Automorphisms of K3 surfaces, holomorphic symplectic manifolds and Enriques varieties

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Automorphisms of K3 surfaces

Let X be a K3 surface and σ an automorphism of finite order $d \in \mathbb{N}$.

$$\sigma : X \longrightarrow X \qquad \sigma^d = \mathrm{id}\,.$$

We call σ *(purely) non-symplectic* if

$$\sigma^* \omega_X = \zeta \omega_X, \qquad \zeta = e^{\frac{2\pi i}{d}}$$

where $H^0(X, \Omega_X^2) = \mathbb{C}\omega_X$.

Their study was started essentially by Nikulin in 1980.

Fixed locus

First question: How to describe the topology of the fixed locus?

The Lefschetz number

$$L(\sigma) = \sum_{i=0}^{2} (-1)^i \operatorname{tr}(\sigma^* | H^i(X, \mathcal{O}_X)) = 1 + \zeta^{-1}.$$

together with the topological Lefschetz formula gives:
$L(\sigma) \neq 0$ then $X^\sigma \neq \emptyset$.

Nikulin: complete classification of non-symplectic involutions (about 75 cases).

On K3 surfaces the only automorphisms without fixed points are *non-symplectic involutions* ($i^* \omega_X = -\omega_X$).

Enriques surfaces

Let X be a K3 surface complete intersection of three quadrics in $\mathbb{P}^5(\mathbb{C})$:

$$Q_i : \; p_i(x_0, x_1, x_2) + q_i(y_0, y_1, y_2) = 0, \qquad i = 1, 2, 3$$

and an involution

$$\begin{array}{rccc} i: & \mathbb{C}^6 & \longrightarrow & \mathbb{C}^6 \\ & (x_0, x_1, x_2, y_0, y_1, y_2) & \mapsto & (-x_0, -x_1, -x_2, y_0, y_1, y_2) \end{array}$$

induces an involution on X without fixed points (generic choice of p_i and q_i).

$Y := X/i$ is an *Enriques surface*: compact, complex, smooth surface such that

$$q(Y) = p_g(Y) = 0, \quad 2K_Y = 0, \; (K_Y \neq 0).$$

In particular we have $\chi(\mathcal{O}_Y) = \frac{1}{2}\chi(\mathcal{O}_X) = 1$ and $\pi_1(Y) = \mathbb{Z}/2\mathbb{Z}$.

Enriques varieties

How to generalize in higher dimension?

Definition

A *Calabi-Yau manifold* (CY) is a complex, compact, smooth, Kähler manifold X such that the canonical bundle is trivial and

$$H^0(X, \Omega^l_X) = 0 \quad \text{for} \quad 0 < l < \dim X$$

Definition

An *irreducible holomorphic symplectic manifold* (IHS) is a complex, compact, smooth, Kähler manifold X simply connected, such that

$$H^0(X, \Omega^2_X) = \mathbb{C}\omega_X$$

where ω_X is an everywhere non-degenerate holomorphic symplectic form.

- Let X be a CY variety, $\dim X = m$ then $\chi(\mathcal{O}_X) = 1 + (-1)^m$: if m is even a quotient variety Y by a fixed point free involution has $\chi(\mathcal{O}_Y) = 1$.
- Let X be an IHS, $\dim X = 2n - 2$, then we can consider quotients Y by fixed point free automorphisms of order d, we get:

  $$\chi(\mathcal{O}_Y) = \frac{1}{d}\left(\frac{\dim X}{2} + 1\right) = \frac{n}{d}$$

  hence take $d = n$ to get $\chi(\mathcal{O}_Y) = 1$.

Observe that the automorphisms must be (purely) non-symplectic (use the holomorphic Lefschetz formula).

In both cases we get $dK_Y = 0$ for some $d \in \mathbb{N}$.

Basic facts

Definition (Boissiére/Nieper-Wisskirchen/Sarti 2010)

1) A connected, compact, smooth, Kähler manifold Y is called an *Enriques variety* if there exists $d \geq 2$ (the *index* of Y) such that $dK_Y = 0$ in $\mathrm{Pic}(Y)$ (and $d'K_Y \neq 0$ for any $0 \neq d' < d$), $\chi(\mathcal{O}_Y) = 1$ and $\pi_1(Y)$ is cyclic of order d.
2) An Enriques variety is called *irreducible* if the holonomy group of its universal cover is irreducible.

