SIX LECTURES ON COMPACT COMPLEX SURFACES

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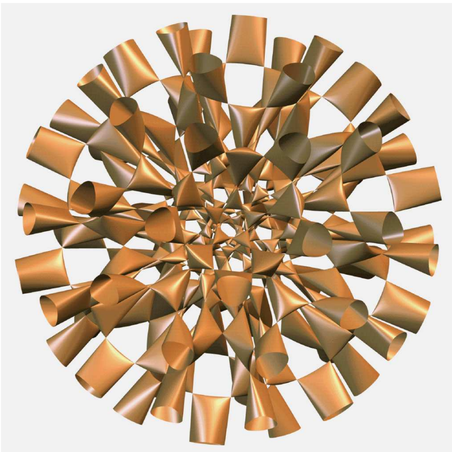
Abstract. This is a formulation of my six lectures (about 12h) on *Superficies Algebraicas Complejas* during my stay at the UBA, the university of Buenos Aires in june/july 2010. I would like to thank all the students and collegues attending my lectures. Particular thanks go to my hosts Fernando Cukiermann and Alicia Dickenstein, to Armando Treibich and the PREMER programm making my stay at UBA possible.

 [1]

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[1] $A_5 \times A_5$-symmetric surface of degree 12 with 600 nodes

1. Introduction

1.1. **Classify.** When studying a geometric problem, one has:

- to define the family of objects one wants to investigate,
- to define the notion of equivalence between the objects,
- to introduce invariants to understand if the objects are equivalent.

In this way one wants to *classify* the objects. For example in *topology* objects are topological spaces and one studies them up to *homeomorphism*. Let $(X_1, \mathcal{B}_1)$ and $(X_2, \mathcal{B}_2)$ be two topological spaces, then X_1 and X_2 are homeomorphic if there exist continuous maps $f_1 : X_1 \longrightarrow X_2$ and $f_2 : X_2 \longrightarrow X_1$ such that $f_1 \circ f_2 = \mathrm{id}_{X_2}$ and $f_2 \circ f_1 = \mathrm{id}_{X_1}$. One can then introduce the notion of *homotopy*, with the same notations as before, $(X_1, \mathcal{B}_1)$ and $(X_2, \mathcal{B}_2)$ are homeotopic equivalent if the continuous maps f_1 and f_2 are such that $f_1 \circ f_2$ is homotopic to id_{X_2} and $f_2 \circ f_1$ is homotopic to id_{X_1}.
Let us restrict our study to connected, compact topological spaces. The next step in the classification is to enrich the structure. There are different ways, particularly important for us is to consider those topological spaces over which one can perform algebra and geometry. We arrive in this way at the notion of manifold.
A *real manifold* is a topological space M, which is Hausdorff, admits a countable basis for the topology and admits an open cover $\{U_i\}_{i \in I}$ such that for any i there is an homeomorphism $\varphi : U_i \longrightarrow V_i \subset \mathbb{R}^n$, (where n is fixed and $\mathbb{R}^n$ is endoved with the euclidian topology) we call n the real topological dimension of M. Moreover there are maps

$$f_{ij} := \varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \longrightarrow \varphi_i(U_i \cap U_j).$$

Asking the maps f_{ij} to be differentiable, of class $\mathcal{C}^k$ or of class $\mathcal{C}^\infty$ one get the notion of differentiable, $\mathcal{C}^k$-differentiable or $\mathcal{C}^\infty$-differentiable manifold. One can then introduce the notion of continuous, $\mathcal{C}^k$-differentiable, or $\mathcal{C}^\infty$-differentiable map between two real manifolds M_1 and M_2 and hence the notion of equivalence (up to diffeomorphism, up to $\mathcal{C}^k$-differentiable or $\mathcal{C}^\infty$-differentiable diffeomorphism).

In a similar way a *complex manifold* is a topological space X which is Hausdorff, has a countable basis for the topology and admits an open cover $\{U_i\}_{i \in I}$ such that for all $i \in I$ there is an homeomorphism $\varphi_i : U_i \longrightarrow V_i \subset \mathbb{C}^n$ (where n is fixed and $\mathbb{C}^n$ is endoved with the euclidian topology). We define n to be the complex topological dimension of X (the real topological dimension is $2n$). Moreover there are maps

$$f_{ij} := \varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \longrightarrow \varphi_i(U_i \cap U_j).$$

Asking these maps to be holomorphic, one get the notion of holomorphic manifold (or complex manifold). One can then introduce the notion of holomorphic map between two such manifolds X_1 and X_2 and so the notion of equivalence: two complex manifolds are equivalent if there is a biholomorphic map between them. Using the notion of sheaf one can go a step further and enrich the structure. For example let X be a complex manifold then for any open subset U in X one can define a ring

$$\mathcal{O}_X(U) = \{f : U \longrightarrow \mathbb{C} \,|\, f \text{ is holomorphic}\}.$$

This assignement defines a preshef which is a sheaf $\mathcal{O}_X$ of rings on X and the couple $(X, \mathcal{O}_X)$ is called a *ringed space*. We will work in the sequel with these kind of objects, in the case that $\dim X = 2$.

Let me go back now to the classification: if two spaces are equivalent then one first important invariant is the dimension, which must be the same. Take for example complex manifold, then we have:

dimension 0: these are points, not very interesting!
dimension 1: curves, these are the Riemann surfaces, i.e. compact connected manifolds of real dimension 2. Recalling that the *genus* is the number of holes in the topological model we have a rough classification of Riemann surfaces:

- $g = 0$: rational curves,
- $g = 1$: elliptic curves,
- $g \geq 2$: curves of general type.

1.2. **Curves.** As seen before one can classify curves by the genus. One important property is that they all admit an embedding in $\mathbb{P}^3(\mathbb{C})$ (this means that all complex curves are projective) and they are all birational to curves in $\mathbb{P}^2(\mathbb{C})$ with at most nodes as singularities (cf. [5, Ch. IV, Cor. 3.6 and Cor. 3.11]). Fixing the genus one can classify curves up to biholomorphism. If C is a complex smooth curve and $g = 0$ then $C \cong \mathbb{P}^1(\mathbb{C})$, i.e. C is isomorphic to $\mathbb{P}^1$ (recall that a birational map between two smooth curves is an isomorphism, hence to say that C is birational or isomorphic to $\mathbb{P}^1$ is the same). If $g(C) = 1$ then C is called an elliptic curve and there is an immersion of C in $\mathbb{P}^2$ such that the equation of C is (in affine coordinates):

$$y^2 = x(x-1)(x-\lambda), \;\; \lambda \in \mathbb{C}$$

this exhibits the curve as a double cover of $\mathbb{P}^1$ ramified at $0, 1, \lambda$ and ∞. In this case the Hurwitz formula reads $2g(C) - 2 = 2(2g(\mathbb{P}^1) - 2) + 4$.

One can define the j-invariant of C:

$$j(C) := \frac{2^8(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2},$$

then two elliptic curves C and C' are isomorphic if and only if $j(C) = j(C')$ (recall that the birapports of the points $0, 1, \lambda, \infty$ define all the same j-invariant, [5, Ch. IV, Prop. 4.6]). If $g(C) \geq 2$ is more complicated to give a classification, but in the case of genus 2 one can do a similar description as in the case of elliptic curves. In fact the curve can be written as a double cover of $\mathbb{P}^1$ ramified over $0, 1, \infty, \lambda_1, \lambda_2, \lambda_3$, where $\lambda_i \in \mathbb{C}$, $i = 1, 2, 3$.

$$z^2 = x(x-1)(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)$$

One can show more precisely that there is a bijection between isomorphism classes of genus two curves and triples $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ modulo the action of the permutation group Σ_6 (cf.[5, Ch. IV, Ex. 2.2]). If $g(C) \geq 3$ one can study linear systems on curves to understand maps to projective spaces. In fact an important tool in the study of varieties are *linear systems*. I recall here some basic facts about them.

Let X be a projective smooth variety and D on X be a Weil(=Cartier, since X is smooth) divisor, i.e. a finite linear combination $D = \sum n_i Z_i$, where the Z_i are irreducible subvarieties of codimension one (for example on a curve a divisor is a linear combination of points) and $n_i \in \mathbb{Z}$. One can then define

$$|D| = \{D' \text{ effective divisor s.t. } D' \sim D\}$$

(a divisor is effective if the n_i in the above sum are all non negative), here $\sim$ denote the linear equivalence of divisors. The previous set has the structure of a projective space and is called the *complete linear system* associated to D. Finally a *linear system* on X is a projective linear subspace $\delta \subset |D|$.
Linear systems can define rational maps (eventually morphisms, embeddings) from the variety to projective spaces; we will see this more in details later.
Another important tool that we know for curves, is the *Riemann-Roch theorem*. Let X be a smooth curve, $D = \sum n_i Z_i$ a divisor on it and $g(X)$ the genus of the curve then we have

$$h^0(\mathcal{O}_X(D)) = \deg D - g(X) + h^1(\mathcal{O}_X(D))$$

where $g(X) = h^1(\mathcal{O}_X) = h^0(\omega_X)$ (using Serre duality) and ω_X is the canonical sheaf (sheaf of holomorphic 1-forms) on the curve; $\deg D := \sum n_i$. The sheaf $\mathcal{O}_X(D)$ is the sheaf associated to the divisor D. If D is defined by a compatible system $\{(U_j, f_j)\}$ (this means that $\{U_j\}$ is an open cover of X, f_j is a local equation of D on U_j, $f_j \in K^*(U_j)$, i.e. f_j is a rational function not identically zero on U_j, $f_i f_j^{-1} \in \mathcal{O}^*(U_i \cap U_j)$, i.e. $f_i f_j^{-1}$ is a regular function which is never zero on $U_i \cap U_j$) then

$$\mathcal{O}_X(D)(U) = \{h \in K(U) \mid hf_j \in \mathcal{O}_X(U \cap U_j)\}$$

where $K(U)$ denotes the sheaf of rational functions on U (more in general we denote by $K(X)$ the sheaf of rational functions on X).

1.3. **Surfaces.** One wants now to classify complex suraces of dimension 2. Already in the 19th century people started to study surfaces: Cayley, Kummer, Steiner; then the italians: Bertini, Cremona, del Pezzo, Segre, Veronese. The first classification of smooth projective surfaces was then obtained by Enriques and Castelnuovo in 1910. In the case of surfaces there are new problems that one does not have in case of curves.

- There exist compact, complex surfaces which are not projective.
- It is not possible to make a biholomorphic classification of surfaces, even only for the projective ones. In fact Enriques and Castelnuovo discovered a

new phenomena: the blow-ups. One can only give a birational classification of smooth projective surfaces.

The classification by Enriques and Castelnuovo was then made clearer by Zariski, Weil and van der Waerden in the case of projective surfaces and by Hodge, de Rham and Lefschetz in the general case (about 1940). The main techniques in algebraic geometry were carried out by Serre, Grothendieck and Hirzebruch in 1950–1960, who introduced the notion of sheaf, cohomology, scheme and their properties. Using these tools Kodaira was able to complete the classification of complex surfaces, also singular and non projective ones.
Nowadays the classification is called *Enriques-Kodaira classification of compact complex surfaces*. The Kodaira classification uses importat invariants called *plurigenera*, these are strictly connected to the *Kodaira dimension* which roughly speaking count the numbers of independent global differential forms on a surface. We will see that the possible values are $-\infty, 0, 1, 2$.

1. Kodaira dimension $-\infty$:
   - rational surfaces (birational to $\mathbb{P}^2$),
   - ruled surfaces (are the projectivized of a rank 2 vector bundle on a smooth projective curve of genus ≥ 0).
2. Kodaira dimension 0:
   - abelian surfaces (the quotients $\mathbb{C}^2/\Lambda$ where Λ is a rank 4 lattice, are called complex tori. The projective ones are the abelian surfaces-but not all the complex tori are projective!-),
   - K3 surfaces,
   - Enriques surfaces,
   - bielliptic surfaces.
3. Kodaira dimension 1: proper elliptic surfaces
4. Kodaira dimension 2: surfaces of general type

This is a coarse classification as in the case of curves, in fact using the Kodaira dimension one has for smooth curves:

1 Kodaira dimension $-\infty$: rational curves (isomorphic to $\mathbb{P}^1$), $g = 0$.
2 Kodaira dimension 0: elliptic curves (are all of the form $\mathbb{C}/\Lambda$), $g = 1$.
3 Kodaira dimension 1: curves of general type, $g \geq 2$.

2. The Picard Group and the Riemann–Roch theorem

In this section and in the next sections, by "surface" I mean always a smooth projective surface over the complex numbers and I will use the Zariski topology. By Serre's GAGA one can work in the algebraic setting (regular, rational functions or maps, ...) or in the analytic setting (holomorphic, meromorphic functions and maps, ...).

