PROJECTIVE MODELS OF K3 SURFACES WITH AN EVEN SET

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ABSTRACT. The aim of this paper is to describe algebraic K3 surfaces with an even set of rational curves or of nodes. Their minimal possible Picard number is nine. We completely classify these K3 surfaces and after a careful analysis of the divisors contained in the Picard lattice we study their projective models, giving necessary and sufficient conditions to have an even set. Moreover we investigate their relation with K3 surfaces with a Nikulin involution.

0. INTRODUCTION

It is a classical problem in algebraic geometry to determine when a set of (-2)-rational curves on a surface is even. This means the following: let $L_1, \ldots, L_N$ be rational curves on a surface X then they form an *even set* if there is $\delta \in Pic(X)$ such that

$$L_1 + \ldots + L_n \sim 2\delta.$$

This is equivalent to the existence of a double cover of X branched on $L_1 + \ldots + L_n$. This problem is related to the study of even sets of nodes, in fact a set of nodes is even if the (-2)-rational curves in the minimal resolution are an even set. In particular the study of even sets on surfaces plays an important role in determining the maximal number of nodes a surface can have (cf. e.g. [Be], [JR]). Here we restrict our attention to K3 surfaces.

In a famous paper of 1975 [N1] Nikulin shows that an even set of disjoint rational curves (resp. of distinct nodes) on a K3 surface contains 0, 8 or 16 rational curves (nodes). If the even set on the K3 surface X is made up by sixteen rational curves, the surface covering X is birational to a complex torus A and X is the Kummer surface of A. This situation is studied by Nikulin in [N1]. If the even set on X is made up by eight rational curves then the surface covering X is also a K3 surface. There are some more general results about even sets of curves not necessarily disjoint. More recently in [B1] Barth studies the case of even sets of rational curves on quartic surfaces (i.e. K3 surfaces in $\mathbb{P}^3$) also in the case that the curves meet each other and he finds sets containing six or ten lines too.

In the paper [B2] he discusses some particular even sets of disjoint lines and nodes on K3 surfaces whose projective models are a double cover of the plane, a quartic in $\mathbb{P}^3$ or a double cover of the quadric $\mathbb{P}^1 \times \mathbb{P}^1$, and he gives necessary and sufficient conditions to have an even set.

Our purpose is to study algebraic K3 surfaces admitting an even set of eight disjoint rational curves. We investigate their Picard lattices, moduli spaces and projective models. The minimal possible Picard number is nine, and we restrict our study to the surfaces with this Picard number. The techniques used by Barth in his article are mostly geometric, here we use lattice theory: we investigate first the Picard lattices of the K3 surfaces and the ampleness of certain divisors, then we study the projective models. We find again the cases studied by Barth and we discuss many new cases, with a special attention to complete intersections. We

give also an explicit relation between the Picard lattice of an algebraic K3 surface with an even set and the Picard lattice of the K3 surface which is its double cover. More precisely if X admits an even set of eight disjoint rational curves, then by [N1], it is the desingularization of the quotient of a K3 surface by a Nikulin involution (i.e. a symplectic automorphism of order two). The Nikulin involutions are well known and are studied by Morrison in [M] and by van Geemen and Sarti in [vGS]. In [vGS] the authors describe also some geometric properties of the quotient by a Nikulin involution and so of K3 surfaces with an even set of eight nodes. In [N1] Nikulin proves that a sufficient condition on a K3 surface to be a Kummer surface (and so to have an even set made up by sixteen disjoint rational curves) is that a particular lattice (the so called Kummer lattice) is primitively embedded in the Néron Severi group of the surface. Here we prove a similar result: a sufficient condition on a K3 surface X to be the desingularization of the quotient of another K3 surface with a Nikulin involution (and so to have an even set made up by eight disjoint rational curves) is that a particular lattice (the so called Nikulin lattice) is primitively embedded in the Néron Severi group of X. This result is essential to describe the coarse moduli space of a K3 surface with an even set of eight disjoint rational curves.

In the Section 1 we recall some known results on even sets on surfaces, in particular on K3 surfaces. In the Section 2 we study algebraic K3 surfaces X with Picard number nine. If X admits an even set of eight disjoint rational curves, then its Néron Severi group has rank at least nine (it has to contain the eight rational curves of the even set and a polarization, because the K3 surface is algebraic). The main results of this section (and also two of the main results of this paper) are the complete description of the possible Néron Severi groups of rank nine of algebraic K3 surfaces admitting an even set and the complete description of the coarse moduli space of the algebraic K3 surfaces with an even set of eight disjoint rational curves. Moreover using the results of [vGS] we describe the relation between the Néron Severi group of an algebraic K3 surface Y admitting a Nikulin involution ι and the Néron Severi group of a K3 surface admitting an even set, which is the desingularization of Y/ι (Corollary 2.2). In the Section 3 we analyze the ampleness of some divisors (or more in general the nefness). These classes are used in the Section 4 to describe projective models of algebraic K3 surfaces with an even set of eight disjoint rational curves. In particular we describe the following projective models:

• **double covers of** $\mathbb{P}^2$: these branch along a sextic with eight nodes (Paragraph 4.1) or along a smooth sextic (Paragraph 4.4, a)) (these two situations are studied also by Barth in [B2], first and second cases), or along a sextic with four nodes (Paragraph 4.7);

• **quartic surfaces in** $\mathbb{P}^3$: these have an even set of nodes (Paragraph 4.2) or an even set of lines (Paragraph 4.5, a)) (these two situations are studied also by Barth in [B2], third and forth cases), or it has a mixed even set of nodes and conics (Paragraph 4.8, b));

• **double covers of a cone :** these branch along a conic and a sextic on the cone, which intersect in six points (Paragraph 4.3);

• **complete intersections of a hyperquadric and a cubic hypersurface in** $\mathbb{P}^4$: these have an even set of nodes (Paragraph 4.4, b)) or an even set of lines (Paragraph 4.7, a));

• **complete intersections of three hyperquadrics in** $\mathbb{P}^5$: these have an even set of nodes (Paragraph 4.5, b) and Paragraph 4.6, b)) or an even set of lines (Paragraph 4.8, a) and Paragraph 4.9, a));

• **double covers of a smooth quadric**: these branch along a curve of bidegree $(4,4)$ (Paragraph 4.6) (this case is studied also by Barth in [B2], sixth case);

• We study also the following complete intersections (c.i.) of hypersurfaces of bidegree (a, b) in $\mathbb{P}^n \times \mathbb{P}^m$:

space	c.i.	paragraph
$\mathbb{P}^1 \times \mathbb{P}^2$	$(2, 3)$	4.9 b)
$\mathbb{P}^4 \times \mathbb{P}^2$	$(2, 0), (1, 1), (1, 1), (1, 1)$	4.4 c)
$\mathbb{P}^2 \times \mathbb{P}^2$	$(1, 2), (2, 1)$	4.10
$\mathbb{P}^3 \times \mathbb{P}^3$	$(1, 1), (1, 1), (1, 1), (1, 1)$	4.11

In Section 4 we describe moreover geometric properties of these K3 surfaces with an even set. In Section 5 we use these properties to give sufficient conditions for a K3 surface to have an even set.

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1. K3 SURFACES WITH AN EVEN SET OF NODES AND OF RATIONAL CURVES

Definition 1.1. *Let X be a surface. A set of m disjoint (-2)-rational smooth curves, N_1, ...,N_m, on X, is an* even set of rational curves *if there is a divisor $\delta \in Pic(X)$ such that*

$$N_1 + \ldots + N_m \sim 2\delta,$$

where $''\sim''$ denotes linear equivalence.

Definition 1.2. *Let $\bar{X}$ be a surface and let $\mathcal{N} = \{p_1, \ldots, p_m\}$ be a set of nodes on $\bar{X}$. Let $\tilde{\beta} : X \longrightarrow \bar{X}$ be the minimal resolution of the nodes of X and let $N_i = \tilde{\beta}^{-1}(p_i)$, $i = 1, \ldots, m$. These are (-2)-rational curves on X. The set $\mathcal{N}$ is an* even set of nodes *if $N_1, \ldots, N_m$ are an even set of rational curves.*

In the case of $K3$ surfaces linear equivalence is the same as algebraic equivalence (which we denote by $\equiv$) and $Pic(X) = NS(X)$.

The existence of an even set $N_1, \ldots, N_m$ on a surface X is equivalent to the existence of a double cover $\pi : \widetilde{Y} \to X$ from a surface $\widetilde{Y}$ to X branched on $N_1 + \ldots + N_m$ [BPV, Lemma 17.1].

Let Y be a surface and ι be an involution on Y with exactly m distinct fixed points $q_1, \ldots, q_m$ and let $\widetilde{Y}$ be the blow up of Y at the points $q_1, \ldots, q_m$. The involution ι induces an involution $\widetilde{\iota}$ on $\widetilde{Y}$. Let $\bar{X}$ be the quotient surface Y/ι and $\pi' : Y \to \bar{X}$ be the projection. The surface $\bar{X}$ has m nodes in $\pi'(q_i)$, $i = 1, \ldots, m$. Let $\tilde{\beta} : X \to \bar{X}$ be the minimal resolution of $\bar{X}$. Then the following diagram commutes

$$\begin{array}{ccc} \widetilde{Y} & \xrightarrow{\beta} & Y \\ \pi \downarrow & & \downarrow \pi' \\ X & \xrightarrow{\tilde{\beta}} & \bar{X}. \end{array} \tag{1}$$

The double cover $\pi : \widetilde{Y} \to X$ is branched on $N_1 + \ldots + N_m$ where N_i are the (-2)-curves such that $\widetilde{\beta}(N_i) = \pi'(q_i)$, $i = 1, \ldots, m$ and these form an even set.

Conversely if $\pi : \widetilde{Y} \to X$ is a double cover of X branched on the divisor $N_1 + \ldots + N_m$ where N_i are (-2)-rational curves, then there is a diagram as (1).

We recall some facts about even sets on K3 surfaces:

- If $N_1, \ldots, N_m$ is an even set of disjoint curves on a K3 surface, by a result of Nikulin [N1, Lemma 3] we have $m = 0, 8$ or 16.
- If $m = 16$ the surface Y in the diagram (1) is a torus of dimension two ([N1, Theorem 1]), the involution ι is defined on Y as $y \mapsto -y$, $y \in Y$ and has sixteen fixed points. So X is the Kummer surface associated to the surface Y (a Kummer surface is by definition the K3 surface obtained as the desingularization of the quotient of a torus Y by the involution $y \mapsto -y$, $y \in Y$). If Y is an algebraic torus (so an Abelian surface), then X is an algebraic K3 surface and its Picard number is $\rho \geq 17$.
- If $m = 8$ then the surface Y is a K3 surface and the cover involution has eight isolated fixed points (it is a *Nikulin involution*, cf. Definition 1.3 below). If Y is an algebraic K3 surface, then X is algebraic and its Picard number is $\rho \geq 9$.

Definition 1.3. *Let Y be a K3 surface. Let ι be an involution of Y. The involution ι is called* Nikulin involution *if $\iota_{|H^{2,0}(X,\mathbb{C})} = id_{|H^{2,0}(X,\mathbb{C})}$.*

We recall some facts about the Nikulin involutions:

- An involution ι on a K3 surface is a Nikulin involution if and only if it has eight isolated fixed points [N2, Section 5].
- The Nikulin involutions are the unique involutions on a K3 surface Y such that the desingularization of $Y/\iota = \bar{X}$ is a K3 surface. In fact let $\widetilde{Y}$ be the blow up of Y on the fixed points of the involution ι. In this way we obtain more algebraic classes on $\widetilde{Y}$, but the transcendental classes are the same and so $H^{2,0}(Y)^{\iota^*} = H^{2,0}(\widetilde{Y})^{\tilde{\iota}^*}$. Since an automorphism of a K3 surface induces a Hodge isometry on the second cohomology group we have $\iota^*(H^{2,0}(Y)) = H^{2,0}(Y) \simeq \mathbb{C}$ and since $H^{2,0}(\widetilde{Y})^{\tilde{\iota}^*} = H^{2,0}(X) \simeq \mathbb{C}$ it follows that ι^* is the identity on $H^{2,0}(Y)$, so ι is a symplectic automorphism.

2. Even sets and Nikulin involutions

Let $N_1, \ldots, N_8$ be an even set of eight disjoint smooth rational curves on a K3 surface X, then by adjunction $N_i^2 = -2$ and Morrison shows in [M, Lemma 5.4] that the minimal primitive sublattice of $H^2(X, \mathbb{Z})$ containing these (-2)-curves is isomorphic to the Nikulin lattice:

Definition 2.1. [M, Definition 5.3] *The* Nikulin lattice *is an even lattice N of rank eight generated by $\{N_i\}_{i=1}^{8}$ and $\hat{N} = \frac{1}{2}\sum N_i$, with bilinear form induced by*

$$N_i \cdot N_j = -2\delta_{ij}.$$

Observe that $\hat{N}^2 = -4$ and $\hat{N} \cdot N_i = -1$. This lattice is a negative definite lattice of discriminant 2^6 and discriminant group $(\mathbb{Z}/2\mathbb{Z})^{\oplus 6}$.

From now on X is an algebraic K3 surface. A K3 surface has an even set of eight disjoint rational curves if there are eight disjoint rational curves spanning a copy of N in $NS(X)$ (then rank $NS(X) \geq 8$). Since X is algebraic the signature of the Néron Severi group $NS(X)$ is $(1, \rho - 1)$, where ρ is the Picard number of X (i.e. the rank of $NS(X)$). So the Néron Severi group of X has signature $(1, \rho - 1)$ and has to contain the negative lattice N of rank eight, so $NS(X)$ contains also a class with positive self intersection. Clearly $\rho \geq 9$ and we will see that the generic algebraic K3 surface with an even set has $\rho = 9$ and that the number of moduli is $20 - 9 = 11$ (Corollary 2.3). Here we study the case of algebraic K3 surfaces with Picard number nine.

Proposition 2.1. *Let X be an algebraic K3 surface with an even set of eight disjoint rational curves and with Picard number nine, let L be a divisor generating $N^{\perp} \subset NS(X)$, $L^2 > 0$.*

Let d be a positive integer such that $L^2 = 2d$ and let

$$\mathcal{L}_{2d} = \mathbb{Z}L \oplus N.$$

Then
(1) if $L^2 \equiv 2 \mod 4$ then $NS(X) = \mathcal{L}_{2d}$,
(2) if $L^2 \equiv 0 \mod 4$ then either $NS(X) = \mathcal{L}_{2d}$ or $NS(X) = \mathcal{L}'_{2d}$, where $\mathcal{L}'_{2d}$ is generated by $\mathcal{L}_{2d}$ and by a class $(L/2, v/2)$, with

- $v^2 \in 4\mathbb{Z}$,
- $v \cdot N_i \in 2\mathbb{Z}$ ($v \in N$ but $v/2 \notin N$),
- $L^2 \equiv -v^2 \mod 8$.

