential factor is irrelevant for L.

In the same way $\mathbf{Y}_{m,0}$ $(m \geq 0)$ is tangent to the space of symmetric tridiagonal matrices, because the solution to

$$\frac{\partial\Psi}{\partial s_m} = \mathbf{Y}_{m,0}\Psi = -(L^m)_b\Psi = (-L^m + (L^m)_s)\Psi = -z^m\Psi + 2\frac{\partial\Psi}{\partial t_m}, \tag{2.3}$$

is given by

$$\Psi = e^{\frac{1}{2}\Sigma t_i z^i}\frac{\tau(t+2s-[z])}{\tau(t)}.$$

Not only the vector fields $\mathbf{Y}_{\ell,0}$, but also the $\mathbf{Y}_{\ell,1}$'s $(\ell \geq -1)$ are tangent to the space of symmetric tridiagonal matrices, because

$$\begin{aligned} \mathbf{Y}_{\ell,1}(L) &= [-(ML^{\ell+1})_b, L]\ , \quad \text{having the form } a_1\delta + \sum_0^\infty a_{-i}\delta^{-i} \\ &= [-ML^{\ell+1}, L] + [(ML^{\ell+1})_s, L] \\ &= L^{\ell+1} + [(ML^{\ell+1})_s, L] = \text{ symmetric matrix for } \ell \geq -1. \end{aligned} \tag{2.4}$$

With these data in mind, Lemma 2.1 implies Theorem 2.2.

3 The action of the symmetries on τ-functions

The main purpose of this section is to show that the symmetry vector fields $\mathbf{Y}$ defined on the manifold of wave function Ψ induce certain precise vector fields on τ, given by the coefficients of the vertex operator expansion. The precise statement is contained in theorem 3.2. Before entering these details, we need a general statement:

Lemma 3.1. *Any vector field $\mathbf{Y}$ defined on the manifold of wave functions Ψ and commuting with the Toda vector fields induce vector fields $\hat{\mathbf{Y}}$ on the manifold of τ-functions; they are related as follows, taking into account the fact that $\mathbf{Y}\log$ acts as a logarithmic derivative:*

$$\mathbf{Y}\log\Psi_n = (e^{-\eta}-1)\ \hat{\mathbf{Y}}\log\tau_n + \frac{1}{2}\hat{\mathbf{Y}}\log\frac{\tau_n}{\tau_{n+1}} \qquad (3.1)$$

$$\equiv \sum_1^\infty a_i^{(n)} z^{-i} + a_0^{(n)}$$

where $\hat{\mathbf{Y}}$ is a vector field acting on τ-functions; the part of (3.1) containing $e^{-\eta}-1$ is a power series in z^{-1} vanishing at $z=\infty$, whereas the other part is z-independent. For any two vector fields

$$[\mathbf{Y},\mathbf{Y}']\log\Psi_n = (e^{-\eta}-1)\ [\hat{\mathbf{Y}},\hat{\mathbf{Y}}']\log\tau_n + \frac{1}{2}[\hat{\mathbf{Y}},\hat{\mathbf{Y}}']\log\frac{\tau_n}{\tau_{n+1}}, \qquad (3.2)$$

showing that the map above from the algebra of vector fields on wave functions to the algebra of vector fields on τ-functions is homomorphism.

Proof: In the computation below we use $\gamma_n = \sqrt{\frac{\tau_{n+1}}{\tau_n}}$ and the fact that $\mathbf{Y} := \dot{}$ commutes with the Toda flows $\partial/\partial t_n$ and thus with η:

$$\begin{aligned}
(\log\Psi_n)^{\cdot} &= (\log\frac{e^{-\eta}\tau_n}{\tau_n} - \log\gamma_n)^{\cdot} \quad \text{see (1.21)} \\
&= \frac{(e^{-\eta}\dot{\tau}_n)}{e^{-\eta}\tau_n} - \frac{\dot{\tau}_n}{\tau_n} + \frac{1}{2}(\log\frac{\tau_n}{\tau_{n+1}})^{\cdot} \\
&= (e^{-\eta}-1)(\log\tau_n)^{\cdot} + \frac{1}{2}(\log\frac{\tau_n}{\tau_{n+1}})^{\cdot}, \quad \text{using } [\eta,\mathbf{Y}]=0.
\end{aligned} \qquad (3.3)$$

Applying the second vector field $\mathbf{Y}'$ to relation (3.3) yields

$$\mathbf{Y}'\mathbf{Y}\log\Psi_n = (e^{-\eta}-1)\mathbf{Y}'(\hat{\mathbf{Y}}\log\tau_n) + \frac{1}{2}\mathbf{Y}'(\hat{\mathbf{Y}}\log\frac{\tau_n}{\tau_{n+1}})$$

with

$$\begin{aligned}\mathbf{Y}'(\hat{\mathbf{Y}}\log f) &= \mathbf{Y}'(\frac{\hat{\mathbf{Y}}f}{f})\\ &= \frac{\hat{\mathbf{Y}}'(\hat{\mathbf{Y}}f)}{f} - \frac{(\hat{\mathbf{Y}}'f)(\hat{\mathbf{Y}}f)}{f^2}\\ &= \frac{\hat{\mathbf{Y}}'(\hat{\mathbf{Y}}f)}{f} - \frac{(\hat{\mathbf{Y}}'f)(\hat{\mathbf{Y}}f)}{f^2}.\end{aligned}$$

Using the relation above twice leads at once to (3.2), ending the proof of Lemma 3.1.

The vector fields $\mathbf{Y}_{\ell,n}$ on Ψ induce certain precise vector fields on τ, constructed from expressions $W_\ell^{(n+1)}$ appearing in the vertex operator expansion; notice the expressions $W_\ell^{(n+1)}$ differ slightly from the customary ones, because of the 1/2 multiplying t but not $\partial/\partial t$:

(3.4)
$$\begin{aligned}X(t,\lambda,\mu)\tau &= e^{\frac{1}{2}\sum_1^\infty t_i(\mu^i-\lambda^i)}e^{\sum_1^\infty(\lambda^{-i}-\mu^{-i})\frac{1}{i}\frac{\partial}{\partial t_i}}\tau\\ &= e^{\frac{1}{2}\Sigma t_i(\mu^i-\lambda^i)}\tau(t+[\lambda^{-1}]-[\mu^{-1}])\\ &= \sum_{k=0}^\infty \frac{(\mu-\lambda)^k}{k!}\sum_{\ell=-\infty}^\infty \lambda^{-\ell-k}W_\ell^{(k)}(\tau).\end{aligned}$$

For instance

(3.5)
$$W_n^{(0)} = J_n^{(0)} = \delta_{n,0}, \qquad W_n^{(1)} = J_n^{(1)}, \qquad W_n^{(2)} = J_n^{(2)} - (n+1)J_n^{(1)}$$

with

$$J_n^{(1)} = \begin{cases}\frac{\partial}{\partial t_n} & \text{for } n>0\\ 0 & \text{for } n=0\\ \frac{1}{2}(-n)t_{-n} & \text{for } n<0\end{cases} \qquad J_n^{(2)} = \begin{cases}\sum_{\substack{i+j=n\\ i,j\geq 1}}\frac{\partial^2}{\partial t_i\partial t_j} + \sum_{\substack{-i+j=n\\ i,j\geq 1}} it_i\frac{\partial}{\partial t_j} & \text{for } n\geq 0\\ \frac{1}{4}\sum_{\substack{i+j=-n\\ i,j\geq 1}}(it_i)(jt_j) + \sum_{i-j=-n} it_i\frac{\partial}{\partial t_j} & \text{for } n\leq 0.\end{cases}$$

We shall also need

(3.6)
$$\mathcal{L}_{\ell,1}^j := W_\ell^{(2)} + 2jW_\ell^{(1)} + (j^2-j)W_\ell^{(0)} = J_\ell^{(2)} + (2j-\ell-1)J_\ell^{(1)} + (j^2-j)J_\ell^{(0)}$$

$$\mathcal{L}_{\ell,0}^j := 2W_\ell^{(1)} + 2jW_\ell^{(0)} = 2J_\ell^{(1)} + 2jJ_\ell^{(0)}.$$

We also introduce $\tilde{W}_n^{(2)}$, which differs from $W_n^{(2)}$ above by a factor 1/2,

$$\tilde{W}_n^{(2)} := J_n^{(2)} - \frac{n+1}{2} J_n^{(1)}, \text{ with } \tilde{W}_n^{(2)} = W_n^{(2)} = J_n^{(2)} \text{ for } n = -1, 0, \tag{3.7}$$

and an operator B_m in z and t

$$B_m := -z^{m+1} \frac{\partial}{\partial z} + \sum_{n > \max(-m,0)} n t_n \frac{\partial}{\partial t_{n+m}}, \quad m \in \mathbf{Z}, \tag{3.8}$$

which restricted to functions $f(t_1, t_2, \ldots)$ of $t \in \mathbf{C}^\infty$ only yields

$$B_m f = J_m^{(2)} f \text{ for } m = -1, 0, 1. \tag{3.9}$$

The expressions $J_n^{(2)}$ form a Virasoro algebra with central extension

$$[J_\ell^{(1)}, J_m^{(1)}] = \frac{\ell}{2} \delta_{\ell+m} \quad [J_\ell^{(1)}, J_m^{(2)}] = \ell J_{m+\ell}^{(1)} \tag{3.10}$$

$$[J_\ell^{(2)}, J_m^{(2)}] = (\ell - m) J_{\ell+m}^{(2)} + \frac{\ell^3 - \ell}{12} \delta_{\ell+m}$$

and consequently the following Virasoro commutation relations hold, upon setting

$$V_\ell = J_\ell^{(2)} + (a\ell + b) J_\ell^{(1)} \quad , \quad c_\ell = \frac{\ell^3 - \ell}{12} + \frac{\ell(b^2 - a^2\ell^2)}{2}, \tag{3.11}$$

$$[V_\ell, V_m] = (l - m) V_{\ell+m} + c_\ell \delta_{\ell+m,0}.$$

In particular, observe

$$[W_\ell^{(1)}, W_m^{(2)}] = \ell W_{\ell+m}^{(1)} + \frac{\ell^2 - \ell}{2} \delta_{\ell+m}, \quad [W_\ell^{(2)}, W_m^{(2)}] = (\ell - m) W_{\ell+m}^{(2)} - 5 \frac{(\ell^3 - \ell)}{12} \delta_{\ell+m}$$

and[11]

$$[\mathcal{L}_{\ell,0}^{(j)}, \mathcal{L}_{m,1}^{(j)}] = \ell \mathcal{L}_{\ell+m,0}^{(j)} + c_{\ell,j} \delta_{\ell+m}, \quad [\mathcal{L}_{\ell,1}^{(j)}, \mathcal{L}_{m,1}^{(j)}] = (\ell - m) \mathcal{L}_{\ell+m,1}^{(j)} + c'_{\ell,j} \delta_{\ell+m}. \tag{3.12}$$

The purpose of this section is to prove the following relationship between the action of the symmetries on Ψ and τ.

