

NON-SYMPLECTIC AUTOMORPHISMS OF ORDER MULTIPLE OF SEVEN ON K3 SURFACES

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ABSTRACT. In this paper we present a complete classification of non-symplectic automorphisms of K3 surfaces whose order is a multiple of seven by describing the topological type of their fixed locus. In the case of purely non-symplectic automorphisms, we provide new results for order 14 and alternative proofs for orders 21, 28 and 42, so that we can unify in the same paper the results on these automorphisms. For each of these orders we also consider not purely non-symplectic automorphisms and obtain a complete characterization of their fixed loci. Several results of our paper were obtained independently in the recent paper [12] by Brandhorst and Hofmann, but the methods used in the two papers are completely different.

1. INTRODUCTION

An automorphism of a K3 surface induces an action on the one-dimensional space of holomorphic 2-forms on the surface, so there are two kinds of automorphisms of K3 surfaces: symplectic and non-symplectic ones. The automorphism is called *symplectic* if the induced action on the 2-form is trivial. Otherwise, it is called *non-symplectic*, in which case one distinguishes between *purely non-symplectic* automorphisms, meaning the action on the volume form is given by multiplication by a primitive root of unity, and *not purely non-symplectic* automorphisms, meaning some (non-trivial) power of the automorphism is symplectic.

It is known [27, Theorem 0.1] that the rank of the transcendental lattice of a K3 surface carrying a purely non-symplectic automorphism of order n is divisible by the Euler totient function of n , which implies $\varphi(n) \leq 20$. Moreover, all positive integers $n \neq 60$ satisfying such property occur as orders of purely non-symplectic automorphisms by [24, Main Theorem 3]. For each possible n , it is thus a natural and fundamental problem to obtain a complete classification of non-symplectic automorphisms of order n in terms of their fixed locus, and many people have contributed to the development of the subject.

A classification of non-symplectic automorphisms of prime order p was completed by Nikulin in [28] when $p = 2$, and by Artebani, Sarti and Taki in [7], [29], [9] when $p > 2$. The study of non-symplectic automorphisms of composite order is much more intricate, one of the reasons being that lattice theory works less well in these cases. Results for some possible orders can be found in [8], [13], [5], [6], [11], [1], [3], [2] and [12], among others.

In this paper, we contribute to the classification of non-symplectic automorphisms of orders that are multiples of seven by describing the topological type of their fixed locus. For purely non-symplectic automorphisms, we provide new results for order 14 and alternative proofs for orders 21, 28 and 42, recovering the results in [11]. Observe that 42 is the maximum possible order which is a multiple of seven. We also consider the not purely non-symplectic case and obtain a complete characterization for each possible order, which is completely new.

Our main result in the case of purely non-symplectic automorphisms is summarized below in Theorem A. We point the reader to Propositions 3.1, 4.1, 5.1 and 6.1 for the details.

Theorem A. *Let σ_n be a purely non-symplectic automorphism of order $n \in \{14, 21, 28, 42\}$ on a K3 surface X . Then the fixed locus of σ_n is not empty, and $\text{Fix}(\sigma_n)$ and the fixed loci of its powers are described by Tables 2, 6, 8, 9.*

We observe that an analogue of Theorem A has also been obtained independently, and via a different method, by Brandhorst and Hofmann in [12, Theorem 1.4]. The approach we use here is more geometric. In particular, we show the different possibilities for the fixed loci are indeed realizable by explicitly constructing examples that have the desired topological types. Examples

Date: November 21, 2022.

2010 Mathematics Subject Classification. 14J28, 14J10, 14J50.

Key words and phrases. K3 surfaces, automorphisms.

in Sections 4, 5 and 6 were already given in [11], but we provide here a different proof and a more detailed description.

In the not purely non-symplectic case, we also consider automorphisms of orders 14, 21, 28, and 42 and again we provide a complete classification. In each case we show that not every power of the automorphism can be symplectic and our main result in this direction is given by Theorem B below. The details are explained in Section 7.

Theorem B. *Let σ_n be a non-symplectic automorphism of order $n \in \{14, 21, 28, 42\}$ on a K3 surface X .*

- (i) *If $n = 14$, then both its square and its 7-th power can be symplectic. In each case, the fixed loci of σ_{14} and its powers are described in Propositions 7.2 and 7.5.*
- (ii) *If $n = 21$, its cube is necessarily non-symplectic, whereas σ_{21}^7 can be symplectic and the fixed loci of σ_{21} and its powers in this case are described in Proposition 7.10.*
- (iii) *If $n = 28$, then σ_n is necessarily purely non-symplectic.*
- (iv) *If $n = 42$, then every power σ_{42}^k is necessarily non-symplectic except for $k = 14$. In this case, the fixed loci of σ_{42} and its powers are described in Proposition 7.13.*

To prove Theorems A and B we apply a unified approach to all orders. A central idea consists in observing that the study of the fixed locus of σ_n can be reduced to a local analysis of the fixed loci of (some of) its powers. In particular, we rely on the classification result for order 7 in [9], and some of the tools we use are the Hodge index theorem and the holomorphic and topological Lefschetz formulas (2) and (3). Moreover, the examples we construct are often given in terms of elliptic fibrations (see Definition 2.3).

The structure of the paper is the following: Section 2 is devoted to presenting background material, introducing notation and recalling some standard results on automorphisms on K3 surfaces. In Section 3 we classify purely non-symplectic automorphisms of order 14 in terms of the topological type of their fixed locus. Our main result is outlined in Proposition 3.1 and Tables 2 and 4. Moreover, we show the different possibilities indeed occur giving explicit examples. Section 4 (resp. 5, 6) provides the classification of purely non-symplectic automorphism of order 21 (resp. 28, 42). The topology of their fixed locus is summarized in Tables 6 (resp. 8, 9). In Section 7 we then consider the case of not purely non-symplectic automorphisms and obtain a complete characterization for each possible order (14, 21, 28 and 42). Finally, in Section 8 we study the Néron-Severi lattice of a K3 surface carrying a purely non-symplectic automorphism of order a multiple of seven.

All computations in this paper are carried out using MAGMA [10] and we work over \mathbb{C} throughout.

ACKNOWLEDGEMENTS

This work started during the workshop Women in Algebraic Geometry, held virtually at ICERM Providence in July 2020. We thank ICERM for this opportunity. We thank S. Brandhorst for useful comments. P.C. has been partially supported by Proyecto Fondecyt Iniciación N.11190428 and Proyecto Fondecyt Regular N.1200608. P.C. and A.S. have been partially supported by Programa de cooperación científica ECOS-ANID C19E06 and Math AmSud-ANID 21 Math 02. A.S. was partially supported by ANR project ANR-20-CE40-0026-0.

2. BACKGROUND AND NOTATION

A K3 surface is a compact, complex surface which is simply connected and has trivial canonical bundle. An automorphism of finite order on a K3 surface is called non-symplectic if it acts non-trivially on the volume form. The automorphism is called purely non-symplectic if the action is given by multiplication by a primitive n -th root of unity.

Notation 2.1. *Throughout the paper we will adopt the following notations:*

- ω_X will denote a nowhere vanishing holomorphic 2-form on a K3 surface X ;
- ζ_n will denote an n -th root of unity;
- σ_n will denote an automorphism of (finite) order n on a K3 surface X . In particular, given σ_n , if m divides n , we will also denote $\sigma_n^{\frac{n}{m}}$ by σ_m ;
- U will denote the unique even unimodular hyperbolic lattice of rank 2;
- $A_i, D_j, E_6, E_7, E_8, i \geq 1, j \geq 4$ will denote the even, negative definite lattices associated with the Dynkin diagrams of the corresponding types;

- K_7 will denote the lattice of rank 2 whose bilinear form is given by the matrix $\begin{pmatrix} -4 & 1 \\ 1 & -2 \end{pmatrix}$;
- given a lattice L , $L(k)$ will denote the lattice having as bilinear form the one on L multiplied by k , $k \in \mathbb{Z}$;
- $S(\sigma_n)$ will denote the invariant lattice: $\{x \in H^2(X, \mathbb{Z}) \mid (\sigma_n)^*(x) = x\}$, which is primitively embedded in the Néron–Severi lattice $\text{NS}(X)$ of the surface X , by [27].

Given any purely non-symplectic automorphism σ_n with $n \geq 3$, by the Hodge Index Theorem, its fixed locus $\text{Fix}(\sigma_n)$ consists of a disjoint union of smooth curves and isolated points:

$$(1) \quad \text{Fix}(\sigma_n) = C_{g_n} \sqcup R_1 \sqcup \dots \sqcup R_{k_n} \sqcup \{p_1, \dots, p_{N_n}\}$$

where C_{g_n} is a smooth curve of genus $g_n \geq 0$ and R_i are rational curves and p_i are isolated fixed points, whose total number is N_n .

By [27], the action of σ_n can be locally linearized and diagonalized around a fixed point so that σ_n acts as multiplication by the matrix

$$A_{i,n} := \begin{bmatrix} \zeta_n^{1+i} & 0 \\ 0 & \zeta_n^{n-i} \end{bmatrix} \text{ such that } 0 \leq i < n,$$

and we say that such a fixed point is of type $A_{i,n}$. The total number of fixed points of type $A_{i,n}$ will be denoted by $m_{i,n}$. Observe that if $i = 0$, one of the eigenvalues of $A_{0,n}$ is 1, thus the fixed point is not isolated but it belongs to a fixed curve.

We may use the holomorphic Lefschetz formula for σ_n to compute the Lefschetz number $L(\sigma_n)$ in two ways. First of all, we have:

$$L(\sigma_n) = \sum_{i=0}^2 (-1)^i \text{tr}(\sigma_n^*|_{H^i(X, \mathcal{O}_X)}) = 1 + \zeta_n^{n-1}$$

where we are assuming $\sigma_n^* \omega_X = \zeta_n \omega_X$. On the other hand, we have:

$$L(\sigma_n) = \sum_{i=1}^{n-2} \frac{m_{i,n}}{\det(I - \sigma_n^*|_{T_X})} + \alpha_n \frac{1 + \zeta_n}{(1 - \zeta_n)^2}.$$

where $\alpha_n := \sum_{C \subset \text{Fix}(\sigma_n)} (g(C) - 1)$. Equating these two expressions we obtain a linear system of equations that allows us to determine the possible values for $m_{i,n}$ and α_n :

$$(2) \quad 1 + \zeta_n^{n-1} = \sum_{i=1}^{n-2} \frac{m_{i,n}}{(1 - \zeta_n^{1+i})(1 - \zeta_n^{n-i})} + \alpha_n \frac{1 + \zeta_n}{(1 - \zeta_n)^2}.$$

The topological Lefschetz formula, in turn, can be used to compute the Euler characteristic of the fixed locus of σ_n :

$$(3) \quad \chi_n \doteq \chi(\text{Fix}(\sigma_n)) = 2 + \text{tr}(\sigma_n^*|_{H^2(X, \mathbb{R})}).$$

Both (2) and (3) will be used extensively throughout the paper in order to perform a local analysis of the action of non-symplectic automorphisms with order a multiple of seven. It is this local analysis that will lead us to a complete classification of such automorphisms, in terms of the topological type of their fixed locus.

We will also make extensive use of the already known classification of non-symplectic automorphisms of order seven, given by Theorem 2.2 below:

Theorem 2.2. [9, Section 6] *If X is a K3 surface and σ_7 a non-symplectic automorphism of order 7, then the possibilities for the fixed locus of σ_7 and the invariant lattice $S(\sigma_7)$ are listed in Table 1 and all cases exist.*

	$m_{1,7}$	$m_{2,7}$	$m_{3,7}$	g_7	k_7	$S(\sigma_7)$
A	2	1	0	1	0	$U \oplus K_7$
†	2	1	0	-	-	$U(7) \oplus K_7$
B	4	3	1	1	1	$U \oplus E_8$
C	4	3	1	0	0	$U(7) \oplus E_8$
D	6	5	2	0	1	$U \oplus E_8 \oplus A_6$

TABLE 1. Order 7

For each possibility in our classification, the existence of a K3 surface carrying an automorphism with fixed locus having the desired topological type will then be obtained via the construction of explicit examples. Most of the examples will arise from elliptic fibrations. Therefore, we also recall some generalities about elliptic K3 surfaces, and we refer the reader to [25] for details.

Definition 2.3. *An elliptic fibration on a projective surface X consists of a surjective proper morphism $\pi : X \rightarrow C$ (with connected fibers) such that the generic fiber is a smooth curve of genus one, and we further assume there exists a section $s : C \rightarrow X$ (i.e. $\pi \circ s = \text{id}_C$).*

A K3 surface X admits an elliptic fibration if and only if there exists a primitive embedding of the hyperbolic lattice U into $NS(X)$, the Néron–Severi lattice of the surface. Any elliptic fibration can be reconstructed from its Weierstrass model, and in the case of K3 surfaces such model is given by an equation of the form:

$$(4) \quad y^2 = x^3 + A(t)x + B(t), \quad t \in \mathbb{P}^1$$

where $A(t)$ and $B(t)$ are polynomials in $\mathbb{C}[t]$ of degrees 8 and 12, respectively.

Given an elliptic fibration, a chosen section $s : C \rightarrow X$ is called the zero section; and one identifies the map s with the curve $s(C)$ on X . In the model given by (4), the zero section is $t \mapsto (0 : 1 : 0)$.

We further observe that, using (4), the volume form can be written locally as

$$\frac{dx \wedge dt}{2y}$$

Moreover, the discriminant of the fibration is the polynomial of degree 24:

$$\Delta(t) = 4A(t)^3 + 27B(t)^2$$

and each zero of $\Delta(t)$ corresponds to a singular fiber of the fibration. The possible singular fibers have been classified by Néron and Kodaira [26], [20], [21].

3. ORDER 14

Let σ_{14} be a purely non-symplectic automorphism of order 14. As described in Section 2, the local actions of σ_{14} at fixed points are of seven types. Points of type $A_{0,14}$ lie on a fixed curve, and isolated fixed points are of type $A_{i,14}$ for $i = 1, \dots, 6$. Thus, the fixed locus of σ_{14} can contain both fixed curves and isolated fixed points of six different types. The goal of this section is to prove the following classification result:

Proposition 3.1. *The fixed locus of a purely non-symplectic automorphism of order 14 on a K3 surfaces is not empty and it consists of either:*

- (i) *The union of N_{14} isolated points, where $N_{14} \in \{3, 4, 5, 6, 7\}$; or*
- (ii) *The disjoint union of a rational curve and N_{14} isolated points, where $N_{14} \in \{6, 11, 12\}$.*

Moreover, all these possibilities occur, and in each case $\sigma_7 \doteq \sigma_{14}^2$ fixes at least one curve. A more detailed description is given in Tables 2 and 5 below, where σ_2 denotes the involution σ_{14}^7 .

	$\text{Fix}(\sigma_{14})$	$\text{Fix}(\sigma_7)$	$\text{Fix}(\sigma_2)$	Example
$A1(9,1)$	$\{p_1, \dots, p_7\}$	$E \sqcup \{p_1, p_2, p_3\}$	$C_9 \sqcup R$	3.20
$A1(3,2)$	$\{p_1, \dots, p_7\}$	$E \sqcup \{p_1, p_2, p_3\}$	$C_3 \sqcup R_1 \sqcup R_2$	3.21
$A2$	$\{p_1, \dots, p_5\}$	$E \sqcup \{p_1, q_1, q_2\}$	C_9	3.22
$B3$	$R \sqcup \{p_1, \dots, p_{12}\}$	$E \sqcup R \sqcup \{p_1, \dots, p_8\}$	$C_6 \sqcup R \sqcup R_1 \sqcup \dots \sqcup R_4$	3.23
$C1(6,1)$	$\{p_1, \dots, p_6\}$	$R \sqcup \{p_1, \dots, p_4, q_1, \dots, q_4\}$	$C_6 \sqcup R'$	3.24
$C1(7,2)$	$\{p_1, \dots, p_6\}$	$R \sqcup \{p_1, \dots, p_4, q_1, \dots, q_4\}$	$C_7 \sqcup R_1 \sqcup R_2$	3.25
$C1(0,2)$	$\{p_1, \dots, p_6\}$	$R \sqcup \{p_1, \dots, p_4, q_1, \dots, q_4\}$	$R_1 \sqcup R_2 \sqcup R_3$	3.26
$C2$	$\{p_1, \dots, p_4\}$	$R \sqcup \{p_1, p_2, q_1, \dots, q_6\}$	C_6	3.27
$C3$	$R \sqcup \{p_1, \dots, p_6\}$	$R \sqcup \{p_1, \dots, p_6, q_1, q_2\}$	$C_6 \sqcup R \sqcup R'$	3.28
$D2$	$\{p_1, p_2, p_3\}$	$R_1 \sqcup R_2 \sqcup \{p_1, \dots, p_{13}\}$	C_3	3.29
$D3$	$\{p_1, \dots, p_7\}$	$R_1 \sqcup R_2 \sqcup \{p_1, p_2, p_3, q_1, \dots, q_{10}\}$	$C_3 \sqcup R' \sqcup R''$	3.30
$D8$	$R \sqcup \{p_1, \dots, p_{11}\}$	$R \sqcup R' \sqcup \{p_1, \dots, p_9, q_1, \dots, q_4\}$	$C_3 \sqcup R \sqcup R_1 \sqcup \dots \sqcup R_4$	3.31

TABLE 2. Order 14

The proof of Proposition 3.1 is done in several steps. First, in Section 3.1 we use formulas (2) and (3) in order to generate Table 4, which provides a list of possibilities for the fixed locus of σ_{14} and its powers. In Section 3.2 we then exclude many of these possibilities using geometric arguments, and produce a new table - Table 5. Finally, in Section 3.3 we show all the remaining cases listed in Table 5 are indeed admissible by constructing explicit examples that have the desired topological types.

