Introduction to algebraic surfaces

Lecture Notes for the course at the
University of Mainz

Wintersemester 2009/2010

Arvid Perego

(preliminary draft)

October 30, 2009

Contents

Introduction

In this series of lectures I will present an introduction to the classification of compact complex surfaces. This is a quite old subject in geometry, and involves several elements coming from different areas of mathematics, like differential geometry, topology, algebraic geometry (classical and modern). It is, to my opinion, a very beautiful piece of mathematics, which is nowadays considered classical, and which is very useful to modern research in geometry.

As usual when dealing with a geometric problem, one needs to define the family of objects one wants to investigate, and to define a notion of equivalence between them. Moreover, in order to understand if two objects of the chosen family are equivalent, one studies their invariants. If one considers as objects topological spaces, i. e. couples (X, τ) where X is a set and τ is a topology on it, then one can talk about continuous maps between two topological spaces, and can introduce the notion of homeomorphism: namely, two topological spaces (X_1, τ_1) and (X_2, τ_2) are *homeomorphic* if there are two continuous maps $f_1 : X_1 \longrightarrow X_2$ and $f_2 : X_2 \longrightarrow X_1$ such that $f_2 \circ f_1 = id_{X_1}$ and $f_1 \circ f_2 = id_{X_2}$. One can then try to classify topological spaces up to homeomorphism, even if this problem seems somehow impossible to solve, as every set can be considered as a topological space (e.g. with the discrete topology). One can even change the notion of equivalence, introducing, as instance, the homotopic equivalence: first of all, one defines the notion of *homotopy* between two maps $f, g : X \longrightarrow Y$; then one says that two topological spaces (X_1, τ_1) and (X_2, τ_2) are *homotopic equivalent* if there are two continuous maps $f_1 : X_1 \longrightarrow X_2$ and $f_2 : X_2 \longrightarrow X_1$ such that $f_2 \circ f_1$ is homotopic to id_{X_1} and $f_1 \circ f_2$ is homotopic to id_{X_2}. Still, the problem seems very hard to solve. For simplicity's sake, we will consider only compact and connected topological spaces.

The next step in the classification is then to enrich the structure of a topoligical space. This can be done in different ways: first of all, one only considers topological spaces over which one can, in some way, perform analysis or algebra. A first approach is to define the notion of a *manifold*: this can be done for different purposes. A *real manifold* will be a topological space M which is Hausdorff, admits a countable basis for the topology, and admits an open cover $\{U_i\}_{i \in I}$, where U_i is an open subset of M and for every $i \in I$ there is a homeomorphism $\varphi_i : U_i \longrightarrow \mathbb{R}^n$ for some fixed n (here $\mathbb{R}^n$ is a topological space with the euclidean topology). This n is called the *(topological) real dimension* of M. If one considers $i, j \in I$, then we have a map

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n.$$

Asking for these maps to be differentiable, of class C^k, of class C^∞, analytic, one gets the notion of *differentiable manifold*, C^k-*differentiable manifold*, $C^\infty-$*differentiable manifold*, *analytic manifold*. Moreover, one can introduce the notion of continuous map, C^k-, $C^\infty-$differentiable map or analytic map between two manifolds M_1 and M_2, and hence of equivalence: one gets the notions of *homeomorphic manifolds* and of *diffeomorphic* (C^k-, $C^\infty-$)*differentiable* or *analytic manifolds*.

In a similar way one can introduce the notion of *complex manifold*, which is a topological space X which is Hausdorff, admits a countable basis for the topology, and admits an open cover $\{U_i\}_{i\in I}$, where U_i is an open subset of X and for every $i \in I$ there is a homeomorphism $\varphi_i : U_i \longrightarrow \mathbb{C}^n$ for some fixed n (here $\mathbb{C}^n$ is a topological space with the euclidean topology). This n is called the *(topological) complex dimension* of X. Again, for every $i, j \in I$ we have a map

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \subseteq \mathbb{C}^n \longrightarrow \mathbb{C}^n.$$

Asking for these maps to be holomorphic, one gets the notion of *holomorphic manifold*, or simply of *complex manifold*. As in the previous case, one can introduce the notion of holomorphic map between two complex manifolds, and hence of equivalence: two complex manifolds X_1 and X_2 are equivalent if they are *biholomorphic*.

Using the notion of sheaf, one can still enrich the structure of a manifold, attaching to it a sheaf of rings: if M is a differentiable or analytic manifold, and $U \subseteq M$ is an open subset, then one can define the ring

$$\mathscr{O}_M(U) := \{f : U \longrightarrow \mathbb{R} \,|\, f \texttt{ is differentiable/analytic}\}.$$

This assignement defines a sheaf of rings over M, and the couple $(M, \mathscr{O}_M)$ is called a *differentiable* or *analytic space*. In a similar way one defines the notion of a *complex space* $(X, \mathscr{O}_X)$.

There is a different approach, relying on algebra instead of analysis. To any commutative ring A with unity 1_A, we can associate a topological space $Spec(A)$, whose underlying set is the set of prime ideals of A, and whose topology is defined using ideals of A, and is called *Zariski topology*. Moreover, using localizations of A, one can define a sheaf of rings $\mathscr{O}_{Spec(A)}$, and the couple $(Spec(A), \mathscr{O}_{Spec(A)})$ is called an *affine scheme*. More generally one can define the notion of a *scheme*: this is simply a couple $(X, \mathscr{O}_X)$, where X is a topological space, $\mathscr{O}_X$ is a sheaf of commutative rings with unity, and admitting an open cover by affine schemes. Again, one can define the notion of *morphism* between two schemes, and hence of equivalence: two schemes are equivalent if they are isomorphic. As for manifolds, one can define the notion of *(algebraic) dimension* of a scheme, relying on the notion of Krull dimension of a commutative ring.

In any case, the notion of dimension is crucial: if two spaces (or schemes) are equivalent, they have the same dimension. This is a first important invariant, and we can classify these objects at first with respect to their dimension. In dimension 0, there is only the point, so there is no classification needed. In dimension 1, things are more complicated: in the complex universe, these objects are called *Riemann surfaces*. These are, in particular, compact connected real manifolds of dimension 2, whose classification started in the second half of the XIX century by Moebius, Klein and others. Anyway, they contain more structure, namely the complex structure: using the theory of holomorphic functions, Riemann was able to give a classification of these objects by means of the *genus*:

roughly speaking, we can divide them in rational curves (i. e., genus 0 Riemann surfaces), elliptic curves (i. e. genus 1 Riemann surfaces) and curves of general type (i. e. Riemann surfaces with genus at least 2).

There are some crucial facts in the theory of Riemann surfaces: first of all, they are all projective, i. e. any Riemann surface can be realized as a curve in some projective space; second, classification of Riemann surfaces is with respect to biholomorphism, i. e. two Riemann surfaces are equivalent if they are homeomorphic as topological spaces and if they carry biholomorphic complex structures.

Once the classification of curves was achieved, it was natural to try to classify $2-$dimensional objects. Already at the end of the 1850's there were important studies on low degree surfaces in the $3-$dimensional projective space by people like Cayley, Kummer and Steiner. During the 1870's the first generation of italian algebraic geometers (Bertini, Cremona, del Pezzo, Segre, Veronese) started to look at surfaces embedded in higher dimensional projective spaces; independently, Noether was able to show some important properties, like what is now known as the Noether formula, even if the proof was not complete. The proof was corrected by Enriques, one of the major exponents of the second generation of algebraic geometers, which together with Castelnuovo was finally able to produce at the end of the 1910's a classification of smooth projective complex surfaces.

Anyway, their classification was lacking of background material, which was yet to be developed. Anyway, they pointed out two of the main problems of the classification of compact complex surfaces. As told before, classification of Riemann surfaces is a biholomorphic classification of smooth projective complex curves. For surfaces, this is not the case: first of all, there are compact complex surfaces which are not projective. Second, it is impossible to carry out a biholomorphic classification of surfaces, even only for the projective ones: this was pointed out by Enriques and Castelnuovo, who realized the existence of an importan phenomenon, nowadays known as blow-up. They then introduced the notion of rational morphism, and they were able to perform (even lacking of the foundations of algebraic geometry) a birational classification of smooth projective complex surfaces.

The first attempt to formulate in a clearer way the results obtained by Enriques and Castelnuovo were made by Zariski, Weil and van der Waerden in the classical projective geometry, and by Hodge, de Rham and Lefschetz on the analytic side (during the 1940's). The main techniques in algebraic geometry were carried out by Serre, Grothendieck and Hirzebruch in the 1950's and 1960's, who introduced the notion of sheaf, of cohomology, of scheme and their properties.

Using these tools, Kodaira was able to complete the classification of compact complex surfaces, including even singular and non-projective ones. The obtained classification goes under the name of *Enriques-Kodaira birational classification of compact complex surfaces*, and which is one of the most important results in the whole complex geometry. During the 1970's, the work of Kodaira was generalized to compact surfaces defined over the an algebraically closed field of characteristic p, by Bombieri and Mumford: these classification is the same as the Kodaira-Enriques one if $p > 3$, but presents some important differencies if $p = 2, 3$. The classification of smooth projective complex surfaces as presented by Kodaira is obtained in the following way: one needs to attach to any surface some invariants, which Enriques and Castelnuovo called *plurigenera*, and which nowadays goes under the name of *Kodaira dimension*. Roughly speaking, then count the number of independent differential forms one has on a surface, and we have 4 possible values of it: $-\infty$, 0, 1 and 2. This invariant gives a first classification of surfaces, which can be completed as follows:

1. Kodaira dimension $-\infty$:
   - rational surfaces (birational to $\mathbb{P}^2$);
   - ruled surfaces (projectivized of a rank 2 vector bundle on a smooth projective curve of genus $g \geq 1$);
2. Kodaira dimension 0:
   - abelian surfaces (projective quotients of $\mathbb{C}^2$ by a maximal rank lattice);
   - K3 surfaces;
   - Enriques surfaces;
   - bielliptic surfaces;
3. Kodaira dimension 1: proper elliptic surfaces;
4. Kodaira dimension 2: surfaces of general type.

As one can see easily from this picture, the birational classification of surfaces is quite coarse, in the same spirit of the classification of Riemann surfaces: using Kodaira dimension, the classification for Riemann surfaces is:

1. Kodaira dimension $-\infty$: rational curves ($\mathbb{P}^1$);
2. Kodaira dimension 0: elliptic curves (quotient of $\mathbb{C}$ by a maximal rank lattice);
3. Kodaira dimension 1: curves of general type.

In this sense, the classification of surfaces as given by Kodaira and Enriques is not complete: as for curves, the family of curves of general type is very coarse, the same can be said for surfaces of general type, and up to now there is no good understanding of this huge family.

Let me conclude this introduction by some mentions about MMP. Starting from the end of the 1970's and the beginning of the 1980's, there were some attempts to classify higher dimensional smooth projective complex varieties, imitating the process used to classify surfaces. The obtained results incouraged the development of what is nowaday known as *Minimal Model Program*, or simply *MMP*. This generalization, which is still to be completed, introduced new techniques to adapt the methods of the classification of surfaces to higher dimensional varieties. Using notions like nefness and cones, one is able to simplify the proof of the classification of surfaces. In the following lectures I will try to present this more modern approach to the theory of surfaces.

Chapter 1

Background material

This first chapter is dedicated to basic facts that will be needed all along these lectures. First, I will recall the notions of complex manifold, of analytic variety and of projective variety, linking these two notions via the Chow Theorem. Then, I will briefly recall the definition and important properties of vector bundles on complex manifolds, giving useful examples.

I will continue with a brief recall about sheaves and their cohomology: this will be done in axiomatic terms and by introducing the notion of Cech cohomology, but will be enough to define the notions of singular cohomology and the de Rham and the Dolbeault Theorems. I will conclude with the statement and corollaries of the GAGA Theorem, following Serre.

The next topic will be Hodge theory: I will recall the notion of hermitian metric and of complex structure, in order to state the Hodge Theorem. I will introduce the notion of Kähler manifold and of Hodge decomposition. Main results will concern Hodge numbers, projectivity of Riemann surfaces and examples on non-Kähler surfaces, like the Hopf surface.

1.1 Complex and projective manifolds

In this first section I will recall two basic notions: complex manifolds and basic properties, and projective manifolds. I will present important examples that will be useful all along the lectures.

Before going on with definitions, let me recall some notations in complex analysis that I will use in the following. If $n \in \mathbb{N}$, I will use $z = \{z_1, ..., z_n\}$ for a coordinate system on the complex affine space $\mathbb{C}^n$. For every $j = 1, ..., n$, I will write $z_j = x_j + iy_j$, the usual decomposition of the complex coordinate z_j into real and imaginary part. For every $j = 1, ..., n$ we can define the two differential operator

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \qquad \frac{\partial}{\partial \overline{z}_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

In a similar way, one can define

$$dz_j := dx_j + idy_j, \qquad d\overline{z}_j = dx_j - idy_j.$$

Using this definitions, if $U \subseteq \mathbb{C}^n$ is an open subset (in the euclidean topology) and $f : U \longrightarrow \mathbb{C}$ is a differentiable function, one has

$$df = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_j + \sum_{j=1}^{n} \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j =: \partial f + \overline{\partial} f.$$

Recall now the following definition: let $U \subseteq \mathbb{C}^n$ be an open subset and $f : U \longrightarrow \mathbb{C}$ a $C^\infty-$function, that we can decompose in real and imaginary part as $f = u + iv$, where $u, v : U \longrightarrow \mathbb{R}$ are two $C^\infty-$functions.

Definition 1.1.1. The function f is *holomorphic on U* if one of the following (equivalent) conditions is verified:

1. *Cauchy-Riemann condition*: for every $j = 1, ..., n$ the two following equations are satisfied on U:

$$\frac{\partial u}{\partial x_j} = \frac{\partial v}{\partial y_j}, \qquad \frac{\partial u}{\partial x_j} = -\frac{\partial v}{\partial y_j}.$$

2. *Antiholomorphic condition*: on U we have $\overline{\partial} f = 0$.

3. *Analytic condition*: there is a power series which absolutely converges to f on U.

If $f = (f_1, ..., f_m) : U \longrightarrow \mathbb{C}^m$, then f is holomorphic if and only if f_i is holomorphic for every $i = 1, ..., m$. Notice that one provide the following matrix, called the *Jacobian* of f:

$$J(f) := [\partial f_i / \partial z_j]_{i=1,...,m;\, j=1,...,n}.$$

This is a $n \times m$ matrix whose enters are holomorphic functions over U. If $n = m$ and $z_0 \in U$, we say that $J(f)$ is *non-singular* at z_0 if $det(J(f)(z_0)) \neq 0$.

1.1.1 Complex manifolds

Let us start with the following basic definition:

Definition 1.1.2. An *$n-$dimensional complex manifold* is a topological space M such that

1. M is Hausdorff;

2. M admits a countable basis for its topology;

3. there is a family $\mathscr{U} = \{U_i, \varphi_i\}_{i \in I}$, called *atlas*, where $\{U_i\}_{i \in I}$ is an open cover of M, for every $i \in I$ we have $\varphi_i : U_i \longrightarrow \mathbb{C}^n$ is a homeomorphism and for every $i, j \in I$ such that $U_{ij} := U_i \cap U_j \neq \emptyset$ the map

$$\varphi_{ji} := \varphi_j^{-1} \circ \varphi_i : \varphi_j(U_{ij}) \longrightarrow \varphi_i(U_{ij})$$

is holomorphic.

Definition 1.1.3. Let M be an $n-$dimensional complex manifold with atlas $\mathscr{U} = \{U_i, \varphi_i\}_{i\in I}$, and let $U \subseteq M$ be an open subset of M. A continuous map $f : U \longrightarrow \mathbb{C}$ is called *holomorphic on* U if for every $i \in I$ such that $U_i \cap U \neq \emptyset$ the map

$$f \circ \varphi_i^{-1} : \varphi_i(U_i \cap U) \longrightarrow \mathbb{C}$$

is holomorphic on $\varphi_i(U_i \cap U)$. A map $f = (f_1, ..., f_m) : U \longrightarrow \mathbb{C}^m$ is *holomorphic on* U if and only if every $f_i : U \longrightarrow \mathbb{C}$ is holomorphic on U. More generally, if M is an $n-$dimensional complex manifold with atlas $\mathscr{U} = \{U_i, \varphi_i\}_{i\in I}$ and N is an $m-$dimensional complex manifold with atlas $\mathscr{V} = \{V_j, \psi_j\}_{j\in J}$, a continuous map $f : M \longrightarrow N$ is *homolorphic* if for every $i \in I$ and for every $j \in J$ such that $V_j \cap f(U_i) \neq \emptyset$, the map

$$\psi_j \circ f \circ \varphi_i : U_i \longrightarrow \psi(V_j \cap f(U_i)) \subseteq \mathbb{C}^m$$

is holomorphic.

One can generalize the notion of Jacobian matrix for any holomorphic map between two complex manifolds.

Example 1.1.1. A first trivial example of $n-$dimensional complex manifold is simply any open subset U of $\mathbb{C}^n$. This is an example of non-compact complex manifold.

Example 1.1.2. Let n be any integer, and let us define $\mathbb{P}^n_{\mathbb{C}} := \mathbb{C}^{n+1}/ \simeq$, where $z = (z_0, ..., z_n)$ and $w = (w_0, ..., w_n)$ are equivalent under $\simeq$ if and only if there is $\lambda \in \mathbb{C}^*$ such that $\lambda z = w$. Let the topology on $\mathbb{P}^n_{\mathbb{C}}$ be the quotient topology. This is the *$n-$dimensional complex projective space*, and its points are denoted as $z = (z_0 : ... : z_n)$. It is easy to see that it is an $n-$dimensional complex manifold: for every $i = 0, ..., n$, let

$$U_i := \{z = (z_0 : ... : z_n) \,|\, z_i \neq 0\}.$$

It is easy to see that U_i is an open subset of $\mathbb{P}^n_{\mathbb{C}}$, and that $\mathbb{P}^n_{\mathbb{C}} = \bigcup_{i=0}^n U_i$. Moreover, the map

$$\varphi_i : U_i \longrightarrow \mathbb{C}^n, \qquad \varphi_i(z_0 : ... : z_n) := \left(\frac{z_0}{z_i}, ..., \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, ..., \frac{z_n}{z_i}\right)$$

is clearly a homeomorphism and holomorphic. This is a very important example of compact complex manifold.

Example 1.1.3. Let $\gamma_i, ..., \gamma_{2n} \in \mathbb{C}^n$ be $2n$ $\mathbb{R}-$linearly independent elements, and let $\Gamma := \mathbb{Z}\gamma_1 \oplus ... \oplus \mathbb{Z}\gamma_{2n} \subseteq \mathbb{C}^n$ be the lattice generated by them. Let $T_\Gamma := \mathbb{C}^n/\Gamma$, and let its topology be the quotient topology. This is the *$n-$dimensional complex torus*, and it is not difficult to see that this is a compact $n-$dimensional complex manifold.

Example 1.1.4. Let $\sim$ be the equivalence relation on $\mathbb{C}^n$ generated by the homothety sending z to $2z$. The quotient topological space $\mathbb{C}^n/\sim$ is the *$n-$dimensional complex Hopf manifold.* It is not hard to see that this is a compact complex manifold.

Example 1.1.5. Any $1-$dimensional compact connected complex manifold is called *Riemann surface.* A $2-$dimensional compact connected complex manifold is called *complex surface.*

Definition 1.1.4. Let M be a complex manifold with atlas $\mathscr{U} = \{U_i, \varphi_i\}_{i\in I}$, and let $m \in M$. The *holomorphic tangent space at M in m* is the complex vector space $T^h_m(M)$ spanned by $\partial/\partial z_1, ..., \partial/\partial z_n$, where $z = (z_1, ..., z_n)$ are local coordinate at m. The *antiholomorphic tangent space at M in m* is the complex vector $T^a_m(M)$ space spanned by $\partial/\partial \overline{z}_1, ..., \partial/\partial \overline{z}_n$. The *complex tangent space at M in m* is

$$T_{\mathbb{C},m}(M) := T^h_m(M) \oplus T^a_m(M).$$

The (*holomorphic*, *antiholomorphic*, *complex*) *cotangent space at M* in m is the dual of the (holomorphic, antiholomorphic, complex) tangent space.

Proposition 1.1.1. *Any complex manifold has a natural orientation.*

Proof. Let U be an open subset of $\mathbb{C}^n$, and let $f = (f_1, ..., f_n)$ be a holomorphic map. Let $f_j = u_j + iv_j$ for every $j = 1, ..., n$, and let $f_{\mathbb{R}} = (u, v) : U \longrightarrow \mathbb{R}^{2n}$: this is a differentiable map, and one can calculate its jacobian $J(f_{\mathbb{R}})$, which is a $2n \times 2n$ matrix whose entries are $\partial u_j/\partial x_k$ and $\partial v_j/\partial y_k$. As f is holomorphic, it is easy to see that $det(J(f_{\mathbb{R}})) = |det(J(f))|^2$: if f is invertible, then $det(J(f_{\mathbb{R}})) > 0$. Now, if one considers a complex manifold M equipped with an atlas $\mathscr{U} = \{U_i, \varphi_i\}_{i\in I}$, for every $i, j \in i$ such that $U_{ij} \neq \emptyset$ one has the map φ_{ji} which is holomorphic. Then $J(\varphi_{ij})$ has positive determinant, implying that M has a natural orientation. $\square$

Let M be an $n-$dimensional complex manifold, $m < n$ and let $f : M \longrightarrow \mathbb{C}^m$ be a holomorphic function. We define

$$V(f) := f^{-1}(0),$$

and we call it the *zero locus* of f. Using the Inverse Function and the Implicit Function Theorems, one can easily show that if the rank of $J(f)$ is always equal to m, then $V(f)$ is an $(n-m)-$dimensional complex manifold. This can be generalized in the following:

Definition 1.1.5. Let M be an $n-$dimensional complex manifold with atlas $\mathscr{U} = \{U_i, \varphi_i\}_{i\in I}$, and let $N \subseteq M$ be a closed subset. Then N is called an *analytic subset* of M if for every $i \in I$ such that $N_i := N \cap U_i \neq \emptyset$ there is a holomorphic map $f : U_i \longrightarrow \mathbb{C}^m$ such that $N_i = V(f_i)$.

If N is irreducible (i. e. it is not union of two or more proper analytic subsets of M), then N is called *analytic subvariety* of M.

If $m < n$ is constant and the rank of $J(f_i)$ is m at every point, then N is called a *complex submanifold* (or *smooth* or *non-singular subvariety*) of M, of dimension $n - m$.

If $M = \mathbb{P}^n_{\mathbb{C}}$, all these are called *projective*.

Example 1.1.6. It is an important fact that every Riemann surface is projective. This is not the case for complex surfaces: the Hopf surface is not projective. We will see these in the following, as a consequence of Hodge theory.

1.1.2 The algebraic world

One can specialize the holomorphic functions allowed.

Definition 1.1.6. Let $U = \mathbb{C}^{n+1}$, and $f = (f_1, ..., f_m) : U \longrightarrow \mathbb{C}^m$ be such that $f_i \in \mathbb{C}[z_0, ..., z_n]$, i. e. f_i is a polynomial for every $i = 1, ..., m$. Then its zero locus $V(f) \subseteq \mathbb{C}^{n+1}$ is called *affine algebraic subset* of $\mathbb{C}^{n+1}$.

As polynomials are holomorphic functions, any affine algebraic subset of $\mathbb{C}^{n+1}$ is an analytic subset. A particular property of affine algebraic subsets of $\mathbb{C}^{n+1}$ is the following: they do not depend on $f_1, ..., f_m$, but only on the ideal $I = (f_1, ..., f_m)$ spanned by $f_1, ..., f_m$ in the commutative ring $\mathbb{C}[z_0, ..., z_n]$. For this reason, one writes $V(I) := V(f)$.

Definition 1.1.7. If $V(I)$ is irreducible, i. e. it is not the union of two or more proper affine subsets of $\mathbb{C}^{n+1}$, then it is called *affine variety.*

Example 1.1.7. It is not difficult to see that if I is a prime ideal, then $V(I)$ is an affine variety.

Definition 1.1.8. Let $V \subseteq \mathbb{C}^{n+1}$ be an affine algebraic variety. A *rational map* f on V is a map $f = (f_1, ..., f_m)$ such that for every $i = 1, ..., m$ there are $p_i, q_i \in \mathbb{C}[z_0, ..., z_n]$ such that $f_i = p_i/q_i$, with q_i is not identically trivial on V. If q_i is everywhere non trivial on V for every $i = 1, ..., m$, then f is called *regular.* A *rational* (resp. *regular*) map between two affine algebraic varieties $V \subseteq \mathbb{C}^n$ and $W \subseteq \mathbb{C}^m$ is simply a rational (resp. regular) map on V taking values in W.

If $f_1, ..., f_m$ are homogenous polynomials in $z_0, ..., z_n$, then $V(f)$ is closed under multiplication by every element of $\mathbb{C}^*$: one can then consider its projection to $\mathbb{P}^n_{\mathbb{C}}$, i. e. the subset of those $z = (z_0 : ... : z_n) \in \mathbb{P}^n_{\mathbb{C}}$ such that $f_i(z) = 0$ for every $i = 1, ..., m$. This is clearly a closed subset $V(f)$ of $\mathbb{P}^n_{\mathbb{C}}$, which clearly depends only on the homogenous ideal I spanned by $f_1, ..., f_m$, and it is hence denoted $V(I)$. It is clear that $V(I)$ is a projective analytic set.

Definition 1.1.9. Any such a projective analytic set is called *projective algebraic set*. If $V(I)$ is irreducible, i. e. it is not union of two or more projective algebraic sets, it is called *projective algebraic variety*.

Remark 1.1.1. Any projective algebraic variety is a projective analytic variety.

Definition 1.1.10. A projective algebraic variety is called *projective algebraic manifold* if it is a complex manifold.

Example 1.1.8. The $n-$dimensional complex projective space is a projective algebraic variety. If F is a homogenous polynomial of degree d in $z_0, ..., z_n$, then $V(F) := V((F))$ is a projective algebraic variety called *hypersurface of degree* d in $\mathbb{P}^n_{\mathbb{C}}$.

Definition 1.1.11. Let V be a projective algebraic variety. A *rational map* f on V is a map $f = (f_1, ..., f_m)$ such that for every $i = 1, ..., m$ there are p_i, q_i homogenous polynomials of the same degree such that q_i is not identically trivial on V and $f_i = p_i/q_i$. If for every $i = 1, ..., m$ the homogenous polynomial is everywhere non trivial, then f is called *regular*.

Notice that any rational map $f = (p_1/q_1, ..., p_m/q_m)$ on V defines in a clear way a map

$$f : V \setminus \bigcup_{i=1}^{m} V(q_i) \longrightarrow \mathbb{P}^n_{\mathbb{C}},$$

and a regular map is defined over the whole V. This map is called *rational* (resp. *regular*) *morphism* on V.

Definition 1.1.12. Let $V \subseteq \mathbb{P}^n_{\mathbb{C}}$ and $W \subseteq \mathbb{P}^m_{\mathbb{C}}$ be two projective algebraic varieties. A *rational morphism* (resp. *regular morphism* or simply *morphism*) between V and W is simpla a rational (resp. regular) map $f : V \longrightarrow \mathbb{P}^m_{\mathbb{C}}$ taking values in W.

Remark 1.1.2. Notice that a regular morphism is a holomorphic map between projective analytic varieties.

On $\mathbb{P}^n_{\mathbb{C}}$, and hence on every projective algebraic set, one can define a new topology, called the *Zariski topology*: an open subset is the complement of a projective algebraic set. Any regular morphism is continuous with respect to this topology.

In conclusion, we have the following picture:

$$\begin{array}{lclcl}
\text{proj. analytic manifolds} & \subset & \text{proj. analytic varieties} & \subset & \text{complex manifolds} \\
\cup & & \cup & & \\
\text{proj. algebraic manifolds} & \subset & \text{proj. algebraic varieties} & &
\end{array}$$

A priori, there is no reason why a projective analytic variety, or manifold, is projective. Anyway, we have there impressive result, that can be seen as a corollary of the GAGA Theorem.

Theorem 1.1.2. *(Chow Theorem). Every projective analytic variety is algebraic. Moreover, any holomorphic map between two projective algebraic varieties is a regular morphism.*

In general, moreover, there is no reason why a complex manifold should be projective. In dimension 1, one has the following:

Theorem 1.1.3. *Any Riemann surface is projective, hence algebraic.*

This is not the case in higher dimension:

Theorem 1.1.4. *The Hopf surface is not projective.*

This is a very important point: to classify Riemann surfaces, we can consider them to be projective. To classifiy compact complex surfaces, one needs to choose between the whole family of compact complex surfaces, and the family of those which are projective.

