Mathematisches Forschungsinstitut Oberwolfach

Report No. 51/2015

DOI: 10.4171/OWR/2015/51

Mini-Workshop: Singular Curves on $K3$ Surfaces and Hyperkähler Manifolds

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8 November – 14 November 2015

Abstract. The workshop focused on Severi varieties on $K3$ surfaces, hyperkähler manifolds and their automorphisms. The main aim was to bring researchers in deformation theory of curves and singularities together with researchers studying hyperkähler manifolds for mutual learning and interaction, and to discuss recent developments and open problems.

Mathematics Subject Classification (2010): Primary: 14H10, 14H20, 14H51, 14C20, 14J28, 14J50; Secondary: 14B07, 14E30.

Introduction by the Organisers

The workshop was attended by 15 participants with broad geographic and thematic representation. Its main aim was to bring together researchers in deformation theory of curves and singularities, especially working on Severi varieties of singular curves on $K3$ surfaces, together with researchers studying hyperkähler manifolds and their automorphisms.

Severi varieties take their name from the mathematician who introduced them at the beginning of last century. Let S be a smooth complex projective surface and $|D|$ a linear system on S containing smooth irreducible curves. The *Severi variety of δ-nodal curves* $V^S_{|D|,\delta} \subseteq |D|$ is defined as the locally closed subset of $|D|$ parametrizing irreducible curves with only δ nodes as singularities. Curves on smooth surfaces, their moduli and their enumerative geometry have been fundamental topics of algebraic geometry from the beginning of the previous century until today, thanks to the contribution of Severi, Segre, Zeuthen, Albanese, Enriques, Castelnuovo, Zariski, Arbarello, Cornalba, Harris, Shustin, Greuel and many others. An important breakthrough was made by Harris [18], who proved that Severi varieties of nodal plane curves are irreducible, as stated by Severi. Some years later, Kontsevich and Manin [23], by using Gromow-Witten theory, computed the degree of the Severi variety of rational plane curves. Their formulas were generalized by Caporaso and Harris [10], who found a recursive formula for the degree of Severi varieties of nodal plane curves of any genus, using only classical techniques. Later on, great progress was made in the study of the enumerative geometry of $V^S_{|D|,\delta}$, by among others Pandharipande, Vakil, Ran, Göttsche, Yau, Zaslow, Vainsencher, Tzeng and Thomas. Although a lot of work has been made on Severi varieties, many interesting problems remain open, especially in the case of $K3$ surfaces, as explained in the abstracts of **Ciliberto–Flamini** and **Dedieu**.

At the same time, the Brill-Noether theory of smooth curves on $K3$ surfaces has received a lot of attention in the last couple of decades, from the seminal papers of Lazarsfeld and Green [24, 17] to the more recent works on the Green conjecture and divisors on the moduli space of curves of Voisin, Farkas, Popa and Aprodu [26, 25, 14, 1]. Very recently, two conjectures about syzygies of curves, the *Green-Lazarsfeld secant conjecture* and the *Prym-Green conjecture* were (essentially) solved by Farkas and Kemeny in [12, 13] using curves on $K3$ surfaces, and an account of this is given in **Kemeny**'s abstract. Similarly, two outstanding conjectures by Wahl were established in [2], where it is proved that a Brill-Noether-Petri curve of genus ≥ 12 lies on a polarised $K3$ surface or on a limit of such if and only if the Wahl map for C is not surjective. An account of related open problems is made in **Sernesi**'s abstract.

The recent paper [11] starts the study of Brill-Noether theory of singular curves on a $K3$ surface S. Besides its intrinsic interest, the study is related to Mori theory of hyperkähler manifolds: indeed, curves on S with normalizations carrying pencils of degree k define rational curves on the Hilbert scheme $S^{[k]}$ of k points on the surface, one of the few examples known (together with its deformations) of hyperkähler manifolds. The other known examples are Albanese fibers of Hilbert schemes of points on abelian surfaces, called *generalized Kummer varieties*, (and their deformations), as well as two examples of O'Grady in dimensions 6 and 10. We recall that a (compact) *hyperkähler manifold* (or *irreducible holomorphic symplectic manifold*) is a simply-connected compact complex Kähler manifold X such that $H^0(X, \Omega^2_X)$ is spanned by a nowhere degenerate two-form. The interest in hyperkähler manifolds stems from Bogomolov's decomposition theorem for compact, complex Kähler manifolds with trivial canonical bundle in the 70s: up to finite étale cover they all decompose into products of Calabi-Yau, hyperkähler manifolds and tori. The birational geometry of hyperkähler manifolds is determined by their rational curves; in particular, rational curves determine their nef and ample cones, just like for $K3$s. Many years of research on this topic, passing in particular through several works and conjectures of Hassett and Tschinkel, culminated recently in the work of Bayer and Macrì [5] using *Bridgeland stability*, which determines (up to numerical computations) the extremal rays of the Mori cone of the Hilbert schemes of points on a $K3$ surface.

Despite recent advances by different methods, the study of curves on $K3$ or abelian surfaces with normalizations carrying special pencils still seems to be the most efficient way of concretely producing rational curves on hyperkähler manifolds. The results in [11] were recently extended to abelian surfaces in [21]. Some consequences of the results in [11, 21] on the birational geometry of the associated hyperkähler manifolds are obtained in [22] and the results and some open problems are given in **Knutsen**'s abstract.

Many of the recent results on singular curves on $K3$ (and abelian) surfaces have been proved by degenerating the surfaces. It is therefore natural to ask whether one can find similar degenerations of hyperkähler manifolds, as is done in **Galati**'s abstract, which also gives a brief account on the $K3$ case.

Another way of producing rational curves on $S^{[k]}$ is through automorphisms, as in e.g. [15]: the idea is to start with a special $K3$ surface such that $S^{[k]}$ contains a family of rational curves not present on the general projective deformation of it, use an automorphism of $S^{[k]}$ to produce another family of rational curves, and prove that the latter can be preserved under deformation. This is an interesting point of view, but one needs automorphisms of $S^{[k]}$ *not* coming from automorphisms of S, i.e. non-natural, and at the moment only one such example is known: the involution of Beauville on $S^{[2]}$ when S is a quartic. Thus one is in need of new such constructions. But the construction of new non-natural automorphisms on $S^{[k]}$ and more generally on other hyperkähler manifolds is an interesting and very active research topic on its own. The interest in automorphisms of hyperkähler manifolds has grown tremendously the last years. The foundational work on $K3$ surfaces by Nikulin, Mukai and Morrison was followed by classification results of Sarti with coauthors [3, 4, 16] and the recent work of Huybrechts [20]. Finally, the study of non-symplectic automorphisms on $K3$ surfaces has found a recent application in the study of Chow groups of $K3$ surfaces in particular in relation to the study of rational curves and the Bloch-Beilinson conjecture [19, 20]. Very little is known in higher dimensions, again there are results of Sarti, Boissière and coauthors [6, 7, 8, 9]. The abstract of **Boissière** gives an overview of results on automorphisms of special hyperkähler manifolds; more precise results and some open problems are formulated in the abstracts of **Camere** and **Cattaneo**, concerning existence of automorphisms and moduli spaces.