2.1. **The Picard group.** Let S be a smooth variety. I denote by $\mathrm{Pic}(S)$ the group of line bundles (i.e. vector bundles of rank 1) modulo isomorphism or (which is equivalent) invertible sheaf (i.e. locally free sheaf of rank 1) modulo isomorphism. Recall that a sheaf $\mathcal{F}$ on S is called locally free of rank one if $\mathcal{F}_{|U} \cong \mathcal{O}_{S|U}$, where U is an open subset of S. We denote by $A^1(S)$ the group of divisor on S modulo linear equivalence.
If D is a divisor on S we can associate an invertible sheaf $\mathcal{O}_S(D)$ and the converse is also true: to any invertible sheaf we can associate a divisor. We get an isomorphism

$\mathrm{Pic}(S) \cong A^1(S)$. Let now $f : S \longrightarrow X$ be a morphism of smooth varieties which is surjective or such that $f(S)$ is dense on X (i.e. $\overline{f(S)} = X$), then we can define the *pull-back*:

$$f^* : \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(S).$$

If D is a divisor in $\mathrm{Pic}(X)$ defined by a compatible system $\{U_i, g_i\}$ then f^*D is defined by $\{f^{-1}(U_i), f^*g_i\}$, where f^*g_i denotes the pull-back of the function g_i (cf. e.g. [11, Ch. III, §1.2]).

Definition 2.1. *(Direct image or push down). Let X be a projective surface, Y a smooth projective variety and*

$$f : X \longrightarrow Y$$

a generically finite morphism of degree d (i.e. it is finite over a dense open set) then for an irreducible curve $C \in \mathrm{Pic}(X)$ (not necessarily smooth) we define $f_*(C) = 0$ if $f(C)$ is a point and $f_*(C) = rf(C)$ if r is the degree of the restriction map $C \longrightarrow f(C)$ induced by f.

By linearity we can define $f_*(D)$ for any Weil divisor on X. One can easily show that if $D \sim D'$ then $f_*D \sim f_*D'$ and $f_*f^*D \sim dD$.
A next basic ingredient in the classification of surfaces is intersection theory which is the topic of the next section.

2.2. **Intersection theory on surfaces.**

Definition 2.2. *Let C, C' be two irreducible curves on a surface X. Let $x \in C \cap C'$, let $f, g \in \mathcal{O}_{X,x}$ local equations for C and C' then the* intersection multiplicity *of C and C' at x is*

$$\mathrm{mult}_x(C \cap C') = \dim_{\mathbb{C}} \mathcal{O}_{X,x}/\langle f, g\rangle$$

Recall that we have $\dim_{\mathbb{C}} \mathcal{O}_{X,x}/\langle f, g\rangle$ finite iff the zero set $V(\langle f, g\rangle)$ is a finite number of points (cf. [4, Ch. II, Proposition 6 and Corollay 1]).

If $\mathrm{mult}_x(C \cap C') = 1$ we say that C, C' are *transverse* (or *meet transversally*) at x. This means that f, g generate the maximal ideal m_x in $\mathcal{O}_{X,x}$, in fact recall that $\mathbb{C} \cong \mathcal{O}_{X,x}/m_x \subset \mathcal{O}_{X,x}/\langle f, g\rangle$. In this case f and g are a system of local coordinates at x.

Definition 2.3. *The* intersection multiplicity *of C and C' is*

$$C \cdot C' = \sum_{x \in C \cap C'} \mathrm{mult}_x(C \cap C')$$

Example 2.1. Let C be the cuspical cubic curve in the plane, i.e. $C = \{y^2 - x^3 = 0\}$ and $L = \{y = 0\}$ a line. Then we have $C \cap L = \{(0, 0)\}$ and $\mathrm{mult}_0(C \cap L) = \dim \mathbb{C}[x, y]/(y^2 - x^3, y) = \dim_{\mathbb{C}}\langle 1, x, x^2\rangle = 3$. Now take $L' = \{x = 0\}$ then $\mathrm{mult}_0(C \cap L') = \dim \mathbb{C}[x, y]/(y^2 - x^3, x) = \dim_{\mathbb{C}}\langle 1, y\rangle = 2$. Finally with $C' = \{y - x^2 = 0\}$ we get $C \cdot C' = \dim \mathbb{C}[x, y]/(y^2 - x^3, y - x^2) = \dim_{\mathbb{C}}\langle 1, x, x^2, x^3\rangle = 4$

Observe that Definiton 2.3 is not good enough, in fact we can not extend it by linearity to any divisor, since we do not know how to define $C \cdot C$. We will see in the next definition how to solve this inconvenient.

Definition 2.4. *Let $\mathcal{L}$, $\mathcal{M} \in \mathrm{Pic}(X)$ be two line bundles, then*

$$\mathcal{L} \cdot \mathcal{M} := \chi(\mathcal{O}_X) - \chi(\mathcal{L}^{-1}) - \chi(\mathcal{M}^{-1}) + \chi(\mathcal{L}^{-1} \otimes \mathcal{M}^{-1})$$

is called intersection product *of $\mathcal{L}$ and $\mathcal{M}$.*

Recall that $\chi(\mathcal{F}) = \sum_{i=0,1,2}(-1)^i h^i(X,\mathcal{F}) = h^0(\mathcal{F}) - h^1(\mathcal{F}) + h^2(\mathcal{F})$ is the Euler characteristic and $\mathcal{F}^{-1} = \mathcal{F}^\vee$ is the dual sheaf defined by $U \mapsto Hom_{\mathcal{O}_{X|U}}(\mathcal{F}_{|U}, \mathcal{O}_{|U})$, here $\mathcal{F} \in \mathrm{Pic}(X)$.
We will show that the intersection product is well defined and bilinear. Then, since it is clearly symmetric, it is an intersection pairing.

Lemma 2.1. *Let $C, C' \subset X$ irreducible curves then*

$$\mathcal{O}_X(C) \cdot \mathcal{O}_X(C') = C \cdot C'.$$

Proof. We have an exact sequence

$$0 \longrightarrow \mathcal{O}_X(-C-C') \xrightarrow{(t,-s)} \mathcal{O}_X(-C) \oplus \mathcal{O}_X(-C') \xrightarrow{(s,t)} \mathcal{O}_X \longrightarrow \mathcal{O}_{C\cap C'} \longrightarrow 0$$

where $s \in H^0(X, \mathcal{O}_X(C))$ is a non–zero section vanishing on C and $t \in H^0(X, \mathcal{O}_X(C'))$ is a non-zero section vanishing on C' (cf. [2, Lemma I.5]). The exactness of the sequence implies that:

$$\chi(\mathcal{O}_X(-C-C')) + \chi(\mathcal{O}_X) = \chi(\mathcal{O}_X(-C) \oplus \mathcal{O}_X(-C')) + \chi(\mathcal{O}_{C\cap C'}).$$

By using the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-C) \longrightarrow \mathcal{O}_X(-C) \oplus \mathcal{O}_X(-C') \longrightarrow \mathcal{O}_X(-C') \longrightarrow 0$$

we get

$$\chi(\mathcal{O}_X(-C) \oplus \mathcal{O}_X(-C')) = \chi(\mathcal{O}_X(-C)) \oplus \chi(\mathcal{O}_X(-C'))$$

and since $\mathcal{O}_X(-C) = \mathcal{O}_X(C)^\vee$ and $\mathcal{O}_X(-C-C') = \mathcal{O}_X(C)^\vee \otimes \mathcal{O}_X(C')^\vee$ we get

$$\mathcal{O}_X(C) \cdot \mathcal{O}_X(C') = \chi(\mathcal{O}_X(C)^\vee \otimes \mathcal{O}_X(C')^\vee) + \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(C)^\vee) - \chi(\mathcal{O}_X(C')^\vee)$$
$$= \chi(\mathcal{O}_{C\cap C'}) = h^0(\mathcal{O}_{C\cap C'})$$

Now recall recall that $h^0(\mathcal{O}_{C\cap C'}) = \sum_{x\in \mathbb{C}\cap\mathbb{C}'} \dim \mathcal{O}_{X,x}/\langle f_x, g_x\rangle = C \cdot C'$ (with the same notations as in Definition 2.2), so we are done. □

Recall that if $\mathcal{L} = \mathcal{O}_X(D)$ for some divisor D on X and we have the inclusion $i : C \hookrightarrow X$, then we define $\mathcal{L}_{|C} = i^*\mathcal{L}$ which is also $\mathcal{L}_{|C} = \mathcal{O}_C(i^*D)$ and $\deg(\mathcal{L}_{|C})$ is then defined as $\deg(i^*D)$.

Lemma 2.2. *Let $C \subset X$ be a smooth irreducible curve, $\mathcal{L} \in \mathrm{Pic}(X)$. Then*

$$\mathcal{O}_X(C) \cdot \mathcal{L} = \deg(\mathcal{L}_{|C})$$

Proof. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-C) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0$$

then $\chi(\mathcal{O}_C) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-C))$ and tensoring with $\mathcal{L}^{-1}$ we get also $\chi(\mathcal{L}^{-1}_{|C}) = \chi(\mathcal{L}^{-1}) - \chi(\mathcal{L}^{-1} \otimes \mathcal{O}_X(-C))$. By definition we have

$$\mathcal{O}_X(C) \cdot \mathcal{L} = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-C)) - \chi(\mathcal{L}^{-1}) + \chi(\mathcal{L}^{-1} \otimes \mathcal{O}_X(-C))$$

By using the previous equalities one get

$$\mathcal{O}_X(C) \cdot \mathcal{L} = \chi(\mathcal{O}_C) - \chi(\mathcal{L}^{-1}_{|C}).$$

Finally the Riemann-Roch theorem for curves gives

$$\chi(\mathcal{O}_C) = 1 - g(C), \quad \chi(\mathcal{L}_{|C}) = \deg(\mathcal{L}_{|C}) + 1 - g(C).$$

Hence one gets the equality. □

Theorem 2.1. *The intersection product*

$$\cdot : \mathrm{Pic}(X) \times \mathrm{Pic}(X) \longrightarrow \mathbb{Z}$$

defined as in Definition 2.4 is a symmetric bilinear form s.t. $\mathcal{O}_X(C) \cdot \mathcal{O}_X(C') = C \cdot C'$ *for any* C, C' *distinct irreducible curves on* X.

Proof. (sketch) It is clear that it is symmetric. Consider now $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3 \in \mathrm{Pic}(X)$ and let

$$s(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) := \mathcal{L}_1 \cdot (\mathcal{L}_2 \otimes \mathcal{L}_3) - \mathcal{L}_1 \cdot \mathcal{L}_2 - \mathcal{L}_1 \cdot \mathcal{L}_3.$$

If $\mathcal{L}_i = \mathcal{O}_X(C_i)$, $i = 1, 2, 3$ and C_i is an irreducible curve, then with

$$\mathcal{L}' := \mathcal{O}_X(C_2) \otimes \mathcal{O}_X(C_3) = \mathcal{O}_X(C_2 + C_3),$$

we have:

$$\begin{aligned} \mathcal{L}_1 \cdot \mathcal{L}' &= \deg(\mathcal{L}'_{|C_1}) = \deg(\mathcal{O}_X(C_2)_{|C_1}) + \deg(\mathcal{O}_X(C_3)_{|C_1}) \\ &= \mathcal{O}_X(C_2) \cdot \mathcal{O}_X(C_1) + \mathcal{O}_X(C_3) \cdot \mathcal{O}_X(C_1). \end{aligned}$$

so we are done in this case. In fact a similar computation works also in the case that only one of the $\mathcal{L}_i$ is of the previous form. In the general case we will use the A Theorem of Serre: *Let* $\mathcal{L} \in \mathrm{Pic}(X)$ *then there exists* $n >> 0$ *s.t.* $\mathcal{L} \otimes \mathcal{O}_X(n)$ *is very ample.* From here it follows that any Weil divisor D can be written as $D = A - B$ where A and B are smooth irreducible curves, in fact it is enough to choose $B \in |nH|$ and $A \in |D+nH|$, (H denotes the class of the hyperplane section). Since for n big enough these two systems are very amples by Bertini's theorem (cf.[5, Ch. II, Theorem 8.18]) the generic element is smooth and irreducible.
Now take $\mathcal{L}, \mathcal{M} \in \mathrm{Pic}(X)$ and write $\mathcal{M} = \mathcal{O}_X(A-B)$, A, B smooth curves. By the previous remark we have:

$$0 = s(\mathcal{L}, \mathcal{M}, \mathcal{O}_X(B)) = \mathcal{L} \cdot (\mathcal{O}_X(A-B) \otimes \mathcal{O}_X(B)) - \mathcal{L} \cdot \mathcal{M} - \mathcal{L} \cdot \mathcal{O}_X(B).$$

The statement follows then by a direct computation using Lemma 2.1 and Lemma 2.2. □

Remark 2.1.

1. Let D and D' be Weil divisors then we have $D \cdot D' = \mathcal{O}_X(D) \cdot \mathcal{O}_X(D')$. In particular if $D = D'$ we have $D^2 := D \cdot D$ is well defined.
2. The above theorem tells us that if $D' \sim \tilde{D}$ then $D \cdot D' = D \cdot \tilde{D}$ for any Weil divisor D, in particular $\tilde{D}^2 = \tilde{D} \cdot D'$.

I will formulate in the next proposition two application of the intersection product.

Proposition 2.1. *Let* X *and* Y *be two smooth projective surfaces and* C *a smooth curve. Then:*

1) *Let* $f : X \longrightarrow C$ *be a surjective morphism and* F *be the class of a fiber, then* $F^2 = 0$.
2) *Let* $g : X \longrightarrow Y$ *be a generically finite morphism of degree d (i.e there exists an open dense subset* $U \subset Y$ *s.t. the restriction* $g^{-1}(U) \longrightarrow U$ *is finite of degree d) and* D, D' *Weil divisors on* Y. *Then* $g^*D \cdot g^*D' = d(D \cdot D')$.