3.1. Generation of table of possibilities. Since σ_{14} is purely non-symplectic, its square $\sigma_7 := \sigma_{14}^2$ is a non-symplectic automorphism of order 7. Moreover, $\text{Fix}(\sigma_{14}) \subseteq \text{Fix}(\sigma_7)$ and in particular each curve contained in $\text{Fix}(\sigma_{14})$ is also contained in $\text{Fix}(\sigma_7)$.

Now, for all $i = 1, \dots, 6$ we have that $(A_{i,14})^2 = A_{j,7}$ for some $j \in \{0, 1, 2, 3\}$. For instance, $A_{1,14}^2 = A_{1,7}$. Thus, fixed points of σ_{14} that are of type $A_{1,14}$ are also points of type $A_{1,7}$ for σ_7 . Similarly:

- points of type $A_{5,14}$ for σ_{14} are of type $A_{1,7}$ for σ_7 ,
- points of types $A_{2,14}$ and $A_{4,14}$ for σ_{14} are of type $A_{2,7}$ for σ_7 , and
- points of type $A_{3,14}$ for σ_{14} are of type $A_{3,7}$ for σ_7 .

In particular, the following inequalities hold:

$$(5) \quad \begin{cases} m_{1,14} + m_{5,14} & \leq m_{1,7} \\ m_{2,14} + m_{4,14} & \leq m_{2,7} \\ m_{3,14} & \leq m_{3,7} \end{cases}$$

And we further observe the following:

Remark 3.2. Note that $A_{6,14}^2 = A_{0,7}$, which shows that points of type $A_{6,14}$ lie on a curve fixed by σ_7 . Therefore, if $m_{6,14} \neq 0$, then there are curves in $\text{Fix}(\sigma_7)$ which are not in $\text{Fix}(\sigma_{14})$.

Remark 3.3. A rational curve R invariant for an automorphism σ_n is either pointwise fixed or R admits two isolated fixed points. In the latter case, the points are of consecutive types, i.e., if one point is of type $A_{i,n}$, then the other is of type $A_{i+1,n}$. If $n = 14$, as in [8, Lemma 4], one can prove that, given a tree of rational curves invariant for σ_{14} , the distribution of types of isolated fixed points is as shown in Figure 1. This can be done in a similar way for $n = 21, 28, 42$.

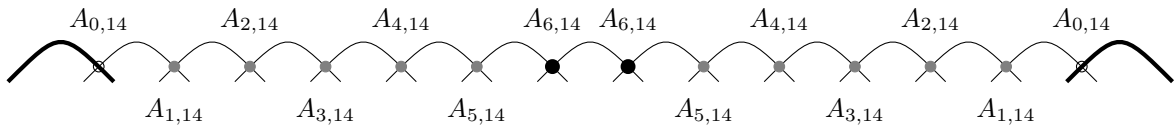


FIGURE 1. Actions of σ_{14} and σ_7 on a tree of rational curves. Thin curves are invariant but not pointwise fixed. Thick curves are pointwise fixed by σ_{14} . The gray points are isolated fixed points for both σ_{14} and σ_7 , and the two black points in the middle lie on a curve fixed by σ_7 only.

As a consequence, from (5) and the previous remarks, if we apply formula (2) to σ_{14} we obtain the following linear system of equations:

$$(6) \quad \begin{cases} m_{1,14} & = 4\alpha_{14} - 2m_{4,14} + m_{5,14} \\ m_{2,14} & = 1 - 2m_{5,14} + 3m_{4,14} \\ m_{6,14} & = 8m_{4,14} + 4 - 2m_{3,14} - 2\alpha_{14} - 4m_{5,14} \end{cases}$$

This allows us to prove the following two Lemmas.

Lemma 3.4. *The value of α_{14} is either 0 or 1.*

Proof. Since $\text{Fix}(\sigma_{14}) \subset \text{Fix}(\sigma_7)$, a curve that is pointwise fixed by σ_{14} must be contained in $\text{Fix}(\sigma_7)$. Thus, according to Table 1, we must have $\alpha_{14} \in \{0, 1, 2\}$. Assume $\alpha_{14} = 2$. Then σ_{14} fixes at least two rational curves. Therefore the fixed locus under σ_7 is described by the last row of Table 1, and both rational curves in $\text{Fix}(\sigma_7)$ are fixed by σ_{14} . By Remark 3.2, $m_{6,14} = 0$. But plugging in $\alpha_{14} = 2$ and $m_{6,14} = 0$ with the inequalities (5) with the values of $m_{1,7}, m_{2,7}, m_{3,7}$ from last line of Table 1 into (6) yields an unsolvable system. Therefore α_{14} can only be equal 0 or 1. \square

Lemma 3.5. *There is no purely non-symplectic automorphism σ_{14} of order 14 such that the fixed locus of σ_7 is described by the second row of Table 1.*

Proof. In this case, no curves are fixed by σ_7 and hence no curves are fixed by σ_{14} . Thus $\alpha_{14} = 0$. But a MAGMA calculation shows that in this case a solution of (6) would have $m_{6,14} = 4$, which would imply that σ_{14} fixes a curve, and so this case cannot occur. \square

In fact we can completely describe what are the possible solutions to (6), i.e. what are the possibilities for the vector $\underline{m} = (m_{1,14}, m_{2,14}, m_{3,14}, m_{4,14}, m_{5,14}, m_{6,14})$ and for the value of α_{14} . Organizing the possibilities according to the fixed locus of $\sigma_7 = \sigma_{14}^2$, we prove:

Proposition 3.6. *If σ_{14} is a purely non-symplectic automorphism of order 14 on a K3 surface, then the possible vectors $\underline{m} = (m_{1,14}, m_{2,14}, m_{3,14}, m_{4,14}, m_{5,14}, m_{6,14})$ satisfying (6) are listed in Table 3 below. The symbol $*$ means that the action on the elliptic curve E is a translation.*

In particular, we obtain a list of possibilities for the fixed locus of σ_{14} .

	$m_{1,14}$	$m_{2,14}$	$m_{3,14}$	$m_{4,14}$	$m_{5,14}$	$m_{6,14}$	α_{14}	curves fixed by σ_{14}
A1	0	0	0	1	2	4	0	\emptyset
A2	0	1	0	0	0	4	0	\emptyset
B1	0	0	1	1	2	2	0	E
B1*	0	0	1	1	2	2	0	\emptyset
B2	0	1	1	0	0	2	0	E
B2*	0	1	1	0	0	2	0	\emptyset
B3	3	2	1	1	1	4	1	R
B4	4	1	1	0	0	0	1	$R \sqcup E$
B4*	4	1	1	0	0	0	1	R
C1	0	0	1	1	2	2	0	\emptyset
C2	0	1	1	0	0	2	0	\emptyset
C3	4	1	1	0	0	0	1	R
D1	0	0	2	1	2	0	0	\emptyset
D2	0	1	2	0	0	0	0	\emptyset
D3	0	0	0	1	2	4	0	\emptyset
D4	0	1	0	0	0	4	0	\emptyset
D5	4	0	0	1	2	2	1	R
D6	4	1	0	0	0	2	1	R
D7	3	1	2	2	3	2	1	R
D8	3	2	2	1	1	2	1	R

TABLE 3.

Proof of Proposition 3.6. We consider each row of Table 1:

Case A This corresponds to the case in which the fixed locus of σ_7 is described by the first row of Table 1 and $\text{Fix}(\sigma_7)$ consists of a genus one curve E , so we only need to determine whether σ_{14} itself fixes E . In both cases, $\alpha_{14} = 0$ and by (5) $m_{3,14} = 0$. A MAGMA calculation shows that the only vectors \underline{m} which satisfy (6) with $\alpha_{14} = m_{3,14} = 0$ are $(0, 0, 0, 1, 2, 4)$ and $(0, 1, 0, 0, 0, 4)$. By Remark 3.2, σ_{14} does not fix E .

Case B When $\text{Fix}(\sigma_7)$ is described by the third row of Table 1, the automorphism σ_7 fixes a genus one curve E and a rational curve R . We analyze this case by considering the possibilities for α_{14} and $m_{6,14}$.

First, suppose $\text{Fix}(\sigma_{14})$ contains no curves, so σ_{14} fixes neither R nor E ; in this case, $\alpha_{14} = 0$. Since σ_{14} acts as an involution on E , by the Riemann-Hurwitz formula it has either four fixed points (coming from $P \mapsto -P$ after a choice of point at infinity) or no fixed points (coming from $P \mapsto P + T$ where T is a 2-torsion point). The action on R has 2 fixed points, so $m_{6,14}$ is either 6 or 2. A MAGMA calculation applying the constraints from (6) shows that the possibilities for \underline{m} are $(0, 0, 1, 1, 2, 2)$ and $(0, 1, 1, 0, 0, 2)$.

Second, suppose that $E \subset \text{Fix}(\sigma_{14})$ and $R \not\subset \text{Fix}(\sigma_{14})$; in this case, $\alpha_{14} = 0$ and $m_{6,14} = 2$. The possibilities for \underline{m} in this case are $(0, 0, 1, 1, 2, 2)$ and $(0, 1, 1, 0, 0, 2)$.

Next, if $R \subset \text{Fix}(\sigma_{14})$ and $E \not\subset \text{Fix}(\sigma_{14})$, σ_{14} fixes either none or four points on E , so $\alpha_{14} = 1$ and $m_{6,14} = 0$ or 4, and the possibilities for \underline{m} are $(3, 2, 1, 1, 1, 4)$ and $(4, 1, 1, 0, 0, 0)$.

Lastly, if $E \sqcup R \subset \text{Fix}(\sigma_{14})$, all curves fixed under σ_7 are also fixed under σ_{14} , so $\alpha_{14} = 1$ and $m_{6,14} = 0$, and the only possibility is $\underline{m} = (4, 1, 1, 0, 0, 0)$.

Case C In this case, the only curve fixed by σ_7 is a rational curve R . If σ_{14} does not fix R , then $\alpha_{14} = 0$ and $m_{6,14} = 2$ and the solutions of (6) for \underline{m} are $(0, 0, 1, 1, 2, 2)$ and $(0, 1, 1, 0, 0, 2)$.

On the other hand, if σ_{14} fixes R , then $\alpha_{14} = 1$ and $m_{6,14} = 0$, and the only possibility for \underline{m} is $(4, 1, 1, 0, 0, 0)$.

Case D Finally, if the fixed locus of σ_7 is described by the last row of Table 1, the curves fixed by σ_7 are two rational curves $R_1 \sqcup R_2$. First, suppose neither R_1 nor R_2 is fixed by σ_{14} , thus $\alpha_{14} = 0$. Then, either σ_{14} exchanges R_1 and R_2 , or σ_{14} acts nontrivially on R_1 and R_2 . If R_1 and R_2 are exchanged (hence fixing no points on either curve), then $m_{6,14} = 0$ and the possibilities for \underline{m} are $(0, 0, 2, 1, 2, 0)$ and $(0, 1, 2, 0, 0, 0)$. Otherwise, there are a total of 4 points fixed on these curves, so $m_{6,14} = 4$ and the possibilities for \underline{m} are $(0, 0, 0, 1, 2, 4)$ and $(0, 1, 0, 0, 0, 4)$.

If σ_{14} fixes one rational curve and acts nontrivially on the other, $\alpha_{14} = 1$ and $m_{6,14} = 2$. Possibilities for \underline{m} are $(0, 0, 1, 1, 2, 2)$, $(0, 1, 1, 0, 0, 2)$, $(3, 2, 2, 1, 1, 2)$ and $(3, 1, 2, 2, 3, 2)$. By Lemma 3.4, σ_{14} does not fix both R_1 and R_2 .

□

We also observe the following:

Proposition 3.7. *If σ_{14} is a purely non-symplectic automorphism on a K3 surface such that $\sigma_7 = \sigma_{14}^2$ is of type B (see Table 1), then σ_{14} is of type B3.*

Proof. Let X be a K3 surface and σ_{14} a purely non-symplectic automorphism of order 14 acting on X . Assume we are in case B so that σ_7 fixes a genus 1 curve, a rational curve and eight isolated points. By [9, Thm. 6.3] X admits an elliptic fibration with a reducible fiber of types II^* at $t = \infty$, a smooth fiber at $t = 0$ and 14 singular fibers of type I_1 . The automorphism σ_7 fixes the fiber over 0 and the central component of the fiber II^* ; all eight isolated points of σ_7 lie on the fiber II^* .

Since σ_7 fixes the genus one curve, the fibration is σ_7 -invariant. Thus the fibers over $t = 0$ and $t = \infty$ are preserved. The II^* fiber does not admit a reflection, and so we can conclude that the central component must be fixed by σ_{14} . Moreover, the eight isolated fixed points of σ_7 are also isolated and fixed by σ_{14} . Table 3 shows that the only case with $N_{14} \geq 8$ is case B3. We also observe that, because $m_{6,14} = 4$, the automorphism σ_{14} acts as an involution on the genus one curve with four fixed points. □

Now, in order to better understand the different fixed loci listed in Table 3, the next step in our approach consists in further studying the fixed locus of the involution σ_{14}^7 , and the eigenspaces of σ_{14}^* in $H^2(X, \mathbb{C})$. We use the following notation:

$$d_i := \dim H^2(X, \mathbb{C})_{\zeta_i}, i = 1, 2, 7, 14.$$

In particular, we have

$$22 = 6d_{14} + 6d_7 + d_2 + d_1.$$

Remark 3.8. *Observe that $\text{rk } S(\sigma_{14}) = d_1$ and $\text{rk } S(\sigma_7) = d_2 + d_1$ and $\text{rk } S(\sigma_2) = 6d_7 + d_1$.*

Moreover, by applying the topological Lefschetz formula (3) to the fixed loci of σ_{14} and its powers, we obtain the following system of equations:

$$(7) \quad \begin{cases} \chi_{14} := \chi(\text{Fix}(\sigma_{14})) = 2 + d_{14} - d_7 - d_2 + d_1 \\ \chi_7 := \chi(\text{Fix}(\sigma_7)) = 2 - d_{14} - d_7 + d_2 + d_1 \\ \chi_2 := \chi(\text{Fix}(\sigma_2)) = 2 - 6d_{14} + 6d_7 - d_2 + d_1 \end{cases}$$

Using (7) and Table 3 we can thus obtain a list of possibilities for (d_{14}, d_7, d_2, d_1) as well as the corresponding Euler characteristics $(\chi_{14}, \chi_7, \chi_2)$. We present our results in Table 4 below.

	N_{14}	α_{14}	χ_{14}	χ_7	χ_2	(d_{14}, d_7, d_2, d_1)	Possible (g_2, k_2)
A1	7	0	7	3	-14 0	(3,0,1,3) (2,1,0,4)	(8,0), (9,1) (1,0), (2,1), (3,2), (4,3), (5,4), (6,5)
A2	5	0	5	3	-16 -2 12	(3,0,2,2) (2,1,1,3) (1,2,0,4)	(9,0), (10,1) (2,0), (3,1), (4,2), (5,3), (6,4) (0,5), (1,6), (2,7)
B3	12	1	14	10	0	(2,0,0,10)	(1,0), (2,1), (3,2), (4,3), (5,4), (6,5)
C1	6	0	6	10	-8 6	(2,0,4,6) (1,1,3,7)	(5,0), (6,1), (7,2) (0,2), (1,3), (2,3), (3,5)
C2	4	0	4	10	-10 4	(2,0,5,5) (1,1,4,6)	(6,0), (7,1) (0,1), (1,2), (2,3), (3,4), (4,5)
C3	6	1	8	10	-6 8	(2,0,3,7) (1,1,2,8)	(4,0), (5,1), (6,2) (0,3), (1,4), (2,5), (3,6)
D1	5	0	5	17	-2	(1,0,7,9)	(2,0), (3,1), (4,2), (5,3), (6,4)
D2	3	0	3	17	-4	(1,0,8,8)	(3,0), (4,1), (5,2), (6,3)
D3	7	0	7	17	0	(1,0,6,10)	(1,0), (2,1), (3,2), (4,3), (5,4), (6,5)
D4	5	0	5	17	-2	(1,0,7,9)	(2,0), (3,1), (4,2), (5,3), (6,4)
D5	9	1	11	17	4	(1,0,4,12)	(0,1), (1,2), (2,3), (3,4), (4,5)
D6	7	1	9	17	2	(1,0,5,11)	(0,0), (1,1), (2,2), (3,3), (4,4), (5,5)
D7	13	1	15	17	8	(1,0,2,14)	(0,3), (1,4), (2,5), (3,6)
D8	11	1	13	17	6	(1,0,3,13)	(0,2), (1,3), (2,3), (3,5)

TABLE 4.