The abstracts of **Lehn**, **Saccà** and **Markushevich** explain other fundamental topics related to hyperkähler manifolds such as the construction of new manifolds, computation of Hodge numbers and Lagrangian fibrations. Finally, the abstract of **Ohashi** explains results on the automorphism group of Enriques surfaces and curve configurations. The study of the automorphism group of Enriques surfaces is very natural when studying automorphisms of $K3$ surfaces.

To promote interaction, the participants were asked to focus their talks on background results and open problems. Most talks were given in the first two days of the workshop to have time to discuss the proposed problems. We present the abstracts in chronological order and end with a few lines about the discussed open questions.

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Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, "US Junior Oberwolfach Fellows".

Workshop: Mini-Workshop: Singular Curves on $K3$ Surfaces and Hyperkähler Manifolds

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Abstracts

Recent progress on the classification of automorphisms of hyperkähler manifolds

Samuel Boissière

The group $\mathrm{Aut}(X)$ of biholomophisms of an irreducible holomorphic symplectic manifold X is a discrete complex Lie group. Let us focus on finite subgroups of $\mathrm{Aut}(X)$ where X is a $K3$ surface or a deformation of the Hilbert scheme $\mathrm{K3}^{[n]}$ of n points on a $K3$ surface.

The symplectic automorphisms are those acting trivially on the symplectic two-form. All finite groups of symplectic automorphims of $K3$ surfaces have been classified by Nikulin [12] and Mukai [11]. A generalisation of this classification for deformations of $\mathrm{K3}^{[2]}$ was obtained by Mongardi [10] and Höhn–Mason [9]. Of special interest is the solution of a conjecture of Camere [7] in [10], namely that every symplectic involution can be deformed to an involution of a Hilbert scheme of two points induced by a symplectic involution of the underlying $K3$ surface.

Let now G be any finite subgroup of $\mathrm{Aut}(X)$. Looking at the action on the symplectic form, one gets an exact sequence

$$0 \longrightarrow G_0 \longrightarrow G \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow 0$$

where G_0 contains only symplectic automorphisms and m is the non-symplectic index of G. If X is non-projective then $m = 1$ (see [3]). Otherwise, a bound for m is given by $\varphi(m) \leq b_2(X) - \rho(X)$, where φ is the Euler's totient function, $b_2(X)$ is the second Betti number of X and $\rho(X) \geq 1$ is its Picard number. The situation is particularly interesting when $m = p$ is a prime number: if X is a $K3$ surface, then $p \leq 19$; but if X is a deformation of $\mathrm{K3}^{[n]}$ one has $p \leq 23$. Boissière–Camere–Mongardi–Sarti [4] have shown the existence of a unique variety in this deformation class that contains a non-symplectic automorphism of order 23. Non-symplectic involutions have been studied and classified by Beauville [2] and Ohashi–Wandel [13].

We give the main ingredients of the classification, obtained by Boissière–Camere–Sarti [5], of non-symplectic groups G of automorphisms of prime order p such that $3 \leq p \leq 19$, acting on a deformation of $\mathrm{K3}^{[2]}$. The classification is governed by two primitive sublattices of the second cohomology lattice $H^2(X, \mathbb{Z})$, equipped with the Beauville-Bogomolov–Fujiki quadratic form: the sublattice T(G) invariant by the automorphism, and its orthogonal complement S(G). In order to get a classification of the pair of lattices $(\mathrm{T(G)}, \mathrm{S(G)})$, we use on one side deep lattice theorical results on existence of lattices with given signature and discriminant and of embedding of lattices with given orthogonal complement, and on the other side topological information on the fixed locus X^G which is closely related to some numerical invariants of the lattices T(G) and S(G): the Euler characteristics of X^G is computed by applying the Lefschetz topological fixed point formula, and the sum of the dimensions of the mod p cohomology of X^G is computed by using a formula of Boissière–Nieper-Wißkirchen–Sarti [6] originated in Smith theory and making use of the classification theorem of finite-dimensional $\mathbb{F}_p[G]$-modules of Diederichsen and Reiner. This classification reveals two interesting features:

- contrary to the classification of non-symplectic automorphisms on $K3$ surfaces [12, 1], in our situation the pair $(\mathrm{T(G)}, \mathrm{S(G)})$ does not uniquely determine the geometry of the fixed locus;
- contrary to the situation for symplectic involutions on deformations of $\mathrm{K3}^{[2]}$ recalled above, it is not true that non-symplectic automorphisms of prime order $p \geq 3$ are deformations of natural automorphisms on $\mathrm{K3}^{[2]}$.

To illustrate these features we present geometric examples using Hilbert schemes of points on $K3$ surfaces and Fano varieties of lines on cubic fourfolds, endowed with a non-symplectic automorphism of order three.

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Non-symplectic involutions on the Hilbert scheme of points on a $K3$ surface

Andrea Cattaneo

(joint work with S. Boissière, M. Nieper-Wisskirchen and A. Sarti)

Let S be a generic projective $K3$ surface, whose Picard group is generated by a divisor H with $H^2 = 2t$, and consider the Hilbert scheme $S^{[2]}$ of 2 points on S. We describe the automorphism group $\operatorname{Aut} S^{[2]}$ giving an account of results in [1].

We prove that $\operatorname{Aut} S^{[2]}$ has at most one non-trivial element, which is not induced by an automorphism of S (except for the case $t = 1$) and corresponds to a non-symplectic involution. We give necessary and sufficient conditions on t for the existence of such involution. These conditions are related to the solutions of certain Pell's equations, and have a nice translation in the geometry of $S^{[2]}$: they reflect the presence of an ample class on $S^{[2]}$ of Beauville–Bogomolov degree 2.

Since any automorphism of $S^{[2]}$ induces an isometry of the Néron–Severi lattice $\mathrm{NS}(S^{[2]})$ of $S^{[2]}$, the first step is to analyse the orthogonal group $O(\mathrm{NS}(S^{[2]}))$ and then the homomorphism $\Psi : \operatorname{Aut} S^{[2]} \longrightarrow O(\mathrm{NS}(S^{[2]}))$ associating to each automorphism its isometry. We show that Ψ is injective.

The main step in the study of $\operatorname{Aut} S^{[2]}$ is then to relate the isometries in $O(\mathrm{NS}(S^{[2]}))$ with the geometry of $S^{[2]}$, and in particular we show that the presence of non-trivial automorphisms depends on the generators of the ample cone of $S^{[2]}$.