Proof. Cf. [8, Proposition 2.4.4] □

Example 2.2.

1. We consider $X = \mathbb{P}^2$ and we describe the intersection product. We have $\mathrm{Pic}(\mathbb{P}^2) \cong \mathbb{Z}$ and it is generated by the hyperplane section of $\mathbb{P}^2$ which is a line L (or equivalently is generated by the line bundle $\mathcal{O}_{\mathbb{P}^2}(1)$). The line L is obtained as the zero set of a section $s \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$. We have that $L^2 = L_1 \cdot L_2$, where the $L_i's$ are two lines of $\mathbb{P}^2$ meeting transversally. Clearly $L^2 = 1$. Since any irreducible curve is linearly equivalent to a positive multiple of L, for $C_i \sim d_i L$, $i = 1, 2$, we have $C_1 \cdot C_2 = d_1 d_2$.
2. Consider the surface $X = \mathbb{P}_1 \times \mathbb{P}_1$ and define the following curves:
$$h_1 := \{(0:1)\} \times \mathbb{P}_1; \quad h_2 := \mathbb{P}_1 \times \{(0:1)\}$$
Let $U := X \backslash (h_1 \cup h_2)$, then one can easily show that $U \cong \mathbb{C}^2$. Since $\mathrm{Pic}(\mathbb{C}^2) = 0$ (cf.[5, Proposition 6.2]) we get $\mathrm{Pic}(U) = 0$. Hence if D is a Weil divisor on X we have that $D_{|U} = (f)$ where $f \in K(U)$ is a rational function on U. Hence there are $n_1, n_2 \in \mathbb{Z}$ such that $D = (f) + n_1 h_1 + n_2 h_2$ and we get
$$\mathrm{Pic}(X) \cong A^1(X) \cong \mathbb{Z} \oplus \mathbb{Z}$$
generated by the classes h_1 and h_2. We denote $\mathcal{O}_X(D) := \mathcal{O}_X(n_1, n_2)$ for a line bundle $\mathcal{O}_X(D)$ with D a divisor on X as above. The couple (n_1, n_2) is called the *bidegree* of D.
To describe the intersection pairing we need to know the products $h_i \cdot h_j$, $i, j = 1, 2$. We have $h_1 \cdot h_2 = |h_1 \cap h_2| = 1$. The projection $p_1 : X \longrightarrow \mathbb{P}_1$ to the first factor is surjective and has h_1 as fiber over $(0 : 1)$, hence by Proposition 2.1, $h_1^2 = 0$. In a similar way $h_2^2 = 0$, hence the matrix of the intersection pairing is
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

2.3. **The Riemann-Roch theorem for surfaces.** Recall the Riemann-Roch theorem for curves: Let C be a smooth curve and D a Weil divisor on C, then we have:
$$h^0(\mathcal{O}_C(D)) - h^1(\mathcal{O}_C(D)) = \chi(\mathcal{O}_C(D)) = \deg(D) - g(C) + 1$$
we want now to show an analogous formula for surfaces:

Theorem 2.2. *(Riemann-Roch) Let X be a smooth projective surface, $\mathcal{L} \in \mathrm{Pic}(X)$, then*
$$\chi(\mathcal{L}) = \chi(\mathcal{O}_X) + \frac{1}{2}(\mathcal{L}^2 - \mathcal{L} \cdot \mathcal{O}_X(K_X)).$$

Proof. By the first definition of intersection pairing
$$\mathcal{L}^{-1} \cdot (\mathcal{L} \otimes \mathcal{O}_X(-K_X)) = \chi(\mathcal{O}_X) - \chi(\mathcal{L}) - \chi(\mathcal{L}^{-1} \otimes \mathcal{O}_X(K_X)) + \chi(\mathcal{O}_X(K_X)).$$
By Serre duality: $h^0(\mathcal{O}_X) = h^2(\mathcal{O}_X(K_X))$, $h^1(\mathcal{O}_X) = h^1(\mathcal{O}_X(K_X))$, $h^2(\mathcal{O}_X) = h^0(\mathcal{O}_X(K_X))$. Finally $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_X(K_X))$ and $\chi(\mathcal{L}) = \chi(\mathcal{L}^{-1} \otimes \mathcal{O}_X(K_X))$. Putting all together, we get
$$\mathcal{L}^{-1} \cdot (\mathcal{L} \otimes \mathcal{O}_X(-K_X)) = 2(\chi(\mathcal{O}_X) - \chi(\mathcal{L}))$$
so
$$\chi(\mathcal{L}) = \chi(\mathcal{O}_X) - \frac{1}{2}\mathcal{L}^{-1} \cdot (\mathcal{L} \otimes \mathcal{O}_X(-K_X)).$$
Now by bilinearity of the intersection pairing:
$$\mathcal{L}^{-1} \cdot (\mathcal{L} \otimes \mathcal{O}_X(K_X)) = \mathcal{L}^{-1} \cdot \mathcal{L} + \mathcal{L}^{-1} \cdot \mathcal{O}_X(K_X) = -\mathcal{L}^2 + \mathcal{L} \cdot \mathcal{O}_X(K_X).$$

Substituting in the previous formula we get the statement. $\square$

Remark 2.2. We can write the Riemann-Roch formula by using divisors. Let D be a Weil divisor such that $\mathcal{L} = \mathcal{O}_X(D)$ then

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \frac{1}{2}(D^2 - D \cdot K_X).$$

Observe that since $\chi(\mathcal{O}_X(D))$ is an integer we get $D^2 - D \cdot K_X \in 2\mathbb{Z}$, in particular if $D \cdot K_X = 0$ then D^2 is even (we will see that this is the case when X is a K3 surface).

Definition 2.5. *The* irregularity *of a smooth projective suface X is $q(X) := h^1(\mathcal{O}_X) = h^{0,1}(X)$ (where one get the last equality using Dolbeault cohomology). The* geometric genus *of X is*

$$p_g(X) := h^2(\mathcal{O}_X) = h^{0,2}(X).$$

With these definitions we get

$$\chi(\mathcal{O}_X) = 1 - q(X) + p_g(X).$$

We recall now two very useful results:

Proposition 2.2. *Let $f : X \longrightarrow Y$ be a morphism between projective varieties such that he natural map $\mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ is an isomorphism. Then the fibers of f are connected and non-empty. Conversely, if f is surjective, Y is normal and the fibers are connected, one has an isomorphism $\mathcal{O}_Y \xrightarrow{\sim} f_*\mathcal{O}_X$*

Proof. [9, Ch. IV, §8, Proposition 4] $\square$

Corollary 2.1. *Suppose that $f : X \longrightarrow Y$ is a surjective morphism between projective varieties, that Y is normal and that the fibres of f are connected. For any line bundle $\mathcal{L}$ on Y there is a natural isomorphism*

$$f^* : H^0(Y, \mathcal{L}) \xrightarrow{\sim} H^0(X, f^*\mathcal{L})$$

given by $f^*(t) = t \circ f$.

Proof. [9, Ch. IV, §8, Corollary 5] $\square$

Example 2.3.

1. If $X = \mathbb{P}^2$ we get $q(X) = p_g(X) = 0$, hence $\chi(\mathcal{O}_{\mathbb{P}^2}) = 1$ and so the Theorem of Riemann-Roch gives

$$\chi(\mathcal{O}_{\mathbb{P}^2}(d)) = 1 + \frac{1}{2}(d^2 + 3d)$$

   recall that $\mathcal{O}_{\mathbb{P}^2}(K_{\mathbb{P}^2}) = \mathcal{O}_{\mathbb{P}^2}(-3)$.
2. If $X = \mathbb{P}^1 \times \mathbb{P}^1$ let $p_i : X \longrightarrow \mathbb{P}^1$, $i = 1, 2$ be the projections. Then by Proposition 2.2 we have $\mathcal{O}_{\mathbb{P}^1} \cong p_{i*}\mathcal{O}_X$ and by [5, Ch. III, Exercice 8.4] $R^j p_{i*}\mathcal{O}_X = 0$ for all $j > 0$, then using the projection formula as in [5, Ch. III, Exercice 8.1], we get

$$H^i(X, \mathcal{O}_X) = H^i(\mathbb{P}^1, p_{i*}\mathcal{O}_X) = H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0, \text{ for all } i > 0$$

   Hence $q(X) = p_g(X) = 0$, (we will see another way to compute $q(X)$ and $p_g(X)$ in Section 4, Proposition 4.2, when we will describe ruled surfaces, cf. also [5, Ch. II, Example 6.6.1]). Now for the tangent bundle $T_X = p_1^* T_{\mathbb{P}^1} \oplus$

$p_2^* T_{\mathbb{P}^1}$ hence, for the canonical sheaf we have $\mathcal{O}_X(K_X) = \mathcal{O}_X(-2,-2)$. This together with $\chi(\mathcal{O}_X) = 1$ gives for the Riemann-Roch Theorem on X:

$$\chi(\mathcal{O}_X(n,m)) = 1 + \frac{1}{2}(2nm + 2n + 2m) = 1 + n + m + nm.$$

Let me recall two more formulas

Theorem 2.3. *(Noether's formula). Let X be a smooth projective surface and $e(X)$ the topological Euler–Poincaré characteristic of X, then:*

$$\chi(\mathcal{O}_X) = \frac{1}{12}(K_X^2 + e(X))$$

where $e(X) = \sum_{i=0}^{4}(-1)^i b_i$ and $b_i := b_i(X) := \dim_{\mathbb{R}} H^i(X,\mathbb{R})$ is the i-th Betti number.

Proof. [1, Ch. I, Theorem 5.4] □

Theorem 2.4. *(Genus formula). Let X be a smooth projective surface, $C \subset X$ be an irreducible curve, then:*

$$g(C) = 1 + \frac{1}{2}(C^2 + C \cdot K_X).$$

Proof. By definition $g(C) = h^1(\mathcal{O}_C)$. By using the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-C) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0$$

we get $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_X(-C)) + \chi(\mathcal{O}_C)$ and using the Riemann-Roch theorem we obtain

$$\chi(\mathcal{O}_X(-C)) = \chi(\mathcal{O}_X) + \frac{1}{2}(C^2 + C \cdot K_X)$$

then $\chi(\mathcal{O}_C) = -\frac{1}{2}(C^2 + C \cdot K_X)$. Combining with the Riemann-Roch Theorem for curves, i.e. $\chi(\mathcal{O}_C) = 1 - g(C)$, one obtains the assertion. □

Theorem 2.5. *(Adjunction formula). Let X be a smooth projective surface, $C \subset X$ be an irreducible curve, then we have for the canonical divisors:*

$$K_C \sim (K_X + C)_{|C}$$

Proof. [1, Ch. I, Theorem 6.3] □

Taking the degrees of the previous divisors one obtains:

$$2g(C) - 2 = \deg(K_C) = \deg((K_X + C)_{|C}) = (K_X + C) \cdot C = C^2 + K_X \cdot C$$

this implies again the genus formula.

Example 2.4.

1. If $X = \mathbb{P}^2$ we get the well known formula for the genus of a plane curve $C \sim dL$:
$$g(C) = 1 + \frac{1}{2}(d^2 - 3d) = \frac{(d-1)(d-2)}{2}.$$
If $d = 1, 2$ then $g(C) = 0$ and so C is rational, if $d = 3$ then $g(C) = 1$, C is elliptic. Finally for $d \geq 4$ we get $g(C) \geq 3$ (in particular a curve of genus 2 can not be embedded as a smooth curve in $\mathbb{P}^2$).
2. If $X = \mathbb{P}_1 \times \mathbb{P}_1$ and C has bidegree (m, n) then we have $g(C) = 1 + nm - n - m$.

3. Birational Geometry

In this chapter we introduce blow-ups, (-1)-curves, minimal models and the Kodaira-dimension.

3.1. **Rational maps and linear systems.** *1. Rational maps.* Let X and Y be projective surfaces a *rational map* $\varphi: X \dashrightarrow Y$ is a morphism from an open subset $U \subset X$ to Y which cannot be extended to a larger open subset. We say that ϕ is defined at $x \in X$ if $x \in U$. This definition works also for non-smooth projective surfaces. With the same notations we have:

Lemma 3.1. *If X is smooth then $X \backslash U := F$ is finite.*

Proof. The problem is local and we can work in a neighbourhood of a point $x \in X$. We can write $\varphi = (f_0 : \ldots : f_n)$ where the f_i are rational functions. Observe that without changing φ we can multiply the f_i by a common factor $g \in K(X)$ and we can assume $f_i \in \mathcal{O}_{X,x}$ for all i and the f_i do not have common factor in $\mathcal{O}_{X,x}$ (since X is smooth $\mathcal{O}_{X,x}$ is UFD), now φ is not regular at the points where $f_0 = \ldots = f_n = 0$. Let Z be a subvariety (hence irreducible) of codimension 1 of X. Again since $\mathcal{O}_{X,x}$ is UFD, the variety Z is defined by an irreducible polynomial $g = 0$. If Z is contained in $\{f_0 = \ldots = f_n = 0\}$ then g is a common factor of the f_i in $\mathcal{O}_{X,x}$ but this is not possible. □

Some remarks:

i. If $C \subset X$ is an irreducible curve, we can talk about the image of C, $\varphi(C) = \overline{\varphi(C-F)}$ which is the Zariski closure of $\varphi(C-F)$. In the same way we can talk about the image of X, $\varphi(X) := \overline{\varphi(X-F)}$. A rational map is *dominant* if $\overline{\varphi(X)} = Y$.

ii. The restriction induces an isomorphism between $\mathrm{Pic}(X)$ and $\mathrm{Pic}(X-F)$. Hence we can talk about inverse image under φ of a divisor D on Y (or of a linear system on Y or of a sheaf on Y), we denote it by $\varphi^* D$.