1.2 Vector bundles

Vector bundles are one of the main objects of study in complex geometry: in particular, line bundles will be one of the main tools in this series of lectures, and it seems useful to recall basic facts and definitions. Anyway, I prefer to posticipate the discussion about properties of line bundles, which will be the content of the second chapter of these notes. As in the following we will not only use vector bundles over complex manifolds, but even on real manifolds, we will start our discussion on a differentiable manifold M.

Definition 1.2.1. A *differentiable vector bundle* E on M is a differentiable manifold E together with a differentiable morphism $p : E \longrightarrow M$ such that

1. for every point $m \in M$, the fibre $E_m := p^{-1}(m)$ is a real vector space;

2. there is an open cover $\mathscr{U} = \{U_i\}$ of M, a real vector space T and a diffeomorphism
$$\varphi_i : p^{-1}(U_i) \longrightarrow U_i \times T,$$
called *trivialization*, such that for every $p \in U_i$ $\varphi_{U_i|E_p} : E_p \longrightarrow \{p\} \times T$ is an isomorphism of vector spaces.

For every $i, j \in I$ such that $U_{ij} \neq \emptyset$ there is a differentiable map

$$\varphi_{ij} : U_{ij} \longrightarrow GL(T),$$

called *transition function*, such that for every $p \in U_{ij}$ and every $t \in T$

$$\varphi_j^{-1}(p,t) = \varphi_i^{-1}(p, \varphi_{ij}(p)(t))$$

and for every $i, j, k \in I$ such that $U_{ijk} := U_i \cap U_j \cap U_k \neq \emptyset$ we have

$$\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki} = id.$$

Definition 1.2.2. Let $p : E \longrightarrow M$ be a differentiable vector bundle on M. If the rank of T is d, then E is called a *differentiable vector bundle of rank d.* A differentiable vector bundle of rank 1 is called *differentiable line bundle.*

Definition 1.2.3. Let $p_E : E \longrightarrow M$ and $p_F : F \longrightarrow M$ be two differentiable vector bundles A *morphism of vector bundles* is a differentiable map $f : E \longrightarrow F$ such that

1. $p_F \circ f = p_E$
2. $f_{|E_p} : E_p \longrightarrow F_p$ is a linear map for every $p \in M$.

It is not difficult to show that the category V_M of differentiable vector bundles on M is an abelian category.

Definition 1.2.4. Let $p : E \longrightarrow M$ be a differentiable vector bundle on M. A *differentiable section* of E is a differentiable map $s : M \longrightarrow E$ such that $p_E \circ s = id_M$. The set of differentiable sections of E is denoted $\Gamma(M, E)$ or $H^0(M, E)$, and has a natural structure of real vector space.

All the definitions we have given up to now go through if one replaces real by complex and differentiable by holomorphic: in this way one gets the notion of *holomorphic vector bundle*, *morphism between two holomorphic vector bundles* and *holomorphic section.*

Example 1.2.1. Let T be a real (resp. complex) vector space, M a differentiable (resp. complex) manifold. Then $T \times M$ is clearly a differentiable (resp. holomorphic) vector bundle, called the *trivial bundle.*

Example 1.2.2. Let M be a differentiable manifold, $m \in M$. Let $T_M := \bigcup_{m \in M} T_m(M)$ with the natural differentiable structure coming from M and the natural projection $p : T_M \longrightarrow M$. Then T_M is a differentiable vector bundle of rank equal to the dimension of M, and it is called *differentiable tangent bundle.* A section of the differentiable tangent bundle is called *differentiable vector field.*

If M is a complex manifold, one can define in a similar manner the *holomorphic tangent bundle* and *holomorphic vector fields.*

There are important constructions one can perform using vector bundles.

Definition 1.2.5. If $p : E \longrightarrow M$ is a differentiable (resp. holomorphic) vector bundle, a *differentiable* (resp. *holomorphic*) *sub-vector bundle* of E is a closed submanifold $F \subseteq E$ such that $p_{|F} : F \longrightarrow M$ is a vector bundle. If F is a differential sub-vector bundle of E, then the *quotient vector bundle* is the differentiable (resp. holomorphic) vector bundle whose fiber over the point $m \in M$ is the quotien vector space E_m/F_m.

Definition 1.2.6. If $p : E \longrightarrow M$ is a differentiable (resp. holomorphic) vector bundle, then the *dual vector bundle* is $E^* := \bigcup_{m \in M} E_m^*$, with the clear differentiable (resp. holomorphic) structure and the clear projection to M.

Example 1.2.3. If M is a differentiable manifold, the *differentiable cotangent bundle* if $T_M^* := (T_M)^*$. Similarly, if M is a complex manifold, one has the notions of *holomorphic cotangent bundle* Ω_M.

Definition 1.2.7. If $p_i : E_i \longrightarrow M$ is a differentiable (resp. holomorphic) vector bundle, $i = 1, 2$, then one can define easily the *direct sum*, *tensor product*, *wedge product vector bundle*, denoted, $E_1 \oplus E_2$, $E_1 \otimes E_2$ and $E_1 \wedge E_2$. If E is a vector bundle of rank k, we define the *determinant line bundle* of E as

$$det(E) := \bigwedge^k E.$$

Example 1.2.4. Let M be a differentiable manifold. The *canonical differentiable line bundle* of M is

$$K_M := det(T_M^*).$$

If M is a complex manifold, the *canonical holomorphic line bundle* is

$$K_M := det(\Omega_M),$$

where Ω_M is the holomorphic cotangent bundle of M.

Example 1.2.5. Let M be a differentiable (resp. complex manifold), and let $N \subseteq M$ be a differentiable (resp. complex) submanifold. The *differentiable* (resp. *holomorphic*) *normal bundle to* N in M is the quotient bundle

$$N_{N/M} := T_{M|N}/T_N.$$

In the algebraic setting, one can define *algebraic vector bundles* and their sections in a completely analogue way, only replacing complex manifold by algebraic manifold, and holomorphic by regular. Every algebraic vector bundle is then clearly holomorphic and, as we will see, on a projective manifold even the converse is true.

1.3 Sheaves and cohomology

In this section I will recall the basic notion of sheaf on a topological space, and the fundamental theory of cohomology. In particular, I will define the notions of presheaf, separated presheaf and sheaf, with the properties that we will use in the following lectures, and I will present some important examples and constructions.

Next, I will concentrate in cohomology: this will be introduce in a complete axiomatic way, and the main aim will be to define some important cohomology theories. Namely, I will present the singular (co)homology of a manifold, introducing Betti numbers, the Euler charcateristic and the Poincaré duality. Then I will define the Čech chomology and its properties, aiming to a way to calculate cohomology of sheaves on a manifold: as a consequence I will state the de Rham and the Dolbeault Theorems, linking singular cohomology to differential forms.

To conclude, I will present a brief discussion on properties of cohomology of sheaves on projective varieties, namely the Serre Vanishing and Finiteness Theorem, the Serre Duality Theorem and other properties.

To conclude I will state the GAGA Theorem, linking the algebraic and the complex worlds, and present come corollaries.

1.3.1 Sheaves

Sheaves theory is one of the main technical tools in modern mathematics. We will use it almost everywhere in the next lectures, and I will present it in a very elemenatry and general way in this section. In the following, let M be a topological space, whose topology is denotet τ, and R a principal domain (i. e. a commutative ring with unity, without zero divisors and such that every ideal admits one generator).

Definition 1.3.1. A *presheaf $\mathscr{F}$ of $R-$modules on M* is given by the following data:

1. an $R-$module $\mathscr{F}(U)$ for every open subset U of M;
2. a morphism of $R-$modules $\rho^U_V : \mathscr{F}(U) \longrightarrow \mathscr{F}(V)$, called *restriction morphism*, for every $V \subseteq U$ open subsets of M, such that
   - $\rho^U_U = id_{\mathscr{F}(U)}$ for every open subset U of M;
   - $\rho^V_W \circ \rho^U_V = \rho^U_W$ for every $W \subseteq V \subseteq U$ open subsets of M.

The elements in $\mathscr{F}(U)$ are called *sections* of $\mathscr{F}$ on U.

Definition 1.3.2. A *separated presheaf $\mathscr{F}$ of $R-$modules on M* is a presheaf $\mathscr{F}$ of $R-$modules on M such that for every open subset $U \subseteq M$ and every open cover $\mathscr{U} = \{U_i\}_{i\in I}$ of U, if $f \in \mathscr{F}$ is such that $\rho^U_{U_i}(f) = 0$ for every $i \in I$, then $f = 0$.

Definition 1.3.3. A *sheaf* $\mathscr{F}$ *of* $R-$*modules on* M is a separated presheaf of $R-$modules on M such that for every open subset U of M, every open cover $\mathscr{U} = \{U_i\}_{i\in I}$ of U and every collection $\{f_i\}_{i\in I}$, where $f_i \in \mathscr{F}(U_i)$ are such that for every $i, j \in I$ such that $U_{ij} \neq \emptyset$ we have $\rho^{U_i}_{U_{ij}}(f_i) = \rho^{U_j}_{U_{ij}}(f_j)$, there is $f \in \mathscr{F}(U)$ such that $\rho^U_{U_i}(f) = f_i$.

Definition 1.3.4. Let $\mathscr{F}$ and $\mathscr{G}$ be two presheaves of $R-$modules on M. A *morphism of presheaves* from $\mathscr{F}$ to $\mathscr{G}$ is a collection $f = \{f_U\}_{U\in\tau}$, where $f_U : \mathscr{F}(U) \longrightarrow \mathscr{G}(U)$ is a morphism of $R-$modules for every open subset U of M, and for every $V \subseteq U$ open subsets of M we have $\rho^U_V \circ f_U = f_V \circ \rho^U_V$.

If $\mathscr{F}$ and $\mathscr{G}$ are separated presheaves or sheaves, then a morphism between them is simply a morphism of presheaves.

Definition 1.3.5. Let $\mathscr{F}$ be a (pre)sheaf of $R-$modules on M, and let $U \subseteq M$ be an open subset. The *restriction* of $\mathscr{F}$ to U is the (pre)sheaf $\mathscr{F}_{|U}$ such that for every $V \subseteq U$ open we have $\mathscr{F}_{|U}(V) := \mathscr{F}(V)$, with the obvious restriction morphisms.

Now, let $\mathscr{F}$ and $\mathscr{G}$ be two (pre)sheaves of $R-$modules on M. For every open subset $U \subseteq M$, the set $Hom(\mathscr{F}_{|U}, \mathscr{G}_{|U})$ has a natural structure of $R-$module, and if $V \subseteq U$ is open, there is an obvious restriction morphism. We have then $\mathscr{H}om(\mathscr{F}, \mathscr{G})$, which is the (pre)sheaf defined as

$$\mathscr{H}om(\mathscr{F}, \mathscr{G})(U) := Hom(\mathscr{F}_{|U}, \mathscr{G}_{|U})$$

and obvious restriction morphisms.

Definition 1.3.6. The (pre)sheaf $\mathscr{H}om(\mathscr{F}, \mathscr{G})$ is called *(pre)sheaf of morphisms* from $\mathscr{F}$ to $\mathscr{G}$.

There are important definitions that are useful in sheaf theory. First of all, notice that if $m \in M$ and τ_m is the set of the open subsets of M containing m, then the family $\{\mathscr{F}(U)\}_{U\in\tau_m}$ is a direct system of $R-$modules: on τ_m one has a partial order relation, given by $V \leq U$ if and only if $V \subseteq U$, and if $V \leq U$ one has a map $\rho^U_V : \mathscr{F}(U) \longrightarrow \mathscr{F}(V)$.

Definition 1.3.7. Let $m \in M$ and $\mathscr{F}$ be a presheaf of $R-$modules on M. The *stalk* of $\mathscr{F}$ at m is

$$\mathscr{F}_m := \varinjlim_{U\in\tau_m} \mathscr{F}(U).$$

If $\mathscr{F}$ is a separated presheaf or a sheaf, then its stalk at m is simply its stalk at m as presheaf.

Notice that for every $U \in \tau_m$ there is a canonical map of $R-$modules $\rho_m^U : \mathscr{F}(U) \longrightarrow \mathscr{F}_m$. Moreover, if $f : \mathscr{F} \longrightarrow \mathscr{G}$ is a morphism of presheaves, then it induces a map

$$f_m : \mathscr{F}_m \longrightarrow \mathscr{G}_m.$$

There are two important construcions one can perform on presheaves or sheaves. First of all, let $\mathscr{F}$ be a presheaf of $R-$modules on M, and let U be an open subset of M. Define: $Sh(\mathscr{F})(U)$ as the set of maps $s : U \longrightarrow \coprod_{x\in U} \mathscr{F}_x$ verifying these two properties:

1. $s(x) \in \mathscr{F}_x$ for every $x \in U$;
2. for every $x \in U$ there is $V \subseteq U$ open neighborhood of x and an element $s_V \in \mathscr{F}(V)$ such that $s(y) = \rho_y^V(s_V)$ for every $y \in V$.

Clearly, $Sh(\mathscr{F})(U)$ is an $R-$module for every open subset U of M. Moreover, if $V \subseteq U$ are open subsets of M, define

$$\rho_V^U : Sh(\mathscr{F})(U) \longrightarrow Sh(\mathscr{F})(V), \qquad \rho_V^U(s) := s_{|V}.$$

It is easy to show that such a map is a well defined morphism of $R-$modules, and that $Sh(\mathscr{F})$ is a sheaf.

Definition 1.3.8. Let $\mathscr{F}$ be a presheaf of $R-$modules on M. The sheaf $Sh(\mathscr{F})$ is the *sheaf associated to* $\mathscr{F}$.

Remark 1.3.1. If $m \in M$, it is easy to see that $\mathscr{F}_m = Sh(\mathscr{F})_m$.

Now, let M and N be two topological spaces, and let $f : M \longrightarrow N$ be a continuous map. Let $\mathscr{F}$ be a (pre)sheaf of $R-$modules on M, and let U be an open subset of N. Define

$$f_*\mathscr{F}(U) := \mathscr{F}(f^{-1}(U)),$$

and if $V \subseteq U$ are open subsets of N, define $\rho_V^U := \rho_{f^{-1}(V)}^{f^{-1}(U)}$. It is easy to see that if $\mathscr{F}$ is a sheaf, then $f_*\mathscr{F}$ is a sheaf too.

Definition 1.3.9. The (pre)sheaf $f_*\mathscr{F}$ is called *push-forward* of $\mathscr{F}$ by f.

Let now $\mathscr{F}$ be a (pre)sheaf of $R-$modules on N, and $U \subseteq M$ be an open subset. Define:

$$f^{-1}\mathscr{F}(U) := \varinjlim_{f(U)\subseteq V} \mathscr{F}(V),$$

with obvious restriction morphisms. In general, this is not a sheaf, even if $\mathscr{F}$ is.

Definition 1.3.10. The *inverse image* of $\mathscr{F}$ by f is the sheaf associated to the presheaf $f^{-1}\mathscr{F}$. This will still be denoted $f^{-1}\mathscr{F}$.

Other important constructions are those of kernel, cokernel and image of a morphism of sheaves of $R-$modules. Let $\mathscr{F}$ and $\mathscr{G}$ be two (pre)sheaves of $R-$modules on M, and let $f : \mathscr{F} \longrightarrow \mathscr{G}$ be a morphism. For every open subset U of M let

$$\ker f(U) := \ker(f_U)$$

with the obvious restriction morphism. If $\mathscr{F}$ is a sheaf, then $\ker f$ is a sheaf, called the *kernel* of f. Similarily one can define the cokernel and the image of f, but in order to get a sheaf one has to consider the associated sheaf to the naif presheaf. It is not difficult then to see that the category whose objects are sheaves of $R-$modules and morphisms are morphisms of presheaves between them is abelian.

Definition 1.3.11. Let $\mathscr{F}, \mathscr{G}$ and $\mathscr{H}$ be three presheaves of $R-$modules on M, and let

$$\mathscr{F} \xrightarrow{f} \mathscr{G} \xrightarrow{g} \mathscr{H}.$$

This sequence is *exact* if and only if for every open subset U of M the sequence

$$\mathscr{F}(U) \xrightarrow{f_U} \mathscr{G}(U) \xrightarrow{g_U} \mathscr{H}(U)$$

is an exact sequence of $R-$modules. If $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{H}$ are sheaves, then the sequence above is *exact as sequence of sheaves* if and only if for every $m \in M$ the sequence

$$\mathscr{F}_m \xrightarrow{f_m} \mathscr{G}_m \xrightarrow{g_m} \mathscr{H}_m$$

is an exact sequence of $R-$modules.

Remark 1.3.2. As we will see, an exact sequence of sheaves is not, in general, an exact sequence of presheaves.

Clearly one can perform constructions as direct sums, tensor products and wedge products even with sheaves, getting new sheaves. To conclude this section, we present some important examples.

Example 1.3.1. Let M be a topological space, R a principal domain and G any $R-$module. Then one can define the presheaf G as follows: for every open subset U of M let $G(U) := G$ and restriction morphisms are simply the identity of G. This in general is not a sheaf (e. g. when M is not connected). The associated presheaf is denoted G_M and is called *constant sheaf* (of value G). Important examples are when $R = \mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$, getting as instance the sheaves $\mathbb{Z}_M$, $\mathbb{Q}_M$, $\mathbb{R}_M$ and $\mathbb{C}_M$.

Example 1.3.2. Let M be a differentiable manifold. For every open subset U of M, let

$$\mathscr{O}_M(U) := \{f : U \longrightarrow \mathbb{R} \,|\, f \texttt{ is differentiable}\},$$

with clear restriction functions. It is easy to see thar $\mathscr{O}_M$ is a sheaf, called the *structural sheaf* of M. In a similar way, if M is a complex manifold, one can define the structural sheaf of M as the sheaf of holomorphic functions. More generally, one can define the sheaf $\mathscr{M}_M$ of meromorphic functions on M.

If M is an affine algebraic variety in $\mathbb{C}^n$, one can define the structural sheaf $\mathscr{O}_M$ simply considering polynomials: if U is a Zariski open subset of M, whose complementary is $V(f)$ for some polynomial f, then we have

$$\mathscr{O}_M = \{p/f^k \mid k \in \mathbb{Z},\ p \in \mathbb{C}[M]\},$$

where $\mathbb{C}[M]$ is the ring of regular functions on M. Similar definitions hold if M is a projective algebraic variety.

Example 1.3.3. Let M be a (differentiable, complex) manifold, and $p : E \longrightarrow M$ be a (differentiable, holomorphic) vector bundle. For every open subset U of M define

$$\mathscr{O}_M(E)(U) := \Gamma(U, E),$$

with obvious restriction morphisms. It is easy to see that $\mathscr{O}_M(E)$ is a sheaf, called the *sheaf of sections associated to* E.

Definition 1.3.12. Let M be a differentiable (resp. complex) manifold, and let $\mathscr{F}$ be a sheaf of $\mathbb{R}-$vector spaces (resp. $\mathbb{C}-$vector spaces) on M. The $\mathscr{F}$ is an $\mathscr{O}_M-$*module* if there is a morphism of sheaves $c : \mathscr{O}_M \longrightarrow \mathscr{F}$. A *morphism of* $\mathscr{O}_M-$*modules* is a morphism of sheaves respecting the structure of $\mathscr{O}_M-$modules. Using this morphisms, one can define the $\mathscr{O}_M-$module $\mathscr{H}om_{\mathscr{O}_M}(\mathscr{F}, \mathscr{G})$.

Remark 1.3.3. Notice that if $\mathscr{F}$ is an $\mathscr{O}_M-$module, there is a natural isomorphism of $\mathscr{O}_M(M)-$modules

$$h : \mathscr{F}(M) \longrightarrow Hom_{\mathscr{O}_M}(\mathscr{O}_M, \mathscr{F}), \qquad h(f)(s) := s \cdot f.$$

Definition 1.3.13. Let M be a (differentiable, complex) manifold, and let $\mathscr{F}$ be an $\mathscr{O}_M-$module. The *dual* of $\mathscr{F}$ is the $\mathscr{O}_M-$module $\mathscr{F}^* := \mathscr{H}om_{\mathscr{O}_M}(\mathscr{F}, \mathscr{O}_M)$.

Notice that if $\mathscr{F}$ and $\mathscr{G}$ are two $\mathscr{O}_M-$modules, then one can define $\mathscr{F} \otimes_{\mathscr{O}_M} \mathscr{G}$ in the obvious way, getting a new $\mathscr{O}_M-$module.

Definition 1.3.14. Let M and N be two complex (resp. algebraic) manifolds. A *morphism* between $(M, \mathscr{O}_M)$ and $(N, \mathscr{O}_N)$ is a couple $(f, f^\sharp)$, where $f : M \longrightarrow N$ is a holomorphic (res. regular) map, and $f^\sharp : f^{-1}\mathscr{O}_N \longrightarrow \mathscr{O}_M$ is a map of sheaves.

In the following, by a *morphism* between two complex (resp. algebraic) manifolds (or varieties) we will always mean a morphism in the sense of the previous definition. As we have a map $f^\sharp : f^{-1}\mathscr{O}_N \longrightarrow \mathscr{O}_M$, we can think of $\mathscr{O}_M$ as an $f^{-1}\mathscr{O}_N-$module. Then, the following definition makes sense:

Definition 1.3.15. Let M and N be two complex (resp. algebraic) manifolds, and let $f: M \longrightarrow N$ be a holomorphic (res. regular) map between them. Let $\mathscr{F}$ be an $\mathscr{O}_N-$module. The *pull-back of* $\mathscr{F}$ by f is the $\mathscr{O}_M-$module

$$f^*\mathscr{F} := f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{O}_N} \mathscr{O}_M.$$

Remark 1.3.4. Notice that the pull-back of any $\mathscr{O}_N-$module has a natural structure of $\mathscr{O}_M-$module. If $\mathscr{F}$ is an $\mathscr{O}_M-$module, then $f_*\mathscr{F}$ has a natural structure of $f_*\mathscr{O}_M-$module. Now, $f_*\mathscr{O}_M$ has a natural structure of $\mathscr{O}_N-$module, as by adjunction the morphism $f^\sharp$ gives a morphism

$$f^\flat : \mathscr{O}_N \longrightarrow f_*\mathscr{O}_M.$$

In conclusion, $f_*\mathscr{F}$ has a natural structure of $\mathscr{O}_N-$module.

Definition 1.3.16. An $\mathscr{O}_M-$ module $\mathscr{F}$ is *free of rank* r if $\mathscr{F} \simeq \bigoplus_{i=1}^r \mathscr{O}_M$. It is *locally free of rank* r if there is an open cover $\{U_i\}_{i\in I}$ of M such that for every $i \in I$ there is an isomorphism $\mathscr{F}_{|U_i} \simeq \bigoplus_{i=1}^r \mathscr{O}_{M|U_i}$. It is called *coherent* if there are $n, m \in \mathbb{N}$ and an open cover $\{U_i\}_{i\in I}$ of M such that for every $i \in I$ there is an exact sequence of sheaves

$$\mathscr{O}_{M|U_i}^{\oplus m} \longrightarrow \mathscr{O}_{M|U_i}^{\oplus n} \longrightarrow \mathscr{F}_{|U_i} \longrightarrow 0.$$

Remark 1.3.5. Notice that any locally free sheaf is coherent.

Example 1.3.4. Let $p: E \longrightarrow M$ be a vector bundle of rank r. It is easy to verify that the sheaf $\mathscr{O}_M(E)$ of sections associated to E is a locally free $\mathscr{O}_M-$module of rank r. Moreover, one can verify that to any locally free $\mathscr{O}_M-$module $\mathscr{F}$ of rank r one can associate a vector bundle $p_{\mathscr{F}} : E_{\mathscr{F}} \longrightarrow M$ such that $\mathscr{F}$ is canonically isomorphic (as $\mathscr{O}_M-$module) to $\mathscr{O}_M(E_{\mathscr{F}})$, and $E_{\mathscr{O}_M(E)}$ is canonically isomorphic to E as vector bundles.

Similarily, to any morphism of vector bundles one can associate a morphism of $\mathscr{O}_M-$modules between the corresponding sheaves of sections, and viceversa. In this way one sees that the category of vector bundles on M and the category of locally free $\mathscr{O}_M-$modules are equivalent.

In particular, any line bundle can be viewed as a locally free sheaf of rank 1: using the previous notations, one can write $\mathscr{O}_M(K_M)$ for the canonical line bundle.

Definition 1.3.17. Let $\mathscr{F}$ be an $\mathscr{O}_M-$module. A *resolution* of $\mathscr{F}$ is an exact sequence of sheaves

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}^1 \longrightarrow \mathscr{F}^2 \longrightarrow ...$$

Example 1.3.5. Let M be a differentiable manifold, $p \in \mathbb{N}$, and let

$$\mathscr{E}_M^p := \bigwedge^p T_M^*.$$

This is the sheaf of differentiable $p-$forms on M: it is easy to see that this is a vector bundle whose rank is $\binom{n}{p}$, where $n = dim(M)$. Using the local coordinates and the definition of differential of a $p-$form, one can produce a morphism of sheaves

$$d^p : \mathscr{E}^p_M \longrightarrow \mathscr{E}^{p+1}_M.$$

It is well known that $d^{p+1} \circ d^p = 0$ for every $p \in \mathbb{N}$. Moreover, there is a clear injective map $\mathbb{R}_M \longrightarrow \mathscr{E}^1_M$ whose image is in the kernel of d^1, so that we get a complex of sheaves

$$0 \longrightarrow \mathbb{R}_M \longrightarrow \mathscr{O}_M \xrightarrow{d^0} \mathscr{E}^1_M \xrightarrow{d^1} \mathscr{E}^2_M \xrightarrow{d^2} ...$$

By the Poincaré Lemma, this complex is a resolution of $\mathbb{R}_M$.

If M is a complex manifold, one can consider Ω^p_M, the vector bundle of holomorphic $p-$forms on M. More generally, one can consider the sheaf $\Omega^{p,q}_M$ of $(p,q)-$forms on M. Using the local coordinate and the definition of the $\overline{\partial}-$differential, one can produce a morphism of sheaves

$$\overline{\partial}^q : \Omega^{p,q}_M \longrightarrow \Omega^{p,q+1}_M.$$

Again, it is well known that $\overline{\partial}^{q+1} \circ \overline{\partial}^q = 0$ for every $q \in \mathbb{N}$. Moreover, there is a clear injective map $\Omega^p_M \longrightarrow \Omega^{p,0}_M$ whose image lies in the kernel of $\overline{\partial}^1$, so that we get a complex of sheaves

$$0 \longrightarrow \Omega^p_M \longrightarrow \Omega^{p,0}_M \xrightarrow{\overline{\partial}^0} \Omega^{p,1}_M \xrightarrow{\overline{\partial}^1} \Omega^{p,2}_M \xrightarrow{\overline{\partial}^2} ...$$

By the $\overline{\partial}-$Poincaré Lemma, this complex is a resolution of Ω^p_M.

To conclude this section, let me define two important classes of sheaves.

Definition 1.3.18. A sheaf $\mathscr{F}$ is *fine* if for every locally finite open cover $\{U_i\}_{i\in I}$ of M, for every $i \in I$ there is an endomorphism

$$h_i : \mathscr{F} \longrightarrow \mathscr{F}$$

whose support (i. e. the set of $m \in M$ such that $h_{i,m} \neq 0$) is contained in U_i.

A sheaf $\mathscr{F}$ is *flasque* if for every $V \subseteq U$ open subsets of M, the restriction morphism ρ^U_V is surjective.

Example 1.3.6. All the sheaves $\mathscr{E}^p_M$ are fine for a differentiable manifold M. Similarily, all the sheaves $\Omega^{p,q}_M$ are fine for a complex manifold M. In particular, the resolutions presented in the previous example are fine resolutions, i. e. resolutions whose sheaves are fine.

Definition 1.3.19. Let $\mathscr{F}$ be a sheaf of $R-$modules on M. The *sheaf of discontinuous sections of* $\mathscr{F}$ is the sheaf associating to any open subset U of M the $R-$module

$$\mathscr{C}^0(\mathscr{F})(U) := \{s : U \longrightarrow \coprod_{x\in U} \mathscr{F}_x \mid s(x) \in \mathscr{F}_x\},$$

with obvious restriction maps.

Remark 1.3.6. The sheaf of discontinuous sections of a sheaf $\mathscr{F}$ is a flasque sheaf. In particular, this implies that any sheaf admits a flasque resolution: we have an injective map

$$s : \mathscr{F} \longrightarrow \mathscr{C}^0(\mathscr{F}),$$

sending $f \in \mathscr{F}(U)$ to

$$s_U(f) : U \longrightarrow \coprod_{x \in U} \mathscr{F}_x, \qquad s_U(f)(y) := \rho_y^U(f).$$

The fact that the map is injective follows from the fact that $\mathscr{F}$ is a sheaf. Now, let $\mathscr{Z}^1(\mathscr{F}) := \mathscr{C}^0(\mathscr{F})/\mathscr{F}$, and let $\mathscr{C}^1(\mathscr{F}) := \mathscr{C}^0(\mathscr{Z}^1(\mathscr{F}))$. Clearly, there is a well-defined map $d^0 : \mathscr{C}^0(\mathscr{F}) \longrightarrow \mathscr{C}^1(\mathscr{F})$. Going on by induction, one gets a complex of sheaves

$$0 \longrightarrow \mathscr{F} \xrightarrow{s} \mathscr{C}^0(\mathscr{F}) \xrightarrow{d^0} \mathscr{C}^1(\mathscr{F}) \xrightarrow{d^1} ...$$

and it is easy to show that this is a resolution of $\mathscr{F}$.