We show under which conditions it is possible to extend an isometry of $\mathrm{NS}(S^{[2]})$ to a Hodge isometry of $H^2(S^{[2]}, \mathbb{Z})$ and finally, using Hodge-theoretic Torelli theorems, how to understand if such extensions are induced by automorphisms.

The first case of a surface S such that $\operatorname{Aut}(S^{[2]}) \simeq \mathbb{Z}/2\mathbb{Z}$ is for $H^2 = 4$ (i.e. $t = 2$), and was geometrically described by Beauville. Here S is a quartic surface in $\mathbb{P}^3$, two points on S define a line that cuts S in two other points, and this sets up the non-symplectic involution on $S^{[2]}$.

The next case happens for $H^2 = 20$ (i.e. $t = 10$).

Question 1. *How can one find a geometric description of the corresponding involution on $S^{[2]}$?*

Up to now there is no such description, and there are several reasons to search for such a description. First of all because it will be the first *explicit* description after Beauville of an involution on $S^{[2]}$ and not on some of its deformations. Then because it provides a link with many other objects of their own interest: Grassmannians, the secant variety of S, moduli spaces of sheaves on S and on curves (in the linear system $|H|$) of genus eleven (this last link provided in [2]).

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Complex ball quotients from four-folds of $K3^{[2]}$-type

Chiara Camere
(joint work with S. Boissière, A. Sarti)

In [4] we study complex ball quotients arising from four-folds of $K3^{[2]}$-type endowed with a non-symplectic automorphism of prime order p with given invariant sublattice T. Such automorphisms have order $3 \leq p \leq 23$ and have been completely classified in [3], [2] and [8].

Let (X, ι) be a T-polarized IHS manifold and $G = \langle \bar{\sigma} \rangle$ a cyclic group of prime order $p \geq 3$ acting non-symplectically on X. Let $\rho : G \longrightarrow O(L)$ be a group homomorphism such that $T = L^{\rho} := \{x \in L \,|\, \rho(g)(x) = x, \forall g \in G\}$. We define a (ρ, T)-*polarization* of (X, ι) as a marking $\eta\colon L \to H^2(X, \mathbb{Z})$ such that $\eta_{|T} = \iota$ and $\bar{\sigma}^* = \eta \circ \rho(\bar{\sigma}) \circ \eta^{-1}$. Let $S(\xi)$ be the eigenspace relative to a primitive p-th root of the unity ξ inside $T^{\perp} \otimes \mathbb{C}$; the period of (X, η) belongs to the space $\Omega_T^{\rho,\xi} := \{x \in \mathbb{P}(S(\xi)) \,|\, q(x + \bar{x}) > 0\}$, which is a complex ball of dimension $\dim S(\xi) - 1$ if $\dim S(\xi) \geq 2$.

Amerik–Verbitsky's [1] theory of algebraic MBM classes and their description of the Kähler cone $\mathcal{K}_X$ of an IHS manifold X allow us to study the image and the locus of injectivity of the period map. Let $\Delta(L)$ be the set of $\delta \in L$ such that there exists a marked pair (X, ϕ) for which $\phi(\delta) \in \mathrm{NS}(X)$ is MBM.

Theorem 2. *The period of a (ρ, T)-polarized pair (X, ϕ) belongs to $\Omega_T^{\rho,\xi} \setminus \Delta$, where*

$$\Delta := \bigcup_{\delta \in \Delta(L) \cap T^{\perp}} (H_\delta \cap \Omega_T^{\rho,\xi}).$$

Given a chamber $K(T)$ of the decomposition of the positive cone of $T \otimes \mathbb{R}$ given by walls in $\Delta(L) \cap T$, a (ρ, T)-polarized pair (X, ϕ) is $K(T)$-*general* if $\phi(K(T)) = \mathcal{K}_X \cap (\phi(T) \otimes \mathbb{R})$. Let $\mathcal{M}_{K(T)}^{\rho,\xi}$ be the set of $K(T)$-general (ρ, T)-polarized pairs.

Theorem 3. *The period map gives a bijection $\mathcal{P} : \mathcal{M}_{K(T)}^{\rho,\xi} \to \Omega := \Omega_T^{\rho,\xi} \setminus (\Delta \cup \Delta')$, where Δ' is a locally finite union of hyperplanes. Moreover, there exist arithmetic subgroups $\mathcal{G}$ and Γ of $O(L)$ and $O(T^{\perp})$ such that the complex ball quotient $\mathcal{M}_{K(T)}^{\rho,\xi}/\mathcal{G} \cong \Omega/\Gamma$ is a quasi-projective variety of dimension* $\dim S(\xi) - 1$.

Once the construction of an algebraic moduli space is established, one of the first natural questions that one can ask is:

Question 4. *What is its Kodaira dimension? In which cases is it rational?*

In the case of complex ball quotients arising from $K3$ surfaces, the pioneering works [6, 5] started the study of the ball quotients corresponding to some non-symplectic automorphisms of order three; later, [7] established rationality for many cases of order three. For natural automorphisms of order three corresponding to the cases studied in the cited papers, the question of rationality can hopefully be answered, since the complex ball quotients studied are the same. For higher orders, rationality seems to be an open problem also in the case of the underlying $K3$ surfaces.

Some of the known non-natural automorphisms of order three appear on Fano varieties of special cubic four-folds. The question is how one can understand whether the corresponding ball quotients are rational.

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Curve configurations on Enriques surfaces and the automorphism groups

Hisanori Ohashi

(joint work with S. Mukai)

For $K3$ and Enriques surfaces, curve configurations play a central role in studying their automorphism groups since the configurations determine a reflection group acting on the cohomology lattice that is a kind of complement to the automorphism group in the total orthogonal group of the hyperbolic lattice. They are described by the dual graph of (-2)-curves. Vinberg [5] developed an effective procedure to measure the size of the reflection group and Nikulin [2, 3, 4] and Kondō [1] classified $K3$ and Enriques surfaces with finite automorphism groups in detail.

We give a further example of Enriques surfaces with a good (-2)-configuration, which is used to determine their infinite automorphism groups precisely by the explicit generators and relations. The new feature here is that the configuration consists of classes of (-2)-curves and centers of numerically reflective involutions.

The resulting dual graph coincides with that of type V surfaces of Kondō [1].