2. Linear systems. For a divisor D on a surface X we can define the *complete linear system* associated to D, which is

$$|D| = \{D' \text{ is effective, } D' \sim D\}.$$

If $s \in H^0(\mathcal{O}_X(D)) = \{f \in K(X) \mid (f) + D \geq 0\}$ is a global section ($K(X)$ denotes the sheaf of rational functions) then $(s)_0 := (s) + D$ is an effective divisor linearly equivalent to D. Given $D' \geq 0$ with $D' \sim D$ then $D' = D + (f)$, for some rational function f, so $D' = (f)_0$. Since f is defined up to a scalar multiplication we have

$$|D| = \mathbb{P}(H^0(\mathcal{O}_X(D))).$$

A *linear system* is a projective sublinear space $\delta \subset |D|$. A curve C is a *fixed component* of δ if for any $D \in \delta$, we have $C \subset \mathrm{supp}(D)$. The *fixed part* of δ is the biggest divisor F that is contained in any $D \in \delta$. For any $D \in \delta$ the system $|D - F|$ has no fixed part. A point $p \in X$ is said a *base point* (or *fixed point*) of δ, if any divisor $D \in \delta$ contains p. If the system δ has no fixed part, then it has only a finite number of base points, say b, then clearly $0 \leq b \leq D^2$ for any $D \in \delta$, (eventually $b = 0$, if there are no base points).

3. Rational maps and linear systems. Let S be a surface then there is a bijection of sets:

(i) $\{$rational maps $\varphi: S \dashrightarrow \mathbb{P}^n$, s.t. $\varphi(S)$ is not contained in an hyperplane$\}$,

(ii) {linear systems on S without fixed part and of dimension n}.

The correspondence goes as follows:

- to φ we can associate the linear system $\varphi^*|H|$ where $|H|$ is the linear system of hyperplanes of $\mathbb{P}^n$.
- Conversely let δ be a linear system on S with no fixed part and let $\check{\delta}$ be the projective dual to δ. Now define a rational map
$$\varphi:\ S \dashrightarrow \check{\delta}$$
by $x \mapsto \{D \in \delta \mid x \in \operatorname{supp}(D)\}^\vee$, the latter is an hyperplane of δ and so a point of $\check{\delta}$, which is a projective space. The map φ is not defined at the base points of δ.

One can define the previous map φ also using sections. For simplicity let $\delta = |D|$ a complete linear system and $s_0, \dots, s_n$ be a basis of $H^0(S, \mathcal{O}_S(D))$ then we have a rational map
$$\varphi_{|D|}:\ S \dashrightarrow \mathbb{P}(H^0(S, \mathcal{O}_S(D)))^\vee\ ,\ \ x \mapsto (s_0(x); \dots ; s_n(x)).$$
Recall that a linear system $|D|$ is *very ample* (cf. [5, Ch. II, §5]) if $\varphi_{|D|}$ is an embedding.

Example 3.1. Consider the complete linear system of quadrics in $\mathbb{P}^2$ then its dimension is $h^0(\mathcal{O}_{\mathbb{P}^2}(2)) - 1 = 5$. A basis is generated by the monomials x_0^2, x_1^2, x_2^2, x_0x_1, x_0x_2, x_1x_2. The linear system is clearly very ample and we have an embedding
$$\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$$
which is called a *Veronese embedding*, we will see this kind of morphism later when we will describe rational surfaces.

Definition 3.1. *A* birational map *between varieties $f:\ X \dashrightarrow Y$ is a rational map amitting an inverse rational map $f^{-1}:\ Y \dashrightarrow X$ such that $f \circ f^{-1}$ and $f^{-1} \circ f$ are the identity (as rational maps). If there is a birational map from X to Y we say that X and Y are* birationally equivalent *or* birational*. Clearly birational varieties have the same dimension.*

Example 3.2. Consider $\mathbb{P}^1 \times \mathbb{P}^1$ and the *Segre embedding*:
$$s : \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3,\ \ ((x_0 : x_1), (y_0 : y_1)) \mapsto (x_0y_0 : x_0y_1 : x_1y_0 : x_1y_1)$$
it is well defined and injective. Putting $u := x_0y_0$, $v := x_0y_1$, $w := x_1y_0$, $z := x_1y_1$, we get $X := \operatorname{Im}(s) = V(uz - vw)$ which is a smooth ruled quartic.
Consider the projection from the point $(0 : 0 : 0 : 1) \in X$:
$$f : X - \{(0 : 0 : 0 : 1)\} \longrightarrow \mathbb{P}^2 = \{z = 0\},\ \ (u : v : w : z) \mapsto (u : v : w)$$
this defines a rational map from X to $\mathbb{P}^2$, and the inverse is
$$g:\ \mathbb{P}^2 \dashrightarrow \mathbb{P}^3\ ,\ \ (z_0 : z_1 : z_2) \mapsto (z_0^2 : z_0z_1 : z_0z_2 : z_1z_2).$$
This map is not defined at $z_0 = z_1 = 0$, which is the point $(0 : 0 : 1)$, and at $z_0 = z_2 = 0$, which is the point $(0 : 1 : 0)$. Observe that all the points of the line $z_0 = 0$ are mapped to the point $(0 : 0 : 0 : 1)$. Clearly $f \circ g$ and $g \circ f$ are the identity as rational maps, hence $\mathbb{P}^1 \times \mathbb{P}^1$ is birational to $\mathbb{P}^2$ but it is not isomorphic. There are many ways to see this, since we have introduced the Picard group one can argue

that $\mathrm{Pic}(\mathbb{P}^2) = \mathbb{Z}$ and $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Z} \oplus \mathbb{Z}$, hence they can not be isomorphic. Or even more easy: on $\mathbb{P}^2$ two curves meet always, on $\mathbb{P}^1 \times \mathbb{P}^1$ it is not the case, since two lines in the same ruling do not meet.

A very useful theorem when working in birational geometry is the Zariski Main Theorem, we formulate it for surfaces but it works also for normal projective varieties (for the proof, see [5, Ch. III, Corollary 11.4]).

Theorem 3.1. *(Zariski Main Theorem). Let X, Y be smooth projective surfaces and $f : X \longrightarrow Y$ a birational morphism, then the fibers of f are connected.*

3.2. **Blow-up of a point.** Let X be a complex surface, $q \in U \subset X$ an open neighborhood and (x, y) local coordinates s.t. $q = (0, 0)$ in this coordinate system. Define

$$\tilde{U} := \{((x, y), (z : w)) \in U \times \mathbb{P}^1 \mid xw = yz\}$$

we have the projection onto the first factor:

$$p_U : \tilde{U} \longrightarrow U, \ \ ((x, y), (z : w)) \mapsto (x, y)$$

if $(x, y) \neq (0, 0)$ then $p_U^{-1}((x, y)) = ((x, y), (x : y))$ and for $q = (0, 0)$ we have $p_U^{-1}(q) = \{q\} \times \mathbb{P}^1 \cong \mathbb{P}^1$. Hence the restriction:

$$p_U : p_U^{-1}(U - \{q\}) \longrightarrow U - \{q\}$$

is an isomorphism and $p_U^{-1}(q)$ is a curve contracted by p_U to a point. We get then a surface $\tilde{X}$ with a morphism $p : \tilde{X} \longrightarrow X$, where p induces an isomorphism $\tilde{X} - p^{-1}(q) \cong X - \{q\}$ and $p(\mathbb{P}^1) = q$ (is a copy of $\mathbb{P}^1$ contracted to a point).

Definition 3.2. *The morphism $p : \tilde{X} \longrightarrow X$ is called the* blow-up *of X along q. We often write $Bl_q(X) := \tilde{X}$. The curve $E := p^{-1}(q) \cong \mathbb{P}^1$ is called* exceptional curve *or* exceptional divisor *of the blow-up.*

Remark 3.1. If X is projective $\tilde{X} := Bl_q(X)$ is projective too: in fact one can define the blow-up of $\mathbb{P}^n$ at $(0 : \ldots : 0 : 1)$ as a subvariety

$$\tilde{\mathbb{P}}_n \subset \mathbb{P}^n \times \mathbb{P}^{n-1}$$

in an analogous way. Let $(x_0 : \ldots : x_n)$ and $(y_0 : \ldots : y_{n-1})$ be the projective coordinates and take the variety defined by $x_i y_j = x_j y_i$, for all $i, j = 0, \ldots, n-1$. Next take the Segre embedding

$$\mathbb{P}^n \times \mathbb{P}^{n-1} \longrightarrow \mathbb{P}^{n^2+n-1}.$$

Now every subvariety of $\mathbb{P}^n \times \mathbb{P}^{n-1}$ odefined by bihomogeneous equation correspond to a projective subvariety of $\mathbb{P}^{n^2+n-1}$ via the Segre embedding. One can then easily show that the blow-up of a projective variety is projective.

Definition 3.3. *Let $C \subset X$ be a curve, the* strict transform *of C under the blow-up $p : \tilde{X} \longrightarrow X$ at q is the Zariski closure*

$$\tilde{C} := \overline{p^{-1}(C - \{q\})}$$

in $\tilde{X}$.

Remark 3.2. The strict transform of a curve is always a curve. If $q \notin C$ then $\tilde{C} = p^{-1}(C)$ so that $\tilde{C} = p^*(C)$ (in this case $\tilde{C}$ and C are isomorphic).

More in general one has

Lemma 3.2. *Let C be an irreducible curve passing through q with multiplicity m, then*

$$p^*(C) = \tilde{C} + mE.$$

Proof. Clearly we have $p^*(C) = \tilde{C} + kE$, $k \in \mathbb{Z}$. Use local coordinates (x, y) around q (in the subset U). On the open subset $\tilde{U} \cap (U \times \{z \neq 0\})$ we can consider coordinates $u = x$ and $v = \frac{w}{z}$ (then, since $xw = yz$, we get $y = xw/z = uv$). In these coordinates we see that $p(u, v) = (u, uv)$. Let f be a local equation of C at q, as C has multiplicity m at q we have:

$$f = f_m(x, y) + o(m)$$

where $o(m)$ denote terms of degree $> m$. Hence looking at the equation of $p^*(C)$ we have $f \circ p(u, v) = f_m(u, uv) + o(m) = u^m(f_m(1, v) + o(m))$ and we see that the component E has multiplicity m. □

Example 3.3. Consider the curve $C : y^2 = x^2(x + 1)$ which is $y^2 - x^2 - x^3 = 0$, the equation of an ordinary node. Blow up $\mathbb{C}^2$ at $(0, 0)$ then $p^*(C)$ is given by $x^2w^2 - x^2 - x^3 = 0$ with $y = xw$. Hence one get $x^2(w^2 - 1 - x) = 0$, so $p^*(C) = \tilde{C} + 2E$.

Proposition 3.1. *Let X be a smooth projective surface, $p : \tilde{X} \longrightarrow X$ be the blow up of X along $q \in X$, $E := p^{-1}(q)$ the exceptional curve.*

1. *The morphism of groups:*

$$\mathrm{Pic}(X) \oplus \mathbb{Z} \longrightarrow \mathrm{Pic}(\tilde{X}), \ \ (\mathcal{O}_X(D), n) \mapsto \mathcal{O}_{\tilde{X}}(p^*D + nE)$$

   is an isomorphism.
2. *For every two divisors D, D' on X we have*

$$p^*D \cdot p^*D' = D \cdot D', \ \ p^*D \cdot E = 0, \ \ E \cdot E = -1.$$

3. $K_{\tilde{X}} = p^*K_X + E$.