We remark that by [28], the fixed locus of a non-symplectic involution is either empty; or it consists of two disjoint elliptic curves; or

$$(8) \quad \text{Fix}(\sigma_2) = C_{g_2} \sqcup R_1 \sqcup \dots \sqcup R_{k_2}$$

where C_{g_2} is a smooth curve of genus $g_2 \geq 0$ and R_i are rational curves, and all possibilities for the pair of invariants (g_2, k_2) are classified (see for example [9, Figure 1]).

In our case, it follows from (6) that $\text{Fix}(\sigma_2)$ cannot be empty. Any possible solution to (6) gives us that $\text{Fix}(\sigma_{14})$ contains at least one fixed point. In fact, this also implies $\text{Fix}(\sigma_2)$ cannot be the union of two elliptic curves either. If the latter occurs, then the action of σ_{14} on each elliptic curve would be without fixed points. Since $\text{Fix}(\sigma_{14}) \subset \text{Fix}(\sigma_2)$, again we would have no fixed points in $\text{Fix}(\sigma_{14})$, contradicting (6). As a consequence, for each line of Table 4, we know that $\text{Fix}(\sigma_2)$ is of the form (8). Moreover, there is more than one possible pair of invariants (g_2, k_2) .

3.2. Excluding cases. We will now show many cases of Table 4 can actually be excluded for geometric reasons. We prove a series of Lemmas in this direction.

Notation 3.9. *In what follows, we will use the following notation: $A1(8,0)$ means that $\text{Fix}(\sigma_7)$ is as in line A of Table 1, $\text{Fix}(\sigma_{14})$ is described in the line A1 of Table 4 and $(g_2, k_2) = (8,0)$. Similarly for all other cases.*

Remark 3.10. *Observe that cases $B3(1,0)$ and $D6(0,0)$ are not admissible because in both cases, the fixed locus $\text{Fix}(\sigma_{14})$ contains a rational curve while $\text{Fix}(\sigma_2)$ does not.*

Remark 3.11. *$\text{Fix}(\sigma_2)$ does not contain a curve of genus 2, 4 or 5. This is a direct consequence of the following Lemma.*

Lemma 3.12 ([19]). *Let C be a curve of genus $g \geq 2$ that admits an automorphism of prime order q where $q > g$. Then either $q = g + 1$ or $q = 2g + 1$.*

Lemma 3.13. *The following cases are not admissible:*

$$\begin{aligned} & A1(6,5), A1(8,0), A1(1,0), A1(6,5), A2(6,4), A2(0,5), A2(1,6), B3(3,2), \\ & C1(3,5), C2(3,4), C3(3,6), D1(6,4), D2(6,3), D3(1,0), D3(6,5), \\ & D4(6,4), D5(0,1), D5(1,2), D6(1,1), D7(0,3), D7(1,4), D8(0,2), D8(1,3). \end{aligned}$$

Proof. Consider case $A1(6,5)$. By Riemann-Hurwitz's formula, the automorphism σ_{14} acts on the curve $C_4 \subset \text{Fix}(\sigma_2)$ fixing four points, and it also acts on each of the five rational curves in $\text{Fix}(\sigma_2)$, fixing two points on each. Therefore σ_{14} fixes a total of 14 points. By a previous computation,

the fixed locus $\text{Fix}(\sigma_{14})$ consists of seven points. Therefore this case is not admissible. A similar argument can be used to exclude the other cases. \square

Lemma 3.14. *Case C1(1,3) is not admissible.*

Proof. Let E be the elliptic curve fixed by the involution $\sigma_2 = \sigma_{14}^7$. The curve E is preserved by σ_{14} . Moreover, E is not fixed by σ_7 pointwise but it is invariant for σ_7 because we are in Case C. Thus, since E is elliptic, the automorphism σ_{14} acts as a translation on E . Let \mathcal{E} be the elliptic fibration induced by E , with fiber E over $t = 0$. Since fixed curves do not meet, the zero section is not fixed by the involution σ_2 . The involution fixes three rational curves since $k_2 = 3$ and they are contained in the fiber F_∞ over $t = \infty$. The only possible types of singular fibers that can contain three curves fixed by the involution are the types I_6 , or III^* , or I_4^* .

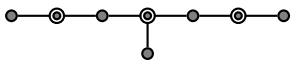


FIGURE
2. Fiber III^*



FIGURE
3. Fiber I_4^*

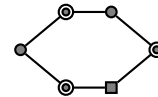


FIGURE
4. Fiber I_6

If F_∞ is of type III^* , then the three curves which are fixed by σ_2 are represented by the three double circles in Figure 2. The zero section would meet the external component of the fiber III^* and thus it would be fixed by σ_2 , which we already observed is impossible. By a similar argument, we may exclude the case when F_∞ is of type I_4^* , as shown in Figure 3.

Suppose that F_∞ is of type I_6 . By analyzing the types of points, it can be seen that one of the curves of the fiber I_6 which is not fixed by σ_2 must be fixed by σ_7 . Such a curve is represented by a square in Figure 4. Since σ_7 must preserve the fiber, this is impossible. \square

Lemma 3.15. *The following cases are not admissible:*

$$A2(10, 1), A2(3, 1), C2(1, 2), C2(0, 1), C3(1, 4), C3(0, 3), D4(3, 1), D5(3, 4), D6(3, 3).$$

Proof. Observe that in Case A2(10,1), $\text{Fix}(\sigma_2) = C_{10} \cup R$, where C_{10} a curve of genus 10 and R a rational curve, and neither of these curves are fixed by σ_{14} . The automorphism σ_{14} fixes five isolated points, two of which lie on R . As observed in Remark 3.3, isolated points on a rational curve are of consequent types but this is in contradiction with the types of points for A2 (see Table 3). The other cases can be excluded by a similar argument. \square

Lemma 3.16. *Suppose that the involution σ_2 fixes a curve C_7 of genus seven. Then the curve C_7 contains two fixed points by σ_{14} , which cannot be of the same type.*

Proof. First, note that σ_{14} acts with order seven on C_7 . Thus, by Riemann-Hurwitz it has exactly two fixed points, which we call p and q .

Considering the line bundle L associated to $8p$, by Riemann-Roch we have $h^0(C_7, L) \geq 2$ so that we obtain a finite (surjective) morphism $f : C_7 \rightarrow \mathbb{P}^1$ of degree $d \leq 8$. Now, because σ fixes p , σ and f induce an automorphism $\tilde{\sigma}$ (of order 7) on \mathbb{P}^1 . This automorphism has two fixed points, say \tilde{p} and \tilde{q} , and we must have (up to relabeling) $f^{-1}(\tilde{p}) = p$ and $f^{-1}(\tilde{q}) = q$. Moreover, we can assume $\tilde{p} = (0 : 1)$ and $\tilde{q} = (1 : 0)$.

We can thus choose local coordinates z on \mathbb{P}^1 centered on \tilde{p} so that the action of $\tilde{\sigma}$ on \tilde{p} is given by multiplication by ζ_{14}^{2j} and on \tilde{q} it is given by multiplication by ζ_{14}^{14-2j} (for some j). Note that $1/z$ is then a local coordinate centered on \tilde{q} . In fact we can choose local coordinates on C_7 which are compatible with the above so that f is given by $z \mapsto z^d$ around \tilde{p} (and analogously for \tilde{q}). Using this, we see that the local action of σ on p must be given by multiplication by $\zeta_{14}^{2j/d}$ and on q it is given by multiplication by $\zeta_{14}^{(14-2j)/d}$.

The local action of σ on p and q as points in X can thus be diagonalized so that p is a point of type $A_{i,14}$ where $i = 2j/d - 1$ or $2j/d$, and q is a point of type $A_{j,14}$ where $k = (14 - 2j)/d - 1$ or $(14 - 2j)/d$. In any case, $i \not\equiv k \pmod{14}$ so that p and q cannot be of the same type. \square

As a consequence we can prove:

Lemma 3.17. *Case C2(7,1) is not admissible.*

Proof. According to Table 3, in case C2 the automorphism σ_{14} fixes exactly one point of type $A_{2,14}$, one point of type $A_{3,14}$, and two points of type $A_{6,14}$. Two of these are on the rational curve fixed by σ_2 and two are on the genus seven curve C_7 fixed by σ_2 . Since the fixed points on R must be of consecutive types (see Remark 3.3), the two points of type $A_{6,14}$ lie on C_7 . This contradicts Lemma 3.16. \square

Thanks to [12], we also prove:

Lemma 3.18. *Cases D1 and D7 are not admissible.*

Proof. By [12, Corollary 1.3], there are exactly 12 distinct deformation classes of K3 surfaces X carrying a purely non-symplectic automorphism σ of order 14. In Section 3.3, we show all 12 cases listed in Table 5 indeed occur. Therefore, it suffices to observe the different cases determine different deformation classes.

In fact, looking at the eigenvalues of the induced isometry σ^* on $H^2(X, \mathbb{Z})$ we see that different cases determine at least 11 deformation classes. With the exception of cases $C1(6, 1)$ and $C1(7, 2)$, the different cases determine 11 distinct vectors (d_{14}, d_7, d_2, d_1) (see Table 5). So we analyze these two cases separately.

By [14, Theorem 1.5.2], if (X, σ) is of type $C1(6, 1)$ and $(\tilde{X}, \tilde{\sigma})$ is of type $C1(7, 2)$, then the invariant lattices $S(\sigma^7)$ and $S(\tilde{\sigma}^7)$ do not lie in the same genus. And, since the deformation class of a pair (X, σ) is determined by the collection of genera of the lattices $S(\sigma^j)$ by [12, Theorem 1.4], we conclude these two cases indeed determine two distinct deformation classes. \square

Using Table 4 and combining Lemmas 3.13, 3.14, 3.15, 3.17 and 3.18 we have thus proved:

Proposition 3.19. *Let σ_{14} be a purely non-symplectic automorphism on a K3 surface. Then the admissible cases according to the possible fixed locus are listed in Table 5.*

	N	α_{14}	χ_{14}	χ_7	χ_2	(g_2, k_2)	(d_{14}, d_7, d_2, d_1)
A1	7	0	7	3	-14	(9, 1)	(3, 0, 1, 3)
					0	(3, 2)	(2, 1, 0, 4)
A2	5	0	5	3	-16	(9, 0)	(3, 0, 2, 2)
B3	12	1	14	10	0	(6, 5)	(2, 0, 0, 10)
C1	6	0	6	10	-8	(6, 1), (7, 2)	(2, 0, 4, 6)
					6	(0, 2)	(1, 1, 3, 7)
C2	4	0	4	10	-10	(6, 0)	(2, 0, 5, 5)
C3	6	1	8	10	-6	(6, 2)	(2, 0, 3, 7)
D2	3	0	3	17	-4	(3, 0)	(1, 0, 8, 8)
D3	7	0	7	17	0	(3, 2)	(1, 0, 6, 10)
D8	11	1	13	17	6	(3, 5)	(1, 0, 3, 13)

TABLE 5.

3.3. Realization by examples. It remains to show each case listed in Table 5 is indeed realizable. For each possibility, we construct explicit examples of K3 surfaces carrying a purely non-symplectic automorphism σ_{14} (of order 14) that has the desired type of fixed locus.

Example 3.20. (Case A1(9,1)) Consider $(X_{a,b}, \sigma_{14})$, taking $X_{a,b}$ to be the elliptic K3 surface with Weierstrass equation

$$y^2 = x^3 + (at^7 + b)x + (t^7 - 1), \quad t \in \mathbb{P}^1$$

where $a, b \in \mathbb{C}$, as in [9, Example 6.1], and letting σ_{14} be the purely non-symplectic order 14 automorphism:

$$\sigma_{14} : (x, y, t) \mapsto (x, -y, \zeta_7^4 t)$$

where ζ_7 denotes a primitive 7-th root of unity.

If a and b are generic, then $X_{a,b}$ contains a fiber of type III at $t = (1 : 0)$ and 21 singular fibers of type I_1 . One can show that the fixed locus of σ_{14} is such that $m = (0, 0, 0, 0, 1, 2, 4)$. In fact it is of type A1(9, 1). It can be described as follows: the four isolated points of type $A_{6,14}$ lie on a curve which is fixed by σ_7 , namely the fiber at $t = (0 : 1)$; the other three points lie on the fiber of type III: the tangency point, along with one other point on each of the two components. Moreover, the involution σ_2 fixes the zero section (which is rational) and the trisection (which has genus 9).

Example 3.21. (Case A1(3,2)) Consider (X, σ_{14}) , where X is the elliptic K3 surface with Weierstrass equation given by

$$y^2 = x(x^2 + (t^7 + 1)), \quad t \in \mathbb{P}^1,$$

and $\sigma_{14}: (x, y, t) \mapsto (x, -y, \zeta_7^4 t)$ is a purely non-symplectic automorphism of order 14. We note that X contains eight singular fibers of type III. The fixed locus of σ_7 is given by an elliptic curve at $t = (0 : 1)$ and three points that lie on the fiber of type III at $t = (1, 0)$. On the fiber of type III, one of the three points is the tangency point, while the remaining two lie on different components. Therefore, we are in case A (for σ_7).

The fixed locus of σ_{14} is such that $m = (0, 0, 0, 0, 1, 2, 4)$ and in fact we can check it is of type A1(3,2). On the elliptic curve σ_{14} acts as an involution and we obtain 4 fixed points there, the other 3 fixed points are again in the fiber of type III at $t = (1, 0)$ distributed as above. The involution σ_2 fixes the bisection which has genus 3, and two rational curves: the zero section and the two torsion section given by $x = y = 0$. Therefore, $(g_2, k_2) = (3, 2)$.

Example 3.22. (Case A2(9,0)) Let us consider (X, σ_{14}) the elliptic K3 surface X together with the automorphism $\sigma = \sigma_{14}$ from Example 3.21.

The translation τ given by $(x, y, t) \mapsto ((y/x)^2 - x, (y/x)^3 - y, t)$ (which is the translation by the 2-torsion section) is a symplectic involution that commutes with σ . As a consequence, the composition $\sigma' := \sigma \circ \tau$ is also a purely non-symplectic automorphism of order 14. We remain in case A for σ_7 and the fixed locus of σ' is such that $m = (0, 0, 1, 0, 0, 0, 4)$. Indeed, σ' acts as an involution on the elliptic curve E at $t = (0, 1)$ and E contains four fixed points. Due to the fact that τ has only eight fixed points, which are precisely the tangency points on the singular fibers of type III, we only have one additional fixed point lying on the fiber at $t = (1, 0)$. The involution does not fix any rational curves and therefore we are in case $(g_2, k_2) = (9, 0)$. We note that this case is also presented in [15, Section 7.2, p.19].

Example 3.23. (Case B3) Consider $(X_{a,b}, \sigma_{14})$, where we let $X_{a,b}$ be the elliptic K3 surface in Example 3.20 with $a = 0$. $X_{0,b}$ contains a fiber of type II^* at $t = (1 : 0)$, a smooth fiber at $(0 : 1)$, and 14 singular fibers of type I_1 . With the order 14 automorphism σ_{14} given in (3.20), the component of multiplicity 6 on the II^* fiber is fixed by σ and the action on the fiber over $t = (0 : 1)$ is an involution, so it has 4 fixed points. Checking types of fixed points, we find $m = (3, 2, 1, 1, 1, 4)$ with $\alpha_{14} = 1$.

Example 3.24. (Case C1(6,1)) Consider $(X_{a,b}, \sigma_{14})$ from Example 3.20, with a generic and b such that $b^3 = -\frac{27}{4}$. Then $X_{a,b}$ contains a fiber of type III at $t = (1 : 0)$, a fiber of type I_7 at $t = (0 : 1)$ and 14 singular fibers of type I_1 . In this case the fixed locus of σ_{14} is such that $m = (0, 0, 0, 1, 1, 2, 2)$. The trisection $\{y = 0\}$ is a curve of genus 6 and it is fixed by the involution, as well as the zero section. Thus the invariants of the fixed locus of the involution σ_2 are $(g_2, k_2) = (6, 1)$.

Example 3.25. (Case C1(7,2)) Let (X, σ_{14}) , the elliptic K3 surface with Weierstrass equation

$$y^2 = x^3 + 4t^4(t^7 - 1), \quad t \in \mathbb{P}^1,$$

together with the order 14 purely non-symplectic automorphism σ_{14} given by

$$\sigma_{14}(x, y, t) = (\zeta_7^4 x, -\zeta_7^6 y, \zeta_7^3 t).$$

We note that the singular fibers are of type IV^* over $t = (0 : 1)$ and type II over $t = (1 : 0)$, in addition to seven fibers of type II. The square of σ_{14} fixes the component of multiplicity 3 on the fiber of type IV^* , so this example falls under case C. The involution σ_2 acts as a reflection on this fiber, and so the fixed locus $\text{Fix}(\sigma_{14})$ only contains points. The 3-section $\{y = 0\}$ has genus seven and it is fixed by the involution, as well as the zero section and one rational component of the fiber IV^* . Thus the invariant of the fixed locus of the involution are $(g_2, k_2) = (7, 2)$. This surface appears in [11, Table 3], with a non-symplectic automorphism of a different order.