Question 5. *Is there a method to classify the abstract good dual graphs in the Enriques lattice? Can we find surfaces with the (-2)-configurations?*

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Rational curves in hyperkähler manifolds

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(joint work with C. Ciliberto; M. Lelli-Chiesa, G. Mongardi)

Following notation in [4], we let $\varepsilon = 0$ (resp., $\varepsilon = 1$) when S is a $K3$ (resp., abelian) surface, and we denote by $S_\varepsilon^{[k]}$ the Hilbert scheme of k points on S when $\varepsilon = 0$ and the $2k$-dimensional generalised Kummer variety on S when $\varepsilon = 1$. It is well-known that there is a canonical decomposition $N_1(S_\varepsilon^{[k]}) \cong N_1(S) \oplus \mathbb{Z}[\mathfrak{r}_k]$, where $\mathfrak{r}_k$ is the class of a general rational curve contracted by the Hilbert-Chow morphism $S_\varepsilon^{[k]} \to \mathrm{Sym}^{k+\varepsilon}(S)$, and the embedding $N_1(S) \subset N_1(S_\varepsilon^{[k]})$ is given by mapping an irreducible curve $C \subset S$ to the class of the curve $\{x \cup x_1 \cup \cdots \cup x_{k+\varepsilon-1} \mid x \in C\}$ for fixed points $x_1, \ldots, x_{k+\varepsilon-1} \in S \setminus C$. Rational curves in $S_\varepsilon^{[k]}$ correspond to (possibly reducible) curves on S with a partial normalization admitting a linear system of type $g^1_{k+\varepsilon}$. The main results concerning irreducible such curves and the rational curves they define in $S_\varepsilon^{[k]}$ are given in [2, 3]:

Theorem 6. *Let (S, L) be a general polarized $K3$ or abelian surface of genus $p := p_a(L)$. Let g and k be integers satisfying $2\varepsilon \le g \le p$ and $k + \varepsilon \ge 2$. Then the locus of irreducible curves of genus g of class $[L]$ carrying a $g^1_{k+\varepsilon}$ is nonempty if and only if $\rho(p, \alpha, (k+\varepsilon)\alpha + p - g) + \varepsilon\alpha(\alpha+2) \ge 0$, where $\alpha = \left\lfloor \frac{g-\varepsilon}{2(k-1+2\varepsilon)} \right\rfloor$.*

Furthermore, whenever nonempty, this locus is equidimensional of dimension $\min\{g, 2(k-1+\varepsilon)\}$ and the normalization of a general element in each component carries a $\max\{0, \rho(g, 1, k+\varepsilon)\}$-dimensional family of $g^1_{k+\varepsilon}$s.

Theorem 7. *The family of rational curves in $S_\varepsilon^{[k]}$ obtained from curves on S with normalizations carrying a $g^1_{k+\varepsilon}$ is precisely $2(k-1)$-dimensional and there is at least one irreducible component yielding rational curves of class $L-(g+k-1+\varepsilon)\mathfrak{r}_k$.*

We note that $2(k-1)$ is the expected dimension of any family of rational curves in a $2k$-dimensional hyperkähler manifold, by a result of Ran [5]. As an interesting consequence, the family is preserved under any small deformation X_t of $X_0 = S_\varepsilon^{[k]}$ keeping their class algebraic. Another interesting consequence is, using a result of Amerik and Verbitsky [1], that any irreducible components of the locus $W \subset S_\varepsilon^{[k]}$

of the family satisfies the property that the dimension of the fibers of the rational quotient of its desingularization equals the codimension of W in $S_\varepsilon^{[k]}$, and as such, is an *algebraically coisotropic subvariety*, cf. [6]. Also in view of recent conjectures of Voisin in [6], some interesting open questions are:

Question 8. *How can one compute* $\dim W$ *(in terms of properties of* (S, L)*)?*

This question is answered in some particular cases in [4].

Question 9. *When deforming* $X_0 = S_\varepsilon^{[k]}$ *keeping the class of the rational curves algebraic, does (some component of) the locus they cover keep its dimension?*

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Nodal curves on $K3$ surfaces: state of the art and open problems

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The following questions came up from discussions with T. Dedieu, C. Galati, A. L. Knutsen and E. Sernesi.

Let $\mathcal{K}_p$ be the moduli space of primitively polarised $K3$ surfaces (X, L), with $L^2 = 2p - 2 > 0$ and let $\mathcal{V}_{p,n,\delta}$ be the (p, n, δ)–*universal Severi variety* parametrizing all triples (X, L, C) with $(X, L) \in \mathcal{K}_p$ and $C \in |nL|$ (with $n \geq 1$) irreducible and with only δ nodes as singularities, with

$$0 \leq \delta \leq \pi_a(p, n) := n^2(p - 1) + 1.$$

Consider the projection

$$\phi_{p,n,\delta} : \mathcal{V}_{p,n,\delta} \to \mathcal{K}_p$$

whose fiber over any $(X, L) \in \mathcal{K}_p$ is denoted by $V_{n,\delta}(X)$: if nonempty , this is the *Severi variety* of δ-nodal irreducible curves in $|nL|$ on X. The variety $V_{n,\delta}(X)$, if nonempty , is well-known to be smooth, pure of dimension $g_{p,n,\delta} := \pi_a(p, n) - \delta = n^2(p - 1) + 1 - \delta$, and this is the case if $(X, L) \in \mathcal{K}_p$ is general, so also $\mathcal{V}_{p,n,\delta}$ is smooth, pure and each irreducible components of it dominates $\mathcal{K}_p$. We often write g for $g_{p,n,\delta}$, which is the *geometric genus* of curves in $\mathcal{V}_{p,n,\delta}$.

Question 10. *Is* $\mathcal{V}_{p,n,\delta}$ *irreducible?*

Question 11. *Let $(X, L) \in \mathcal{K}_p$ be general and $0 < \delta < \pi_a(p, n)$. Is $V_{n,\delta}(X)$ irreducible?*

Question 12. *Construct examples, if any, of $(X, L) \in \mathcal{K}_p$ and $\delta < \pi_a(p, n)$ such that $V_{n,\delta}(X)$ is empty or reducible.*

Of course an affirmative answer to Question 11 implies an affirmative answer to Question 10 for $\delta < \pi_a(p, n)$. The state of the art , cf. [7], is that Question 10 has an affirmative answer for $n = 1$, $3 \le p \le 9$ or $p = 11$ and any (admissible) δ.

The following particular case of Questions 10 and 11 should be easier:

Question 13. *Let $(X, L) \in \mathcal{K}_p$ be general and $\delta \le \frac{\pi_a(p,n)}{3}$. Is $V_{n,\delta}(X)$ irreducible?*

Partial results in this direction can be found in [18, 19] .

Question 14. *Let $(X, L) \in \mathcal{K}_p$ be general and $\delta \le \pi_a(p, n)$. Compute the Hilbert polynomial of the closure of $V_{n,\delta}(X)$ in $|nL|$.*

State of the art: the degree of $V_{1,\delta}(X)$ is known [5] and the *putative* degree of the 0–dimensional $V_{n,\pi_a(p,n)}(X)$ (i.e., the number of rational curves in $|nL|$) is known for any $n \ge 1$ (this is the *Yau–Zaslow formula* [3]). The case of $V_{n,1}(X)$, i.e., the *dual variety* of X embedded in projective space via $|nL|$, is trivial.