Proof. First we show 2. Let C, C' be curves, $q \notin C \cap C'$ then $C \cdot C' = p^*C \cdot p^*C' = \tilde{C} \cdot \tilde{C}'$ where the first equality follows by Proposition 2.1, since the blow-up on C is a finite morphism of degree one. Moreover we have $p^*C \cdot E = 0$ as E is contracted to q (use the projection formula). For E^2 consider the following: let C be a curve containing q with multiplicity one (since X is projective take smooth hyperplanes sections). Now $\tilde{C} = p^*C - E$ so that $(p^*C - E) \cdot E = \tilde{C} \cdot E = 1$ hence using the fact that $p^*C \cdot E = 0$ one get $E^2 = -1$. One obtain in an easy way the case of $q \in C \cap C'$ and the general case with divisors.
We show 1. Every curve on $\tilde{X}$ is either a multiple of E or it is not. If not, then it is not contracted to a point and it is the proper transform of a curve on X. This shows that the map

$$\mathrm{Pic}(X) \oplus \mathbb{Z} \longrightarrow \mathrm{Pic}(\tilde{X}), \ \ (C, n) \mapsto \tilde{C} \,\text{or}\, (0, n) \mapsto nE$$

is surjective.
To show injectivity: suppose $p^*D + nE \sim 0$ then $0 = (p^*D + nE) \cdot E = -n$ hence $p^*D \sim 0$ now $p_*p^*D = D$ and so $D \sim 0$. To conclude with 3. Let ω be a meromorphic 2–form on X which is holomorphic in a neighborhood of $q \in X$ and $\omega(p) \neq 0$. Away from E the zeros and poles of $p^*\omega$ are the same as the zeros and

poles of ω. Hence $\operatorname{div}(p^*\omega) = p^*(\omega) + kE$, i.e. $K_{\tilde{X}} = p^*K_X + kE$ for some $k \in \mathbb{Z}$. Now use adjunction (with $E \cong \mathbb{P}^1$) to get:

$$-2 = \deg K_E = \deg((K_{\tilde{X}} + E)_{|E}) = K_{\tilde{X}} \cdot E + E^2 = p^*K_X \cdot E + kE^2 + E^2 = -(k+1)$$

so $k = 1$ and we are done. □

3.3. **Indeterminacies.** In this section we prove a fundamental result of surface theory: every birational morphism is a composition of blow-ups and isomorphisms. Consider

$$f: \; S - - \rightarrow \mathbb{P}^n$$

and let $\operatorname{Ind}_f \subset S$ the indeterminacy locus of f, then $\operatorname{codim}(\operatorname{Ind}_f) \geq 2$, so for surfaces Ind_f contains only points. We denote as usual by $f(S) := \overline{f(S - \operatorname{Ind}_f)}$ the image of S in $\mathbb{P}^n$.

Proposition 3.2. *(Elimination of indeterminacy). Let $f: \; S - - \rightarrow \mathbb{P}^n$ be a rational map. There is a finite sequence*

$$S_n \xrightarrow{p_n} S_{n-1} \xrightarrow{p_{n-1}} \ldots \xrightarrow{p_1} S_0 := \; S - - \rightarrow \mathbb{P}^n$$

where for each n the map p_n is the blow–up of S_{n-1} along a point, such that the map:

$$f_n := f \circ p_1 \circ \ldots \circ p_n : S_n \longrightarrow \mathbb{P}^n$$

is a morphism.

Proof. We can assume that $f(S)$ is not contained in an hyperplane of $\mathbb{P}^n$. If one considers a system of hyperlanes of $\mathbb{P}^n$ one get by pull-back a linear system $\delta \subset |D|$ on S associated to a divisor D, whose base locus is contained in Ind_f. For simplicity we give the proof for complete linear systems (i.e. $\delta = |D|$), it is not difficult to generalize it.
If $\operatorname{Ind}_f = \emptyset$, then f is a morphism and there is nothing to prove. Suppose $\operatorname{Ind}_f \neq \emptyset$ and $q \in \operatorname{Ind}_f$. Let

$$S_1 := Bl_{q_0}(S) \xrightarrow{p_1} S = S_0$$

the blow-up at q_0, then $p_1^*D_0 = D_1 + m_1E_1$ where $D_0 := D$, D_1 is the proper transform of D and E_1 is the exceptional divisor of p_1. Then E_1 is a fixed component of the system p_1^*D (since q_0 is a base point), but then the system $|D_1| = |p_1^*D - m_1E_1|$ has no base components. If $|D_1|$ has no base points we are done, if $Bs(D_1) \neq \emptyset$ then let $q_1 \in Bs(D_1)$ and we go on (observe that one may need more than one blow-up to get one base point less in the base locus). After k steps we get a surface $S_k = Bl_{q_{k-1}}(S_{k-1})$ with $p_k : S_k \longrightarrow S_{k-1}$ and on S_k we have a divisor D_k s.t. $p_k^*D_{k-1} = D_k + m_kE_k$, $m_k \in \mathbb{Z}$ and $|D_k|$ has no fixed components. By the properties of the blow-up we get

$$0 \leq D_k^2 = D_{k-1}^2 - m_k^2 < D_{k-1}^2.$$

The first inequality follows since D_k does not contain curves in the fixed locus. Hence the sequence $\{D_k^2\}$ must stabilize for some $k >> 0$. Hence for some $k >> 0$ we have that $|D_k|$ has no base locus and we are done. □

Theorem 3.2. *(Universal property of blow-up). Let X, Y be smooth projective surfaces , $f : X \longrightarrow Y$ be a birational morphism. If $y \in Y$ is a point where the birational inverse $g: \; Y - - \rightarrow X$ is not defined, then there is a morphism*

$h : X \longrightarrow Bl_y(Y)$ which is birational and such that $p \circ h = f$, where p is the blow–up of Y at y.

Proof. Cf. [2, Proposition II.8] □

Remark 3.3.

1. On can show that in fact $f^{-1}(y)$ is a curve not necessarily irreducible but connected (a consequence of Zariski Main Theorem!).
2. One has a similar result for $f : X \dashrightarrow Y$ a birational map, such that the inverse is not defined at $y \in Y$. In fact there exists a curve (not necessarily irreducible) s.t. $f(D) = y$ (the fibers are not necessarily connected in this case, hence one can not use directly the Zariski Main Theorem, for a proof see [2, Lemma II.10]).

Theorem 3.3. *Let $f : X \longrightarrow Y$ be a birational morphism, X and Y be smooth projective surfaces. Then f is the composition of a finite number of blow-ups and isomorphisms.*

Proof. Suppose that f is not an isomorphism, the there is a point $y_1 \in Y$ over which the inverse map f^{-1} is not defined. By Theorem 3.2 there is a birational morphism $f_1 : X \longrightarrow Bl_{y_1}(Y)$ such that $p_1 \circ f_1 = f$. The map f contracts all the curves contracted by f_1 and the curves mapped from f_1 to the exceptional divisor E_1 of p_1 (observe that the number of curves contracted is over the points where f^{-1} is not defined, hence it is a Zariski closed set, i.e. it consists of a finite number of points). The number of curves contracted by f_1 is strictly less than the number of curves contracted by f. If f_1 does not contract any curve, then f^{-1} is everywhere defined and f_1 is an isomorphism, so we are done. If f_1 contracts curves there exists $y_2 \in Y_1$ s.t. f_1^{-1} is not defined and we can proceed as before. Notice that this process has to finish, since the number of curves contracted strictly decrease. □

Corollary 3.1. *Let X, Y be smooth projective surfaces and $f : X \dashrightarrow Y$ be a birational map. Then there is a smooth projective surface Z and two morphisms $g : Z \longrightarrow X$, and $h : Z \longrightarrow Y$ which are composition of blow-ups and isomorphisms, such that $h = f \circ g$.*

Proof. By applying Proposition 3.2 and Theorem 3.3 we can find the maps g and h as in the statement. □

Corollary 3.2. *If X and Y are birational surfaces then $q(X) = q(Y)$, $p_g(X) = p_g(Y)$, $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y)$*

Proof. By Corollary 3.1 it is enough to show that q, p_g and χ are invariant under blow-up. Hence let $p : Y \longrightarrow X$ be the blow-up of X at some point. Then by Proposition 2.2 $p_*\mathcal{O}_Y = \mathcal{O}_X$, moreover by [5, Ch. V, Proposition 3.4] $R^ip_*\mathcal{O}_X = 0$ for all $i > 0$, hence using the projection formula as in [5, Ch. 3, Exercice 8.4] one get:

$$H^i(X, \mathcal{O}_X) = H^i(X, p_*\mathcal{O}_Y) = H^i(Y, \mathcal{O}_Y)$$

hence we get the assertion. □

3.4. **Castelnuovo's contraction theorem.** We have seen that birational maps between smooth projective surfaces are combinations of isomorphisms and blow-ups of points. Let $x \in X$, if we blow-up we get a surface $\tilde{X}$ containing a (-1)-curve E, i.e. E is rational and $E^2 = -1$. We will see that the contrary is also true, i.e. if a surface $\tilde{X}$ contains a smooth rational irreducible curve C, with $C^2 = -1$, (i.e. a (-1)-curve), then $\tilde{X}$ is the blow-up of a surface X and C is the exceptional divisor of the blow-up.

Theorem 3.4. *(Castelnuovo's contraction). Let X be a smooth projective surface and suppose that there is a (-1)-curve E. Then there is a smooth projective surface Y and a point $y \in Y$ such that $X \cong Bl_y(Y)$ and $E = p^{-1}(y)$ where $p : X \longrightarrow Y$ is the blow-up morphism at y.*

Proof. For the complete proof see [2, Theorem II.17]. I recall only how to define the map p. Let H be a very ample divisor on X (this is possible since X is projective). In the proof one needs also $H^1(X, \mathcal{O}_X(H)) = 0$, this is easy up to change H with some multiple mH, $m >> 0$, then the B Theorem of Serre tells us that $h^i(\mathcal{O}_X(mH)) = 0$ for any $i > 0$, since $\mathcal{O}_X(H)$ is a coherent sheaf. Hence suppose that H already satisfy this condition. Let $d := H \cdot E$ and $H' := H + dE$ then $H' \cdot E = 0$. Now consider the map $\varphi_{|H'|} :\ X \dashrightarrow \varphi_{|H'|}(X)$ defined by the linear system $|H'|$ on $X - Bs(|H'|)$. One then shows that $\varphi_{|H'|}$ is defined on the whole X and has the properties of a blow–up. □

Conclusion. Whenever a smooth projective surface X contains a (-1)–curve we can contract it smoothly, getting a surface Y with $\mathrm{rank}(\mathrm{Pic}(Y)) =: \rho(Y) = \rho(X) - 1$.

Definition 3.4. *A surface X is called* minimal *if every birational morphism $f : X \longrightarrow Y$ is an isomorphism. A* minimal model *for a surface X is a surface X' which is minimal and birational to X.*

Corollary 3.3.

1. *A surface is minimal if and only if it contains no (-1)-curve.*
2. *In particular every surface has a minimal model.*

Proof. 1. If X is a smooth projective minimal surface and there exists a (-1)-curve, one can blow it down and we get a birational morphism which is not an isomorphism. Suppose that X does not contain any (-1)-curve, but X is not minimal, then there is a birational morphism $f : X \longrightarrow Y$ which is not an isomorphism. By Theorem 3.3, f is a composition of isomorphisms and blow-ups, and there is at least one blow-up. As f is defined on X then X contains a (-1)-curve, a cotraddiction.
2. Let X be a smooth projective surface. We can assume that X is not already minimal, then from part 1, there exists a (-1)-curve $E \subset X$. By Castelnuovo's theorem we can contract the curve E and we get a surface X_1 with $\rho(X) = \rho(X_1) + 1$, if X_1 is minimal we are done, otherwise we go on and we get a sequence of blow-ups

$$X = X_0 \xrightarrow{p_0} X_1 \xrightarrow{p_1} \dots \xrightarrow{p_{n-1}} X_n \xrightarrow{p_n} \dots$$

with $\rho(X_n) = \rho(X) - n$ and $\rho(X_n) > 0$, this sequence must finish since $\rho(X_i) > 0$ (since all the X_i are projective). Hence there is an n_0 such that X_{n_0} does not contain any (-1)-curve and so it is a minimal model for X. □

Remark 3.4.

1. Classify minimal surfaces allow us to classify surfaces up to birational equivalence.
2. One natural question is how many minimal models there are? I'll give you an example.

Example 3.4. Let $X = \mathbb{P}^2$ and $Y = \mathbb{P}^1 \times \mathbb{P}^1$. Then X and Y are birational but not isomorphic and they are both minimal. In fact for any irreducible curve $C \subset X$ we have $C^2 > 0$ and for any $C \subset Y$ we have C^2 even. Hence a smooth surface S birational to $\mathbb{P}^2$ can have at least two non-isomorphic birational minimal models.

3.5. **Kodaira dimension.** The *pluricanonical system* $|nK_X|$ is very important for a surface X, using it we can define the Kodaira dimension.

Definition 3.5. *Let X be a smooth projective variety, K_X a canonical divisor on X. Let*

$$\varphi_{|nK_X|} : X \dashrightarrow \mathbb{P}^N$$

be the rational map defined by $|nK_X|$. The Kodaira dimension *of X, denoted $\kappa(X)$ is the maximum dimension of the images $\varphi_{|nK_X|}(X)$ for $n \geq 1$. If* $\dim(X) = 2$ *we have $\kappa(X) = -\infty, 0, 1$ or 2.*

Remark 3.5.

1. The system $|nK_X|$ can have fixed components then one can take them away and consider the rational map defined by the new system, which has eventually only fixed points and has the same dimension of the system with which we have started.
2. If $|nK_X| = \emptyset$ for all n then $\varphi_{|nK_X|}(X) = \emptyset$ and we say that $\kappa(X) = -\infty$.
3. For a curve recall that $g = h^1(\mathcal{O}_C) = h^0(\mathcal{O}_C(K_C))$ and we have (cf. [2, Chapter VII])
   - $\kappa(C) = -\infty$ iff $g = 0$,
   - $\kappa(C) = 0$ iff $g = 1$,
   - $\kappa(C) = 1$ iff $g \geq 2$.
4. For a surface we have
   - $\kappa(X) = -\infty$ iff $h^0(nK_X) = 0$ for all $n \geq 1$,
   - $\kappa(X) = 0$ iff $h^0(nK_X) = 0$ or $h^0(nK_X) = 1$ and there exists N such that $h^0(NK_X) = 1$,
   - $\kappa(X) = 1$ iff there exists N such that $h^0(NK_X) \geq 2$ and $\varphi_{nK_X}(X)$ is at most a curve for all n,
   - $\kappa(X) = 2$ iff $\varphi_{NK_X}(X)$ is a surface for some N.

