Generalized Kummer surfaces and a configuration of conics in the plane (joint w. D. Kohl and X. Roulleau)

§ Introduction

Motivation for the talk: result of N. Nikulin:

Let X be a K3 surface projective, X contains 16 disjoint smooth rational curves $E_1, \dots, E_{16}$

(then we have $\sum E_i = 2\delta \in \mathrm{Pic}(X)$)

$\Rightarrow \exists$ A abelian surface and a diagram:

$$\begin{tikzcd}
 & A \arrow[loop right, "i"] \arrow[d, "2:1"] \\
\mathrm{Km}(A) = X \arrow[r, "\text{contract the 16 } (-2)\text{-curves}"'] & \overline{X}
\end{tikzcd}$$

2:1 ← ramified on the 16 nodes

$\overline{X}$ ← 16 nodes

Kummer Surface.

$\overline{X} = A/\langle i \rangle$, $\quad i : A \longrightarrow A$, $(x,y) \longmapsto (-x,-y)$

In 1997 Barth has shown the following generalisation:

Theorem E_i, E_i' $i=1,\dots,9$ 18 smooth rational curves on a K3 surface s.t. $E_i \cdot E_i' = 1$, $E_i^2 = E_i'^2 = -2$ and 0 otherwise. (We have 9 A_2-configurations.)

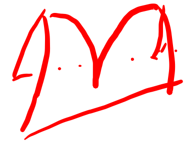
$\Rightarrow \exists$ integers $a_i, a_i' \in \{1,2\}$, $a_i \neq a_i'$ s.t.

$$\sum_{i=1}^{9} (a_i E_i + a_i' E_i') = 3\delta \in \mathrm{Pic}(X)$$

then we have a diagram:

$$\begin{tikzcd} & A \arrow[d, "\text{3:1 cover ramified on the cusps}"] \\ X \arrow[r, "\text{contraction of } 9A_2"'] & \bar{X} \end{tikzcd}$$

$\bar{X} \leftarrow$ 9 cusps



A = abelian surface

$\bar{X} = A/G$, $G \curvearrowright A$ symplectically, $G = \mathbb{Z}/3\mathbb{Z}$

(locally $(x,y) \longmapsto (\omega x, \omega^2 y)$, ω = third root of unity).

$X = Km_3(A)$ is called generalized Kummer.

Def. A generalized Kummer str. on a K3 surface X is the data of (A, G) with A an abelian surface and $G \curvearrowright A$ symplectically, $G \cong \frac{\mathbb{Z}}{3\mathbb{Z}}$ s.t.

$X = Km_3(A)$ (NB rank $Pic(X) \geq 18$)

Q How many generalized Kummer str. can have a K3 surface?

In 1977 Shioda asked the same question for Kummer surfaces.

Proposition Let X be a generalized Kummer.

We have a bijection: $(X = \mathrm{Km}_3(A))$.

$$\underbrace{\{\text{gen. Kummer str.}\}}_{\text{iso}} \xleftrightarrow{1:1} \underbrace{\{S A_2\text{-config}\}}_{\text{auto.}}$$

* X. Roulleau (last week): "computed" the number of this structure.

Aim of the talk: (1) $\to$ Study some $S A_2$-configuration on K3 double planes.

(2) $\to$ give a method to construct new (not iso) configurations.

§2 K3 double planes

We start here with a construction studied by Birkenhake–Lange 1996:

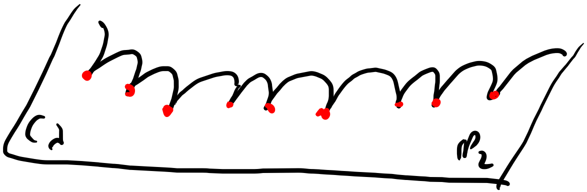
Consider the pencil of cubics (Hesse pencil):

$$E_\lambda : x^3 + y^3 + z^3 - 3\lambda xyz = 0 \subset \mathbb{P}_2(\mathbb{C})$$

Consider the dual curve: it is a $\boxed{\text{sextic}}$ in $\mathbb{P}_2 = \mathbb{P}_2(\mathbb{C})^\vee$

$$C_\lambda = E_\lambda^\vee \subset \mathbb{P}_2 = \mathbb{P}_2^\vee$$

Fact C_λ has 9 cusps coming from the 9 inflectional pts of E_λ.



$$\mathbb{P}^2 \xleftarrow{2:1} X_\lambda \xleftarrow[\text{res.}]{\text{min.}} \tilde{X}_\lambda$$

$X \cdot - \Delta$ — $9A_2$

9 cusps

1-dim family of K3-surfaces w/ a $9A_2$-config.