1.3.2 Cohomology

In this section I will introduce one of the main tools we are going to use, namely sheaf and singular cohomology. I will introduce it in a pure axiomatic way, and then introduce some examples. Consider a topological space M.

Definition 1.3.20. A *cohomology theory* on M is given by the following data:

1. for every sheaf of $R-$modules on M define an $R-$module $H^q(M, \mathscr{F})$ for every $q \in \mathbb{Z}$,

2. for every morphism of sheaves of $R-$modules $f : \mathscr{F} \longrightarrow \mathscr{G}$ define a morphism of $R-$modules

$$H^q(f) : H^q(M, \mathscr{F}) \longrightarrow H^q(M, \mathscr{G}),$$

   for every $q \in \mathbb{Z}$,

such that the following properties are satisfied:

- for every sheaf of $R-$modules on M and $q < 0$ we have $H^q(M, \mathscr{F}) = 0$; for $q = 0$ there is an isomorphism of $R-$modules

$$i_{\mathscr{F}} : H^0(M, \mathscr{F}) \longrightarrow \mathscr{F}(M)$$

  such that for every $f : \mathscr{F} \longrightarrow \mathscr{G}$ morphism of sheaves of $R-$modules on M we have $i_{\mathscr{G}} \circ H^0(f) = H^0(g) \circ i_{\mathscr{F}}$;

- if $\mathscr{F}$ is flasque or fine, then $H^q(M,\mathscr{F})=0$ for every $q>0$;
- for every sheaf $\mathscr{F}$ of $R-$modules we have $H^q(id_{\mathscr{F}})=id_{H^q(M,\mathscr{F})}$ for every $q\in\mathbb{Z}$, and for every morphism $f:\mathscr{F}\longrightarrow\mathscr{G}$ and $g:\mathscr{G}\longrightarrow\mathscr{H}$ we have $H^q(g\circ f)=H^q(g)\circ H^q(f)$ for every $q\in\mathbb{Z}$.
- for every exact sequence of sheaves of $R-$modules

$$0\longrightarrow\mathscr{F}\xrightarrow{f}\mathscr{G}\xrightarrow{g}\mathscr{H}\longrightarrow 0$$

  and every $q\in\mathbb{Z}$ there is $\delta^q:H^q(M,\mathscr{H})\longrightarrow H^{q+1}(M,\mathscr{F})$, morphism of $R-$modules called *connecting morphism*, such that the sequence

$$...\xrightarrow{\delta^{q-1}}H^q(M,\mathscr{F})\xrightarrow{H^q(f)}H^q(M,\mathscr{G})\xrightarrow{H^q(g)}H^q(M,\mathscr{H})\xrightarrow{\delta^q}...$$

  is exact. This is called *long exact sequence induced in cohomology.*

This abstract definition does not say anything about the existence of cohomology theory. There are anyway several ways of producing them. A first way is to consider a flasque resolution of a sheaf $\mathscr{F}$, as

$$0\longrightarrow\mathscr{F}\longrightarrow\mathscr{F}^0\xrightarrow{d^0}\mathscr{F}^1\xrightarrow{d^1}...$$

Then take the complex

$$0\longrightarrow\Gamma(M,\mathscr{F}^0)\xrightarrow{d^0_M}\Gamma(M,\mathscr{F}^1)\xrightarrow{d^1_M}...$$

and let

$$H^q(M,\mathscr{F}):=\ker(d^q_M)/im(d^{q-1}_M).$$

One can replace flasque resolutions with fine resolutions, and in this way, up to providing the existence of a flasque or fine resolution for every sheaf, one produces a cohomology theory.

Example 1.3.7. Let M be a differentiable manifold: then there is a fine resolution of the sheaf $\mathbb{R}_M$, which is

$$0\longrightarrow\mathbb{R}_M\longrightarrow\mathscr{O}_M\xrightarrow{d^0}\mathscr{E}^1_M\xrightarrow{d^1}\mathscr{E}^2_M\xrightarrow{d^2}...$$

Then one can calculate the cohomology of $\mathbb{R}_M$ as just mentioned, getting the *de Rham cohomology* $H^p_{dR}(M)$.

If M is a complex manifold, for every $p\in\mathbb{N}_0$ there is a fine resolution of the sheaf Ω^p_M, which is

$$0\longrightarrow\Omega^p_M\longrightarrow\Omega^{p,0}_M\xrightarrow{\overline{\partial}^0}\Omega^{p,1}_M\xrightarrow{\overline{\partial}^1}\Omega^{p,2}_M\xrightarrow{\overline{\partial}^2}...$$

One can then calculate the cohomology of Ω^p_M as before, getting the *Dolbeault* of $\overline{\partial}-$*cohomology* $H^{p,q}_{\overline{\partial}}(M):=H^q_{\overline{\partial}}(M,\Omega^p_M)$.

Another way to produce a cohomology theory is Čech cohomology. Let $\mathscr{U} = \{U_i\}_{i\in I}$ be an open cover of M. For every $i_1, ..., i_n \in I$, where $i_j \neq i_k$ for $i \neq k$, let

$$U_{i_1,...,i_n} := \bigcap_{j=1}^{n} U_{i_j}.$$

Then define

$$C^p(\mathscr{U}, \mathscr{F}) := \prod_{i_0,...,i_p \in I} \mathscr{F}(U_{i_0,...,i_p}),$$

and the map

$$d^p : C^p(\mathscr{U}, \mathscr{F}) \longrightarrow C^{p+1}(\mathscr{U}, \mathscr{F})$$

as

$$(d^p(\alpha))_{i_0,...,i_{p+1}} := \left(\sum_{j=0}^{p+1} (-1)^j (\alpha)_{i_0,...,i_{j-1},i_{j+1},...,i_{p+1}} \right)_{|U_{i_0,...,i_{p+1}}},$$

where if $\beta \in C^q(\mathscr{U}, \mathscr{F})$, we denote $(\beta)_{i_0,...,i_q}$ its component in $\mathscr{F}(U_{i_0,...,i_q})$. It is not difficult to show that $d^{p+1} \circ d^p = 0$ for every $p \in \mathbb{N}_0$, so that we get a complex

$$0 \longrightarrow C^0(\mathscr{U}, \mathscr{F}) \xrightarrow{d^0} C^1(\mathscr{U}, \mathscr{F}) \xrightarrow{d^1} ...$$

Now define

$$\check{H}^q(M, \mathscr{U}, \mathscr{F}) := \ker(d^q)/im(d^{q-1}).$$

Now, let us consider two open covers $\mathscr{U} = \{U_i\}_{i\in I}$ and $\mathscr{V} = \{V_j\}_{j\in J}$ of M. We have a partial order relation on the family of open covers of M, putting $\mathscr{V} \leq \mathscr{U}$ if and only if for every $j \in J$ there is $i \in I$ such that $V_j \subseteq U_i$. In particular, then for every $j_1, ..., j_p \in J$ there are $i_1, ..., i_p \in I$ such that $V_{j_1,...,j_p} \subseteq U_{i_1,...,i_p}$. Hence, we have a restriction map

$$\rho^{\mathscr{U}}_{\mathscr{V}} : C^p(\mathscr{U}, \mathscr{F}) \longrightarrow C^p(\mathscr{V}, \mathscr{F}), \qquad \rho^{\mathscr{U}}_{\mathscr{V}|\mathscr{F}(U_{i_1,...,i_p}} := \rho^{U_{i_1,...,i_p}}_{V_{j_1,...,j_p}},$$

which, as easily seen, pass through cohomology. From this, one sees that the family of $\check{H}^p(M, \mathscr{U}, \mathscr{F})$ forms a direct system, and we can define

$$\check{H}^p(M, \mathscr{F}) := \varinjlim_{\mathscr{U}} \check{H}^p(M, \mathscr{U}, \mathscr{F}).$$

An important result is that if M is Hausdorff and admits a countable basis for its topology, then one can show that Čech cohomology is a cohomology theory.

Example 1.3.8. For any topological space M one can define the notion of *singular cohomology of* M. To do that, one needs the notion of $p-$*simplex*, which is the topological space

$$\Delta_p := \{(x_1, ..., x_p) \in \mathbb{R}^p \,|\, x_i \geq 0,\ x_1 + ..., x_p = 1\}.$$

A $p-$*simplex* in M is simply a continuous map $f_p : \Delta_p \longrightarrow M$. One can then define the group $S_p(M)$, which is the free abelian group generated by $p-$simplexes in M, and $S^p(M) := Hom_{\mathbb{Z}}(S_p(M), \mathbb{Z})$. Moreover one has morphisms

$$d_p : S_p(M) \longrightarrow S_{p-1}(M), \qquad d^p : S^p(M) \longrightarrow S^{p+1}(M),$$

such that $d_p \circ d_{p+1} = 0$ and $d^{p+1} \circ d^p = 0$. Finally, one has

$$H_p(M, \mathbb{Z}) := \ker(d_p)/im(d_{p+1}), \qquad H^p(M, \mathbb{Z}) := ker(d^p)/im(d^{p-1}).$$

These are called the *singular homology* and *cohomology* of M. One can modify the contruction in order to get $H^p(M, \mathbb{Q})$ (resp. $H^p(M, \mathbb{R})$, $H^p(M, \mathbb{C})$): by the Universal Coefficient Theorem, this is simply $H^p(M, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ (resp. tensoring by $\mathbb{R}$, by $\mathbb{C}$). If M is Hausdorff and admits a countable basis for its topology, one can show that

$$H^p(M, \mathbb{Z}) \simeq \check{H}^p(M, \mathbb{Z}_M),$$

and similar for $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.3.21. Let M be a topological space. The $i-th$ *Betti number* of M is

$$b_i(M) := rk(H^i(M, \mathbb{Z})).$$

The *topological Euler characteristic* of M is

$$e(M) := \sum_i (-1)^i b_i(M).$$

These two examples are indeed strictly linked. We have the following:

Theorem 1.3.1. ***(de Rham, Dolbeault)***. *Let M be a differentiable manifold. For every $p \in \mathbb{Z}$ we have*

$$H^p(M, \mathbb{R}) \simeq H^p_{dR}(M).$$

If M is a complex manifold, then for every p, q we have

$$H^p(M, \Omega^q_M) \simeq H^{p,q}_{\overline{\partial}}(M).$$

To conclude this section, I just want to mention an important property of singular cohomology of a complex manifold. First of all, recall that any complex manifold admits an orientation. We have the following:

Proposition 1.3.2. *Let M be a connected compact real manifold of dimension n. Then M admits an orientation if and only if $H_n(M, \mathbb{Z}) \simeq \mathbb{Z}$.*

From this, one sees that if M is a complex manifold of real dimension $2n$, then $b_{2n}(M) = 1$. A generator of $H_{2n}(M, \mathbb{Z})$ is called *fundamental class* of M, and is denoted η_M.

Now, let M be a real manifold of dimension n. For every integer p by definition we have

$$S^p(M) \times S_p(M) \longrightarrow \mathbb{Z}$$

which is a well defined pairing, passing through cohomology, getting hence a pairing, called *Kronecker pairing*

$$(.,.) : H^p(M,\mathbb{Z}) \times H_p(M,\mathbb{Z}) \longrightarrow \mathbb{Z}.$$

Passing to $\mathbb{Q}$, one has a perfect pairing, so that $H^p(M,\mathbb{Q}) \simeq H_p(M,\mathbb{Q})^*$.

One can introduce a $\mathbb{Z}-$graded ring structure on $H^*(M,\mathbb{Z}) := \bigoplus_p H^p(M,\mathbb{Z})$: the grading is clear, while the product is the following, called *cup product*. Let $f \in S^p(X)$, $g \in S^q(X)$, and $\alpha \in S_{p+q}(X)$. Let α_p be the $p-$simplex obtained considering only the first $p+1$ elements of α, and α_q be the $q-$simplex obtained by the remaining elements. Then let

$$f \cup g(\alpha) := f(\alpha_p)g(\alpha_q).$$

This product passes through cohomology, and we get the desired cup product on $H^*(M,\mathbb{Z})$, still denoted $\cup$.

Now we can define the following

$$D_M : H^p(M,\mathbb{Z}) \longrightarrow H_{n-p}(M,\mathbb{Z}), \qquad (a, D_M(b)) = (a \cup b, \eta_M).$$

Theorem 1.3.3. ***(Poincaré duality)***. *For every p the morphism D_M is an isomorphism.*

Corollary 1.3.4. *Let M be a real manifold of dimension n. Then $b_i(M) = b_{n-i}(M)$. In particular $b_i(M) = 0$ if $i < 0$ or $i > n$.*

Example 1.3.9. If M is a Riemann surface, then $b_0(M) = b_2(M) = 1$. For $b_1(M)$, recall that $H^1(M,\mathbb{Z}) \simeq \mathbb{Z}^{2g(M)}$, where $g(M)$ is the genus of M. In conclusion, we have $b_1(M) = 2g(M)$.

1.3.3 Cohomology of coherent sheaves

It seems now useful to recall some basic facts about the cohomology of coherent sheaves. In particular, I will present some results on the cohomology of coherent sheaves on affine varieties and on $\mathbb{P}^n_{\mathbb{C}}$. As a conclusion of this, I will introduce an important theorem, due to Serre, about the vanishing and the dimension of cohomology $\mathbb{C}-$vector spaces of coherent sheaves. Let me start with coherent sheaves on $\mathbb{C}^n$. In the following, let

$$R := \mathbb{C}[z_1, ..., z_n].$$

As we are concerned with $\mathbb{C}^n$ as an affine variety, we put on it the Zariski topology: in this way, we see that the structural sheaf of regular functions on $\mathbb{C}^n$ is completely determined by R: if $f \in R$ is a regular map on $\mathbb{C}^n$ and $U(f) = \mathbb{C}^n \setminus V(f)$ is the open subset of $\mathbb{C}^n$ associated to f, we have

$$\mathscr{O}_{\mathbb{C}^n}(U(f)) = R_f,$$

where R_f is the localization of R at f (i. e. $R_f = \{p/f^k \mid f \in R,\ k \in \mathbb{N}_0\}$). More generally, if M is any $R-$module, one can define the following sheaf: for any open subset $U(f)$ as before, define

$$\widetilde{M}(U(f)) := M_f,$$

where again M_f is the localisation of M at f. Notice that if $U(g) \subseteq U(f)$, this implies that $V(f) \subseteq V(g)$, meaning that f is contained in the ideal generated by g. Then there is $k_0 \in \mathbb{N}$ such that $f = g^{k_0}$, so that there is a well defined map

$$\rho^{U(f)}_{U(g)} : \widetilde{M}(U(f)) \longrightarrow \widetilde{M}(U(g)).$$

It is easy to verify that $\widetilde{M}$ is an $\mathscr{O}_{\mathbb{C}^n}-$module, called *quasi-coherent sheaf associated to* M. If M and N are two $R-$modules and $h : M \longrightarrow N$ is a morphism of $R-$modules, then we can define a morphism of $\mathscr{O}_{\mathbb{C}^n}-$modules

$$\widetilde{h} : \widetilde{M} \longrightarrow \widetilde{N}.$$

Remark 1.3.7. It is easy to verify that $\widetilde{M}$ is coherent if and only if M is a finitely generated $R-$module: indeed, M is a finitely generated $R-$module if and only if its localization M_f is a finitely generated R_f-module, i. e. a quotient of some $\bigoplus_{i=1}^n R_f$ by the submodule of relations, for every $f \in R$. But this is equivalent to ask that $\widetilde{M}$ is a coherent $\mathscr{O}_{\mathbb{C}^n}-$module.

Conversely, to any $\mathscr{O}_{\mathbb{C}^n}-$module $\mathscr{F}$ one can associate $\Gamma(\mathbb{C}^n, \mathscr{F})$, which has a natural structure of $R-$module.

Proposition 1.3.5. *The functor sending any $R-$module M to $\mathscr{O}_{\mathbb{C}^n}-$module $\widetilde{M}$ and any morphism h to $\widetilde{h}$ is an equivalence of between the category of $R-$modules and the category of quasi-coherent $\mathscr{O}_{\mathbb{C}^n}-$modules. The inverse functor is the one sending a quasi-coherent sheaf $\mathscr{F}$ to the $R-$module of its global sections. Moreover, this functor is exact, i. e. if*

$$0 \longrightarrow \mathscr{F} \xrightarrow{f} \mathscr{G} \xrightarrow{g} \mathscr{H} \longrightarrow 0$$

is an exact sequence of quasi-coherent sheaves on $\mathscr{C}^n$, the sequence

$$0 \longrightarrow \Gamma(\mathbb{C}^n, \mathscr{F}) \xrightarrow{f_{\mathbb{C}^n}} \Gamma(\mathbb{C}^n, \mathscr{G}) \xrightarrow{g_{\mathbb{C}^n}} \Gamma(\mathbb{C}^n, \mathscr{H}) \longrightarrow 0$$

is exact.

All these constructions and results can be generalized to any affine variety. In this generality, one can prove the following:

Proposition 1.3.6. *Let X be an affine variety in $\mathbb{C}^n$, and $\mathscr{F}$ a quasi-coherent sheaf on X. Then $\check{H}^p(X, \mathscr{F}) = 0$ for every $p > 0$.*

Let us pass to coherent sheaves on $\mathbb{P}^n_{\mathbb{C}}$. Let

$$S := \mathbb{C}[z_0, ..., z_n],$$

and consider it as a $\mathbb{Z}-$graded ring, where every z_i has degree 1 and $\alpha \in \mathbb{C}$ has degree 0. Let $d \in \mathbb{Z}$ and $S(d)$ be the same as S, but the degree is shifted by d, i. e. $\alpha \in \mathbb{C}$ has degree $-d$ and z_i has degree $1-d$. As we are concerned with $\mathbb{P}^n_{\mathbb{C}}$ as a projective algebraic variety, we put on it the Zariski topology. In this way, we see that the structural sheaf of regular functions on $\mathbb{P}^n_{\mathbb{C}}$ is completely determined by S. Indeed, if f is a homogenous polynomial in S and $U(f) = \mathbb{P}^n_{\mathbb{C}} \setminus V(f)$ is the open subset of $\mathbb{P}^n_{\mathbb{C}}$ determined by f, then

$$\mathscr{O}_{\mathbb{P}^n_{\mathbb{C}}}(U(f)) = S_f,$$

where

$$S_f := \{p/f^k \,|\, p \in S,\ k \in \mathbb{Z},\ deg(p) = deg(f^k)\}.$$

More generally, if M is a $\mathbb{Z}-$graded $S-$module, one can define the following $\mathscr{O}_{\mathbb{P}^n_{\mathbb{C}}}-$module: for every open subset U of $\mathbb{P}^n_{\mathbb{C}}$ as before, let

$$\widetilde{M}(U(f)) := M_f.$$

As in the previous case, if $U(g) \subseteq U(f)$, then one can define a restriction map

$$\rho^{U(f)}_{U(g)} : \widetilde{M}(U(f)) \longrightarrow \widetilde{M}(U(g)).$$

This is the *quasi-coherent sheaf associated to M*.

Remark 1.3.8. As in the previous case, $\widetilde{M}$ is coherent if and only if M is finitely generated.

Example 1.3.10. Let $d \in \mathbb{Z}$ and consider $S(d)$, which is a graded $S-$module. The associated sheaf is

$$\mathscr{O}(d) = \mathscr{O}_{\mathbb{P}^n_{\mathbb{C}}}(d) := \widetilde{S(d)}.$$

It is not difficult to see that for every $d \in \mathbb{Z}$, the sheaf $\mathscr{O}_d$ is a locally free sheaf of rank 1 on $\mathbb{P}^n_{\mathbb{C}}$. Notice that if $d_1, d_2 \in \mathbb{Z}$, then

$$\mathscr{O}(d_1) \otimes_{\mathscr{O}_{\mathbb{P}^n_{\mathbb{C}}}} \mathscr{O}(d_2) \simeq \mathscr{O}(d_1 + d_2).$$

These locally free sheaves are very important in the study of $\mathbb{P}^n_{\mathbb{C}}$ and, in general, of projective varieties. An important result is the following, describing their cohomology:

Proposition 1.3.7. *The following hold (here the cohomology is the Čech cohomology):*

1. *The natural map* $S \longrightarrow \bigoplus_{d\in\mathbb{Z}} H^0(\mathbb{P}^n_{\mathbb{C}}, \mathscr{O}(d))$ *is a graded isomorphism.*

2. *For every* $d \in \mathbb{Z}$ *and every* $0 < i < n$ *we have* $H^i(\mathbb{P}^n_{\mathbb{C}}, \mathscr{O}(d)) = 0$.

3. *We have* $H^n(\mathbb{P}^n_{\mathbb{C}}, \mathscr{O}(-n-1)) \simeq \mathbb{C}$, *and* $H^n(\mathbb{P}^n_{\mathbb{C}}, \mathscr{O}(d)) = 0$ *for every* $d > -n-1$.

4. *For every* $d \geq 0$ *the natural map*
$$H^0(\mathbb{P}^n_{\mathbb{C}}, \mathscr{O}(d)) \times H^n(\mathbb{P}^n_{\mathbb{C}}, \mathscr{O}(-k-n-1)) \longrightarrow H^n(\mathbb{P}^n_{\mathbb{C}}, \mathscr{O}(-n-1)) \simeq \mathbb{C}$$
is a perfect pairing. This is called Serre duality for $\mathbb{P}^n_{\mathbb{C}}$.

5. *Let* $K_{\mathbb{P}^n_{\mathbb{C}}}$ *be the canonical line bundle of* $\mathbb{P}^n_{\mathbb{C}}$. *Then*
$$\mathscr{O}_{\mathbb{P}^n_{\mathbb{C}}}(K_{\mathbb{P}^n_{\mathbb{C}}}) \simeq \mathscr{O}(-n-1).$$

6. *Let* $\mathscr{F}$ *be a coherent sheaf on* $\mathbb{P}^n_{\mathbb{C}}$. *Then there are* $k \in \mathbb{N}$ *and* $n_1, ..., n_k \in \mathbb{Z}$ *such that there exists a surjective morphism*
$$\bigoplus_{i=1}^{k} \mathscr{O}(n_i) \longrightarrow \mathscr{F} \longrightarrow 0.$$

From this one can deduce the following important theorem:

Theorem 1.3.8. ***(Serre's Vanishing and Finiteness)***. *Let* X *be a projective algebraic variety in* $\mathbb{P}^n_{\mathbb{C}}$, *and let* $j : X \longrightarrow \mathbb{P}^n_{\mathbb{C}}$ *be the inclusion. Let* $\mathscr{F}$ *be a coherent* $\mathscr{O}_X-$*module, and for every* $d \in \mathbb{Z}$ *let* $\mathscr{F}(d) := \mathscr{F} \otimes_{\mathscr{O}_X} j^*\mathscr{O}(d)$. *Then*

1. *for every* $i \in \mathbb{Z}$ *the* $\mathbb{C}-$*vector space* $H^i(X, \mathscr{F})$ *is finite dimensional;*

2. *there is* $n_0 = n_0(\mathscr{F}) \in \mathbb{Z}$ *such that* $H^i(X, \mathscr{F}(m)) = 0$ *for every* $m > n_0$ *and every* $i > 0$*;*

3. *if* $dim(X) = n$, *then* $H^p(X, \mathscr{F}) = 0$ *for* $p > n$.

To conclude, let me recall some basic properties of pull-back and push-forward of coherent sheaves. First, let me introduce the following notion:

Definition 1.3.22. Let M and N be two topological spaces, $f : M \longrightarrow N$ a continuous map, and let R be a principal domain. Let $\mathscr{F}$ be a sheaf of $R-$modules on M and $q \in \mathbb{Z}$. The *q−th direct image sheaf* of $\mathscr{F}$ by f is the sheaf $\mathbb{R}^q f_*\mathscr{F}$ associated to the presheaf $H^q(\mathscr{F})$ defined as follows: for every open subset U of N, let
$$H^q(\mathscr{F})(U) := H^q(f^{-1}(U), \mathscr{F}_{|f^{-1}(U)}),$$
with clear restriction maps.

Remark 1.3.9. Notice that $\mathbb{R}^q f_* \mathbb{F} = 0$ if $q < 0$, and $\mathbb{R}^0 f_* \mathscr{F} = f_* \mathscr{F}$.

Proposition 1.3.9. *Let X and Y be two projective algebraic varieties, and let $f : X \longrightarrow Y$ be a morphism.*

1. *If $\mathscr{F}$ is a coherent $\mathscr{O}_Y-$module, then $f^* \mathscr{F}$ is a coherent $\mathscr{O}_X-$module.*
2. *If $\mathscr{F}$ is a coherent $\mathscr{O}_X-$module, then $\mathbb{R}^q f_* \mathscr{F}$ is a coherent $\mathscr{O}_Y-$module for every $q \in \mathbb{Z}$.*
3. *Let $\mathscr{F}$ be a coherent $\mathscr{O}_X-$module and E a locally free $\mathscr{O}_Y-$module of finite rank. Then there is a canonical isomorphism*
$$\mathbb{R}^q f_* (\mathscr{F} \otimes_{\mathscr{O}_X} f^* E) \xrightarrow{\sim} \mathbb{R}^p f_* \mathscr{F} \otimes_{\mathscr{O}_Y} E$$
for every $p \in \mathbb{Z}$. This is called projection formula.

1.3.4 The GAGA Theorem

Up to now we have been dealing with two a priori different worlds: the complex one, with complex manifolds, analytic varieties, holomorphic maps and holomorphic vector bundles; and the algebraic one, with projective algebraic varieties, regular maps and algebraic vector bundles. we have already seen that every projective analytic variety (or manifold) is a projective algebraic variety (or manifold), and that holomorphic maps between two projective algebraic varieties are regular. In this section we want to mention that the same kind of properties hold even for vector bundles, sheaves and cohomology.

If X is a projective variety, it can clearly be viewed as a complex subvariety X^h of $\mathbb{P}^n_{\mathbb{C}}$: one of the main difference is that on X one has the Zariski topology, which is not Hausdorff, while on X^h we have the ordinary topology, which is Hausdorff. Similarily, if X and Y are two projective varieties and $f : X \longrightarrow Y$ is a regular map, it can be viewed as an holomorphic map $f^h : X^h \longrightarrow Y^h$.

Now, on X one has a natural structure sheaf of regular functions on X, which is $\mathscr{O}_X$. Similarily, on X^h we have the sheaf of holomorphic functions on X, which is $\mathscr{O}_{X^h}$. Now, consider the identity map
$$i : X^h \longrightarrow X,$$
which is continuous, and induces a natural map
$$i^\sharp : i^{-1} \mathscr{O}_X \longrightarrow \mathscr{O}_{X^h},$$
as regular maps are holomorphic.

More generally, if $\mathscr{F}$ is an $\mathscr{O}_X-$module, then one can consider $i^* \mathscr{F}$, which has a natural structure of $\mathscr{O}_{X^h}-$module. If $\mathscr{F}$ is coherent, we can define the following: as $\mathscr{F}$ is coherent, there is a Zariski open cover $\{U_i\}_{i\in I}$ of X such that for every $i \in I$ there is an exact sequence

$$\mathscr{O}_{U_i}^{\oplus m} \xrightarrow{M_i} \mathscr{O}_{U_i}^{\oplus n} \longrightarrow \mathscr{F}_{|U_i} \longrightarrow 0.$$

In other words, we have $\mathscr{F}_{|U_i} = coker(M_i)$. If one considers U_i^h, the analytic variety associated to U_i, it still makes sense to consider $\mathscr{O}_{U_i^h}$, and we clearly have an analogue map M_i^h. We can then define the sheaf $\mathscr{F}^h$ as the $\mathscr{O}_{X^h}-$module such that for every $i \in I$ we have

$$\mathscr{O}_{U_i^h}^{\oplus m} \xrightarrow{M_i^h} \mathscr{O}_{U_i^h}^{\oplus n} \longrightarrow \mathscr{F}_{|U_i^h} \longrightarrow 0.$$

Then $\mathscr{F}^h$ is a coherent $\mathscr{O}_{X^h}-$module. It is easy to see that $\mathscr{F}^h \simeq i^*\mathscr{F}$. In a similar way, to any morphism $f : \mathscr{F} \longrightarrow \mathscr{G}$ of $\mathscr{O}_X-$modules on can associate a morphism $f^h : \mathscr{F}^h \longrightarrow \mathscr{G}^h$ of $\mathscr{O}_{X^h}-$modules.