Proof. The discriminant group of $\mathcal{L}_{2d} = \mathbb{Z}L \oplus N$ is $(\mathbb{Z}/2d\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})^{\oplus 6}$, hence an element in the Néron Severi group of X but not in $\mathcal{L}_{2d}$ is of the form $(\alpha L/2d, v/2)$ with $\alpha \in \mathbb{Z}$, $v \in N$. Since $2 \cdot (\alpha L/2d, v/2) - v \in NS(X)$ we can assume that $\alpha = d$ and so the element is $(L/2, v/2)$. We can write $v = \sum \alpha_i N_i + \beta \hat{N}$, $\beta \in \{0, 1\}$, we have

$$\left(\frac{L}{2}, \frac{v}{2}\right) \cdot N_i \in \mathbb{Z}.$$

Hence by doing the computations it follows

$$\frac{1}{2}(-2\alpha_i - \beta) \in \mathbb{Z},$$

hence $\beta \in 2\mathbb{Z}$, and so we may assume $\beta = 0$. We have also

$$\left(\frac{L}{2}, \frac{v}{2}\right) \cdot \hat{N} \in \mathbb{Z}$$

so

$$-\frac{1}{2}\left(\sum \alpha_i\right) \in \mathbb{Z}$$

hence $\alpha_1 + \ldots + \alpha_8 \in 2\mathbb{Z}$ and so $\alpha_1^2 + \ldots + \alpha_8^2 \in 2\mathbb{Z}$ too. We have

$$\begin{aligned} v^2 &= -2\sum \alpha_i^2 - 4\beta^2 - 2\beta \sum \alpha_i \\ &= -2\left(\sum \alpha_i^2\right). \end{aligned}$$

It follows that $v^2 \in 4\mathbb{Z}$ and $v \cdot N_i \in 2\mathbb{Z}$.
Since the Néron Severi lattice of a K3 surface is even we have

$$\left(\frac{L}{2}, \frac{v}{2}\right)^2 = \frac{L^2 + v^2}{4} \in 2\mathbb{Z}$$

which gives $L^2 \in 4\mathbb{Z}$, so d must be even and $L^2 + v^2 \equiv 0 \mod 8$.
Assume now that there is another class $(L/2, v'/2) \in NS(X)$, then the class $(L/2, v/2) - (L/2, v'/2) = (v - v')/2 \in NS(X)$ too. Since N is primitive $(v - v')/2 \in N$. So there is a $\delta \in N$ s.t. $v - v' = 2\delta$. So $(L/2, v'/2) \in NS(X)$ if and only if $(L/2, v'/2) = (L/2, v/2) + \delta$ for certain $\delta \in N$. This concludes the proof of the proposition. $\square$

Proposition 2.2. *Under the assumptions of the Proposition 2.1, $\mathcal{L}'_{2d}$ is the unique even lattice (up to isometry) such that $[\mathcal{L}'_{2d} : \mathcal{L}_{2d}] = 2$ and N is a primitive sublattice of $\mathcal{L}'_{2d}$.*

Proof. We describe briefly the group $O(N)$ of isometries of N. These must preserve the intersection form, so the image of each (-2)-vector under an isometry is a (-2)-vector. The only (-2)-vectors in the Nikulin lattice N up to the sign are the eight vectors N_i and so if $\sigma \in O(N)$ then $\sigma(N_i) = \pm N_j$, $i, j = 1, \ldots, 8$. In particular the group of permutation of eight elements Σ_8 is contained in $O(N)$. This group fixes the class $\hat{N}$.

Each class v in N is $v = \sum_{i=1}^{8} \alpha_i N_i + a\hat{N}$, α_i, $a \in \mathbb{Z}$. We consider two different elements v and v' such that $\mathcal{L}_{2d}$ together with the class $(L/2, v/2)$ or with the class $(L/2, v'/2)$ generate an overlattice of $\mathcal{L}_{2d}$. We want to prove that there exists an isometry σ of N such that $\sigma(v) = v'$. From the conditions given on v, or v', (in particular from the fact that $v \cdot N_i \in 2\mathbb{Z}$), $v = \sum_{i=1}^{8} \alpha_i N_i$, $\alpha_i \in \mathbb{Z}$ and $v' = \sum_{i=1}^{8} \beta_i N_i$, $\beta_i \in \mathbb{Z}$, we may assume that $\alpha_i, \beta_i \in \{0, 1\}$. The only possibilities for v^2 (or v'^2) are $-4, -8, -12$, (by the condition on v^2 given in the previous proof) and this depends only on the number of $\alpha_i's$ (resp. $\beta_i's$) equal to one.
We distinguish two different cases: $v^2 = v'^2$ and $v^2 \neq v'^2$ but $v^2 \equiv v'^2 \mod 8$ (since $-v'^2 \equiv L^2 \equiv -v^2 \mod 8$).
The case $v^2 = v'^2$. This condition implies that there are the same number of α_i and β_i equal to one. Hence there is a permutation $\sigma \in \Sigma_8 \subset O(N)$ of the N_i, s.t. $\sigma(v) = v'$.
Observe that if $L^2 \equiv 0 \mod 8$, then it is clear from the description above that $v^2 = v'^2 = -8$ and so we are in this case.
The case $v^2 \neq v'^2$, $v^2 \equiv v'^2 \mod 8$ and $L^2 \equiv 4 \mod 8$. If $L^2 \equiv 4 \mod 8$ then v^2 and v'^2 are -4 or -12. So we can assume $v^2 = -4$ and $v'^2 = -12$ and $v = N_1 + N_2$ $v' = N_3 + N_4 + N_5 + N_6 + N_7 + N_8$ (up to isometry of the lattice). Observe that $v'/2 = \hat{N} - v/2$ hence the lattice generated by $\mathcal{L}_{2d}$ and by $(L/2, v/2)$ or by $\mathcal{L}_{2d}$ and by $(L/2, v'/2)$ are the same. $\qquad \square$

Corollary 2.1. *Let $L^2 \equiv 0 \mod 4$ and $NS(X) = \mathcal{L}'_{2d}$. Then there are two possibilities:*
- *$L^2 \equiv 4 \mod 8$. In this case one can assume that $v = -N_1 - N_2$ and $(L - N_3 - \ldots - N_8)/2 = (L + v)/2 + \hat{N} - (N_3 + \cdots + N_8)$ is in $NS(X)$ too.*
- *$L^2 \equiv 0 \mod 8$. In this case one can assume that $v = -(N_1 + N_2 + N_3 + N_4)$ and $(L - N_5 - N_6 - N_7 - N_8)/2$ is in $NS(X)$ too.*

Proposition 2.3. *Let $\Gamma = \mathcal{L}_{2d}$ or $\mathcal{L}'_{2d}$ then there exists a K3 surface X with an even set of eight disjoint rational (-2)-smooth curves, such that $NS(X) = \Gamma$.*

In the proof of this proposition we will use the relations between the Néron Severi group of a K3 surface Y with a Nikulin involution and the Néron Severi group of a K3 surface X which is the desingularization of the quotient of Y by the Nikulin involution. Here we recall the two following Propositions of [vGS] in which the properties of the Néron Severi group of a K3 surface with a Nikulin involution are described (we use the notation of the Diagram 1).
Proposition [vGS, Proposition 2.2] *Let Y be an algebraic K3 surface admitting a Nikulin involution and with Picard number nine. Let M be a divisor generating $E_8(-2)^{\perp} \subset NS(Y)$, $M^2 = 2d' > 0$ and let*

$$\mathcal{M}_{2d'} = \mathbb{Z}M \oplus E_8(-2).$$

Then M is ample, and
(1) if $M^2 \equiv 2 \mod 4$ then $NS(Y) = \mathcal{M}_{2d'}$,
(2) if $M^2 \equiv 0 \mod 4$ then either $NS(Y) = \mathcal{M}_{2d}$ or $NS(Y) = \mathcal{M}'_{2d'}$, where $\mathcal{M}'_{2d'}$ is generated by $\mathcal{M}_{2d'}$ and by a class $(M/2, v/2)$, with $v \in E_8(-2)$.
Proposition [vGS, Proposition 2.7] *(1) Assume that $NS(Y) = \mathbb{Z}M \oplus E_8(-2) = \mathcal{M}_{2d'}$. Let $E_1, \ldots, E_8$ be the exceptional divisors on $\tilde{Y}$. Then: (i) In case $M^2 = 4n + 2$, there exist line bundles $L_1, L_2 \in NS(X)$ such that for a suitable numbering of these E_i we have:*
$\beta^* M - E_1 - E_2 = \pi^* L_1$, $\qquad \beta^* M - E_3 - \ldots - E_8 = \pi^* L_2$.
The decomposition of $H^0(Y, M)$ into ι^*-eigenspaces is:
$H^0(Y, M) \cong \pi^* H^0(X, L_1) \oplus \pi^* H^0(X, L_2)$, $\qquad (h^0(L_1) = n + 2, h^0(L_2) = n + 1)$
and the eigenspaces $\mathbb{P}^{n+1}, \mathbb{P}^n$ contain six, respectively two, fixed points.

(ii) In case $M^2 = 4n$, for a suitable numbering of the E_i we have:
$\beta^* M - E_1 - E_2 - E_3 - E_4 = \pi^* L_1, \qquad \beta^* M - E_5 - E_6 - E_7 - E_8 = \pi^* L_2$ with $L_1, L_2 \in NS(X)$.
The decomposition of $H^0(Y, M)$ into ι^*-eigenspaces is:
$H^0(Y, M) \cong \pi^* H^0(X, L_1) \oplus \pi^* H^0(X, L_2), \qquad (h^0(L_1) = h^0(L_2) = n + 1)$.
and each of the eigenspaces $\mathbb{P}^n$ contains four fixed points.
(2) Assume $NS(Y) = \mathcal{M}'_{2d'}$. Then there is a line bundle $L \in NS(X)$ such that:
$\beta^* M \cong \pi^* L$. *The decomposition of $H^0(Y, M)$ into ι^*-eigenspaces is:*
$H^0(Y, M) \cong H^0(X, L) \oplus H^0(X, L - \hat{N}), \qquad (h^0(L) = n + 2, h^0(L - \hat{N}) = n)$
and all fixed points map to the eigenspace $\mathbb{P}^{n+1} \subset \mathbb{P}^{2n+1}$.

Proof of Proposition 2.3. First observe that the lattices $\mathcal{L}_{2d}$ and $\mathcal{L}'_{2d}$ are primitively embedded in the K3 lattice by [N3, Theorem 1.14.1], so we can identify them with sublattices of $U^3 \oplus E_8(-1)^2$.
1) We consider first the case of $\Gamma = \mathcal{L}_{2d} = \mathbb{Z}L \oplus N$ in this case $L^2 \equiv 2 \mod 4$ or $L^2 \equiv 0 \mod 4$. We show that there exists a K3 surface with an even set of (-2)-smooth curves s.t. $NS(X) = \mathcal{L}_{2d}$. Let Y be a K3 surface with $\rho(Y) = 9$, with Nikulin involution and Néron Severi group containing the lattice $\mathbb{Z}M \oplus E_8(-2)$ with index two and with $M^2 \equiv 0 \mod 4$; such a K3 surface exists by [vGS, Proposition 2.2, 2.3] (in this case its Néron-Severi group is $\mathcal{M}'_{2d'}$, for some non-negative integer d'). We have a diagram like Diagram 1, and so a K3 surface X, which is the minimal resolution of the quotient of Y by the Nikulin involution. Since $\rho(Y) = 9$ then $\rho(X) = 9$ too. By [vGS, Proposition 2.7] there is a line bundle L, $L \in NS(X)$ with $\pi^* L = \beta^* M$. By the properties of the map π^*, $2L^2 = (\pi^* L)^2 = (\beta^* M)^2 = M^2 \equiv 0 \mod 4$ and so $L^2 \equiv 2 \mod 4$ or $L^2 \equiv 0 \mod 4$. Moreover X has an even set made up by the eight curves in the resolutions of the nodes of the quotient $\bar{X}$.
If $L^2 \equiv 2 \mod 4$ then by the Proposition 2.1 $NS(X) = \mathcal{L}_{2d}$, where $L^2 = 2d$, as required.
If $L^2 \equiv 0 \mod 4$ we must exclude that $NS(X) = \mathcal{L}'_{2d}$. Assume that we have an element $L_1 = (L - N_1 - N_2)/2 \in NS(X)$. If $NS(Y) = \mathcal{M}'_{2d'}$ the primitive embedding of $NS(Y)$ in $U^3 \oplus E_8(-1)^2$ is unique up to isometry. Assume that $M^2 = 4n$ and choose an $\alpha \in E_8(-1)$ with $\alpha^2 = -2$ if n is odd and $\alpha^2 = -4$ if n is even. Let $v \in E_8(-2) \subset U^3 \oplus E_8(-1)^2$ be $v = (0, \alpha, -\alpha)$ and let M be $M = (2u, \alpha, \alpha) \in U^3 \oplus E_8(-1)^2$ where $u = e_1 + \frac{(n+1)}{2} f_1$ if n is odd, and $u = e_1 + (\frac{n}{2} + 1) f_1$ if n is even (here e_1, f_1 denotes the standard basis of the first copy of U). Then $M^2 = 4n$ and $(M + v)/2 = (u, \alpha, 0) \in U^3 \oplus E_8(-1)^2$. This gives a primitive embedding of $NS(Y)$ in $U^3 \oplus E_8(-1)^2$, which extends the standard one of $E_8(-2) \subset U^3 \oplus E_8(-1)^2$. Now we can assume that $L = (u, 0, \alpha) \in U(2) \oplus N \oplus E_8(-1) \subset H^2(X, \mathbb{Z})$, so by [vGS, Proposition 1.8] we have $\beta^* M = \pi^* L$. Now $(L - N_1 - N_2)/2 = (u, -N_1 - N_2, \alpha)/2 \in NS(X)$. By using [vGS, Proposition 1.8] again we obtain $\pi^*((L - N_1 - N_2)/2) = (u, \frac{\alpha}{2}, \frac{\alpha}{2}, -E_1 - E_2) \in NS(\tilde{Y})$ and so $(u, \frac{\alpha}{2}, \frac{\alpha}{2}) \in NS(Y)$, this means that $M/2 \in NS(Y)$ which is not the case. Hence $(L - N_1 - N_2)/2 \notin NS(X)$, in a similar way one shows that $(L - N_1 - N_2 - N_3 - N_4)/2 \notin NS(X)$ and so we conclude that $NS(X) = \mathcal{L}_{2d}$.
2) Assume now that $\Gamma = \mathcal{L}'_{2d}$. In this case we have either
a) $L^2 \equiv 4 \mod 8$ and so $(L - N_1 - N_2)/2$ and $(L - N_3 - \ldots - N_8)/2$ are in Γ or
b) $L^2 \equiv 0 \mod 8$ and so $(L - N_1 - N_2 - N_3 - N_4)/2$ and $(L - N_5 - N_6 - N_7 - N_8)/2$ are in Γ. We do the proof assuming that we are in case a), for the case b) the proof is very similar. Let Y be a K3 surface with $\rho(Y) = 9$, Nikulin involution, Néron Severi group $NS(Y) = \mathbb{Z}M \oplus E_8(-2)$ (which is $\mathcal{M}_{2d'}$ for a non-negative integer d') and $M^2 = 4n + 2$, such a K3 surface exists by [vGS, Proposition 2.2, 2.3]. Moreover by [vGS, Proposition 2.7] there are line bundles L_1 and L_2 in $NS(X)$ with $\beta^* M - E_1 - E_2 = \pi^* L_1$, $\beta^* M - E_3 - \ldots - E_8 = \pi^* L_2$. Since the

embedding of $\mathbb{Z}L \oplus E_8(-2)$ in the K3 lattice is unique we may assume that $M = e_1 + (2n+1)f_1$ and $L_1 = (e_1 + (2n+1)f_1 + N_1 + N_2)/2 - N_1 - N_2 \in NS(X)$, by [vGS, Proposition 1.8] we have $\beta^* M - E_1 - E_2 = \pi^* L_1$. The class $U(2) \ni (e_1 + (2n+1)f_1) = 2L_1 + N_1 + N_2$ is in $NS(X)$, is orthogonal to the N_i and has self intersection $8n' + 4$, we call it L. By Proposition 2.1 we have $NS(X) = \mathcal{L}'_{2d}$ with $d = 4n + 2$, so we are done. $\square$

Remark. By using the *surjectivity of the period map* one can show the existence of a K3 surface X with $NS(X) = \Gamma$, it is however difficult to show that there is an embedding of the classes N_i as irreducible (-2)-smooth curves in $NS(X)$. This is assured by the previous proposition.