Example 3.26. (Case C1(0,2)) Let us consider (X, σ_{14}) , the elliptic K3 surface X with Weierstrass equation given by

$$y^2 = x^3 + t^2x + t^{10}, \quad t \in \mathbb{P}^1,$$

and the order 14 purely non-symplectic automorphism $\sigma_{14}: (x, y, t) \mapsto (\zeta_7 x, \zeta_7^5 y, -\zeta_7 t)$. Note that X contains a fiber of type IV at $t = (1 : 0)$, a fiber of type I_0^* at $t = (0 : 1)$ and 14 singular fibers of type I_1 . The fixed locus of σ_7 fixes one rational curve, the non-reduced component, and eight points,

so this example falls under Case C. Because the involution σ_2 fixes only three rational curves, we see that σ_{14} is of type C1 with $(g_2, k_2) = (0, 2)$.

Example 3.27. (Case C2(6,0)) Consider (X, σ_{14}) , where X is the K3 surface with equation

$$y^2 = x^7 s - t^2(t - s^2)(t - 2s^2)$$

in $\mathbb{P}(4, 2, 1, 1)_{(y,t,x,s)}$ and $\sigma_{14}: (y, t, x, s) \mapsto (-y, t, \zeta_7 x, s)$ is a purely non-symplectic automorphism of order 14. One can see that the points $(-1 : 1 : 0 : 0)$ and $(1 : 1 : 0 : 0)$ are of type A_1 . Moreover, at the point $(0 : 0 : 0 : 1)$ we have a singularity of type A_6 . Since σ_7 fixes the rational curve $C_x := \{x = 0\}$ and eight points, this example falls under Case C. The only curve fixed by the involution σ_2 is C_y , which has genus six.

Example 3.28. (Case C3) Consider (X, σ_{14}) , the elliptic surface X with Weierstrass equation

$$y^2 = x^3 + t^3(t^7 + 1), \quad t \in \mathbb{P}^1,$$

and the order 14 purely non-symplectic automorphism σ_{14}

$$\sigma_{14}(x, y, t) = (\zeta_7^2 x, -\zeta_7^3 y, \zeta_7^3 t).$$

The singular fibers consist of a type I_0^* fiber over $t = (0 : 1)$, a type IV fiber over $t = (1 : 0)$, and seven type II fibers (cusps). We call R the non-reduced component of the I_0^* fiber. The involution σ_2 fixes the zero section, the rational curve R and the 3-section C given by $y = 0$. The curve C passes through the center of the IV fiber and through the cusps, and so C has genus six by Riemann-Hurwitz. Thus the invariants of the involution are $(g_2, k_2) = (6, 2)$ which corresponds to Case C3. The fixed locus of σ_{14} consists of the curve R and six points.

Another example for Case C3 is given as follows. Let X be the K3 surface with equation

$$x^2 + y^3 z + z^7 + w^{14} = 0$$

and weights $(7, 4, 2, 1)$. Singularities can occur only at singularities of $\mathbb{P}(7, 4, 2, 1)$ and one can see that the point $(0 : 1 : 0 : 0)$ is an A_3 singularity and $(0 : \zeta_6^j : 1 : 0)$, $j = 1, 3, 5$ is an A_1 singularity.

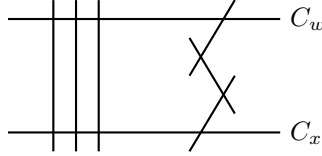


FIGURE 5. C3

After resolving the singularities, the curve $C_w := \{w = 0\}$ has genus zero, while the transform of $C_x := \{x = 0\}$ has genus six. The automorphism $\sigma_{14}: (x, y, z, w) \mapsto (x, y, z, \zeta_{14} w)$ is a purely non-symplectic automorphism of order 14 and it fixes the rational curve C_w . Its square σ_7 fixes C_w as well, so that this example falls under Case C. Moreover, the involution σ_2 fixes C_w and C_x and the central fiber of the resolution of the A_3 (another rational curve). Therefore, σ_{14} is of type C3 and the invariants of the involution are $(g_2, k_2) = (6, 2)$.

Example 3.29. (Case D2) Let (X, σ_{14}) be the K3 surface X with equation

$$y^2 = x^7 s - t^2(t - s^2)^2$$

in $\mathbb{P}(4, 2, 1, 1)_{(y,t,x,s)}$ and the order 14 purely non-symplectic automorphism $\sigma: (y, t, x, s) \mapsto (-y, t, \zeta_7 x, s)$.

The points $(-1 : 1 : 0 : 0)$ and $(1 : 1 : 0 : 0)$ are of type A_1 . Moreover, at the points $(0 : 0 : 0 : 1)$ and $(0 : 1 : 0 : 1)$, we have singularities of type A_6 . Since σ_7 fixes two rational curves C_1 and C_2 , appearing when $x = 0$, this example falls under Case D. The only curve fixed by the involution σ_2 is C_y , which has genus three. See also [15, Section 7.3].

Example 3.30. (Case D3) Consider (X, σ_{14}) , where X is the K3 surface with equation

$$x^2 = w^7 y + y^4 + z^7$$

in $\mathbb{P}(14, 7, 4, 3)_{(x,y,z,w)}$ given in [4], and σ_{14} the order 14 purely non-symplectic automorphism $\sigma_{14}: (x, y, z, w) \mapsto (-x, y, z, \zeta_7 w)$.

We have the following: point $(1 : 0 : 1 : 0)$ of type A_1 ; points $(1 : 1 : 0 : 0)$ and $(-1 : 1 : 0 : 0)$, both of type A_6 ; and point $(0 : 0 : 0 : 1)$ of type A_2 (Figure 6).

Since σ_7 fixes the rational curves $C_z := \{z = 0\}$ and $C_w := \{w = 0\}$, we are in case D. The involution σ_2 fixes the curve $C_x := \{x = 0\}$ of genus 3 and two rational curves given by the

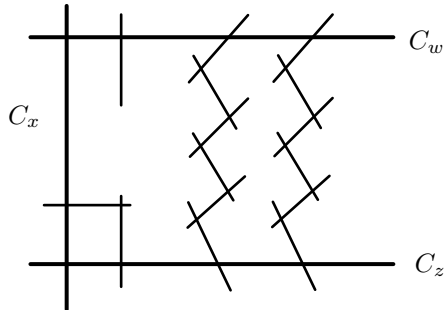


FIGURE 6. D3

component of the A_1 and one of the components of the A_2 . The involution σ_2 also exchanges the two A_6 points.

Example 3.31. (Case D8) Again, consider $(X_{a,b}, \sigma_{14})$, the K3 surface together with the automorphism from Example 3.20. If $a = 0$ and b is such that $b^3 = -\frac{27}{4}$, it follows that $X_{a,b}$ contains a fiber of type II^* at $t = (1 : 0)$, a fiber of type I_7 at $t = (0 : 1)$, and seven singular fibers of type I_1 . The fixed locus of σ_{14} is of type D8.

Observe that the surface $y^2 = x^3 + t^3x + t^8$, $t \in \mathbb{P}^1$ given in [22, Example 7.5] admits the purely non-symplectic order 14 automorphism $\sigma_{14} : (x, y, t) \mapsto (\zeta_7^3x, -\zeta_7y, \zeta_7^2t)$ and corresponds to case D8 as well.

4. ORDER 21

Purely non-symplectic automorphisms of order 21 on K3 surfaces have been classified in [11]. Here we present a new proof and a more detailed description of Brandhorst's result. More precisely, using the same kind of approach from the previous section, we show that the examples of [11, Table 3] fit the invariants of Table 6 below, and we prove:

Proposition 4.1. *The fixed locus of a non-symplectic automorphism of order 21 on a K3 surface is not empty and it consists of either:*

- (i) *The union of N_{21} isolated points, where $N_{21} \in \{4, 7\}$; or*
- (ii) *The disjoint union of a rational curve and N_{21} isolated points, where $N_{21} \in \{8, 11\}$.*

Moreover, all these possibilities occur, and a more detailed description is given in Table 6 below, where $\sigma_7 \doteq \sigma_{21}^3$ and $\sigma_3 \doteq \sigma_{21}^7$.

	$\text{Fix}(\sigma_{21})$	$\text{Fix}(\sigma_7)$	$\text{Fix}(\sigma_3)$	Example
$C(3,2,3)$	$R \sqcup \{p_1, \dots, p_8\}$	$R \sqcup \{p_1, \dots, p_8\}$	$C_3 \sqcup R \sqcup R' \sqcup \{p_1, p_2, p_3\}$	4.4
$C(3,1,2)$	$\{p_1, \dots, p_7\}$	$R \sqcup \{p_1, \dots, p_5, q_1, q_2, q_3\}$	$C_3 \sqcup R \sqcup \{p_1, q_1\}$	4.5
$C(3,0,1)$	$\{p_1, \dots, p_4\}$	$R \sqcup \{p_1, \dots, p_8\}$	$C_3 \sqcup \{p_1\}$	4.6
$B(3,3,4)$	$R \sqcup \{p_1, \dots, p_{11}\}$	$E \sqcup R \sqcup \{p_1, \dots, p_8\}$	$C_3 \sqcup R \sqcup R' \sqcup R'' \sqcup \{p_1, \dots, p_4\}$	4.7

TABLE 6. Order 21

In order to prove Proposition 4.1, we first note that, as we observed in Section 2, at any fixed point a purely non-symplectic automorphism σ_{21} of order 21 acts as multiplication by the matrix $A_{i,21}$ for some i , with

$$A_{i,21} := \begin{pmatrix} \zeta_{21}^{1+i} & 0 \\ 0 & \zeta_{21}^{21-i} \end{pmatrix}, \quad 0 \leq i \leq 10.$$

Thus, the holomorphic Lefschetz formula (2) applied to σ_{21} gives us the following linear system of equations:

$$(9) \quad \begin{cases} 3m_{6,21} &= 3 + 4m_{1,21} - 5m_{2,21} - 4m_{4,21} + 8m_{5,21} \\ 3m_{7,21} &= 3 - 5m_{1,21} + 4m_{2,21} - 13m_{4,21} + 17m_{5,21} \\ m_{8,21} &= 1 - 2m_{1,21} + 2m_{2,21} - 5m_{4,21} + 6m_{5,21} \\ m_{9,21} &= 3 - 4m_{1,21} + 4m_{2,21} - 3m_{3,21} - 3m_{4,21} + 7m_{5,21} \\ 2m_{10,21} &= 2 - 3m_{1,21} + 3m_{2,21} - 2m_{3,21} - 3m_{4,21} + 6m_{5,21} \\ 6\alpha_{21} &= m_{1,21} + m_{2,21} - m_{4,21} + 2m_{5,21} \end{cases}$$

where $\alpha_{21} := \sum 1 - g(C)$ and the sum is taken over all curves C fixed by σ_{21} .

Moreover, considering the non-symplectic automorphism $\sigma_7 = \sigma_{21}^3$ of order 7, we know that

$$\begin{cases} m_{1,21} + m_{5,21} + m_{8,21} &\leq m_{1,7} \\ m_{2,21} + m_{4,21} + m_{9,21} &\leq m_{2,7} \\ m_{3,21} + m_{10,21} &\leq m_{3,7} \end{cases}$$

We note also that points of type $A_{6,21}$ and $A_{7,21}$ lie on a curve fixed by σ_7 (but not fixed by σ_{21}) and points of type $A_{j,21}$, where $j = 2, 3, 5, 6, 8, 9$, lie on a curve fixed by $\sigma_3 = \sigma_{21}^7$ (but not fixed by σ_{21}). For this reason, we choose $r := m_{6,21} + m_{7,21}$. Using MAGMA, we obtain the following four possibilities for the vector $(m_{1,21}, \dots, m_{10,21}; \alpha_{21}, r)$:

$$\begin{array}{ll} v_1 = (3, 3, 1, 0, 0, 0, 0, 1, 0, 0; 1, 0) & v_2 = (0, 0, 0, 0, 0, 0, 1, 1, 1, 3, 1; 0, 2) \\ v_3 = (0, 0, 1, 0, 0, 1, 1, 1, 0, 0; 0, 2) & v_4 = (3, 2, 1, 1, 1, 3, 0, 0, 0, 0; 1, 3) \end{array}$$

Furthermore, we observe the following:

Lemma 4.2. *If the fixed locus of σ_{21} is described by one of the vectors v_1, v_2, v_3 , then the fixed locus of $\sigma_7 = \sigma_{21}^3$ is as in Case C of Table 1. If it is described by the vector v_4 , then the fixed locus of σ_7 is as in Case B.*

Proof. We first observe that σ_7 cannot be of type A. Assume we are in Case A. We know that $\text{Fix}(\sigma_{21}) \subseteq \text{Fix}(\sigma_7)$. By the Riemann-Hurwitz formula, the genus one curve in $\text{Fix}(\sigma_7)$ would contain either none or three isolated points fixed by σ_{21} , and thus $r = 0$ or 3 . But the cases with these values of r both have $\alpha_{21} = 1$, which is not possible in Case A (recall that in Case A, a fixed curve must have genus 1, as shown in Table 1).

Case D for σ_7 is not admissible either. In fact, if σ_7 is as in case D, then $\text{Fix}(\sigma_7)$ contains two rational curves. If they were both pointwise fixed by σ_{21} , this would give $\alpha_{21} = 2$. If one curve is pointwise fixed and the other one is invariant, then $\alpha_{21} = 1$ and $r = 2$. If both curves are invariant but not pointwise fixed, then $\alpha_{21} = 0$ and $r = 4$. These cases do not appear among the admissible ones. Therefore we conclude that σ_7 must fall under Case B or Case C.

We now observe that the situation described by the vector v_4 is only possible in Case B: since $r = 3$ in this case, it means that there are three points on curves fixed by σ_7 and they are not fixed by σ_{21} . Thus there must be an elliptic curve in $\text{Fix}(\sigma_7)$. As we observed in Lemma 3.7, if σ_7 fixes an elliptic curve and a rational curve as in Case B, the surface admits an elliptic fibration with a fiber of type II^* and 14 fibers of type I_1 . Since the fiber of type II^* does not admit a symmetry of order three, σ_{21} fixes the central curve of this fiber and eight points that lie on it.

As for vector v_2 (respectively v_3), the fixed locus of σ_{21} consists of seven (respectively four) points. Thus σ_7 cannot belong to Case B, since by the previous remark, it would fix too many points.

Assume now that we are in Case B and the vector v_1 describes the action of σ_{21} . Then the fixed locus of σ_{21} is the union of a rational curve and eight points; since $r = 0$, the action of σ_{21} on the elliptic curve in $\text{Fix}(\sigma_7)$ is a translation. But then the action should be a translation on the fiber II^* , and this is not the case. \square

At last, we are now in position to prove Proposition 4.1:

Proof of Proposition 4.1. Consider the induced action of σ_{21} on $H^2(X, \mathbb{R})$ and recall the definition of $d_i := \dim H^2(X, \mathbb{R})_{\zeta_i}$ for $i = 1, 3, 7, 21$.

For each $i = 3, 7, 21$ we let χ_i denote the Euler characteristic of the fixed locus of $\sigma_i = \sigma^{\frac{21}{i}}$. By applying the topological Lefschetz formula (3) to σ_{21}, σ_7 and σ_3 , we obtain:

$$(10) \quad \begin{cases} \chi_{21} &= 2 + d_{21} - d_7 - d_3 + d_1 \\ \chi_7 &= 2 - (2d_{21} + d_7) + 2d_3 + d_1 \\ \chi_3 &= 2 - 6d_{21} + 6d_7 - d_3 + d_1 \end{cases}$$

Moreover, we know that

$$22 = \dim H^2(X, \mathbb{R}) = 12d_{21} + 6d_7 + 2d_3 + d_1.$$

Combining these equations one gets the possibilities given in Table 7.

Type σ_7	χ_7	χ_{21}	χ_3	(d_{21}, d_7, d_3, d_1)	$(m_{1,21}, \dots, m_{10,21}, \alpha_{21}, r)$	Fix(σ_{21})
C	10	10	3	(1,0,1,8)	(3,3,1,0,0,0,0,1,0,0,1,0)	$R \sqcup$ 8 pts
	10	7	0	(1,0,2,6)	(0,0,0,0,0,1,1,1,3,1,0,2)	7 pts
	10	4	-3	(1,0,3,4)	(0,0,1,0,0,1,1,1,0,0,0,2)	4 pts
B	10	13	6	(1,0,0,10)	(3,2,1,1,1,3,0,0,0,0,1,3)	$R \sqcup$ 11pts

TABLE 7.