[11] with $c_{\ell,j} = \ell(\ell + 2j - 1) \quad c'_{\ell,j} = -\frac{\ell}{12}(5\ell^2 + 24j^2 + 7)$

Fundamental Theorem 3.2. *The following relationship holds*

$$\mathbf{Y}_{\ell,n}\log\Psi=(e^{-\eta}-1)\mathcal{L}_{\ell,n}\log\tau+\frac{1}{2}\Big(\mathcal{L}_{\ell,n}\log\tau-(\mathcal{L}_{\ell,n}\log\tau)_\delta\Big) \tag{3.13}$$

$$\text{for}\quad n=0,\quad \text{all } \ell\in\mathbf{Z}$$
$$n=1,\quad \text{all } \ell\geq -1$$

where $\mathbf{Y}_{\ell,n}\log$ *and* $\mathcal{L}_{\ell,n}\log$ *act as logarithmic derivatives and where*

$$(\tau_\delta)_j=\tau_{j+1}\qquad (\mathcal{L}_{\ell,n}\tau)_j=\mathcal{L}^j_{\ell,n}\tau_j\qquad \Big((\mathcal{L}_{\ell,n})_\delta\Big)_j=\mathcal{L}^{j+1}_{\ell,n}.$$

Corollary 3.2.1. *The following holds for* $m\geq -1$:

$$-\Big(ML^{m+1}+\frac{m+1}{2}L^m\Big)_b\Psi=\Big(z^j\Psi_j((e^{-\eta}-1)\frac{(J^{(2)}_m+2jJ^{(1)}_m+j^2J^{(0)}_m)\tau_j}{\tau_j}$$
$$+\frac{1}{2}\Big(\frac{(J^{(2)}_m+2jJ^{(1)}_m+j^2J^{(0)}_m)\tau_j}{\tau_j}-\frac{(J^{(2)}_m+2(j+1)J^{(1)}_m+(j+1)^2J^{(0)}_m)\tau_{j+1}}{\tau_{j+1}}\Big)\Big)\Big)_{j\in\mathbf{Z}},$$

where $J^{(i)}_\ell$ *is defined in (3.5).*

Remark that the statements of the theorem and the corollary are equally valid for semi-infinite matrices. Before giving the proof we need three lemmas.

Lemma 3.3. *The operator* B_m *defined in (3.8) interacts with* $\eta=\sum_1^\infty\frac{z^{-i}}{i}\frac{\partial}{\partial t_i}$ *and* $\Sigma=\sum_1^\infty t_iz^i$ *as follows*[12]

$$[B_m,\eta]=\sum_{k=1}^{m}z^{m-k}\frac{\partial}{\partial t_k}\quad\text{and}\quad B_m\Sigma=-\sum_{k=1}^{-m}z^{m+k}kt_k \tag{3.14}$$

$$[B_m,e^{-\eta}]=-e^{-\eta}[B_m,\eta];$$

thus

$$[B_m,\eta]=0\ \text{when}\ m\leq 0\ \text{and}\ B_m\Sigma=0\ \text{when}\ m\geq 0.$$

[12]with the understanding that

$$\sum_{j=1}^{\alpha}=0\quad\text{if}\quad\alpha<1.$$

Moreover given an arbitrary function $f(t_1,t_2,\dots)$ of $t\in\mathbf{C}^\infty$, define

$$\Phi = e^{\Sigma/2}\frac{e^{-\eta}f}{f}. \tag{3.15}$$

Then we have for $m\geq 0$

$$\begin{array}{lll} (i) & -\frac{1}{2}z^{-m}\Phi = \Phi(e^{-\eta}-1)\frac{W^{(1)}_{-m}(f)}{f}, & m>0 \\ (ii) & (B_{-m}+\frac{mt_m}{2})\Phi = \Phi(e^{-\eta}-1)\frac{\tilde W^{(2)}_{-m}(f)}{f}, & \\ (iii) & B_m\Phi = \Phi\Big((e^{-\eta}-1)\frac{(B_m-[B_m,\eta])f}{f}-\frac{[B_m,\eta]f}{f}\Big). & \end{array} \tag{3.16}$$

Proof: The first commutation relation (3.14) follows from a straightforward computation:

$$\begin{aligned} [B_m,\eta] &= \Big[-z^{m+1}\frac{\partial}{\partial z},\eta\Big]+\Big[\sum_{k>\max(-m,0)}kt_k\frac{\partial}{\partial t_{k+m}},\eta\Big] \\ &= \sum_1^\infty z^{m-i}\frac{\partial}{\partial t_i}-\sum_1^\infty z^{-j}\frac{\partial}{\partial t_{j+m}}=\sum_1^m z^{m-i}\frac{\partial}{\partial t_i}. \end{aligned}$$

The third commutation relation (3.14) follows at once from the fact that the bracket $[B_m,.]$ is a derivation and that $[[B_m,\eta],\eta]$. Also

$$B_m\Sigma = -\sum_1^\infty kt_kz^{k+m}+\sum_{k>\max(-m,0)}kt_k\,z^{k+m}=-\sum_1^{\max(-m,0)}kt_k\,z^{k+m}.$$

Next, given Φ defined as (3.15), we have that (using B_m is a derivation and (3.14)),
(3.17)

$$\begin{aligned} B_m\Phi &= B_me^{\Sigma/2}\frac{e^{-\eta}f}{f} \\ &= \frac{e^{-\eta}f}{f}B_me^{\Sigma/2}+e^{\Sigma/2}B_m\frac{e^{-\eta}f}{f} \\ &= \Phi\Big((e^{-\eta}-1)\frac{B_mf}{f}+\frac{1}{2}B_m\Sigma-e^{-\eta}\frac{[B_m,\eta]f}{f}\Big) \\ &= \Phi\Big((e^{-\eta}-1)\frac{(B_m-[B_m,\eta])f}{f}+\frac{1}{2}B_m\Sigma-\frac{[B_m,\eta]f}{f}\Big) \end{aligned}$$

which yields (3.16) (iii) for $m \geq 0$, taking into account the fact that $B_m\Sigma = 0$ for $m \geq 0$. Equation (3.16) (i) is straightforward, using $W^{(1)}_{-m} = \frac{1}{2}mt_m$.

Proving (3.16) (ii) is a bit more involved; indeed first observe that for $m \geq 0$

$$\begin{aligned}\tilde{W}^{(2)}_{-m}f &= \Big(\sum_{n>m} nt_n \frac{\partial}{\partial t_{n-m}} + \frac{1}{4}\sum_{n=1}^{m-1} nt_n(m-n)t_{m-n} + \frac{1}{4}(m-1)mt_m\Big)f \\ &= B_{-m}f + \frac{1}{4}\sum_{n=1}^{m-1} nt_n(m-n)t_{m-n}f + \frac{1}{4}(m-1)mt_m f;\end{aligned}$$

therefore

$$(e^{-\eta}-1)\frac{B_{-m}(f)}{f} \tag{3.19}$$

$$\begin{aligned}&= (e^{-\eta}-1)\Big(\frac{\tilde{W}^{(2)}_{-m}f}{f} - \frac{1}{4}\sum_{n=1}^{m-1} nt_n(m-n)t_{m-n} - \frac{1}{4}m(m-1)t_m\Big) \\ &= (e^{-\eta}-1)\frac{\tilde{W}^{(2)}_{-m}(f)}{f} \\ &\quad -\frac{1}{4}\sum_{n=1}^{m-1} n(m-n)((t_n - \frac{1}{n}z^{-n})(t_{m-n} - \frac{1}{m-n}z^{-m+n}) - t_n t_{m-n}) \\ &\quad -\frac{1}{4}m(m-1)((t_m - \frac{1}{m}z^{-m}) - t_m) \\ &= (e^{-\eta}-1)\frac{\tilde{W}^{(2)}_{-m}(f)}{f} + \frac{1}{2}\sum_{n=1}^{m-1} nt_n z^{n-m}.\end{aligned}$$

Using expression (3.17) for $B_m\Phi$, $-m \leq 0$, and $[B_m, \eta] = 0$ when $m \leq 0$, we find

$$\begin{aligned}(B_{-m} + \frac{mt_m}{2})\Phi &= \Phi\Big((e^{-\eta}-1)\frac{B_{-m}f}{f} + \frac{1}{2}B_{-m}\Sigma + \frac{mt_m}{2}\Big) \\ &= \Phi\Big((e^{-\eta}-1)\frac{B_{-m}f}{f} - \frac{1}{2}\sum_{k=1}^{m-1} z^{-m+k}kt_k\Big), \text{ using (3.14)} \\ &= \Phi(e^{-\eta}-1)\frac{\tilde{W}^{(2)}_{-m}f}{f}, \text{ using (3.19)},\end{aligned}$$

ending the proof of (3.14) (ii) and thus of Lemma 3.3.

Lemma 3.4. *The following holds* ($\nu \equiv \varepsilon\partial =$ *diag* $(\cdots, \nu_i = i, \cdots)$)

(i)
$$(PL^2)_s = \nu L_s + L_-$$

(ii)
$$\frac{(PL^2)_s\Psi}{\Psi} = (e^{-\eta}-1)(2\nu - I)\frac{\partial}{\partial t_1}\log\tau + \nu\Big(z - \frac{\partial}{\partial t_1}\log\frac{\tau_s}{\tau}\Big).$$

Proof: First consider

(3.20)
$$PL^{m+1} = S\varepsilon\delta^{m+1}S^{-1} = S\nu\delta^m S^{-1} = [S,\nu]\delta^m S^{-1} + \nu S\delta^m S^{-1} = [S,\nu]\delta^m S^{-1} + \nu L^m.$$

Since $\nu = \varepsilon\delta$ is diagonal, since $S \in \mathcal{D}_{-\infty,0}$ and so $[S,\nu] \in \mathcal{D}_{-\infty,-1}$, we have

$$[S,\nu]\delta S^{-1} \in \mathcal{D}_{-\infty,0} \quad \text{and thus} \quad ([S,\nu]\delta S^{-1})_s = 0.$$

This combined with the above observation leads to[13]

(3.21)
$$\begin{aligned}(PL^2)_s = (\nu L)_s &= \nu L_{++} - (\nu L_{++})^\top \\ &= \nu L_{++} - L_-\nu \\ &= \nu(L_{++} - L_-) + \nu L_- - L_-\nu \\ &= \nu L_s + [\nu, L_-] \\ &= \nu L_s + L_-, \quad \text{since } [\nu, L_-] = L_-,\end{aligned}$$

since L_- has one subdiagonal. In order to evaluate $(PL^2)_s\Psi$, we need to know $L_s\Psi$ and $L_-\Psi$. Anticipating (3.25), we have by (2.3) and (3.3)

$$\begin{aligned}\mathbf{Y}_{1,0}\Psi = -L_b\Psi &= 2\Psi(e^{-\eta}-1)\frac{\partial}{\partial t_1}\log\tau - \Psi\frac{\partial}{\partial t_1}\log\frac{\tau_\delta}{\tau} \\ &= 2\Psi(e^{-\eta}-1)\frac{\partial}{\partial t_1}\log\tau - L_0\Psi, \qquad \text{by (1.16)},\end{aligned}$$

and, since $L_b = 2L_- + L_0$,

(3.22)
$$L_-\Psi = \frac{1}{2}(L_b - L_0)\Psi = -\Psi(e^{-\eta}-1)\frac{\partial}{\partial t_1}\log\tau,$$

whereas, using

$$\Psi = e^{\frac{1}{2}\Sigma t_i z^i}\Big(z^j\gamma_j^{-1}\frac{\tau_j(t-[z^{-1}])}{\tau_j(t)}\Big),$$

[13] L_{++} denotes the strictly uppertriangular part of L, i.e. the projection of L onto $\mathcal{D}_{1,\infty}$

we have (using the logarithmic derivative)

$$L_s\Psi = 2\frac{\partial\Psi}{\partial t_1} = 2\Psi(e^{-\eta}-1)\frac{\partial}{\partial t_1}\log\tau + z\Psi - \Psi\frac{\partial}{\partial t_1}\log\frac{\tau_\delta}{\tau}. \tag{3.23}$$

Using these two formulas (3.22) and (3.23), in (3.21) we have

$$\begin{aligned}(PL^2)_s\Psi &= \nu L_s\Psi + L_-\Psi \\ &= \Psi(e^{-\eta}-1)(2\nu - I)\frac{\partial}{\partial t_1}\log\tau + z\nu\Psi - \nu\Psi\frac{\partial}{\partial t_1}\log\frac{\tau_\delta}{\tau},\end{aligned}$$

ending the proof of Lemma 3.4.

This lemma proves the main statement about symmetries for the $s\ell(2,\mathbf{C})$ part of the Virasoro symmetry algebra; this is the heart of the matter.