From the Proposition 2.3 follows a relation between the Néron Severi group of the K3 surface Y admitting a Nikulin involution and the Néron Severi group of a K3 surface X which is the desingularization of the quotient.

Corollary 2.2. *Let Y be an algebraic K3 surface with $\rho(Y) = 9$ admitting a Nikulin involution, and let X be the desingularization of its quotient.*
(1) $NS(Y) = \mathcal{M}_{2d}$ if and only if $NS(X) = \mathcal{L}'_{4d}$;
(2) $NS(Y) = \mathcal{M}'_{4d}$ if and only if $NS(X) = \mathcal{L}_{2d}$.

Proof. The proof follows from [vGS, Proposition 2.7] and Proposition 2.3. We sketch it briefly. The proof of the direction $\Leftarrow$ of the statement follows immediately from the proof of Proposition 2.3. For the other direction we distinguish three cases (we use the notation of *loc. cit.*):
(a) Case (1), (*i*). Clearly $(\beta^* M - E_1 - E_2)^2 = (\pi^* L_1)^2$ and in the proofs of Proposition 2.3, case (2), and of [vGS, Proposition 2.7] it is proved that

$$(2) \qquad L_1 = (L - N_1 - N_2)/2, \quad L_2 = (L - N_3 - \ldots - N_8)/2.$$

Since π is a $2:1$ map to X the previous equality becomes $4n + 2 - 1 - 1 = \frac{1}{2}(L - N_1 - N_2)^2 = \frac{1}{2}(L^2 - 4)$ and so $L^2 = 2(4n + 2)$. By the Proposition 2.1, where we describe the possible Néron-Severi groups of K3 surfaces with an even set, we obtain that $NS(X) = \mathcal{L}'_{2d}$, $d \equiv 2$ mod 4.
(b) Case (1), (*ii*). As before, in the proof of [vGS, Proposition 2.7] it is proved that:

$$(3) \qquad L_1 = (L - N_1 - \ldots - N_4)/2, \ L_2 = (L - N_5 - \ldots - N_8)/2.$$

So we obtain $4n - 4 = (\beta^* M - E_1 - E_2 - E_3 - E_4)^2 = 2((L - N_1 - \ldots - N_4)/2)^2 = \frac{1}{2}(L^2 - 8)$ and so $L^2 = 2(4n)$. By the Proposition 2.1 we obtain that $NS(X) = \mathcal{L}'_{2d}$, $d \equiv 0 \mod 4$.
(c) Case (2), $M^2 = 2L^2$, and so by an argumentation as in the proof of the Proposition 2.3, case (1), we have $NS(X) = \mathcal{L}_{2d}$. $\square$

Some explicit correspondence between the K3 surfaces Y and X are shown in the Table 1.
Remark. Let X be a K3 surface such that the lattice $\Gamma = \mathcal{L}_{2d}$ or $\mathcal{L}'_{2d}$ is primitively embedded in $NS(X)$ and $\rho(X) \geq 9$. There exists a deformation of the K3 surface $\{X_t\}$ such that $X_{\bar{t}} = X$ and X_0 is such that $NS(X_0) = \Gamma$. Let Y_0 be the K3 surface such that the desingularization of its quotient by a Nikulin involution is X_0. The Néron Severi group of Y_0 is either $\mathcal{M}'_{4d}$ or $\mathcal{M}_d$. The deformation on X induces a deformation $\{Y_t\}$ of Y_0 such that the surface $Y_{\bar{t}}$ admits a Nikulin involution and the desingularization of its quotient by the Nikulin involution is $X_{\bar{t}}$. This means that $X_{\bar{t}}$ admits an even set of eight disjoint rational curves.
In particular if X is an algebraic K3 surface such that $\mathcal{L}_{2d}$ (resp. $\mathcal{L}'_{2d}$) is primitively embedded in $NS(X)$, then X is the minimal resolution of the quotient of a K3 surface Y such that $\mathcal{M}'_{4d}$ (resp. $\mathcal{M}_d$) is primitively embedded in $NS(Y)$.

Corollary 2.3. *The coarse moduli space of Γ-polarized K3 surfaces (cf. [D, p.5] for the definition) is the quotient of*

$$\mathcal{D}_\Gamma = \{\omega \in \mathbb{P}(\Gamma^\perp \otimes_\mathbb{Z} \mathbb{C}) : \omega^2 = 0, \ \omega\bar{\omega} > 0\}$$

by an arithmetic group $O(\Gamma)$ and has dimension eleven. The generic K3 surface with an even set of eight disjoint rational curves has Picard number nine.

Proof. By Proposition 2.1, each K3 surface with an even set is contained in this space, on the other hand, by Proposition 2.3 each point of this space corresponds to a K3 surface with an even set of irreducible (-2)-curves. $\square$

3. AMPLENESS AND NEFNESS OF SOME DIVISORS ON X

Our next aim (cf. Section 4) is to describe projective models of K3 surfaces with an even set of eight disjoint rational curves. Here we give some results on ampleness and on nefness of divisors on such K3 surfaces. We prove moreover that the associated linear systems have no base points. These properties guarantee that the maps induced by the linear systems are regular (in fact birational) maps.

Definition 3.1. *A divisor L on a surface S is:*

- **nef**   *if $L^2 \geq 0$ and $L \cdot C \geq 0$ for each irreducible curve C on S,*
- **pseudo ample**   *if $L^2 > 0$ and $L \cdot C \geq 0$ for each irreducible curve C on S,*
  *(or **big and nef**)*
- **ample**   *if $L^2 > 0$ and $L \cdot C > 0$ for each irreducible curve C on S.*

If X is a K3 surface with a line bundle L such that $L^2 \geq 0$, the condition $L \cdot C \geq 0$ for each irreducible curve C on X is equivalent to the condition $L \cdot \delta \geq 0$ for each irreducible (-2)-curve δ on X (cf. [BPV, Proposition 3.7]).

Let H be an effective divisor on a K3 surface. The intersection of H with each curve C is non-negative except when C is a component of H and C is a (-2)-curve. If the linear system $|H|$ does not have fixed components and if $H^2 > 0$, then the generic element in $|H|$ is smooth and irreducible and H is a pseudo ample divisor (cf. [SD, Proposition 2.6]). The fixed components of a linear system on a K3 surface are always (-2)-curves [SD, Paragraph 2.7.1]. Recall that by [R, Theorem p.79] if H is pseudo ample (or ample) then either $|H|$ has no fixed components or $H = aE + \Gamma$ where $|E|$ is a free pencil and Γ is an irreducible (-2)-curve such that $E\Gamma = 1$. Finally in [SD, Corollary 3.2] Saint-Donat proves that a linear system on a K3 surface has no base points outside its fixed components.

Let now H be a pseudo ample divisor on X. If $|H|$ has a fixed component B, then $H = B + M$, where M is the moving part of the linear system $|H|$. The linear system $|H|$ defines a map ϕ_H and if $H = M + B$ then $\phi_H = \phi_M$. Now we assume that $|H|$ has no fixed components (and hence no base points). The system $|H|$ defines the map:

$$\phi_H : X \longrightarrow \mathbb{P}^{p_a(H)}$$

where $p_a(H) = H^2/2 + 1$ and there are two cases (cf. [SD, Paragraph 4.1]):
(i) either ϕ_H is of degree two and its image has degree $p_a(H) - 1$ (ϕ_H is *hyperelliptic*),
(ii) or ϕ_H is birational and its image has degree $2p_a(H) - 2$.
In particular in the second case if H is ample (i.e. does not contract (-2)-curves) ϕ_H is an embedding, so H is very ample.

Proposition 3.1. *Let X be as in the Proposition 2.1. Then we may assume that L is pseudo ample and it has no fixed components.*

Proof. Since X and Y are algebraic, by using the notations of the Diagram 1, the surface $\bar{X}$ is embedded in some projective space and has eight nodes. The generic hyperplane section of $\bar{X}$ is a smooth and irreducible curve (it does not pass through the nodes). Its pull back on X is then orthogonal to $N_1, \ldots, N_8$, we call it $\mathcal{H}$, observe that $\mathcal{H} = \alpha L$ for some integer α. Since $\mathcal{H}$ is pseudo ample then L is pseudo ample too, in particular observe that $L\Gamma > 0$ for each (-2)-curve which is not one of the N_i's. If L has fixed components then by [R, Theorem p.79] it is $L = aE + \Gamma$ where $|E|$ is a free pencil and Γ an irreducible (-2)-curve such that $E\Gamma = 1$. If $\Gamma \neq N_i$ for each $i = 1, \ldots, 8$, then $0 < L\Gamma = a - 2$, which gives $a > 2$. Now $0 = LN_i = aEN_i + \Gamma N_i$, since $a > 2$, $EN_i \geq 0$, $\Gamma N_i \geq 0$ we obtain $EN_i = 0$ and $\Gamma N_i = 0$ for each i, so Γ is in $(N)^{\perp}$ which is not possible. If $\Gamma = N_i$ for some i, then $0 = LN_i = a - 2$ so $a = 2$ then $L = 2E + N_i$ and so $(L - N_i)/2$ is in the Néron Severi group too which is not the case. So by [SD, Proposition 2.6] we can assume that L is smooth and irreducible. $\qquad\square$

Proposition 3.2. *Let X be as in the Proposition 2.1. If $d \geq 3$, i.e. $L^2 \geq 6$, then the class $L - \hat{N}$ in the Néron Severi group is an ample class.*

Proof. The self intersection of $L - \hat{N}$ is $(L - \hat{N})^2 = 2d - 4$, which is positive for each $d \geq 3$. So to prove that $L - \hat{N}$ is ample we have to prove that for each irreducible (-2)-curve C the intersection number $C \cdot (L - \hat{N})$ is positive.

In the proof we use the inequality:

$$(4) \qquad\qquad \left(\sum_{i=1}^{n} x_i\right)^2 \leq n \sum_{i=1}^{n} x_i^2$$

which is true for every $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Suppose that there exists an effective irreducible curve C such that $C \cdot (L - \hat{N}) \leq 0$, then we prove that $C \cdot C < -2$.

We observe that each element in the Néron Severi group is a linear combination of L and N_i with coefficients in $\frac{1}{2}\mathbb{Z}$. We consider the curve $C = aL + \sum_{i=1}^{8} b_i N_i$ where $a, b_i \in \frac{1}{2}\mathbb{Z}$. If $a = 0$ the only possible (-2)-curves are the N_i's and $N_i \cdot (L - \hat{N}) = 1$. So we can assume that $a \neq 0$. Since C is an irreducible curve, it has a non-negative intersection with all effective divisors. Hence $C \cdot L = 2da \geq 0$, so $a > 0$, and $C \cdot N_i = -2b_i \geq 0$, so $b_i \leq 0$.

Now we assume that $C \cdot (L - \hat{N}) \leq 0$, then

$$\left(aL + \sum_{i=1}^{8} b_i N_i\right) \cdot (L - \hat{N}) = 2da + \sum_{i=1}^{8} b_i \leq 0.$$

Since $b_i \leq 0$, $2da - \sum_{i=1}^{n} |b_i| \leq 0$ and so $2da \leq \sum_{i=1}^{n} |b_i|$, where each member is non negative. So it is possible to pass to the square of the relation, obtaining $4d^2a^2 \leq (\sum_{i=1}^{n} |b_i|)^2$. Using the relation (4) one has

$$(5) \qquad\qquad 4d^2a^2 \leq \left(\sum_{i=1}^{n} |b_i|\right)^2 \leq 8 \sum_{i=1}^{8} b_i^2.$$

Now we compute the square of C and we use the inequality (5) to estimate it:

$$C \cdot C = 2da^2 - 2 \sum_{i=1}^{8} (b_i^2) \leq 2da^2 - d^2a^2.$$

If $d \geq 5$ then $\sqrt{\frac{2}{d^2-2d}} < \frac{1}{2}$, and so for $d \geq 5$ we have $C \cdot C < -2$ (because $a \geq \frac{1}{2}$). This proves the theorem in the case $d \geq 5$.

More in general for each $d \geq 3$, $\sqrt{\frac{2}{d^2-2d}} < 1$, so for the cases $d = 3$ and $d = 4$ one has to study only the case $a = \frac{1}{2}$.

For $d = 3$, then $L^2 = 6$ and so in $NS(X)$ all the elements are of the form $aL + \sum_{i=1}^{8} b_i N_i$ with $a \in \mathbb{Z}$ (and not in $\frac{1}{2}\mathbb{Z}$). Then the theorem is proved exactly in the same way as before. Let $d = 4$. By the previous computations follows that the only possible irreducible (-2)-curves with a negative intersection with $(L - \hat{N})$ are of the form $\frac{1}{2}(L + N_1 + N_2 + N_3 + N_4) + \sum_{i=1}^{8} \beta_i N_i$ with $\beta_i \in \mathbb{Z}$, $\beta_1, \ldots, \beta_4 \leq -1$ and $\beta_5, \ldots, \beta_8 \leq 0$. It is easy to see that the only (-2)-curves of this type are $\frac{L+N_1+N_2+N_3+N_4}{2} - N_1 - N_2 - N_3 - N_4 - N_j$, $j = 5, 6, 7, 8$ and these curves have a positive intersection with $L - \hat{N}$. Then the proposition is proved also for $d = 4$. $\qquad\square$

Proposition 3.3. *In the situation of Proposition 3.2, $m(L - \hat{N})$ and $mL - \hat{N}$ for $m \in \mathbb{Z}_{>0}$, are ample. If $d = 2$, i.e. $(L - \hat{N})^2 = 0$, then $m(L - \hat{N})$ is nef and $mL - \hat{N}$ is ample for $m \geq 2$.*

Proof. It is a similar computation as in the proof of Proposition 3.2. $\qquad\square$

Proposition 3.4. *The divisors $L - \hat{N}$, $mL - \hat{N}$ and $m(L - \hat{N})$, $m \in \mathbb{Z}_{>0}$, do not have fixed components for $d \geq 2$.*

Proof. We proof the proposition for the divisor $L - \hat{N}$. The proof in the other cases is essentially the same.