Thus, it remains to look at the fixed locus of σ_3 , which by [7] consists of N_3 isolated fixed points, a curve of genus $g_3 \geq 0$, and k_3 rational curves, where by [7, Theorem 2.2] the following relation holds:

$$1 - g_3 + k_3 = N_3 - 3.$$

In particular, $\chi_3 = N_3 + 2(1 - g_3 + k_3) = 3N_3 - 6$, and we can list the possibilities for (g_3, k_3, N_3) according to the value of N_3 .

If $\chi_3 = 3$, then $N_3 = 3$ and by [7] we have the following possibilities for the invariants (g_3, k_3, N_3) of Fix(σ_3):

$$(g_3, k_3, N_3) = (-, -, 3), (1, 0, 3), (2, 1, 3), (3, 2, 3).$$

Similarly,

- if $\chi_3 = 0$, then $N_3 = 2$ and the possibilities are $(g_3, k_3, N_3) = (2, 0, 2), (3, 1, 2), (4, 2, 2)$.
- if $\chi_3 = -3$ then $N_3 = 1$ and the possibilities are $(g_3, k_3, N_3) = (3, 0, 1), (4, 1, 1)$.
- if $\chi_3 = 6$, then $N_3 = 4$ and the possibilities are: $(g_3, k_3, N_3) = (3, 3, 4), (2, 2, 4), (1, 1, 4), (0, 0, 4)$.

Next, we observe that we can actually eliminate most of these possibilities.

As in Lemma 3.12, the automorphism σ_{21} acts with order seven on Fix(σ_3), and thus C_{g_3} should admit an automorphism of order seven. But if $g_3 \geq 2$ and if ϕ is an automorphism of prime order p , we must have $p \leq 2g_3 + 1$. Then we may eliminate the case where $g_3 = 2$.

A curve of genus four does not admit an automorphism of order seven by [23], and thus $g_3 \neq 4$.

Finally, if $\chi_3 = 3$, then Fix(σ_{21}) consists of a fixed rational curve plus eight points. Since Fix(σ_{21}) \subseteq Fix(σ_3), using Riemann-Hurwitz we can also eliminate the triples $(g_3, k_3, N_3) = (-, -, 3), (1, 0, 3)$. The argument is similar for triples $(g_3, k_3, N_3) = (1, 1, 4), (0, 0, 4)$ with $\chi_3 = 6$.

Therefore, the possible cases are the ones listed in Table 6. \square

Remark 4.3. Note that in the proof of Proposition 4.1, we have

$$\text{rk } S(\sigma_{21}) = d_1, \text{rk } S(\sigma_7) = 2d_3 + d_1, \text{rk } S(\sigma_3) = 6d_7 + d_1.$$

We end this section by showing the examples in [11] are indeed compatible with the invariants of Table 6, as claimed.

Example 4.4. (Case C(3,2,3)) Let (X, σ_{21}) be the following elliptic K3 surface with the non-symplectic automorphism σ_{21} of order 21:

$$y^2 = x^3 + 4t^4(t^7 - 1), t \in \mathbb{P}^1 \quad \sigma_{21} : (x, y, t) \mapsto (\zeta_7^6 \zeta_3 x, \zeta_7^2 y, \zeta_7 t).$$

The collection of singular fibers of the elliptic fibration consist of a fiber of type IV* over $t = 0$, a fiber of type II over $t = \infty$, and 7 of type II over the zeros of $t^7 - 1$. The fixed locus of σ_7 consist of the central component R of the fiber IV*, six isolated points on the fiber IV* and two points on the fiber II over $t = \infty$. The automorphism σ_{21} has the same fixed locus as σ_7 . The fixed locus of σ_3 consists of the zero section, the curve R and the 3-section $y = 0$, which has genus three and 3 additional points.

In particular, the invariants of Fix(σ_{21}^j), $j = 1, 3, 7$ are as in the first row of Table 6.

Example 4.5. (Case C(3,1,2)) Let (X, σ_{21}) be the following elliptic K3 surface with the non-symplectic automorphism σ_{21} of order 21:

$$y^2 = x^3 + t^3(t^7 + 1), t \in \mathbb{P}^1 \quad \sigma_{21} : (x, y, t) \mapsto (\zeta_7^3 \zeta_3 x, \zeta_7 y, \zeta_7^3 t).$$

The singular fibers of the elliptic fibration are $I_0^* + IV + 7II$. The fixed locus $\text{Fix}(\sigma_7)$ consists of the central component R of the fiber I_0^* , four points on I_0^* , and four points on IV . The automorphism σ_{21} does not fix R and only fixes isolated points. The automorphism σ_3 exchanges three of the non-central components of the fiber I_0^* and acts on the remaining one, and thus $(g_3, k_3, N_3) = (3, 1, 2)$.

The conclusion is that the invariants of $\text{Fix}(\sigma_{21}^j)$, $j = 1, 3, 7$ are as in the second row of Table 6.

Example 4.6. (Case C(3,0,1)) Let X be the K3 surface whose equation in \mathbb{P}^3 is

$$x_0^3 x_1 + x_1^3 x_2 + x_0 x_2^3 + x_0 x_3^3 = 0.$$

This surface admits the purely non-symplectic automorphism of order 21

$$\sigma_{21} : (x_0, x_1, x_2, x_3) \mapsto (\zeta_7 x_0, \zeta_7^5 x_1, x_2, \zeta_3 x_3)$$

whose fixed locus consists of the four standard coordinate points. The fixed locus of σ_3 consists of the genus three curve $\{x_3 = 0\} \cap X$ and the point $p_1 = (0 : 0 : 0 : 1)$.

In particular, we see that the invariants of $\text{Fix}(\sigma_{21}^j)$, $j = 1, 3, 7$ are as described in the third row of Table 6.

Example 4.7. (Case B(3,3,4)) Let (X, σ_{21}) be the following elliptic K3 surface with the non-symplectic automorphism σ_{21} of order 21:

$$y^2 = x^3 + t^5(t^7 - 1), t \in \mathbb{P}^1 \quad \sigma_{21} : (x, y, t) \mapsto (\zeta_{21}^2 x, \zeta_7 y, \zeta_7^6 t).$$

The collection of singular fibers consists of a type II^* fiber at $t = \infty$ and seven type II fibers over the zeros of $t^7 + 1$. The order seven automorphism σ_7 fixes the following: the smooth fiber E of genus one over $t = 0$, the central component R of the II^* fiber, and eight isolated points on the same fiber II^* . The automorphism σ_{21} fixes R as well and acts on E as an automorphism of order three, fixing three points. The fixed locus of σ_3 consists of R , along with another rational curve in the fiber II^* , the zero section, the genus three 3-section $X \cap \{y = 0\}$, and four isolated points on the fiber II^* .

Therefore, the invariants of $\text{Fix}(\sigma_{21}^j)$, $j = 1, 3, 7$ are as in the fourth row of Table 6.

Another example of this type of automorphism is given by the following. Consider the equation $y^3 = z^7 + x^2 w + x w^{11}$ in the weighted projective space $\mathbb{P}(10, 7, 3, 1)_{x,y,z,w}$, and consider the order 21 automorphism

$$\sigma_{21} : (x, y, z, w) \mapsto (x, \zeta_3 y, z, \zeta_7 w).$$

The curve $C_y := \{y = 0\}$ has genus three and is fixed by σ_3 , and the curve $C_w := \{w = 0\}$ has genus one and is fixed by σ_7 . The rational curve fixed by σ_{21} is a rational component in the resolution of the A_9 singularity $(1 : 0 : 0 : 0)$.

5. ORDER 28

We now prove a classification theorem for purely non-symplectic automorphisms of order 28 recovering the results in [11]. Our result is the following:

Proposition 5.1. *The fixed locus of a purely non-symplectic automorphism of order 28 on a K3 surfaces is not empty and it consists of either:*

- (i) *The union of N_{21} isolated points, where $N_{21} \in \{3, 5\}$; or*
- (ii) *The disjoint union of a rational curve and 10 isolated points.*

Moreover, all these possibilities occur. The examples of [11, Table 3] fit the invariants of Table 8 below, which provides a more detailed description of the possible different fixed loci of σ_{28} and its powers.

$\text{Fix}(\sigma_{28})$	$\text{Fix}(\sigma_{14})$	$\text{Fix}(\sigma_7)$	$\text{Fix}(\sigma_4)$	$\text{Fix}(\sigma_2)$	Example
$\{p_1, \dots, p_5\}$	$\{p_1, \dots, p_5, p_6, p_7\}$	$E \sqcup \{p_1, p_2, p_3\}$	$\{q_1, \dots, q_7, p_1\} \sqcup R_1 \sqcup R_2$	$C_3 \sqcup R_1 \sqcup R_2$	5.3
$\{p_1, p_2, p_3\}$	$\{p_1, \dots, p_7\}$	$E \sqcup \{p_1, q_1, q_2\}$	C_3	$C_3 \sqcup R_1 \sqcup R_2$	5.4
$R \sqcup \{p_1, \dots, p_{10}\}$	$R \sqcup \{p_1, \dots, p_{10}, p_{11}, p_{12}\}$	$E \sqcup R \sqcup \{p_1, \dots, p_8\}$	$\{p_1, \dots, p_8\} \sqcup R \sqcup R_1$	$C_6 \sqcup R \sqcup R_1 \sqcup \dots \sqcup R_4$	5.5

TABLE 8. Order 28

Proof. As explained in Section 2, at any fixed point the automorphism σ_{28} acts as multiplication by $A_{i,28} := \begin{pmatrix} \zeta_{28}^{i+1} & 0 \\ 0 & \zeta_{28}^{28-i} \end{pmatrix}$ for $0 \leq i \leq 13$, and we denote the number of points of type $A_{i,28}$ by $m_{i,28}$.

The holomorphic Lefschetz formula (2) applied to σ_{28} gives us the following linear system of equations

$$\begin{cases} 3m_{6,28} &= 3 + 4m_{1,28} - 5m_{2,28} - 4m_{4,28} + 8m_{5,28} \\ 3m_{7,28} &= 3 - 5m_{1,28} + 4m_{2,28} - 13m_{4,28} + 17m_{5,28} \\ m_{8,28} &= 1 - 2m_{1,28} + 2m_{2,28} - 5m_{4,28} + 6m_{5,28} \\ m_{9,28} &= 3 - 4m_{1,28} + 4m_{2,28} - 3m_{3,28} - 3m_{4,28} + 7m_{5,28} \\ 2m_{10,28} &= 2 - 3m_{1,28} + 3m_{2,28} - 2m_{3,28} - 3m_{4,28} + 6m_{5,28} \\ 6\alpha_{28} &= m_{1,28} + m_{2,28} - m_{4,28} + 2m_{5,28} \end{cases}$$

where $\alpha_{28} := \sum(1 - g(C))$ and the sum runs over all curves C which are fixed by σ_{28} .

Moreover, considering the automorphism $\sigma_7 = \sigma_{28}^4$ which has order seven, we further know that

$$\begin{cases} m_{1,28} + m_{5,28} + m_{8,28} + m_{12,28} &\leq m_{1,7} \\ m_{2,28} + m_{4,28} + m_{9,28} + m_{11,28} &\leq m_{2,7} \\ m_{3,28} + m_{10,28} &\leq m_{3,7} \end{cases}$$

Note that

- points of type $A_{13,28}$ lie on a curve fixed by σ_{14} (but not by σ_{28});
- points of type $A_{7,28}$, $A_{8,28}$ and $A_{13,28}$ lie on a curve fixed by σ_7 (but not fixed by σ_{28});
- points of type $A_{j,28}$, $j = 3, 4, 7, 8, 11, 12$ lie on a curve fixed by σ_4 (but not fixed by σ_{28}).

Because of the observations listed above, letting $r := m_{6,28} + m_{7,28} + m_{13,28}$ we obtain the following four possibilities for $(m_{1,28}, \dots, m_{13,28}; \alpha_{28}, r)$:

$$\begin{aligned} w_1 &= (0, 0, 0, 0, 0, 0, 2, 2, 1, 0, 0, 0, 0; 0, 2) & w_3 &= (0, 0, 0, 1, 0, 0, 2, 0, 0, 0, 0, 0; 0, 2) \\ w_2 &= (3, 2, 1, 0, 1, 2, 0, 2, 1, 0, 0, 0; 1, 2) & w_4 &= (3, 2, 1, 1, 1, 2, 0, 0, 0, 0, 0, 0; 1, 2) \end{aligned}$$

We now consider the induced action of σ_{28} on $H^2(X, \mathbb{R})$ and as in Section 2 we let

$$d_i := \dim H^2(X, \mathbb{R})_{\zeta_i}, \quad i = 28, 14, 7, 4, 2, 1$$

For each $i = 2, 4, 7, 14, 28$ we let χ_i denote the Euler characteristic of the fixed locus of the power of σ_{28} which has order i . Applying the topological Lefschetz formula (3) to $\sigma_{28}, \sigma_{14}, \sigma_7, \sigma_4$ and σ_2 we obtain:

$$(11) \quad \begin{cases} \chi_{28} &= 2 + d_{14} - d_7 - d_2 + d_1 \\ \chi_{14} &= 2 + 2d_{28} - d_{14} - d_7 - 2d_4 + d_2 + d_1 \\ \chi_7 &= 2 - 2d_{28} - d_{14} - d_7 + 2d_4 + d_2 + d_1 \\ \chi_4 &= 2 - 6d_{14} + 6d_7 - d_2 + d_1 \\ \chi_2 &= 2 - 12d_{28} + 6d_{14} + 6d_7 - 2d_4 + d_2 + d_1 \end{cases}$$

Moreover, we know that

$$22 = \dim H^2(X, \mathbb{R}) = 12d_{28} + 6d_{14} + 6d_7 + 2d_4 + d_2 + d_1$$

Using (11) one gets the following possibilities, according to the four vectors w_i :

w_i	χ_{28}	(d_{28}, \dots, d_1)	χ_{14}	χ_7	χ_4	χ_2
w_1	5	(1,1,0,1,0,2)	3	3	-2	-4
		(1,1,0,0,1,3)	7	3	-2	0
		(1,0,1,0,0,4)	7	3	12	0
		-	-	-	-	-
w_2	14	-	-	-	-	-
w_3	3	(1,1,0,1,0,1)	3	3	-4	-4
		(1,0,1,1,0,2)	3	3	10	-4
		(1,1,0,0,2,2)	7	3	-4	0
		(1,0,1,0,1,3)	7	3	10	0
w_4	12	(1,0,0,0,0,10)	14	10	12	0

Observe that vector w_2 does not give any admissible case and χ_{14} cannot be 3 by our classification of Section 3. This implies either $\chi_7 = 10$, the vector of types of points is w_4 and $\sigma_{14} = \sigma_{28}^2$ is of type B3 of Table 2 or $(\chi_{14}, \chi_7, \chi_2) = (7, 3, 0)$. In the latter case, σ_{14} is of type A1(3,2) of Table 2

and $\text{Fix}(\sigma_2) = C_3 \cup R_1 \cup R_2$. Recalling that $\text{Fix}(\sigma_4) \subseteq \text{Fix}(\sigma_2)$ and that σ_4 acts with order 1 or 2 on $\text{Fix}(\sigma_2)$, with Riemann-Hurwitz formula we can conclude that $\chi_4 \in \{-4, 0, 4, 8, 12\}$.

This leaves only the cases highlighted in gray.

We now study the action of σ_4 on $\text{Fix}(\sigma_2)$. If $\chi_4 = -4$, then $\text{Fix}(\sigma_4) = C_3$ and R_1 and R_2 are exchanged by σ_4 . If $\chi_4 = 12$, then σ_4 can only fix rational curves and [8, Proposition 1] implies σ_4 fixes exactly two rational curves and 8 isolated points. \square

Remark 5.2. *As in the order 21 case, note that the following relations hold:*

$$\text{rk } S(\sigma_7) = 2d_4 + d_2 + d_1, \quad \text{rk } S(\sigma_4) = 6d_7 + d_1, \quad \text{rk } S(\sigma_2) = 6(d_{14} + d_7) + d_2 + d_1.$$

We now show the examples in [11, Table 3] are consistent with the invariants listed on Table 8.

Example 5.3. *The elliptic K3 surface with Weierstrass equation*

$$y^2 = x^3 + (t^7 + 1)x$$

admits the following order 28 purely non-symplectic automorphism

$$\sigma_{28}(x, y, t) = (x - (y/x)^2, i(y - (y/x)^3), \zeta_7 t),$$

The elliptic fibration admits a smooth fiber over $t = 0$, a fiber of type II over $t = \infty$ and 7 fibers of type II over the roots of $\Delta = 4(t^7 + 1)^3$. One can check the invariants of $\text{Fix}(\sigma_{28}^j)$, $j = 1, 2, 4, 14$ are as in the first row of Table 8. In particular, the automorphism $\sigma_{14} = \sigma_{28}^2$ is of type A1(3, 2) in our classification of Section 3. Moreover, given that $\text{Fix}(\sigma_2) = C_3 \sqcup R_1 \sqcup R_2$, we have that σ_4 does not exchange R_1 and R_2 and fixes the 8 tangential points of the fibers of type II lying on C_3 . Therefore, σ_{28} fixes the same three points in the fiber over $t = \infty$ and two additional points in smooth fiber over $t = 0$.

