NOTES ON THE TALK IN BASEL-DIJON-EPFL JOINT SEMINAR

SOKRATIS ZIKAS

Abstract. The Cremona group $\mathrm{Cr}_n(\mathbf{k})$ is the group of birational transformations of the projective n-space $\mathbb{P}^n$ over a field $\mathbf{k}$. The study of these groups dates back to the 19th century with some of the central questions still being open. In the recent years new techniques, based on the Minimal Model Program, have been developed to answer some of these questions when $\mathbf{k} = \mathbb{C}$.

In this talk, utilizing these techniques, I will explain how to construct families of birational involutions on $\mathbb{P}^3$ which do not fit in an elementary relation of Sarkisov links. Using these involutions, we can construct new homomorphisms from $\mathrm{Cr}_3(\mathbb{C})$, effectively reproving non-simplicity, and show that it admits a free product structure. Furthermore, using the free product structure, we will show that the group $\mathrm{Aut}(\mathrm{Cr}_3(\mathbb{C}))$ is not generated by inner and field automorphisms. Similar constructions also apply to the study of the group of birational transformations of a cubic threefold, where we obtain counterpart results.

1. The Cremona group

The Cremona group $\mathrm{Cr}_n(\mathbf{k}) = \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n)$ is the group of birational transformations of the projective space $\mathbb{P}^n$ over a field $\mathbf{k}$.

1.1. Simplicity.

One of the central questions regarding the structure of this group is the following:

Question. *Is the Cremona group $\mathrm{Cr}_n(\boldsymbol{k})$ a simple group (i.e. does it admit no non-trivial homomorpisms to other groups)?*

The question is settled in many cases and in all those cases the answer is negative:

- $n = 2$ for any field $\mathbf{k}$ by [CL13] and [Lon16];
- $n = 2$ over a perfect field $\mathbf{k}$ with some extra restrictions, [LZ20] (there exists a Galois orbit of size 8) and [Sch21] ($[\bar{\mathbf{k}} : \mathbf{k}] > 2$) constructed non-trivial homomorphisms from $\mathrm{Cr}_2(\mathbf{k})$;
- similarly, for $n = 3$ over $\mathbb{C}$, by [BLZ21].

The recipe of [BLZ21] for the construction of the homomorphism goes as follows: First we get a presentation for the groupoid $\mathrm{BirMori}(\mathbb{P}^n)$, that is the groupoid whose objects are Mori fiber spaces birational to $\mathbb{P}^n$ and morphisms between them are birational maps between them. Then define a morphisms (of groupoids) from $\mathrm{BirMori}(\mathbb{P}^n)$ to a group G and restrict it to Cr_n.

We will construct families birational involutions of $\mathbb{P}^3$ which:

1. lie in a natural set of generators;
2. do not appear in any non-trivial relation in $\mathrm{BirMori}(\mathbb{P}^3)$ (will be made more precise later).

Using these involutions we obtain the following results:

Date: November 25, 2021.

Theorem 1. *There exists a surjective group homomorphism*

$$\psi\colon \mathrm{Cr}_3(\mathbb{C}) \to \mathop{\ast}_{I} \mathbb{Z}/2\mathbb{Z},$$

where

- *the indexing set I is uncountable (and actually parametrizes the aforementioned links);*
- *explicit elements of order as low as 19, not contained in the kernel.*

Furthermore, we have an isomorphism

$$\mathrm{Cr}_3(\mathbb{C}) \cong G * \left(\mathop{\ast}_{J} \mathbb{Z}/2\mathbb{Z} \right),$$

where J is uncountable.

This effectively reproves the non-simpicity of Cr_3. The distinct advantage of this construction over previous ones, is that it is quite explicit.

The free product structure also gives a surjective group homorphism to a free product of $\mathbb{Z}/2\mathbb{Z}$ by projecting to the second factor. However, the kernel of the projection is much larger than that of ψ.

1.2. **Generation by inner and field automorphisms.**

Question. *Let $\phi\colon \mathrm{Cr}_n(\boldsymbol{k}) \to \mathrm{Cr}_n(\boldsymbol{k})$ be a group automorphism. Is ϕ the composition of a field automorphism of $\boldsymbol{k}$ with an inner automorphism?*

Recall that if σ is a field automorphism of $\mathbf{k}$ then it acts on $\mathrm{Cr}_n(\mathbf{k})$ as follows: if $f \in \mathrm{Cr}_n(\mathbf{k})$ then it is of the form $f = (f_0, \dots, f_n)$, where the f_i are homogeneous polynomials of the same degree. Then σ acts on f by acting on the coefficients of the f_i's.