Lemma 3.5. *The vector fields $\{\mathbf{Y}_{-1,1},\mathbf{Y}_{0,1},\mathbf{Y}_{1,1}\}$ form a representation of $s\ell(2,\mathbf{C})$ and induce vector fields on τ as follows (in the notation (3.6)):*

$$\mathbf{Y}_{\ell,1}\log\Psi = \frac{-(ML^{\ell+1})_b\Psi}{\Psi} = (e^{-\eta}-1)\frac{\mathcal{L}_{\ell,1}\tau}{\tau} + \frac{1}{2}\left(\frac{\mathcal{L}_{\ell,1}\tau}{\tau} - (\frac{\mathcal{L}_{\ell,1}\tau}{\tau})_\delta\right) \tag{3.24}$$

for $\ell = -1,0,1$.

Also

$$\mathbf{Y}_{\ell,0}\log\Psi = \frac{-(L^{\ell})_b\Psi}{\Psi} = (e^{-\eta}-1)\frac{\mathcal{L}_{\ell,0}\tau}{\tau} + \frac{1}{2}\left(\frac{\mathcal{L}_{\ell,0}\tau}{\tau} - (\frac{\mathcal{L}_{\ell,0}\tau}{\tau})_\delta\right) \tag{3.25}$$

for $\ell\in\mathbf{Z}$.

Proof of Lemma 3.5: Relation (3.25) will first be established for $\ell = -m < 0$:

$$\begin{aligned}\mathbf{Y}_{-m,0}\Psi &= -(L^{-m})_b\Psi = -L^{-m}\Psi, \quad \text{since} \quad L^{-m}\in\mathcal{D}_b \quad \text{for} \quad m\geq 0 \\ &= -z^{-m}\Psi \\ &= 2\gamma^{-1}\chi\left(-\frac{1}{2}z^{-m}(e^{\Sigma/2}\frac{e^{-\eta}\tau_n}{\tau_n})\right)_{n\in\mathbf{Z}} \\ &= 2\Psi(e^{-\eta}-1)\frac{W^{(1)}_{-m}\tau}{\tau}, \quad \text{applying (3.16)(i) componentwise to} \\ &\qquad\qquad \Phi = e^{\Sigma/2}\frac{e^{-\eta}\tau_n}{\tau_n} \\ &= \Psi\Big((e^{-\eta}-1)\frac{2W^{(1)}_{-m}\tau}{\tau} + \frac{1}{2}(\frac{2W^{(1)}_{-m}\tau}{\tau} - (\frac{2W^{(1)}_{-m}\tau}{\tau})_\delta)\Big),\end{aligned}$$

the difference in brackets vanishing identically. The same will now be established for $\ell = m > 0$; indeed

$$\begin{aligned}
\mathbf{Y}_{m,0}\Psi &\equiv -(L^m)_b\Psi \\
&= (-L^m + (L^m)_s)\Psi \\
&= -z^m\Psi + 2\frac{\partial}{\partial t_m}\Psi\ , \qquad \text{using (1.16)} \\
&= -z^m\Psi + z^m\Psi + 2e^{\Sigma/2}\chi\frac{\partial}{\partial t_m}(\frac{e^{-\eta}\tau}{\tau}\gamma^{-1}), \\
&\qquad\qquad \text{using } \Psi = e^{\Sigma/2}\frac{e^{-\eta}\tau}{\tau}\gamma^{-1}\chi \\
&= 2e^{\Sigma/2}\chi\gamma^{-1}\frac{e^{-\eta}\tau}{\tau}\Big((e^{-\eta}-1)\frac{\partial}{\partial t_m}\log\tau + \gamma\frac{\partial}{\partial t_m}\gamma^{-1}\Big), \quad \text{using (3.3)} \\
&= \Psi\Big((e^{-\eta}-1)\frac{2W_m^{(1)}\tau}{\tau} + \frac{1}{2}(\frac{2W_m^{(1)}\tau}{\tau} - (\frac{2W_m^{(1)}\tau}{\tau})_\delta)\Big) \\
&\quad \text{since } \gamma_j = \sqrt{\frac{\tau_{j+1}}{\tau_j}}, \text{ and } \gamma\frac{\partial}{\partial t_m}\gamma^{-1} = -\frac{\partial}{\partial t_m}\log\gamma.
\end{aligned}$$

Relation (3.25) for $\ell = 0$ is obvious, since

$$\frac{\mathbf{Y}_{0,0}\Psi}{\Psi} = \frac{-(L^0)_b\Psi}{\Psi} = -I = \frac{1}{2}(2\nu - 2(\nu)_\delta) = \frac{1}{2}\left(\frac{2\nu\tau}{\tau} - (\frac{2\nu\tau}{\tau})_\delta\right) = \frac{1}{2}\left(\frac{\mathcal{L}_{0,0}\tau}{\tau} - (\frac{\mathcal{L}_{0,0}\tau}{\tau})_\delta\right)$$

To prove (3.24), consider now

$$\begin{aligned}
(3.26)\quad & -(ML^{m+1})_b\Psi \\
&= (-ML^{m+1} + (ML^{m+1})_s)\Psi \\
&= \Big(-z^{m+1}\frac{\partial}{\partial z} + \frac{1}{2}\sum_1^\infty kt_k(L^{k+m})_s + (PL^{m+1})_s\Big)\Psi \\
&= \Big(-z^{m+1}\frac{\partial}{\partial z} + \frac{1}{2}\sum_{k>\max(-m,0)} kt_k(L^{k+m})_s + (PL^{m+1})_s\Big)\Psi, \qquad \text{using (1.28)} \\
&\qquad\qquad \text{since } (L^\alpha)_s = 0 \text{ for } \alpha \leq 0 \\
&= \Big(-z^{m+1}\frac{\partial}{\partial z} + \sum_{k>\max(-m,0)} kt_k\frac{\partial}{\partial t_{k+m}}\Big)\Psi + (PL^{m+1})_s\Psi \\
&= B_m\Psi + (PL^{m+1})_s\Psi \qquad \text{using the definition (3.8) of } B_m, \\
&= (z^j\gamma_j^{-1}B_m(e^{\Sigma/2}\frac{e^{-\eta}\tau_j}{\tau_j}))_{j\in\mathbf{Z}} - z^m\nu\Psi - \Psi B_m\log\gamma + (PL^{m+1})_s\Psi,
\end{aligned}$$

remembering the definition (1.21) of Ψ, using the definition (3.8) of B_m, and using the fact that B_m is a derivation.

1. For $m=-1,0$, we have

$$PL^{m+1}\in\mathcal{D}_{-\infty,0}\quad\text{and thus}\quad (PL^{m+1})_s=0.$$

We set $m\mapsto -m$ with $m\geq 0$. Componentwise, the above expression reads, by adding and subtracting $mt_m/2$ ($m\geq 0$) and using definition (3.8) of B_{-m} and $\nu=\operatorname{diag}(\cdots,i,\cdots)$:

$$
\begin{aligned}
&(-(ML^{-m+1})_b\Psi)_j\\
&= z^j\gamma_j^{-1}\left(B_{-m}+\frac{mt_m}{2}-jz^{-m}\right)e^{\Sigma/2}\frac{e^{-\eta}\tau_j}{\tau_j}-z^j\Psi_j\left(B_{-m}(\log\gamma_j)+\frac{mt_m}{2}\right)\\
&= z^j\gamma_j^{-1}\left(B_{-m}+\frac{mt_m}{2}+2j(1-\delta_{0,m})(-\frac{z^{-m}}{2})\right)(e^{\Sigma/2}\frac{e^{-\eta}\tau_j}{\tau_j})\\
&\quad -z^j\Psi_j\left(B_{-m}\log\gamma_j+\frac{mt_m}{2}\right)-\frac{1}{2}z^j\Psi_j 2j\delta_{0,m}\\
&= z^j\Psi_j(e^{-\eta}-1)(\frac{\tilde W^{(2)}_{-m}(\tau_j)}{\tau_j}+2j\frac{W^{(1)}_{-m}(\tau_j)}{\tau_j})\quad\text{(using (3.16)(i) and (ii))}\\
&\quad +\frac{1}{2}z^j\Psi_j\big(-\frac{\tilde W^{(2)}_{-m}(\tau_{j+1})}{\tau_{j+1}}+\frac{\tilde W^{(2)}_{-m}(\tau_j)}{\tau_j}\big)\quad\text{(using }\gamma_j=\sqrt{\frac{\tau_{j+1}}{\tau_j}}\text{, (3.9) and (3.7))}\\
&\quad +\frac{1}{2}z^j\Psi_j\left(-2(j+1)\frac{W^{(1)}_{-m}(\tau_{j+1})}{\tau_{j+1}}+2j\frac{W^{(1)}_{-m}(\tau_j)}{\tau_j}\right)\quad\text{(using }\frac{mt_m}{2}=W^{(1)}_{-m})\\
&\quad +\frac{1}{2}z^j\Psi_j(-2j\delta_{0,m})\\
&= z^j\Psi_j\Big((e^{-\eta}-1)\frac{(\tilde W^{(2)}_{-m}+2jW^{(1)}_{-m}+(j^2-j)W^{(0)}_{-m})\tau_j}{\tau_j}\\
&\quad +\frac{1}{2}\big(\frac{(\tilde W^{(2)}_{-m}+2jW^{(1)}_{-m}+(j^2-j)W^{(0)}_{-m})\tau_j}{\tau_j}\\
&\qquad -\frac{(\tilde W^{(2)}_{-m}+2(j+1)W^{(1)}_{-m}+((j+1)^2-(j+1))W^{(0)}_{-m})\tau_{j+1}}{\tau_{j+1}}\big)\Big),
\end{aligned}
$$

since, in the last line, $-2j=(j^2-j)-((j+1)^2-(j+1))$ and $\delta_{0,m}=W^{(0)}_{-m}$. Thus relation (3.24) for $m=-1$ and 0 follows from the simple observation (3.7) that $\tilde W^{(2)}_{-m}=W^{(2)}_{-m}$ for $m=0,1$.

2. For $m=1$ in (3.24), we use the identities in Lemma 3.3(iii) and Lemma 3.4(ii) in (3.26):

$$
\begin{aligned}
&-(ML^2)_b\Psi\\
&= \Psi(e^{-\eta}-1)\frac{(B_1-[B_1,\eta])\tau}{\tau}-\Psi\frac{[B_1,\eta]\tau}{\tau}-z\nu\Psi-\frac{1}{2}\Psi B_1\log\frac{\tau_\delta}{\tau}+(PL^2)_s\Psi\\
&\qquad\text{using (3.16)(iii)}\\
&= \Psi(e^{-\eta}-1)(J_1^{(2)}-\frac{\partial}{\partial t_1})\log\tau-\Psi\frac{\partial}{\partial t_1}\log\tau-z\nu\Psi-\frac{1}{2}\Psi J_1^{(2)}\log\frac{\tau_\delta}{\tau},\\
&\qquad +\Psi(e^{-\eta}-1)(2\nu-I)\frac{\partial}{\partial t_1}\log\tau+z\nu\Psi-\nu\Psi\frac{\partial}{\partial t_1}\log\frac{\tau_\delta}{\tau}\\
&\qquad\qquad \text{using (3.9), } [B_1,\eta]=\frac{\partial}{\partial t_1} \text{ (see (3.14)), and Lemma 3.4(ii),}\\
&= \Psi\Big((e^{-\eta}-1)(J_1^{(2)}+(2\nu-2)\frac{\partial}{\partial t_1})\log\tau+\frac{1}{2}(J_1^{(2)}+(2\nu-2)\frac{\partial}{\partial t_1})\log\tau\\
&\qquad\qquad -\frac{1}{2}(J_1^{(2)}+2\nu\frac{\partial}{\partial t_1})\log\tau_\delta\Big),\\
&= \Psi\left((e^{-\eta}-1)\frac{\mathcal{L}_{1,1}\tau}{\tau}+\frac{1}{2}\left(\frac{\mathcal{L}_{1,1}\tau}{\tau}-(\frac{\mathcal{L}_{1,1}\tau}{\tau})_\delta\right)\right),
\end{aligned}
$$

using in the last line, the definition (3.6) of $\mathcal{L}_{1,1}$ and $\mathcal{L}^{j+1}_{1,1}\tau_{j+1}=((\mathcal{L}_{1,1}\tau)_\delta)_j$, establishing (3.24) for $m=1$, thus ending the proof of Lemma 3.5.