Example 3.5. For $\mathbb{P}^2$ we have $\mathcal{O}(K_{\mathbb{P}^2}) = \mathcal{O}_{\mathbb{P}^2}(-3)$ and $\mathcal{O}(nK_{\mathbb{P}^2}) = \mathcal{O}_{\mathbb{P}^2}(-3n)$, so $h^0(\mathcal{O}_{\mathbb{P}^2}(-3n)) = 0$ which gives $\kappa(\mathbb{P}^2) = -\infty$.

Proposition 3.3. *Let $S_{d_1,\ldots,d_r}$ be a surface in $\mathbb{P}^{r+2}$ which is a smooth complete intersection of r hypersurfaces of degrees $d_1, \ldots, d_r$ then:*

- *The surfaces $S_2, S_3, S_{2,2}$ (which are* rational*) have $\kappa = -\infty$.*
- *The surfaces $S_4, S_{2,3}, S_{2,2,2}$ have canonical class $K \sim 0$ and so $\kappa = 0$ (these are* K3 surfaces*),*
- *all other surfaces $S_{d_1,\ldots,d_r}$ have $\kappa = 2$ (are* surfaces of general type*)*

Proof. By using the adjunction formula (Theorem 2.5) one computes for $S_2, S_3, S_4 \subset \mathbb{P}^3$:

$$\mathcal{O}_{S_i}(K_{S_i}) = \mathcal{O}_{\mathbb{P}^3}(K_{\mathbb{P}^3 + S_i})_{|S_i} = (\mathcal{O}_{\mathbb{P}^3}(-4 + \deg S_i))_{|S_i} = \mathcal{O}_{\mathbb{P}^3}(i-4)_{|S_i} = \mathcal{O}_{S_i}(i-4).$$

In general for $S := S_{d_1,\dots,d_r}$ we get $\mathcal{O}_S(K_S) = \mathcal{O}_S((\sum d_i) - r - 3)$. Hence one get immediately that $\kappa(S_2) = \kappa(S_3) = \kappa(S_{2,2}) = -\infty$ (the systems $|nK_S|$ are empty). We have $K_{S_4} \sim 0$, $K_{S_{2,3}} \sim 0$, $K_{S_{2,2,2}} \sim 0$. In the other cases K_S is a positive multiple of the hyperplane section, hence $\kappa = 2$. □

Remark 3.6. A surface $S_4 = \{f_4(x_0, x_1, x_2, x_3) = 0\}$ is the zero set in $\mathbb{P}^3$ of a homogeneous polynomials of degree 4 (for general choice of the coefficients of f_4, the zero set is smooth). As seen before it has $K_{S_4} \sim 0$ and it is a first easy example of a K3 surface. We will return on K3 surfaces in Section 5.2.

Let now $P_m(X) := h^0(X, \mathcal{O}_X(mK_X))$ be the *plurigenera*.

Theorem 3.5. *Let X, Y be birational smooth projective surfaces then $P_m(X) = P_m(Y)$, for all $m \geq 0$ and so $\kappa(X) = \kappa(Y)$.*

Proof. By Corollary 3.1 there exists a projective surface Z and two maps $g : Z \longrightarrow X$ and $h : Z \longrightarrow Y$ composition of blow-ups and isomorphisms s.t. $h = f \circ g$. It is hence enough to show that the Kodaira dimension does not change under blow-up. Let X be a projective surface, $p : \tilde{X} \longrightarrow X$ be the blow-up at $x \in X$, $E \subset \tilde{X}$ be the exceptional divisor. By the properties of the blow-up (Proposition 3.1) we have $K_{\tilde{X}} = p^*K_X + E$. Assume that there is an effective divisor D on $\tilde{X}$, $D \sim mK_{\tilde{X}}$ for some m, then we have:

$$D \cdot E = mK_{\tilde{X}} \cdot E = mp^*K_X \cdot E + mE^2 = -m.$$

Since D is a divisor on $\tilde{X}$, then one can write $D = D' + kE$ with $k \geq 0$, $\mathrm{supp}(D')$ does not contains E and $D' \geq 0$. Hence we get:

$$0 \leq D' \cdot E = D \cdot E - kE^2 = -m + k, \text{ hence } k \geq m.$$

So $D - mE = D - kE + (k - m)E$ is still effective, as $D \sim mK_{\tilde{X}}$ then $D - mE \sim mK_{\tilde{X}} - mE$. So we get a bijective map:

$$|mK_{\tilde{X}}| \longrightarrow |mK_{\tilde{X}} - mE| = |p^*(mK - X)|, \;\; D \mapsto D - mE$$

So that

$$\begin{aligned} h^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(mK_{\tilde{X}})) &= h^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(m(K_{\tilde{X}} - E))) \\ &= h^0(\tilde{X}, p^*\mathcal{O}_X(mK_X)) = h^0(X, p^*\mathcal{O}_X(mK_X)) \end{aligned}$$

For the last inequality we apply Corollary 2.1. Hence $\kappa(X) = \kappa(\tilde{X})$. □

Theorem 3.6. *(Enriques-Castelnuovo's birational classification). Let X be a smooth projective surface. A minimal model of X is one of the following:*

1. *Kodaira dimension $-\infty$:*
   - *rational surfaces,*
   - *ruled surfaces.*
2. *Kodaira dimension 0:*
   - *abelian surfaces,*
   - *K3 surfaces,*
   - *Enriques surfaces,*
   - *bielliptic surfaces.*
3. *Kodaira dimension 1: proper elliptic surfaces.*
4. *Kodaira dimension 2: surfaces of general type.*

We will not prove here the Enriques Castelnuovo's theorem (cf. [1, Ch. VI]), we will more show properties of the surfaces in the classification.

4. Kodaira dimension $-\infty$

4.1. Ruled and rational surfaces.

Definition 4.1.

1. *Let X be a smooth projective surface, then X is* ruled *if there exists a birational map*
$$X \dashrightarrow C \times \mathbb{P}^1$$
where C is a smooth curve. If $C \cong \mathbb{P}^1$ then X is said rational.
2. *A smooth projective surface X is* geometrically ruled *if there exists a surjective morphism $p : X \longrightarrow C$, where C is a smooth curve, so that the fibers are isomorphic to $\mathbb{P}^1$ and p admits a section, i.e. a morphism $\sigma : C \longrightarrow X$ such that $p \circ \sigma = \mathrm{id}$.*

Remark 4.1.

1. One can show that a geometrically ruled surface is ruled, i.e. X is birational to $C \times \mathbb{P}^1$ (this follows from a theorem of Noether and Enriques, cf. [2, Theorem III.4]). In particular for every $x \in X$ there exists an open subset $U \subset C$ and $x \in V := p^{-1}(U) \subset X$ s.t. $V \cong U \times \mathbb{P}^1$ (the isomorphism is locally trivial).

2. Any ruled surface is birational (not necessarily isomorphic!) to a geometrically ruled surface (consider the projection $p : \mathbb{P}^1 \times C \longrightarrow C$).

Example 4.1. Let E be a rank 2 vector bundle on a curve C, one can consider the projective bundle $\mathbb{P}_C(E)$ associated to E. It is a surface which is fibered over C,
$$\mathbb{P}_C(E) \longrightarrow C.$$
The fiber over $x \in C$ is $\mathbb{P}(E_x) \cong \mathbb{P}^1$. Since E is locally trivial $\mathbb{P}_C(E)$ is locally isomorphic to $U \times \mathbb{P}^1$ so it is a ruled surface (one can easily show the existence of a section). More in general we have:

Theorem 4.1. *Let $p : X \longrightarrow C$ be a geometrically ruled surface. Then there exist a rank 2 vector bundle on C and an isomorphism*
$$f : X \longrightarrow \mathbb{P}(E)$$
such that $\pi \circ f = p$, where $\pi : \mathbb{P}(E) \longrightarrow C$ is the projection. The geometrically ruled surfaces $\mathbb{P}(E)$ and $\mathbb{P}(E')$ over C are isomorphic iff there exist a line bundle L over C such that $E \cong E' \otimes L$.

Proof. Consider the sheaf $Gl(2, \mathcal{O}_C)$ defined by
$$Gl(2, \mathcal{O}_C)(U) := Gl(2, \mathcal{O}_C(U))$$
these are the 2×2-matrices with coefficients in $\mathcal{O}_C(U)$. We can consider furthermore the quotient sheaf $Gl(2, \mathcal{O}_C)/\mathcal{O}_C^* = \mathbb{P}Gl(2, \mathcal{O}_C)$, where the action of $\mathcal{O}_C^*$ on $Gl(2, \mathcal{O}_C)$ is by a scalar multiplication of the coefficients. One can show that the isomorphism classes of rank 2 vector bundles on C are classified by
$$H^1(C, Gl(2, \mathcal{O}_C)) = \frac{\{\text{rank 2 vector bundles on } C\}}{\text{isomorphism}},$$

(recall that $H^1(C, Gl(1, \mathcal{O}_C)) = H^1(C, \mathcal{O}_C^*) = \mathrm{Pic}(C)$ the line bundles on C) and

$$H^1(C, \mathbb{P}Gl(2, \mathcal{O}_C)) = \frac{\{\mathbb{P}^1 - \text{bundles on } C\}}{\text{isomorphism}}.$$

There is an exact sequence

$$1 \longrightarrow \mathcal{O}_C^* \longrightarrow Gl(2, \mathcal{O}_C) \longrightarrow \mathbb{P}Gl(2, \mathcal{O}_C) \longrightarrow 1$$

inducing an exact sequence in cohomology (one has to be careful here, since the sheaves are not of abelian groups, cf. [2, Proposition III.7] but also [10, Ch. VII, Annexe, Proposition I] for the existence of the exact sequence in cohomology):

$$H^1(C, \mathcal{O}_C^*) \longrightarrow H^1(C, Gl(2, \mathcal{O}_C)) \xrightarrow{p} H^1(C, \mathbb{P}Gl(2, \mathcal{O}_C)) \longrightarrow H^2(C, \mathcal{O}_C^*) = 0$$

where p is defined as $p(E) = \mathbb{P}(E)$. Hence we have the assertion. $\square$

4.2. **Invariants of (geometrically) ruled surfaces.** I recall the definiton of the *Néron-Severi group* that we will need in the sequel (cf. [1, Ch. I, Section 6]). Let X be a complex manifold then we have the *exponential sequence* of sheaves

$$0 \longrightarrow \mathbb{Z}_X \xrightarrow{i} \mathcal{O}_X \xrightarrow{j} \mathcal{O}_X^* \longrightarrow 0$$

where i is just the inclusion and j associates $\exp(f)$ to a germ of holomorphic function f. Then we have the exponential cohomology sequence:

$$\ldots \longrightarrow H^1(\mathbb{Z}_X) \longrightarrow H^1(\mathcal{O}_X) \longrightarrow H^1(\mathcal{O}_X^*) \xrightarrow{c_1} H^2(\mathbb{Z}_X) \longrightarrow H^2(\mathcal{O}_X) \longrightarrow \ldots$$

Recall that $H^1(\mathcal{O}_X^*) = \mathrm{Pic}(X)$ and c_1 is the first Chern class of a line bundle. Let $\mathrm{Pic}^0(X) := \ker(c_1) = \{\text{line bundles which are equivalent to } 0\}$ then $\mathrm{Pic}(X)/\mathrm{Pic}^0(X) \cong \mathrm{Im}(c_1) \subset H^2(X, \mathbb{Z})$ is called the *Néron-Severi group* of X, denoted by $NS(X)$. This is also defined as $\{\text{Divisor}\}/\{\text{algebraic equivalence}\}$.
Let $C_0 = \sigma(C)$ where $p : X \longrightarrow C$ is a geometrically ruled surface and $\sigma : C \longrightarrow X$ is a section.