Next we want to propose questions concerning the *moduli map*

$$\psi_{p,n,\delta} : \mathcal{V}_{p,n,\delta} \longrightarrow \mathcal{M}_g,$$

where $\mathcal{M}_g$ is the moduli space of smooth genus–g curves, sending a curve C to the class of its normalization. Of course, one can consider its restriction

$$\psi_{X,n,\delta} : V_{n,\delta}(X) \longrightarrow \mathcal{M}_g.$$

Naive dimension counts:
- ensure that no dominance is allowed for $\psi_{X,n,\delta}$, whereas
- suggest that $\psi_{p,n,\delta}$ should be

$$\begin{array}{ll} \text{dominant if} & g \le 11, \\ \text{generically finite onto its image if} & g \ge 11. \end{array} \tag{1}$$

Question 15. *Let $(X, L) \in \mathcal{K}_p$ be general and $\delta < \pi_a(p, n)$. Let V be an irreducible component of $V_{n,\delta}(X)$. Is $\psi_{X,n,\delta|V}$ generically finite or even birational (if $g > 1$) to its image?*

State of the art: Let $(X, L) \in \mathcal{K}_p$. For any component $V \subseteq V_{n,\delta}(X)$ containing irreducible rational nodal curves in its Zariski closure, the restriction $\psi_{X,n,\delta|V}$ is generically finite [11]. In particular, for a general $(X, L) \in \mathcal{K}_p$, the moduli map is generically finite on at least one component of the Severi variety $V_{n,\delta}(X)$. The last statement follows from the fact that $V_{n,\pi_a(p,n)}(X)$ is nonempty [6]. In [17] one proves that if $\mathrm{Pic}(X) = \mathbb{Z}\langle L\rangle$ and $C \in V_{n,\delta}(X)$ is such that either

$$n = 1, 2 \quad \text{and} \quad \delta \le \frac{\pi_a(p,n)}{2} - 25, \quad \text{or} \quad n \ge 3 \text{and} \quad \delta \le 2(n-1)(p-1) - 25,$$

then the pull back of T_X to C' is stable, where C' is the normalization of C. This implies that $\psi_{X,n,\delta|V}$ generically finite onto its image in these cases.

Question 16. *Let $\mathcal{V}$ be an irreducible component of $\mathcal{V}_{p,n,\delta}$ [resp. V be an irreducible component of $V_{n,\delta}(X)$ where $(X,L) \in \mathcal{K}_p$ is general]. Does the differential of $\psi_{p,n,\delta|\mathcal{V}}$ [resp. $\psi_{X,n,\delta,|V}$] have maximal rank?*

State of the art: Results have been obtained recently by Kemeny [20] and in [8]. We quote the latter result, which is optimal for $n \geq 5$, but noting that Kemeny's results are stronger in case (B) for $n \leq 4$:

Theorem 17. (A) *For the following values of $p \geq 3$, n and $g = \pi_a(p,n) - \delta$ there is an irreducible component $\mathcal{V}$ of $\mathcal{V}_{p,n,\delta}$, such that the moduli map $\mathcal{V} \to \overline{\mathcal{M}}_g$ is dominant:*

- *$n = 1$ and $0 \leq g \leq 7$;*
- *$n = 2$, $p \geq g - 1$ and $0 \leq g \leq 8$;*
- *$n = 3$, $p \geq g - 2$ and $0 \leq g \leq 9$;*
- *$n = 4$, $p \geq g - 3$ and $0 \leq g \leq 10$;*
- *$n \geq 5$, $p \geq g - 4$ and $0 \leq g \leq 11$.*

(B) *For the following values of p, n and $g = \pi_a(p,n) - \delta$ there is an irreducible component $\mathcal{V}$ of $\mathcal{V}_{p,n,\delta}$, such that the moduli map $\mathcal{V} \to \overline{\mathcal{M}}_g$ is generically finite onto its image:*

- *$n = 1$ and $p \geq g \geq 15$;*
- *$2 \leq n \leq 4$, $p \geq 15$ and $g \geq 16$;*
- *$n \geq 5$, $p \geq 7$ and $g \geq 11$.*

Note that for $3 \leq p \leq 11$, $n = 1$ and $0 \leq \delta \leq p - 2$, the moduli map $\psi_{p,n,\delta}$ is dominant on any component $\mathcal{V} \subseteq \mathcal{V}_{p,n,\delta}$ by [16]. From [21, 22], the map $\psi_{p,1,0}$ is dominant for $p \leq 9$ and $p = 11$ but not for $p = 10$, the latter against expectation (1). Indeed, [13] proved that $\mathcal{S}_{10} := \mathrm{Im}(\psi_{10,1,0})$ is a divisor in $\mathcal{M}_{10}$ consisting of those curves of genus 10 lying in the primitive linear system of surfaces in $\mathcal{K}_{10}$: the exceptional behaviour is due to the fact that the general $K3$ surface of genus 10 is a codimension-3 linear section of a suitable 5-fold and the existence of Fano threefolds of that genus, as Mukai showed. Then [25, 14] gave another interesting realization of the divisor $\mathcal{S}_{10}$: it is the divisorial component of the locus in $\mathcal{M}_{10}$ of curves for which $\mathrm{SU}_C(2, K_C, 7) \neq \emptyset$, i.e. carrying a semistable, rank-two vector bundle with canonical determinant and at least 7 independent global sections.

On the other hand, in [23] it is proved that $\psi_{p,1,0}$ is generically finite for $p = 11$ and $p \geq 13$ but not for $p = 12$ (once again the latter against expectation (1)). More precisely, [12] proved that $\psi_{p,1,0}$ is birational for $p = 11$ and $p \geq 13$; then [24], for $p = 11$, and more recently [1], for any $p > 11$ congruent to 3 (mod 4), explicitely construct a rational inverse of $\psi_{p,1,0}$.

Question 18. *In the cases not covered by Theorem 17 (or Kemeny's result* [20]*), is it possible to find cases in which expectation* (1) *is contradicted, as happens for $\mathcal{V}_{p,1,0}$ for $p = 10$ and $p = 12$? If yes, what kind of geometric reasons are behind this fact? Is there any chance to get a divisorial image for $g \leq 11$?*

Question 19. *Is there any chance to describe some of the images* $\mathrm{Im}(\psi_{p,n,\delta})$ *via higher-rank Brill-Noether theory as in* [25, 14]*?*

Question 20. *Let $\mathcal{V}$ be an irreducible component of $\mathcal{V}_{p,n,\delta}$ [resp. V be an irreducible component of $V_{n,\delta}(X)$ where $(X,L) \in \mathcal{K}_p$ is general] such that the differential $\psi_{p,n,\delta|\mathcal{V}}$ [resp. $\psi_{X,n,\delta,|V}$] is of maximal rank. Is then $\psi_{p,n,\delta|\mathcal{V}}$ [resp. $\psi_{X,n,\delta,|V}$] birational onto its image? What is the Kodaira dimension of this image? If $\psi_{p,n,\delta|\mathcal{V}}$ is birational onto its image, is it possible to construct a rational inverse, as in* [24, 1] *(Torelli type theorem)?*