Proof of Theorem 3.2: The only remaining point is to establish (3.13) for $n=1$ and all $\ell\geq 2$. To do this we use the underlying Lie algebra structure. First we have the following identity, using (3.2) and (3.1),

$$
\begin{aligned}
&(e^{-\eta}-1)\Big[\hat{\mathbf{Y}}_{z^\alpha(\frac{\partial}{\partial z})^\beta},\hat{\mathbf{Y}}_{z^{\alpha'}(\frac{\partial}{\partial z})^{\beta'}}\Big]\log\tau_n+\frac{1}{2}\Big[\hat{\mathbf{Y}}_{z^\alpha(\frac{\partial}{\partial z})^\beta},\hat{\mathbf{Y}}_{z^{\alpha'}(\frac{\partial}{\partial z})^{\beta'}}\Big]\log\frac{\tau_n}{\tau_{n+1}}\\
&= \Big[\mathbf{Y}_{z^\alpha(\frac{\partial}{\partial z})^\beta},\mathbf{Y}_{z^{\alpha'}(\frac{\partial}{\partial z})^{\beta'}}\Big]\log\Psi_n\\
&= \mathbf{Y}_{\left[z^{\alpha'}(\frac{\partial}{\partial z})^{\beta'},z^\alpha(\frac{\partial}{\partial z})^\beta\right]}\log\Psi_n\\
&= (e^{-\eta}-1)\hat{\mathbf{Y}}_{\left[z^{\alpha'}(\frac{\partial}{\partial z})^{\beta'},z^\alpha(\frac{\partial}{\partial z})^\beta\right]}\log\tau_n+\frac{1}{2}\hat{\mathbf{Y}}_{\left[z^{\alpha'}(\frac{\partial}{\partial z})^{\beta'},z^\alpha(\frac{\partial}{\partial z})^\beta\right]}\log\frac{\tau_n}{\tau_{n+1}}.
\end{aligned}
$$

The terms containing $(e^{-\eta}-1)$ in the first and last expressions are power series in z^{-1}, with no constant term; the second terms are independent of z. Therefore, equating constant terms yield:

$$(i)\qquad \left(\left[\hat{\mathbf{Y}}_{z^{\alpha}(\frac{\partial}{\partial z})^{\beta}}, \hat{\mathbf{Y}}_{z^{\alpha'}(\frac{\partial}{\partial z})^{\beta'}}\right] - \hat{\mathbf{Y}}_{\left[z^{\alpha'}(\frac{\partial}{\partial z})^{\beta'}, z^{\alpha}(\frac{\partial}{\partial z})^{\beta}\right]}\right)\log\frac{\tau_n}{\tau_{n+1}} = 0$$

and thus also

$$(e^{-\eta}-1)\left(\left[\hat{\mathbf{Y}}_{z^{\alpha}(\frac{\partial}{\partial z})^{\beta}}, \hat{\mathbf{Y}}_{z^{\alpha'}(\frac{\partial}{\partial z})^{\beta'}}\right] - \hat{\mathbf{Y}}_{\left[z^{\alpha'}(\frac{\partial}{\partial z})^{\beta'}, z^{\alpha}(\frac{\partial}{\partial z})^{\beta}\right]}\right)\log\tau_n = 0.$$

Since $(e^{-\eta}-1)f=0$ implies $f=$ constant, there exists a constant c depending on $\alpha,\beta,\alpha',\beta'$ and n such that

$$(ii)\qquad \left(\left[\hat{\mathbf{Y}}_{z^{\alpha}(\frac{\partial}{\partial z})^{\beta}}, \hat{\mathbf{Y}}_{z^{\alpha'}(\frac{\partial}{\partial z})^{\beta'}}\right] - \hat{\mathbf{Y}}_{\left[z^{\alpha'}(\frac{\partial}{\partial z})^{\beta'}, z^{\alpha}(\frac{\partial}{\partial z})^{\beta}\right]}\right)\log\tau_n = c_{\alpha\beta\alpha'\beta',n};$$

relation (i) says $c_{\alpha\beta\alpha'\beta'}$ is independent of n; hence (ii) reads

The two relations (i) and (ii) combined imply

$$(3.27)\qquad \left(\left[\hat{\mathbf{Y}}_{z^{\alpha}(\frac{\partial}{\partial z})^{\beta}}, \hat{\mathbf{Y}}_{z^{\alpha'}(\frac{\partial}{\partial z})^{\beta'}}\right] - \hat{\mathbf{Y}}_{\left[z^{\alpha'}(\frac{\partial}{\partial z})^{\beta'}, z^{\alpha}(\frac{\partial}{\partial z})^{\beta}\right]} - c_{\alpha\beta\alpha'\beta'}\right)\tau_n = 0$$

with

$$c_{\alpha\beta\alpha'\beta'}\quad \text{independent of } n.$$

Applying (3.27) to

$$\left[z^{m+1}\frac{\partial}{\partial z}, z^{\ell}\right] = \ell z^{m+\ell} \quad\text{and}\quad \left[z^{\ell+1}\frac{\partial}{\partial z}, z^{m+1}\frac{\partial}{\partial z}\right] = (m-\ell)z^{m+\ell+1}\frac{\partial}{\partial z},$$

leads to

$$(3.28)\qquad \left(\left[\hat{\mathbf{Y}}_{\ell,0}, \hat{\mathbf{Y}}_{m,1}\right] - \ell\hat{\mathbf{Y}}_{\ell+m,0} - c_{\ell,m}\right)\tau_n = 0$$

and

$$\left(\left[\hat{\mathbf{Y}}_{m,1}, \hat{\mathbf{Y}}_{\ell,1}\right] - (m-\ell)\hat{\mathbf{Y}}_{m+\ell,1} - c'_{m,\ell}\right)\tau_n = 0.$$

By virtue of (3.12), we have

$$(3.29)\qquad [\mathcal{L}_{\ell,0}, \mathcal{L}_{m,1}] - \ell\mathcal{L}_{m+\ell,0} - \text{ constant } = 0.$$

According to (3.25) we have

$$\hat{\mathbf{Y}}_{\ell,0} = \mathcal{L}_{\ell,0}$$

implying by subtracting (3.29) from (3.28)

$$[\mathcal{L}_{\ell,0}, \hat{\mathbf{Y}}_{m,1} - \mathcal{L}_{m,1}] = \text{ constant, for } \ell, m \in \mathbf{Z}, m \geq -1.$$

The only operator commuting (modulo constant) with all $\mathcal{L}_{\ell,0} = 2\frac{\partial}{\partial t_1} + (-\ell)t_{-\ell}$ is given by linear combinations of a constant, t_α and $\partial/\partial t_\alpha$, i.e.:

$$\begin{aligned} \tilde{\mathbf{Y}}_{m,1} - \mathcal{L}_{m,1} &= \sum_{j=-\infty}^{\infty} c_j^{(m)} J_{j+m}^{(1)}, \text{ for } m \geq 2, \quad (c_{-m}^{(m)} = 0) \\ &= 0 \text{ for } m = -1, 0, 1. \end{aligned} \tag{3.30}$$

Putting $\tilde{\mathbf{Y}}_{m,1}$ from (3.30) into the second relation of (3.28) implies (modulo constants)

$$\Big[\mathcal{L}_{m,1} + \sum_{j=-\infty}^{\infty} c_j^{(m)} J_{j+m}^{(1)}, \mathcal{L}_{\ell,1} + \sum_{j=-\infty}^{\infty} c_j^{(\ell)} J_{j+\ell}^{(1)}\Big] = (m-\ell)(\mathcal{L}_{m+\ell,1} + \Sigma c_j^{(m+\ell)} J_{j+m+\ell}^{(1)})$$

which also equals by explicit computation, using (3.12):

$$= (m-\ell)\mathcal{L}_{m+\ell,1} - \Sigma c_j^{(\ell)}(j+\ell) J_{m+j+\ell}^{(1)} + \Sigma c_j^{(m)}(j+m) J_{m+j+\ell}^{(1)}.$$

Comparing the coefficients of the $J^{(1)}$'s in two expressions on the right hand side yields

$$(m-\ell)c_j^{(m+\ell)} = (m+j)c_j^{(m)} - (\ell+j)c_j^{(\ell)} \quad \text{provided } m+j+\ell \neq 0 \tag{3.31}$$

with $c_j^{(m)} = 0$ for $m = -1, 0, 1$, all $j \in \mathbf{Z}$.

Setting $\ell = 0$ in (3.31), yields

$$j(c_j^{(m)} - c_j^{(0)}) = 0 \quad \text{and thus} \quad c_j^{(m)} = c_j^{(0)} \quad \text{for} \quad j \neq 0, -m,$$

implying

$$c_j^{(m)} = 0 \quad \text{all } m \geq -1 \quad \text{and} \quad j \neq 0, -m.$$

Also, setting $j = 0$ and $\ell = -1$ in (3.31) yields

$$mc_0^{(m)} = (m+1)c_0^{(m-1)} - c_0^{(-1)}, \qquad \text{for } m \geq 2,$$

implying by induction, since $c_0^{(-1)} = c_0^{(1)} = 0$

$$c_0^{(m)} = 0 \quad \text{for all} \quad m \geq -1,$$

concluding the proof.

Proof of Corollary 3.2.1: According to theorem 3.2, the vector field

$$\mathbf{Y}_{m,1} + \frac{m+1}{2}\mathbf{Y}_{m,0} = -(ML^{m+1} + \frac{m+1}{2}L^m)_b,$$

acting on Ψ, induces on τ_j the vector field

$$\begin{aligned}\mathcal{L}^j_{m,1} + \frac{m+1}{2}\mathcal{L}^j_{m,0} &= J^{(2)}_m + (2j-m-1)J^{(1)}_m + (j^2-j)\delta_{m,0} \\ &\quad + \frac{m+1}{2}(2J^{(1)}_m + 2j\delta_{m,0}) \\ &= J^{(2)}_m + 2jJ^{(1)}_m + j^2 J^{(0)}_m\end{aligned}$$

establishing the corollary.

Example : symmetries at $t = 0$.

By (2.4), the symmetries take on the following form

$$\mathbf{Y}_{\ell,1}L = L^{\ell+1} + [(ML^{\ell+1})_s, L],$$

with (see (1.23))

$$M = S\varepsilon S^{-1} + \frac{1}{2}\sum_1^\infty kt_k L^{k-1} = P + \frac{1}{2}\sum_1^\infty kt_k L^{k-1}.$$

Therefore, at $t = 0$,

$$\mathbf{Y}_{\ell,1}L\Big|_{t=0} = L^{\ell+1} + [(PL^{\ell+1})_s, L],$$

with

$$(PL^{\ell+1})_s = ([S,\nu]\delta^\ell S^{-1})_s + (\nu L^\ell)_s,$$

upon using (3.20). In the formula above, the wave operator $S = \gamma^{-1}\tilde{S}$ is given by (1.9) and one finds, by a computation similar to (3.21),

$$(\nu L^\ell)_s = \nu(L^\ell)_s + [\nu, (L^\ell)_-].$$

If b_i and a_i stand for the diagonal and off-diagonal elements of L respectively, i.e. $L = \delta^{-1}a + b\delta^0 + a\delta$, we have, in view of Lemma 3.4,

$$\mathbf{Y}_{-1,1}L\Big|_{t=0} = I, \qquad \mathbf{Y}_{0,1}L\Big|_{t=0} = L$$

and

$$\mathbf{Y}_{1,1}L\Big|_{t=0} \ : \ \begin{cases} \dot{b}_i = b_i^2 + (2i+1)a_i^2 - (2i-3)a_{i-1}^2 \\ \dot{a}_i = a_i((i+1)b_{i+1} - (i-1)b_i). \end{cases}$$

The subsequent symmetry vector fields can all be computed and are non-local; for instance $\mathbf{Y}_{2,1}L$ involves the coefficients $\frac{\partial}{\partial t_1}\log\tau_n = \sum_0^{n-1} b_i$ of δ^{-1} in $\gamma^{-1}\tilde{S}$ (see (1.9)).