Proposition 4.1. *Let $p : X = \mathbb{P}(E) \longrightarrow C$ be a geometrically ruled surface. Then:*

1. *$\mathrm{Pic}(X) \cong p^* \mathrm{Pic}(C) \oplus \mathbb{Z}[C_0]$, $H^2(X, \mathbb{Z}) = NS(X)$ and is generated by F and C_0 (F is the class of the fiber of p).*
2. *If $d = \deg(E)(= \deg(\det(E)))$ then $C_0^2 = d$, $(\det(E) = \wedge^2 E)$.*
3. *$c_1(K_X) \sim -2C_0 + (2g(C) - 2 + d)F$ in $H^2(X, \mathbb{Z})$ and $K_X^2 = 8(1 - g(C))$.*

Proof. (sketch). 1. We have $C_0 \cdot F = 1$ (C_0 is the class of the section) and $F^2 = 0$ (two fibers do not meet, cf. Proposition 2.1). We have a map:

$$\begin{array}{ccc} p^* \mathrm{Pic}(C) \oplus \mathbb{Z}[C_0] & \longrightarrow & \mathrm{Pic}(X) \\ (p^* D, nC_0) & \mapsto & p^* D + nC_0 \end{array}$$

We show injectivity: if $p^* D + nC_0 \sim p^* D' + mC_0$ then $p^*(D - D') \sim (m - n)C_0$ but this is not possible since C_0 is not contracted to a point. Hence $n = m$ and $p^* D \sim p^* D'$. For the surjectivity see [2, Proposition III.18]. We show that $H^2(X, \mathbb{Z})$ coincides with the Néron-Severi group of X and it is generated by the classes of C_0 and F. By using the exponential sequence we obtain

$$\ldots \longrightarrow H^1(\mathcal{O}_X) \longrightarrow \mathrm{Pic}(X) = p^* \mathrm{Pic}(C) \oplus \mathbb{Z}[C_0] \xrightarrow{c_1} H^2(\mathbb{Z}) \longrightarrow H^2(\mathcal{O}_X) \longrightarrow \ldots$$

Two points in $\mathrm{Pic}(C)$ have the same image in $H^2(C, \mathbb{Z}) \cong \mathbb{Z}$ (here c_1 is just the degree), moreover observe that F and C_0 are independent in $H^2(X, \mathbb{Z})$. In fact since $F^2 = 0$ and $F \cdot C_0 = 1$ the associated matrix has rank 2. We are left to show that

$H^2(X, \mathcal{O}_X) = 0$. We have $F \cdot K_X = F \cdot (F + K_X) = \deg K_F = \deg K_{\mathbb{P}^1} = -2$. If $H^2(X, \mathcal{O}_X) \neq 0$ then $H^0(X, \mathcal{O}_X(K_X)) \neq 0$ so $|K_X|$ contains an equivalent effective divisor D. Now we can write $D = D' + nF$, where D' is effective and its support does not contain F and $n \geq 0$. Then we obtain $-2 = D \cdot F = D' \cdot F + nF^2 = D' \cdot F \geq 0$, a contradiction. Hence $H^2(X, \mathcal{O}_X) = 0$.
2. See [2, Proposition III.18].
3. Is a direct computation. □

Remark 4.2. From here immediately follows that $p_g(X) = h^2(\mathcal{O}_X) = 0$.

Proposition 4.2. *Let X be a ruled surface then $p_g(X) = p_m(X) = 0$ for all $m > 0$ and $q(X) = g(C)$. In particular $\kappa(X) = -\infty$.*

Proof. We already computed $p_g(X)$. For $p_m(X)$, $m > 0$ the computation is similar. For $q(X) = h^1(\mathcal{O}_X)$ we use the fact that X is birational to $C \times \mathbb{P}^1$ and $q(X)$ is a birational invariant. Let p and q be the projections of $C \times \mathbb{P}^1$ to C, respectively $\mathbb{P}^1$, then

$$\Omega^1_{C \times \mathbb{P}^1} \cong p^* \Omega_C \oplus q^* \Omega_{\mathbb{P}^1}$$

and so

$$\begin{aligned} q(X) &= h^1(X, \mathcal{O}_X) = h^0(X, \Omega^1_X) = h^0(C \times \mathbb{P}^1, \Omega_{C \times \mathbb{P}^1}) \\ &= h^0(C \times \mathbb{P}^1, p^* \Omega_C) + h^0(C \times \mathbb{P}^1, q^* \Omega_{\mathbb{P}^1}) = h^0(C, \Omega^1_C) + h^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}) = g(C) \end{aligned}$$

where we have used the fact that the irregularity is a birational invariant and we have used Corollary 2.1. □

4.3. **Rational surfaces.** Recall that a rational surface X is a ruled surface birational to $\mathbb{P}^1 \times \mathbb{P}^1$ hence X is in particular birational to $\mathbb{P}^2$ and one can compute its numerical invariants: $q(X) = p_g(X) = 0$ and $p_m(X) = 0$ for all $m > 0$. A rational surface is then <u>birational</u> to a geometrically ruled surface. If it is <u>isomorphic</u> to a geometrically ruled surface then it is of the form $\mathbb{P}(E)$ where E is a rank 2 vector bundle on $\mathbb{P}^1$. Now recall the

Theorem 4.2. *(Grothendieck) Let E be a vector bundle of rank n on $\mathbb{P}^1$, then there are integers $a_1, \ldots, a_n \in \mathbb{Z}$ such that $E \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n)$.*

Proof. See e.g.[9, Ch. III, Lemma 7]. □

Hence if X is rational and geometrically ruled then

$$X \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2))$$

now tensoring by a line bundle $\mathcal{O}_{\mathbb{P}^1}(-a_1)$ one get $X \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$, $n \geq 0$.

Definition 4.2. *The surfaces $\mathbb{F}_n := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$, $n \geq 0$ are called* Hirzebruch surfaces

Remark 4.3. Hence a geometrically ruled surface over $\mathbb{P}^1$ is an Hirzebruch surface.

4.4. **Minimal ruled surfaces.**

Proposition 4.3. *(Noether-Enriques). Let X be a minimal surface and suppose that there exists a surjective morphism:*

$$p : X \longrightarrow C$$

where C is smooth and the generic fiber is a smooth rational curve. Then p gives X the structure of a geometrically ruled surface.

Proof. One must show that all the fibers are irreducible. See [2, Lemma III.8]. □

From this proposition it follows:

Theorem 4.3. *Let X be a ruled surface which is not rational (X is birational to $C \times \mathbb{P}^1$, $g(C) > 0$) then X is geometrically ruled if and only if X is minimal.*

Proof. We show first that geometrically ruled non rational surfaces are minimal. Assume that there exists $E \subset X$, a (-1)–curve. Since $E^2 = -1$ then E is not contained in a fiber (here the fibers are all irreducible). Since $\mathrm{Pic}(X) = p^* \mathrm{Pic}(C) \oplus \mathbb{Z}[C_0]$ then $p(E) = p_*(E) = C$. Using Hurwitz formula one sees that C should be rational, which is not the case. Assume now that X is ruled minimal (non rational) then there exists a smooth curve C with $g(C) > 0$ s.t. X is birational to $C \times \mathbb{P}^1$. Hence we get a rational map:

$$p : \ X \dashrightarrow C \times \mathbb{P}^1 \longrightarrow C$$

if p is not a morphism then we blow–up the indeterminacies of p to get a birational morphism $\tilde{p} : \tilde{X} \longrightarrow C$. On $\tilde{X}$ there are (-1)–curves all mapped to points of C (otherwise the images would be a curve, hence C, but C is not rational). This is not possible since we are resolving indeterminacies (one could extend the definition of p at the points where it is not defined!). Hence p is a surjective morphism such that the generic fiber is a smooth rational curve. Since X is minimal then X is geometrically ruled by Theorem 4.3. □

Remark 4.4.

1. In conclusion the minimal models of ruled surfaces, which are not rational, are geometrically ruled surfaces, i.e. of the form $\mathbb{P}(E)$, where E is a rank 2 vector bundle on a smooth curve C, $g(C) > 0$.
2. Classifying minimal models of non-rational ruled surfaces is equivalent to classify rank 2 vector bundles on curves.
3. There are minimal rational ruled surfaces which are not geometrically ruled, for example $\mathbb{P}^2$. If $\mathbb{P}^2$ was geometrically ruled, then there exists n such that $\mathbb{P}^2 \cong \mathbb{F}_n$. But $\mathrm{Pic}(\mathbb{P}^2) \cong \mathbb{Z}$ and $\mathrm{Pic}(\mathbb{F}_n) \cong \mathrm{Pic}(\mathbb{P}^1) \oplus \mathbb{Z} \cong \mathbb{Z}^{\oplus 2}$. However recall that $\mathbb{P}^2$ is birational to $\mathbb{P}^1 \times \mathbb{P}^1 = \mathbb{F}_0$, which is birational to any $\mathbb{F}_n$, $n > 0$.
4. One can give a list of minimal rational surfaces. First of all one has the following:

Proposition 4.4. *The Hirzebruch surface $\mathbb{F}_1 \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ is isomorphic to the blow–up of $\mathbb{P}^2$ along a point. In particular $\mathbb{F}_1$ is not minimal.*

Proof. By Proposition 4.1 there exists a rational curve C_0, with $C_0^2 = -1$, hence it is not minimal. Let now S denote the blow–up of $\mathbb{P}^2$ in a point and E denotes the exceptional divisor. By considering the set of lines through the point we get a morphism $S \longrightarrow E \cong \mathbb{P}^1$, such that all the fibers are isomorphic to $\mathbb{P}^1$. Hence S is geometrically ruled. Since there is a curve E with $E^2 = -1$ then $S \cong \mathbb{F}_1$ (we will show that all the Hirzebruch surfaces $\mathbb{F}_n$ are minimal for $n \neq 1$). □

Theorem 4.4. *Let X be a minimal rational surface. If it geometrically ruled, then $X \cong \mathbb{F}_n$, $n \neq 1$.*

Proof. We show that $\mathbb{F}_n$ is minimal for $n \neq 1$. One has $\mathbb{F}_0 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}) = \mathbb{P}^1 \times \mathbb{P}^1$ which is minimal. Suppose now $n > 1$ and $p_n : \mathbb{F}_n \longrightarrow \mathbb{P}^1$ be the canonical projection ($\mathbb{F}_n$ is geometrically ruled). On $\mathbb{F}_n$ there are two natural type of sections of p_n. A section is a morphism

$$\mathbb{P}^1 \longrightarrow \mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)), \ \ x \mapsto (\lambda, p(x))$$

$$\lambda \in \mathbb{C}, \ p(x) \text{ homogeneous polynomial of degree } n$$

We have the sections

$$\bar{s} : \mathbb{P}^1 \longrightarrow \mathbb{F}_n \ \ x \mapsto (1, 0)$$

and

$$s_p : \mathbb{P}^1 \longrightarrow \mathbb{F}_n \ \ x \mapsto (0, p(x))$$

$\bar{s}$ and s_p define curves in $\mathbb{F}_n$ and $s_p \sim s_q$ in $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n))$, ($p$ and q are homogeneous of degree n) for $p \neq q$. So let $s := s_p$ for some homogeneous polynomial p of degree n. Then

- $\bar{s}$ and s do not intersect,
- two equivalent sections s_p and s_q meet at n points, the zeros of $p(x) = q(x)$, hence $s^2 = n$

We compute $\bar{s}^2$. By Proposition 4.1 we can write $\bar{s} = \alpha s + \beta F$ in $H^2(\mathbb{F}_n, \mathbb{Z})$. Since $1 = \bar{s} \cdot F = \alpha$ we get $\bar{s} = s + \beta F$. Now $0 = s \cdot \bar{s} = n + \beta$. Finally $\bar{s} = s - nF$ and so $\bar{s}^2 = -n$. As $n > 1$, $\bar{s}$ is not a (-1)-curve. Let now $C \subset \mathbb{F}_n$ be an irreducible curve, $C \neq \bar{s}$ and $C = as + bF$. We have $0 \leq C \cdot s = a + nb$ and $0 \leq C \cdot F = a$. We have also $0 \leq C \cdot \bar{s} = b$ (since $C \neq \bar{s}$). Now $C^2 = na^2 + 2ab \geq 0$. So that the only irreducible curve with negative self-intersection is $\bar{s}$. So $\mathbb{F}_n$ is minimal for all $n > 0$. □

Remark 4.5.

1. We have seen that $\mathbb{P}^2$ is not isomorphic to $\mathbb{F}_n$ for all n and that $\mathbb{F}_n$ is not isomorphic to $\mathbb{F}_m$ for all $n \neq m$ (in fact $\mathbb{F}_n$ contains a curve of self-intersecton $-n$ and $\mathbb{F}_m$, $m \neq n$ does not contain such a curve).
2. For rational surfaces there are many minimal models (cf. Theorem 4.4).

Theorem 4.5. *Let X be a minimal surface, $\kappa(X) = -\infty$ then X is either geometrically ruled or $X \cong \mathbb{P}^2$.*

Proof. see [9, Corollary 3, paragragh 14]. □

Remark 4.6.

1. From the previous theorem immediately follows that the minimal models of rational surfaces are $\mathbb{P}^2$ or $\mathbb{F}_n$, $n \neq 1$ (i.e. if X is a rational ruled minimal surface then $X \cong \mathbb{P}^2$ or $X \cong \mathbb{F}_n$).
2. I will continue with an interesting remark on minimal models. Let X be a smooth projective surface, then there is a birational morphism to a minimal model but not necessarily a birational morphism to any minimal model. For example if one blow ups $\mathbb{P}^2$ in one point one get $\mathbb{F}_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$. Blowing up a point on $\mathbb{F}_1$ not on the exceptional curve and contracting a (-1)-curve on $\mathbb{F}_1$ one get $\mathbb{P}^1 \times \mathbb{P}^1$. But there is no birational morphism of $\mathbb{F}_1$ to $\mathbb{P}^1 \times \mathbb{P}^1$ (only a birational map).

4.5. **Some example of rational surfaces.** Let $S \subset \mathbb{P}^n$ be a rational surface and choose a birational map $\mathbb{P}^2 \dashrightarrow S$, then we get a rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^n$ and so a linear system on $\mathbb{P}^2$ with no fixed components. One can then consider e.g. linear systems of conics/cubics and study rational surfaces in $\mathbb{P}^n$.