As for the very last question, we would expect that, except maybe for very few cases that should be possible to classify, the image is of general type as soon as $3\delta > \pi(p,n)$. One may also consider the intersection of the images of the maps with Brill-Noether loci $\mathcal{M}^r_{g,d}$ in the moduli space $\mathcal{M}_g$ of curves:

Question 21. *What can one say about $\mathrm{Im}(\psi_{X,n,\delta}) \cap \mathcal{M}^r_{g,d}$ and $\mathrm{Im}(\psi_{p,n,\delta}) \cap \mathcal{M}^r_{g,d}$? In the first case, which consequences can one draw on the Mori theory of the Hilbert scheme of points $X^{[d]}$?*

State of the art: For $\delta > 0$ and $n = r = 1$ rather complete results are found in [10] (extending preliminary results in [15]), see the abstract of Knutsen.

For a smooth, irreducible curve C one has the *Wahl map*

$$w_C : \wedge^2 H^0(C, \omega_C) \to H^0(C, \omega_C^{\otimes 3}).$$

If C sits on a $K3$ surface, then w_C is not surjective (see [4, 26]). The corank of w_C for C general in the image of $\psi_{p,n,0}$ has been studied in [12] and subsequent papers on the subject. Moreover, if C has general moduli of genus at least 11, but different from 12, then w_C is surjective (see [9]).

Wahl formulated in [27] the conjecture that if C is a smooth curve of genus $p \geq 8$ which is Brill–Noether general, then C is in the image of $\psi_{p,1,0}$ if and only if w_C is not surjective. This had been proved for $p = 10$ in [13]. In the recent paper [2] Wahl's conjecture has been *essentially* proved, i.e.: if C is a smooth irreducible curve of genus $p \geq 19$ which is Brill–Noether–Petri general, then $C \in L$, with $(X,L) \in \mathcal{K}_p$ (i.e., C is in the image of $\psi_{p,1,0}$), or is a limit of such a curve (also see the abstract of Sernesi).

For nodal curves C one can consider suitable modifications of the Wahl map. Two different versions are given by Kemeny [20] and Halic [17]. We call them the *Kemeny–Wahl map* kw_C and the *Halic–Wahl map* hw_C. For both maps, what is *essentially* true is that, like the ordinary Wahl map, they are not surjective for curves on $K3$ surfaces but are surjective for a *general nodal curve*, cf. [20, 17].

Question 22. *Compare kw_C and hw_C.*

Question 23. *Let C be general in a component of $\mathcal{V}_{p,n,\delta}$. What are the coranks of kw_C and hw_C?*

Question 24. *Does it make sense to extend Wahl's conjecture for nodal curves C using kw_C and/or hw_C, and to try to prove the analogue of Arbarello–Bruno–Sernesi's theorem* [2] *in such a setting?*

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The problem of the density of nodal curves in equigeneric families

Thomas Dedieu

(joint work with E. Sernesi)

Let S be a smooth projective surface, ξ a class in its Néron–Severi group, and g an integer. We consider V_g^ξ the locally closed subset of the Hilbert scheme of curves on S parametrizing reduced curves of genus g in the class ξ. The question is whether the subset of V_g^ξ corresponding to nodal curves (i.e. curves with only ordinary double points as possible singularities) is dense in V_g^ξ. The answer is known to be 'yes' when S is the projective plane [2, 1, 7], or a Del Pezzo or Hirzebruch surface [4], see also [3]. For $K3$ and abelian surfaces, only partial results are known [3, 5]; in particular, the general element of V_g^ξ is a curve with immersed singularities when $g > 0$ and $g > 2$ respectively.

Most of these results are obtained by a careful local study of the space $M_g^{\xi,\mathrm{bir}}(S)$ of birational morphisms from smooth genus g curves to S with image in the class ξ, taking advantage of the fact that the deformation theory of such objects is well understood, and of a key observation [1] relating the geometry of $M_g^{\xi,\mathrm{bir}}(S)$ to that of V_g^ξ (see [3, Lem. 2.5]). This method gives optimal results to prove that the general member of V_g^ξ is immersed, but is unfortunately artificially too expensive in terms of positivity of the canonical class K_S to answer the question in general.

One should therefore approach the question directly by studying deformations of integral curves $C \subset S$. Those preserving the geometric genus (resp. the topological type of the singularities) of C are governed by $N_{C/S} \otimes A$ (resp. $N_{C/S} \otimes I$), where A is the adjoint ideal of C and I is the equisingular ideal introduced by J. Wahl [6]. Following [3], we propose:

Question 25. *Use the fact that $I \subsetneq A$ if and only if C has singularities worse than nodes to answer the question of the density of nodal curves.*

We also discuss implications of this question in the enumerative geometry of curves on surfaces.

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Moduli spaces of hyperkähler manifolds and compactification problems. What do we know?

Concettina Galati

The following problem came up from discussions with C. Ciliberto, F. Flamini and A. L. Knutsen.

Besides its intrinsic interest, deformation theory of singular varieties has wide application in algebraic geometry. In particular, several recent papers about Severi varieties on $K3$ surfaces are based on degeneration arguments of curves and surfaces (cf., e.g., [1, 5, 7, 8, 9]). Semi-stable degenerations of $K3$ surfaces have been classified by Kulikov and Persson-Pinkham [6, 10] and are of three types. By [4, 3], we also know that there exists a *partial* compactification $\overline{\mathcal{K}_p}$ of the moduli space $\mathcal{K}_p$ of polarized $K3$ surfaces of genus p, such that its boundary is a smooth divisor whose points correspond to "stable" Type II degenerations (see [3, (4.9) and (4.10)] and [2, Sect. 3] for details). The existence of this partially compactified moduli space $\overline{\mathcal{K}_p}$ is a key ingredient in [2] and is, more in general, useful for the study of the universal Severi varieties on $K3$ surfaces. Finally, [4, Thm. 5.10] provides a really interesting description of the versal deformation space of a degenerate $K3$ surface of type II and more generally of a "d-semi-stable" $K3$ surface.