4 Orthogonal polynomials, matrix integrals, skew-symmetric matrices and Virasoro constraints

Remember from the introduction the orthogonal (orthonormal) polynomial basis of $\mathcal{H}^+ = \{1, z, z^2, \dots\}$ on the interval $[a, b]$, $-\infty \leq a < b \leq \infty$,
(4.1)

$$\tilde{p}_r(t,z) = z^r + \dots \text{ (monic) and } p_r(t,z) = \frac{1}{\sqrt{h_r(t)}}\tilde{p}_r(t,z) \text{ (orthonormal)}, \; r \geq 0,$$

with regard to the t-dependent inner product (via the exponential $e^{\sum t_i z^i}$) :

$$\langle u, v\rangle_t = \int_a^b uv\rho_t dz, \text{ with } \rho_t(z) = e^{-V_0(z)+\sum_1^\infty t_i z^i} = \rho_0(z)e^{\sum t_i z^i}; \tag{4.2}$$

i.e.,

$$\langle \tilde{p}_i, \tilde{p}_j\rangle_t = h_i\delta_{ij} \text{ and } \langle p_i, p_j\rangle_t = \delta_{ij}.$$

Then the semi-infinite vector (of $\langle\,,\,\rangle_0$-orthonormal functions)

$$\Psi(t,z) := e^{\frac{1}{2}\Sigma t_i z^i} p(t,z) := e^{\frac{1}{2}\Sigma t_i z^i}(p_0(t,z), p_1(t,z), \dots)^\top, \tag{4.3}$$

satisfies the orthogonality relations

$$\langle (\Psi(t,z))_i, (\Psi(t,z))_j\rangle_0 = \langle p_i(t,z), p_j(t,z)\rangle_t = \delta_{ij}, \tag{4.4}$$

The weight is assumed to have the following property[14]:
(4.5)

$$-\frac{\rho_0'}{\rho_0} = V_0' = \frac{\sum_0^\infty b_i z^i}{\sum_0^\infty a_i z^i} = \frac{h_0(z)}{f_0(z)} \quad \text{with} \quad \rho_0(a)f_0(a)a^k = \rho_0(b)f_0(b)b^k = 0, \quad k \geq 0.$$

Define semi-infinite matrices L and P such that

$$zp(t,z) = L(t)p(t,z), \quad \frac{\partial}{\partial z}p(t,z) = Pp(t,z) \tag{4.6}$$

The ideas of Theorem 4.1 are due to Bessis-Itzykson-Zuber [BIZ] and Witten [W].

[14] the choice of f_0 is not unique. When V_0' is rational, then picking $f_0 =$ (polynomial in the denominator) is a canonical choice.

Theorem 4.1. *The semi-infinite vector $\Psi(t,z)$ and the semi-infinite matrices $L(t)$ (symmetric), and $M(t) := P(t) + \frac{1}{2}\sum_1^\infty kt_k L^{k-1}$, satisfy*

$$z\Psi(t,z) = L(t)\Psi(t,z), \quad \text{and} \quad \frac{\partial}{\partial z}\Psi(t,z) = M\Psi(t,z) \tag{4.7}$$

and

$$\frac{\partial L}{\partial t_n} = \frac{1}{2}[(L^n)_s, L], \quad \frac{\partial M}{\partial t_n} = \frac{1}{2}[(L^n)_s, M], \text{ and } \frac{\partial \Psi}{\partial t_n} = \frac{1}{2}(L^n)_s, \Psi; \tag{4.8}$$

the wave vector $\Psi(t,z)$ and the L^2-norms $h_n(t)$ admit the representation

$$\Psi(t,z) = e^{\frac{1}{2}\Sigma t_i z^i}\Big(z^n \frac{\tau_n(t-[z^{-1}])}{\sqrt{\tau_n(t)\tau_{n+1}(t)}}\Big)_{n\geq 0} \text{ and } h_n(t) = \frac{\tau_{n+1}(t)}{\tau_n(t)} \tag{4.9}$$

with

$$\tau_n(t) = \frac{1}{\Omega_n n!}\int_{\mathcal{M}_n(a,b)} e^{-Tr\, V_0(Z) + \Sigma t_i Tr\, Z^i} dZ; \tag{4.10}$$

the integration is taken over the space $\mathcal{M}_n(a,b)$ of $n \times n$ Hermitean matrices with eigenvalues $\in [a,b]$.

Proof: Step 1. Suppose $\dot\Psi = \mathcal{B}\Psi$ and $P(z, \frac{\partial}{\partial z})\Psi = \mathcal{P}\Psi$ where $\mathcal{B}$ and $\mathcal{P}$ are matrices and P is a polynomial with constant coefficients. Then

$$\dot{\mathcal{P}} = [\mathcal{B}, \mathcal{P}].$$

Indeed, this follows from differentiating $P(z, \frac{\partial}{\partial z})\Psi = \mathcal{P}\Psi$, and observing that

$$P\mathcal{B}\Psi = P\dot\Psi = \dot{\mathcal{P}}\Psi + \mathcal{P}\dot\Psi = \dot{\mathcal{P}}\Psi + \mathcal{P}\mathcal{B}\Psi.$$

Step 2. The matrix $L \in \mathcal{D}_{-\infty,1}$ is symmetric because the operation of multiplication by z is symmetric with respect to $\langle\, ,\, \rangle$ on $\mathcal{H}^+$ and is represented by L in the basis p_i. Moreover, $P \in \mathcal{D}_{-\infty,-1}$ and $[L,P] = [L,M] = 1$. Also for $k \geq 0$,

$$\frac{\partial \Psi}{\partial z} = \frac{\partial p}{\partial z} e^{\frac{1}{2}\sum_1^\infty t_i z^i} + \frac{1}{2}\sum_1^\infty i t_i z^{i-1} p e^{\frac{1}{2}\sum_1^\infty t_i z^i} = \left(P + \frac{1}{2}\Sigma i t_i L^{i-1}\right)\Psi = M\Psi. \tag{4.11}$$

establishing (4.7).

Step 3. We now prove the first statement of (4.8). Since $\partial p_k/\partial t_i$ is again a polynomial of the same degree as p_k, we have:

$$\frac{\partial p_k}{\partial t_i} = \sum_{0\le \ell\le k} A^{(i)}_{k\ell} p_\ell, \quad A^{(i)} \in \mathcal{D}_b. \tag{4.12}$$

The precise nature of $A^{(i)}$ is found as follows: for $\ell < k$,

$$\begin{aligned}
0 &= \frac{\partial}{\partial t_i}\int p_k(z)p_\ell(z)\rho_t(z)dz \\
&= \int \frac{\partial p_k}{\partial t_i} p_\ell \rho_t(z)dz + \int p_k \frac{\partial p_\ell}{\partial t_i}\rho_t(z)dz + \int dz(\frac{\partial}{\partial t_i} e^{-V_0+\Sigma t_j z^j}) p_k p_\ell, \quad \text{using } \langle p_i, p_j\rangle = \delta_{ij} \\
&= \int \sum_{j\le k} A^{(i)}_{kj} p_j p_\ell \rho_t(z)dz + \int \sum_j (L^i)_{kj} p_j p_\ell \rho_t(z)dz, \quad \text{using (4.12) and (4.6)} \\
&= A^{(i)}_{k\ell} + (L^i)_{k\ell}
\end{aligned}$$

and for $\ell = k$,

$$\begin{aligned}
0 = \frac{\partial}{\partial t_i}\int (p_k(z))^2 \rho_t(z)dz &= 2\int \sum_{m\le k} A^{(i)}_{km} p_m p_k \rho_t(z)dz + \int \sum_j (L^i)_{kj} p_j p_k \rho_t(z)dz \\
&= 2A^{(i)}_{kk} + (L^i)_{kk},
\end{aligned}$$

implying

$$A^{(i)} = -(L^i)_- - \frac{1}{2}(L^i)_0 = -\frac{1}{2}(L^i)_b, \tag{4.13}$$

and thus

$$\frac{\partial \Psi}{\partial t_i} = \frac{\partial}{\partial t_i} e^{\Sigma/2} p = \frac{1}{2} e^{\Sigma/2} z^i p - \frac{1}{2} e^{\Sigma/2}(L^i)_b p = \frac{1}{2}(L^i - (L^i)_b)\Psi = \frac{1}{2}(L^i)_s \Psi.$$

Now using step 1, we have immediately (4.8). So Ψ satisfies the Toda equations (1.16) and behaves asymptotically as:

$$\Psi = e^{\frac{1}{2}\Sigma t_i z^i}\Big(\frac{1}{\sqrt{h_n}} z^n (1 + O(z^{-1}))\Big)_{n\in\mathbf{Z}} \tag{4.14}$$

Therefore in view of (1.15), we must have

$$\sqrt{h_n} = \gamma_n = \sqrt{\frac{\tau_{n+1}}{\tau_n}}.$$

Step 4. The integration (4.10) is taken with respect to the invariant measure

$$dZ=\prod_{1\le i\le n}dZ_{ii}\prod_{1\le i<j\le n}d(ReZ_{ij})d(ImZ_{ij}). \tag{4.15}$$

Since the integrand only depends on the spectrum of Z and since the measure separates into an "angular" and a "radial" part, one first integrates out the former, accounting for the Ω_n and next the latter, in terms of the monic orthogonal polynomials $\tilde p_i$:

$$\begin{aligned}I_n&=\Omega_n\int_{[a,b]^n}dz_1\dots dz_n\prod_{1\le i<j\le n}(z_i-z_j)^2\prod_{i=1}^n e^{-V(z_i)},\\&=\Omega_n\int_{[a,b]^n}dz_1\dots dz_n(\det(\tilde p_{i-1}(z_j))_{1\le i\le j\le n})^2\prod_1^n e^{-V(z_i)}\\&=\Omega_n n!\int_a^b dz_1\rho_t(z_1)\tilde p_0(z_1)^2\dots\int_a^b dz_n\rho_t(z_n)\tilde p_{n-1}(z_n)^2\\&=\Omega_n n!h_0\dots h_{n-1}\\&=\Omega_n n!\tau_0\frac{\tau_1}{\tau_0}\frac{\tau_2}{\tau_1}\dots\frac{\tau_n}{\tau_{n-1}}=\Omega_n n!\tau_n,\ \text{ using }\tau_0=1\end{aligned} \tag{4.16}$$

ending the proof of Theorem 4.1.

Next we show that the wave vector constructed from the orthogonal polynomial basis is a fixed point for an algebra of symmetries, which in turn implies Virasoro-like constraints on τ. The skew-symmetry of the matrix Q below had been pointed out by E. Witten [W] in the context of Hermite polynomials. The Virasoro constraints for the matrix integrals with the weight $\rho_0=e^{-z^2}$ had been computed by E. Witten [W], Gerasimov, Marshakov, Mironov, Morozov and Orlov [GMMMO]; they relate to the deformations of Hermite polynomials. The case of deformations of Laguerre polynomials was worked out by Haine & Horozov [HH] and applied to questions of highest weight representation of the Virasoro algebra.