Next step is to compare cohomologies. Let $\mathscr{F}$ be a coherent $\mathscr{O}_X-$module on the projective variety X, and let $p \in \mathbb{Z}$. As $\mathscr{F}^h \simeq i^*\mathscr{F}$, one can easily produce a natural map

$$H^p(i) : H^p(X, \mathscr{F}) \longrightarrow H^p(X^h, \mathscr{F}^h)$$

for every p (here we use Čech cohomology).

The most surprising facts about these constructions, is that one can go in the opposite direction:

Theorem 1.3.10. ***(Serre's GAGA)****. Let X be a projective variety.*

1. *For any coherent $\mathscr{O}_{X^h}-$module $\mathscr{F}$ there is a coherent $\mathscr{O}_X-$module $\mathscr{F}^a$ such that $(\mathscr{F}^a)^h \simeq \mathscr{F}$. Moreover, if $\mathscr{F}$ is a coherent $\mathscr{O}_X-$module, then $(\mathscr{F}^h)^a \simeq \mathscr{F}$.*

2. *For every morphism $f : \mathscr{F} \longrightarrow \mathscr{G}$ of coherent $\mathscr{O}_{X^h}-$modules there is a morphism $f^a : \mathscr{F}^a \longrightarrow \mathscr{G}^a$ of coherent $\mathscr{O}_X-$modules such that*

   - *if $\mathscr{F} = \mathscr{G}$ and $f = id_{\mathscr{F}}$, then $f^a = id_{\mathscr{F}^a}$;*
   - *if $f : \mathscr{F} \longrightarrow \mathscr{G}$ and $g : \mathscr{G} \longrightarrow \mathscr{H}$ are two morphisms of coherent $\mathscr{O}_{X^h}-$modules, then $(g \circ f)^a = g^a \circ f^a$;*
   - *we have $(f^a)^h = f$. If $f : \mathscr{F} \longrightarrow \mathscr{G}$ is a morphism of $\mathscr{O}_X-$modules, then $(f^h)^a = f$.*

3. *the natural maps $H^p(i)$ are isomorphisms for every $p \in \mathbb{Z}$.*

By this Theorem one then sees that if we are in the projective world, it is the same thing to work in the algebraic or in the analytic setting: from now on, as soon as we are in the projective world, we will only talk about projective varieties, coherent sheaves and cohomology, without specifying if this is to be considered in the analytic or in the algebraic sense.

1.4 Hodge theory

An important piece of theory we are going to use at several points in what follows is the nowadays called Hodge theory. The aim of this section is to present basic material on this interesting subject. In particular, we will introduce the notion of Hermitian metrics, Hodge operators and harmonic forms on a compact complex manifold. Then we will pass to the case of Kähler manifolds, among which sit projective manifolds, and to state the main result of the section, namely the Hodge decomposition theorem. This will lead us to the notion of the Hodge numbers and of the Hodge diamond, and we will state one of the main tools of algebraic geometry, which is the Serre Duality Theorem. To conclude, I will present some remarks about Riemann surfaces and complex surfaces.

1.4.1 Harmonic forms

Let M be a compact complex manifold of dimension n, and let $p : E \longrightarrow M$ be a holomorphic vector bundle.

Definition 1.4.1. A *Hermitian metric* on E is a $C^\infty-$function $h : E \times E \longrightarrow \mathbb{C}$ such that for every $m \in M$ the map $h_m := h_{|E_m \times E_m}$ is a Hermitian inner product. A holomorphic vector bundle equipped with a Hermitian metric is called *Hermitian vector bundle*. The complex manifold M is called *Hermitian manifold* if $T_M \otimes \overline{T_M}$ is a Hermitian vector bundle.

Now, let M be a Hermitian manifold with Hermitian metric h on $T_M \otimes \overline{T_M}$. Let $m \in M$ and consider $\{z_1, ..., z_n\}$ to be a coordinate system at m. Then we can define a basis for $T_{M,m}$, which is given by $\{\partial/\partial z_1, ..., \partial/\partial z_n\}$, and a basis for $\overline{T_M}$, which is $\{\partial/\partial \overline{z}_1, ..., \partial/\partial \overline{z}_n\}$. It is then clear that for every $j, k = 1, ..., n$ there must be $C^\infty-$functions h_{jk} (defined on a neighborhood of m) such that locally at m we have

$$h = \sum_{j,k=1}^{n} h_{jk} dz_j \otimes d\overline{z}_k.$$

We can associate to the Hermitian metric h a real $(1,1)-$form ω_h, which locally at m is

$$\omega_h = \frac{i}{2} \sum_{j,k=1}^{n} h_{jk} dz_j \wedge d\overline{z}_k.$$

Now, as h is a Hermitian metric on $T_M \otimes \overline{T_M}$, it induces a metric on every associated tensor bundle: in particular, if $p, q \in \mathbb{N}_0$, we have a Hermitian metric on

$$\mathscr{E}_M^{p,q} := \Big(\bigwedge^p T_M^*\Big) \otimes \Big(\bigwedge^q \overline{T_M}^*\Big).$$

Let us now define

$$\mathscr{E}_M^l := \bigoplus_{p+q=l} \mathscr{E}_M^{p,q}.$$

We have then a Hermitian metric on $\mathscr{E}^l_M$ for every $l \in \mathbb{N}_0$. Let

$$vol_h := \bigwedge^n \omega_h,$$

which is a $2n-$form on M, and let the *Hodge inner product* be

$$(.,.) := \int_M (.,.)_m vol_h,$$

where $(.,.)_m$ is the Hermitian inner product induced by h on the fiber over m. With respect to the Hodge inner product, the decomposition of $\mathscr{E}^l_M$ is orthogonal for every l.

Definition 1.4.2. Let $l \in \mathbb{Z}$. The *Hodge $*-$operator of index l* is the linear map

$$* : \bigwedge^l T^*_M \longrightarrow \bigwedge^{2n-l} T^*_M$$

defined by the following formula: for every point $m \in M$, and every α, β we have $\alpha \wedge *\beta = (\alpha, \beta) vol_h(m)$.

The Hodge $*-$operators induce linear operators

$$* : \mathscr{E}^{p,q}_M \longrightarrow \mathscr{E}^{n-p,n-q}_M, \qquad * : \mathscr{E}^l_M \longrightarrow \mathscr{E}^{2n-l}_M.$$

The corresponding conjugate operators are

$$\overline{*} : \mathscr{E}^{p,q}_M \longrightarrow \mathscr{E}^{n-q,n-p}_M, \qquad \overline{*}(\alpha) := *(\overline{\alpha}).$$

As we have inner products on $\mathscr{E}^l_M$ and $\mathscr{E}^{p,q}_M$, one can define a formal adjoint operator to the differentials d, ∂ and $\overline{\partial}$, which will be denoted d^*, ∂^* and $\overline{\partial}^*$.

Definition 1.4.3. The *$d-$Laplacian* (with respect to h) is

$$\Delta_d := dd^* + d^*d.$$

Analogously, one defines *$\partial-$Laplacian* Δ_∂ and *$\overline{\partial}-$Laplacian* $\Delta_{\overline{\partial}}$. An $m-$form (resp. $(p,q)-$form) α is *$d-$harmonic* (resp. *$\partial-$harmonic*, *$\overline{\partial}-$harmonic*) if $\Delta_d(\alpha) = 0$ (resp. if $\Delta_\partial(\alpha) = 0$, $\Delta_{\overline{\partial}}(\alpha) = 0$). The complex vector space of $d-$harmonic (resp. $\overline{\partial}-$harmonic) $m-$forms (resp. $(p,q)-$forms) is denoted $Harm^m(M)$ (resp. $Harm^{p,q}(M)$).

Now, assume that the complex vector spaces $\mathscr{E}^{p,q}_M(M)$ are finite dimensional (the infinite dimensional case is more complicated, but the results still hold), and consider the complex

$$\mathscr{E}^{p,q-1}_M(M) \xrightarrow{\overline{\partial}^{p,q-1}_M} \mathscr{E}^{p,q}_M(M) \xrightarrow{\overline{\partial}^{p,q}_M} \mathscr{E}^{p,q+1}_M(M).$$

By linear algebra one gets a direct sum decomposition

$$\mathscr{E}^{p,q}_M(M) = (\ker(\overline{\partial}^{p,q}) \cap \ker(\overline{\partial}^{p,q*})) \oplus im(\overline{\partial}^{p,q-1}) \oplus im(\overline{\partial}^{p,q-1*}).$$

Notice that the first summand consists of $\overline{\partial}-$harmonic forms, and $\Delta_{\overline{\partial}}$ is an isomorphism on the other two direct summands. The main result of this section are the following:

Theorem 1.4.1. *Let M be a compact differentiable manifold equipped with an Hermitian metric of dimension n, and let $m \in \mathbb{N}_0$.*

1. *The complex vector space $Harm^m(M)$ has finite dimension.*
2. *Let $H : \mathscr{E}^m_M(M) \longrightarrow Harm^m(M)$ be the orthogonal projection to the complex vector space of $\overline{\partial}-$harmonic $m-$forms. There is a unique linear endomorphism G of $\mathscr{E}^m_M(M)$ such that*
   - *$Harm^m(M) \subseteq \ker(G)$;*
   - *$H + \Delta_{\overline{\partial}} \cdot G = id_{\mathscr{E}^m_M(M)}$.*

   In particular there is a direct sum decomposition

   $$\mathscr{E}^m_M(M) = Harm^m(M) \oplus \overline{\partial}\,\overline{\partial}^* G\mathscr{E}^m_M(M) \oplus \overline{\partial}^*\overline{\partial} G\mathscr{E}^m_M(M).$$
3. *The map H induces an isomorphism $Harm^m(M) \simeq H^m_{dR}(M)$.*

Theorem 1.4.2. *Let M be a Hermitian manifold of dimension n, and let $p, q \in \mathbb{N}_0$.*

1. *The complex vector space $Harm^{p,q}(M)$ has finite dimension.*
2. *Let $H : \mathscr{E}^{p,q}_M(M) \longrightarrow Harm^{p,q}(M)$ be the orthogonal projection to the complex vector space of $\overline{\partial}-$harmonic $(p,q)-$forms. There is a unique linear endomorphism G of $\mathscr{E}^{p,q}_M(M)$ such that*
   - *$Harm^{p,q}(M) \subseteq \ker(G)$;*
   - *$H + \Delta_{\overline{\partial}} \cdot G = id_{\mathscr{E}^{p,q}_M(M)}$.*

   In particular there is a direct sum decomposition

   $$\mathscr{E}^{p,q}_M(M) = Harm^{p,q}(M) \oplus \overline{\partial}\,\overline{\partial}^* G\mathscr{E}^{p,q}_M(M) \oplus \overline{\partial}^*\overline{\partial} G\mathscr{E}^{p,q}_M(M).$$
3. *The map H induces an isomorphism $Harm^{p,q}(M) \simeq H^{p,q}_{\overline{\partial}}(M)$.*

Corollary 1.4.3. *Let M be a Hermitian manifold and $p, q \in \mathbb{N}_0$. Then the complex vector spaces $H^p(M, \Omega^q_M)$ are finite dimensional.*

Definition 1.4.4. Let M be a Hermitian manifold and $p, q \in \mathbb{N}_0$. The $(p,q)-th$ *Hodge number of* M is

$$h^{p,q}(M) := dim_{\mathbb{C}}(H^p(M, \Omega^q_M)).$$

Corollary 1.4.4. ***(Serre Duality)***. *Let M be a Hermitian manifold of dimension n, $p, q \in \mathbb{N}_0$. Then*

1. *There is an isomorphism $H^n(M, \Omega^n_M) \simeq \mathbb{C}$, so that $h^{n,n}(M) = 1$.*
2. *The Hodge $*-$operator defines an isomorphism of complex vector spaces*
$$H^p(M, \Omega^q_M(E)) \longrightarrow H^{n-p}(M, \Omega^{n-q}_M(E^*))^*.$$
In particular $h^{p,q}(M) = h^{n-p,n-q}(M)$.

Notice that as $p+q > 2n$ we have $\mathscr{E}^{p,q}_M(M) = 0$, we can consider the following, called *Hodge diamond* of M:

$$\begin{array}{c} h^{n,n}(M) \\ h^{n,n-1}(M) \quad h^{n-1,n}(M) \\ ... \\ h^{n,0}(M) \quad h^{n-1,1}(M) \quad ... \quad h^{1,n-1}(M) \quad h^{0,n}(M) \\ ... \\ h^{1,0}(M) \quad h^{0,1}(M) \\ h^{0,0}(M) \end{array}$$

Notice that by Serre Duality, one just needs to known the lower half of the Hodge diamond in order to determine it completely.

1.4.2 Kähler manifolds and Hodge decomposition

If one restricts to particular families of Hermitian manifolds, then one can get more informations about the Hodge numbers, which will play a fundamental role in the classification of surfaces.

Definition 1.4.5. Let M be a compact complex manifold. A Hermitian metric h on M is called *Kähler metric* if the associated real $(1,1)-$form ω_h is closed. A compact complex manifold admitting a Kähler metric is called *Kähler manifold.*

First of all, let me give some examples of important Kähler manifolds.

Example 1.4.1. Let M be a Riemann surface. Then, any Hermitian metric h on M is Kähler: this is clear, as ω_h is a $(1,1)-$form over a $1-$dimensional manifold, so it is closed.

Example 1.4.2. Let $n \in \mathbb{N}$, and set for every $z \in \mathbb{C}^{n+1} \setminus \{0\}$:

$$h(z) := \frac{1}{2\pi} \partial\overline{\partial} \log \parallel z \parallel^2 .$$

It is easy to verify that h is an Hermitian metric which is $\mathbb{C}^*-$invariant. In particular, this induces an Hermitian metric on $\mathbb{P}^n_{\mathbb{C}}$, which is still denoted h. One can even show that h is a Kähler metric, called *Fubini-Study metric.*

Example 1.4.3. Let M be a Kähler manifold, and N an analytic submanifold of M. Then N is Kähler: indeed, any Hermitian metric h on M clearly defines an Hermitian metric $h_{|N}$ on N. If the induced $(1,1)-$form ω_h is closed, then $\omega_{h_{|N}}$ is closed, hence N is Kähler. In particular, every projective manifold is Kähler.

The reason why Kähler manifolds are so important is the following: let M be a Kähler manifold with Kähler metric h, and let

$$L^{p,q} : \mathscr{E}_M^{p,q} \longrightarrow \mathscr{E}_M^{p+1,q+1}, \qquad L(\alpha) := \omega_h \wedge \alpha.$$

Moreover, let $\Lambda^{p,q} : \mathscr{E}_M^{p+1,q+1} \longrightarrow \mathscr{E}_M^{p,q}$ be its formal adjoint.

Theorem 1.4.5. *(Kähler identities). Let M be a Kähler manifold, and let $p, q \in \mathbb{Z}$. The following identities are verified:*

$$\partial^{p,q*} = i[\Lambda^{p,q}, \overline{\partial}^{p,q}],$$

$$\overline{\partial}^{p,q*} = -i[\Lambda^{p,q}, \partial^{p,q}].$$

From these identities, one has the following:

Corollary 1.4.6. *Let M be a Kähler manifold. Then $\Delta_d = 2\Delta_{\overline{\partial}}$. In particular, the Laplacian is real, preserves type (i. e. the Laplacian of a $(p,q)-$form is still a $(p,q)-$form) and preserves $\overline{\partial}-$harmonic forms.*

Proof. The proof is easy: by Theorem 1.4.5 we have

$$i(\partial\overline{\partial}^* + \overline{\partial}^*\partial) = \partial[\Lambda, \partial] + [\Lambda, \partial]\partial = 0.$$

Hence $\partial\overline{\partial}^* + \overline{\partial}^*\partial = 0$. Then

$$\Delta_d = (\partial + \overline{\partial})(\partial^* + \overline{\partial}^*) + (\partial^* + \overline{\partial}^*)(\partial + \overline{\partial}) = \Delta_\partial + \Delta_{\overline{\partial}}.$$

But now

$$i\Delta_\partial = -\partial[\Lambda, \overline{\partial}] - [\Lambda, \overline{\partial}]\partial = \overline{\partial}[\Lambda, \partial] + [\Lambda, \partial]\overline{\partial} = i\Delta_{\overline{\partial}},$$

so that $\Delta_d = 2\Delta_{\overline{\partial}}$. □

As a corollary one gets:

Theorem 1.4.7. *(**Hodge Decomposition**). Let M be a Kähler manifold, and let $k \in \mathbb{Z}$. Then we have the following direct sum decomposition*

$$H_{dR}^k(M) \otimes \mathbb{C} = \bigoplus_{p+q=k} H_{\overline{\partial}}^{p,q}(M).$$

Moreover, for every $p, q \in \mathbb{Z}$ we have $H_{\overline{\partial}}^{p,q}(M) = \overline{H_{\overline{\partial}}^{q,p}(M)}$.

Proof. As the Laplacian preserves type, we have

$$Harm^k(M) = \bigoplus_{p+q=k} Harm^{p,q}(M).$$

From the Hodge Theorems 1.4.1 and 1.4.2 we get the first statement. As the Laplacian is real, the last statement follows. □

As a conclusion, for Kähler manifolds one has $h^{p,q} = h^{q,p}$, so that the Hodge diamond is known if one knows $h^{p,q}$ for $0 \leq p \leq n$ and $0 \leq q \leq p$. Moreover, recall that by the de Rham Theorem we have $H^k_{dR}(M) \simeq H^k(M, \mathbb{R})$, so that tensoring with $\mathbb{C}$ we finally have

$$H^k(M, \mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,q}_{\overline{\partial}}(M).$$

It even follows that for every $k \in \mathbb{Z}$ we have

$$b_k(M) = \sum_{p+q=k} h^{p,q}(M).$$

As a simple corollary we have then:

Corollary 1.4.8. *Let M be a Kähler manifold. Then $b_1(M)$ is even.*

Proof. As M is a Kähler manifold, we have $h^{1,0}(M) = h^{0,1}(M)$ and $b_1(M) = h^{1,0}(M) + h^{0,1}(M)$, and we are done. □

In particular, using this we are finally able to show that there are compact complex surfaces which are not Kähler, hence non projective: one example is given by the Hopf surface X described in section 1. Topologically, X is homeomorphic to $S^1 \times S^3$, so that

$$b_1(X) = b_1(S^1) + b_1(S^3).$$

by the Künneth Formula. But $b_1(S^1) = 1$ and $b_1(S^3) = 0$, so that $b_1(X) = 1$, which is not even. By Corollary 1.4.8 we can then conclude that X is not Kähler.

Chapter 2

Line bundles

In this chapter we introduce the most important tool in the study of the geometry of surfaces: line bundles and their properties. First, I will recall the notion of Picard group and of Néron-Severi group of a surface: the first is the group of isomorphism classes of line bundles on a surface, with respect to the composition law given by the tensor product; the second is strictly related to the Picard group, and we are going to see various definitions of it.

Then I will introduce the notion of divisor on a surface. There two ways of doing this: one is the notion of Weil divisor, which is simply a formal sum of curves; the other one is the notion of Cartier divisor, namely the locus of zeroes and poles of a meromorphic function. These two notions are strictly related: if the surface is smooth, they are equivalent. Moreover, these are strictly related to line bundles: to any line bundle one can associate a Cartier, or Weil divisor, and conversely.

The next step will be to introduce basic facts about intersection theory on surfaces, very ampleness, ampleness, nefness and bigness. Among these we are going to define the nef cone, the ample cone and we are going to state the Kleiman Criterion for ampleness and nefness, the Hodge Index Theorem, the Kodaira Vanishing Theorem, the Riemann-Roch Theorem, the Noether Formula and so on. These are very fundamental tools in the theory of surfaces.

2.1 The Picard group

As seen in the first chapter, line bundles can be viewed in two equivalent ways: vector bundles of rank 1 or locally free sheaves of rank 1. We are going to switch from one point of view to the other. Moreover, as our goal is to work on projective surfaces, we are not going to specify if a line bundle is considered to be holomorphic or algebraic, this would be clear from the setting. In this section, X will be a (not necessarily projective) complex surface, and $\mathscr{O}_X$ will be the structural sheaf.

A line bundle $\mathscr{L}$ on X is then a locally free $\mathscr{O}_X-$module of rank 1, and if $\mathscr{L}'$ is another line bundle, we can define their tensor product $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{L}'$. This is again a locally free $\mathscr{O}_X-$module (as easily seen) whose rank is clearly 1. Consider the following set

$$Pic(X) := \{\mathscr{L}, |, \mathscr{L} \texttt{ is a line, bundle}\}/\simeq,$$

i. e. the set of isomorphism classes of line bundles on X: this is closed under tensor product of $\mathscr{O}_X-$modules.

Proposition 2.1.1. *For every X, $Pic(X)$ is an abelian group with respect to the tensor product.*

Proof. As $Pic(X)$ consists of isomorphism classes of line bundle, it is easy to see that $\otimes_{\mathscr{O}_X}$ is associative and commutative. Moreover, $\mathscr{O}_X$ is a line bundle, and for every $\mathscr{L} \in Pic(X)$ we have $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{O}_X \simeq \mathscr{O}_X \otimes_{\mathscr{O}_X} \mathscr{L} \simeq \mathscr{L}$, i. e. $Pic(X)$ is an abelian modoid. In order to show that $Pic(X)$ is an abelian group, we only need to provide an inverse element to any line bundle $\mathscr{L}$: indeed, consider

$$\mathscr{L}^{-1} := \mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{O}_X) = \mathscr{L}^*.$$

It is easy to show that $\mathscr{L}^{-1}$ is a locally free $\mathscr{O}_X-$module of rank 1, and we have

$$\mathscr{L}^{-1} \otimes_{\mathscr{O}_X} \mathscr{L} \simeq \mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{L}) = \mathscr{O}_X.$$

$\square$

Definition 2.1.1. For every ringed space $(X, \mathscr{O}_X)$, the group $Pic(X)$ is called *Picard group*.

Before going on with the properties of the Picard group, let me introduce some notation that will be useful in the following. Let X be a complex manifold, and let $\mathscr{L}$ be a line bundle. Then we set

$$h^i(X, \mathscr{L}) := dim_{\mathbb{C}} H^i(X, \mathscr{L}).$$

Definition 2.1.2. The *Euler characteristic* of $\mathscr{L}$ is

$$\chi(X, \mathscr{L}) := \sum_{i=0}^{dim(X)} (-1)^i h^i(X, \mathscr{L}).$$

A very useful theorem that we are going to use several times in the following, and that we present here without proof, is the following:

Theorem 2.1.2. ***(Serre duality)**. Let X be a smooth projective manifold, and let $\mathscr{L} \in Pic(X)$. Then $H^n(X, \mathscr{O}_X(K_X))$ is a 1−dimensional vector space. Moreover, for every $0 \leq i \leq n$ there is a perfect pairing*

$$H^i(X, \mathscr{L}) \otimes H^{n-i}(X, \mathscr{L}^{-1} \otimes \mathscr{O}_X(K_X)) \longrightarrow H^n(X, \mathscr{O}_X(K_X)) \simeq \mathbb{C}.$$

In particular $h^i(X, \mathscr{L}) = h^{n-i}(X, \mathscr{L} \otimes \mathscr{O}_X(K_X))$ for every i.

One of the main properties of the Picard group of the ringed space $(X, \mathscr{O}_X)$ is that it has a purely cohomological description in terms of Čech cohomology. We have indeed the following:

Proposition 2.1.3. *Let $(X, \mathscr{O}_X)$ be a ringed space. Then there is an isomorphism of groups $Pic(X) \simeq H^1(X, \mathscr{O}_X^*)$.*

Proof. To prove this, let $\mathscr{L}$ be a line bundle on X. As this is a locally free $\mathscr{O}_X$−module of rank 1, there is an open cover $\mathscr{U} = \{U_i\}_{i \in I}$ of X such that for every $i \in I$ there is an isomorphism $\varphi_i : \mathscr{O}_{U_i} \longrightarrow \mathscr{L}_{|U_i}$. For every $i, j \in I$ let $\varphi_{ij} := \varphi_i^{-1} \circ \varphi_j$, which is an automorphism of $\mathscr{O}_{U_{ij}}$, i. e. an element in $\mathscr{O}_X^*(U_{ij})$. Notice that for every $i, j, k \in I$ we have $\varphi_{ii} = id$, $\varphi_{ij} = \varphi_{ji}^{-1}$ and $\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki} = id$, i. e. the family $\{\varphi_{ij}\}_{i,j \in I}$ defines a 1−cocycle: its cohomology class defines then an element in $H^1(X, \mathscr{O}_X^*)$. It is moreover easy to see that if $\mathscr{L}$ and $\mathscr{M}$ are isomorphic line bundles, their corresponding classes in $H^1(X, \mathscr{O}_X^*)$ are equal. In conclusion, we defined a group morphism

$$Pic(X) \longrightarrow H^1(X, \mathscr{O}_X^*).$$

We need to show that this is an isomorphism. For the injectivity, let us suppose that the class corresponding to $\mathscr{L}$ is trivial: without loss of generality we can suppose $\varphi_{ij} = id$ for every $i, j \in I$. But this implies that $\varphi_{i|U_{ij}} = \varphi_{j|U_{ij}}$: we can then glue them together, getting an isomorphism $\varphi : \mathscr{O}_X \longrightarrow \mathscr{L}$, and injectivity is shown.

For surjectivity, let $\gamma = \{\varphi_{ij}\}_{i,j \in I}$ be a 1−cocyle, where $\varphi_{ij} : \mathscr{O}_{U_{ij}} \longrightarrow \mathscr{O}_{U_{ij}}$ is an isomorphism. As it is a 1−cocyle, for every $i, j, k \in I$ we have $\varphi_{ii} = id$, $\varphi_{ij} = \varphi_{ji}^{-1}$ and $\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki} = id$. We can then glue together the sheaves $\mathscr{O}_{U_i}$ along this gluing datum, getting a sheaf $\mathscr{L}$ such that for every $i \in I$ we have $\mathscr{L}_{|U_i} \simeq \mathscr{O}_{U_i}$ via some isomorphism φ_i. Then $\mathscr{L}$ is a line bundle whose corresponding class in $H^1(X, \mathscr{O}_X^*)$ is the class of γ. □

Notice that if X, Y are two complex spaces and $f : X \longrightarrow Y$ is a map between them, if $\mathscr{L}$ is a line bundle on Y, then $f^*\mathscr{L}$ is a line bundle on X. We have then

$$f^* : Pic(Y) \longrightarrow Pic(X),$$

which is a group morphism.

An important application of this description is via the exponential sequence, i. e. the short exact sequence of sheaves

$$0 \longrightarrow \mathbb{Z}_X \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_X^* \longrightarrow 0.$$

It induces a long exact sequence in cohomology

$$\cdots H^1(X, \mathscr{O}_X) \longrightarrow H^1(X, \mathscr{O}_X^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathscr{O}_X) \cdots$$

By Proposition 2.1.3, we have then the exact sequence

$$\cdots H^1(X, \mathscr{O}_X) \longrightarrow Pic(X) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathscr{O}_X) \cdots \tag{2.1}$$

Definition 2.1.3. The morphism c_1 in the exact sequence (2.1) is called *first Chern class*.

Let us see how one can apply this sequence:

Example 2.1.1. If X is a connected curve, then $H^2(X, \mathscr{O}_X) = 0$ and $H^2(X, \mathbb{Z}) \simeq \mathbb{Z}$. Choosing a generator for $H^2(X, \mathbb{Z})$, we can consider then $H^2(X, \mathbb{Z}) = \mathbb{Z}$, and $c_1 : Pic(X) \longrightarrow \mathbb{Z}$ is rather called the *degree*, and denoted *deg*.

Example 2.1.2. If $X = \mathbb{P}^n$, then one has $H^2(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}) = H^1(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}) = 0$ by Proposition 1.3.7. Then c_1 is an isomorphism. In the next section we prove that $Pic(\mathbb{P}^n) \simeq \mathbb{Z}$: this will imply even $b_2(\mathbb{P}^n) = 1$.

Example 2.1.3. More generally, if X is simply connected then c_1 is injective: indeed, in this case one has $H^1(X, \mathscr{O}_X) = 0$. Similarily, if $H^2(X, \mathscr{O}_X) = 0$, then c_1 is surjective.