Question 26. *What happens for hyperkähler manifolds of higher dimension? Is it possibile to classify their degenerations? How many examples of degenerate hyperkähler manifolds are known and what do we know about their versal deformation space? Finally, in which cases is it possible to provide (possible partial) compactifications of moduli spaces of hyperkähler manifolds?*

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Problems related to "fake" $K3$ surfaces

Edoardo Sernesi

In [1] we consider a certain class of rational and ruled surfaces, sometimes called *fake $K3$ surfaces*. The relation between the smoothability of such surfaces to $K3$ surfaces and the existence of Petri general curves on them is investigated. The analysis in [1] suggests some interesting questions not yet investigated and related with moduli of curves, $K3$ surfaces and with classical problems on Del Pezzo surfaces of degree 1. Perhaps the most interesting one is:

Question 27. *Assume $g \geq 12$. Is the locus $(K3)_g \subset \mathcal{M}_g$ of curves lying on a $K3$ surface closed?*

For instance, there are specific curves lying on smoothable fake $K3$s for which we could not decide whether they are Petri general or not. The study of such curves can be translated into questions concerning Severi varieties on Del Pezzo surfaces of degree 1. The all subject has connections with the study of moduli spaces of stable $K3$ surfaces. Another vague but suggestive problem is the following one:

Question 28. *Is it possible to generalize in some way Lazarsfeld's proof* [2] *of Gieseker-Petri's theorem to curves lying on a fake $K3$ surface?*

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Geometry of O'Grady's 6 dimensional example

Giulia Saccà
(joint work with G. Mongardi, A. Rapagnetta)

There are not many known examples of hyperkähler manifolds. Two series of examples appear in dimension $2n$, for every $n > 1$, and are related to the Hilbert scheme of points on a $K3$ or an abelian surface; and in dimension 6 and 10 there is one extra, or exceptional, deformation class, each of which was found by O'Grady. While considerable work has been devoted to studying hyperkähler manifolds belonging to the first two deformation classes, not much is known for the exceptional deformation classes. We present results from [2] regarding the geometry of O'Grady's six dimensional example [3]. We use Lehn and Sorger's local analytic description of O'Grady's singularity [1], and results from [4] to realize these examples as "quotients" of another hyperkähler manifold by a birational involution. Since this hyperkähler manifold turns out to be deformation equivalent to the Hilbert scheme of 3 points on a $K3$ surface, we are able to compute all the Hodge numbers and study properties of their moduli spaces.

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Syzygies of curves and $K3$ surfaces

Michael Kemeny

(joint work with G. Farkas)

$K3$ surfaces have recently been used to make significant progress on several conjectures concerning syzygies of curves on a $K3$ surface. We discuss recent work in [5] and [6].

Let C be a curve and L an ample line bundle. Following Green, define the Koszul group $K_{i,j}(C,L)$ as the middle cohomology of

$$\bigwedge^{i+1} H^0(C,L)\otimes H^0(C,(j-1)L)\to \bigwedge^{i} H^0(C,L)\otimes H^0(C,jL)\to \bigwedge^{i-1} H^0(C,L)\otimes H^0(C,(j+1)L).$$

One says (C,L) satisfies *property* (N_p) if we have the vanishings

$$K_{i,j}(C,L)=0 \text{ for } i\leq p,\ j\geq 2.$$

Then $\phi_L : C \hookrightarrow \mathbb{P}^r$ is projectively normal if and only if (C,L) satisfies (N_0), whereas the ideal of C is generated by quadrics if, in addition, it satisfies (N_1).

The line bundle L is called *p-very ample* if and only if for every effective divisor D of degree $p+1$ the evaluation map

$$ev : H^0(C,L)\to H^0(D,L_{|D})$$

is surjective. Equivalently, L is *not* p-very ample if and only if $C\subseteq \mathbb{P}^r$ admits a $(p+1)$-secant $(p-1)$-plane.

The Secant Conjecture of Green–Lazarsfeld then states:

Conjecture 29 (Secant Conjecture)**.** *Let L be a globally generated line bundle of degree d on a curve C of genus g such that*

$$d\geq 2g+p+1-2h^1(C,L)-\mathit{Cliff}(C).$$

Then (C,L) fails property (N_p) if and only if L is not $p+1$-very ample.

We discuss the recent proof of the Secant Conjecture for general curves. The proof uses moduli spaces of polarised $K3$ surfaces and relies on Voisin's proof of Green's conjecture for curves on suitably general $K3$ surfaces [3, 4] . It is rather straightforward to see that if L is not $p+1$ very ample, that is, L admits a $(p+2)$-secant p-plane, then $K_{p,2}(C,L)$ is nonzero. The difficulty in establishing the above conjecture is thus to go in the other direction, that is, to construct a secant plane out of a syzygy in $K_{p,2}(C,L)$

The second conjecture we consider is the *Prym–Green Conjecture.* Consider a smooth curve C of genus g, and fix a nontrivial torsion line bundle η of order l. A *paracanonical curve* is the embedded curve

$$\phi_{\omega_C\otimes\eta}\ :\ C\hookrightarrow\mathbb{P}^{g-2}.$$

For a general *canonical* curve

$$\phi_{\omega_C}\ :\ C\hookrightarrow\mathbb{P}^{g-1}$$

the famous work of Voisin [3, 4] suffices to describe the shape of the free resolution of the homogeneous coordinate ring of C. The Prym–Green conjecture likewise predicts a roughly similar shape for the homogeneous coordinate ring of a *general* paracanonical curve. We describe our proof of this when g is odd and l is either two (the classical case) or l large. The proof works by degenerating curves on special $K3$ surfaces and uses work of van Geemen–Sarti [2] resp. Barth–Verra [1] in the $l=2$ resp. large l cases.

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Twisted cubics on a cubic fourfold and in involution on the associated 8-dimensional symplectic manifold

Manfred Lehn

(joint work with Ch. Lehn, Ch. Sorger, D. van Straten; N. Addington; and I. Dolgachev)

A by now classical theorem of Beauville and Donagi states that the Fano variety of lines on a smooth cubic fourfold Y is an irreducible holomorphic symplectic manifold of dimension 4 that is deformation equivalent to the second Hilbert scheme of a $K3$ surface. With the intention to generalise this construction to curves of higher degree, we consider the moduli space of generalised twisted cubics on Y. If the fourfold Y does not contain a plane, this moduli space $M=M_3(Y)$ turns out to be smooth projective of dimension 10. The natural morphism $M\to G:=\mathrm{Grass}(3,\mathbb{P}^5)$ that sends a curve $C\subset Y\subset\mathbb{P}^5$ to the three-dimensional linear span $\langle C\rangle$, fibres as follows: $M\to Z'\to Z_{\mathrm{Stein}}\to G$, where $Z_{\mathrm{Stein}}\to G$ is the finite part of the Stein factorisation, $Z' \to Z_{\text{Stein}}$ is a resolution of singularities, and, most importantly, $M \to Z'$ is a $\mathbb{P}^2 \to Z'$-bundle.

According to a result of Piene and Schlessinger, twisted cubic curves in 3-space come in two flavours: the general curve is arithmetically Cohen-Macaulay and defined by three quadrics that are the minors of a (3×2)-matrix A_0 with linear entries. Points in a smooth divisor represent non-Cohen-Macaulay curves. Such curves are singular plane cubics with an embedded point at a singularity. Non-Cohen-Macaulay curves also form a smooth divisor in M, and this divisor is contracted to a smooth divisor $D \subset Z'$. Finally, there is a divisorial contraction $Z' \to Z = Z(Y)$ to an 8-dimensional holomorphic symplectic manifold Z. Under this map, the divisor D is mapped to a copy of Y in Z, and curve C with embedded point $p \in C$ is mapped to this point $p \in Y$.