Theorem 4.2. *Consider the semi-infinite wave vector $\Psi(t,z)$, arising in the context of orthogonal polynomials with a weight $\rho_t(z)$ as in (4.2) and satisfying (4.5). Then $\Psi(t,z)$ is a fixed point for a Lie algebra of symmetry vector fields; that is*

$$\mathbf{V}_m\Psi:=-(V_m)_b\Psi=0,\quad \textit{for}\quad m\ge -1; \tag{4.17}$$

the symmetries $\mathbf{V}_m$ form a (non-standard) representation of Diff$(S^1)^+$:

$$[\mathbf{V}_m,\mathbf{V}_n]=(m-n)\sum_{i\ge 0}a_i\mathbf{V}_{m+n+i},\qquad -1\le n,m<\infty, \tag{4.18}$$

and are defined by the semi-infinite matrices[15] [16]

$$V_m := \{Q, L^{m+1}\} = QL^{m+1} + \frac{m+1}{2} L^m f_0(L), \text{ with } Q := Mf_0(L) + g_0(L), \tag{4.19}$$

which are skew-symmetric on the locus of $\Psi(t,z)$ *above. Moreover* Q *is a solution of the "string equation"*

$$[L, Q] = f_0(L), \tag{4.20}$$

and the τ*-vector satisfies the Virasoro constraints*

(4.21)
$$\mathcal{V}_m^{(n)} \tau_n = \sum_{i \geq 0} \left(a_i (J^{(2)}_{i+m} + 2n\, J^{(1)}_{i+m} + n^2\, J^{(0)}_{i+m}) - b_i (J^{(1)}_{i+m+1} + n\, J^{(0)}_{i+m+1}) \right) \tau_n = 0,$$

$$\text{for } m = -1, 0, 1, \dots, \; n = 0, 1, 2, \dots,$$

with the $\mathcal{V}_m^{(n)}, m \geq -1$ *(for fixed* $n \geq 0$*) satisfying the same Virasoro relations as (4.18), except for an additive constant.*

In preparation of the proof we give some elementary lemmas.

Lemma 4.3. *Consider operators* S_1 *and* S_f *acting on a suitable space of functions of* z*, such that* $[S_1, z] = 1$ *and* $S_f = \sqrt{f} S_1 \sqrt{f}$*; then the following holds*[17]*:*
(i) $S_f = \{S_1, f\}$ *and* $[S_f, z] = f$,
(ii) $[\{S_1, h_1\}, \{S_1, h_2\}] = \{S_1, (h_1, h_2)\}$
(iii)$\{S_1, h_1 h_2\} = \{\{S_1, h_1\}, h_2\} = \{\{S_1, h_2\}, h_1\}$
(iv) $[\{S_1, fz^{m+1}\}, \{S_1, fz^{n+1}\}] = (n-m)\{S_1, f^2 z^{m+n+1}\} = (n-m)\{\{S_1, fz^{m+n+1}\}, f\}$
(v) the operators $\{S_1, z^{m+1}\},\ m \in \mathbf{Z}$ *form a representation of Diff*(S^1),

$$[\{S_1, z^{m+1}\}, \{S_1, z^{n+1}\}] = (n-m)\{S_1, z^{m+n+1}\}$$

(vi) given $f(z) = \sum_{i \geq 0} a_i z^i$*, the* $\{S_1, fz^{m+1}\},\ m \in \mathbf{Z}$ *also form a representation of Diff*(S')*:*

$$[\{S_1, fz^{m+1}\}, \{S_1, fz^{n+1}\}] = (n-m) \sum_{i \geq 0} a_i \{S_1, fz^{m+n+i+1}\},$$

[15] in terms of the anticommutator (0.20)
[16] set $g_0 := \frac{(f_0 \rho_0)'}{2\rho_0} = \frac{f_0' - h_0}{2}$, with $h_0 := -\frac{f_0 \rho_0'}{\rho_0}$
[17] $(h_1, h_2) = h_1 h_2' - h_2 h_1'$ denotes the Wronskian

the map to the standard generators, $f^{-1}(z)=\sum_{i\geq -k}\bar{a}_i z^i$*,*

$$\{S_1, fz^{m+1}\}\mapsto\{S_1,z^{m+1}\}=\{S_1,f^{-1}\cdot fz^{m+1}\}=\sum_{i\geq -k}\bar{a}_i\{S_1,fz^{m+i+1}\}.$$

Proof: $[S_1,z]=1$ implies $[S_1,z^n]=nz^{n-1}$, since $[S_1,.]$ is a derivation, and thus $[S_1,h]=h'$, which leads to

$$S_f=\sqrt{f}S_1\sqrt{f}=\sqrt{f}[S_1,\sqrt{f}]+fS_1=\frac{1}{2}(2fS_1+f')=\frac{1}{2}(2fS_1+[S_1,f])=\{S_1,f\}.$$

The second part of (i),(ii) and (iii) follows by direct computation; (iv) is an immediate consequence of (ii), (iii) and the Wronskian identity

$$(fz^{m+1},fz^{n+1})=(n-m)f^2z^{m+n+1}.$$

The Virasoro relations (v) follow immediately from (iv), whereas (vi) follows from the argument:

$$\begin{aligned}[\{S_1,fz^{m+1}\},\{S_1,fz^{n+1}\}]&=(n-m)\{S_1,f^2z^{m+n+1}\}\\&=(n-m)\{\{S_1,fz^{m+n+1}\},f\},\quad\text{by }(iii)\\&=(n-m)\sum_{i\geq 0}a_i\{\{S_1,fz^{m+n+1}\},z^i\}\\&=(n-m)\sum_{i\geq 0}a_i\{S_1,fz^{m+n+i+1}\},\quad\text{by }(iii)\end{aligned}$$

ending the proof of lemma 4.3.

Lemma 4.4. *Consider the function space* $\mathcal{H}=\{\dots,z^{-1},1,z,\dots\}$ *with a real inner product* $\langle u,v\rangle_\rho=\int_a^b uv\rho dz$*,* $-\infty\leq a<b\leq\infty$ *with regard to the weight* ρ*, with* $\rho(a)=\rho(b)=0$*; also consider an arbitrary function* $f=\sum_{i\geq 0}a_iz^i$ *with* $f(a)\rho(a)a^m=f(b)\rho(b)b^m=0$*, for* $m\in\mathbf{Z}$*. Then the first-order differential operator from* $\mathcal{H}$ *to* $\mathcal{H}$

$$S=f\frac{d}{dz}+g,$$

is skew-symmetric for $\langle\ ,\rangle_\rho$ *if and only if* S *takes on the form*

$$S=\sqrt{\frac{f}{\rho}}\frac{d}{dz}\sqrt{f\rho}=\{S_1,f\},\ \text{where } S_1=\frac{1}{\sqrt{\rho}}\frac{d}{dz}\sqrt{\rho}=\frac{d}{dz}+\frac{1}{2}\frac{\rho'}{\rho}\qquad(4.22)$$

So, the operators $\{S_1,z^{m+1}\}$ *and* $\{S_1,fz^{m+1}\}$*, for* $m\in\mathbf{Z}$*, form representations of Diff*(S^1) *in the space of skew-symmetric operators* $so(\mathcal{H},<,>_\rho)$*.*

Proof: First compute the expressions:

$$\begin{aligned}\langle Su, v\rangle &= \int_a^b f\frac{du}{dz}v\rho\, dz + \int_a^b guv\rho dz \\ &= uvf\rho\,|_a^b - \int_a^b u\frac{d}{dz}(vf\rho)dz + \int_a^b guv\rho dz \\ &= \int_a^b \rho u\Big[\rho^{-1}(-\frac{d}{dz}f + g)(\rho v)\Big]dz, \text{ using } f\rho(a) = f\rho(b) = 0\end{aligned}$$

and

$$\langle u, Sv\rangle = \int_a^b \rho u(f\frac{d}{dz} + g)v dz.$$

Imposing S skew, i.e., $\langle Su, v\rangle = \langle u, S^T v\rangle = -\langle u, Sv\rangle$, leads to the operator identity

$$\rho^{-1}(-\frac{d}{dz}f + g)\rho = -(f\frac{d}{dz} + g),$$

in turn, leading to $g = \frac{1}{2}\rho^{-1}(f\rho)'$; thus S takes on the form (4.22). The last part of the proof of Lemma 4.4 follows at once from the above and Lemma 4.3 (*iv*) and (*v*).

Lemma 4.5. *Consider the above inner-product $\langle u, v\rangle$ in the space $\mathcal{H}^+ = \{1, z, z^2, \ldots\}$, for the weight ρ having a representation of the form*

$$-\frac{\rho'}{\rho} = \frac{\sum_{i\geq 0} b_i z^i}{\sum_{i\geq 0} a_i z^i} \equiv \frac{h}{f}\ . \tag{4.23}$$

Let $\mathcal{H}^+$ have an orthonormal basis of functions $(\varphi_k)_{k\geq 0}$; then the operators $\{S_1, fz^{m+1}\}$ for $m \geq -1$ are maps from $\mathcal{H}^+$ to $\mathcal{H}^+$ and its representing matrices in that basis (i.e., $(\langle\{S_1, fz^{m+1}\}\varphi_k, \varphi_\ell\rangle)_{k,\ell\geq 0}$), for $m \geq -1$ form a closed Lie algebra $\subset so(0,\infty)$[18].

Remark: The operators $\{S_1, z^{m+1}\}$ do not map $\mathcal{H}^+$ in $\mathcal{H}^+$.

Proof: The operators,

$$\{S_1, fz^{m+1}\} = \Big\{\{S_1, f\}, z^{m+1}\Big\} : \mathcal{H} \to \mathcal{H}, \quad \text{for} \quad m \geq -1,$$

[18] $so(0,\infty)$ denotes the Lie algebra of semi-infinite skew-symmetric matrices.

which are skew-symmetric by Lemma 4.4, preserve the subspace $\mathcal{H}^+$, since by virtue of (4.22),

$$\{S_1, f\} = f\frac{d}{dz} + \frac{(f\cdot\rho)'}{2\rho} = f\frac{d}{dz} + \frac{f'-h}{2} \tag{4.24}$$

contains holomorphic series f and $f'-h$, by (4.23). In a basis of functions $(\varphi_k)_{k\geq 0}$, orthonormal with respect to $\langle\, ,\, \rangle_\rho$, the corresponding matrices will also be skew-symmetric.

Proof of Theorem 4.2: According to Lemma 4.5, the operators

$$T_m := T_m^{(\rho_0,f_0)} := \{\frac{1}{\sqrt{\rho_0}}\frac{d}{dz}\sqrt{\rho_0}, f_0 z^{m+1}\} = \big\{\{\frac{1}{\sqrt{\rho_0}}\frac{d}{dz}\sqrt{\rho_0}, f_0\}, z^{m+1}\big\}, \quad m\geq -1 \tag{4.25}$$

map $\mathcal{H}^+$ into $\mathcal{H}^+$ and form an algebra with structure constants:

$$[T_m, T_n] = (n-m)\sum_{i\geq 0} a_i T_{m+n+i}, \quad m,n\geq -1. \tag{4.26}$$

Under the map ϕ (Theorem 2.2), the operators T_m get transformed into matrices $V_m = \phi(T_m)$, such that

$$T_m\Psi(t,z) = V_m\Psi(t,z); \tag{4.27}$$

namely (see footnote 15)

$$Q := V_{-1} = \phi(T_{-1}) = \phi\big(\{\frac{1}{\sqrt{\rho_0}}\frac{d}{dz}\sqrt{\rho_0}, f_0\}\big) = \phi\left(f_0\frac{d}{dz} + g_0\right) = Mf_0(L) + g_0(L) \tag{4.28}$$

and

$$\begin{aligned} V_m :&= \phi(T_m) \\ &= \phi(\{T_{-1}, z^{m+1}\}) \\ &= \phi(z^{m+1}T_{-1} + \frac{1}{2}[T_{-1}, z^{m+1}]) \\ &= \phi(z^{m+1}T_{-1} + \frac{1}{2}[f_0\frac{d}{dz}, z^{m+1}]) \\ &= \phi(z^{m+1}T_{-1} + \frac{m+1}{2}f_0 z^m) \end{aligned} \tag{4.29}$$

$$= QL^{m+1} + \frac{m+1}{2} L^m f_0(L)$$
$$= \sum_{i\geq 0} a_i M L^{i+m+1} + \sum_{i\geq 0} \frac{(i+1)a_{i+1} - b_i}{2} L^{i+m+1} + \frac{m+1}{2} \sum_{i\geq 0} a_i L^{i+m}$$
$$= \sum_{i\geq 0} a_i (M L^{i+m+1} + \frac{i+m+1}{2} L^{i+m}) - \sum_{i\geq 0} \frac{b_i}{2} L^{i+m+1},$$

where we used

$$f_0 = \sum_{i\geq 0} a_i z^i \quad \text{and} \quad g_0 = \frac{(f_0\rho_0)'}{2\rho_0} = \frac{1}{2}\sum_{i\geq 0}((i+1)a_{i+1} - b_i) z^i.$$