For $n = 2$ and $\mathbf{k} = \mathbb{C}$, we have an affirmative answer by [Dés06]. Similarly, if $n \geq 2$ and $\mathbf{k}$ is a field of characteristic 0, under the additional assumption that ϕ is a homeomorphism (with respect to the Zariski topology) then the answer is again yes.

Using the free product structure on $\mathrm{Cr}_3(\mathbb{C})$ we obtain the following:

Theorem 2. *There exists uncountably many automorphism of $\mathrm{Cr}_3(\mathbb{C})$ of arbitrary order which are not generated by inner and field automorphisms.*

Consequently, these automorphisms are not continuous (with respect to the Zariski topology).

This serves as a negative answer to the aforementioned question. Moreover, it provides the first examples on non-continuous group automorphisms of the Cremona group.

2. Presentation of the groupoid $\mathrm{BirMori}(X)$

2.1. **Generators of $\mathbf{BirMori}(X)$.**

Definition 2.1. *A Sakrisov link between two Mori fiber spaces X/B and X'/B' over a base R is a diagram of the form*

$$\begin{tikzcd}
Y \arrow[rr, dotted, "\chi"] \arrow[d] & & Y' \arrow[d] \\
X \arrow[d] & & X' \arrow[d] \\
B \arrow[dr] & & B' \arrow[dl] \\
& R &
\end{tikzcd}$$

which satisfies the following properties:

(1) X, X', Y, Y' *are birational and* χ *is an isomorphism or pseudo-isomorphism (i.e. both* χ *and it's inverse are isomorphisms after removing subsets of codimension greater that* 1*);*
(2) *vertical arrows are isomorphisms or extremal contractions (i.e. morphism with connected fibers);*
(3) *varieties of maximal dimension are mildly singular (*$\mathbb{Q}$*-factorial and terminal);*
(4) ***the relative Picard rank*** $\rho(Z/R)$ ***of any variety*** Z ***in the diagram is at most*** 2***.***

Properties (1) and (2) imply that $Y \to X$ and $Y' \to X'$ are either divisorial contractions or isomorphisms, since all varieties on top have the same dimension. Property (3) implies that exactly one of the two vertical arrows in each side is an isomorphism. There are 4 configurations depending on the position of the isomorphisms, namely Bottom-Top/B-B/T-B/T-T, corresponding to the Sarkisov diagrams of **Type I/II/III/IV**.

Type I

$$\begin{tikzcd} Y \arrow[r, dashed] \arrow[d] & X' \arrow[d, "\psi"] \\ X \arrow[d, "\phi"'] & B' \arrow[dl] \\ R & \end{tikzcd}$$

Type II

$$\begin{tikzcd} Y \arrow[r, dashed] \arrow[d] & Y' \arrow[d] \\ X \arrow[d, "\phi"'] & X' \arrow[d, "\psi"] \\ B \arrow[r, "\sim"] & B' \end{tikzcd}$$

Type III

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[d, "\phi"'] & Y' \arrow[d] \\ B \arrow[dr] & X' \arrow[d, "\psi"] \\ & R \end{tikzcd}$$

Type IV

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[d, "\phi"'] & & X' \arrow[d, "\psi"] \\ B \arrow[dr] & & B' \arrow[dl] \\ & R & \end{tikzcd} .$$

Property (4) is the defining property of Sarkisov diagram: it implies that the whole diagram can be recovered just from the data Y/B (or by Y'/B) via the 2-ray game. Indeed, $\rho(Y/B) = 2$ and thus the cone of curves only has 2 extremal rays. One corresponds to the contraction $Y \to X$ (or $Y \to B$ if $Y \to X$ is an isomorphism). The 2nd ray corresponds to another contraction which may be divisorial or of flipping type. In the former case, we are done, while in the latter we repeat the same process with the flipped variety.

Theorem 2.1 ([Cor95] in dimension 3, [HM13] in any dimension)**.** *Any birational map between Mori fiber spaces can be decomposed as a sequence of Sarkisov links.*

This implies that Sarkisov links generate the groupoid $\mathrm{BirMori}(X)$.