For $d = 2$ we have $(L - \hat{N})^2 = 0$ and is nef by the Proposition 3.3 so by [R, Theorem p. 79, (b)] $L - \hat{N} = aE$ where $|E|$ is a free pencil, and so the assertion is proved in this case.

Assume $d \geq 3$, then for [R, Theorem p. 79, (d)] we have either $L - \hat{N}$ has no fixed components or $L - \hat{N} = aE + \Gamma$, where $|E|$ is a free pencil and Γ is an irreducible (-2)-curve such that $E\Gamma = 1$. We assume we are in the second case, then since $L - \hat{N}$ is ample we have

$$0 < \Gamma(L - \hat{N}) = a - 2$$

and so $a > 2$. We distinguish two cases:

1. $\Gamma = \alpha L + \sum \beta_j N_j$, $\Gamma \neq N_i$ for each i, so $\alpha \neq 0$. For each i we have:

$$1 = N_i(L - \hat{N}) = aEN_i + \Gamma N_i$$

Since $EN_i \geq 0$ and $a > 2$ then $EN_i = 0$ and so

$$1 = \Gamma N_i = (\alpha L + \sum_j \beta_j N_j)N_i = -2\beta_i.$$

We obtain $\beta_j = -1/2$ for all j, so

$$\Gamma = \alpha L - \frac{N_1 + \ldots + N_8}{2} = \alpha L - \hat{N}.$$

By considering the self-intersection of Γ we obtain

$$-2 = \alpha^2 2d - 4 \geq 6\alpha^2 - 4$$

which is positive since α is a non zero integer. So this case is not possible.

2. $\Gamma = N_i$ for some $i = 1, \ldots, 8$. We have

$$1 = N_i(L - \hat{N}) = N_i(aE + N_i) = aEN_i - 2 = a - 2$$

so $a = 3$, $L - \hat{N} = 3E + N_i$. For $j \neq i$ we have

$$1 = (L - \hat{N})N_j = 3EN_j$$

but this is impossible. Hence $L - \hat{N}$ has no base components. $\square$

Lemma 3.1. *The map $\phi_{L-\hat{N}}$ is*

- *an embedding if $L^2 \geq 10$,*
- *a 2:1 map to $\mathbb{P}^1 \times \mathbb{P}^1$ if $L^2 = 8$,*
- *a 2:1 map to $\mathbb{P}^2$ if $L^2 = 6$.*

Proof. By the Proposition 3.2 $L - \hat{N}$ is ample and by the Proposition 3.4 $|L - \hat{N}|$ has no fixed components; for a K3 surface this implies that $|L - \hat{N}|$ has no base points too (cf. [SD, Corollary 3.2]), and it defines a map $\phi_{L-\hat{N}}$. The assertion for $L^2 = 6$ is clear since $(L - \hat{N})^2 = 2$ and hence the map $\phi_{L-\hat{N}}$ defines a double cover of $\mathbb{P}^2$. We show that in the case $L^2 = 2d \geq 10$, i.e. $d \geq 5$, the map is not hyperelliptic. By [SD, Theorem 5.2] $L - \hat{N}$ is hyperelliptic iff (i) there is an elliptic irreducible curve E with $E \cdot (L - \hat{N}) = 2$ or (ii) there is an irreducible curve B, with $p_a(B) = 2$ and $L - \hat{N} = \mathcal{O}(2B)$. The case (ii) would implies $L - \hat{N} \equiv 2B$ and so $\frac{1}{2}(L - \hat{N}) \in NS(X)$ which is not possible by the description of $NS(X)$ of Proposition 2.1. We have to exclude (i). We argue in a similar way as in Proposition 3.2. Assume that there is $E = aL + \sum b_i N_i$ an irreducible curve with $E \cdot (L - \hat{N}) = 2$. Then we show $E^2 \neq 0$. Since E is the class of an irreducible curve, $a \in \frac{1}{2}\mathbb{Z}_{>0}$ and $b_i \in \frac{1}{2}\mathbb{Z}_{\leq 0}$. We have $2 = E \cdot (L - \hat{N}) = 2da + \sum_{i=1}^{8} b_i$ and so $2da - 2 = -\sum_{i=1}^{8} b_i$ which gives together with the inequality (4):

$$4(da - 1)^2 = (\sum_{i=1}^{8} |b_i|)^2 \leq 8 \sum_{i=1}^{8} |b_i|^2$$

and so $(da - 1)^2 \leq 2 \sum_{i=1}^{8} b_i^2$. On the other hand we have

$$E^2 = 2da^2 - 2\sum_{i=1}^{8} b_i^2 \leq 2da^2 - (da - 1)^2 = 2da^2 - d^2a^2 - 1 + 2da.$$

We have $E^2 < 0$ for $a < \frac{d-\sqrt{2d}}{d(d-2)}$ or $a > \frac{d+\sqrt{2d}}{d(d-2)}$, since $a > 1/2$ and $\frac{d-\sqrt{2d}}{d(d-2)} < \frac{1}{2}$ for each $d \geq 5$ and $\frac{d+\sqrt{2d}}{d(d-2)} < \frac{1}{2}$ for each $d \geq 6$, we obtain $E^2 < 0$ for $d \geq 6$. We analyze the case of $d = 5$. Here $L^2 = 10$ and so $a \in \mathbb{Z}_{>0}$, for $d = 5$ we have $\frac{d+\sqrt{2d}}{d(d-2)} = \frac{5+\sqrt{10}}{15} < 1$. In conclusion for each $d \geq 5$ we obtain $E^2 < 0$. In the case of $d = 4$, then we have $L^2 = 8$ and the classes $E_1 = \frac{L - N_1 - N_2 - N_3 - N_4}{2}$, $E_2 = \frac{L - N_5 - N_5 - N_7 - N_8}{2}$ are in the Néron Severi group. We have $E_1^2 = E_2^2 = 0$, $E_1 \cdot E_2 = 2$ and $L - \hat{N} = E_1 + E_2$, so $\phi_{L-\hat{N}}$ defines a 2:1 map to a quadric in $\mathbb{P}^1 \times \mathbb{P}^1$ (cf. [SD, Proposition 5.7]). $\square$

Proposition 3.5. *1) Let D be the divisor $D = L - (N_1 + \ldots + N_r)$ (up to relabel the indices), $1 \leq r \leq 8$.*

- *If $NS(X) = \mathcal{L}_{2d}$, then D is pseudo ample for $d > r$;*
- *if $NS(X) = \mathcal{L}'_{2d}$, then D is nef for $d = r + 4$ and pseudo ample for $d > r + 4$,*
- *if D is pseudo ample and $NS(X) = \mathcal{L}_{2d}$ then it does not have fixed components.*

2) Let $NS(X) = \mathcal{L}'_{2d}$. Let $\bar{D} = (L - (N_1 + \ldots + N_r))/2$ with $r = 2, 6$ if $2d \equiv 4 \mod 8$ and $r = 4$ if $2d \equiv 0 \mod 8$. Then

- the divisor $\bar{D}$ is nef and is pseudo ample whenever it has positive self intersection,
- if $\bar{D}$ is pseudo ample then it does not have fixed components, if $\bar{D}^2 = 0$ then the generic element in $|\bar{D}|$ is an elliptic curve.

Proof. The arguments are similar as those used in the the proof of the Proposition 3.2 for the ampleness properties and in the proof of the Proposition 3.4 for the absence of fixed components. $\square$

Corollary 3.1. *Let D and $\bar{D}$ be divisors as in the Proposition 3.5. We suppose that $D^2 > 0$, $\bar{D}^2 > 0$. Let C be a (-2)-curve with $C \cdot D = 0$ or $C \cdot \bar{D} = 0$. Then $C = N_i$ for some $i = 1, \ldots, 8$.*

Lemma 3.2. *With the same notation as in Proposition 3.5, we have:*
- *for $NS(X) = \mathcal{L}_{2d}$ and $D^2 \geq 4$ the map ϕ_D is birational,*
- *for $\bar{D}^2 \geq 4$ the map $\phi_{\bar{D}}$ is birational.*

Proof. The proof is very similar to the proof of Lemma 3.1 and is left to the reader. $\square$

We prove the following proposition which is a generalization of [C, Proposition 2.6] to the case of surfaces in $\mathbb{P}^n$.

Proposition 3.6. *Let F be a surface in $\mathbb{P}^n$ and let $\mathcal{N}$ be a subset of the set of nodes of F. Let $G \subset \mathbb{P}^n$ be a hypersurface s.t. $div_F(G) = 2C$ (here $div_F(G)$ denotes the divisor cut out by G on F), with C a divisor on F which is not Cartier at the points of $\mathcal{N}$. Then $\mathcal{N}$ is an even set of nodes iff G has even degree. Conversely if $\mathcal{N}$ is an even set of nodes then there is an hypersurface G as above.*

Proof. The proof is identical to the proof of [C, Proposition 2.6], we recall it briefly.
Let $\tilde{F} \longrightarrow F$ be the minimal resolution of the singularities of F, let H denote the pull-back on $\tilde{F}$ of the hyperplane section on F and let $\deg G = m$ then

$$mH \sim 2\tilde{C} + \sum \alpha_i N_i$$

where $\tilde{C}$ denotes the strict transform of C on $\tilde{F}$ and the N_i's denote the exceptional curves over the nodes. Since C is not Cartier at the singular points, the α_i are odd. Hence

$$\sum N_i \sim \delta H + 2 \left([\frac{m}{2}]H - \tilde{C} - \sum [\frac{\alpha_i}{2}]N_i \right)$$

where $\delta = 0, 1$ according to m even or odd. Now if $\sum N_i$ is an even set then $\delta = 0$ and m is even. If m is even then $\delta = 0$ and so $\sum N_i$ is an even set.
On the other hand, if $\mathcal{N}$ is an even set then

$$2B \sim \sum N_i.$$

For $B \in Pic(\tilde{F})$ choose r such that $rH - B$ is linearly equivalent to an effective divisor $\tilde{C}$. Then

$$2rH \sim 2B + 2\tilde{C} = 2\tilde{C} + \sum N_i$$

so there is a hypersurface G ($\sim 2rH$) with the properties of the statement. $\square$

From this follows a geometrical characterization of even set of nodes on K3 surfaces.

Corollary 3.2. *Let $\bar{X} \subset \mathbb{P}^{d+1}$, $d \geq 2$, be a surface of degree $2d$ with a set of m nodes $\mathcal{N}$, s.t. its minimal resolution is a K3 surface. Then*
(i) if $m = 8$, then $\mathcal{N}$ is even iff G is a quadric,
(ii) if $m = 16$ and $d \geq 3$, then $\mathcal{N}$ is even iff G is a quadric.

Proof. (i) Let $L := H \cap \bar{X}$ be the generic hyperplane section with $2d = L^2$ and let $\mathcal{N}$ be an even set of nodes. Then the lattice $\mathbb{Z}L \oplus N \subset NS(X)$ and we have $2\hat{N} \equiv \sum N_i$. Since the self-intersection of $L - \hat{N}$ is $2d - 4 \geq 0$ by the theorem of Riemann-Roch $L - \hat{N}$ or $-(L - \hat{N})$ is effective. Since $(L - \hat{N}) \cdot L \geq 0$, $L - \hat{N}$ is effective, so $2L \equiv 2(L - \hat{N}) + \sum N_i$. And so $G \in |2L|$ and $div_{\bar{X}}(G) = 2C = 2(L - \hat{N})$. The converse follows from the Proposition 3.6.

(ii) The proof in this case is essentially the same. We use the Kummer lattice K instead of the Nikulin lattice N. The lattice K is generated over $\mathbb{Q}$ by the sixteen disjoint rational (-2)-curves $K_1, \ldots, K_{16}$ and it contains the class $(K_1 + \ldots + K_{16})/2$, which we use in the proof above instead of the class $\hat{N}$ (for a precise definition of the lattice K see [N1]). $\square$

Remark. In particular this means that if $\bar{X}$ is a K3 surface with an even set of nodes, then there exists a quadric cutting a curve on $\bar{X}$ with multiplicity two, passing through the even set of nodes (this is the condition $div_F(G) = 2C$ of the theorem).

4. PROJECTIVE MODELS

In this section we determine projective models of K3 surfaces with an even set of nodes and Picard number nine. These were already partially studied by Barth in [B2]. Here we recover, with different methods, some of these examples and we discuss many new examples. Observe that some of the cases that Barth describes require Picard number at least ten (these are case five and case four in his list (cf. Paragraph 4.5 below)).

In Section 3 we proved that the divisors L, $L - \hat{N}$, and $(L - N_1 - \ldots - N_m)/2$, $m = 2$ or $m = 4$ or $m = 6$ on X define regular maps. We use these divisors to give projective models of a K3 surface X with Néron Severi group isometric to $\mathcal{L}_{2d}$ or to $\mathcal{L}'_{2d}$ and in general we study the projective models of the same surface by using different polarizations. In particular in each case one can use as polarization L or $L - \hat{N}$, if $L^2 > 4$. The first polarization contracts the curves of the even sets to eight nodes on the surface, the second one sends these curves to lines on the projective model.

In case (1) of the Corollary 2.2 it is also possible to study the projective models given by the maps ϕ_{L_1}, resp. ϕ_{L_2} (for $L_i^2 > 0$) where L_1 and L_2 are the divisors defined in (2) or in (3). They give projective models of X in the projective space $\mathbb{P}(H^0(X, L_1))$, resp. $\mathbb{P}(H^0(X, L_2))$ or give $2 : 1$ maps to the images of X in these spaces. If the maps are not $2 : 1$, the image of X contains nodes and lines, which on X form an even set. The image of X under $\phi_{L_1} \times \phi_{L_2} : X \to \mathbb{P}(H^0(X, L_1)) \times \mathbb{P}(H^0(X, L_2))$ is a surface, which is the image of $Y \hookrightarrow \mathbb{P}^{h^0(L_1)+h^0(L_2)-1}$ under the projection to the eigenspaces: $\mathbb{P}^{h^0(L_1)+h^0(L_2)-1} \longrightarrow \mathbb{P}(H^0(X, L_1)) \times \mathbb{P}(H^0(X, L_2))$. Indeed put $h^0(L_1) = m_1 + 1$, $h^0(L_2) = m_2 + 1$, then $h^0(M) = m_1 + m_2 + 2$ and we have a commutative diagram:

$$
\begin{array}{ccccc}
Y & \longrightarrow & \mathbb{P}^{m_1+m_2+1} & \xrightarrow{\ v\ } & \mathbb{P}^r \supset V_{m_1+m_2+1} \\
\downarrow & & & & \downarrow p \\
X & \longrightarrow & \mathbb{P}^{m_1} \times \mathbb{P}^{m_2} & \xrightarrow{\ s\ } & \mathbb{P}^{r'} \supset S_{m_1+m_2}
\end{array}
$$

Here the rational map between Y and X follows from Diagram 1, v is the Veronese embedding and so $r = \frac{(m_1+m_2+2)(m_1+m_2+3)}{2}$, s is the Segre embedding and so $r' = (m_1 + 1)(m_2 + 1) - 1$, p is the projection, $V_{m_1+m_2+1}$ is the image of $\mathbb{P}^{m_1+m_2+1}$ in $\mathbb{P}^r$ and $S_{m_1+m_2}$ is the image of $\mathbb{P}^{m_1} \times \mathbb{P}^{m_2}$ in $\mathbb{P}^{r'}$. The Nikulin involution on $\mathbb{P}^{m_1+m_2+1}$ operates as:

$$(x_0 : \ldots : x_{m_1} : y_0 : \ldots : y_{m_2}) \mapsto (x_0 : \ldots : x_{m_1} : -y_0 : \ldots : -y_{m_2})$$

which induces an operation on the coordinates of $\mathbb{P}^r$ as

$$(x_0^2 : x_1^2 : \ldots : y_0^2 : y_1^2 : \ldots : y_{m_2-1}y_{m_2} : x_0y_1 : x_0y_2 : \ldots : x_{m_1}y_{m_2}) \mapsto$$
$$(x_0^2 : x_1^2 : \ldots : y_0^2 : y_1^2 : \ldots : y_{m_2-1}y_{m_2} : -x_0y_1 : -x_0y_2 : \ldots : -x_{m_1}y_{m_2}).$$

The projection p goes to the invariant space $\mathbb{P}^{r'}$ with coordinates $(x_0y_1 : x_0y_2 : \ldots : x_{m_1}y_{m_2})$ and so the equations of the image of Y in $V_{m_1+m_2+1}$ in these coordinates give equations for the image of the surface X in $S_{m_1+m_2}$ (cf. also [vGS, Proposition 2.7]).