1. Recall that $h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) = \binom{n+k}{k}$.
2. The map ϕ is in general not everywhere defined. If not then blow-up the base points of the system δ, which defines the map. For simplicity assume that it is enough to blow-up one time. We get a diagramm:

$$\begin{tikzcd} \tilde{S} \arrow[d, "\epsilon"'] \arrow[dr, "f"] & \\ \mathbb{P}^2 \arrow[r, dashed] & \mathbb{P}^n \end{tikzcd}$$

   Where $\tilde{S}$ is the blow up at $p_1, \dots, p_r \in \mathbb{P}^2$, ϵ is the blow-up and $E_i := \epsilon^{-1}(p_i)$. If m_i is the multiplicity of the elements of δ at p_i then $\bar{\delta} \subset |dL - \sum m_i E_i|$, $L = \epsilon^* l$.
3. A particularly interesting case is when $\bar{\delta}$ is very ample i.e. induces an embedding ($\tilde{S} \cong f(\tilde{S})$).
4. If f is an embedding then $f(\tilde{S}) \cong \tilde{S}$ is rational ($\tilde{S}$ is birational to $\mathbb{P}^2$).
5. One can then study the geometry of $S' := f(\tilde{S})$:
   - The Picard group: a basis is given by $L = \epsilon^* l$, E_i, $i = 1, \dots, r$ where $L^2 = 1$, $E_i^2 = -1$, $L \cdot E_i = 0$, $E_i \cdot E_j = 0$ for $i \neq j$. An hyperplane section of S' is then $H = dL - \sum m_i E_i$.
   - The degree of S' is equal to $H^2 = d^2 - \sum m_i^2$.
   - One can study the lines on S': i.e. curves D such that $H \cdot D = 1$.
   - Equations for S'.

4.6. **Linear system of conics.** The complete linear system of conics $|2l| = \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^2}(2)))$ has no base points and projective dimension 5, hence we have an embedding

$$j : \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$$

Let $V := j(\mathbb{P}^2)$ be the Veronese surface. We have $\deg(V) = 4$ (since for a conic $C^2 = 4$) and in particular if $D \subset V$ is an irreducible curve then $H \cdot D \in 2\mathbb{Z}$. Hence V does not contain lines but a 2-dimensional linear system of conics, which are the images of the lines of $\mathbb{P}^2$. One can also consider projections of V to get other rational surfaces (cf. [2, Ch. IV]).

4.7. **Linear system of cubics.** Let $p_1, \dots, p_r$, $r \leq 6$ distinct points on $\mathbb{P}^2$. They are said to be in *general position* if no 3 are on a line and no 6 are on a conic. Let $\epsilon : P_r \longrightarrow \mathbb{P}^2$ be the blow-up of $p_1, \dots, p_r$. Set $d := 9 - r$.

Proposition 4.5. *1. The linear system of cubics through $p_1, \dots, p_r$ defines an embedding $j : P_r \hookrightarrow \mathbb{P}^d$, the surface $S_d := j(P_r)$ has $\deg(S_d) = 9 - r$ and it is called a* Del Pezzo surface of degree *d.*
2. S_3 is a cubic in $\mathbb{P}^3$, S_4 is a complete intersection of 2 quadrics in $\mathbb{P}^4$.
3. S_d contains a finite number of lines which are the images under j of the following curves of P_r:

- *E_i,*
- *strict transform of the lines $< p_i p_j >$, $i \neq j$,*

- *strict transform of the conics through five of the p_i.*

Proof. cf. [2, Proposition IV.9]. □

Remark 4.7. If $S \subset \mathbb{P}^3$ is a smooth cubic surface, then S is a del Pezzo surface S_3 (i.e. S is isomorphic to $\mathbb{P}^2$ with 6 points blown up) and contains 27 lines.
If $S \subset \mathbb{P}^4$ is a complete intersection of 2 quadrics then S is a del Pezzo surface S_4 (the blow up of $\mathbb{P}^2$ at 5 points).

5. Kodaira dimension 0

5.1. **Unicity of the minimal model.**

Theorem 5.1. *Let X be a smooth projective surface, $\kappa(X) \geq 0$, then all minimal models of X are isomorphic.*

Before to give the proof we need a first characterisation of (-1)-curves.

Proposition 5.1. *Let $C \subset X$ be an irreducible curve then C is a (-1)-curve if and only if $C^2 < 0$ and $K_X \cdot C < 0$.*

Proof. One implication is clear using the genus formula. Assume the $C^2 < 0$ and $K_X \cdot C < 0$ then

$$2g(C) - 2 = \deg(\omega_C) = (K_X + C) \cdot C = K_X \cdot C + C^2 < 0$$

hence $g(C) = 0$, so $0 = 2 + (C^2 + C \cdot K_X)$ so $C^2 = -1$ and $C \cdot K_X = -1$. □

Proposition 5.2. *Let X be a smooth projective surface, $\kappa(X) \geq 0$, D an effective divisor on X s.t. $K_X \cdot D < 0$, then D contains a (-1)-curve*

Proof. It sufficies to show that if D is an irreducible curve with $K_X \cdot D < 0$ then D is a (-1)-curve (in fact since D is effective and $D \cdot K_X < 0$ then there exists an irreducible curve D' in the support of D with $D' \cdot K_X < 0$). By assumption since $\kappa(X) \geq 0$ there exists $n \geq 1$, such that $K := nK_X = \sum c_i C_i$, $c_i \geq 0$. Since $nK_X \cdot D < 0$ there exists j such that $C_j = D$, w.l.o.g. we assume $j = 0$ so $C_0 = D$. Hence $D \cdot (K - c_0 D) \geq 0$ and so we must have $D^2 < 0$. We have $D \cdot K_X < 0$ and $D^2 < 0$, hence by Proposition 5.1 D is a (-1)-curve. □

Recall the following

Definition 5.1. *A divisor D is called* numerically effective, *or simply* nef *if for any irreducible curve C we have $D \cdot C \geq 0$.*

Corollary 5.1. *Let X be a smooth projective surface such that K_X is not nef. Then either $\kappa(X) = -\infty$ or X contains a (-1)-curve. In particular K_X is nef if X is a minimal surface with $\kappa(X) \geq 0$.*

Proof. (of Theorem 5.1). We show that if $f : X \dashrightarrow Y$ is a birational map between minimal surfaces X and Y with $\kappa(X) \geq 0$ and $\kappa(Y) \geq 0$ then f is an isomorphism. First we know that K_X and K_Y are nef. Now observe that if $\sigma : \tilde{X} \longrightarrow X$ is the blow up of X at some point $q \in X$ and $E = \sigma^{-1}(q)$, we have for any irreducible curve $\tilde{C} \subset X$, with $\sigma(\tilde{C}) = C$ a curve:

$$\begin{aligned} K_{\tilde{X}} \cdot \tilde{C} &= (\sigma^* K_X + E) \cdot (\sigma^* C - mE) \\ &= K_X \cdot C + m \geq K_X \cdot C \end{aligned}$$

and $K_{\tilde{X}} \cdot E = (\sigma^* K_X + E) \cdot E = -1$. Blow up now X to resolve the indeterminacies of f

$$X' \longrightarrow X \dashrightarrow Y$$

take a minimal number of blow ups (one needs at least one blow up). Let E be the exceptional curve of the last blow up and f' be the morphism $X' \longrightarrow Y$. Then $f'(E) := C$ must be a curve (otherwise one does not need to blow it up). Now $K_{X'} \cdot E = -1$ then one get $-1 = K_{X'} \cdot E \geq K_Y \cdot C$, which is impossible since K_Y is nef, hence f is a morphism. Similarly K_X is nef and so f^{-1} is a morphism. In conclusion f is an isomorphism. □

Remark 5.1.

1. Observe that the proof works also for those surfaces X with $K_X \sim 0$ as K3 surfaces.
2. In particular any birational map between minimal surfaces with $\kappa(X) \geq 0$ is an isomorphism.

5.2. **K3 surfaces, Enriques surfaces.**

Definition 5.2. *Let X be a projective smooth surface then:*

1) *X is a* K3 surface *if $q(X) = h^1(\mathcal{O}_X) = 0$, $K_X = 0$ $(p_g(X) = 1)$.*

2) *X is an* Enriques surface *if $q(X) = p_g(X) = 0$ and $2K_X = 0$ $(K_X \neq 0)$.*

3) *X is an* Abelian surface *if $q(X) = 2$, $p_g(X) = 1$ $(K_X = 0)$.*

4) *X is a* bielliptic surface *if $q(X) = 1$, $p_g(X) = 0$ (in this case we have $4K_X = 0$ or $6K_X = 0$,* [2, Corollary VIII.7]*)*

Example 5.1. 1. K3 surfaces. Let $f_4(x_0, x_1, x_2, x_3)$ be a homogeneous polynomial of degree four. Then if $S_4 := \{f_4(x_0, x_1, x_2, x_3) = 0\}$ is smooth, then it is a K3 surface. In fact $q(S_4) = h^1(\mathcal{O}_{S_4}) = (1/2)b_1(S_4)$, where b_1 is the first Betti number (S_4 is a kähler manifold, cf. [1, Ch. IV, Theorem 2.6 and Corollary 2.10]). Using now Barth-Lefschetz theorem on hyperplane sections as $\mathbb{P}^3$ is simply connected then S_4 is simply connected too, so $b_1(S_4) = 0$. By adjunction one sees easily that $K_X = 0$. In a similar way the complete intersections $S_{2,3} \subset \mathbb{P}^4$, $S_{2,2,2} \subset \mathbb{P}^4$ are K3 surfaces.
2. Enriques surfaces. Let X be a K3 surface and $i : X \longrightarrow X$ be an involution on X without fixed points. Consider the quotient $Y := X/i$, the map $p : X \longrightarrow Y$ is a degree two étale covering. If $C \subset Y$ is a curve then $p_* p^* C = 2C$ hence by linearity for any divisor we have $p_* p^* D = 2D$ in particular we have

$$p_* p^* K_Y = 2K_X$$

but $p^* K_Y = K_X \sim 0$ (the cover is unramified), then $2K_X \sim 0$ and $\chi(\mathcal{O}_X) = 2\chi(\mathcal{O}_Y)$. Since X is a K3 surface $q(X) = 0$, $p_g(X) = 1$ so $\chi(\mathcal{O}_X) = 2$ and $\chi(\mathcal{O}_Y) = 1$. From here it follows that $q(Y) = p_g(Y)$. Since the cover is unramified we have $h^1(\mathcal{O}_Y) \leq h^1(\mathcal{O}_X) = 0$ hence $p_g(Y) = q(Y) = 0$. We conclude that Y is an Enriques surface. In fact one can show that all Enriques surfaces are obtained in this way. In particular any K3 surface with involution as i is projective (cf. e.g. the result of Nikulin [7, Theorem 3.1]) and so any Enriques surface is projective (but one can show directly that Enriques surfaces are projective, without using the result of Nikulin on non-symplectic involutions on K3 surfaces, see [3] for an extended description of Enriques surfaces). We want to give a concrete example. Let $X \subset \mathbb{P}^5$, $X = S_{2,2,2}$ be the complete intersection of three quadrics of the form

$$P_i(x_0, x_1, x_2) + Q_i(x_3, x_4, x_5) = 0$$

$i = 1, 2, 3$, P_i, Q_i homogeneous of degree 2. For a generic choice of P_i and Q_i one get that X is smooth (by Bertini's thorem) and there is a natural involution:

$$i : (x_0, x_1, x_2, x_3, x_4, x_5) \mapsto (x_0, x_1, x_2, -x_3, -x_4, -x_5).$$

The fixed points are the intersections of $\{x_0 = x_1 = x_2 = 0\} \cong \mathbb{P}^2$ with $Q_1 = Q_2 = Q_3 = 0$. These are three conics in $\mathbb{P}^2$ and they do not meet in general. In the same way we have the intersection of the plane $\{x_3 = x_4 = x_5 = 0\} \cong \mathbb{P}^2$ with $P_1 = P_2 = P_3 = 0$, this is again the intersection of three quadrics, which is empty in general. Hence choosing P_j and Q_j generic, the involution i does not have fixed points on $S_{2,2,2}$. Hence the quotient X/i is an Enriques surface (and in fact it can be shown that the generic Enriques surface can be obtained in this way).
3. Abelian surfaces. These are complex tori $\mathbb{C}^2/\Lambda$, (Λ is a rank four lattice) with an embedding in a projective space.
4. Bielliptic surfaces. These are quotients $(E \times F)/G$ where E, F are elliptic curves, G is a finite group of translation on E acting on F such that $F/G \cong \mathbb{P}^1$.
Bagnera and de Franchis (cf. [2, List VI.20]) give a complete list of such surfaces: for example let $F_i := \mathbb{C}/\mathbb{Z} \oplus i\mathbb{Z}$ with i a primitive 4th-root of the unity, E an elliptic curve and let $G = \mathbb{Z}/4\mathbb{Z}$ be a group acting on F_i by multiplication by i, i.e. $x \mapsto ix$ and it acts on E by translation (on an elliptic curve we have n^2 points of n-torsion).
Finally we say that a surface is *elliptic* if there is a surjective morphism

$$\pi : X \longrightarrow B$$

where B is a smooth curve, whose generic fiber is an elliptic curve. Then bielliptic surfaces are elliptic having two elliptic fibrations (the contrary is not true: for example there are elliptic rational surfaces or elliptic K3 surfaces, cf. [6], for a beautiful survey on elliptic fibrations).

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