The definition 2) means in particular that Y is not a product (the product of two Enriques varieties of index prime to each other is again an Enriques variety).

Oguiso and Schröer (2010) defined Enriques varieties as quotients of IHS.

Properties of Enriques varieties

Theorem (B/N-W/S)

1) Every Enriques variety is even dimensional.
2) If Y is an irreducible Enriques variety of index 2, then there exists a CY variety X, $\dim X = 2r$ and an involution ι on X without fixed points such that $Y = X/\langle \iota \rangle$.
3) If Y is an irreducible Enriques variety of index > 2, then there exists an IHS variety X, $\dim X = 2d - 2$ and an automorphism f without fixed points, $f^d = \mathrm{id}$ such that $Y = X/\langle f \rangle$.
4) Every irreducible Enriques variety is projective.

Important ingredient: Bogomolov decomposition theorem for compact Kähler manifolds with first Chern class $c_1 = 0$.

If Y is an Enriques variety its universal cover X has $c_1 = 0$ and it is simply connected. We have

$$X \cong \prod_j V_j \times \prod_i W_i$$

with V_j =CY (of even dimension) and W_i =IHS.

Examples

It is not difficult to produce examples of Enriques varieties of index 2 (use Calabi-Yau's).

For the index > 2: we use generalized Kummer varieties. Let A be a complex torus of dimension 2.

$$\mathrm{Km}_n(A) = s^{-1}(0) \subset \mathrm{Hilb}^n(A) \xrightarrow{s} A$$

Then $\mathrm{Km}_n(A)$ is a *generalized Kummer variety* of dimension $2n - 2$. (Beauville 1983: $\mathrm{Km}_n(A)$ is an IHS).

Let $A = E \times E$, E be an elliptic curve with an automorphism of order $n \in \{3, 4\}$,

$$h := \begin{pmatrix} \zeta_n & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{Aut}_{\mathbb{Z}}(A) \qquad \zeta_n = e^{\frac{2\pi i}{n}}$$

and $a_i \in E$, $i = 1, 2$ points of order n, $a := (a_1, a_2)$. The composition

$$\psi := t_a \circ h$$

induces an automorphism $\psi^{[n]}$ on $\mathrm{Hilb}^n(A)$.

Proposition (B/N-W/S)

- One can choose a such that $\psi^{[n]}$ has no fixed points on $\mathrm{Km}_n(A)$.
- There exists Enriques varieties of index 3 and 4.

Remark: The same construction does not work to produce an example of an Enriques variety of index 6, we always get fixed points.

One can not use $\mathrm{Hilb}^n(X)$, X a K3 surface, and automorphisms of K3 suraces to produce examples of Enriques varieties (they always have fixed points).

Problem: Study the automorphisms of $\mathrm{Hilb}^n(X)$ and $\mathrm{Km}_n(A)$ *not* induced by automorphisms of X or A.

Further results

Theorem (B/N-W/S)

If Y is an Enriques variety, $\dim Y = 2d-2$, d odd, $d \geq 3$, then Y is irreducible (and so projective).

This is false if d is even:
Let V be the 6-dimensional CY variety in $\mathbb{P}^{13}$ complete intersection of 7 quadrics:

$$Q_j(x_0, \ldots, x_6) - P_j(y_0, \ldots, y_6) = 0 \qquad j = 1, \ldots, 7.$$

The involution $\iota : (x, y) \mapsto (x, -y)$ does not have fixed points on V. Let $W := \mathrm{Km}_3(A)$ then the 10 dimensional quotient:

$$V \times W / \langle \iota \times \psi^{[3]} \rangle := Y$$

is an Enriques variety cannot be obtained as the quotient of a CY or an IHS by a fixed points free automorphism.

Kim's generalization

Theorem (Kim 2010)

1) Let Y be an Enriques variety, $\dim Y = 2n-2$, $n = 2m$, m prime, and $\pi_1(Y)$ cyclic of order n.
   Then Y is the quotient of a product of a Calabi-Yau manifold of dimension $2m$ and an IHS of dimension $2m-2$ by an automorphism f of order n acting freely.
2) The variety Y and its universal cover are projective.