In addition, according to Lemma 4.5, the z-operators T_m are skew-symmetric with regard to $\langle\,,\rangle_0$ and thus form a representation of $\mathrm{Diff}(S^1)^+$

$$\mathrm{Diff}(S^1)^+ \longrightarrow so(\mathcal{H}^+, \langle\,,\rangle_0) := \left\{ \begin{array}{l} \text{skew-symmetric operators} \\ \text{on } \mathcal{H}^+, \langle\,,\,\rangle_0 \end{array} \right\},$$

with structure constants given by (4.18). The components $e^{\sum t_i z^i} p_n(t,z)$, $n \geq 0$ of $\Psi(t,z)$ form an orthonormal basis of $\mathcal{H}^+$, $\langle\,,\,\rangle_0$, with regard to which the operators T_m are represented by semi-infinite skew-symmetric matrices; i.e., the anti-homomorphism ϕ restricts to the following map

$$\phi : so(\mathcal{H}^+, \langle\,,\,\rangle_0) \longrightarrow so(0,\infty) \quad \text{(anti-homomorphism)}.$$

Hence the matrices V_m, $m \geq -1$ are skew-symmetric (i.e., $(V_m)_b = 0$) and thus, using (4.29), we have

(4.30) $$0 = \mathbf{V}_m \Psi = -(V_m)_b \Psi$$
$$= \left(\sum_{i\geq 0} a_i (M L^{i+m+1} + \frac{i+m+1}{2} L^{i+m})_b - \sum_{i\geq 0} \frac{b_i}{2} (L^{i+m+1})_b \right) \Psi.$$

In the final step, Theorem 3.2 and Corollary 3.2.1 lead to the promised τ-constraints (4.21), modulo a constant, i.e.,

(4.31) $$\mathcal{V}_m^{(k)} \tau_k = c_m^{(k)} \tau_k \qquad m \geq -1, k \geq 0.$$

By (3.27), this constant is independent of k, i.e.,

$$c_m^{(k)} = c_m^{(0)};$$

upon evaluating (4.31) at $k = 0$ and upon using $\tau_0 = 1$, we conclude $c_m^{(0)} = 0$, yielding (4.21), as claimed. Finally the map

$$T_m \longmapsto \mathcal{V}_m, \qquad m \geq -1$$

is an anti-homomorphism (modulo constants) by Lemma 3.1; we also have

$$\phi : [T_{-1}, z] = f_0(z) \longmapsto [L, Q] = f_0(L), \tag{4.32}$$

which is the "string equation", concluding the proof of Theorem 4.2.

Remark 4. : Note that, if $f_0^{-1} = \sum_{i\geq -k} \bar{a}_i z^i$, the map

$$T_m \mapsto \bar{T}_m = \sum_{i\geq -k} \bar{a}_i T_{m+i}, \quad m \geq k-1 \tag{4.33}$$

sends T_m into the standard representation of Diff(S^1):

$$[\bar{T}_m, \bar{T}_n] = (n-m)\bar{T}_{m+n}, \quad m, n \geq k-1, \tag{4.34}$$

according to Lemma 4.3.

Example. In the next section we shall consider the classical orthogonal polynomials; we consider here, for a given polynomial $q(z)$, in the interval $[a, b]$ the weight

$$\rho_0 = (z-a)^\alpha e^{q(z)} (z-b)^\beta, \quad \text{with} \quad f_0 = (z-a)(z-b) \quad \text{and} \quad \alpha, \beta \in \mathbf{Z}, \geq 1.$$

It implies that (see footnote 15)

$$f_0(a)\rho_0(a) = f_0(b)\rho_0(b) = 0$$

and that both f_0 and

$$g_0 = \frac{(f_0\rho_0)'}{2\rho_0} = z - \frac{(z-a)(z-b)}{2}\left(\frac{\alpha}{z-a} + q' + \frac{\beta}{z-b}\right) - \frac{a+b}{2},$$

are polynomial. Then $Q = Mf_0(L) + g_0(L)$ is skew-symmetric and $[L, Q] = f_0(L)$. The Virasoro constraints (4.21) are then given by a finite sum.

5 Classical orthogonal polynomials

It is interesting to revisit the classical orthogonal polynomials, from the point of view of this analysis. As a main feature, we note that, in this case, not only is L (multiplication by z) symmetric and tridiagonal, but there exists another operator, a first-order differential operator, which yields a *skew-symmetric and tridiagonal matrix*. It is precisely given by the matrix Q!

The classical orthogonal polynomials are characterized by Rodrigues' formula,

$$p_n = \frac{1}{K_n \rho_0} (\frac{d}{dz})^n (\rho_0 X^n),$$

K_n constant, $X(z)$ polynomial in z of degree ≤ 2, and $\rho_0 = e^{-V_0}$.Compare Rodrigues' formula for $n = 1$ with the one for g_0, (see theorem 4.2, footnote 15)

$$\frac{K_1}{2} p_1 = \frac{1}{2\rho_0} \frac{d}{dz} (\rho_0 X) \quad \text{and} \quad g_0 = \frac{1}{2\rho_0} \frac{d}{dz} (\rho_0 f_0),$$

which leads to the natural identification

$$g_0 = \frac{K_1}{2} p_1 \text{ and } f_0 = X$$

and thus

$$T_{-1} = f_0 \frac{d}{dz} + g_0 = X \frac{d}{dz} + \frac{K_1 p_1}{2} = X \frac{d}{dz} + \frac{X' - XV_0'}{2}.$$

Since both (degree X) ≤ 2 and (degree $(X' - XV_0')$) ≤ 1, as will appear from the table below, we have that T_{-1}, acting on polynomials, raises the degree by at most 1:

$$T_{-1} p_k(0, z) = \sum_{i \leq k+1} Q_{ki} p_i(0, z),$$

while, since Q is skew-symmetric,

$$(6.1) \qquad T_{-1} p_k(0, z) = (Qp(0, z))_k = -Q_{k,k-1} p_{k-1}(0, z) + Q_{k,k+1} p_{k+1}(0, z),$$

together with

$$(6.2) \qquad z p_k(0, z) = L_{k-1,k} p_{k-1}(0, z) + L_{k,k} p_k(0, z) + L_{k,k+1} p_{k+1}(0, z).$$

This implies at the level of the flag

$$\ldots \supset W_{k-1}^t \supset W_k^t \equiv \text{ span}\{(\Psi)_k, (\Psi)_{k+1}, \ldots\} \supset W_{k+1}^t \supset \ldots$$

that

$$zW_k^t \subset W_{k-1}^t \quad \text{and} \quad T_{-1}W_k^t \subset W_{k-1}^t.$$

Thus the recursion operators z and T_{-1} serve to characterize the flag and so the wave vector Ψ. It is interesting to speculate on considering "(p,q)-cases", where, for instance,

$$z^p W_k^t \subset W_{k-p}^t \quad \text{and} \quad T_{-1}W_k^t \subset W_{k-q}^t.$$

The existence of two operators, a symmetric and a skew-symmetric one, both represented by tridiagonal matrices, probably characterize the orthogonal polynomials on the line. Related, it is interesting to point out a conjecture by Karlin and Szegö and a precise formulation by Al-Salam and Chihara, were classical orthogonal polynomials are characterized by orthogonality and the existence of a differentiation formula of the form

$$f_0(z)p_n'(z) = (\alpha_n z + \beta_n)p_n(z) + \gamma_n p_{n-1}(z).$$

We now have the following table:

	Hermite	Laguerre	Jacobi
$e^{-V_0(z)}dz$	$e^{-z^2}dz$	$e^{-z}z^\alpha dz$	$(1-z)^\alpha(1+z)^\beta dz$
(a,b)	$(-\infty,\infty)$	$(0,\infty)$	$(-1,1)$
$T_{-1} = f_0\frac{d}{dz} + g_0$	$\frac{d}{dz} - z$	$z\frac{d}{dz} - \frac{1}{2}(z-\alpha-1)$	$(1-z^2)\frac{d}{dz}$ $-\frac{1}{2}((\alpha+\beta+2)z + (\alpha-\beta))$
string equation	$[L,Q]=1$	$[L,Q]=L$	$[L,Q]=1-L^2$

We now give a detailed discussion for each case:

(a) weight $e^{-z^2}dz$.

The corresponding (monic) orthogonal Hermite polynomials satisfy the classic relations

$$z\tilde{p}_n = \frac{n}{2}\tilde{p}_{n-1} + \tilde{p}_{n+1}, \quad \text{and} \quad \frac{d}{dz}\tilde{p}_n = n\tilde{p}_{n-1}.$$

Therefore the matrices defined by

$$z\tilde{p}_n = \frac{n}{2}\tilde{p}_{n-1} + \tilde{p}_{n+1} \qquad (\frac{d}{dz} - z)\tilde{p}_n = \frac{n}{2}\tilde{p}_{n-1} - \tilde{p}_{n+1}$$

can be turned simultaneously into symmetric and skew-symmetric matrices L and $Q = M - L$ respectively, by an appropriate diagonal conjugation. The string equation reads $[L, Q] = 1$ and the matrix integrals τ_n satisfy

$$\mathcal{V}_m^{(n)}\tau_n = (J_m^{(2)} + 2nJ_m^{(1)} - 2J_{m+2}^{(1)} + n^2\delta_{m,0})\tau_n = 0, m = -1, 0, 1, \ldots$$

upon using formula (7.12) for $a_0 = 1$, $b_1 = 2$ and all other $a_i, b_i = 0$; this captures the original case of Bessis-Itzykson-Zuber and Witten []; Witten had pointed out in his Harvard lecture that $M - L$ is a skew-symmetric matrix.

(b) weight $e^{-z}z^{\alpha}dz$.
Again the classic relations for (monic) Laguerre polynomials,

$$\begin{aligned} z\tilde{p}_n &= n(n+\alpha)\tilde{p}_{n-1} + (2n+\alpha+1)\tilde{p}_n + \tilde{p}_{n+1} \\ z\frac{d}{dz}\tilde{p}_n &= n(n+\alpha)\tilde{p}_{n-1} + n\tilde{p}_n \end{aligned}$$

yield symmetric and skew-symmetric matrices L and Q, after conjugation of

$$\begin{aligned} z\tilde{p}_n &= n(n+\alpha)\tilde{p}_{n-1} + (2n+\alpha+1)\tilde{p}_n + \tilde{p}_{n+1} \\ (2z\frac{\partial}{\partial z} - (z - \alpha - 1))\tilde{p}_n &= n(n+\alpha)\tilde{p}_{n-1} + 0.\tilde{p}_n - \tilde{p}_{n+1}. \end{aligned}$$

Setting $a_1 = 1$, $b_0 = -\alpha$, $b_1 = 1$ and all other $a_i = b_i = 0$, yields

$$\mathcal{V}_m^{(n)}\tau_n = (J_m^{(2)} + 2nJ_m^{(1)} + \alpha J_m^{(1)} - J_{m+1}^{(1)} + n(n+\alpha)\delta_{m,0})\tau_n = 0, m = 0, 1, 2, \ldots,$$

and the string equation $[L, Q] = L$; this case was investigated by Haine and Horozov [HH].