Observe that the sum of the divisors L_1 and L_2 is exactly $L-\hat{N}$. From now on if $NS(X) = \mathcal{L}'_{2d}$ and $d/2$ is odd then $L_1 := (L - N_1 - N_2)/2$, $L_2 := (L - N_3 - \ldots - N_8)/2$, if $NS(X) = \mathcal{L}'_{2d}$ and $d/2$ is even then $L_1 := (L - N_1 - \ldots - N_4)/2$, $L_2 := (L - N_5 - \ldots - N_8)/2$.

In the case $NS(X) = \mathcal{L}_{2d}$ the construction above holds if instead of L_1 and L_2 we take L and $L - \hat{N}$.

4.1. **The case of $L^2 = 2$, $NS(X) = \mathcal{L}_2$, the polarization L.** Since L is pseudo ample by the Proposition 3.1 the linear system $|L|$ defines a $2 : 1$ map $X' \longrightarrow \mathbb{P}^2$ ramified on a sextic curve with eight nodes where X' is the surface X after contraction of the (-2)-curves. More precisely we have a commutative diagram:

$$
\begin{array}{ccc}
X & \longrightarrow & X' \\
\downarrow & & \downarrow \\
\widetilde{\mathbb{P}^2} & \longrightarrow & \mathbb{P}^2
\end{array}
$$

where $\widetilde{\mathbb{P}^2}$ is the blow up of $\mathbb{P}^2$ at the eight double points of the sextic. By general results on cyclic coverings the pull back of the branching sextic on X is $3L - (N_1 + \ldots + N_8) = 3L - 2\hat{N} = (L - \hat{N}) + (2L - \hat{N})$. Now $(L - \hat{N})^2 = 2 - 4 = -2$, $L \cdot (L - \hat{N}) = 2$ and so by using Riemann-Roch Theorem the divisor $L - \hat{N}$ is effective, and is a rational curve of degree two on $\mathbb{P}^2$. Observe that its image is an irreducible conic, in fact we are assuming that X' has exactly eight nodes and no other singularities. On the other hand $(2L - \hat{N})^2 = 8 - 4 = 4$ so by Proposition 3.4 and by [SD, Proposition 2.6] the generic member in $|2L - \hat{N}|$ is an irreducible curve of genus three, and its image in $\mathbb{P}^2$ is a curve of degree four (and in fact genus $3 = (4-1)(4-2)/2$). In both cases we have $(L - \hat{N}) \cdot N_i = (2L - \hat{N}) \cdot N_i = 1$ and so the curves intersect at the points which are the images of the curves N_i in $\mathbb{P}^2$. This is the first case in the paper of Barth, [B2].

4.2. **The case of $L^2 = 4$, $NS(X) = \mathcal{L}_4$, the polarization L.** By the Proposition 3.1 the linear system $|L|$ defines a birational map ϕ_L from X to a quartic surface in $\mathbb{P}^3$, the curves N_i are contracted to nodes. In this case $(L - \hat{N})^2 = 0$ and by the Proposition 3.4 $|L - \hat{N}|$ has no base components, so the generic member in the system is an irreducible elliptic curve (observe that by the structure of the Néron Severi group it cannot be the multiple of an elliptic curve). Since $L \cdot (L - \hat{N}) = 4$ the elliptic curve is sent to a quartic curve in $\mathbb{P}^3$ and is a complete intersection of two quadrics (observe that it cannot be a plane quartic since this has genus three). Moreover since $(L - \hat{N}) \cdot N_i = 1$, the quartic contains the nodes. There is a third quadric passing through the nodes, in fact $h^1(L - \hat{N}) = 0$ ([SD, Proposition 2.6]) hence $h^0(L - \hat{N}) = 2$ and again by loc. cit. $h^0(2(L - \hat{N})) = 3$. Now $2(L - \hat{N}) = 2L - (N_1 + \ldots + N_8)$ and the image of these divisors are precisely the quadrics which vanish on the eight singular points (cf. Corollary 3.2). Let s_1, s_2 be a basis of $H^0(L - \hat{N})$ then s_1^2, s_1s_2, s_2^2 is a basis of $H^0(2(L - \hat{N}))$ and these are the three quadrics through the nodes. This is the case three of Barth [B2] and by the Table 1 it corresponds to the case of $\mathcal{M}'_8$ of [vGS].

TABLE 1. Néron Severi lattices and projective models

	X		Y
	$NS(X) = \mathcal{L}_2$		$NS(Y) = \mathcal{M}'_4$
ϕ_L	double plane (singular sextic)	ϕ_M	smooth quartic in $\mathbb{P}^3$
	$NS(X) = \mathcal{L}_4$		$NS(Y) = \mathcal{M}'_8$
ϕ_L	quartic with even set of nodes	ϕ_M	complete intersection in $\mathbb{P}^5$
	$NS(X) = \mathcal{L}'_4$		$NS(Y) = \mathcal{M}_2$
ϕ_L	double cover of a cone	ϕ_M	double plane
ϕ_{L_1}	elliptic fibration		
	$NS(X) = \mathcal{L}_6$		$NS(Y) = \mathcal{M}'_{12}$
ϕ_L	singular complete intersection in $\mathbb{P}^4$	ϕ_M	projective model in $\mathbb{P}^7$
$\phi_{L-\hat{N}}$	double plane (smooth sextic)		
$\phi_L \times \phi_{L-\hat{N}}$	complete intersection in $\mathbb{P}^4 \times \mathbb{P}^2$		
	$NS(X) = \mathcal{L}_8$		$NS(Y) = \mathcal{M}'_{16}$
ϕ_L	singular complete intersection in $\mathbb{P}^5$	ϕ_M	projective model in $\mathbb{P}^9$
$\phi_{L-\hat{N}}$	smooth quartic in $\mathbb{P}^3$		
	$NS(X) = \mathcal{L}'_8$		$NS(Y) = \mathcal{M}_4$
ϕ_L	singular complete intersection in $\mathbb{P}^5$	ϕ_M	smooth quartic in $\mathbb{P}^3$
$\phi_{L-\hat{N}}$	double cover of a quadric		
	$NS(X) = \mathcal{L}_{10}$		$NS(Y) = \mathcal{M}'_{20}$
$\phi_{L-\hat{N}}$	smooth complete intersection in $\mathbb{P}^4$	ϕ_M	projective model in $\mathbb{P}^{11}$
$\phi_{L-\sum_{i=1}^4 N_i}$	double cover of a plane		
	$NS(X) = \mathcal{L}_{12}$		$NS(Y) = \mathcal{M}'_{24}$
$\phi_{L-\hat{N}}$	smooth complete intersection in $\mathbb{P}^5$	ϕ_M	projective model in $\mathbb{P}^{13}$
$\phi_{L-\sum_{i=1}^4 N_i}$	singular quartic in $\mathbb{P}^3$		
	(mixed even set with conics)		
	$NS(X) = \mathcal{L}'_{12}$		$NS(Y) = \mathcal{M}_6$
$\phi_{L-\hat{N}}$	smooth complete intersection in $\mathbb{P}^5$	ϕ_M	complete intersection in $\mathbb{P}^4$
$\phi_{L_2} \times \phi_{L_1}$	surface of bidegree $(2,3)$ in $\mathbb{P}^1 \times \mathbb{P}^2$		
	$NS(X) = \mathcal{L}'_{16}$		$NS(Y) = \mathcal{M}_8$
$\phi_{L_1} \times \phi_{L_2}$	complete intersection in $\mathbb{P}^2 \times \mathbb{P}^2$	ϕ_M	complete intersection in $\mathbb{P}^5$
	$NS(X) = \mathcal{L}'_{24}$		$NS(Y) = \mathcal{M}_{12}$
$\phi_{L_1} \times \phi_{L_2}$	complete intersection in $\mathbb{P}^3 \times \mathbb{P}^3$	ϕ_M	complete intersection in $\mathbb{P}^7$

4.3. **The case of $L^2 = 4$, $NS(X) = \mathcal{L}'_4$.**
(a) The polarization L. We may assume that the class $(L/2, v/2)$ is equal to $(L/2, (-N_1 - N_2)/2)$. By the Proposition 3.5 and [SD, Proposition 2.6] this defines a pencil of elliptic curves which we denote by E. Observe that $L = 2E + N_1 + N_2$ with $N_i \cdot E = 1$, hence by [SD, Proposition 5.7, (iii), a)] L defines a $2:1$ map to a cone of $\mathbb{P}^3$. The pencil $|E|$ corresponds to the system of lines through the vertex of the cone under this map. The class $C_2 := E - \hat{N} + N_1 + N_2 = L/2 + (-N_3 - \ldots - N_8)/2$ is effective with $C_2 \cdot N_i = 1$, similarly we may assume that the class $C_6 := 3L/2 + (-N_3 - \ldots - N_8)/2$ is an irreducible curve (it

follows by Proposition 3.5), with $C_6 \cdot N_i = 1$, moreover $C_2 \cdot C_6 = 0$, $C_2 \cdot L = 2$ and $C_6 \cdot L = 6$. Let $c_2 := \varphi_L(C_2)$ and $c_6 := \varphi_L(C_6)$. These two curves meet on the cone at the images of N_i, $i = 3, \ldots, 8$. Their union is a curve of degree eight, which is the branch divisor of the covering. In fact if C_2 is not a component of the branch divisor then $\varphi_L(C_2)$ has degree one and so is a line. But this means that $C_2 \in |E|$ which is not the case. Hence C_2 is a component of the branch divisor and c_2 is a conic. If now C_6 is not in the branch divisor, we have $\deg c_6 = 3$, and $c_2 \cdot c_6 = 6$, but then c_6 is contained in the plane of c_2 and on the cone too, which is impossible. Hence C_6 is also a component of the branch divisor. Finally observe that $\varphi_L(N_i) = Q$, for $i = 1, 2$ where Q is the vertex of the cone. This surface is also described in [vGS, Paragraph 3.2].

(b) An elliptic fibration. Now we describe the elliptic fibration on X defined by the divisor E. We consider the rational curve $C_2 = L/2 + (-N_3 - \ldots - N_8)/2$, it has intersection one with the class of the fiber E. So C_2 is a section of the fibration $\phi_E : X \to \mathbb{P}^1$ and the classes E and C_2 generate a lattice isometric to U.

Since the six (-2)-curves N_3, $N_4, \ldots, N_8$ are orthogonal to E, they are the components of some reducible fibers. All these curves intersect the section C_2 so they are components of six different reducible fibers. The rational curve N_1 is another section of the fibration (because its intersection with E is one). The Néron Severi group is generated over $\mathbb{Q}$ by the classes E of the fiber, by C_2, by the components N_i, $i = 3, \ldots, 8$ of the reducible fibers and by the other section N_1. The Néron Severi group of an elliptic fibration admitting a section is generated by the class of the fiber, by the zero section, by the irreducible components of the reducible fibers (not meeting the zero section) and by other sections. Since the Picard number is nine the six reducible fibers containing N_i, $i = 3, \ldots, 8$ are the only reducible fibers of the fibration, they are all of type I_2 (two rational curves meeting in two distinct points). The Euler characteristic of a K3 surface is 24 and is the sum of the Euler characteristics of the singular fibers. The singular irreducible fibers in the generic case are of type I_1 (singular irreducible curve with a node). Each fiber of type I_1 has Euler characteristic one, and each fiber of type I_2 has Euler characteristic equal to two. By the computation on the Euler characteristic it is clear that there are twelve singular fibers of type I_1 and six of type I_2. There are two independent sections, so the rank of the Mordell-Weil lattice is one. One of these sections (the zero section) is the curve C_2, mapped by ϕ_L to the conic in the branch locus on the cone. Other sections correspond to the curves N_1 and N_2, these are both mapped to the vertex of the cone.

Since X is a double cover of a cone, it admits an involution j. This involution fixes the classes N_i, $i = 3, \ldots, 8$, because they correspond to the intersection points between the conic and the sextic in the branch locus on the cone; it fixes the class C_2, and switches the classes N_1 and N_2. The involution j fixes also the class L which defines the double cover ϕ_L. On the fibration the involution j fixes the class of the fiber E and so it acts on the base, $\mathbb{P}^1$, of the fibration as the indentity, it fixes the zero section, which corresponds to the class C_2, and switches the other two independent sections N_1 and N_2. On the reducible fiber the involution j clearly fixes the component N_i and it fixes the other component $E - N_i$ too, since it fixes the fiber and a reducible fiber has two components. On $E - N_i$ the involution j switches the points P_1 and P_2, which are the points of intersection between the fiber and the sections N_1, respectively N_2.

The surface X admits an even set of eight disjoint rational curves, so it is the minimal resolution of the quotient of a K3 surface Y by a Nikulin involution. The elliptic fibration of X on $\mathbb{P}^1$ induces a fibration of $\widetilde{Y}$ (the blow up of Y) on $\mathbb{P}^1$ and so of Y (cf. Diagram 1). Let E denote the generic fiber of the fibration on X and A the generic fiber of the fibration on

$\widetilde{Y}$. By the Hurwitz formula, we have

$$2g(A) - 2 = 2(2g(E) - 2) + \deg R$$

where R is the branch divisor. Since X has an elliptic fibration we have $g(E) = 1$ and $\deg R = 2$ because the involution ramifies on the points of intersection $E \cap N_1$ and $E \cap N_2$. So we find $2g(A) - 2 = 2(2 - 2) + 2$, hence the generic fiber of the fibration $\widetilde{Y} \to \mathbb{P}^1$ is hyperelliptic of genus two.