2.2. **Relations in BirMori**(X)**.** To obtain a presentation of $\mathrm{BirMori}(X/B)$, we would need to know the relations among the Sarkisov links. This is achieved by the theory of rank r fibrations, developed by Blanc-Lamy-Zimmermann based on ideas of Kaloghiros.

Definition 2.2. *Let* $r \geq 1$ *be an integer. A morphism* $\eta\colon X \longrightarrow B$ *is a* ***rank r fibration*** *if the following conditions hold:*

(1) $\dim X > \dim B$ *and* $\rho(X/B) = r$*;*
(2) X/B *is a Mori Dream Space*[1]*;*
(3) X *is* $\mathbb{Q}$*-factorial and terminal and for any divisor* D *on* X*, the output of any* D*-MMP over* B *is still* $\mathbb{Q}$*-factorial and terminal.*
(4) *There exists an effective* $\mathbb{Q}$*-divisor* Δ_B *such that the pair* (B, Δ_B) *is klt.*

[1] For our purposes it is enough to keep in mind that for any divisor D on X, we can run any D-MMP over B, i.e. all relevant contractions/flips exist, any sequence of flips terminates and if at some point D becomes nef then it also becomes semi-ample.

(5) *The anticanonical divisor of X is η-big (can be written as a sum of an η-ample divisor with an effective divisor).*

*We say that a rank r fibration X/B **dominates** a rank r' fibration X'/B' if we have a commutative diagram*

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr, dashed] & & B \\
& X' \arrow[r] & B' \arrow[u]
\end{tikzcd}$$

where $X \dashrightarrow X'$ is a birational contraction and $B' \longrightarrow B$ is a morphism with connected fibres.

Remark 2.2. *A rank 1 fibration corresponds to a terminal Mori fibre space while a rank 2 fibration corresponds to a Sarkisov links between two Mori fibre spaces.*

Proposition 2.3. *Let $X \longrightarrow B$ be a rank 3 fibration. Then there are only finitely many rank 2 fibrations, corresponding to Sarkisov links χ_i, dominated by X/B, and they fit in a relation*

$$\chi_t \circ \cdots \circ \chi_1 = id.$$

*A relation arising from a rank 3 fibration is called an **elementary relation**.*

Proof. Since X/B is an MDS, there exist finitely many pseudo-isomorphisms $g_i\colon X \dashrightarrow X_i$ such that the movable cone $\mathrm{Mov}(X/B)$ is the union of the $g_i^*(\mathrm{Nef}(X_i))$. Moreover, each X_i/B is an MDS, thus $\mathrm{NE}(X_i/B)$ are polyhedral cones and every extremal ray can be contracted to some Y_i. By doing all these contractions we get a diagram

$$\begin{tikzcd}
& & Y_1 \arrow[dl, dashed, "\chi_1"'] & & \\
& Y_2 \arrow[dl, dashed, "\chi_2"'] & & Y_t \arrow[ul, dashed, "\chi_t"'] & \\
Y_3 \arrow[dr, dashed, "\chi_3"'] & & X \arrow[uu, dashed] \arrow[ul, dashed] \arrow[ur, dashed] \arrow[ll, dashed] \arrow[rr, dashed] \arrow[dl, dashed] \arrow[dd, dashed] & & Y_{t-1} \arrow[ul, dashed, "\chi_{t-1}"'] \\
& Y_4 \arrow[dr, dashed, "\chi_4"'] & & \iddots \arrow[ur, dashed, "\chi_{t-2}"'] & \\
& & Y_5 \arrow[ur, dashed, "\chi_5"'] & &
\end{tikzcd}$$

where $X \dashrightarrow Y_i$ are birational contractions and Y_i/B are rank 2 fibrations, which correspond to Sarkisov links. □

Theorem 2.4 ([BLZ21])**.** *Let X/B be a terminal Mori fibre space. Any relation between Sarkisov links in* $\mathrm{BirMori}(X)$ *is generated elementary relations.*

Combining Theorems 2.1 and 2.4 we get that $\mathrm{BirMori}(X/B)$ is of the form

$$\mathrm{BirMori}(X/B) = \left\langle \begin{array}{c} \text{Sarkisov links between} \\ \text{Mfs's birational to } X \end{array} \;\middle|\; \begin{array}{c} \text{relations dominated} \\ \text{by rank 3 fibrations} \end{array} \right\rangle$$

and we can use this presentation to construct morphisms from $\mathrm{BirMori}(X/B)$.