Q: Can we find more $9A_2$ configurations on $\tilde{X}_\lambda$?

Prop $\operatorname{Pic}(\tilde{X}_\lambda) = \mathbb{Z}L \oplus K_3$ where L is big + nef. $L^2 = 2$ and $K_3 \subseteq H^2(\tilde{X}_\lambda, \mathbb{Z})$ is the min. primitive lattice $\supseteq 9A_2$

Remarque (Beauville 1988)

K_3 is generated by the $8\, A_2$ classes E_i, E_i', $i=1,\dots,8$

and by $v_1 = \frac{1}{3}\sum_{i=1}^{8}(E_i+2E_i')$ and two classes

v_2, v_3 of the form $\frac{1}{3}\sum_{j=1}^{6}(A_j+2B_j)$ with $A_j \neq B_j$

$A_j, B_j \in \{E_i, E_i', i=1,\dots,8\}$, $A_j \cdot B_j = 1$

To fix ideas let:

$$v_2 = \frac{1}{3}\sum_{i=1}^{6}(E_i+2E_i')$$

§ Special conics in the plane

Consider the divisor: $L - v_2 = 0$ $\quad (L - v_2)^2 = 2 - \frac{6}{9} \cdot 6 = -2$

& $\quad L.(L - v_2) = 2$

So is a good candidate to be a rational curve.

Our conjecture

$$\begin{tikzcd} E_i, E_i' \subset \tilde{X}_\lambda \arrow[r] & X_\lambda \arrow[r] & {} \arrow[from=1-1, to=1-3, bend left, "|L|"] \end{tikzcd}$$

$L - v_2$ is sent to a conic C by $|L|$. and C contains 6 cusps of C_λ. (on $\tilde{X}_\lambda$ C splits in two rational curves)

Let $\mathcal{P}_9 = \{$ set of cusps of $C\,\partial \} = \{p_1, \dots, p_9\}$

$C_{12} = \{$ conics of $\mathbb{P}^2$ containing 6 pts $\}$

Theorem (KRS 2021)

We have that $|C_{12}| = 12$ each conic contains exactly 6 pts of $\mathcal{P}_9$ and through each pt. of $\mathcal{P}_9$ there are exactly 8 conics, so that we get a $\boxed{(9_8, 12_6)\text{-configuration.}}$

← discovered indep. by Dolgachev - Laface - Persson - Urzúa (2020)

← Pokora - Szemberg (2020)

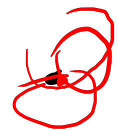
Chilean configuration of conics.

Interesting Fact

Fix a pt of $p_1 \in \mathcal{P}_8$

- $\Rightarrow \exists\, 8$ conics through p_1.

Let $\boxed{\mathcal{P}_2 := \mathcal{P}_8 \setminus \{p_1\}}$ if $C \in \mathcal{C}_1 =$ set of conics through p_2



$\Rightarrow C$ contains 5 pts of $\mathcal{P}_2$

if $p \in \mathcal{P}_2 \Rightarrow \exists\, 5$ conics of $\mathcal{C}_1$ cont. it.

We get a $(8_5, 8_5)$ configuration

- $\exists!\; C' \in \mathcal{C}_1$ s.t. $C \cap C' = 1$ pt. outside $\mathcal{P}_2$

Prop 2 The strict transform on $\tilde{X}_\lambda$ of the 8 conics in $\mathcal{C}_1$ form a 8 A_2 configuration.

One need an A_2 config. more to get a 9 A_2-config:

Consider singular quartics in $\mathbb{P}_2$, through the 8 cusps. by pulling back $\to$ one get a 9 A_2-config.

By repeating the construction we get 9 A_2-config. $\mathcal{A}_1, \dots, \mathcal{A}_9$ & these all equivalent to our original configuration. $\leftarrow \mathcal{A}_0$

Xavier Roulleau: in this case $\exists!$ structure.