(c) weight $(1-z)^{\alpha}(1+z)^{\beta}dz$.
The matrices L and Q will be defined by the operators acting on (monic) Jacobi polynomials

$$\begin{aligned} z\tilde{p}_n &= A_{n-1}\tilde{p}_{n-1} + B_n\tilde{p}_n + \tilde{p}_{n+1} \\ -(\frac{1}{n+1} + \frac{\alpha+\beta}{2})^{-1}((1-z^2)\frac{d}{dz} + \frac{(f_0\rho_0)'}{2\rho_0})\tilde{p}_n &= -A_{n-1}\tilde{p}_{n-1} + \tilde{p}_{n+1} \end{aligned}$$

with

$$\begin{aligned} A_{n-1} &= \frac{4n(n+\alpha+\beta)(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)(2n+\alpha+\beta-1)} \\ B_n &= -\frac{\alpha^2-\beta^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}. \end{aligned}$$

Setting

$$a_0 = 1, a_1 = 0, a_2 = -1, b_0 = \alpha - \beta, b_1 = \alpha + \beta$$

and all other $a_i = b_j = 0$ leads to the constraints

$$(J_m^{(2)} - J_{m-2}^{(2)} - 2nJ_{m-2}^{(1)} + 2nJ_m^{(1)} + (\alpha-\beta)J_{m-1}^{(1)} + (\alpha+\beta)J_m^{(1)} - n^2\delta_{m,2} + n(\alpha-\beta)\delta_{m,1})\tau_n = 0,\ m = 1, 2, 3, \ldots$$

and the string equation $[L, Q] = 1 - L^2$. Gegenbauer ($\alpha = \beta = \lambda - 1/2$) and Legendre ($\alpha = \beta = 0$) polynomials are limiting cases of Jacobi polynomials.

6 Appendix: Virasoro constraints via the integrals

The Virasoro constraints for the integrals can be shown in a direct way:

Lemma A.1. *Given $f_0(z) = \sum_{j\geq 0} a_j z^j$, the following holds for $k \geq -1$:*

$(A.1)$

$$e^{\sum_1^\infty t_i Z^i} \frac{\partial}{\partial \varepsilon} e^{\varepsilon f_0(Z) Z^{k+1} \frac{\partial}{\partial Z}} dZ \Big|_{\varepsilon=0} = dZ \sum_{r\geq 0} a_r \Big(\sum_{i+j=r+k} \frac{\partial^2}{\partial t_i \partial t_j} + 2n \frac{\partial}{\partial t_{r+k}} + n^2 \delta_{r+k} \Big) e^{\sum_1^\infty t_i Z^i},$$

where $\partial/\partial t_j = 0$ for $j \leq 0$.

Proof: We break up the proof of this Lemma in elementary steps, involving the diagonal part of dZ, i.e.,

$$dZ = dz_1 \ldots dz_n \Delta(z)^2 \times \text{ angular part.}$$

At first, we compute for $k \geq 0$:

$$(A.2) \qquad \frac{2}{\Delta(z)} \frac{\partial}{\partial \varepsilon} \Big(e^{a\varepsilon \sum_{i=1}^n z_i^{k+1} \frac{\partial}{\partial z_i}} \Big) \Delta(z)$$

$$\begin{aligned} &= 2 \frac{\partial}{\partial \varepsilon} \log \Delta(z_1 + a\varepsilon z_1^{k+1}, \ldots, z_n + a\varepsilon z_n^{k+1}) \Big|_{\varepsilon=0} \\ &= 2a \left(\sum_{\substack{1\leq\alpha<\beta\leq n \\ i+j=k \\ i,j>0}} z_\alpha^i z_\beta^j + (n-1) \sum_{1\leq\alpha\leq n} z_\alpha^k \right) - an(n-1)\delta_{k,o} \end{aligned}$$

and for $k \geq -1$

$$(A.3) \qquad \frac{\frac{\partial}{\partial \varepsilon} e^{a\varepsilon \sum_{i=1}^n z_i^{k+1} \frac{\partial}{\partial z_i}} dz_1 \ldots dz_n \Big|_{\varepsilon=0}}{dz_1 \ldots dz_n}$$

$$= a\frac{\partial}{\partial\varepsilon}\prod_{\alpha=1}^{n}(1+\varepsilon(k+1)z_\alpha^k)\Big|_{\varepsilon=0}$$
$$= a(k+1)\sum_{1\leq\alpha\leq n} z_\alpha^k.$$

Note that both expressions (A.2) and (A.3) vanish for k=-1. Also, we have for $k \geq 1$,

$$(A.4) \qquad \frac{\sum\limits_{\substack{i+j=k\\ i,j>0}} \frac{\partial^2}{\partial t_i \partial t_j} e^{\sum\limits_{\substack{1\leq i\leq\infty\\ 1\leq\alpha\leq n}} t_i z_\alpha^i}}{e^{\sum\limits_{\substack{1\leq i\leq\infty\\ 1\leq\alpha\leq n}} t_i z_\alpha^i}} = 2\sum_{\substack{1\leq\alpha<\beta\leq n\\ i+j=k\\ i,j>0}} z_\alpha^i z_\beta^j + (k-1)\sum_{1\leq\alpha\leq n} z_\alpha^k$$

and

$$(A.5) \qquad (2n\frac{\partial}{\partial t_k}\log)\ e^{\sum\limits_{\substack{1\leq i\leq\infty\\ 1\leq\alpha\leq n}} t_i z_\alpha^i} = 2n\sum_{1\leq\alpha\leq n} z_\alpha^k.$$

Summing up (A.2) and (A.3) yields for $k \geq 0$:

$$\frac{\frac{\partial}{\partial\varepsilon} e^{a\varepsilon\sum_{i=1}^n z_i^{k+1}\frac{\partial}{\partial z_i}} \Delta^2(z)dz_1\dots dz_n\Big|_{\varepsilon=0}}{\Delta^2(z)dz_1\dots dz_n} = a\left(2\sum_{\substack{1\leq\alpha<\beta\leq n\\ i+j=k\\ i,j>0}} z_\alpha^i z_\beta^j + (2n+k-1)\sum_{1\leq\alpha\leq n} z_\alpha^k - n(n-1)\delta_k\right);$$

this expression vanishes for $k=-1$. This expression equals the sum of (A.4) and (A.5); thus

$$\frac{\frac{\partial}{\partial\varepsilon} e^{a\varepsilon\sum_{i=1}^n z_i^{k+1}\frac{\partial}{\partial z_i}} \Delta^2(z)dz_1\dots dz_n\Big|_{\varepsilon=0}}{\Delta^2(z)dz_1\dots dz_n} = \frac{a\left(\sum\limits_{\substack{i+j=k\\ i,j>0}}\frac{\partial^2}{\partial t_i\partial t_j} + 2n\frac{\partial}{\partial t_k} + \delta_k n^2\right) e^{\sum\limits_{\substack{1\leq i\leq\infty\\ 1\leq\alpha\leq n}} t_i z_\alpha^i}}{e^{\sum\limits_{\substack{1\leq i\leq\infty\\ 1\leq\alpha\leq n}} t_i z_\alpha^i}}$$

establishing Lemma A.1.

Theorem A.2. *Let the weight* $\rho_0 = e^{-V_0}$ *have a representation (not necessarily unique) of the form*

$$V_0' = \frac{\sum_{i\geq 0} b_i z^i}{\sum_{i\geq 0} a_i z^i} \equiv \frac{h_0}{f_0}.$$

Then the matrix integral $\tau_n = \frac{I_n}{\Omega_n n!}$ satisfies the KP equation, and Virasoro-like constraints:

$$\sum_{i\geq 0}\Big(a_i(J^{(2)}_{i+m} + 2n\,J^{(1)}_{i+m} + n^2\,J^{(0)}_{i+m}) - b_i(J^{(1)}_{i+m+1} + n\,J^{(0)}_{i+m+1})\Big)\,\tau_n = 0 \quad \textit{with } m \geq -1.$$

Proof: Shifting the integration variable Z by means of

$$Z \mapsto Z + \varepsilon f_0(Z)Z^{m+1}, \quad m \geq -1$$

and using the notation

$$\Phi(Z) = e^{-Tr\,V_0(Z)+\Sigma t_i Tr\,Z^i},$$

we compute, since the integral remains unchanged, that

$$\begin{aligned}
0 &= \frac{\partial}{\partial\varepsilon}\int_{\mathcal{M}_n} e^{\varepsilon f_0(Z)Z^{m+1}\frac{\partial}{\partial Z}}\,\Phi(Z)dZ\Big|_{\varepsilon=0}\\
&= \frac{\partial}{\partial\varepsilon}\int_{\mathcal{M}_n} e^{-Tr\,V_0(Z+\varepsilon f_0(Z)Z^{m+1})}e^{\sum_{\ell=1}^{\infty} t_\ell Tr(Z+\varepsilon f_0(Z)Z^{m+1})^\ell}\,e^{\varepsilon f_0(Z)Z^{m+1}\frac{\partial}{\partial Z}}(dZ)\Big|_{\varepsilon=0}\\
&= \frac{\partial}{\partial\varepsilon}\int_{\mathcal{M}_n} e^{\varepsilon(-Tr\,V_0'(Z)f_0(Z)Z^{m+1}+\sum_{\ell=1}^{\infty}\ell t_\ell Tr\,f_0(Z)Z^{m+\ell})+O(\varepsilon^2)}\Phi(Z)e^{\varepsilon f_0(Z)Z^{m+1}\frac{\partial}{\partial Z}}dZ\Big|_{\varepsilon=0}\\
&= \int_{\mathcal{M}_n}\Bigg(-Tr\,h_0(Z)Z^{m+1} + \sum_{\ell=1}^{\infty}\ell t_\ell Tr\,f_0(Z)Z^{m+\ell}\\
&\qquad\qquad + \sum_{i\geq 0}a_i\Big(\sum_{\alpha+\beta=i+m}\frac{\partial^2}{\partial t_\alpha\partial t_\beta} + 2n\frac{\partial}{\partial t_{i+m}} + n^2\delta_{i+m}\Big)\Bigg)\Phi(Z)dZ\\
&\qquad\qquad\qquad \text{using Lemma A.1.}\\
&= \int_{\mathcal{M}_n}\Bigg(-\sum_{i\geq 0}b_i Tr\,Z^{i+m+1}\\
&\qquad\qquad + \sum_{i\geq 0}a_i\Big(\sum_{\ell=1}^{\infty}\ell t_\ell Tr\,Z^{i+m+\ell} + \sum_{\alpha+\beta=i+m}\frac{\partial^2}{\partial t_\alpha\partial t_\beta} + 2n\frac{\partial}{\partial t_{i+m}} + n^2\delta_{i+m}\Big)\Bigg)\Phi(Z)dZ\\
&= \int_{\mathcal{M}_n}\Bigg(-\sum_{i\geq 0, i>-m-1}b_i\frac{\partial}{\partial t_{i+m+1}} - nb_0\delta_{m+1}\\
&\qquad\qquad + \sum_{i\geq 0}a_i\Big(\sum_{\ell=1}^{\infty}\ell t_\ell\frac{\partial}{\partial t_{i+m+\ell}} + \sum_{\alpha+\beta=i+m}\frac{\partial^2}{\partial t_\alpha\partial t_\beta} + 2n\frac{\partial}{\partial t_{i+m}} + n^2\delta_{i+m}\Big)\Bigg)\Phi(Z)dZ\\
&= \sum_{i\geq 0}\Big(a_i(J^{(2)}_{i+m} + 2n\,J^{(1)}_{i+m} + n^2\,J^{(0)}_{i+m}) - b_i(J^{(1)}_{i+m+1} + n\,J^{(0)}_{i+m+1})\Big)\,\tau_n,
\end{aligned}$$

ending the proof of Theorem A.2.

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