3. The involutions

Proposition 3.1. *Let C be a general element of $\mathcal{H}_{g,d}$, with $(g,d) \in \{(2,8),(6,9),(10,10),(11,14)\}$. Denote by X the blowup of $\mathbb{P}^3$ along C. They we have a diagram*

$$\begin{tikzcd} X \arrow[rr, dotted] \arrow[d] & & X \arrow[d] \\ \mathbb{P}^3 \arrow[rr, dashed, "\chi"] \arrow[dr] & & \mathbb{P}^3 \arrow[dl] \\ & \mathrm{Spec}(\mathbb{C}) & \end{tikzcd}$$

where χ is an involution.

Proof. The generality condition on C ensures that X is smooth and weak-Fano. Moreover we have the formula

$$(-K_X)^3 = (-K_{\mathbb{P}^3})^3 + 2K_{\mathbb{P}^3} \cdot C - 2 + 2g = 2.$$

By the Hirzebruch-Riemann-Roch and the Kawamata-Viehweg vanishing we get

$$h^0(X, -nK_X) = \frac{n(n+1)(2n+1)}{12}(-K_X^3) + 2n + 1 = \frac{n(n+1)(2n+1)}{6} + 2n + 1.$$

Computing the dimensions of $h^0(X, -nK_X)$ we get that

$$R(X, -K_X) := \bigoplus_{n \geq 0} H^0(X, -nK_X) = \mathbb{C}[x_0, \ldots, x_3, t]/(t^2 - f_6(x_0, \ldots, x_3))$$

where the grading on the right is given by $(1,1,1,1,3)$. Taking the Proj we conclude.

If we denote by $\pi: X \to \mathbb{P}^3$ the blowup along C and by $r: X \to Z$ the anti-canonical morphism then the induced involution is the composition

$$\mathbb{P}^3 \xdashrightarrow{\pi^{-1}} X \xrightarrow{r} Z \xrightarrow{\eta} Z \xdashrightarrow{r^{-1}} X \xrightarrow{\pi} \mathbb{P}^3.$$

□

The aforementioned involutions are actually Sarkisov links dominated by rank the 2 fibrations $X \to \mathrm{Spec}(\mathbb{C})$. They are thus completely determined by X and consequently by C. Thus in what follows, χ_C will denote the involution induced by the blowup of C.

Proposition 3.2. *There is no rank 3 fibration dominating $X \to \mathrm{Spec}(\mathbb{C})$. Consequently, there is no elementary relation involving χ_C.*

Proof. Let $W \to X \to \mathrm{Spec}(\mathbb{C})$ be such a rank 3 fibration. Then $W \to X$ is a divisorial contraction with W terminal and X smooth. Thus $W \to X$ is either a weighted blowup of a point or the (regular) blowup of a curve. We will distinguish cases and in all, we will show that $-K_W$ is not big, contradicting Property (5) of Definition 2.2.

Case 1: $E \subset W \longrightarrow p \in X$ is a $(1,a,b)$-blowup.

We will only give a rough idea in the baby case $a = b = 1$, i.e. a regular blowup. Assume for contradiction that $-K_W$ is big and let $S_W \in |-nK_W|$ be a general element, $n \gg 0$ and $H_X \in |-K_X|$ be an element of H_X containing p. Denote by $S_X \in |-nK_X|$ the image of S_W in X. Let C_X be the complete intersection of S_X and H_X. We have

$$-K_X \cdot C_X = -K_X \cdot S_X \cdot H_X = n(-K_X)^3 = 2n.$$

Moreover, if C_W denotes the strict transform of C_X, then

$$E \cdot C_W = v_E(S_W) \cdot v_E(H_X) = 2n$$

and a quick calculation yields

$$-K_W \cdot C_W = (-K_X - 2E) \cdot C_W = 2n - 4n = -2n < 0.$$

By varying S_W and H_X we may find $-K_X$-negative curves which cover W, which contradicts the bigness of $-K_W$.
Case 2: $E \subset W \longrightarrow z \in X$ is the blowup of a curve contracted by $X \to Z$.