Example 5.4. *The elliptic K3 surface with Weierstrass equation*

$$y^2 = x^3 + (t^7 + 1)x, \quad t \in \mathbb{P}^1,$$

admits the following order 28 purely non-symplectic automorphism

$$\sigma_{28}(x, y, t) = (-x, iy, -\zeta_7 t).$$

One can check that the invariants of $\text{Fix}(\sigma_{28}^j)$, $j = 1, 2, 4, 14$ are as in the second row of Table 8. In particular, the automorphism $\sigma_{14} = \sigma_{28}^2$ is of type A1(3, 2) in our classification of Section 3. Given that $\text{Fix}(\sigma_2) = C_3 \sqcup R_1 \sqcup R_2$, the automorphism σ_4 exchanges R_1 and R_2 and fixes C_3 . As a consequence, σ_{28} fixes the tangential point in the fiber of type II over $t = \infty$ and two additional points in the smooth fiber.

Another example of this type of automorphism is given by the following. Consider the K3 surface in $\mathbb{P}(7, 3, 2, 2)$ which is the zero locus of the quasi-smooth polynomial $x^2 + y^4 z + z^7 + w^7$. It admits the purely non-symplectic automorphism of order 28

$$\sigma_{28}(x, y, z, w) = (x, iy, z, \zeta_7 w).$$

Resolving the singularity of type A_2 at $(0 : 1 : 0 : 0)$ and the seven singularities of type A_1 at $(0 : 0 : \zeta_{14}^i, 1)$, $i = 1, 3, \dots, 13$ we see that the different fixed loci of the powers of σ_{28} are as in the second row of Table 8.

Example 5.5. *The elliptic K3 surface with Weierstrass equation*

$$y^2 = x^3 + x + t^7, \quad t \in \mathbb{P}^1,$$

admits the order 28 purely non-symplectic automorphism

$$\sigma_{28}(x, y, t) = (-x, iy, -\zeta_7 t).$$

The elliptic fibration admits a smooth fiber over $t = 0$, a fiber of type II^ over $t = \infty$ and 14 nodal curves over the roots of $\Delta = 4 + 27t^{14}$. The automorphism $\sigma_{14} = \sigma_{28}^2$ is of type B3 in our classification in Section 3 and we can check the invariants of $\text{Fix}(\sigma_{28}^j)$, $j = 1, 4, 14$ indeed agree with the third row of Table 8. Moreover, since $\text{Fix}(\sigma_2) = C_6 \sqcup R \sqcup R_1 \sqcup \dots \sqcup R_4$, we have that σ_4 fixes two rational curves including R , two points in C_6 and six additional points in the other rational curves. As a consequence, σ_{28} fixes R and ten additional points, two of them on the smooth fiber over $t = 0$.*

6. ORDER 42

In [11], Brandhorst classifies purely non-symplectic automorphisms of order 42 on K3 surfaces. Here, we provide a different and more geometric view of his result. We prove:

Proposition 6.1. *The fixed locus of a purely non-symplectic automorphism of order 42 on a K3 surfaces is not empty and it consists of either:*

- (i) *The union of N_{21} isolated points, where $N_{21} \in \{5, 6\}$; or*
- (ii) *The disjoint union of a rational curve and 9 isolated points.*

Moreover, all these possibilities occur, and a more detailed description is given in Table 9 below.

Type σ_{14}	$\text{Fix}(\sigma_{42})$	$\text{Fix}(\sigma_7)$	$\text{Fix}(\sigma_{21})$	$\text{Fix}(\sigma_3)$	Example
$C1$	$\{p_1, \dots, p_6\}$	$R \sqcup \{p_1, \dots, p_8\}$	$R \sqcup \{p_1, \dots, p_8\}$	$C_3 \sqcup R \sqcup R' \sqcup \{p_1, p_2, p_3\}$	6.5
$C3$	$\{p_1, \dots, p_5\}$	$R \sqcup \{p_1, \dots, p_8\}$	$\{p_1, \dots, p_7\}$	$C_3 \sqcup R \sqcup \{p_1, p_2\}$	6.6
$B3$	$R \sqcup \{p_1, \dots, p_9\}$	$E \sqcup R \sqcup \{p_1, \dots, p_8\}$	$R \sqcup \{p_1, \dots, p_{11}\}$	$C_3 \sqcup R \sqcup R' \sqcup R'' \sqcup \{p_1, \dots, p_4\}$	6.7

TABLE 9. Order 42

Proof. Let σ_{42} be a purely non-symplectic automorphism of order 42. Thus its square is a purely non-symplectic automorphism of order 21 and we use the classification of Section 4.

Observe that isolated fixed points for σ_{42} of type $A_{20,42}$ lie on curves fixed by σ_{21} and not fixed by σ_{42} . Thus, if σ_{21} has invariants as in the first or fourth rows of Table 6, it must be the case that $m_{20,42}$ is either 0 or 2, according to the fact the the rational curve $R \subset \text{Fix}(\sigma_{21})$ is fixed by σ_{42} or not.

We also have the following inequalities

$$\begin{aligned}
 m_{1,42} + m_{19,42} &\leq m_{1,21}, & m_{2,42} + m_{18,42} &\leq m_{2,21}, & m_{3,42} + m_{17,42} &\leq m_{3,21}, & m_{4,42} + m_{16,42} &\leq m_{4,21}, \\
 m_{5,42} + m_{15,42} &\leq m_{5,21}, & m_{6,42} + m_{14,42} &\leq m_{6,21}, & m_{7,42} + m_{13,42} &\leq m_{7,21}, & m_{8,42} + m_{12,42} &\leq m_{8,21}, \\
 m_{9,42} + m_{11,42} &\leq m_{9,21}, & m_{10,42} &\leq m_{10,21}
 \end{aligned}$$

According to this, we look for possible solutions $m = (m_{1,42}, \dots, m_{20,42}; \alpha_{42})$ of the Lefschetz holomorphic formula (2) applied to σ_{42} . Using MAGMA we get the following:

- if σ_{21} is as in the first row of Table 6, there is no possible solution m with $\alpha_{42} = 1$. If $\alpha_{42} = 0$ one gets the vector $m = (0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 2; 0)$. Thus $\text{Fix}(\sigma_{42})$ consists of 6 isolated points, two of which are contained in the rational curve fixed by σ_{21} .
- if σ_{21} is as in the second row of Table 6, then α_{42} is necessarily 0 and $m_{20,42} = 0$. There is one solution $m = (0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0; 0)$. Thus $\text{Fix}(\sigma_{42})$ consists of 5 isolated points.
- if σ_{21} is as in the third row of Table 6, α_{42} is necessarily 0 and $m_{20,42} = 0$. There is no solution in this case.
- if σ_{21} is as in the fourth of Table 6, α_{42} can be 0 or 1. If $\alpha_{42} = 0$ and $m_{20,42} = 2$ there are no solutions. If $\alpha_{42} = 1$ and $m_{20,42} = 0$ by MAGMA we get only one solution $(m_{1,42}, \dots, m_{20,42}; \alpha_{42}) = (3, 2, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0; 1)$. Thus $\text{Fix}(\sigma_{42})$ consists of a rational curve and 9 isolated points.

Thus, there are three possibilities for $\text{Fix}(\sigma_{42})$. As before, let $d_i := \dim H^2(X, \mathbb{R})_{\zeta_i}$ for $i = 42, 21, 14, 7, 6, 3, 2, 1$. We have

$$22 = 12d_{42} + 12d_{21} + 6d_{14} + 6d_7 + 2d_6 + 2d_3 + d_2 + d_1$$

By the topological Lefschetz formula (3) applied to the powers of σ_{42} we get the following linear system of equations

$$(12) \quad \begin{cases} \chi_{42} &= 2 - d_{42} + d_{21} + d_{14} - d_7 + d_6 - d_3 - d_2 + d_1 \\ \chi_{21} &= 2 + d_{42} + d_{21} - d_{14} - d_7 - d_6 - d_3 + d_2 + d_1 \\ \chi_{14} &= 2 + (2d_{42} + d_{14}) - (2d_{21} + d_7) - (2d_6 + d_2) + 2d_3 + d_1 \\ \chi_7 &= 2 - (2d_{42} + 2d_{21} + d_{14} + d_7) + 2d_6 + 2d_3 + d_2 + d_1 \\ \chi_6 &= 2 + (6d_{42} + d_6) - (6d_{21} + d_3) - (6d_{14} + d_2) + 6d_7 + d_1 \\ \chi_3 &= 2 - (6d_{42} + 6d_{21} + d_6 + d_3) + 6d_{14} + 6d_7 + d_2 + d_1 \\ \chi_2 &= 2 - (12d_{42} + 6d_{14} + 2d_6 + d_2) + 12d_{21} + 6d_7 + 2d_3 + d_1 \end{cases}$$

Considering the different possible solutions we can compute the values of the Euler characteristics of the fixed locus of σ_{42} and its powers:

Type σ_{14}	χ_{42}	$\text{Fix}(\sigma_{42})$	(d_{42}, \dots, d_1)	χ_{21}	χ_{14}	χ_7	χ_6	χ_3	χ_2
C1	6	6 pts	(1,0,0,0,1,0,2,6)	10	6	10	13	3	-8
C3	5	5 pts	(1,0,0,0,1,1,1,5)	7	8	10	12	0	-6
B3	11	$R \sqcup 9$ pts	(1,0,0,0,0,0,0,10)	13	14	10	18	6	0

□

Remark 6.2. Observe the following relations hold:

$$\begin{aligned} \text{rk } S(\sigma) = d_1 \geq 1, \quad \text{rk } S(\sigma_7) = 2d_6 + 2d_3 + d_2 + d_1, \quad \text{rk } S(\sigma_6) = 6d_7 + d_1, \\ \text{rk } S(\sigma_3) = 6d_{14} + 6d_7 + d_2 + d_1, \quad \text{rk } S(\sigma_2) = 6(2d_{21} + d_7) + 2d_3 + d_1. \end{aligned}$$

Remark 6.3. The complete description of $\text{Fix}(\sigma_3)$ follows from Proposition 4.1.

Remark 6.4. The possible values of χ_6 obtained in the proof of Proposition 6.1 and the classification in [13] allow us to also completely describe $\text{Fix}(\sigma_6)$. The description is as follows:

If σ_{42} is as in the first row of Table 9, then we must have $m_{2,6} = 10$ and $m_{1,6} = 1$. Moreover, σ_6 fixes 1 rational curve. With our notations, there are 8 fixed points under σ_6 lying on C_3 , σ_6 fixes p_1 , it also fixes R and it has 2 more fixed points lying on R' .

Now, if σ_{42} is as in the second row of Table 9, then $m_{2,6} = 8$ and $m_{1,6} = 2$. Moreover, σ_6 fixes 1 rational curve. There are 8 points fixed under σ_6 lying on C_3 , σ_6 fixes p_1 and p_2 , and it also fixes R .

Finally, if σ_{42} is as in the last row of Table 9, then $m_{2,6} = 10$ and $m_{1,6} = 4$. Moreover, σ_6 fixes 2 rational curves. There are 8 points fixed under σ_6 lying on C_3 , σ_6 fixes p_1, \dots, p_4 , it also fixes R and R' and it has 2 more fixed points lying on R'' .

Observe that Proposition 6.1 is compatible with [11]. In fact the examples in [11, Table 3] agree with the invariants listed on Table 9, as we describe below:

Example 6.5. The K3 surface is the same as in Example 4.4, see [11]. On the same elliptic fibration $y^2 = x^3 + 4t^4(t^7 - 1)$ the order 42 automorphism σ_{42} is given by

$$\sigma_{42} : (x, y, t) \mapsto (\zeta_7^6 \zeta_3 x, -\zeta_7^2 y, \zeta_7 t).$$

The automorphism σ_{42} acts on the fiber of type IV^* as a reflection, moving two legs and leaving the third invariant. Thus on the fiber IV^* σ_{42} fixes 4 isolated points. The 2 isolated points fixed by σ_{21} on the cuspidal fiber over $t = \infty$ are fixed by σ_{42} too. In particular, the invariants of $\text{Fix}(\sigma_{42}^j)$, $j = 1, 2, 3, 6, 7, 14, 21$ are as in the first row of Table 9.

Example 6.6. The K3 surface is the same as in Example 4.5, see [11]. On the same elliptic fibration $y^2 = x^3 + t^3(t^7 + 1)$ the order 42 automorphism σ_{42} is given by

$$\sigma_{42} : (x, y, t) \mapsto (\zeta_7^3 \zeta_3 x, -\zeta_7 y, \zeta_7^3 t).$$

One can check that the invariants of $\text{Fix}(\sigma_{42}^j)$, $j = 1, 2, 3, 6, 7, 14, 21$ are as in the second row of Table 9.

Example 6.7. The K3 surface is the same as in Example 4.7, see [11]. On the same elliptic fibration $y^2 = x^3 + t^5(t^7 + 1)$ the order 42 automorphism σ_{42} is given by

$$\sigma_{42} : (x, y, t) \mapsto (\zeta_{42}^2 x, \zeta_{42}^3 y, \zeta_{42}^{18} t)$$

On the fiber II^* σ_{42} fixes 8 isolated points and the central component R . It also fixes 1 point on the elliptic curve E over $t = 0$. Therefore, the invariants of $\text{Fix}(\sigma_{42}^j)$, $j = 1, 2, 3, 6, 7, 14, 21$ are as in the third row of Table 9.

7. NOT PURELY NON-SYMPLECTIC AUTOMORPHISMS

As we observed in Section 2, a not purely non-symplectic automorphism f is such that its action on the period ω_X is given by multiplication by a non-primitive n -th root of unity (different from 1). As a consequence, at least one power of f is symplectic.

The following are well known results about symplectic automorphisms on K3 surfaces. First, by [27], a symplectic automorphism can only fix isolated points, and its order must be less than or equal to eight. Moreover, according to the possible orders:

Lemma 7.1. (see [16, Prop 1.1],[17, Prop. 5.1], [27]) *Given a symplectic automorphism g on a K3 surface, the number N of isolated fixed points and the rank of the invariant lattice $S(g)$ are shown in the following table.*

$ord(g)$	N	$rk S(g)$	$ord(g)$	N	$rk S(g)$
2	8	14	6	2	6
3	6	10	7	3	4
4	4	8	8	2	2
5	4	6			

In this section we will provide a complete classification of not purely non-symplectic automorphisms of orders 14, 21, 28 and 42 according to which powers of the automorphisms are assumed to be symplectic.

7.1. Order 14. Let σ_{14} be a non-symplectic automorphisms of order 14 such that either $\sigma_7 = \sigma_{14}^2$ or $\sigma_2 = \sigma_{14}^7$ are symplectic. We will study the two cases separately.

7.1.1. σ_7 symplectic. When the square of σ_{14} is symplectic we prove:

Proposition 7.2. *Let σ_{14} be a non-symplectic automorphism of order 14 on a K3 surface X such that $\sigma_7 = \sigma_{14}^2$ is symplectic. Then $\text{Fix}(\sigma_{14})$ consists of 3 isolated points and the possible values of (d_{14}, d_7, d_2, d_1) are $(2, 1, 2, 2), (3, 0, 3, 1)$. In the first case, $\text{Fix}(\sigma_2)$ consists of a curve of genus 3, while in the second case, it consists of a curve of genus 10. Moreover, both possibilities occur.*

Proof. By Lemma 7.1, the fixed locus of σ_7 consists of 3 isolated points and since $\text{Fix}(\sigma_{14}) \subset \text{Fix}(\sigma_7)$, it follows that the number N_{14} of isolated points fixed by σ_{14} is at most 3. Now, by Lemma 7.1, we also know the invariant lattice of σ_7 has rank 4. Therefore, by Remark 3.8, we further know $d_2 + d_1 = 4$ and $d_{14} + d_7 = 3$. Moreover, by the topological Lefschetz formula (3) (applied to σ_{14}) we have

$$\chi_{14} = N_{14} = 2 + d_{14} - d_7 - d_2 + d_1.$$

Further observing that we must have $d_2 > 0$ and $d_1 > 0$, these give the following list of possibilities for $(d_{14}, d_7, d_2, d_1; N_{14})$:

$$(0, 3, 1, 3; 1), (1, 2, 1, 3; 3), (1, 2, 2, 2; 1), (2, 1, 2, 2; 3), (2, 1, 3, 1; 1), (3, 0, 3, 1; 3).$$

As in Section 3.1, using (7) and [28], we can compute χ_2 and the possible invariants (g_2, k_2) of the fixed locus of σ_2 . These are listed in Table 10 below. In particular, we observe that if $(d_{14}, d_7, d_2, d_1) = (0, 3, 1, 3)$, then we would have $\chi_2 = 22$, which is impossible by [28].