There is a more intrinsec description of the first Chern class of a line bundle on a complex manifold X. Let $\mathscr{L}$ be a holomorphic line bundle, and let h be an Hermitian metric on $\mathscr{L}$. Then one can define the following $(1,1)-$form

$$c(\mathscr{L}, h) := -\frac{i}{2\pi}\partial\overline{\partial} \log h.$$

This is a $\overline{\partial}-$closed $(1,1)-$form on X. One of the most important properties of Hermitian metrics on a holomorphic line bundle asserts that if h and h' are two such metrics, then there is a holomorphic function f such that $h' = he^f$: then it is easy to see that

$$c(\mathscr{L}, h') = c(\mathscr{L}, h) + d(\overline{\partial} f).$$

In conclusion, the cohomology class of $c(\mathscr{L}, h)$ does not depend on the chosen Hermitian metric, so that one can associate an element $c(\mathscr{L}) \in H^2_{\overline{\partial}}(X)$ to any line bundle $\mathscr{L}$. Now, recall that there is an isomorphism between $H^2_{\overline{\partial}}(X)$ and $H^2(X, \mathbb{C})$, so that one gets finally an element $c(\mathscr{L}) \in H^2(X, \mathbb{C})$. The main property of this class is the following:

Theorem 2.1.4. *Let X be a complex manifold, and let $\mathscr{L}$ be a holomorphic line bundle on X. Then $c(\mathscr{L}) \in H^2(X,\mathbb{Z})$ and $c(\mathscr{L}) = c_1(\mathscr{L})$. In particular $c_1(\mathscr{L}) \in H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$.*

Now, let $Pic^0(X) := \ker(c_1)$. Using the exponential sequence, it is easy to see that

$$Pic^0(X) \simeq \frac{H^1(X,\mathscr{O}_X)}{im(H^1(X,\mathbb{Z}))}.$$

Notice that $H^1(X,\mathscr{O}_X)$ is a complex vector space, and the image of $H^2(X,\mathbb{Z})$ is a lattice in it. One can prove that it is always a maximal rank lattice: indeed, by the Hodge theory recall that $H^1(X,\mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X)$. So, if $[\alpha] \in H^1(X,\mathscr{O}_X)$ is the class of a $(1,0)-$real form, then $[\alpha] = \overline{[\alpha]}$. Then, two of these classes are linearly independent over $\mathbb{C}$ if and only if they are linearly independent over $\mathbb{R}$. This implies that $im(H^1(X,\mathbb{Z}))$ is a maximal rank lattice in $H^1(X,\mathscr{O}_X)$. In conclusion, $Pic^0(X)$ has the structure of a complex torus.

Definition 2.1.4. The variety $Pic^0(X)$ is called *Picard variety* of X. If X is a curve, then $Pic^0(X)$ is the *Jacobian* of X.

Definition 2.1.5. The group $NS(X) := Pic(X)/Pic^0(X)$ is the *Néron-Severi group* of X.

By the previous Theorem we have $NS(X) \subseteq H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$. We will return on the Néron-Severi group of surfaces in the following sections.

2.2 Cartier and Weil divisors

In this section we introduce two different notions of divisor on a complex manifold. The first one is due to Cartier, and is more recent than the second one, due to Weil. On a singular variety, these two notions can be different: generally, equivalence classes of Cartier divisors are in bijective correspondence with line bundles; this is not true for Weil divisors, since any line bundle defines a Weil divisor but the inverse statement is generally not true. However, as we are usually going to consider only smooth projective surfaces, these notions will be equivalent, giving different interpretation of the notion of line bundle.

2.2.1 Cartier divisors

The notion of Cartier divisor was introduced in the 1950s by Pierre Cartier. Let X be any complex variety (or manifold), and let $\mathscr{O}_X$ be the sheaf of regular functions on it.

Definition 2.2.1. A *Cartier divisor* on X is a global section of the sheaf $\mathscr{M}_X^*/\mathscr{O}_X^*$.

Using Čech cohomology one can represent a Cartier divisor in the following way: let $\mathscr{U} = \{U_i\}_{i\in I}$ be an open cover of X, and for every $i \in I$ let $f_i \in \mathscr{M}_X^*(U_i)$ (i. e. f_i is a non-trivial meromorphic function on U_i if X is a complex manifold, and f_i is a rational function on U_i if X is algebraic). In order to define a Cartier divisor, we need the following property to be verified: for every $i, j \in I$, we have $f_i/f_j \in \mathscr{O}_X^*(U_{ij})$.

In other words, a Cartier divisor is locally the locus of zeroes and poles of some meromorphic functions. Notice that the set of Cartier divisors has a natural structure of abelian group.

Notice that we have a short exact sequence of sheaves

$$0 \longrightarrow \mathscr{O}_X^* \longrightarrow \mathscr{M}_X^* \longrightarrow \mathscr{M}_X^*/\mathscr{O}_X^* \longrightarrow 0,$$

inducing the following exact sequence

$$H^0(X, \mathscr{M}_X^*) \longrightarrow H^0(X, \mathscr{M}_X^*/\mathscr{O}_X^*) \xrightarrow{\alpha} H^1(X, \mathscr{O}_X^*) = Pic(X). \tag{2.2}$$

In particular, then, to every Cartier divisor one can associate an isomorphism class of line bundles on X. Let us see the map α more explicitely: let $D = \{U_i, f_i\}_{i\in I}$ be a Cartier divisor. Then consider the family $\{\varphi_{ij}\}_{i,j\in I}$ where $\varphi_{ij} := f_i/f_j$. As D is a Cartier divisor, we have $\varphi_{ij} \in \mathscr{O}_X^*(U_{ij})$. Moreover, it is easy to see that $\varphi_{ii} = 1$, $\varphi_{ij} = \varphi_{ji}^{-1}$ and $\varphi_{ij}\varphi_{jk}\varphi_{ki} = 1$, so that $\{\varphi_{ij}\}_{i,j\in I}$ is a $1-$cocycle whose class is an element of $H^1(X, \mathscr{O}_X^*)$. The corresponding line bundle is denoted $\mathscr{O}_X(D)$, and it is called *line bundle associated to the Cartier divisor D*. We can be more explicit in the definition of $\mathscr{O}_X(D)$: if $U \subseteq X$ is open, we have

$$\mathscr{O}_X(D)(U) = \{f \in \mathscr{M}_X(U) \,|\, f \cdot f_i \in \mathscr{O}_X(U \cap U_i)\ \forall\, i \in I \texttt{ s. t. } U \cap U_i \neq \emptyset\}.$$

Indeed, for every $i \in I$ the map

$$\mathscr{O}_{U_i} \stackrel{\cdot 1/f_i}{\longrightarrow} \mathscr{O}_X(D)_{|U_i}$$

is well-defined and it is an isomorphism. Hence $\mathscr{O}_X(D)$ is a line bundle and a trivialization is given. The corresponding cycle is then clearly $\{f_i/f_j\}_{i,j\in I}$.

Definition 2.2.2. A Cartier divisor is *principal* if it is in $im(H^0(X, \mathscr{M}_X^*))$. Two Cartier divisors are *linearily equivalent* if their difference is a principal Cartier divisor.

We can then define

$$Car(X) := \frac{H^0(X, \mathscr{M}_X^*/\mathscr{O}_X^*)}{im(H^0(X, \mathscr{M}_X^*))}.$$

Notice that the exact sequence (2.2) gives then an injection

$$\alpha : Car(X) \longrightarrow Pic(X).$$

In many situations, this map is an isomorphism; however, there is an example due to Kleiman where this is not the case. Nakai shows that α is an isomorphism whenever X is projective. We can show the following:

Proposition 2.2.1. *If X is irreducible, then $\alpha : Car(X) \longrightarrow Pic(X)$ is an isomorphism.*

Proof. The main remark is that if X is irreducible, then $\mathscr{M}_X^*$ is simply the constant sheaf $k(X)_X^*$. As this sheaf is flasque, then $H^1(X, \mathscr{M}_X^*) = 0$, and the map α is surjective. □

2.2.2 Weil divisors

The notion of Weil divisor is born first in algebra (this is why we still use the name divisor), where it was introduced by Kummer, and only some decades later was generalized by André Weil to algebraic varieties. Their nature is hence strictly algebraic, and we define them only on algebraic varieties having particular properties (via the GAGA Theorem one can rewrite all this section in analytic terms). In this section, X will be an integral projective variety which is regular in codimension 1, i. e. the local ring $\mathscr{O}_{X,x}$ is regular for every point $x \in X$ of codimension 1 (which means that the Krull dimension of the commutative ring $\mathscr{O}_{X,x}$ is 1).

Definition 2.2.3. A *prime divisor* of X is a closed irreducible subvariety of X of codimension 1. The set of prime divisors of X is denoted $P(X)$. A *Weil divisor* is an element in the free abelian group $Div(X) = \mathbb{Z}^{P(X)}$ generated by $P(X)$.

In particular, a Weil divisor is written in the form

$$D = \sum_{Y \in P(X)} n_Y Y,$$

where $n_Y \in \mathbb{Z}$ for every $Y \in P(X)$, and $n_Y = 0$ for all but a finite number of prime divisors.

If $Y \subseteq X$ is a prime divisor, then Y is irreducible, and it has exactly one generic point η_Y. The local ring $\mathscr{O}_{X,\eta_Y}$ is a 1−dimensional commutative local ring (since the codimension of Y is 1), hence it is regular by our assumptions. By basic commutative algebra, one can show that hence $\mathscr{O}_{X,\eta_Y}$ is a discrete valuation ring whose quotient field is $k(X)$. The discrete valuation is

$$v_Y : k(X)^* \longrightarrow \mathbb{Z}.$$

It is not difficult to show that if $f \in k(X)^*$, then there are at most a finite number of prime divisors $Y \in P(X)$ such that $v_Y(f) \neq 0$. Then, the following definition makes sense:

$$(f) := \sum_{Y \in P(X)} v_Y(f)Y.$$

Definition 2.2.4. A Weil divisor D is *principal* if there is $f \in k(X)^*$ such that $D = (f)$. Two Weil divisors D_1, D_2 are *linearily equivalent*, and we write $D_1 \sim D_2$, if their difference is a principal Weil divisor.

Let us define

$$A^1(X) := Div(X)/\sim .$$

This is an abelian group which is strictly related to the Picard group and to $Car(X)$. Let us begin with this second relation: it is easy to associate to a Cartier divisor $\{U_i, f_i\}_{i \in I}$ a Weil divisor. Indeed, let

$$D := \sum_{Y \in P(X)} v_Y(f_{i(Y)})Y,$$

where $i(Y) \in I$ is chosen among all those $i \in I$ such that $U_i \cap Y \neq \emptyset$. Clearly, there can be many of these: however, if $i, j \in I$ are such that $U_{ij} \cap Y \neq \emptyset$, then $f_i/f_j \in \mathscr{O}^*_X(U_{ij})$, hence $v_Y(f_i) = v_Y(f_j)$. In conclusion, D is a well-defined Weil divisor.

Moreover, if the initial Cartier divisor is principal, then D is principal. Indeed, there is $f \in k(X)^*$ such that $f_i = f_{|U_i}$, and $v_Y(f_i) = v_Y(f)$, so that $D = (f)$. We have then a group morphism

$$\beta : Car(X) \longrightarrow A^1(X).$$

Composing with the previous α^{-1}, we get a map

$$\beta \circ \alpha^{-1} : Pic(X) \longrightarrow A^1(X).$$

Proposition 2.2.2. *If X is regular, then $\beta : Car(X) \longrightarrow A^1(X)$ is an isomorphism.*

More generally, one has that if X is normal and integral, then β is injective. In order to have β an isomorphism, we can only suppose X to be locally factorial, i. e. the local ring $\mathscr{O}_{X,x}$ is a unique factorization domain for every point $x \in X$. In general, however, if X is not regular then β can be not surjective.

Example 2.2.1. If X is a Riemann surface, then a prime divisor on X is simply a point. Hence a Weil divisor is written as $D = \sum_{P \in X} n_P P$, where $n_P \in \mathbb{Z}$ is zero for all but a finite number of points. As a Riemann surface is smooth, we have that $A^1(X) \simeq Pic(X)$, and in this form one can even give an interpretation of the degree of a line bundle: if $\mathscr{L} \in Pic(X)$, then there is a Weil divisor $D \in A^1(X)$ such that $\mathscr{L} \simeq \mathscr{O}_X(D)$. Let $D = \sum_{P \in X} n_P P$: then $deg(\mathscr{L}) = \sum_{P \in X} n_P$.

Example 2.2.2. Let us calculate $A^1(\mathbb{P}^n)$. Let $D = \sum_{Y \in P(\mathbb{P}^n)} n_Y Y$ be a Weil divisor in $\mathbb{P}^n$. As $Y \in P(\mathbb{P}^n)$, we have that Y is an irreducible hypersurface in $\mathbb{P}^n$, i. e. it is the zero locus of an irreducible homogenous polynomial g_Y (more precisely, $Y = V(I)$ where I is a prime ideal generated by an element g_Y). We can then define the *degree* of Y as $deg(Y) := deg(F_Y)$. Using this, we can define the following group morphism

$$deg : Div(\mathbb{P}^n) \longrightarrow \mathbb{Z}, \qquad deg\left(\sum_{Y \in P(\mathbb{P}^n)} n_Y Y\right) := \sum_{Y \in P(\mathbb{P}^n)} n_Y deg(Y).$$

Then, it is easy to see that deg is well-defined on $A^1(\mathbb{P}^n)$: let f be a rational function on $\mathbb{P}^n$, so that there are $p, q \in \mathbb{C}[z_0, ..., z_n]$ homogenous polynomials of the same degree such that $f = p/q$. If $Y \in P(\mathbb{P}^n)$, we have then $v_Y(f) = v_Y(p) - v_Y(q) = 0$ as p and q have the same degree, so that $(f) = 0$. In conclusion, we have a group morphism

$$deg : A^1(\mathbb{P}^n) \longrightarrow \mathbb{Z}.$$

We show that this is an isomorphism: let $g \in \mathbb{C}[z_0, ..., z_n]$ be a homogenous polynomial of degree d. We can write it (in a unique way up to a constant) as $g = g_1^{n_1} ... g_p^{n_p}$, where g_i is irreducible and $n_i \in \mathbb{N}$ for every $i = 1, ..., p$. Then $V(g_i) = Y_i$ is a prime divisor, i. e. an hypersurface of degree d_i, and we have $d = \sum_{i=1}^p n_i d_i$. Let

$$(g) := \sum_{i=1}^p n_i Y_i.$$

This is an effective Weil divisor of degree d. Now, let $D = \sum_{Y \in P(X)} n_Y Y$ be any Weil divisor of degree d, and write it as $D = D_+ - D_-$, where $D_+ = \sum_{Y \in P(X),\ n_Y > 0} n_Y Y$ and $D_- = -\sum_{Y \in P(X),\ n_Y < 0} n_Y Y$. If $d_+ := deg(D_+)$ and $d_- = deg(D_-)$, we have that D_+ and D_- are two effective Weil divisors of respective degree d_+ and d_- and $d = d_+ - d_-$. Let now $Y = V(g_Y)$, where g_Y is an homogenous polynomial, and let $g_+ = \prod_{Y \in P(X),\ n_Y > 0} g_Y^{n_Y}$ and $g_- = \prod_{Y \in P(X)} g_Y^{-n_Y}$. We have then $D_+ = (g_+)$ and $D_- = (g_-)$. Finally, let H be the hyperplane whose equation is $z_0 = 0$, and let $f := g_+ / z_0^d g_-$. Then it is easy to see that f is a rational function on $\mathbb{P}^n$ and that

$$D - dH = (f).$$

In conclusion, if D is a Weil divisor of degree d, we have that $D \sim dH$. Now, simply notice that $deg(H) = 1$, so that deg is an isomorphism.

We can be then more precise about the relationship between Weil divisors and line bundles. Let D be a Weil divisor, and write $D = \sum_{Y \in P(X)} n_Y Y$. Let us suppose that D is actually Cartier, so that it corresponds to a line bundle

$\mathscr{O}_X(D)$. Let us represent D as a Cartier divisor by $\{U_i, f_i\}_{i\in I}$. Recall that if $U \subseteq X$ is open, we have

$$\mathscr{O}_X(D)(U) = \{f \in \mathscr{M}_X(U) \,|\, f \cdot f_i \in \mathscr{O}_X(U \cap U_i)\ \forall\, i \in I \texttt{ s. t. } U \cap U_i \neq \emptyset\}.$$

Now, let $Y \in P(X)$ be such that $Y \cap U \neq \emptyset$, and let $i \in I$ be such that $U_i \cap U \cap Y \neq \emptyset$. As $f \cdot f_i \in \mathscr{O}_X(U_i)$, we have $v_Y(f \cdot f_i) > 0$, so that $v_Y(f) > v_Y(f_i)$. Now, recall that $v_Y(f_i) = n_Y$, so that we have finally

$$\mathscr{O}_X(D)(U) = \{f \in \mathscr{M}_X(U) \,|\, v_Y(f) > n_Y\ \ \forall Y \in P(X) \texttt{ s. t. } U \cap Y \neq \emptyset\}. \quad (2.3)$$

Now, let us introduce the following definition:

Definition 2.2.5. A Weil divisor $D = \sum_{Y\in P(X)} n_Y Y$ is *effective*, and we write $D \geq 0$, if $n_Y \geq 0$ for every $Y \in P(X)$. If D_1, D_2 are two Weil divisors on X, we write $D_1 \geq D_2$ if and only if $D_1 - D_2$ is effective.

Using this, we can describe the space of global sections of the line bundle associated to D: we have

$$\mathscr{O}_X(D)(X) = \{f \in k(X) \,|\, (f) \geq -D\}. \quad (2.4)$$

In other words, if $\mathscr{L}$ is a line bundle and D is an associated Weil divisor, then the space $H^0(X, \mathscr{L})$ is the space of all effective divisors D' which are linearily equivalent to D, i. e. all Weil divisors associated to $\mathscr{L}$. This will be used in the following.

Remark 2.2.1. Let X be a smooth projective variety, and let Y be a prime divisor on X. As Y is an irreducible subvariety of X of codimension 1, it defines in a natural way an element $[c(Y)] \in H_{2n-2}(X, \mathbb{Z})$. By linearity, every Weil divisor D on X defines a class $[c(D)] \in H_{2n-2}(X, \mathbb{Z})$, whose Poincaré dual $D_M^{-1}([c(D)])$ is in $H^2(X, \mathbb{Z})$. Then it is possible to show that $c_1(\mathscr{O}_X(D)) = D_M^{-1}([c(D)])$.

Let me now state some interesting properties of Weil divisors.

Proposition 2.2.3. *Let X be a projective variety which is regular in codimension 1, and let $Z \subseteq X$ be a closed subset whose complementary open subset is U. Let $i : U \longrightarrow X$ be the inclusion of U in X.*

1. *If $codim_X(Z) \geq 2$, then i induces an isomorphism between $A^1(X)$ and $A^1(U)$.*

2. *If $codim_X(Z) = 1$, then we have an exact sequence*

$$\mathbb{Z} \xrightarrow{z} A^1(X) \xrightarrow{i^*} A^1(U) \longrightarrow 0,$$

where $z(1)$ is the class of Z in $A^1(X)$.

This has a very important consequence: let us suppose that X is a normal projective variety. The normality assumption implies that $codim_X(X_{sing}) \geq 2$, so that by the previous Proposition we have $A^1(X) = A^1(X^s)$, where $X^s := X \setminus X_{sing}$. As X^s is smooth, by Proposition 2.2.2 we have $A^1(X^s) \simeq Pic(X^s)$: in conclusion, we have $A^1(X) \simeq Pic(X^s)$, so that every Weil divisor on X is determined by a line bundle on X^s.

In particular this allows us to define a canonical Weil divisor on every normal projective variety. Recall that the canonical line bundle is $\mathscr{O}_X(K_X) := \bigwedge^{dim(X)} \Omega_X$. If X is smooth, then Ω_X is a vector bundle of rank $dim(X)$, so that $\mathscr{O}_X(K_X)$ is a well-defined line bundle. If X is singular, this is no longer true, so we have no canonical line bundle. Anyway, on X^s we have $\mathscr{O}_{X^s}(K_{X^s})$, which is a line bundle: if X is normal, this defines a natural Weil divisor K_X, which is called *canonical Weil divisor* of X.

To conclude this section, let me make the following observation: if D is any Weil divisor (not necessarily Cartier), equation (2.3) still makes sense. In particular, then, we can associate an $\mathscr{O}_X-$module $\mathscr{O}_X(D)$ to every Weil divisor D on X. If D is Cartier, then $\mathscr{O}_X(D)$ is known to be a line bundle; otherwise, it still has some interesting properties.

Definition 2.2.6. A *divisorial sheaf* on X is a coherent $\mathscr{O}_X-$module which verifies the three following properties:

1. $\mathscr{L}$ has no torsion;
2. $rk(\mathscr{L}) = 1$;
3. $\mathscr{L} = \mathscr{L}^{**}$, where $\mathscr{L}^* := \mathscr{H}om(\mathscr{L}, \mathscr{O}_X)$ is the dual of $\mathscr{L}$.

Theorem 2.2.4. *Let X be a normal projective variety.*

- *Let D be a Weil divisor on X. Then $\mathscr{O}_X(D)$ is a divisorial sheaf. Moreover, if $D_1 \sim D_2$ are linearily equivalent Weil divisors, then $\mathscr{O}_X(D_1) \simeq \mathscr{O}_X(D_2)$.*
- *Let $\mathscr{L}$ be a divisorial sheaf on X. Then there is a Weil divisor D on X such that $\mathscr{L} \simeq \mathscr{O}_X(D)$.*

As a conclusion of this section, we add the following theorem:

Theorem 2.2.5. ***(Bertini)**. Let X be a smooth projective variety. A generic hypersurface of X is smooth.*

Proof. Let us consider X as a $d-$dimensional subvariety of $\mathbb{P}^n$ for some $n \in \mathbb{Z}$. Let us consider hypersurfaces of degree k, which are obtained as $H \cap X$, where H is a hypersurface of degree k in $\mathbb{P}^n$. The family of these hypersurfaces is $|\mathscr{O}_{\mathbb{P}^n}(k)|$, which is a projective space of dimension $\binom{n}{k}$. Let

$$B_{X,k} := \{(x, H) \in X \times |\mathscr{O}_{\mathbb{P}^n}(k)| \,|\, T_x(X) \subseteq T_x(H)\},$$

which is the set of couples (x, H) such that H is not transversal to the projectivized tangent space of X at x. As $dim(X) = d$, we have that $\mathbb{P}(T_x(X))$ is has dimension $d-1$, and the possible hyperplanes containing it form a projective space of dimension $\binom{n}{k} - d - 1$. The canonical projection $B \longrightarrow X$ presents B as a projective bundle, whose fiber is then $\mathbb{P}^{\binom{n}{k}-d-1}$. In conclusion $dim(B) = \binom{n}{k} - 1$, so that $p : B \longrightarrow |\mathscr{O}_{\mathbb{P}^n}(k)|$ cannot be surjective. Let then $H \in |\mathscr{O}_{\mathbb{P}^n}(k)| \setminus p(B)$: then $H \cap X$ is a hypersurface of X of degree k which is smooth at every point $x \in X$. $\square$

2.3 Ampleness and very ampleness

Let us start with the following basic definition. Let X be a projective manifold of dimension n, and let $\mathscr{L}$ be a line bundle on X. As X is compact, the complex vector space $H^0(X, \mathscr{L})$ is finite dimensional. Let $dim(H^0(X, \mathscr{L}) = n+1$, and let us suppose that this is not trivial.

Definition 2.3.1. The *complete linear system associated to* $\mathscr{L}$ is the set

$$|\mathscr{L}| := \{D \in Div(X) \,|\, D \geq 0,\ \mathscr{O}_X(D) \simeq \mathscr{L}\}.$$

By equation (2.4), we have that $|\mathscr{L}| = H^0(X, \mathscr{L})$, so that $|\mathscr{L}|$ is a complex vector space. Any linear subspace of $|\mathscr{L}|$ is called *linear system*. This allows us to define the following:

Definition 2.3.2. The *base locus* of $\mathscr{L}$ is the set

$$Bs(\mathscr{L}) := \bigcap_{D \in |\mathscr{L}|} D.$$

Every point in $Bs(\mathscr{L})$ is called *base point* of $\mathscr{L}$.

Using sections, we have that $Bs(\mathscr{L}) = \bigcap_{s \in H^0(X,\mathscr{L})} V(s)$, i. e. it is the locus of common zeroes of global sections of $\mathscr{L}$. Now, let us consider $\{s_0, ..., s_n\}$ a basis of $H^0(X, \mathscr{L})$, and define

$$\varphi_{s_0,\dots,s_n} : X \setminus Bs(\mathscr{L}) \longrightarrow \mathbb{P}^n, \qquad \varphi_{s_0,\dots,s_n}(x) := (s_0(x) : ... : s_n(x)).$$

This map is clearly not defined on $Bs(\mathscr{L})$ and depends on the chosen basis. Anyway, we will always write $\varphi_{\mathscr{L}} := \varphi_{s_0,\dots,s_n}$.

Example 2.3.1. Let X be a projective manifold, and let $f : X \longrightarrow \mathbb{P}^n$. If $\mathscr{L} = f^*\mathscr{O}_{\mathbb{P}^n}(1)$, then $Bs(\mathscr{L}) = \emptyset$ and $\varphi_{\mathscr{L}} = f$ (for the canonical basis of $H^0(X, f^*\mathscr{O}_{\mathbb{P}^n}(1))$). Indeed, we have that $h^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1)) = n+1$ by Proposition 1.3.7, and a canonical basis is given by the homogenous coordinates $z_0, ..., z_n$. Now, a canonical basis for $H^0(X, \mathscr{L})$ is given by $s_0, ..., s_n$, where $s_i = f^*z_i$ for every $i = 0, ..., n$, so that $\varphi_{\mathscr{L}} = f$. If $x \in Bs(\mathscr{L})$, then $s_i(x) = 0$ for every

$i = 0, ..., n$: By definition, this means that $z_i(f(x)) = 0$ for every $i = 0, ..., n$, i. e. the coordinates of the point $f(x)$ are all 0, so that $Bs(\mathscr{L}) = \emptyset$.

Definition 2.3.3. A line bundle $\mathscr{L}$ on X is *globally generated* if $Bs(\mathscr{L}) = \emptyset$. It is called *very ample* if it is globally generated and the map $\varphi_{\mathscr{L}}$ is a closed embedding.

Remark 2.3.1. If X is a complex manifold admitting a very ample divisor, then X is clearly projective. We then write $\mathscr{L} = \mathscr{O}_X(1)$.

Remark 2.3.2. A line bundle $\mathscr{L}$ is globally generated if and only if the evaluation map

$$ev : H^0(X, \mathscr{L}) \otimes \mathscr{O}_X \longrightarrow \mathscr{L}, \qquad ev(s,t) := s(t),$$

is surjective. More precisely, if $s \in H^0(X, \mathscr{L})$, then s corresponds to a morphism $s : \mathscr{O}_X \longrightarrow \mathscr{L}$, which can be evaluated on every section t of $\mathscr{O}_X$. The surjectivity of ev is verified if and only if the corresponding map on stalks is surjective. If $x \in X$, we have $\mathscr{L}_x \simeq \mathscr{O}_{X,x}$, so that ev_x is surjective if and only if it is not trivial. As $ev_x(s,1) = s(x)$, we have that $ev_x \neq 0$ if and only if $x \notin Bs(\mathscr{L})$.

Proposition 2.3.1. *A line bundle $\mathscr{L}$ on X is globally generated (resp. very ample) if and only if there is a map (resp. a closed embedding) $f : X \longrightarrow \mathbb{P}^n$, where $n = h^0(X, \mathscr{L}) - 1$, such that $\mathscr{L} \simeq f^*\mathscr{O}_{\mathbb{P}^n}(1)$.*

There is a very geometric criterion to test if a line bundle is very ample. First of all, let me remark the following: a line bundle $\mathscr{L}$ is globally generated if and only if $\varphi_{\mathscr{L}}$ is everywhere defined. By Remark 2.3.2, this means that for every point $x \in X$ there must be a section $s \in H^0(X, \mathscr{L})$ such that $s(x) \neq 0$. Let m_x be the ideal of $\mathscr{O}_{X,x}$ of those germes of functions vanishing at x: this is the maximal ideal of $\mathscr{O}_{X,x}$. we have then the following exact sequence

$$0 \longrightarrow m_x\mathscr{L} \longrightarrow \mathscr{L} \longrightarrow \mathscr{L}_x/m_x\mathscr{L}_x \longrightarrow 0.$$

Clearly, there is a section s of $\mathscr{L}$ such that $s(x) \neq 0$ if and only if $s_x \notin m_x\mathscr{L}_x$, i. e. if and only if $H^1(X, m_x\mathscr{L}) = 0$ for every $x \in X$.

Definition 2.3.4. A line bundle $\mathscr{L}$ *separates points* if and only if for every $x \neq y \in X$ we have $H^1(X, m_x \cdot m_y\mathscr{L}) = 0$. A line bundle $\mathscr{L}$ *separates tangent directions* if and only if for every $x \in X$ we have $H^1(X, m_x^2\mathscr{L}) = 0$.