In this case, we will show that $(-K_W)^3 = 0$ but $-K_W$ is nef. For the former we only need to notice that all contracted curves are rational and so using the formula for the cube of the anti-canonical we get $(-K_W)^3 = 0$. As for the latter, any curve negative against $-K_W$ is in the (stable) base locus. However, sections of $-K_W$ are pullbacks of sections of hyperplane sections of Z passing through the image of Z. The only base locus of this system is the image of z. Thus $-K_W$-negative curves lie in the exceptional divisor E of $W \to X$. The we may calculate explicitly, that $E \cong \mathbb{F}_0$ or $\mathbb{F}_2$ and using adjunction we may check that $-K_W$ is nef when restricted there.
Case 3: $E \subset W \longrightarrow z \in X$ is the blowup of a curve NOT contracted by $X \to Z$.

Senction of $-nK_W$ are pullback of hypersurface sections of degree n vanishing with multiplicity n at the image of z. Let $I = (f_1, \dots, f_k)$ be the ideal of the image of C and $h = 0$ such a hyperplane section. Then $h \in I^n$ and since $\deg(h) = n$, h is a linear combination of degree n monomials in the linear elements of I. For the system to be big, we need to have at least 4 such elements, which is a contradiction. □

4. Corollaries

4.1. **Group homomorphism and semidirect product structure.** As promised, we will now define a groupoid homomorphism from $\mathrm{BirMori}(\mathbb{P}^3)$ to a free product of $\mathbb{Z}/2\mathbb{Z}$'s.

What we would like to do is define I to be the union of the Hilbert schemes $\mathcal{H}_{g,d}$ for (g,d) as in Proposition 3.1 and then define the homomorphism by sending χ_{C_i} to the 1_i (the non-zero element of the i-th factor of the free product) and every other Sarkisov link to 0. The problem with this approach is that for any automorphism $a \in \mathrm{Aut}(\mathbb{P}^3)$ we have the relation

$$a^{-1} \circ \chi_{a(C)} \circ a = \chi_C.$$

However, this can be easily remedied by taking I to be the elements in the Hilbert schemes up to projective equivalence.

We define a groupoid homomorphism Ψ from $\mathrm{BirMori}(\mathbb{P}^3)$ to $\bigast_I \mathbb{Z}/2\mathbb{Z}$ as follows: On the level of objects, Ψ maps everything to the unique object of $\bigast_I \mathbb{Z}/2\mathbb{Z}$ (when considered as a groupoid). On the level of Sarkisov links and automorphisms, for each $i \in I$, if C_i belongs to the projective equivalence class of i, χ_{C_i} is mapped to the non-zero element of the factor i. All other links and isomorphisms are mapped to the zero element. Using the fact that there are no non-trivial elementary relations among the χ_{C_i}, we see that this map is well defined. Finally we define $\psi\colon \mathrm{Cr}_3 \to \bigast_I \mathbb{Z}/2\mathbb{Z}$ to be the restriction of Ψ to Cr_3.

Theorem 4.1. *The homomorphism $\psi\colon \mathrm{Cr}_3 \to \bigast_I \mathbb{Z}/2\mathbb{Z}$ defined above is clearly surjective and admits a section. Thus we have*

$$\mathrm{Cr}_3 = N \rtimes \bigast_I \mathbb{Z}/2\mathbb{Z},$$

where N is the kernel of ψ.

Proof. For a section, we just choose a curve C_i in each projective equivalence class $i \in I$ and map 1_i to χ_{C_i}. □

4.2. **Free product structure.** With a little bit more care and using the same involutions, one may obtain a free product structure on Cr_3. We would like to say that

$$\mathrm{Cr}_3 \cong G * \left(\mathop{\ast}_{I} \mathbb{Z}/2\mathbb{Z} \right),$$

where G is the subgroup of Cr_3 consisting of elements that admit a decomposition into Sarkisov links that contain no χ_{C_i}, $i \in I$.

However we run into a similar problem again: Let C be a curve corresponding to some index $i \in I$ and $a \in \mathrm{Aut}(\mathbb{P}^3)$ such that $a(C) = C$. Then $a \circ \chi_C \circ a^{-1} = \chi_C$ which should be different words in the free product.