N_{14}	d_{14}	d_7	d_2	d_1	χ_2	(g_2, k_2)
1	0	3	1	3	22	-
1	1	2	2	2	8	$(3, 6), (2, 5), (1, 4), (0, 3)$
1	2	1	3	1	-6	$(6, 2), (5, 1), (4, 0)$
3	1	2	1	3	10	$(2, 6), (1, 5), (0, 4)$
3	2	1	2	2	-4	$(6, 3), (5, 2), (4, 1), (3, 0)$
3	3	0	3	1	-18	$(10, 0)$

TABLE 10.

With computations similar to the ones of Section 3.2, we can actually eliminate many of the other possibilities. In fact we see we must have that $\text{Fix}(\sigma_{14}) = \text{Fix}(\sigma_7)$ consists of 3 isolated points and $\text{Fix}(\sigma_2)$ consists of either a curve of genus 3 or a curve of genus 10. The existence of both cases is shown in the following examples. □

Example 7.3. *Let $f(x_0, x_1, x_2) := x_0^3x_1 + x_1^3x_2 + x_2^3x_0$ and consider the K3 surface*

$$X_f := \{(x_0 : x_1 : x_2 : x_3) : x_3^4 = f(x_0, x_1, x_2)\} \subset \mathbb{P}^3.$$

This surface carries the order 14 automorphism $\sigma_{14} : (x_0 : x_1 : x_2 : x_3) \mapsto (\zeta_7^4x_0 : \zeta_7^2x_1 : \zeta_7x_2 : -x_3)$. We have $\text{Fix}(\sigma_{14}) = \{(1 : 0 : 0 : 0), (0 : 1 : 0 : 0), (0 : 0 : 1 : 0)\}$ and $\text{Fix}(\sigma^7)$ is given by the curve $\{x_3 = 0\}$, which has genus three. Note that σ^2 is symplectic.

Example 7.4. Let X be the surface in $\mathbb{P}(3, 1, 1, 1)_{x,y,z,w}$ given as the zero locus of $x^2 + y^5z + z^5w + w^5y$. X admits the action of the order 14 automorphism

$$\sigma_{14} : (x, y, z, w) \mapsto (-x, \zeta_7^5 y, \zeta_7^3 z, \zeta_7^6 w)$$

whose square is symplectic and fixes the three points $\{(0 : 1 : 0 : 0), (0 : 0 : 1 : 0), (0 : 0 : 0 : 1)\}$. The fixed locus of σ_2 is the genus 10 curve $\{x = 0\} \cap X$.

7.1.2. σ_2 symplectic. We now consider what happens when the involution σ_{14}^7 is symplectic and σ_{14}^2 is non-symplectic. About the fixed loci of σ_{14} and its powers, we can prove the following:

Proposition 7.5. Let σ_{14} be a non-symplectic automorphism of order 14 on a K3 surface X and assume the involution $\sigma_2 = \sigma_{14}^7$ is symplectic. Then $\text{Fix}(\sigma_{14})$ consists of $N_{14} \leq 8$ isolated points and the possible values of N_{14} and (d_{14}, d_7, d_2, d_1) are given in Table 11 below, together with χ_7 in each case.

N_{14}	d_{14}	d_7	d_2	d_1	χ_7
1	0	1	8	8	17
8	1	1	2	8	10
1	1	2	2	2	3

TABLE 11.

Proof. By [27], the fixed locus of the symplectic involution σ_2 consists of 8 isolated points. Since $\text{Fix}(\sigma_{14}) \subseteq \text{Fix}(\sigma_2)$, it follows that $N_{14} \leq 8$. The invariant lattice of σ_2 has rank 14 by Lemma 7.1, thus $6d_7 + d_1 = 14$ and $6d_{14} + d_2 = 8$ by Remark 3.8. Moreover, by the topological Lefschetz formula (applied to σ_{14}) we have

$$\chi(\text{Fix}(\sigma_{14})) = N_{14} = 2 + d_{14} - d_7 - d_2 + d_1.$$

Further observing that we must have $d_7 > 0$ and $d_1 > 0$, this gives the above list of possibilities, that is, if the involution σ_2 is symplectic one has 3 possibilities for (d_{14}, d_7, d_2, d_1) , namely $(0, 1, 8, 8), (1, 1, 2, 8), (1, 2, 2, 2)$. \square

According to [12, Table 3] there are four different deformation families of K3 surfaces with these automorphisms. Assuming general conditions, i.e. that the Picard lattice of the surface coincides with $S(\sigma_7)$, we prove the following Lemma which complements Proposition 7.5 (and Table 11).

Lemma 7.6. Let σ_{14} be a non-symplectic automorphism of order 14 on a K3 surface and assume the involution $\sigma_2 = \sigma_{14}^7$ is symplectic. Under the assumption that the Picard lattice agrees with $S(\sigma_7)$ we have that the order seven automorphism $\sigma_7 = \sigma_{14}^2$ cannot be of type \dagger (here we are referring to the notation in Table 1). In particular, σ_7 must fix a curve.

Proof. Since the K3 surface admits a symplectic involution, the transcendental lattice T_X must be primitively embedded in $E_8(2) \oplus U \oplus U \oplus U$ [18]. By assumption, $T_X = T(\sigma_7)$, and we see that σ_7 cannot be of type \dagger , since in that case $T(\sigma_7) = U(7) \oplus U \oplus E_8 \oplus A_6$. \square

As a consequence of the Lemma, the four deformation families given in [12, Table 3] correspond to families of type A, B, C and D. One can check Table 11 to know N_{14} in each case.

We now exhibit two examples of possibilities in Table 11: one belonging to case C, corresponding to the second line of Table 11, and one belonging to case A and corresponding to the third line of Table 11. Observe that in both cases, the K3 surfaces appear also in the classification of Section 3, showing that these surfaces admit both a symplectic and non-symplectic involution.

Example 7.7. (Case C) Let us consider the elliptic K3 surface X with Weierstrass equation given by $y^2 = x^3 + t^2x + t^{10}$, $t \in \mathbb{P}^1$. The automorphism $\sigma_{14} : (x, y, z, t, s) \mapsto (\zeta_7 x, -\zeta_7^5 y, -\zeta_7 t, s)$ is a non-symplectic automorphism of order 14 and σ_2 is symplectic. Note that $\text{Fix}(\sigma_{14})$ consists of 8 points.

Example 7.8. (Case A) Let X be the elliptic K3 surface given by an equation of the form $y^2 = x(x^2 + t^7 + 1)$, $t \in \mathbb{P}^1$. Then X admits the order 14 purely non-symplectic automorphism $\sigma : (x, y, t) \mapsto (x, -y, \zeta_7 t)$ described in Example 3.21, which corresponds to case A1 with $(g_2, k_2) = (3, 2)$ in our classification of Section 3. Composing $\sigma^2 = \sigma_7 : (x, y, t) \mapsto (x, y, \zeta_7 t)$ and the translation

$$\tau : (x, y, t) \mapsto ((y/x)^2 - x, (y/x)^3 - y, t)$$

by the 2-torsion section produces an automorphism of order 14, say φ , which is not purely non-symplectic. By construction, $\varphi^7 = \tau$ and τ is symplectic. The invariants of φ are as in the third row of Table 11.

7.2. Order 21. Let σ_{21} be a non-symplectic automorphism of order 21 such that either $\sigma_7 = \sigma_{21}^3$ or $\sigma_3 = \sigma_{21}^7$ are symplectic. Again, we will study the two cases separately.

7.2.1. σ_7 symplectic.

Proposition 7.9. *If σ_{21} is a non-symplectic automorphism of order 21 on a K3 surface X , then σ_7 cannot be symplectic.*

Proof. By contradiction, assume σ_7 is symplectic. Then, by Nikulin, the fixed locus of σ_7 consists of 3 isolated points. Since $\text{Fix}(\sigma_{21}) \subseteq \text{Fix}(\sigma_7)$, σ_{21} is acting with order three on $\text{Fix}(\sigma_7)$. So $\text{Fix}(\sigma_{21})$ consists of $N_{21} = 0$ or $N_{21} = 3$ isolated fixed points.

Now, because the invariant lattice of σ_7 has rank 4 (Lemma 7.1), we also know $d_3 + d_1 = 4$ and $d_{21} + d_7 = 3$. Moreover, by the topological Lefschetz formula (3) (applied to σ_{21}, σ_7 and σ_3) we have

$$\begin{cases} \chi_{21} &= N_{21} = 2 + d_{21} - d_7 - d_3 + d_1 \\ \chi_7 &= 2 - (2d_{21} + d_7) + 2d_3 + d_1 = 3 \\ \chi_3 &= 2 - 6(d_{21} - d_7) - d_3 + d_1 \end{cases}$$

By Remark 4.3 and further observing that $d_3, d_1 > 0$ these give the possibilities for (d_{21}, d_7, d_3, d_1) and χ_3 shown in Table 12, but since $\chi_3 = N_3 + 2(1 - g_3 + k_3) = N_3 + 2(N_3 - 3) = 3N_3 - 6$ we can eliminate all cases.

N_{21}	(d_{21}, d_7, d_3, d_1)	χ_3
3	(2,1,2,2)	-4
3	(1,2,1,3)	10
3	(3,0,3,1)	-18

TABLE 12.

□

7.2.2. σ_3 symplectic. Similarly, we can prove:

Proposition 7.10. *Let σ_{21} be a non-symplectic automorphism of order 21 on a K3 surface X and assume σ_3 is symplectic. Then $\text{Fix}(\sigma_{21})$ consists of exactly $N_{21} = 6$ isolated points and the only possible values for (d_{21}, d_7, d_3, d_1) are $(1, 1, 0, 4)$. Moreover, $\text{Fix}(\sigma_7)$ is as in case A of Table 1 and such an automorphism exists (see Example 7.11).*

Proof. By Lemma 7.1, the fixed locus of σ_3 consists of 6 isolated points. Since $\text{Fix}(\sigma_{21}) \subseteq \text{Fix}(\sigma_3)$, it follows that $\text{Fix}(\sigma_{21})$ consists of $N_{21} \leq 6$ isolated fixed points. The invariant lattice of σ_3 has rank 10 by Lemma 7.1, so we also know $6d_7 + d_1 = 10$ and $6d_{21} + d_3 = 6$. Further observing that $d_7, d_1 > 0$, the topological Lefschetz formula (3) (applied to σ_{21}, σ_7 and σ_3) gives $(d_{21}, d_7, d_3, d_1) = (1, 1, 0, 4)$, $N_{21} = 6$ and $\chi_7 = 3$.

Note that since $\text{Fix}(\sigma_{21}) = \{6 \text{ pts}\} \subseteq \text{Fix}(\sigma_7)$, the above implies σ_7 is of type A. That is, $\text{Fix}(\sigma_7) = E \cup 3 \text{ pts}$ and we must have three fixed points under σ_{21} lying on E . □

Example 7.11. *In $\mathbb{P}(3, 2, 1, 1)$ we consider the surface*

$$x^2w + xy^2 + yw^5 + z^7 = 0$$

with the order 21 automorphisms

$$\sigma_{21} : (x, y, z, w) \mapsto (\zeta_3x, \zeta_3y, \zeta_7z, \zeta_3w)$$

The order 7 automorphism σ_7 is non-symplectic and fixes the genus 1 curve $\{z = 0\}$ and 3 more points on the resolutions of the singularities $(1:0:0:0)$ and $(0:1:0:0)$, of type A_2 and A_1 respectively. The automorphism $\sigma^7 = \sigma_3$ is symplectic.

7.3. Order 28. Let σ_{28} be a non-symplectic automorphisms of order 28. We will prove in what follows that no power of σ_{28} can be symplectic.

Proposition 7.12. *If σ_{28} is a non-symplectic automorphism of order 28, then σ_{28} is purely non-symplectic. In other words, no power of σ_{28} is symplectic.*

Proof. We will assume some power of σ_{28} is symplectic. Since there are no symplectic automorphisms of (finite) order bigger than 8 by [27], we have to consider three possibilities:

Case I $\sigma_7 = \sigma_{28}^4$ is a symplectic automorphism of order 7; or

Case II $\sigma_4 = \sigma_{28}^7$ is a symplectic automorphism of order 4; or

Case III $\sigma_2 = \sigma_{28}^{14}$ is a symplectic involution.

Observe that in this last case, $\sigma_4 = \sigma_{28}^7$ is also symplectic.

We refer to Section 5 for the definition of $(d_{28}, d_{14}, d_7, d_4, d_2, d_1)$ and recall the relations given in Remark 5.2:

$$\text{rk } S(\sigma_7) = 2d_4 + d_2 + d_1, \quad \text{rk } S(\sigma_4) = 6d_7 + d_1, \quad \text{rk } S(\sigma_2) = 6(d_{14} + d_7) + d_2 + d_1.$$

We now study the three cases separately.

Case I If σ_7 is a symplectic automorphism of order 7, the action of σ_7^* on the period ω_X of X is trivial. Therefore $\sigma_{28}^* \omega_X = \zeta_4 \omega_X$, which implies that $d_4 = \dim H^2(X, \mathbb{R})_{\zeta_4} \geq 1$. Then $(\sigma_{14})^* \omega_X = \pm \omega_X$ and since there are no symplectic automorphisms of finite order bigger than 8 by [27], then $(\sigma_{14})^* \omega_X = -\omega_X$. By Lemma 7.1, the fixed locus of a symplectic automorphism of order 7 consists of 3 isolated points. Since $\text{Fix}(\sigma_{28}) \subseteq \text{Fix}(\sigma_7)$, then σ_{28} only fixes isolated points and their number is $N_{28} \leq 3$. Moreover, σ_{28} acts with order 1, 2 or 4 on $\text{Fix}(\sigma_7)$, hence $N_{28} = 1$ or 3. By Lemma 7.1, $\text{rk } S(\sigma_7) = 4$; it follows by the above formulas and (11) that

$$\begin{cases} 2d_4 + d_2 + d_1 = 4 \\ 6(2d_{28} + d_{14} + d_7) = 18 \\ \chi_{28} = N_{28} = 2 + d_{14} - d_7 - d_2 + d_1 \end{cases}$$

This gives the following list of possibilities for $(d_{28}, d_{14}, d_7, d_4, d_2, d_1)$:

$$(13) \quad \begin{aligned} & (1, 0, 1, 1, 1, 1), (0, 1, 2, 1, 1, 1), (0, 0, 3, 1, 0, 2), (1, 1, 0, 1, 1, 1), \\ & (0, 2, 1, 1, 1, 1), (1, 0, 1, 1, 0, 2), (0, 1, 2, 1, 0, 2). \end{aligned}$$

Moreover observe that σ_{14} is non-symplectic, σ_7 is symplectic and $\sigma_7 = (\sigma_{14})^2$. Thus we can use the classification of not purely non-symplectic automorphisms of order 14 given in Proposition 7.2. In this case, the possible values of $(a', b', c', d') = (d_{14}, d_7, d_2, d_1)$ are $(2, 1, 2, 2)$, $(3, 0, 3, 1)$ and the relations with $(d_{28}, d_{14}, d_7, d_4, d_2, d_1)$ are

$$d_{28} = a', \quad d_{14} + d_7 = b', \quad c' = d_4, \quad d' = d_2 + d_1.$$

The vectors in (13) do not satisfy the above conditions, thus it is not possible for σ_7 to be symplectic.

Case II Assume σ_4 is symplectic. Since $\sigma_4^* \omega_X = \omega_X$, then $\sigma_{28}^* \omega_X = \zeta_7 \omega_X$, which implies $d_7 \geq 1$. By Lemma 7.1, the fixed locus of σ_4 consists of 4 isolated points. Since $\text{Fix}(\sigma_{28}) \subseteq \text{Fix}(\sigma_4)$, it follows that σ_{28} only fixes N_{28} isolated points with $N_{28} \leq 4$. Moreover, σ_{28} acts with order 1 or 7 on $\text{Fix}(\sigma_4)$, hence $N_{28} = 4$. By Lemma 7.1, $\text{rk } S(\sigma_4) = 8$, thus it follows from the above formulas and (11) that

$$\begin{cases} 6d_7 + d_1 = 8 \\ 12d_{28} + 6d_{14} + 2d_4 + d_2 = 14 \\ \chi_{28} = N_{28} = 2 + d_{14} - d_7 - d_2 + d_1 = 4 \end{cases}$$

The only solution is $(d_{28}, d_{14}, d_7, d_4, d_2, d_1) = (0, 1, 1, 4, 0, 2)$. Observe that in this case $\sigma_{28}^2 = \sigma_{14}$ is non-symplectic. By (11), we can compute $\chi_{14} = -6$ and this is impossible since by [9], the Euler characteristic of the fixed locus of a non-symplectic automorphism of order 14 is bigger than 0. Thus there are no possibilities for $(d_{28}, d_{14}, d_7, d_4, d_2, d_1)$ and hence σ_4 can't be symplectic.