Proposition 2.3.2. *A line bundle $\mathscr{L}$ is very ample if and only if it is globally generated, it separates points and tangent directions.*

Proof. If a line bundle is very ample, then clearly it is globally generated. Moreover, the morphism $\varphi_{\mathscr{L}}$ is an closed embedding. In particular, it is injective, so that for every $x \neq y \in X$ there must be $s, t \in H^0(X, \mathscr{L})$ such that $s(x) = 0$, $t(x) \neq 0$ and $s(y) \neq 0$ ad $t(y) = 0$. Consider the exact sequence

$$0 \longrightarrow m_x \cdot m_y \mathscr{L} \longrightarrow \mathscr{L} \longrightarrow \mathscr{L}_x / m_x \mathscr{L}_x \oplus \mathscr{L}_y / m_y \mathscr{L}_y \longrightarrow 0.$$

Then the two sections above exists if and only if $H^1(X, m_x \cdot m_y \mathscr{L}) = 0$, i. e. if and only if $\mathscr{L}$ separates points.

Moreover, $\varphi_{\mathscr{L}}$ is an embedding if for every $x \in X$ and every cotangent vector $v \in T_x(X)^*$ there is a section $s \in H^0(X, \mathscr{L})$ such that $s(x) = 0$ and $ds(x) = v$. Now, there is a well-defined map

$$d_x : m_x \mathscr{L} \longrightarrow T_x(X)^* \mathscr{L}_x / m_x \mathscr{L}_x, \qquad d_x(s) := ds(x).$$

Moreover we have an exact sequence

$$0 \longrightarrow m_x^2 \mathscr{L} \longrightarrow m_x \mathscr{L} \longrightarrow T_x(X)^* \mathscr{L}_x / m_x \mathscr{L}_x \longrightarrow 0.$$

Then $\varphi_{\mathscr{L}}$ is an embedding if and only if $H^1(X, m_x^2 \mathscr{L}) = 0$, i. e. if and only if $\mathscr{L}$ separates tangent directions. □

Example 2.3.2. As an application of the previous criterion, we can show that if X is a Riemann surface, then X is projective. We already know that every Riemann surface is Kähler, but here we prove something more. Let $\mathscr{L}$ be a line bundle on X: as X is compact, the complex vector spaces $H^0(X, \mathscr{L})$ and $H^1(X, \mathscr{L})$ are finite dimensional. Moreover, let $d = deg(L) \in \mathbb{Z}$: if $d < 0$, then $H^0(X, \mathscr{L}) = 0$. Indeed, if $H^0(X, \mathscr{L}) \neq 0$, there is a section s of $\mathscr{L}$ corresponding to a Weil divisor D which is trivial or effective, such that $\mathscr{L} \simeq \mathscr{O}_X(D)$. Then we have $deg(D) \geq 0$, so that $deg(\mathscr{L}) \geq 0$. By Serre's duality we have $H^1(X, \mathscr{L}) \simeq H^0(X, \mathscr{L}^{-1} \otimes \mathscr{O}_X(K_X))^*$. It is an easy calculation to show that $deg(K_X) = 2g(X) - 2$, so that $H^1(X, \mathscr{L}) = 0$ if $deg(\mathscr{L}) > 2g(X) - 2$.

Now, let $D = (2g(X) + 1)P$ for some $P \in X$, and let $\mathscr{L} := \mathscr{O}_X(D)$. If $Q \in X$ is any point, it is easy to see that $m_Q = \mathscr{O}_X(-Q)$: then $m_Q \mathscr{L} = \mathscr{O}_X((2g(X)+1)P - Q)$, so that $deg(m_Q \mathscr{L}) = 2g(X)$; moreover, if $R \in X$ we have $m_R \cdot m_Q \mathscr{L} = \mathscr{O}_X((2g(X)+1)P - Q - R)$, so that $deg(m_R \cdot m_Q \mathscr{L}) = 2g(X) - 1$. By the previous remark we have $H^1(X, m_Q \mathscr{L}) = 0$ for every point $Q \in X$, so that the map

$$\varphi_{\mathscr{L}} : X \longrightarrow \mathbb{P}^n$$

is well-defined. Moreover $H^1(X, m_R \cdot m_Q \mathscr{L}) = 0$ for every choice of $R, Q \in X$, so that $\mathscr{L}$ is very ample and $\varphi_{\mathscr{L}}$ is a closed embedding. In conclusion, every Riemann surface is projective.

Now, we introduce one of the main definitions in the theory of line bundles.

Definition 2.3.5. A line bundle $\mathscr{L}$ on X is *ample* if there is $n \in \mathbb{N}$ such that $\mathscr{L}^{\otimes n}$ is very ample.

Remark 2.3.3. By Proposition 2.3.1, a line bundle is ample if and only if there is a closed embedding $f : X \longrightarrow \mathbb{P}^N$, for some $N \in \mathbb{N}$, and $n \in \mathbb{N}$ such that $\mathscr{L}^{\otimes n} \simeq f^*\mathscr{O}_{\mathbb{P}^n}(1)$. Clearly, if X admits an ample line bundle, then X is projective, and the embedding is given by $\varphi_{\mathscr{L}^{\otimes n}}$.

Remark 2.3.4. Notice that if two line bundles $\mathscr{L}$ and $\mathscr{M}$ are ample, then $\mathscr{L}\otimes\mathscr{M}$ is ample.

To conclude this section, we show the following important criterion for ampleness:

Theorem 2.3.3. *Let X be a projective manifold and $\mathscr{L}$ be a line bundle on X. The following are equivalent:*

1. *$\mathscr{L}$ is ample;*

2. *for every coherent sheaf $\mathscr{F}$ there is $n_0 \geq 0$ such that for every $n > n_0$ we have $H^i(X, \mathscr{F} \otimes \mathscr{L}^{\otimes n}) = 0$ for every $i > 0$;*

3. *for every coherent sheaf $\mathscr{F}$ there is $m_0 \geq 0$ such that for every $n > m_0$ we have $\mathscr{F} \otimes \mathscr{L}^{\otimes n}$ is globally generated;*

4. *there is $p_0 \geq 0$ such that $\mathscr{L}^{\otimes m}$ is ample for every $m > p_0$.*

Proof. The fact that 1 implies 2 is simply a corollary of Serre's Vanishing: if $\mathscr{L}$ is ample, then $\mathscr{L}^{\otimes k}$ is very ample for some $k \in \mathbb{N}$. Then by Proposition 2.3.1 there is a closed embedding $f : X \longrightarrow \mathbb{P}^n$, for some $n \in \mathbb{N}$, such that $\mathscr{L}^{\otimes k} \simeq f^*\mathscr{O}_{\mathbb{P}^n}(1)$. By Serre's Vanishing, for every coherent sheaf $\mathscr{F}$ there is $l \gg 0$ such that $H^i(X, \mathscr{F} \otimes \mathscr{L}^{\otimes lk}) = 0$ for every $i > 0$. Now, let $n \in \mathbb{B}$, and suppose $n = lk + r$, for $0 \leq r < k$: then we have

$$H^i(X, \mathscr{F} \otimes \mathscr{L}^{\otimes lk+r}) = 0$$

for every $i > 0$, and we are done.

The fact that 2 implies 3 is easy: to show that $\mathscr{F}\otimes\mathscr{L}^m$ is globally generated, we just need to show that for every point $x \in X$ we have $H^1(X, m_x\mathscr{F}\otimes\mathscr{L}^{\otimes m}) = 0$. As $m_x\mathscr{F}$ is coherent, by assumption this is the case for $m \gg 0$. By compactness of X, we finish the proof.

The fact that 3 implies 1 is more complicated. □

As a corollary to this, we have the following:

Corollary 2.3.4. *Let X, Y be two projective manifolds, and let $f : X \longrightarrow Y$ be a finite morphism. If $\mathscr{L}$ is an ample line bundle on Y, then $f^*\mathscr{L}$ is an ample line bundle on X.*

Proof. Let $\mathscr{F}$ be a coherent sheaf on X. As f is finite, the sheaf $f_*\mathscr{F}$ is coherent, and we have $H^i(Y, f_*\mathscr{F} \otimes \mathscr{L}^{\otimes n}) = 0$ for $i > 0$ and $n \gg 0$. By projection formula, we have $f_*\mathscr{F} \otimes \mathscr{L}^n \simeq f_*(\mathscr{F} \otimes f^*\mathscr{L}^n)$, so that

$$H^i(X, \mathscr{F} \otimes f^*\mathscr{L}^{\otimes n}) \simeq H^i(Y, f_*(\mathscr{F} \otimes f^*\mathscr{L}^{\otimes n})) = 0$$

for $i > 0$ and $n \gg 0$, as f is a finite morphism. □

There is another important theorem we need to state:

Theorem 2.3.5. ***(Kodaira's Vanishing)****. Let X be a smooth projective variety of dimension n, and let $p, q \in \mathbb{Z}$. If $\mathscr{L}$ is ample line bundle on X, then*

$$H^q(X, \mathscr{L} \otimes \Omega^p_X) = 0$$

for every $p + q > n$. In particular $H^i(X, \mathscr{L} \otimes \mathscr{O}_X(K_X)) = 0$ for every $i > 0$.

The Kodaira Vanishing Theorem holds in greater generality: one can get rid of the projectivity of M, and consider any complex manifold. To do so, one would have to introduce the notion of positive line bundle, which is the complex analogue of ample line bundle. There are other various generalizations of this theorem, but we will not be concerned with them.

2.4 Intersection theory on surfaces

From now on, X will be a smooth projective surface. We introduce here one of the basic element in the theory of classification of surfaces: the intersection theory of divisors on a surface. Notice that if X is a smooth projective surface, a prime divisor is simply an irreducible curve on X, so that a Weil divisor is simply a formal finite linear combination of irreducible curves.

We introduce the following definition:

Definition 2.4.1. Let X be a smooth projective surface, Y any smooth projective variety and $f : X \longrightarrow Y$ a finite morphism. If $C \in P(X)$ is an irreducible curve, we define $f_*C := 0$ if $f(C)$ is a point, otherwise we define $f_*C := df(C)$, where d is the degree of the map $C \longrightarrow f(C)$ induced by f. By linearity we can define f_*D for every Weil divisor D on X.

Remark 2.4.1. If D is a Weil divisor on X, then D defines a line bundle $\mathscr{O}_X(D)$: as f is finite, we have that $f_*\mathscr{O}_X(D)$ is a coherent sheaf on Y, which in general is not a line bundle. The previous definition is really defined on Weil divisors. It is easily verified that if $D \sim D'$, then $f_*D \sim f_*D'$, so that f induces a map $f_* : A^1(X) \longrightarrow A^1(Y)$.

Remark 2.4.2. If $D \in A^1(Y)$, we have that $f_*f^*D \sim dD$.

We introduce now the basic definition for the intersection of divisors.

Definition 2.4.2. Let C and C' be two distinct irreducible curves on X, and let $x \in C \cap C'$. Let $f, g \in \mathscr{O}_{X,x}$ be two local equations respectively for C and C'. Then the *intersection multiplicity* of C and C' at x is

$$mult_x(C, C') := dim_{\mathbb{C}} \mathscr{O}_{X,x}/(f, g),$$

where $(f, g) \subseteq \mathscr{O}_{X,x}$ is the ideal generated by f and g. If $mult_x(C, C') = 1$, we say that C and C' are *transverse* (or *meet transversally*) at x. The *intersection multiplicity* of C and C' is

$$C \cdot C' := \sum_{x \in C \cap C'} mult_x(C, C').$$

Remark 2.4.3. The fact that $\mathscr{O}_{X,x}/(f, g)$ is a finite dimensional complex vector space comes from the Hilbert Nullstellensatz. The intersection multiplicity is a well-defined integer as $C \cap C'$ is a finite set.

Notice that we cannot define by linearity the intersection multiplicity of two Weil divisors, as we do not know how to define it for C with C itself. We use instead a different approach, following Beauville. Recall that if C is an irreducible curve, then we have $\mathscr{I}_C = \mathscr{O}_X(-C)$. If C and C' are two irreducible curves on X, let

$$\mathscr{O}_{C \cap C'} := \mathscr{O}_X/(\mathscr{O}_X(-C) + \mathscr{O}_X(-C')).$$

This is a skyscraper sheaf supported on $C \cap C'$, and it is easy to verify that if $x \in C \cap C'$ we have

$$(\mathscr{O}_{C \cap C'})_x = \mathscr{O}_{X,x}/(f, g),$$

where f and g are local equations for C and C' respectively. By this, it follows that

$$C \cdot C' = h^0(X, \mathscr{O}_{C \cap C'}).$$

Definition 2.4.3. Let $\mathscr{L}$ and $\mathscr{M}$ be two line bundles on X. Let

$$\mathscr{L} \cdot \mathscr{M} := \chi(\mathscr{O}_X) - \chi(\mathscr{L}^{-1}) - \chi(\mathscr{M}^{-1}) + \chi(\mathscr{L}^{-1} \otimes \mathscr{M}^{-1}),$$

which is called *intersection product* of $\mathscr{L}$ and $\mathscr{M}$.

We want to prove that this is a well-defined bilinear and non-degenerated form, so it is an intersection pairing. Let us start with the following:

Lemma 2.4.1. *Let C and C' be two distinct irreducible curves on X. Then*

$$\mathscr{O}_X(C) \cdot \mathscr{O}_X(C') = C \cdot C'.$$

Proof. The proof is rather easy: consider the following sequence

$$0 \longrightarrow \mathscr{O}_X(-C - C') \xrightarrow{(t,-s)} \mathscr{O}_X(-C) \oplus \mathscr{O}_X(-C') \xrightarrow{(s,t)} \mathscr{O}_X \longrightarrow \mathscr{O}_{C \cap C'} \longrightarrow 0,$$

where $s \in H^0(X, \mathscr{O}_X(C))$ is a section defining C and $t \in H^0(X, \mathscr{O}_X(C'))$ is a section defining C'. It is easy to verify that this sequence is exact: namely, we need to show that for every point $x \in C \cap C'$ the sequence

$$0 \longrightarrow \mathscr{O}_{X,x} \stackrel{(g,-f)}{\longrightarrow} \mathscr{O}_{X,x}^2 \stackrel{(f,g)}{\longrightarrow} \mathscr{O}_{X,x} \longrightarrow \mathscr{O}_{X,x}/(f,g) \longrightarrow 0$$

is exact, where f and g are local equations for C and C' respectively. To do so, we just need to check that if $a, b \in \mathscr{O}_{X,x}$ are such that $af = bg$, then there is $k \in \mathscr{O}_{X,x}$ such that $a = kg$ and $b = kf$. But this is clear: we have that f and g are coprime (since C and C' are two distinct irreducible curves), and $\mathscr{O}_{X,x}$ is a unique factorization domain (as X is smooth). Now, the exactness of the sequence implies that

$$\chi(\mathscr{O}_X(-C - C')) + \chi(\mathscr{O}_X) = \chi(\mathscr{O}_X(-C) \oplus \mathscr{O}_X(-C')) + \chi(\mathscr{O}_{C \cap C'}).$$

It is easy to show that $\chi(\mathscr{O}_X(-C) \oplus \mathscr{O}_X(-C')) = \chi(\mathscr{O}_X(-C)) + \chi(\mathscr{O}_X(-C'))$, and that $\mathscr{O}_X(-C) = \mathscr{O}_X(C)^*$ and $\mathscr{O}_X(-C - C') = \mathscr{O}_X(C) \otimes \mathscr{O}_X(C')$. Now, recall that $\mathscr{O}_{C \cap C'}$ is a skyscraper sheaf, so that $h^i(X, \mathscr{O}_{C \cap C'}) = 0$ for every $i > 0$. Then $\chi(\mathscr{O}_{C \cap C'}) = h^0(X, \mathscr{O}_{C \cap C'}) = C \cdot C'$, and the proof is finished. □

The next property we need to show is the following

Lemma 2.4.2. *Let C be a smooth curve on X and $\mathscr{L} \in Pic(X)$. Then*

$$\mathscr{O}_X(C) \cdot \mathscr{L} = deg(\mathscr{L}_{|C}).$$

Proof. Consider the exact sequence

$$0 \longrightarrow \mathscr{O}_X(-C) \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_C \longrightarrow 0.$$

Then $\chi(\mathscr{O}_C) = \chi(\mathscr{O}_X) - \chi(\mathscr{O}_X(-C))$. Tensoring with $\mathscr{L}^{-1}$ we get $\chi(\mathscr{L}^{-1}_{|C}) = \chi(\mathscr{L}^{-1}) - \chi(\mathscr{L}^{-1} \otimes \mathscr{O}_X(-C))$. Now, by definition we have

$$\mathscr{O}_X(C) \cdot \mathscr{L} = \chi(\mathscr{O}_X) - \chi(\mathscr{O}_X(-C)) - \chi(\mathscr{L}^{-1}) + \chi(\mathscr{O}_X(-C) \otimes \mathscr{L}^{-1}) =$$

$$= \chi(\mathscr{O}_C) - \chi(\mathscr{L}^{-1}_{|C}).$$

Now, use the Riemann-Roch Theorem on curves, i. e. $\chi(\mathscr{O}_C) = 1 - g(C)$ and $\chi(\mathscr{L}_{|C}) = deg(\mathscr{L}_{|C}) + 1 - g(C)$, so that

$$\mathscr{O}_X(C) \cdot \mathscr{L} = deg(\mathscr{L}_{|C}),$$

and we are done. □

We are finally able to show the following:

Theorem 2.4.3. *The intersection product*

$$\cdot : Pic(X) \times Pic(X) \longrightarrow \mathbb{Z}$$

is a symmetryc bilinear form such that $\mathscr{O}_X(C) \cdot \mathscr{O}_X(C') = C \cdot C'$ for every two distinct irreducible curves C, C' on X.

Proof. The symmetry is obvious. Now, consider $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3 \in Pic(X)$, and let

$$s(\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) := \mathscr{L}_1 \cdot \mathscr{L}_2 \otimes \mathscr{L}_3 - \mathscr{L}_1 \cdot \mathscr{L}_2 - \mathscr{L}_1 \cdot \mathscr{L}_3.$$

If $\mathscr{L}_i = \mathscr{O}_X(C)$ for some $i = 1, 2, 3$ and for some irreducible curve C. Then $s(\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) = 0$ by Lemma 2.4.2, so that in this case bilinearity is shown.

For the general case, we need the following fact: let $\mathscr{L} \in Pic(X)$. Then there is $n \geq 0$ such that $\mathscr{L} \otimes \mathscr{O}_X(n)$ is very ample. In particular, if D is any Weil divisor, then there are A, B smooth curves such that $D \sim A - B$ (simply consider B a smooth curve in $|\mathscr{O}_X(n)|$ and A a smooth curve in $|\mathscr{O}_X(D) \otimes \mathscr{O}_X(n)|$).

Now, let $\mathscr{L}, \mathscr{M} \in Pic(X)$. Write $\mathscr{M} = \mathscr{O}_X(A - B)$ for A, B smooth curves on X, so that

$$\mathscr{L} \cdot \mathscr{M} = \mathscr{L} \cdot \mathscr{O}_X(A) - \mathscr{L} \cdot \mathscr{O}_X(B).$$

From this and Lemma 2.4.2 we deduce the linearity in $\mathscr{L}$, and bilinearity is shown. The remaining part is simply Lemma 2.4.1. □

If D and D' are two Weil divisors, let $D \cdot D' := \mathscr{O}_X(D) \cdot \mathscr{O}_X(D')$. In particular, if $D = D'$, we have $D^2 := D \cdot D$. The previous Theorem guarantees that if $D \sim \widetilde{D}$, then $D \cdot D' = \widetilde{D} \cdot D'$ for every Weil divisor D'. In particular, if $D' \sim D$, we have $D \cdot D' = D^2$. Let me now give two first very useful application of this principle.

Proposition 2.4.4. *Let X and Y be two smooth projective surfaces, and let C be a smooth curve.*

1. *Let $f : X \longrightarrow C$ be a surjective morphism, and let F be the class of a fiber. Then $F^2 = 0$.*
2. *Let $g : X \longrightarrow Y$ be a generically finite morphism of degree d, and let D, D' be two Weil divisors on Y. Then $g^*D \cdot g^*D' = dD \cdot D'$.*

Proof. Let us begin with the first point. As X is a smooth surface and C is a smooth curve, the fiber $f^{-1}(P) =: X_P$ of f over the point $P \in C$ is a closed subvariety of X of codimension 1 (it is not empty as f is surjective). It is then a Weil divisor, and we write F for its class in $A^1(X)$. Notice that $P \in C$ is a Weil divisor on X, and $\mathscr{O}_X(X_P) \simeq f^*\mathscr{O}_C(P)$. The Weil divisor P is ample on C, so that there is $n \in \mathbb{N}$ such that nP is very ample. Notice that for $n \gg 0$ we have $h^0(nP) > 1$: indeed, we have $h^0(K_C - nP) = 0$ for $n \gg 0$ (since for $n \gg 0$ we have $deg(K_C - nP) = 2g(C) - 2 - n < 0$). By Serre duality this implies that for $n \gg 0$ we have $h^0(nP) = \chi(nP)$. Using Riemann-Roch for curves, we then have $h^0(nP) = n + 1 - g(C) > 1$, as $n \gg 0$. Hence there is a section $s \in H^0(X, \mathscr{O}_C(nP))$ such that $D = div(s)$ is an effective divisor which is linearily equivalent to nP and P does not appear in D. Notice that if $D = \sum_{Q \in C} n_Q Q$, we have that $f^*\mathscr{O}_C(D) \simeq \mathscr{O}_X(\sum_{Q \in C} n_Q X_Q)$. In conclusion the Weil divisor nX_P is linearly equivalent to $\sum_{Q\in C} n_Q X_Q$, with $n_P = 0$. Clearly, we have $X_Q \cap X_P = \emptyset$ for every $Q \neq P$, so that $X_Q \cdot X_P = 0$ in this case. Hence, notice that

$$n^2F^2 = n^2X_P^2 = nX_P \cdot \sum_{Q\in C} n_Q X_Q = \sum_{Q\neq P} nn_Q X_Q \cdot X_P = 0,$$

and the first item is shown.

For the second point, recall that there are A, B, A', B' smooth curves on Y such that $D \sim A - B$ and $D' \sim A' - B'$. If the statement is true for smooth curves, we have

$$g^*D \cdot g^*D' = g^*A \cdot g^*A' + g^*B \cdot g^*B' - g^*A \cdot g^*B' - g^*A' \cdot g^*B = dD \cdot D'.$$

In conclusion, we can suppose D and D' to be smooth curves. As g is generically finite of degree d, there is an open subset U of Y such that the map $g : g^{-1}(U) \longrightarrow U$ is finite of degree d and not ramified. Up to changing D and D' by linear equivalent ones, we can suppose that D and D' meet transversally at every point, and that $D \cap D' \subseteq U$. Then it is easy to see that

$$g^*D \cdot g^*D' = g^{-1}(D \cap D') = dD \cdot D',$$

and we are done. $\square$

I will conclude this section with two important examples.

Example 2.4.1. We can easily describe the intersection form of $\mathbb{P}^2$. Indeed, we know that $Pic(\mathbb{P}^2) \simeq \mathbb{Z}$, and that a generator for it is the line bundle $\mathscr{O}_{\mathbb{P}^2}(1)$. A Weil divisor corresponding to this line bundle is obtained as the zero locus of a section: As $H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(1))$ is given by homogenous polynomials of degree 1, a Weil divisor is simply a curve of degree 1 in $\mathbb{P}^2$, i. e. a line L. As $L^2 = L_1 \cdot L_2$, where L_1 and L_2 are two lines in $\mathbb{P}^2$, we can choose L_1 and L_2 meeting transversally: take $L_i = (z_i)$, where z_i are homogenous coordinates on $\mathbb{P}^2$, so that

$$L_1 \cdot L_2 = \sharp(L_1 \cap L_2) = 1,$$

as $L_1 \cap L_2 = \{(1 : 0 : 0)\}$. Then $\mathscr{O}_{\mathbb{P}^2}(1) \cdot \mathscr{O}_{\mathbb{P}^2}(1) = 1$, and the intersection product is described by linearity. As a consequence, we simply get Bezout's Theorem: if C_i is a curve of degree d_i in $\mathbb{P}^2$, for $i = 1, 2$, then $C_1 \cdot C_2 = d_1 d_2$.

Example 2.4.2. In this example we introduce a new surface $X = \mathbb{P}^1 \times \mathbb{P}^1$. Define the following curves

$$h_1 := \{(0 : 1)\} \times \mathbb{P}^1, \qquad h_2 := \mathbb{P}^1 \times \{(0 : 1)\}.$$

Moreover, let $U := X \setminus (h_1 \cup h_2)$: it is easy to see that $U \simeq \mathbb{C}^2$. Indeed, if $P = ((x_0 : x_1), (y_0, y_1)) \in U$, then $x_0, y_0 \neq 0$, so that one can associate to P

the point $(x_1/x_0, y_1/y_0)$ of $\mathbb{C}^2$. Conversely, to every $Q = (x, y) \in \mathbb{C}^2$ we can associate the point $((1 : x), (1 : y)) \in U$, and we have the isomorphism.

Now, it is a basic fact that $Pic(\mathbb{C}^2) = 0$, so that $Pic(U) = 0$. Hence, if D is a Weil divisor on X, we have that $D_{|U} = (f)$ for some function f. There must then be $n_1, n_2 \in \mathbb{Z}$ such that $D = (f) + n_1 h_1 + n_2 h_2$. In conclusion, we get that

$$Pic(X) \simeq A^1(X) \simeq \mathbb{Z} \oplus \mathbb{Z},$$

and generators are the classes of h_1 and h_2. The line bundle $\mathscr{O}_X(D)$ will be denoted $\mathscr{O}_X(n_1, n_2)$, and (n_1, n_2) is called the *bidegree* of D.

In order to describe the intersection pairing, we then just need to know $h_i \cdot h_j$ for every i, j. By the very definition of h_1 and h_2, we have that they meet transversally, so that

$$h_1 \cdot h_2 = \sharp(h_1 \cap h_2) = \sharp\{((0 : 1), (0 : 1))\} = 1.$$

We then need only to determine h_i^2. Now, notice that the projection

$$p_1 : X \longrightarrow \mathbb{P}^1$$

to the first factor is surjective and has h_1 as fiber over $(0 : 1)$. Then $h_1^2 = 0$ by Proposition 2.4.4. Similarily for h_2: in conclusion, the intersection pairing is given by $h_1 \cdot h_2 = h_2 \cdot h_1 = 1$ and $h_1^2 = h_2^2 = 0$.

2.4.1 The Riemann-Roch Theorem for surfaces

In this subsection we introduce one of the most important formulas in the whole theory of surfaces. Recall a basic result in the theory of curves: if D is a Weil divisor on a smooth curve C, then we have

$$\chi(\mathscr{O}_C(D)) = deg(D) - g(C) + 1.$$

This is called *Riemann-Roch Theorem for curves*. We now want to introduce an analogue of this for smooth projective surfaces. Here is the statement:

Theorem 2.4.5. ***(Riemann-Roch)***. *Let X be a smooth projective surface, and let $\mathscr{L} \in Pic(X)$. Then*

$$\chi(\mathscr{L}) = \chi(\mathscr{O}_S) + \frac{1}{2}(\mathscr{L}^2 - \mathscr{L} \cdot \mathscr{O}_X(K_X)).$$

Proof. The proof uses the Serre Duality. By the very definition of the intersection pairing, we have:

$$\mathscr{L}^{-1} \cdot (\mathscr{L} \otimes \mathscr{O}_X(-K_X)) = \chi(\mathscr{O}_X) - \chi(\mathscr{L}) - \chi(\mathscr{L}^{-1} \otimes \mathscr{O}_X(K_X)) + \chi(\mathscr{O}_X(K_X)).$$

By Serre duality $\chi(\mathscr{O}_X) = \chi(\mathscr{O}_X(K_X))$ and $\chi(\mathscr{L}) = \chi(\mathscr{L}^{-1} \otimes \mathscr{O}_X(K_X))$, so that we finally get

$$\chi(\mathscr{L}) = \chi(\mathscr{O}_X) - \frac{1}{2}\mathscr{L}^{-1} \cdot (\mathscr{L} \otimes \mathscr{O}_X(K_X)).$$

But now notice that by bilinearity of the intersection pairing we have $\mathscr{L}^{-1} \cdot (\mathscr{L} \otimes \mathscr{O}_X(K_X)) = -\mathscr{L}^2 + \mathscr{L} \cdot \mathscr{O}_X(K_X)$, and we are done. $\square$

Using the additive notations of Weil divisors and the definition of χ, the Riemann-Roch Formula becomes

$$h^0(D) - h^1(D) + h^2(D) = \chi(\mathscr{O}_X) + \frac{1}{2}(D^2 - D \cdot K_X).$$

Moreover, we can use Serre's Duality to replace $h^1(D)$ by $h^1(K_X - D)$ and $h^2(D)$ by $h^0(K_X - D)$. As $h^1(D) \geq 0$ for every Weil divisor D, we have the following inequality

$$h^0(D) + h^0(K_X - D) \geq \chi(\mathscr{O}_X) + \frac{1}{2}(D^2 - D \cdot K_X).$$

It is moreover interesting to note that as $\chi(\mathscr{O}_X(D))$ is an integer, for every Weil divisor we have $D^2 - D \cdot K_X \in 2\mathbb{Z}$.