4.4. **The case of $L^2 = 6$, $NS(X) = \mathcal{L}_6$.**
(a) The polarization $L - \hat{N}$. In this case $(L - \hat{N})^2 = 2$ by Lemma 3.1 it defines a $2 : 1$ map to $\mathbb{P}^2$. The curves N_i are mapped to lines in the plane. Let $l := \phi_{L-\hat{N}}(L - \hat{N})$, then for each curve C in the plane we have the formula $\phi^*_{L-\hat{N}}(l) \cdot \phi^*_{L-\hat{N}}(C) = 2(C \cdot l)$. Since $(L - \hat{N}) \cdot N_i = 1$ the curves N_i are contained in the preimage $\phi^{-1}_{L-\hat{N}}(T_i)$ where T_i are lines which are tritangents to the branch divisor, and so $\varphi^*_{L-\hat{N}}(T_i) = N_i + N'_i$. The curves N_i, N'_i meet in three points. Barth in [B2, Paragraph 2] shows that there is a quartic in $\mathbb{P}^2$ meeting the branch sextic at the tangency points.
(b) The polarization L. We consider the projective model of X as complete intersection of a cubic and a quadric hypersurface, it has eight nodes and the map is $\phi_L : X \to \mathbb{P}^4$. The curve $L - \hat{N}$ (cf. Proposition 3.4) has degree $6 = L \cdot (L - \hat{N})$ and genus $(L - \hat{N})^2/2 + 1 = 2$. Since $(L - \hat{N}) \cdot N_i = 1$, $i = 1, \ldots, 8$ its image in $\mathbb{P}^4$ passes through the eight singular points. This curve is contained in the intersection of three quadrics in $\mathbb{P}^4$, in fact $h^0(2L - (L - \hat{N})) = (L + \hat{N})^2/2 + 2 = 3$. The eight singular points of the surface are contained in three more quadrics, in fact $h^0(2L - (\sum_{i=1}^8 N_i)) = 6$ (cf. Corollary 3.2).
We consider now the linear system $|L - \hat{N}|$ associated to the hyperplane sections passing through the eight singular points of the image of X in $\mathbb{P}^4$. We have $h^0(L - \hat{N}) = 3$ and let l_1, l_2, l_3 be its generators. The six elements $l_1^2, l_2^2, l_3^2, l_1 \cdot l_2, l_1 \cdot l_3, l_2 \cdot l_3$ span $|2L - \sum_{i=1}^8 N_i| \cong \mathbb{P}^5$ (these are the quadrics passing through the nodes).
(c) The map $\phi_L \times \phi_{L-\hat{N}}$. In [vGS, Paragraph 3.9] the K3 surface Y admitting a Nikulin involution with Néron Severi group $\mathcal{M}'_{12}$ is described. Its quotient $\bar{X}$ is birational to a K3 surface which is complete intersection of a hypersurface of bidegree $(2, 0)$ and three hypersurfaces of bidegree $(1, 1)$ in $\mathbb{P}^4 \times \mathbb{P}^2$ (for a more detailed description of this complete intersection see the Section 5). The minimal resolution of the quotient $\bar{X}$ is the K3 surface X with $NS(X) = \mathcal{L}_6$. The projection of X to the first factor is defined by the divisor L and to the second one by $L - \hat{N}$. The first projection contracts eight disjoint rational curves and the same curves are sent to eight lines by the second projection (the $2 : 1$ map to $\mathbb{P}^2$).

4.5. **The case of $L^2 = 8$, $NS(X) = \mathcal{L}_8$.**
(a) The polarization $L - \hat{N}$. We have $(L - \hat{N})^2 = 4$ and the map $\phi_{L-\hat{N}} : X \longrightarrow \mathbb{P}^3$ exhibits X as a quartic surface in $\mathbb{P}^3$ with eight disjoint lines. This case is studied by Barth in [B2]. He describes two conditions to have an even set. The second one is not satisfied in our case, since it requires Picard number at least ten. In fact he shows that in this case there are two skew lines Z_1, Z_2 on the quartic surface with Z_1 meeting four lines and skipping the other four lines, and viceversa for Z_2. An easy computation shows that the intersection matrix of the hyperplane section, of the lines N_i and of Z_1 (or Z_2) has rank ten.
Barth's first condition says that there is an elliptic quartic curve in $\mathbb{P}^3$ which meets in two points four rational curves and skips the other four. In term of classes in the Néron Severi lattice this means that there is a curve $E = \alpha L + \sum b_i N_i$ with $E^2 = 0$, $EN_i = 2$ for $i = 1, \ldots, 4$

and $EN_i = 0$ for $i = 5, \ldots, 8$. By using the intersection products we obtain $\alpha = 1$, $b_i = -1$ for $i = 1, \ldots, 4$ and $b_i = 0$ for $i = 5, \ldots, 8$. So the elliptic curve is $E = L - N_1 - \ldots - N_4$ and in fact $E \cdot (L - \hat{N}) = 4$ which is the degree of E in $\mathbb{P}^3$. Similarly the curve $L - N_5 - \ldots - N_8$ meets the other four curves and skips the first four. Finally observe that these divisors are not studied in Proposition 3.5, with the notation there this is the case $d = r$ and the proof does not work in this case.

We describe briefly the elliptic fibration defined by E. Since $E \cdot N_i = 0$, for $i = 5, \ldots, 8$ these are components of reducible fibers. On the other hand the curves N_i, $i = 1, \ldots, 4$ are bisections of the fibration. The curves $L - N_2 - N_3 - N_4 - N_j - N_k$ with $j \neq k$ and $j, k = 5, 6, 7, 8$ are rational (-2)-curves which meet E in two points and N_j, N_k in two points as well. Hence they are also bisections of the fibration, and since a bisection meets also the singular fiber in two points, the curves $N_5, \ldots, N_8$ are contained in four different singular fibers, which are of type I_2. The remaining singular fibers are of type I_1, and we have 16 of them. This fibration does not admit sections. In this case the even set consists of four bisections and of four components of the singular fibers I_2 (these are all disjoint).

(b) The polarization L. We consider the projective model of X given by the map $\phi_L : X \to \mathbb{P}^5$, this is a complete intersection of three quadrics and has eight nodes. The generic element in $|L - \hat{N}|$ is a curve of degree $8 = L \cdot (L - \hat{N})$ and genus $(L - \hat{N})^2/2 + 1 = 3$ (cf. Proposition 3.4). Since $(L - \hat{N}) \cdot N_i = 1$, $i = 1, \ldots, 8$ the image of the curve $L - \hat{N}$ in $\mathbb{P}^5$ passes through the eight singular points, moreover this divisor is not Cartier at the nodes. By the Corollary 3.2 there exists a quadric G which cuts on the surface the curve $L - \hat{N}$ passing through all the singular points, so this curve is contained in the intersection of four quadrics in $\mathbb{P}^5$, in fact $h^0(2L - (L - \hat{N})) = (L + \hat{N})^2/2 + 2 = 4$. The quadric G must cut the image of $L - \hat{N}$ with multiplicity two since $\deg(X) = 8$ and the intersection has degree 16. The eight singular points of the surface are contained in the intersection of ten quadrics, in fact $h^0(2L - (\sum_{i=1}^8 N_i)) = 10$. We consider the linear system $|L - \hat{N}|$ associated to the hyperplane section passing through the eight singular points. We have $h^0(L - \hat{N}) = 4$ and we call l_1, l_2, l_3, l_4 its generators. The ten elements $l_1^2, l_2^2, l_3^2, l_4^2, l_1 l_2, l_1 l_3, l_1 l_4, l_2 l_3, l_2 l_4, l_3 l_4$ span $|2L - \sum_{i=1}^8 N_i|$.

(c) The map $\phi_L \times \phi_{L - \hat{N}}$. This K3 surface is the minimal resolution of the quotient of a K3 surface Y by a Nikulin involution. The Néron Severi group of Y is $\mathcal{M}'_{16}$ by the Table 1, and M is the ample class on Y with $M^2 = 16$. This gives an immersion of Y in $\mathbb{P}^9$, and the action of the Nikulin involution is induced by $(x_0 : \ldots : x_5 : y_0 : \ldots : y_3) \mapsto (x_0 : \ldots : x_5 : -y_0 : \ldots : -y_3)$. By the projection formula we have $H^0(Y, M) \cong H^0(X, L) \oplus H^0(X, L - \hat{N})$, with $h^0(X, L) = 6$, $h^0(X, L - \hat{N}) = 4$. Now

$$S^2 H^0(Y, M) = (S^2 H^0(X, L) \oplus S^2 H^0(X, L - \hat{N})) \oplus (H^0(X, L) \otimes H^0(X, L - \hat{N})).$$

This has dimension $55 = (21 + 10) + 24$. On the other hand

$$H^0(Y, 2M) \cong H^0(X, 2L) \oplus H^0(X, 2L - \hat{N})$$

and the dimensions are $34 = 18 + 16$. This shows that there are $(21 + 10) - 18 = 13$ invariant quadrics and $24 - 16 = 8$ antiinvariant quadrics $Q_i(x, y)$, $i = 1 \ldots, 8$ in the ideal of Y. Since the quadrics in four variables are only ten, there are three quadrics $q_1(x_0, \ldots, x_5)$, $q_2(x_0, \ldots, x_5)$, $q_3(x_0, \ldots, x_5)$ in the ideal of Y. The map $\phi_L \times \phi_{L - \hat{N}}$ sends X to the product $\mathbb{P}^5 \times \mathbb{P}^3$ and its image is the image of $Y \subset \mathbb{P}^9$ into the product of the eigenspaces, hence it is contained in three quadrics $q_1(x_0, \ldots, x_5)$, $q_2(x_0, \ldots, x_5)$, $q_3(x_0, \ldots, x_5)$ of bidegree $(2, 0)$ and eight quadrics $Q_i(x, y)$, $i = 1, \ldots, 8$, of bidegree $(1, 1)$ in particular it is not a

complete intersection of quadrics. The quadrics $q_1(x_0,\ldots,x_5)$, $q_2(x_0,\ldots,x_5)$, $q_3(x_0,\ldots,x_5)$ define the image of Y in $\mathbb{P}^5$, which is X with the polarization L. Since the fixed points of the Nikulin involution are contained in the space $y_0 = \ldots = y_3 = 0$, then the projection of the ten quadrics of the kind $q(x) - q'(y) = 0$ to $\mathbb{P}^5$ are ten quadrics cutting out the set of nodes on $X \subset \mathbb{P}^5$. The projection to $\mathbb{P}^3$ is X with the polarization $L - \hat{N}$ and is a quartic. One can obtain an equation for the quartic in the following way: a point $x \in X \subset \mathbb{P}_5$ has a non-trivial counterimage if there is a non-trivial solution of $Q_i(x,y) = \sum_{j=0}^{5} a_{ij}(y)x_j = 0$, $i = 1,\ldots,8$ which for a fixed x is a linear system of eight equations in six variables. Hence all the 6×6 minors of the matrix $(a_{ij}(y))$ are zero. Each of these is a sextic surface of $\mathbb{P}^3$ vanishing on $X \subset \mathbb{P}^3$. Since this is a surface of degree four, each of them splits into a product $q(x) \cdot p_4(x)$ where $p_4(x) = 0$ is an equation of $X \subset \mathbb{P}^3$.

4.6. The case of $L^2 = 8$, $NS(X) = \mathcal{L}'_8$.

(a) The polarization $L - \hat{N}$. In this case we have the divisor $E_1 := (L/2, v/2)$ with $v^2 = -8$ and $v = -N_1 - N_2 - N_3 - N_4$, and also the divisor $E_2 := (L/2, v'/2)$ with $v' = -N_5 - N_6 - N_7 - N_8$ so $(L/2, v/2)^2 = (L/2, v'/2)^2 = 0$ and $L - \hat{N} = (L/2, v/2) + (L/2, -v/2)$ is the sum of two elliptic curves (cf. Proposition 3.5). This is a $2 : 1$ map to $\mathbb{P}^1 \times \mathbb{P}^1$ (by Lemma 3.1) and the curves N_i are sent to lines on the quadric. Moreover since $E_1 \cdot N_i = 1$ and $E_2 \cdot N_i = 0$ for $i = 1,2,3,4$ the images of these lines belong to the same ruling on the quadric and the images of N_i, $i = 5,6,7,8$ belong to the other ruling. By a similar computation as in Paragraph 4.4 the curves N_i are one of the two components of the preimage of a curve on the quadric which splits on X, hence $\phi_{L-\hat{N}}(N_i) = T_i$ and these are bitangents to the branch curve of bidegree $(4,4)$. Let $\phi^*_{L-\hat{N}}(T_i) = N_i + N'_i$ then

$$\mathcal{O}_{B'}(N_1 + \ldots + N_8 + N'_1 + \ldots + N'_8) = \mathcal{O}_{B'}(4(L - \hat{N})) = \mathcal{O}_{B'}(2(2(L - \hat{N})))$$

By Proposition 3.4 $2(L - \hat{N})$ is a curve, and has bidegree $(2,2)$ on the quadric. Hence the divisor cut out by $N_1 + \ldots + N'_8$ is two times the divisor cut out by $2(L - \hat{N}) + \mu$ where μ is a 2-torsion element in the Picard group. Barth shows in [B2], case six, that such an element does not exist. This implies that the tangency points of the T_i on the quadric are cut out by a curve of bidegree $(2,2)$.

(b) The polarization L. All the considerations of the Paragraph 4.5, case **(b)** are true. Moreover there are two elliptic curves $L_1 = \frac{L-N_1-N_2-N_3-N_4}{2}$, $L_2 = \frac{L-N_5-N_6-N_7-N_8}{2}$ passing through four of the eight singular points each and not passing through the other four. Obviously also in this case the image of $2(L - \hat{N}) = 2(L_1 + L_2)$ is cut out by a quadric. By the Table 1 this case corresponds to a K3 surface Y with $NS(Y) = \mathcal{M}_4$. After a change of coordinates the surface X can be written in the form

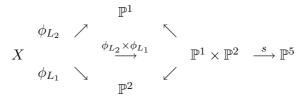$$q(z_0,\ldots,z_5) = 0, \; z_0 z_1 - z_4^2 = 0, \; z_2 z_3 - z_5^2 = 0$$

and there are four singularities on $z_0 = z_1 = z_4 = 0$ and four on $z_2 = z_3 = z_5 = 0$ (the two copies of $\mathbb{P}^2$ which are the vertices of the cones). Now the quadrics of the kind $z_i z_j = 0$ with $i = 0,1$ and $j = 2,3$ meet the K3 surface in two curves C_i, C_j with multiplicity two, hence $2C_i \in |L - (N_1 + \ldots + N_4)|$ and $2C_j \in |L - (N_5 + \ldots + N_8)|$ and so C_i, C_j are in the linear system of L_1, resp. of L_2.

4.7. The case of $L^2 = 10$, $NS(X) = \mathcal{L}_{10}$.

(a) The polarization $L - \hat{N}$. Since $(L - \hat{N})^2 = 6$, then the projective model of X is a complete intersection of a quadric and a cubic hypersurfaces in $\mathbb{P}^4$ with an even set of eight lines (cf. Lemma 3.1).