Every twisted curve C of aCM-type determines an integral cubic surface $S = Y \cap \langle C \rangle$ and a linear determinantal representation $g = \det(A)$ of the equation g of S in the linear hull $\langle C \rangle$. A curve of nonCM-type corresponds to a skew-symmetric (3×3)-matrix A the entries of which define the position of the embedded point.

For surfaces S with at most ADE-singularities, families of twisted cubics are given by equivalence classes of roots in the E_6-type root system of the lattice $K_{\tilde{S}}^{\perp} \subset H^2(\tilde{S}, \mathbb{Z})$ in the integral cohomology of the minimal resolution $\tilde{S} \to S$. Two roots are equivalent if they lie in the same orbit under the action of the Weyl group generated by effective roots, i.e. exceptional curves of the resolution $\tilde{S} \to S$.

The two stages of the contraction $M \to Z' \to Z$ can be interpreted in terms of left mutations from the derived category $\mathcal{D}(Y)$ onto Kuznetsov's subcategory $\mathcal{A} \subset \mathcal{D}(Y)$, and Z can be seen as a moduli space of objects in $\mathcal{A}$.

It is interesting to note that Z admits a non-symplectic involution i that can be described as follows: For any C the sections in $H^0(I_{C/S}(2))$ correspond to quadric Q that contain C. As the intersection $Q \cap S$ is curve of degree 6, it decomposes as $Q \cap S = C \cup C'$, and the involution takes (the family of) C to (the family of) C'. In terms of root systems, the involution is given by $\alpha \mapsto -\alpha$. Finally, in terms of matrices, it is $A \mapsto A^t$. From the last description it is clear that the fixed point locus has two components Y and Y', the former corresponding to skew-symmetric matrices and hence nonCM-curves, the latter corresponding to symmetric matrices and thus symmetric linear determinantal representations. Such representations only exist for surfaces with four A_1-singularities and for degenerations of such surfaces. So roughly, the second fixed point component is given by four-tuples of points on Y with the property that the 3-plane through the points is tangent to Y in each point. We hope that the quotient Y/i can be embedded into $\mathbb{P}^{20}$, by a map that restricts on the first fixed point component Y to the second Veronese map, and that this will allow to describe the quotient directly in terms of the projective geometry of Y leading to a construction of Z analogous to the double EPW-sextics of O'Grady.

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On the problem of compactification of Lagrangian fibrations

Dimitri Markushevich

The following problem is studied: given an abelian fibration, when can one endow its total space with a symplectic structure in such a way that it becomes a Lagrangian fibration (LF), compactifiable to a possibly singular irreducible symplectic variety (ISV)?

Partial answers to this question were obtained for relative Jacobians of families of curves [5, 7] and relative Prymians [6, 1]. We give a survey of results of the thesis by J. Bouali [2] on the compactification of the Lagrangian fibrations in intermediate Jacobians associated to $K3$-Fano flags [4]. Then an approach is presented toward a compactification of the Donagi–Markman LF [3] in intermediate Jacobians of cubic 3-folds that are hyperplane sections of a fixed cubic 4-fold, based upon the following almost-theorem:

Conjecture 30. *Let $X \subset \mathbb{P}^5$ be a generic cubic 4-fold, and let $\{V_t\}_{t\in\mathbb{P}^{5*}}$ denote the family of its hyperplane sections. Then for any $t_0 \in \mathbb{P}^{5*}$, the following properties are verified:*

(1) $\operatorname{Sing}(V_{t_0}) = \{p_1, \dots, p_k\}$ *is finite, the singular points p_i are of types A_{μ_i} ($1 \le \mu_i \le 5$) or D_{μ_i} ($4 \le \mu_i \le 5$), and $\sum \mu_i \le 5$.*

(2) *The family $\mathcal{V} = \{V_t\}_{t\in\mathbb{P}^{5*}} \longrightarrow \mathbb{P}^{5*}$ is locally analytically a multi-versal unfolding of $\operatorname{Sing}(V_{t_0})$ near t_0.*

Thus, the degenerations of the family $\mathcal{V}/\mathbb{P}^{5*}$ are well-behaved and there is a hope to compactify the relative intermediate Jacobian $J = J(\mathcal{V}/\mathbb{P}^{5*})$ to an ISV. An evidence is given by the fact that the relative compactified Jacobian of a family of integral curves with plane singularities is non-singular, provided the family is multi-versal for each of its singular fibers. Remark that the monodromy of the versal deformation of an isolated curve singularity coincides with the monodromy of the versal deformation of the stably equivalent 3-dimensional singularity, which motivates the following question:

Question 31. *Is the relative compactified Jacobian of a family of curves over a disk in $\mathbb{C}$ toroidal, provided that the family is a multi-versal unfolding of singularities of its central fiber, and that those singularities are only simple singularities of types A_{μ_i} ($1 \le \mu_i \le 5$) or D_{μ_i} ($4 \le \mu_i \le 5$), $\sum \mu_i \le 5$?*

An alternative approach using a representation of the intermediate Jacobians as Prym varieties of double covers of curves is discussed.

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Short report on discussion sessions

One group of participants, formed by Ciliberto, Dedieu, Galati, Kemeny, Knutsen, Saccà and Sernesi concentrated on some of the problems proposed by Ciliberto/Flamini and Sernesi. In particular, they figured out a way of possibly answering Question 13 affirmatively. Concerning Question 20, they observed that $V_{n,\delta}(X)$ is never rationally connected if $(X, L) \in \mathcal{K}_p$ is general.

Some time was also devoted to Question 28 towards extending Lazarsfeld's vector bundle methods to fake $K3$ surfaces. A few days after the workshop, the problem was solved in [1], which in particular proves that certain hyperplane sections of fake $K3$ surfaces are indeed Brill-Noether-Petri general curves.

A second group of participants, formed by Boissiére, Camere, Cattaneo, Flamini, Lehn, Markushevich, Ohashi and Sarti focused on finding geometric description of the involution on $S^{[2]}$ as in Question 1 proposed by Cattaneo. Various methods of attack were considered. The ideas developed were very useful and we believe that these will help in future research in finding the geometric description.

Some time was also devoted to discussing other problems, in various constellations of participants. Among those, considerable attention was devoted to degenerations of hyperkähler manifolds as in Question 26 proposed by Galati. Great progress in the study of deformations of degenerate hyperkähler manifolds was made in [2]. Nevertheless, up to now, very few examples of degenerate hyperkähler manifolds are known. Various attempts to find new examples were made.

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