Definition 2.4.4. The *irregularity* of a smooth projective surface X is $q(X) := h^1(\mathscr{O}_X) = h^{0,1}(X)$. The *geometric genus* of X is $p_g(X) := h^2(\mathscr{O}_X) = h^{0,2}(X)$.

Using this classical definitions, we have

$$\chi(\mathscr{O}_X) = 1 - q(X) + p_g(X),$$

and one could even state the Riemann-Roch Formula using this.

Remark 2.4.4. Notice that as X is a smooth projective surface, it has a natural Kähler structure, so that $h^{0,1}(X) = h^{1,0}(X)$ and $h^{2,0}(X) = h^{0,2}(X)$. In particular, then, we see that $b_1(X) = 2q(X)$, and that $b_2(X) = 2p_g(X) + h^{1,1}(X)$. From this, it follows that $q(X)$ is a topological invariant of any smooth projective surface.

Example 2.4.3. If $X = \mathbb{P}^2$, then we have $q(X) = p_g(X) = 0$, so that $\chi(\mathscr{O}_{\mathbb{P}^2}) = 1$. The Riemann-Roch formula is then very easy:

$$\chi(\mathscr{O}_{\mathbb{P}^2}(d)) = 1 + \frac{d^2 + 3d}{2}.$$

Example 2.4.4. If $X = \mathbb{P}^1 \times \mathbb{P}^1$, then we have the following properties: let $p_i : X \longrightarrow \mathbb{P}^1$ be the projection to the $i-$th factor. Then it is easy to see that $p_{i*}\mathscr{O}_X \simeq \mathscr{O}_{\mathbb{P}^1}$, so that by projection formula we get $q(X) = p_g(X) = 0$. Moreover, we have that $T_X \simeq p_1^* T_{\mathbb{P}^1} \oplus p_2^* T_{\mathbb{P}^1}$. Hence $K_X = \mathscr{O}_X(-2,-2)$. In conclusion, we have that $\chi(X) = 1$ and that

$$\chi(\mathscr{O}_X(n,m)) = 1 + n + m + nm.$$

We conclude this section by stating two important properties more.

Theorem 2.4.6. ***(Noether's Formula)***. *Let X be a smooth projective formula, and let $e(X)$ be the topological Euler characteristic of X. Then*

$$\chi(\mathscr{O}_X) = \frac{1}{12}(K_X^2 + e(X)).$$

This formula represents a connection between topological and complex characteristics of a smooth projective surface. An important consequence of the Riemann-Roch formula for surfaces is the following:

Theorem 2.4.7. ***(Genus formula)***. *Let X be a smooth projective surface, and let C be an irreducible curve on X. The genus of C is*

$$g(C) = 1 + \frac{1}{2}(C^2 + C \cdot K_X).$$

Proof. Recall that the genus of a curve is $g(C) = h^1(C, \mathscr{O}_C)$. Using the following exact sequence

$$0 \longrightarrow \mathscr{O}_X(-C) \longrightarrow \mathscr{O}_X \longrightarrow {}_C \longrightarrow 0$$

we get $\chi(\mathscr{O}_X) = \chi(\mathscr{O}_X(-C)) + \chi(\mathscr{O}_C)$. By the Riemann-Roch Formula for X we then get

$$\chi(\mathscr{O}_C) = -\frac{1}{2}(C^2 + C \cdot K_X).$$

Now, using Riemann-Roch for C we have that $\chi(\mathscr{O}_C) = 1 - g(C)$, and we are done. □

Remark 2.4.5. Recall that if C is a curve in X, then we have an exact sequence

$$0 \longrightarrow \mathscr{O}_X(-C)_{|C} \longrightarrow \Omega_{X|C} \longrightarrow \mathscr{O}_C(K_C) \longrightarrow 0.$$

From this it follows easily that $\mathscr{O}_C(K_C) \simeq \mathscr{O}_X(K_X + C)_{|C}$ or, written in Weil divisor form, we have $K_C \sim (K_X + C)_{|C}$. This is called the *Adjunction Formula*. Now, consider the degrees

$$2g(C) - 2 = deg(K_C) = deg((K_X + C)_{|C}) = C^2 + K_X \cdot C,$$

where the last equality follows from 2.4.2. Notice that this again implies the genus formula.

Example 2.4.5. Notice that the genus formula for $X = \mathbb{P}^2$ gives us the following: if C is a smooth curve of degree d in $\mathbb{P}^2$, then we have

$$g(C) = 1 + \frac{d^2 - 3d}{2} = \frac{(d-1)(d-2)}{2},$$

which is the well-known formula relating genus and degree for a smooth projective curve. If $X = \mathbb{P}^1 \times \mathbb{P}^1$ and C is a curve of bidegree (n, m), then we have

$$g(C) = 1 + nm - n - m.$$

2.5 Nef line bundles

In this section we introduce the notion of nefness of a line bundle, which can be seen as a limit of ampleness. This will be made precise in the following, and in order to do this we need to introduce some important material. First of all we discuss the intersection pairing on the Néron-Severi group of a smooth projective surface, and we prove that it is a finite rank lattice. We then introduce the notion of ample cone, effective cone and nef cone, and we study their relations. We will then prove some important criterion for ampleness and nefness (Kleiman, Nakai-Moishezon) which will be used frequently in the following.

2.5.1 Cup product and intersection product

Let us first make some remarks on the intersection pairing. If C and C' are two irreducible curves on a smooth projective surface, and we suppose them to meet transversally, we defined

$$C \cdot C' = \sharp(C \cap C').$$

This definition has to be modified in order to intersect curves which don't meet transversally or which are not irreducible One way of doing this is simply to consider the intersection pairing $\mathscr{O}_X(C) \cdot \mathscr{O}_X(C')$ we defined in the previous section. There is another way of doing this, namely using algebraic cycles: C and C' are two algebraic cycles in X, which is a complex manifold of dimension 2, and their dimension is 1. Let $p \in C \cap C'$, and let $v \in T_p(C)$, $w \in T_p(C')$ be two oriented basis for these two $1-$dimensional complex vector spaces. As C and C' meet transversally at p, we have that v, w is a basis for $T_p(M)$. We define $i_p(C, C') = 1$ if this is an oriented basis (with the natural orientation of M), otherwise $i_p(C, C') = -1$. Then we define

$$^\sharp(C \cdot C') := \sum_{p \in C \cap C'} i_p(C, C').$$

The main point is that as we are on complex manifolds, then for every point $p \in C \cap C'$ we have $i_p(C \cap C') = +1$, so that $^\sharp(C \cdot C') = C \cdot C'$. Another important point is to show is the following:

Lemma 2.5.1. *Let $C \sim \widetilde{C}$ and $C' \sim \widetilde{C}'$ be homologically equivalent cycles, i. e. $[C] = [\widetilde{C}] \in H_2(X, \mathbb{Z})$ and $[C'] = [\widetilde{C}'] \in H_2(X, \mathbb{Z})$. Then $^\sharp(C \cdot C') =^\sharp (\widetilde{C} \cdot \widetilde{C}')$.*

As a consequence we have an intersection pairing

$$\cap : H_2(X, \mathbb{Z}) \times H_2(X, \mathbb{Z}) \longrightarrow \mathbb{Z},$$

which is unimodular. This is the real content of the Poincaré duality: this implies that for every element $\alpha \in H^2(X, \mathbb{Z})$, which corresponds to a linear map

$H_2(X, \mathbb{Z}) \longrightarrow \mathbb{Z}$ by definition, there is $\gamma \in H_2(X, \mathbb{Z})$ such that $\alpha = \gamma \cap$. In this way we can interpret the cup product

$$\cup : H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \longrightarrow \mathbb{Z}.$$

Indeed, we can show that if $\alpha, \beta \in H^2(X, \mathbb{Z})$, then $\alpha \cup \beta = D_M(\alpha) \cap D_M(\beta)$, i. e. the intersection pairing is Poincaré-dual to the cup product.

Now, let us consider $\mathscr{O}_X(D)$ and $\mathscr{O}_X(D')$, for D, D' two Weil divisors on X. Then by the properties of the intersection pairing on line bundles and the one on cycles, we have that

$$\mathscr{O}_X(D) \cdot \mathscr{O}_X(D') = [D] \cap [D'].$$

By the previous duality we have then

$$\mathscr{O}_X(D) \cdot \mathscr{O}_X(D') = D_M^{-1}([D]) \cup D_M^{-1}([D']).$$

Now, by Remark 2.2.1 we have that $D_M^{-1}([D]) = c_1(\mathscr{O}_X(D))$, so that

$$\mathscr{O}_X(D) \cdot \mathscr{O}_X(D') = c_1(\mathscr{O}_X(D)) \cup c_1(\mathscr{O}_X(D')).$$

There is another interpretation of the intersection product which can be obtained from the de Rham Theorem: this result states, in particular, that if X is a smooth projective surface, we have an isomorphism $i : H^2(X, \mathbb{R}) \longrightarrow H^2_{dR}(X)$. Recall that an element in $H^2_{dR}(X)$ is a $2-$form. we have then the following pairing

$$H^2_{dR}(X) \times H^2_{dR}(X) \longrightarrow \mathbb{R}, \qquad (\omega, \omega') \mapsto \int_X \omega \wedge \omega'.$$

The explicit formulation of the de Rham Theorem states that for every $\alpha, \beta \in H^2(X, \mathbb{R})$ we have

$$\alpha \cup \beta = \int_X i(\alpha) \wedge i(\beta).$$

2.5.2 The Néron-Severi group

Let X be a smooth projective surface. We defined the Néron-Severi group of X as

$$NS(X) := Pic(X)/Pic^0(X).$$

In this way, the Néron-Severi group is a group of equivalence classes of line bundles on X. Moreover, we have in injective map $NS(X) \longrightarrow H^2(X, \mathbb{Z})$, so that we can view the Néron-Severi group as a submodule of $H^2(X, \mathbb{Z})$. Moreover, we have that $NS(X) \subseteq H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$. A first important result is the following:

Theorem 2.5.2. ***(Lefschetz)***. *If X is a smooth projective surface, then*

$$NS(X) = H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$$

Proof. Recall that there is a sequence of injections $\mathbb{Z}_X \xrightarrow{i} \mathbb{C}_X \xrightarrow{j} \mathscr{O}_X$. Now, the injection j induces the map $H^k(X,\mathbb{C}) \longrightarrow H^k(X,\mathscr{O}_X) = H^{0,k}(X)$ for every k, which is the projection onto the factor in the Hodge decomposition. In particular we have

$$H^2(X,\mathbb{Z}) \xrightarrow{H^2(i)} H^2(X,\mathbb{C}) \xrightarrow{H^2(j)} H^2(X,\mathscr{O}_X).$$

This combination is exactly the map $H^2(X,\mathbb{Z}) \xrightarrow{h} H^2(X,\mathscr{O}_X)$ coming from the exponential sequence. Hence, the kernel of h is exactly $NS(X)$. Now, let $\alpha \in H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$. Then $H^2(j) \circ H^2(i)(\alpha) = 0$ (as α is of type $(1,1)$), so that $\alpha \in \ker(h) = NS(X)$. $\square$

There is an alternative definition of the Néron-Severi group. First of all, let me define the following:

Definition 2.5.1. Let D be a Weil divisor on X. Then D is *numerically equivalent to* 0 if and only if for every irreducible curve C of X we have $D \cdot C = 0$. Two Weil divisors D, D' are *numerically equivalent*, and we write $D \equiv D'$, if $D - D'$ is numerically equivalent to 0.

Let us define $N^1(X) := Div(X)/\equiv$. Notice that since the intersection pairing depends only on the linear class of a Weil divisor, we have that $N^1(X) = Pic(X)/\equiv$. First of all, notice that if $\mathscr{L} \in Pic^0(X)$, then $\mathscr{L} \equiv 0$: indeed, let C be an irreducible curve on X. Then

$$\mathscr{L} \cdot \mathscr{O}_X(C) = c_1(\mathscr{L}) \cdot c_1(\mathscr{O}_X(C)) = 0,$$

as $c_1(\mathscr{L}) = 0$. Then we have a map $NS(X) \longrightarrow N^1(X)$, which is clearly surjective. Now, consider the following theorem

Theorem 2.5.3. ***(Kleiman)**. A line bundle $\mathscr{L}$ is numerically equivalent to 0 if and only if there is $m \in \mathbb{Z}$, $m \neq 0$ such that $\mathscr{L}^{\otimes m} \in Pic^0(X)$.*

Let $\mathscr{L} \in Pic(X)$ be a numerically trivial line bundle. Then there is $m \in \mathbb{Z}$, $m \neq 0$, such that $\mathscr{L}^{\otimes m} \in Pic^0(X)$. This implies that $c_1(\mathscr{L}^{\otimes m}) = 0$, so that $c_1(\mathscr{L}) \in H^2(X,\mathbb{Z})_{tors}$. By the Lefschetz Theorem on $(1,1)-$classes, we then get

$$N^1(X) = H^2(X,\mathbb{Z})_{free} \cap H^{1,1}(X),$$

so that $N^1(X) = NS(X)/tors$. As a corollary of this we have:

Theorem 2.5.4. *Let X be a smooth projective surface. Then $N^1(X)$ is a finitely generated free abelian group. Hence $NS(X)$ is a finitely generated abelian group.*

Definition 2.5.2. The rank $\rho(X)$ of $NS(X)$ is called *Picard number* of X.

As $NS(X)$ (or $N^1(X)$) is a submodule of $H^2(X, \mathbb{Z})$, and on this one we have the intersection pairing given by the cup product, we can consider the intersection pairing induced on $NS(X)$ (or $N^1(X)$). In this way, $N^1(X)$ becomes a lattice of rank $\rho(X)$. Our aim is to study this intersection pairing. Let me first introduce the following definitions:

Definition 2.5.3. A *Weil* $\mathbb{Q}-$*divisor* on X is an element of $Div_{\mathbb{Q}}(X) := Div(X) \otimes \mathbb{Q}$.

In particular, a Weil $\mathbb{Q}-$divisor D on X can be written as

$$D = \sum_{Y \in P(X)} n_Y Y,$$

where $n_Y \in \mathbb{Q}$ is 0 for all but a finite number of $Y \in P(X)$. A Weil $\mathbb{Q}-$divisor D on X has the following property: there is $n \in \mathbb{Z}$ such that nD is a Weil divisor. On Weil $\mathbb{Q}-$divisors one can define a linear equivalence relation by linearity: namely, two Weil $\mathbb{Q}-$divisors D, D' are linearly equivalent if and only if there are $n, n' \in \mathbb{Z}$ such that nD and $n'D'$ are linearly equivalent Weil divisors. One has then $A^1_{\mathbb{Q}}(X)$, which is naturally isomorphic to $A^1(X) \otimes \mathbb{Q}$. As X is smooth, we have $A^1(X) \simeq Pic(X)$, so that $A^1_{\mathbb{Q}}(X)$ is canonically isomorphic to $Pic(X) \otimes \mathbb{Q}$. Notice that while the objects in the first vector space have a clear geometric interpretation, the second vector space has no real meaning.

Recall, that on $Div(X)$ there is an intersection pairing. By linearity one can then define an intersection pairing

$$\cdot : Div_{\mathbb{Q}}(X) \times Div_{\mathbb{Q}}(X) \longrightarrow \mathbb{Q}.$$

Definition 2.5.4. Two Weil $\mathbb{Q}-$divisors D, D' are *numerically equivalent*, and we write $D \equiv D'$, if and only if for every irreducible curve C on X we have $D \cdot C = D' \cdot C$. We denote by $N^1_{\mathbb{Q}}(X) := Div_{\mathbb{Q}}(X)/ \equiv$.

Clearly, we have $N^1_{\mathbb{Q}}(X) = N^1(X) \otimes \mathbb{Q}$. Moreover, the intersection pairing on $Div_{\mathbb{Q}}(X)$ defines an intersection pairing on $N^1_{\mathbb{Q}}(X)$, which coincides with the linear extension of the one we have on $N^1(X)$. Recall that $N^1(X) \simeq NS(X)/tors$, so that $N^1(X) \otimes \mathbb{Q} \simeq NS(X) \otimes \mathbb{Q}$, so that $N^1_{\mathbb{Q}}(X) \simeq NS(X) \otimes \mathbb{Q}$. This is a $\mathbb{Q}-$vector space of dimension $\rho(X)$.

Using the same formal extension of coefficients, we can define $\mathbb{R}-$divisors and $N^1_{\mathbb{R}}(X)$, which is isomorphic to $NS(X) \otimes \mathbb{R}$, and which is a real vector space of dimension $\rho(X)$. On this vector space we have a non-degenerate symmetryc bilinear form, given by the intersection pairing

$$\cdot : N^1_{\mathbb{R}}(X) \times N^1_{\mathbb{R}}(X) \longrightarrow \mathbb{R}.$$

In a completely similar way, one can extend the cup product on $H^2(X, \mathbb{Z})$ to

$$\cup : H^2(X, \mathbb{R}) \times H^2(X, \mathbb{R}) \longrightarrow \mathbb{R},$$

and get an injection $c_1 : N^1_{\mathbb{R}}(X) \longrightarrow H^2(X,\mathbb{R})$, so that by the Lefschetz Theorem on $(1,1)-$classes we even get $N^1_{\mathbb{R}}(X) = H^2(X,\mathbb{R}) \cap H^{1,1}(X)$. Here is the main result of this section:

Theorem 2.5.5. ***(Hodge Index Theorem)****. The cup product restricts non-degenerately to $N^1(X)$ with signature $(1, \rho(X)-1)$.*

Proof. The proof goes as follows: let ω be the real $(1,1)-$form associated to the Fubini-Study metric (the metric induced on X by some embedding in $\mathbb{P}^N$). This is a Kähler form, hence it is closed, and defines a class $[\omega] \in H^{1,1}(X) \cap H^2(X,\mathbb{R})$, i. e. an element of $N^1_{\mathbb{R}}(X)$. Moreover, the form ω is pointwise positive definite. Now, let

$$H^2_{prim}(X,\mathbb{R}) := \{[\alpha] \mid [\alpha \wedge \omega] = 0\}.$$

As $[\alpha] \cup [\omega] = \int_X \alpha \wedge \omega$, we have that $H^2_{prim}(X,\mathbb{R}) = [\omega]^\perp$ (where the orthogonality is with respect to the cup product). In conclusion we have a direct sum decomposition $H^2(X,\mathbb{R}) = H^2_{prim}(X,\mathbb{R}) \oplus \mathbb{R}[\omega]$, orthogonal with respect to the cup product, so that

$$N^1_{\mathbb{R}}(X) = (H^2_{prim}(X,\mathbb{R}) \cap H^{1,1}(X)) \oplus \mathbb{R}[\omega].$$

As ω is pointwise positive definite, we have $\int_X \omega \wedge \omega > 0$, so that $[\omega] \cup [\omega] > 0$ and the intersection pairing is positive on $\mathbb{R}[\omega]$. Let us study the intersection pairing on $H^2_{prim}(X,\mathbb{R}) \cap H^{1,1}(X)$: let α be a real $(1,1)-$form with $\alpha \wedge \omega = 0$. Choose a $C^\infty-$trivialization of T^*_X by two $1-$forms β_1 and β_2, which is everywhere orthonormal with respect to the Kähler metric corresponding to ω. We then have (locally)

$$\omega = \frac{i}{2}(\beta_1 \wedge \overline{\beta}_1 + \beta_2 \wedge \overline{\beta}_2), \qquad \alpha = \sum_{j,k} \alpha_{jk} \beta_j \wedge \overline{\beta}_k.$$

Since α is real, we must have α_{11} and α_{22} purely imaginary and $\alpha_{12} = \overline{\alpha_{21}}$. The fact that $\alpha \wedge \omega = 0$ implies $\alpha_{11} = -\alpha_{22}$. In conclusion, we have

$$\frac{1}{2}\alpha \wedge \alpha = (|\alpha_{11}|^2 + |\alpha_{12}|^2)\beta_1 \wedge \overline{\beta}_1 \wedge \beta_2 \wedge \overline{\beta}_2 = -(|\alpha_{11}|^2 + |\alpha_{12}|^2)\omega \wedge \omega.$$

Hence $\int_X \alpha \wedge \alpha \leq 0$, and $\alpha \wedge \alpha = 0$ if and only if $\alpha = 0$. But this clearly means that the intersection form is negative definite on $H^2_{prim}(X,\mathbb{R}) \cap H^{1,1}(X)$. To conclude, simply remark that $\mathbb{R}[\omega] \cap N^1(X) \neq \emptyset$, as there is surely the class of an ample divisor. □

Remark 2.5.1. An equivalent reformulation of the Hodge Index Theorem is the following: let D be a Weil divisor such that $D^2 > 0$. If there is a Weil divisor C such that $D \cdot C = 0$, then $C^2 \leq 0$, and we have $C^2 = 0$ if and only if $C \equiv 0$. Indeed, we can produce a basis of $N^1_{\mathbb{R}}(X)$ such that D is an element of it. Then, as $D^2 > 0$, every element C in $N^1_{\mathbb{R}}(X)$ orthogonal to D is such that $C^2 \leq 0$ and it is 0 if and only if $C \equiv 0$. The converse direction is similar.

2.5.3 The Nakai-Moishezon Criterion for ampleness

First of all, let me introduce the following definition:

Definition 2.5.5. A Weil divisor D on X is *ample* if the corresponding line bundle $\mathscr{O}_X(D)$ is ample.

Here is a corollary of the Hodge Index Theorem:

Lemma 2.5.6. *Let D be a Weil divisor on a smooth projective surface X such that $D^2 > 0$. If H is an ample Weil divisor, then $D \cdot H \neq 0$. Moreover, if $D \cdot H > 0$ (resp. $D \cdot H < 0$) then there is $m \in \mathbb{N}$ such that mD (resp. $-mD$) is effective.*

Proof. Let $D^2 > 0$. If $D \cdot H = 0$, then $D \cdot (mH) = 0$ for every $m \in \mathbb{Z}$. We can then suppose H to be very ample. By the Hodge Index Theorem (see Remark 2.5.1) we would have $H^2 \leq 0$. As H is very ample, it is easy to see that $H^2 > 0$, hence $D \cdot H \neq 0$.

Now, let us suppose $D \cdot H > 0$. Consider $m \in \mathbb{N}$: then the linear system $|K_X - mD|$ is empty for $m \gg 0$. Indeed, if there is an effective divisor D' such that $D' \sim K_X - mD$, we have $D' \cdot H > 0$ as D' is effective and H is ample (see Lemma 2.5.7), but

$$D' \cdot H = K_X \cdot H - mD \cdot H < 0$$

for $m \gg 0$ as $D \cdot H > 0$. Hence $h^0(K_X - mD) = 0$ for $m \gg 0$. Using the inequality coming from the Riemann-Roch Theorem we get

$$h^0(mD) + h^0(K_X - mD) \geq \frac{1}{2}D^2m^2 - \frac{1}{2}(D \cdot K_X)m + \chi(\mathscr{O}_X).$$

If $m \geq 0$, as $D^2 > 0$ we have $h^0(mD) > 0$, so that mD is effective, hence D is effective. The other case follows. $\square$

We have the following lemma:

Lemma 2.5.7. *Let X be a smooth projective surface, and let $\mathscr{L}$ be an ample line bundle on X. If C is an irreducible curve on X, then $\mathscr{L} \cdot \mathscr{O}_X(C) > 0$. Equivalently, if a Weil divisor D on X is ample, then $D \cdot C > 0$ for every irreducible curve C on X.*

Proof. The proof is easy: let $i : C \longrightarrow X$ be the inclusion. As the generic curve in the linear system $|\mathscr{O}_X(C)|$ is smooth by Bertini's Theorem, we can suppose C smooth. Since i is a finite morphism, the line bundle $\mathscr{L}_{|C} := i^*\mathscr{L}$ is ample on C. There is then $m \in \mathbb{N}$ such that $\mathscr{L}_{|C}^{\otimes m}$ is very ample, hence there is an inclusion $f : C \longrightarrow \mathbb{P}^n$ such that $f^*\mathscr{O}_{\mathbb{P}^n}(1) = \mathscr{L}_{|C}^{\otimes m}$. Now, notice that $deg(f^*\mathscr{O}_{\mathbb{P}^n}(1)) = H \cdot f(C)$, where H is an hyperplane in $\mathbb{P}^n$. Since $f(C)$ is a curve in $\mathbb{P}^n$ we easily see that $H \cdot f(C) > 0$. In conclusion $deg(\mathscr{L}_{|C}^{m}) > 0$. Now,

$m > 0$, so that $\mathscr{L} \cdot \mathscr{O}_X(C) = deg(\mathscr{L}_{|C}) > 0$, where the first equality is Lemma 2.4.2. $\square$

In general, the converse statement is non true, and we are going to see an example in the following section. Anyway, we have the following important criterion:

Theorem 2.5.8. *(**Nakai-Moishezon's Criterion for Ampleness**). Let X be a smooth projective surface. A line bundle $\mathscr{L}$ on X is ample if and only if $\mathscr{L}^2 > 0$ and $\mathscr{L} \cdot \mathscr{O}_X(C) > 0$ for every irreducible curve C on X.*

Proof. Let D be a Weil divisor corresponding to $\mathscr{L}$. If $\mathscr{L}$ is ample, then $D \cdot C > 0$ for every irreducible curve on X. Moreover, we have $D^2 > 0$: indeed, as D is ample there is $m \in \mathbb{N}$ such that mD is very ample, so that $(mD)^2 > 0$. Hence $m^2 D^2 > 0$, so that $D^2 > 0$. For the converse, let us begin with the following lemma:

Lemma 2.5.9. *Let $D = D_1 + D_2$ be the sum of two effective divisors on X. Then there is an exact sequence*

$$0 \longrightarrow \mathscr{O}_{D_1}(-D_2) \longrightarrow \mathscr{O}_D \longrightarrow \mathscr{O}_{D_2} \longrightarrow 0.$$

Proof. It is sufficient to prove the lemma in the affine case, so let $D_1 = (f_1)$ and $D_2 = (f_2)$ for some $f_1, f_2 \in \mathbb{C}[x, y]$. Then $D = (f_1 f_2)$, so that the restriction map r_2 is the natural surjection

$$r_2 : \mathbb{C}[x, y]/(f_1 f_2) \longrightarrow \mathbb{C}[x, y]/(f_2).$$

Its kernel is $(f_2) \cdot \mathbb{C}[x, y]/(f_1 f_2)$, which is readily seen to be isomorphic to $(f_2) \cdot \mathbb{C}[x, y]/(f_1)$. $\square$

Let H be a very ample divisor. It is then effective, so that $D \cdot H > 0$. Moreover $D^2 > 0$, and we can apply Lemma 2.5.6 to guarantee that there is $m \in \mathbb{N}$ such that mD is effective. Notice that D is ample if and only if mD is ample, so we can suppose D to be effective.

We are noy goint to show that if D' is a divisor supported on D and $n \in \mathbb{N}$, then $H^1(D', \mathscr{O}_{D'}(nD)) = 0$ for $n \gg 0$. Let us suppose D' to be irreducible. Then consider its normalisation $\nu : D'' \longrightarrow D'$. As ν is a finite map, we have $H^1(D'', \nu^* \mathscr{O}_{D'}(nD)) \simeq H^1(D', \mathscr{O}_{D'}(nD))$. Moreover $deg(\nu^* \mathscr{O}_{D'}(nD)) = deg(\mathscr{O}_{D'}(nD))$, and this last is $nD \cdot D'$. As $D \cdot D' > 0$ (since D' is supported on D), for $n \gg 0$ we then have $H^1(D'', \nu^* \mathscr{O}_{D'}(nD)) = 0$, and we are done.

If D' is reducible, we can write it as $D' = D'' + R$, where D'' is irreducible and R is effective. The exact sequence in Lemma 2.5.9 twisted by $\mathscr{O}_{D'}(nD)$ then gives an exact sequence

$$H^1(D'', \mathscr{O}_{D''}(nD - R)) \longrightarrow H^1(D', \mathscr{O}_{D'}(nD)) \longrightarrow H^1(R, \mathscr{O}_R(nD)).$$

Now, the first group is trivial for $n \gg 0$ (follow the same argument as before, as D'' is irreducible). Now we can proceed by induction on the number of components of R to get that the last group is trivial for $n \gg 0$, and we are done.