(b) The polarizations $L - N_1 - N_2 - N_3 - N_4$ **and** $L - N_5 - N_6 - N_7 - N_8$. The divisors $L - N_1 - N_2 - N_3 - N_4$ and $L - N_5 - N_6 - N_7 - N_8$ are pseudo ample classes by the Proposition 3.5. They define two maps $2:1$ to $\mathbb{P}^2$. Each of these maps contracts four curves of the eight rational curves N_i and maps the other four in four conics.

4.8. The case of $L^2 = 12$, $NS(X) = \mathcal{L}_{12}$.
(a)The polarization $L - \hat{N}$. Since $(L - \hat{N})^2 = 8$ the projective model of X is a K3 surface in $\mathbb{P}^5$ with an even set of eight disjoint lines.
(b) The polarizations $L - N_1 - N_2 - N_3 - N_4$ **and** $L - N_5 - N_6 - N_7 - N_8$. The curves $E_1 = L - N_1 - N_2 - N_3 - N_4$ and $E_2 = L - N_5 - N_6 - N_7 - N_8$ have self intersection four, so they define two maps to $\mathbb{P}^3$ (by Lemma 3.2). The map ϕ_{E_1} contracts the four curves N_i, $i = 1, \ldots, 4$ and sends the other in four conics. The map ϕ_{E_2} contracts the other four curves and sends N_i, $i = 1, \ldots, 4$ in conics.

4.9. The case of $L^2 = 12$, $NS(X) = \mathcal{L}'_{12}$.
(a)The polarization $L - \hat{N}$. Observe that the considerations of Paragraph 4.8, **(a)** are true also in this case. Moreover there are two curves $C_1 = \frac{L - N_1 - N_2 - N_3 - N_4 - N_5 - N_6}{2}$ and $C_2 = \frac{L - N_7 - N_8}{2}$ intersecting respectively six and two of the lines N_i in one point. The curve C_1 has degree three and genus one. The curve C_2 has degree five and genus two.
(b) The map $\phi_{L_1} \times \phi_{L_2}$. The intersection properties of L_1 and L_2 are $L_1 \cdot L_1 = 2$, $L_2 \cdot L_2 = 0$ and $L_1 \cdot L_2 = 3$. The K3 surface X is the minimal rsolution of the quotient $\bar{X}$ of a K3 surface Y admitting a Nikulin involution with $NS(Y) = \mathcal{M}_6$ which is described in [vGS, Paragraph 3.3]. The surface $\bar{X}$ has bidegree $(2, 3)$ in $\mathbb{P}^1 \times \mathbb{P}^2$. The maps ϕ_{L_1} and ϕ_{L_2} are respectively the projection to the second and to the first projective space.

The map $\phi_{L_1} : X \to \mathbb{P}^2$ is a 2:1 map. It contracts the six rational curves $N_3, \ldots, N_8$ to six nodes of the branch sextic and the two curves N_1 and N_2 are mapped to lines in $\mathbb{P}^2$ which are tritangent to the branch locus. The map $\phi_{L_2} : X \to \mathbb{P}^1$ is an elliptic fibration, it contracts the two rational curves N_1, N_2, whence the curves $N_3, \ldots, N_8$ are six independent sections of the fibration. This fibration has two reducible fibers of type I_2 (made up by the classes N_1, $E_2 - N_1$ and N_2, $E_2 - N_2$).

The Segre map s sends $\mathbb{P}^1 \times \mathbb{P}^2$ in $\mathbb{P}^5$.

$$
\begin{array}{ccccccc}
 & & \mathbb{P}^1 & & & & \\
 & \phi_{L_2} \nearrow & & \searrow & & & \\
X & & \xrightarrow{\phi_{L_2} \times \phi_{L_1}} & & \mathbb{P}^1 \times \mathbb{P}^2 & \xrightarrow{\; s \;} & \mathbb{P}^5 \\
 & \phi_{L_1} \searrow & & \nearrow & & & \\
 & & \mathbb{P}^2 & & & &
\end{array}
$$

Observe that the map $s \circ (\phi_{L_2} \times \phi_{L_1}) : X \longrightarrow \mathbb{P}^5$ is the map $\phi_{L_1+L_2} = \phi_{L-\hat{N}}$ (since $L_1 + L_2 = L - \hat{N}$). Indeed let s_1, s_2, s_3 be a basis of $H^0(L_1)$, and s_4, s_5 be a basis of $H^0(L_2)$. Then the products $s_1 s_4, s_1 s_5, s_2 s_4, s_2 s_5, s_3 s_4, s_3 s_5$ are linear independent sections in $H^0(L_1 + L_2)$ and define the Segre embedding of X in $\mathbb{P}^1 \times \mathbb{P}^2$. Since $h^0(L_1 + L_2) = 6$ then the map $\phi_{L_1+L_2}$ is exactly the map $s \circ (\phi_{L_2} \times \phi_{L_1})$.

4.10. The case of $L^2 = 16$, $NS(X) = \mathcal{L}'_{16}$, the map $\phi_{L_1} \times \phi_{L_2}$.
The intersection properties of L_1 and L_2 are $L_1 \cdot L_1 = 2$, $L_2 \cdot L_2 = 2$ and $L_1 \cdot L_2 = 4$. In [vGS, Paragraph 3.6] is described the K3 surface Y admitting a Nikulin involution with Néron Severi group $\mathcal{M}_8$. Its quotient $\bar{X}$ is the complete intersection of a hypersurface of bidegree $(1, 1)$ and of a hypersurface of bidegree $(2, 2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$, its minimal resolution is X. A K3 surface which is a complete intersection of a bidegree $(1, 1)$ and a bidegree $(2, 2)$ hypersurface in $\mathbb{P}^2 \times \mathbb{P}^2$ is a *Wehler surface*

(cf. [W]). We describe this surface more in details in the Section 5. The surface X has a projective model in $\mathbb{P}^2 \times \mathbb{P}^2$ and the map associated to the divisors L_1 and L_2 are respectively the projection to the first and to the second projective space. The map $\phi_{L_1} : X \to \mathbb{P}^2$ is a $2 : 1$ map. It contracts the four rational curves $N_5, \ldots, N_8$ to nodes of the branch sextic of the double cover. The four curves $N_1, \ldots, N_4$ are mapped to lines in $\mathbb{P}^2$. Since their intersection with L_1 is equal to one, each of them is one of the two components of the pullback of a line in $\mathbb{P}^2$. So their image under the map ϕ_{L_1} is a line tritangent to the branch curve. The branch curve has degree six and has four nodes so its genus is $(6-1)(6-2)/2 - 4 = 6$. The curve R_1 on X such that $\phi_{L_1}(R_1)$ is the branch curve, has degree six and genus six (because it is the branch curve, so the genus of the curve on X is the genus of its image on $\mathbb{P}^2$), this implies that the curve R_1 has self-intersection ten $(g = R_1^2/2 + 1)$ and its intersection with L_1 is six. The curve R_1 has to intersect the curves N_i, $i = 1, \ldots, 4$ in three points (because N_i are mapped to tritangent to the sextic) and the curves N_i, $i = 5, \ldots, 8$ in two points (because the branching curve has nodes in the points which are the images of these curves). So we find

$$R_1 = 3 \left(\frac{L - N_1 - N_2 - N_3 - N_4}{2} \right) - (N_5 + N_6 + N_7 + N_8).$$

Exactly in the same way one sees that the branch curve of the second projection is

$$R_2 = 3 \left(\frac{L - N_5 - N_6 - N_7 - N_8}{2} \right) - (N_1 + N_2 + N_3 + N_4).$$

The equation of a generic K3 surface which is complete intersection of a $(1,1)$ and a $(2,2)$ hypersurface in $\mathbb{P}^2 \times \mathbb{P}^2$ is given by the system

$$\begin{cases} \sum_{i,j=0,1,2} q_{ij}(x_0 : x_1 : x_2) y_i y_j = 0 \\ \sum_{i=0,1,2} l_i(x_0 : x_1 : x_2) y_i = 0 \end{cases}$$

where q_{ij} and l_i, $i, j = 0, 1, 2$ are homogeneous polynomial of degree respectively two and one in the variables x_j, which are the coordinates of the first copy of $\mathbb{P}^2$ and y_j denote coordinates of the second copy of $\mathbb{P}^2$.

For a generic point $(\overline{x_0} : \overline{x_1} : \overline{x_2})$ of $\mathbb{P}^2$ the system has two solutions in $(y_0 : y_1 : y_2)$ and this gives the 2:1 map to $\mathbb{P}^2$. If the point $(\overline{x_0} : \overline{x_1} : \overline{x_2})$ is such that $\sum_{i=0,1,2} l_i(\overline{x_0} : \overline{x_1} : \overline{x_2}) y_i = 0$ for each $(y_0 : y_1 : y_2)$, then the fiber on it is the quadric $\sum_{i,j=0,1,2} q(\overline{x_0} : \overline{x_1} : \overline{x_2}) y_i y_j = 0$. Otherwise if $\sum_{i,j=0,1,2} q(\overline{x_0} : \overline{x_1} : \overline{x_2}) y_i y_j = 0$ for each $(y_0 : y_1 : y_2)$ then the fiber on $(\overline{x_0} : \overline{x_1} : \overline{x_2})$ is a line.

Since in our case each map to $\mathbb{P}^2$ contracts four lines, then for each copy of $\mathbb{P}^2$ there are four points in which $\sum_{i,j=0,1,2} q(x_0 : x_1 : x_2) y_i y_j = 0$ are identically satisfied. Up to a projective transformation one can suppose that the four points with dimension one fiber are, on each $\mathbb{P}^2$, $(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1), (1 : 1 : 1)$. This implies that the equation $\sum_{i,j=0,1,2} q(x_0 : x_1 : x_2) y_i y_j = 0$ is of the form

$$y_0 y_1 (x_0 x_1 + a x_0 x_2 - (a+1) x_1 x_2) + y_0 y_2 (b x_0 x_1 + c x_0 x_2 - (b+c) x_1 x_2) +$$
$$+ y_1 y_2 (-(1+b) x_0 x_1 - (a+c) x_0 x_2 + (a+b+c+1) x_1 x_2) = 0.$$

and so it depends on three projective parameters. The equation of type $(1,1)$ are

$$x_0 y_0 + d x_0 y_1 + e x_0 y_2 + f x_1 y_0 + g x_1 y_1 + h x_1 y_2 + l x_2 y_0 + m x_2 y_1 + n x_2 y_2 = 0$$

and so depends on eight parameters (we can not apply other projective transformations because we have chosen the points on which there are lines as fibers). So a Wehler K3 surface such that the projection ϕ_{L_1} to $\mathbb{P}^2$ contracts four rational curves of the K3 surface and ϕ_{L_2}

contracts four other rational curves (disjoint from the previous curves) depends exactly on eleven parameters.

4.11. **The case of $L^2 = 24$, $NS(X) = \mathcal{L}'_{24}$, the map $\phi_{L_1} \times \phi_{L_2}$.** The intersection properties of L_1 and L_2 are $L_1 \cdot L_1 = 4$, $L_2 \cdot L_2 = 4$ and $L_1 \cdot L_2 = 6$. Each of them defines a map from X to $\mathbb{P}^3$ (by Lemma 3.2). Each map ϕ_{L_i}, $i = 1, 2$ contracts four rational curves and sends the other in four lines. The curve L_1 is sent by ϕ_{L_2} to a curve of degree six, and viceversa. In [vGS, Paragraph 3.8] it is described the K3 surface Y admitting a Nikulin involution with Néron-Severi group $\mathcal{M}_{12}$. Its quotient is $\bar{X}$ and it is a complete intersection of four varieties of bidegree $(1, 1)$ in $\mathbb{P}^3 \times \mathbb{P}^3$, the minimal resolution is X (cf. Table 1). The projections to the two copies of $\mathbb{P}^3$ are ϕ_{L_1} and ϕ_{L_2}.

5. Geometric conditions to have an even set.

In this section we describe geometrical properties of K3 surfaces which imply the presence of an even set. These even sets can be of eight nodes, of eight rational curves (lines or conics) or of some nodes and some rational curves. The following results are in a certain sense the converse of the results of the previous section, where we supposed that a K3 surface admits an even set and we described its geometry. To prove the existence of an even set on S we will prove that either the lattice $\mathcal{L}_{2d}$ or $\mathcal{L}'_{2d}$ is embedded in $NS(S)$ and that the sublattice N of $\mathcal{L}_{2d}$ (or of $\mathcal{L}'_{2d}$) is generated over $\mathbb{Q}$ by (-2)-irreducible curves. Since rank $\mathcal{L}_{2d} =$ rank $\mathcal{L}'_{2d} = 9$ and since the K3 surfaces with Néron Severi group equal to $\mathcal{L}_{2d}$ or $\mathcal{L}'_{2d}$ have an even set, then the number of moduli of the families of K3 surfaces that we describe here is eleven.

5.1. **Double cover of a cone with an even set.** Let S be a K3 surface which is a double cover of a cone, then by [SD, Proposition 5.7 case iii)] the map from S to the cone is given by a class L' in $NS(S)$ such that either

    a) $L' = 2E' + \Gamma_0 + \Gamma_1$ with $\Gamma_0 \cdot E' = \Gamma_1 \cdot E' = 1$ and $\Gamma_0 \cdot \Gamma_1 = 0$ or

    b) $L' = 2E' + 2\Gamma_0 + \ldots + 2\Gamma_n + \Gamma_{n+1} + \Gamma_{n+2}$, with $E' \cdot \Gamma_0 = \Gamma_i \cdot \Gamma_{i+1} = 1$ $i = 0, \ldots, n-1$, $\Gamma_n \cdot \Gamma_{n+1} = \Gamma_n \cdot \Gamma_{n+2} = 1$ and the other intersections are equal to zero.

The Γ_i's are irreducible (-2)-curves. If we are in the case a), then we can give a sufficient condition for S to have an even set of eight disjoint rational curves.

Proposition 5.1. *Let S be a K3 surface such that there exist a map $\phi_{L'} : S \overset{2:1}{\to} Z$, where Z is a cone and $L' = 2E' + \Gamma_0 + \Gamma_1$ with $\Gamma_0 \cdot E' = \Gamma_1 \cdot E' = 1$ and $\Gamma_0 \cdot \Gamma_1 = 0$. If the branch locus of the double cover is the union of a conic and a sextic meeting in six distinct points and not passing through the vertex of the cone, then S admits an even set of eight disjoint rational curves.*

Proof. We prove that under the hypothesis the lattice $\mathcal{L}'_4$ is embedded in the Néron Severi lattice of S. In particular there exist eight disjoint rational curves in $NS(S)$ generating on $\mathbb{Q}$ a copy of N in the Néron Severi lattice. This implies that S admits an even set made up by these eight disjoint rational curves.