Case III We now show that there is no K3 surface with a non-symplectic automorphism σ_{28} such that $\sigma_4 = \sigma_{28}^7$ is non-symplectic and $\sigma_2 = \sigma_{28}^{14}$ is symplectic. Assume the involution σ_2 is symplectic and σ_4 is non-symplectic. Thus

$$\sigma_2^* \omega_X = \omega_X, \quad \sigma_{28}^* \omega_X = \zeta_{14}^i \omega_X, \quad \sigma_4^* \omega_X \neq \omega_X.$$

Thus we are interested in odd i 's, such that $\sigma_4^* \omega_X = -\omega_X$. In particular, $d_{14} \geq 1$. By Lemma 7.1, the fixed locus of σ_2 consists of 8 isolated points and since $\text{Fix}(\sigma_{28}) \subseteq \text{Fix}(\sigma_2)$, it follows that σ_{28} only fixes N_{28} isolated points and $N_{28} \leq 8$. Moreover, σ_{28} acts on $\text{Fix}(\sigma_2)$ with order 1, 2, 7 or 14; it follows that either N_{28} is even or $N_{28} = 1$. By Lemma 7.1, $\text{rk } S(\sigma_2) = 14$, thus it follows from the above formulas and (11) that

$$\begin{cases} 6d_{14} + 6d_7 + d_2 + d_1 = 14 \\ 12d_{28} + 2d_4 = 8 \\ \chi_{28} = N_{28} = 2 + d_{14} - d_7 - d_2 + d_1 = 4 \end{cases}$$

This gives the following list of possibilities for $(d_{28}, d_{14}, d_7, d_4, d_2, d_1)$:

$$(14) \quad (0, 2, 0, 4, 1, 1), (0, 1, 1, 4, 1, 1), (0, 2, 0, 4, 0, 2), (0, 1, 1, 4, 0, 2), (0, 1, 0, 4, 5, 3).$$

Moreover, observe that σ_{14} is non-symplectic, σ_2 is symplectic and $\sigma_2 = (\sigma_{14})^7$. Thus we can use the classification of not purely non-symplectic automorphisms of order 14 given in Proposition 7.2. In Proposition 7.5 we found three possible vectors $(a', b', c', d') = (d_{14}, d_7, d_2, d_1)$:

$$(0, 1, 8, 8), (1, 1, 2, 8), (1, 2, 2, 2)$$

and the relations with $(d_{28}, d_{14}, d_7, d_4, d_2, d_1)$ are, as before,

$$d_{28} = a', d_{14} + d_7 = b', c' = d_4, d' = d_2 + d_1.$$

The vectors in (14) do not satisfy the above conditions, thus it is not possible for σ_2 to be symplectic.

Therefore, we proved that a non-symplectic automorphism of order 28 is necessarily purely non-symplectic. □

7.4. Order 42. Let σ_{42} be a non-symplectic automorphism of order 42. We will prove in what follows that $\sigma_{42}^{14} = \sigma_3$ can be symplectic, but any other power σ_{42}^k , where $k = 6, 7$ or 21 must be non-symplectic. Note that there are no symplectic automorphisms of (finite) order bigger than 8 by [27].

We first prove the following:

Proposition 7.13. *Let σ_{42} be a non-symplectic automorphism of order 42 on a K3 surface X , and assume $\sigma_3 = \sigma_{42}^{14}$ is symplectic. Then $\text{Fix}(\sigma_{42})$ consists of 2 or 4 isolated points. In the first case $\text{Fix}(\sigma_{14})$ is as in case A2(9,0) of Table 2; and, in the second, it is as in case A1(9,1). Moreover, both cases exist (see Example 7.14 and Example 7.15).*

Proof. Let $\sigma = \sigma_{42}$. If $\sigma^{14} = \sigma_3$ is symplectic. Then $\sigma^2 = \sigma_{21}$ is a non-symplectic automorphism of order 21 such that σ_{21}^7 is symplectic. Therefore, we can apply Proposition 7.10 to conclude that σ_{21} fixes exactly 6 points and $\sigma_7 = \sigma^6$ is of type A of Table 1.

Now, if we let $a = d_{42} + d_{21}, b = d_{14} + d_7, c = d_6 + d_3, d = d_2 + d_1$, and $d_i \doteq \dim H^2(X, \mathbb{R})_{\zeta_i}$ for $i = 42, 21, 14, 7, 6, 3, 2, 1$, then Proposition 7.10 also gives us $(a, b, c, d) = (1, 1, 0, 4)$.

Combining the above with the Topological Lefschetz formula (3) applied to the powers of σ as in (12) (and imposing the relations in Remark 6.2) gives the following list of possible values for $(d_{42}, d_{21}, d_{14}, d_7, d_6, d_3, d_2, d_1)$:

$$(1, 0, 1, 0, 0, 0, 2, 2), (1, 0, 1, 0, 0, 0, 1, 3)$$

Note that $\sigma^* \omega = \zeta_{14}^i \omega$ for some $1 \leq i \leq 13$, and if i is even (resp. $i = 7$), then σ^7 (resp. σ^2) is symplectic, but the latter is impossible by Proposition 7.10 and Lemma 7.1. Thus, i is odd ($\neq 7$) and $d_{14} \geq 1$. In other words, $\sigma^3 = \sigma_{14}$ is purely non-symplectic of order 14. In particular, $\chi_2 \in \{0, -16, -14\}$, by Proposition 3.1.

In addition, note also that using (12), the first vector gives us $(\chi_{42}, \chi_{21}, \chi_{14}, \chi_7, \chi_6, \chi_3, \chi_2) = (2, 6, 5, 3, 2, 6, -16)$, while the second gives $(\chi_{42}, \chi_{21}, \chi_{14}, \chi_7, \chi_6, \chi_3, \chi_2) = (4, 6, 7, 3, 4, 6, -14)$.

Therefore, $\text{Fix}(\sigma)$ consists of 2 or 4 isolated points. In the first case, σ_{14} is of type A(9,0) of Table 2 and, in the second, σ_{14} is of type A(9,1). \square

Example 7.14. In $\mathbb{P}(4, 2, 1, 1)_{x,y,t,s}$ we consider the K3 surface given by

$$x^2 + s^7 t + y^4 + yt^6$$

with the order 42 automorphism

$$\sigma_{42} : (x, y, z, w) \mapsto (-x, y, \zeta_7 \zeta_3 t, \zeta_3^2 s).$$

The order 14 automorphism σ_{14} is purely non-symplectic of type A2(9, 0), i.e. σ_7 fixes the genus 1 curve $\{s = 0\}$ and 3 points, two of them on the resolutions of the singularities of $(1 : -1 : 0 : 0)$ and $(1 : 1 : 0 : 0)$ of type A_1 . Moreover, we have that σ_2 fixes a genus 9 curve $\{x = 0\}$. The automorphism $\sigma^{14} = \sigma_3$ is symplectic.

Example 7.15. In $\mathbb{P}(5, 3, 1, 1)_{x,y,t,s}$ we consider the K3 surface given by

$$x^2 + s^7 y + y^3 t + t^{10}$$

with the order 42 automorphism

$$\sigma_{42} : (x, y, z, w) \mapsto (-x, \zeta_3 y, t, \zeta_7 \zeta_3^2 s).$$

The order 14 automorphism σ_{14} is purely non-symplectic of type A1(9, 1), i.e. σ_7 fixes the genus 1 curve $\{s = 0\}$ and 3 points, two of them on the resolutions of the singularities of $(0 : 1 : 0 : 0)$ of type A_2 . Moreover, we have that σ_2 fixes a genus 9 curve $\{x = 0\}$ and a rational curve. The automorphism $\sigma^{14} = \sigma_3$ is symplectic.

In contrast, we further prove:

Proposition 7.16. Let σ_{42} be a non-symplectic automorphism of order 42 on a K3 surface X . Then σ_{42}^k is non-symplectic for $k = 6, 7, 21$.

Proof. By contradiction, assume $\sigma = \sigma_{42}$ is a non-symplectic automorphism of order 42 such that one of the powers σ_{42}^k is symplectic with $k = 6, 7$ or 21. That is, assume there exists a $k \in \{6, 7, 21\}$ such that the action of σ on ω_X is given by multiplication by ζ_k^i for some $1 \leq i < k$.

k = 6 If $\sigma^6 = \sigma_7$ is symplectic, then $\sigma^2 = \sigma_{21}$ is a non-symplectic automorphism of order 21 such that σ_{21}^3 is symplectic. But this contradicts Proposition 7.9.

k = 7 If σ^7 is symplectic, then σ^{14} is also symplectic. And since $\text{Fix}(\sigma) \subset \text{Fix}(\sigma^7) \subset \text{Fix}(\sigma^{14})$, the proof of Proposition 7.13 implies we must have

$$(d_{42}, d_{21}, d_{14}, d_7, d_6, d_3, d_2, d_1) = (1, 0, 1, 0, 0, 0, 2, 2)$$

where $d_i \doteq \dim H^2(X, \mathbb{R})_{\zeta_i}$ for $i = 42, 21, 14, 7, 6, 3, 2, 1$. But for this vector we do not have $d_7 \geq 1$. Therefore, σ^7 cannot be symplectic.

k = 21 Finally, assume $\sigma^{21} = \sigma_2$ is symplectic. Then $\sigma^i \omega_X = \zeta_{21}^i \omega_X$ for some $1 \leq i \leq 20$. If $i = 7$, then σ^3 is symplectic, which is impossible by [27]. If $i = 3$, then σ^7 would be symplectic, which we showed is not possible (case $k = 7$). Therefore, $d_{21} \geq 1$.

Now, observe $\sigma_2 = \sigma_{14}^7$ so that we can use our classification results from Section 7.1 to conclude σ_2 cannot be symplectic.

More precisely, applying Proposition 7.5, and letting $a = d_{42} + d_{14}, b = d_{21} + d_7, c = d_2 + d_6, d = d_3 + d_1$, we find that the possible values of (a, b, c, d) are $(0, 1, 8, 8), (1, 1, 2, 8)$ or $(1, 2, 2, 2)$. But for these three vectors, if we use (12) and Remark 6.2 together with the fact that $d_{21}, d_1 \geq 1, \chi_2 = 8, \chi_{42} \leq 8$ and

$$22 = 12(d_{42} + d_{21}) + 6(d_{14} + d_7) + 2(d_6 + d_3) + d_2 + d_1$$

then we find no possible solutions for $(d_{42}, d_{21}, d_{14}, d_7, d_6, d_3, d_2, d_1)$. \square

8. THE NÉRON–SEVERI LATTICE

We conclude with a description of the Néron–Severi lattice of a K3 surface X admitting a purely non-symplectic automorphism $\sigma = \sigma_n$ of order $n = 14, 21, 28$ or 42. Under the assumption of generality we have:

$$(15) \quad r \doteq \text{rk } NS(X) = 22 - d_n \cdot \varphi(n)$$

and using the results obtained in the previous sections we are able to describe $NS(X)$ in every case. Since the invariant lattices $S(\sigma^i)$ are all primitively embedded in $NS(X)$ by [27, Section 3], if we can find one power i such that the corresponding invariant lattice has the expected rank r , then we can conclude we have equality $S(\sigma^i) = NS(X)$.

We will call a pair (X, σ_n) satisfying (15) as above a general pair and we will use the classification of automorphisms of prime orders in [9] and [28] in order to describe explicitly the lattices $NS(X)$.

Our results are presented in Propositions 8.1 and 8.4 below:

Proposition 8.1. *Let (X, σ_{14}) be a general pair. For each possibility listed in Table 5, with the exception of case $C1(0, 2)$ (see Remark 8.3), the Néron–Severi lattice $NS(X)$ is as in Table 13 below.*

	χ_{14}	χ_7	χ_2	(d_{14}, d_7, d_2, d_1)	$NS(X)$
$A1$	7	3	-14	$(3, 0, 1, 3)$	$S(\sigma_7) = U \oplus K_7$
$A1$	7	3	0	$(2, 1, 0, 4)$	$S(\sigma_2)$
$A2$	5	3	-16	$(3, 0, 2, 2)$	$S(\sigma_7) = U \oplus K_7$
$B3$	14	10	0	$(2, 0, 0, 10)$	$S(\sigma_7) = U \oplus E_8$
$C1$	6	10	-8	$(2, 0, 4, 6)$	$S(\sigma_7) = U(7) \oplus E_8$
$C2$	4	10	-10	$(2, 0, 5, 5)$	$S(\sigma_7) = U(7) \oplus E_8$
$C3$	8	10	-6	$(2, 0, 3, 7)$	$S(\sigma_7) = U(7) \oplus E_8$
$D2$	3	17	-4	$(1, 0, 8, 8)$	$S(\sigma_7) = U \oplus E_8 \oplus A_6$
$D3$	7	17	0	$(1, 0, 6, 10)$	$S(\sigma_7) = U \oplus E_8 \oplus A_6$
$D8$	13	17	6	$(1, 0, 3, 13)$	$S(\sigma_7) = U \oplus E_8 \oplus A_6$

TABLE 13.

Proof. For $n = 14$, one has $\varphi(14) = 6$ and by Remark 3.8, $\text{rk } S(\sigma_7) = d_2 + d_1$, $\text{rk } S(\sigma_2) = 6d_7 + d_1$. By (15) one has

- if $d_{14} = 3$, $\text{rk } NS(X) = 4$;
- if $d_{14} = 2$, $\text{rk } NS(X) = 10$;
- if $d_{14} = 1$, $\text{rk } NS(X) = 16$.

and we get that the Néron–Severi lattice $NS(X)$ is as in Table 13. \square

Remark 8.2. *If $\sigma = \sigma_{14}$ is a purely non-symplectic automorphism of order 14 on a K3 surface X such that σ^2 is of type D , then $NS(X) = S(\sigma^2)$. In fact, we know that $r = \text{rk } NS(X) \geq 16 = \text{rk } S(\sigma^2)$, hence the rank ℓ of the transcendental lattice $NS(X)^\perp$ is at most 6. But since ℓ must be divisible by $\varphi(14) = 6$, it must be the case that $\ell = 6$ and $r = 16$.*

Remark 8.3. *For a general pair (X, σ_{14}) such that $\text{Fix}(\sigma_{14})$ is of type $C1(0, 2)$, none of the invariant lattices $S(\sigma_{14}^i)$ have the expected rank. Thus we are not able to compute the Néron–Severi lattice of the general K3 surface in this case.*

When $n = 21, 28$ or 42 , we have that $\varphi(21) = \varphi(28) = \varphi(42) = 12$ and for all cases $d_n = 1$, thus $\text{rk } NS(X) = 22 - 12 = 10$. We prove:

Proposition 8.4. *If $n = 21, 28$ or 42 , the description of the lattice $NS(X)$ for a general pair (X, σ_n) is as follows:*

(i) *If $n = 21$, the possibilities are shown in the following table:*

Type σ_{21}	χ_{21}	χ_7	χ_3	(d_{21}, d_7, d_3, d_1)	$NS(X)$
$C(3, 2, 3)$	10	10	3	$(1, 0, 1, 8)$	$S(\sigma_7) = U(7) \oplus E_8$
$C(3, 1, 2)$	7	10	0	$(1, 0, 2, 6)$	$S(\sigma_7) = U(7) \oplus E_8$
$C(3, 0, 1)$	4	10	-3	$(1, 0, 3, 4)$	$S(\sigma_7) = U(7) \oplus E_8$
$B(3, 3, 4)$	13	10	6	$(1, 0, 0, 10)$	$S(\sigma_7) = U \oplus E_8$

(ii) *Similarly, if $n = 28$ we have the following table of possibilities:*

Type σ_{14}	χ_2	χ_4	χ_7	χ_{14}	χ_{28}	$(d_{28}, d_{14}, d_7, d_4, d_2, d_1)$	$NS(X)$
$A1(3,2)$	0	12	3	7	5	$(1,0,1,0,0,4)$	$S(\sigma_2)$
$A1(3,2)$	0	-4	3	7	3	$(1,1,0,0,2,2)$	$S(\sigma_2)$
$B3(6,5)$	0	12	10	14	12	$(1,0,0,0,0,10)$	$S(\sigma_2)$

(iii) And if $n = 42$ we have:

Type σ_{14}	χ_{21}	χ_7	χ_3	$(d_{42}, d_{21}, d_{14}, d_7, d_6, d_3, d_2, d_1)$	$NS(X)$
$C1$	10	10	3	$(1,0,0,0,1,0,2,6)$	$S(\sigma_7) = U(7) \oplus E_8$
$C3$	7	10	0	$(1,0,0,0,1,1,1,5)$	$S(\sigma_7) = U(7) \oplus E_8$
$B3$	13	10	6	$(1,0,0,0,0,0,0,10)$	$S(\sigma_7) = U \oplus E_8$

Proof. It follows from Remarks 4.3, 5.2 and 6.2. \square

Remark 8.5. We observe that when $n = 28$ and $\text{Fix}(\sigma_{14})$ is of type $A1(3, 2)$, then the 2-elementary lattice $S(\sigma_2)$ has invariants $(r, a) = (10, 6)$. But, a priori, the invariant δ is not unique. By [18, Theorem 0.1], we have that $\delta = 0$ if and only if X also admits a symplectic involution.

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