To circumvent this, we define a refinement J of I, as the subset of projective equivalent classes that admit no projective automorphisms. With some work, we can show that J is still uncountable. We have

Theorem 4.2. *We have an isomoprhism*

$$\mathrm{Cr}_3 = G * \left(\mathop{\ast}_{J} \langle \chi_{C_j} \rangle \right) \cong G * \left(\mathop{\ast}_{J} \mathbb{Z}/2\mathbb{Z} \right).$$

Proof. By working with curves up to projective equivalence and only with those that admit no projective automorphisms, the only relations involving the χ_{C_j} are $(\chi_{C_j})^2 = \mathrm{id}$.

Given an $f \in \mathrm{Cr}_3$, we decompose it into Sarkisov links. If there is a link of the form χ_C such that $C = a(C_j)$ for some $a \in \mathrm{Aut}(\mathbb{P}^3)$ and $j \in J$ the we replace χ_C by $a \circ \chi_{C_j} \circ a^{-1}$. We then group up anything in between two χ_{C_j} and $\chi_{C_{j'}}$ to obtain a decomposition into alternating elements of G and elements of the form χ_{C_j}. This shows that the factors generate Cr_3.

Given a relation $R = \mathrm{id}$ in Cr_3, we consider it as a relation in $\mathrm{BirMori}(\mathbb{P}^3)$. Then R is a product of conjugates of elements $(\chi_{C_j})^2$, $j \in j$ and R_G, where R_G is a relation in G. By the rules of the free product, we may cancel $\chi^2_{C_j}$ and thus we are left with product of conjugates of R_G, i.e. a relation in G. This shows that there are no relations in between the factors. □

4.3. **Generation by inner and field automorphisms.** Fix a free product structure on Cr_3 as above. Choose two curves $C = C_j$ and $C' = C_{j'}$, with $j \in J$, such that $g(C) \neq g(C')$. Let $\phi \colon \mathrm{Cr}_3 \to \mathrm{Cr}_3$ the automorphism exchanging the two factors j and j'. We have

Theorem 4.3. *The automorphism ϕ is not generated by inner and field automorphisms.*

Proof. Let $g \in \mathrm{Cr}_3$ and σ be a field automorphism. Suppose that

$$\phi(f) = g^{-1} \circ \sigma(f) \circ g.$$

For $f = \chi_{C_j}$ we get

$$\chi_{C_{j'}} = g^{-1} \circ \sigma(\chi_{C_j}) \circ g.$$

Since the only relation involving $\chi_{C_{j'}}$ is $(\chi_{C_{j'}})^2 = \mathrm{id}$ the only possiblities would be $g = \mathrm{id}$ and $\sigma(\chi_{C_j}) = \chi_{C_{j'}}$. By comparing base loci we would get that $\sigma(C_j) = C_{j'}$. However a field automorphism preserves the Hilbert polynomial which is a contradiction by our choice of C, C'. □

References

[BLZ21] Jérémy Blanc, Stéphane Lamy, and Susanna Zimmermann. Quotients of higher-dimensional Cremona groups. *Acta Math.*, 226(2):211–318, 2021.

[CL13] Serge Cantat and Stéphane Lamy. Normal subgroups in the Cremona group. *Acta Math.*, 210(1):31–94, 2013. With an appendix by Yves de Cornulier.

[Cor95] Alessio Corti. Factoring birational maps of threefolds after Sarkisov. *J. Algebraic Geom.*, 4(2):223–254, 1995.

[Dés06] Julie Déserti. Sur les automorphismes du groupe de Cremona. *Compos. Math.*, 142(6):1459–1478, 2006.

[HM13] Christopher D. Hacon and James McKernan. The Sarkisov program. *J. Algebraic Geom.*, 22(2):389–405, 2013.

[Lon16] Anne Lonjou. Non simplicité du groupe de Cremona sur tout corps. *Ann. Inst. Fourier (Grenoble)*, 66(5):2021–2046, 2016.

[LZ20] Stéphane Lamy and Susanna Zimmermann. Signature morphisms from the Cremona group over a non-closed field. *J. Eur. Math. Soc. (JEMS)*, 22(10):3133–3173, 2020.

[Sch21] Julia Schneider. Relations in the cremona group over perfect fields. *Ann. Inst. Fourier (Grenoble)*, to appear, 2021.