By the hypothesis the classes L', E' and Γ_0 are linearly independent and are in $NS(S)$. The map $\phi_{L'}$ is a $2 : 1$ map to the cone, which contracts the two rational curves Γ_0 and Γ_1 to the vertex of the cone. The (smooth) K3 surface S is the double cover of the blow up of the cone in the vertex and in the six singular points of the ramification locus. On S there are six rational curves Γ_i, $i = 2, \ldots, 7$ on the six singular points of the ramification locus, and the two rational curves Γ_0 and Γ_1 on the blow up of the vertex of the cone (since this is not in the ramification locus we obtain two curves).

Let C_2' be the curve such that $\phi_{L'}(C_2') = c_2'$ is the conic of the branching locus. Since c_2' is a conic, $C_2' \cdot L = 2$ and since it does not pass through the vertex then $C_2' \cdot \Gamma_0 = C_2' \cdot \Gamma_1 = 0$, so $C_2' \cdot E' = 1$, moreover C_2' is a rational curve and so $C_2'^2 = -2$. Since on the cone c_2' passes through the six singular points of the ramification locus, on the K3 surface we have $C_2' \cdot \Gamma_i = 1$, $i = 2, \ldots, 7$.

The classes L', E', Γ_0, C_2, Γ_i, $i = 2, \ldots, 7$ spans a lattice R which is isometric to the lattice $\mathcal{L}_4'$. In fact a basis for $\mathcal{L}_4'$ is given by $(L - N_1 - N_2)/2$, $\hat{N}$ and N_i, $i = 1, \ldots 7$. The map

$$E' \mapsto (L - N_1 - N_2)/2, \quad C_2 + E' - L' \mapsto \hat{N}, \quad \Gamma_i \mapsto N_{i+1} \quad i = 0, \ldots, 7.$$

gives the explicit change of basis from R to $\mathcal{L}_4'$. $\square$

5.2. Complete intersection of one $(2,0)$ and three $(1,1)$ hypersurfaces in $\mathbb{P}^4 \times \mathbb{P}^2$.

If S is a complete intersection of a hypersurface of bidegree $(2,0)$ and three hypersurfaces of bidegree $(1,1)$ in $\mathbb{P}^4 \times \mathbb{P}^2$, by the adjunction formula S is a K3 surface. The Néron Severi group of a generic K3 surface which is a complete intersection of this type is generated by the two divisors D_1 and D_2 associated to the two projections. The family of the K3 surfaces of this type has Picard number two and so it has 18 moduli. To give the complete description of the Néron Severi group we compute the intersection $D_1 \cdot D_2$. We describe here how to find $D_1 \cdot D_2$ as explained in [vG, Section 5]. On the K3 surface the divisors D_1 and D_2 correspond to the restriction to S of the pull back of the hyperplane section of $\mathbb{P}^4$, respectively of $\mathbb{P}^2$. We put $h = \mathbb{P}^3 \times \mathbb{P}^2$ and $k = \mathbb{P}^4 \times \mathbb{P}^1$. It is clear that $h^3 = \{point\} \times \mathbb{P}^2$, and so $h^4 = 0$ because in $\mathbb{P}^4$ it corresponds to the intersection between a point and a space. In the same way one computes that $k^2 = \mathbb{P}^4 \times \{point\}$ (intersection of two lines in $\mathbb{P}^2$) and $k^3 = 0$, $h^3 k^2 = 1$ ($\{point\} \times \{point\}$). The hypersurface of bidegree $(2,0)$ corresponds to $2h$ (has degree two with respect to the first factor, so with respect to h, and zero with respect to the second factor, k) and the hypersurfaces of bidegree $(1,1)$ correspond to the divisor $h + k$. Since X is the complete intersection of one hypersurface of bidegree $(2,0)$ and three hypersurfaces of bidegree $(1,1)$, X corresponds in $\mathbb{P}^4 \times \mathbb{P}^2$ to the divisor $2h(h+k)^3$. We want to compute $D_1 \cdot D_2$ which is $h \cdot k$ restricted to $2h(h+k)^3$. Then $D_1 \cdot D_2$ is equal to $hk(2h)(h+k)^3$ in the six dimensional space $\mathbb{P}^4 \times \mathbb{P}^2$. The terms $h^i k^j$ with $i + j = 6$ correspond to the intersections of codimension six and so are a finite number of points. The sum of the coefficients of these terms is exactly the number of points, so $D_1 \cdot D_2 = 6$.

Hence the general K3 surface which is complete intersection of a $(2,0)$ hypersurface and three $(1,1)$ hypersurfaces in $\mathbb{P}^4 \times \mathbb{P}^2$ has Néron-Severi lattice isometric to $\left\{ \mathbb{Z}^2, \begin{bmatrix} 6 & 6 \\ 6 & 2 \end{bmatrix} \right\}$. This is a sublattice of the Néron Severi lattice of any $K3$ surface which is a complete intersection of a $(2,0)$ hypersurface and three $(1,1)$ hypersurfaces in $\mathbb{P}^4 \times \mathbb{P}^2$.

Proposition 5.2. *Let S be a complete intersection of one hypersurface of bidegree $(2,0)$ and three hypersurfaces of bidegree $(1,1)$ in $\mathbb{P}^4 \times \mathbb{P}^2$. Let ϕ_{A_1} and ϕ_{A_2} be the projections to the first and to the second factor associated to the pseudo ample class A_1, with $A_1^2 = 6$, and to the pseudo ample class A_2, with $A_2^2 = 2$. If there exist eight curves R_i, $i = 1, \ldots, 8$ such that ϕ_{A_1} contracts all these curves to eight nodes of the image and ϕ_{A_2} sends these curves in lines on $\mathbb{P}^2$, then R_i, $i = 1, \ldots, 8$ form an even set.*

Proof. The idea of the proof is similar to the proof of Proposition 5.1 and is based on the presence of certain divisors in $NS(S)$. The divisors A_j, $j = 1, 2$, R_i, $i = 1, \ldots, 8$ are contained in the Néron Severi group. Nine of these classes are linearly independent. The lattice generated by $A_1, A_2, R_1, R_2, R_3, R_4, R_5, R_6, R_7$ is embedded in $NS(S)$ and a computation shows that

it is isometric to the lattice $\mathcal{L}_6$. Since the lattice $\mathcal{L}_6$ contains an even set, also in the Néron Severi group of S there is an even set made up by $R_1, \ldots, R_7, 2A_2 - 2A_1 + R_1 + \ldots + R_7$.$\square$

Remark. Observe that Proposition 5.2 gives a sufficient condition for a K3 surface complete intersection in $\mathbb{P}^4$ to have an even set of nodes (or of eight rational curves in the minimal resolution).

5.3. Complete intersection of three quadrics in $\mathbb{P}^5$ with an even set of nodes.
We give two different sufficient conditions for a K3 surface in $\mathbb{P}^5$ to have an even set of nodes. These two possibilities correspond to the fact that the Néron Severi group of such a K3 surface, with Picard number nine, is equal either to the lattice $\mathcal{L}_8$ or $\mathcal{L}_8'$.

Proposition 5.3. *Let S be a K3 surface admitting two maps ϕ_{A_1}, ϕ_{A_2} associated to the pseudo ample class A_1 with $A_1^2 = 8$ and to the ample class A_2 with $A_2^2 = 4$. If there exist eight curves R_i, $i = 1, \ldots, 8$ such that ϕ_{A_1} contracts all these curves to eight nodes and ϕ_{A_2} sends these curves to lines on the quartic in $\mathbb{P}^3$, then R_i, $i = 1, \ldots, 8$ form an even set.*

Proof. One can prove that $\mathcal{L}_8$ is primitively embedded in $NS(S)$ as in the Propositions 5.1 and 5.2. $\square$

Proposition 5.4. *Let X be a K3 surface in $\mathbb{P}^5$ having eight nodes. These nodes form an even set if X is the complete intersection of a smooth quadric and two quadrics, which are singular in two planes $H \cong \mathbb{P}^2$ and $K \cong \mathbb{P}^2$, $H \cap K = \emptyset$ and four of the points are contained in H and the other four in K.*

Proof. Let $h_0 = h_1 = h_2 = 0$ and $k_0 = k_1 = k_2 = 0$ be the equations defining H resp. K in $\mathbb{P}^5$, then we can write the equations of the two cones as $h_0 h_1 - h_2^2 = 0$ and $k_0 k_1 - k_2^2 = 0$. The quadrics $h_i k_j = 0$, $i = 0, 1$, $j = 0, 1$ meet the K3 surface in two curves C_i, C_j with multiplicity two, which passes through four singular points, resp. to the other four. So $2(C_i + C_j) \in |2L - (N_1 + \ldots + N_8)|$, which shows that $N_1 + \ldots + N_8$ form an even set. $\square$

5.4. Double covers of $\mathbb{P}^2$.
Here we consider two different K3 surfaces with an even set which admit maps $2 : 1$ to $\mathbb{P}^2$. The first one is a Wehler surface, the second one is not. In the first case the curves of the even sets are contracted to singular points of the branch locus or are sent to lines of $\mathbb{P}^2$ which are tritangent to the ramification locus, in the second case they are contracted or sent to conics. Other double covers of $\mathbb{P}^2$ with an even set are described in [B2].

5.4.1. The first case: complete intersections of bidegree $(1,1)$, $(2,2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$.
The complete intersections of bidegree $(1,1)$ and $(2,2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ are the Wehler surfaces. The projections to the two copies of $\mathbb{P}^2$ are 2:1 maps. It is known (but can also be computed as in Section 5.2) that the Néron Severi group of the generic member of this family is the two dimensional lattice $\{\mathbb{Z}^2, \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}\}$. The number of moduli of the family of the Wehler K3 surfaces is 18.

Proposition 5.5. *Let S be a Wehler K3 surface such that the first projection π_1 contracts four rational disjoint curves R_l, $l = 1, \ldots, 4$ on S and the second projection π_2 contracts other four rational disjoint curves R_l, $l = 5, \ldots, 8$. Moreover the map π_1 sends the curves contracted by π_2 to lines on $\mathbb{P}^2$ and viceversa. Then the eight rational curves R_l, $l = 1, \ldots, 8$ form an even set on S.*

Proof. One can prove that $\mathcal{L}_{16}'$ is primitively embedded in $NS(S)$ as in the Propositions 5.1 and 5.2, and that $N \subset \mathcal{L}_{16}'$ is generated over $\mathbb{Q}$ by the curves R_i. $\square$

5.4.2. *The second case.*

Proposition 5.6. *Let S be a K3 surface admitting two maps $2 : 1$ to $\mathbb{P}^2$. If there exist eight curves R_i, $i = 1, \ldots, 8$ such that the map on the first copy of $\mathbb{P}^2$ contracts the curves R_i, $i = 1, \ldots, 4$ and sends the others in four conics and the map on the second copy of $\mathbb{P}^2$ contracts the curves R_i $i = 5, \ldots, 8$ and sends the others in conics, then R_i, $i = 1, \ldots, 8$ is an even set of eight disjoint rational curves.*

Proof. One can prove that $\mathcal{L}_{10}$ is primitively embedded in $NS(S)$ as in the Propositions 5.1 and 5.2. □

5.5. A mixed even set.

Proposition 5.7. *Let S be a K3 surface admitting two maps to $\mathbb{P}^3$. If there exist eight curves R_i, $i = 1, \ldots, 8$ such that the map on the first copy of $\mathbb{P}^3$ contracts the curves R_i, $i = 1, \ldots, 4$ and sends the others in four conics and the map on the second copy of $\mathbb{P}^3$ contracts the curves R_i, $i = 5, \ldots, 8$ and sends the others in conics, then R_i, $i = 1, \ldots, 8$ is an even set of eight disjoint rational curves.*

Proof. One can prove that $\mathcal{L}_{12}$ is primitively embedded in $NS(S)$ as in the Propositions 5.1 and 5.2. □

In this case we have on a quartic in $\mathbb{P}^3$ a *mixed even set*, in fact it consists of four nodes and of four conics.

5.6. Surfaces of bidegree $(2, 3)$ in $\mathbb{P}^1 \times \mathbb{P}^2$.

A K3 surface in $\mathbb{P}^1 \times \mathbb{P}^2$ has bidegree $(2, 3)$ by the adjunction formula. These K3 surfaces are studied in [vG, Paragraph 5.8]. The family has 18 moduli, in fact the Néron Severi group of such a K3 surface has to contain two classes D_1, D_2 giving the regular maps ϕ_{D_1} and ϕ_{D_2}, which correspond to the projections to $\mathbb{P}^1$ and to $\mathbb{P}^2$. We can compute the intersection properties of D_1 and D_2 as in the Paragraph 5.2 (these computations can be found also in [vG, Paragraph 5.8]). The general K3 surface which has bidegree $(2, 3)$ in $\mathbb{P}^1 \times \mathbb{P}^2$ has Néron-Severi lattice isometric to $\{\mathbb{Z}^2, \begin{bmatrix} 0 & 3 \\ 3 & 2 \end{bmatrix}\}$. This is a sublattice of the Néron Severi lattice of all the K3 surfaces which have bidegree $(2, 3)$ in $\mathbb{P}^1 \times \mathbb{P}^2$.

Proposition 5.8. *Let S be a K3 surface of bidegree $(2, 3)$ in $\mathbb{P}^1 \times \mathbb{P}^2$ and such that the projection to the first space p_1 (which gives an elliptic fibration) contracts two disjoint rational curves and the projection p_2 to the second space contracts other six disjoint rational curves. If the curves contracted by p_1 are sent to lines by p_2 and the curves contracted by p_2 are sent by p_1 to two sections of the elliptic fibration, then the eight rational curves on S form an even set.*

Proof. One can prove that $\mathcal{L}'_{12}$ is primitively embedded in $NS(S)$ as in the Propositions 5.1 and 5.2. □

5.7. Complete intersections in $\mathbb{P}^3 \times \mathbb{P}^3$.

The Néron Severi group of a complete intersection of four hypersurfaces of bidegree $(1, 1)$ in $\mathbb{P}^3 \times \mathbb{P}^3$ is generated by the two divisors D_1 and D_2 associated to the two projections. The divisors D_1 and D_2 have self intersection equal to four, computing as before the intersection between D_1 and D_2 one finds $D_1 \cdot D_2 = 6$, so the Néron Severi lattice of the generic K3 surface which is a complete intersection of four bidegree $(1, 1)$ hypersurfaces in $\mathbb{P}^3 \times \mathbb{P}^3$ is $\{\mathbb{Z}^2, \begin{bmatrix} 4 & 6 \\ 6 & 4 \end{bmatrix}\}$.

Proposition 5.9. *Let S be a complete intersection of four bidegree $(1,1)$ hypersurfaces in $\mathbb{P}^3 \times \mathbb{P}^3$. Let A_1 and A_2 be two pseudo ample divisors defining two maps to $\mathbb{P}^3$. If the map ϕ_{A_1}, respectively ϕ_{A_2}, contracts four rational curves R_1, R_2, R_3, R_4, respectively R_5, R_6, R_7, R_8, and sends the others four rational curves in lines, then R_i, $i = 1, \ldots, 8$ is an even set on X.*

Proof. One can prove that $\mathcal{L}'_{24}$ is primitively embedded in $NS(S)$ as in the Propositions 5.1 and 5.2. $\qquad\square$

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