Let us now consider the following exact sequence

$$0 \longrightarrow \mathscr{O}_X((n-1)D) \longrightarrow \mathscr{O}_X(nD) \longrightarrow \mathscr{O}_D(nD) \longrightarrow 0. \tag{2.5}$$

By the previous part, we have that $H^1(D, \mathscr{O}_D(nD)) = 0$ for $n \gg 0$. The long exact sequence induced by (2.5) in cohomology then gives a surjective map

$$H^1(X, \mathscr{O}_X((n-1)D)) \longrightarrow H^1(X, \mathscr{O}_X(nD))$$

for $n \gg 0$. As these two spaces are finite dimensional, this surjective map is an isomorphism for $n \gg 0$, so that the exact sequence induced by (2.5) in cohomology implies that the map

$$H^0(X, \mathscr{O}_X(nD)) \longrightarrow H^0(D, \mathscr{O}_D(nD))$$

is surjective for $n \gg 0$. Using a similar argument as before, one can show that $\mathscr{O}_D(nD)$ is globally generated, so that $\mathscr{O}_X(nD)$ is generated by global sections on points of D. But recall that D is effective, hence there must be a section of $\mathscr{O}_X(D)$ whose zero locus is exactly D: hence $\mathscr{O}_X(nD)$ is globally generated. By properties of globally generated line bundles, we have then a map

$$\varphi_{nD} : X \longrightarrow \mathbb{P}^N.$$

Let us suppose there is a curve C of X such that $\varphi_{nD}(C) = \{P\}$, where $P \in \mathbb{P}^N$. Let L be a hyperplane of $\mathbb{P}^N$ such that $P \notin L$. Hence

$$0 = \mathscr{O}_X(C) \cdot \varphi_{nD}^* \mathscr{O}_{\mathbb{P}^N}(L) = \mathscr{O}_X(C) \cdot \mathscr{O}_X(nD).$$

But this implies that there is a curve C of X such that $D \cdot C = 0$, contradicting the hypothesis. Hence φ_{nD} is a finite map. From Corollary 2.3.4 we then get that nD is ample, so that D is ample. □

Another characterization of ampleness of a line bundle can be given in terms of its first Chern class: if X is a complex manifold, we can define a line bundle $\mathscr{L}$ to be *positive* if and only if $c_1(\mathscr{L})$ is a positive $(1,1)-$form. Hence if X is projective, then it has to carry a positive line bundle by the Nakai-Moishezon Criterion. What is more surprising, is that even the converse is true, and we have the following result, whose proof is omitted:

Theorem 2.5.10. ***(Kodaira's Projectivity Criterion)****. Let X be a compact complex manifold. Then X is projective if and only if there is a positive $(1,1)-$form $\omega \in H^{1,1}(X) \cap H^2(X, \mathbb{Q})$.*

2.5.4 Cones

In the previous section we have shown that if D is an ample divisor on X, then $D \cdot C > 0$ for every irreducible curve on X. We now, define the following definition:

Definition 2.5.6. A line bundle $\mathscr{L}$ on X is *nef* if $\mathscr{L} \cdot \mathscr{O}_X(C) \geq 0$ for every irreducible curve C on X. A Weil divisor D on X is *nef* if the corresponding line bundle $\mathscr{O}_X(D)$ is nef.

It follows from Lemma 2.5.7 that every ample line bundle is nef. Let us moreover introduce the following

Definition 2.5.7. A Weil $\mathbb{Q}-$divisor (resp. $\mathbb{R}-$divisor) D on X is *effective* if $D \sim \sum_{i=1}^k d_i A_i$ for some integer k, where A_i is a Weil divisor and $d_i \in \mathbb{Q}_{>0}$ (resp. $d_i \in \mathbb{R}_{>0}$) for every $i = 1, ..., k$.

A Weil $\mathbb{Q}-$divisor (resp. $\mathbb{R}-$divisor) D on X is *ample* if there are $D \sim \sum_{i=1}^k d_i A_i$ for some integer k, where A_i is an ample Weil divisor and $d_i \in \mathbb{Q}_{>0}$ (resp. $d_i \in \mathbb{R}_{>0}$) for every $i = 1, ..., k$.

A Weil $\mathbb{Q}-$divisor (resp. $\mathbb{R}-$divisor) D on X is *nef* if there are $D \sim \sum_{i=1}^k d_i A_i$ for some integer k, where A_i is a nef Weil divisor and $d_i \in \mathbb{Q}_{>0}$ (resp. $d_i \in \mathbb{R}_{>0}$) for every $i = 1, ..., k$.

Remark 2.5.2. Let D be a Weil $\mathbb{Q}-$divisor, and let us suppose $n \in \mathbb{N}$ such that $nD \in Div(X)$. It is easy to see that D is effective if and only if nD is effective.

Moreover D is ample if and only if nD is ample. Indeed, if nD is ample, simply take $k = 1$, $d_1 = 1/n$ and $A_1 = nD$. If D is ample, then $D = \sum_{i=1}^k d_i A_i$ for $d_i \in \mathbb{Q}_{>0}$ and A_i is ample. Then $nD = \sum_{i=1}^k nd_i A_i$ is a Weil divisor. As A_i is ample and $d_i > 0$ for every $i = 1, ..., k$, we then have $nd_i A_i$ ample for every i, so that D is ample.

Similiarily, D is nef if and only if nD is nef. Indeed, if nD is nef, then do as before. If $D = \sum_{i=1}^k d_i A_i$ for $d_i \in \mathbb{Q}_{>0}$ and A_i is nef. Then $nD = \sum_{i=1}^k nd_i A_i$ is a Weil divisor. Let C be an irreducible curve: then $A_i \cdot C \geq 0$ for every i. As $d_i > 0$, we have $nD \cdot C > 0$, so that nD is nef.

Remark 2.5.3. Let D be a Weil $\mathbb{R}-$divisor. If D is ample, then $D \cdot C > 0$ for every irreducible curve C. Similarily, if D is nef, then $D \cdot C \geq 0$ for every irreducible curve C on X.

Let us now define the following set:

$$QEff(X) := \{\alpha = \sum_i a_i[C_i] \in N^1_{\mathbb{R}}(X) \,|\, C_i \textit{ is an irr. curve, } a_i \geq 0\}.$$

Clearly $QEff(X)$ is a cone in $N^1_{\mathbb{R}}(X)$. Now, let $Eff(X) := \overline{QEff(X)}$, the closure of $QEff(X)$ in the euclidean topology of $N^1_{\mathbb{R}}(X)$. Then $Eff(X)$ is a cone, called the *effective cone* of X. In general we have $QEff)(X) \subseteq Eff(X)$, but not the equality.

Notice that for classes in $N^1_{\mathbb{R}}(X)$ we can define ampleness and nefness: a class $\alpha \in N^1_{\mathbb{R}}(X)$ is *ample* (resp. *nef*) if it is the numerical class of an ample (resp. of a nef) Weil $\mathbb{R}-$divisor. Before going on, we cite the following result:

Theorem 2.5.11. ***(Kleiman's Criterion for nefness).*** *Let X be a smooth projective surface, and let D be a Weil $\mathbb{R}-$divisor. Then D is nef if and only if $D^2 \geq 0$ and $D \cdot C \geq 0$ for every irreducible curve on X.*

We have then

$$Amp(X) := \{\alpha \in N^1_{\mathbb{R}}(X) \,|\, \alpha \textit{ is ample}\},$$

which is called the *ample cone* of X, and

$$Nef(X) := \{\alpha \in N^1_{\mathbb{R}}(X) \,|\, \alpha \textit{ is nef}\},$$

called the *nef cone* of X. Notice that $Amp(X) \subseteq Nef(X)$, and that these are cones.

Proposition 2.5.12. *Let X be a smooth projective surface. Then $Nef(X)$ is the dual cone of $Eff(X)$ (and conversely) with respect to the intersection pairing on $N^1_{\mathbb{R}}(X)$.*

Proof. If V is a real vector space with an intersection form $(.,.)$ and C is a cone in V, then the dual cone of C with respect to $(.,.)$ is

$$C^{\vee} := \{\alpha \in V \,|\, (\alpha, \beta) \geq 0 \;\forall\, \beta \in C\}.$$

Let now $\alpha \in Nef(X)$. Then $\alpha \cdot [C] \geq 0$ for every irreducible curve C of X. This then implies that $\alpha \cdot \gamma \geq 0$ for every $\gamma \in QEff(X)$, hence of $Eff(X)$. In conclusion, $\alpha \in Eff(X)^{\vee}$. Conversely, if $\alpha \in Eff(X)^{\vee}$, then $\alpha \cdot \gamma \geq 0$ for every $\gamma \in Eff(X)$. In particular this is true for every irreducible curve of X, so that by Theorem 2.5.11 we get that α is the class of a nef $\mathbb{R}-$divisor. In conclusion $Nef(X) = Eff(X)^{\vee}$. □

Next, we study the relation between the nef and the ample cone:

Proposition 2.5.13. *Let X be a smooth projective surface. Then $Amp(X) = Nef(X)^{\circ}$ and $Nef(X) = \overline{Amp(X)}$. In particular, the nef cone is closed and the ample cone is open.*

Proof. As ample line bundles are nef, we easily see that $Amp(X) \subseteq Nef(X)$. The fact that $Nef(X)$ is closed is evident, and we need to show that $Amp(X)$ is open. In order to do that, let us prove the following:

Lemma 2.5.14. *Let H be an ample $\mathbb{Q}-$divisor on X, and let E be a $\mathbb{Q}-$divisor. Then $H+\epsilon E$ is ample for every $\epsilon \in \mathbb{Q}$ such that $0<|\epsilon|\ll 1$.*

Proof. We need to show that there is $n \gg 0$ such that $n(H+\epsilon E)$ is an ample divisor. First of all we can clear the denominators, and we can suppose that H and E are Weil divisors, with H ample. Then there is $m \gg 0$ such that $mH+E$ is ample. Indeed, as H is ample, we can choose $m_1, m_2 \in \mathbb{N}$ such that nH is very ample for $n \geq m_1$ and $nH+E$ is globally generated for $n \geq m_2$. Hence, if $n \geq m_1+m_2$, then $nH+E$ is very ample. Now, let us take $H+\frac{1}{n}E$: this is an ample $\mathbb{Q}-$divisor for $n \gg 0$, and we are done. □

Now, let us consider H an ample $\mathbb{Q}-$divisor, and let $E_1,...,E_r$ be arbitrary $\mathbb{Q}-$divisor. It follows from the previous Lemma that $H+\epsilon_1 E_1+...+\epsilon_r E_r$ is ample for $\epsilon_i \in \mathbb{Q}$ and $0<|\epsilon_i|\ll 1$ for every $i=1,...,r$. Notice that we can even suppose $\epsilon_i \in \mathbb{R}$. We can now consider H an ample $\mathbb{R}-$divisor and $E_1,...,E_r$ arbitrary $\mathbb{R}-$divisors. As every E_i is a finite linear combination (with real coefficients) of Weil divisors, we can suppose E_i to be a Weil divisor for every $i=1,...,r$. Let us write $H=\sum_i d_i A_i$, where $d_i \in \mathbb{R}_{>0}$ and A_i is an ample Weil divisor. Let $q \in \mathbb{Q}$ be such that $0<q<d_1$, so that we can write

$$H+\sum_{j=1}^{r}\epsilon_j A_j=(qA_1+\sum_{j=1}^{r}\epsilon_j E_j)+(d_1-q)A_1+\sum_{i>1}d_i A_i.$$

The second and the third summands are ample $\mathbb{R}-$divisors, and the first one is ample provided that $0<|\epsilon_j|\ll 1$ for every i by the previous part, so that $H+\epsilon_1 E_1+...+\epsilon_r E_r$ is an ample $\mathbb{R}-$divisor.

From this it follows easily that $Amp(X)$ is an ample cone. Hence $Amp(X)\subseteq Nef(X)^\circ$ and $Nef(X)\subseteq \overline{Amp(X)}$. We need to show the opposite inclusions.

Lemma 2.5.15. *Let X be a smooth projective surface, and let D be a nef $\mathbb{R}-$divisor on X. If H is an ample $\mathbb{R}-$divisor on X, then $D+\epsilon H$ is ample for every $\epsilon>0$. Conversely, if D and H are two $\mathbb{R}-$divisors such that $D+\epsilon H$ is ample for every $\epsilon>0$, then D is nef*

Proof. Let D and H be $\mathbb{R}-$divisors such that $D+\epsilon H$ is ample for every $\epsilon>0$. Then, if C is any effective curve on X, we have

$$0<(D+\epsilon H)\cdot C=D\cdot C+\epsilon H\cdot C.$$

Then, the limit for ϵ going to 0 is non-negative, and D is nef.

Conversely, let D be nef and H ample. We can replace ϵH by H, so that we just need to prove that $D+H$ is ample. If D and H are $\mathbb{Q}-$divisors, then we can clear denominators and suppose that D is a nef divisor, and H is ample. If C is an irreducible curve, we then get

$$(D+H)\cdot C=D\cdot C+H\cdot C>0,$$

as $D \cdot C \geq 0$ (since D is nef) and $H \cdot C >$ (since H is ample). Moreover, we have

$$(D+H)^2 = D^2 + H^2 + 2D \cdot H > 0,$$

since $D^2 \geq 0$ by the Kleiman Criterion for nefness, $H^2 > 0$ by the Nakai-Moishezon Criterion for ampleness and $D \cdot H \geq 0$ by the Kleiman Criterion for nefness (indeed, we have $D \cdot H \geq 0$ if and only if $D \cdot (nH) \geq 0$ for some $n \in \mathbb{N}$; as H is ample, then there is $n \in \mathbb{N}$ such that nH is very ample, hence nH is effective, and $D \cdot (nH) \geq 0$ as D is nef).

Now, let us suppose D and H to be $\mathbb{R}-$divisors. Then, let $H_1, ..., H_{\rho(X)}$ be ample divisors giving a basis for $N^1_{\mathbb{R}}(X)$ (this is possible as $Amp(X)$ is open). By Lemma 2.5.14 the $\mathbb{R}-$divisor $H(\epsilon_1, ..., \epsilon_{\rho(X)}) := H - \sum_{i=1}^{\rho(X)} \epsilon_i H_i$ is ample for $0 < |\epsilon_i| \ll 1$. We can find $\epsilon_1, ..., \epsilon_{\rho(X)} \in \mathbb{R}$ such that the divisor $D' := D + H(\epsilon_1, ..., \epsilon_{\rho(X)})$ is a $\mathbb{Q}-$divisor, and by the previous part then D' is ample. Finally, simply note that

$$D + H = D' + \epsilon_1 H_1 + ... + \epsilon_{\rho(X)} H_{\rho(X)},$$

which is ample as D' is ample and every H_i is ample. □

From this lemma it follows immediately the Proposition: any nef divisor is the limit of ample divisors, and conversely. □

Now, notice that if D is a nef divisor, then $Nef(X) \subseteq Eff(X)$: indeed, if D is a nef class in $N^1_{\mathbb{R}}(X)$, and H is an ample $\mathbb{R}-$divisor, then $D + \epsilon H$ is ample for every $\epsilon > 0$. Let us suppose D, H be $\mathbb{Q}-$divisors, so that clearing the denominators we can suppose D and H to be Weil divisors. As $D + \epsilon H$ is ample, then there is $n \gg 0$ such that $n(D+\epsilon H)$ is very ample, hence its class in $N^1_{\mathbb{R}}(X)$ is effective. Hence $D + \epsilon H$ is effective, and taking the limit for ϵ going ot 0, we then get that D is a class in $Eff(X)$.

Moreover, notice that $Eff(X) \subseteq Nef(X)$ if and only if for every irreducible curve C we have $C^2 \geq 0$: indeed, every irreducible curve intersects a distinct curve non-negatively, hence C is nef if and only if $C^2 \geq 0$. By Kleiman's Criterion for nefness we then conclude. If there is an irreducible curve C such that $C^2 \leq 0$, then $Eff(X)$ is spanned by $\mathbb{R}_{\geq 0} \cdot C$ and by the following cone

$$Eff(X)_{C \geq 0} := \{\gamma \in Eff(X) \,|\, \gamma \cdot C \geq 0\}.$$

Indeed, if C' is an effective curve on X such that C is not contained in C' as a component, the $C \cdot C' \geq 0$. Now, $C \in Eff(X)$, and $C^2 < 0$, so that $C \notin Eff(X)_{C \geq 0}$, and the assertion is proved.

Moreover, if $C^2 < 0$, then the ray spanned by C is not contained in the nef cone. It then spans an extremal ray in $Eff(X)$.

Example 2.5.1. Let us consider $X = \mathbb{P}^2$. Then the line bundle $\mathscr{O}(d)$ is ample if and only if $d > 0$, and it is nef if and only if $d \geq 0$. In this case we have $N^1_{\mathbb{R}}(X) = \mathbb{R}$. Then $Eff(X) = Nef(X) = \mathbb{R}_{\geq 0}$, and $Amp(X) = \mathbb{R}_{>0}$. Notice that $\mathscr{O}_X(K_X) = \mathscr{O}(-3)$ is not nef.

Example 2.5.2. Let us consider $X = \mathbb{P}^1 \times \mathbb{P}^1$. Then $N^1_{\mathbb{R}}(X) \simeq \mathbb{R}^2$, and we consider as a basis h_1, h_2. A class $\alpha = ah_1 + bh_2$ is effective if and only if $a \geq 0$ and $b \geq$, so that $Eff(X) = \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$. Let now $\mathscr{O}_X(n, m)$ be the line bundle on X of bidegree (n, m). Then $\mathscr{O}_X(n, m) \cdot \alpha = am + bn$, and if $\mathscr{O}_X(n, m)$ then $am + bn \geq 0$ for every $a, b \geq 0$. If $b = 0$, then we get $m \geq 0$, and if $a = 0$ then we get $n \geq 0$. Moreover, $\mathscr{O}_X(n, m)^2 = 2nm$, which is non-negative if $n, m \geq 0$, so that $Nef(X) = Eff(X)$. The ample cone is clearly the interior of $Nef(X)$. Recall that $\mathscr{O}_X(K_X) = \mathscr{O}_X(-2, -2)$, hence K_X is not nef.

Example 2.5.3. We describe now a new example of smooth projective surface. Let C be a smooth curve of genus g, and let E be a rank 2 vector bundle on C. Let $X := \mathbb{P}(E)$ be the projective bundle associated to E, and let $\pi : X \longrightarrow C$ be the projection morphism. Without loss of generality, we can suppose $deg(E) = 0$. Let $\xi := c_1(\mathscr{O}_X(1))$ be the first Chern class of the tautological line bundle on X (i. e. $\mathscr{O}_X(1)$ is the line bundle such that $\pi_*\mathscr{O}_X(1) \simeq E$), and let f be the class of a fiber. Then $N^1_{\mathbb{R}}(X) \simeq \mathbb{R}^2$, where a basis is given by ξ and f (notice that if $P, Q \in C$ are different points, then it is not true in general that $P \sim Q$, so that $f^*P \sim f^*Q$ is not necessarily true. Anyway, $P - Q \in Pic^0$, so that by passing to $N^1(X)$ thew give the same class). We have the following intersection properties: $f^2 = 0$ by Proposition 2.4.4; $\xi \cdot f = 1$, as ξ represents (locally) a section of π; $\xi^2 = deg(E) = 0$. Hence, the intersection form is the same as in Example 2.5.2.

Let us consider $af + b\xi \in N^1_{\mathbb{R}}(X)$. We have $(af + b\xi)^2 = 2ab$, which is non-negative if and only if $a, b \geq 0$ or $a, b \leq 0$. As f is evidently nef, we then get that $Nef(X) \subseteq \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$, and $\xi = 0$ gives a boundary of it. Similarly, if a curve $af + b\xi$ is effective, then $a \geq 0$; moreover, f is effective, so that $Eff(X)$ has $\xi = 0$ as a boundary. The remaining part of these cones depends on the geometry of E.

Case 1. Let us suppose that there is a line bundle $A \subseteq E$ such that $deg(A) = d < 0$. Hence we have $C := \mathbb{P}(A) \subseteq X$ is an effective curve on X, and its class in $N^1(X)$ is $df + \xi$. Indeed, if $C = af + b\xi$, we have $C \cdot f = 1$, so that $b = 1$, and $C \cdot \xi = deg(A)$, so that $a = d$. In particular, notice that $C^2 = 2d < 0$, hence we have an effective divisor which is not nef. Moreover, $Eff(X)$ is the cone spanned by f and $af + \xi$, and $Nef(X)$ is strictly contained in it. In order to determine $Nef(X)$, let $nf + m\xi$ be a nef line bundle. Hence $n, m \geq 0$, and

$$n + md = (nf + m\xi) \cdot (df + \xi) \geq 0.$$

Then $Nef(X)$ is the cone spanned by f and $\xi - df$. Notice in this case that ξ

is an effective divisor such that $\xi \cdot (af + b\xi) > 0$ for every $a, b > 0$, but ξ is not ample: indeed $\xi^2 = 0$.

Case 2. Let us suppose that if $A \subseteq E$ is a line bundle of degree d, then $d \geq 0$. Any effective curve on X arises then by a line bundle of the form $\mathscr{O}_X(m) \otimes \pi^* A$ for some $m \in \mathbb{N}_0$ and some line bundle $A \subseteq E$. The class of this curve is then $df + m\xi$, and here $d \geq 0$, so that $Eff(X) \subseteq \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$. As ξ and f are effective curves, then it is an equality. Now, it is easy to show that ξ is nef, so that $Nef(X) = Eff(X)$.

Example 2.5.4. Let C be a smooth curve of genus g, and let $X := C \times C$. Then X is a smooth surface having two natural projections $p_1, p_2 : X \longrightarrow C$. Let us consider the following curves on X: f_1 is a fiber of p_1, f_2 is a fiber of p_2 and δ is the class of the diagonal. If $g \geq 1$, these classes are independent. Indeed, it is clear that f_1 and f_2 are independent, and that $f_i^2 = 0$ and $f_1 \cdot f_2 = 1$. Now, we have $\delta^2 = 2 - 2g$, and clearly $\delta \cdot f_i = 1$. If there are $n_1, n_2 \in \mathbb{Z}$ such that $\delta = n_1 f_1 + n_2 f_2$, we would then get $n_1 = n_2 = 1$, and $\delta^2 = 2$. Hence $g = 0$, but we supposed $g > 0$, so that δ is independent from f_1 and f_2. We have then $N^1_{\mathbb{R}}(X) \simeq \mathbb{R}^3$ and a basis is given by f_1, f_2, δ. In this case it is more complicated to describe the nef and the effective cone.

We conclude this section with two basic properties of nef line bundles:

Proposition 2.5.16. *Let X be a smooth projective surface, and let $\mathscr{L} \in Pic(X)$. Moreover, let Y be a smooth projective variety and $f : X \longrightarrow Y$ be a morphism.*

1. *If $\mathscr{L}$ is nef, then $f^*\mathscr{L}$ is nef.*
2. *If f is surjective and $f^*\mathscr{L}$ is nef, then $\mathscr{L}$ is nef.*

Proof. For the first item, let C be an irreducible curve on X. If $f(C)$ is a point, then $f^*\mathscr{L} \cdot \mathscr{O}_X(C) = 0$. If $f(C)$ is a curve and $f : C \longrightarrow f(C)$ is a map of degree d, then $f^*\mathscr{L} \cdot \mathscr{O}_X(C) = d\mathscr{L} \cdot \mathscr{O}_Y(f(C)) \geq 0$ as $\mathscr{L}$ is nef and $f(C)$ is an irreducible curve. The second item is evident. ☐

Chapter 3

Birational geometry

In this chapter we start with the investigation on the birational geometry of surfaces. We introduce some important notions we will need in the following, namely the blow-up of a surface in a point, and the birational map between two surfaces. The main role will be played by $(-1)-$curves, i. e. curves E such that $E^2 = -1$. We will show that a surface X admits a $(-1)-$curve if and only if it is a blow-up of another surface at a smooth point, and E is exactly the curve coming from the blow-up. This will allow us to introduce the notion of minimal model of a smooth projective surface, so that the birational classification of smooth projective surfaces is simply the classification of thei minimal models.

In particular we recall some properties of normalizations, and we prove the Zariski Main Theorem about fibers of a birational morphism.

Moreover, we introduce the notion of Kodaira dimension of a line bundle and of a surface, and we prove that this is a birational invariant of a smooth projective surface: the birational classification then will proceed first by dividing surfaces by means of thei Kodaira dimension.

3.1 Contraction of curves

In this section we recall some basic facts about rational and birational maps, blow-ups and related subjects. This will be one of the basic steps in order to understand how the Enriques-Castelnuovo classification works.

3.1.1 Rational and birational maps

Let us start from the following easy lemma:

Lemma 3.1.1. *Let X and Y be two projective varieties, and let $f, g : X \longrightarrow Y$ be two morphisms. If there is a non-empty open subset $U \subseteq X$ such that $f_{|U} = g_{|U}$, then $f = g$.*

Proof. Let $i : Y \longrightarrow \mathbb{P}^n$ be a closed immersion. Clearly, if the Lemma is true for $i \circ f$ and $i \circ g$, then it is true for f and g, so we can suppose $Y = \mathbb{P}^n$. The morphisms f and g give then a map $(f, g) : X \longrightarrow \mathbb{P}^n \times \mathbb{P}^n$, and let Δ be the diagonal of $\mathbb{P}^n \times \mathbb{P}^n$. By hypothesis we have $(f, g)(U) \subseteq \Delta$. As U is dense in X and Δ is closed, we get $(f, g)(X) \subseteq \Delta$, so that $f = g$. $\square$

The idea of a rational map is that we want to allow morphisms which are not defined on the whole X, but only on some dense open subset of X. So, the naif definition should be that a rational map $f : X \dashrightarrow Y$ is a couple (U, f), where U is a dense open subset of X, and $f : U \longrightarrow Y$ is a morphism. Anyway, the correct definition is more complicated.

Definition 3.1.1. Let X and Y be two projective varieties. A *rational map* $f : X \dashrightarrow Y$ is an equivalence class of pairs (U, f), where U is a dense open subset of X and $f : U \longrightarrow Y$ is a morphism, where (U, f) and (U, g) are equivalent if and only if $U \cap V \neq \emptyset$ and $f_{|U\cap V} = g_{|U\cap V}$. A rational map is *dominant* if $f(U)$ is dense in Y for every pair (U, f) representing f.

Notice that the Lemma implies that the relation described is an equivalence relation. We can compose rational (resp. dominant) maps getting another rational (resp. dominant) map.

Definition 3.1.2. A *birational map* $f : X \dashrightarrow Y$ is a rational map adimitting an inverse $f^{-1} : Y \dashrightarrow X$, which is a rational map such that $f \circ f^{-1}$ and $f^{-1} \circ f$ are the identity (as rational maps). If there is a birational map from X to Y, we say that X is *birational* to Y.

Example 3.1.1. Let us describe an important example of birational map. First of all, let us consider $\mathbb{P}^1 \times \mathbb{P}^1$, and then the Segre embedding

$$s : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3, \qquad s((x_0 : x_1), (y_0 : y_1)) := (x_0 y_0 : x_0 y_1 : x_1 y_0 : x_1 y_1),$$

which is a well-defined injective map. The image $X := im(i)$ is then isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Notice that if on $\mathbb{P}^3$ we use coordinates $(u : v : w : z)$, then $X = (uz - vw)$, which is a smooth quadric in $\mathbb{P}^3$. Now, let us consider the following map:

$$f : X \setminus \{(0 : 0 : 0 : 1)\} \longrightarrow \mathbb{P}^2, \qquad f(u : v : w : z) := (u : v : w),$$

which is the projection of X from the point $(0 : 0 : 0 : 1)$. This gives a rational map $f : X \dashrightarrow \mathbb{P}^2$, and we can easily write its inverse map

$$g : \mathbb{P}^2 \longrightarrow \mathbb{P}^3, \qquad g((x_0 : x_1 : x_2)) := (x_0^2 : x_0 x_1 : x_0 x_2 : x_1 x_2).$$

It is clear that $im(g) \subseteq X$, and these two maps are inverse to each other. Then $\mathbb{P}^2$ is birational to $\mathbb{P}^1 \times \mathbb{P}^1$. Notice that these two varieties are anyway not isomorphic (their Picard group are different).

Now, let $\varphi : X \dashrightarrow Y$ be a dominant rational map, and let $f\mathscr{M}_Y(Y)$ be a rational function on Y. Then φ can be represented by a pair (U, φ_U), and f by a pair (V, f_V), where V is an open subset of Y where f is regular. As φ is dominant, the open subset $\varphi_U(U)$ is dense in Y, and $V \cap \varphi_U(U) \neq \emptyset$. Then $\varphi_U^{-1}(V \cap \varphi_U(U))$ is a dense open subset of X, so that $f \circ \varphi_U$ is a regular function on it. Hence, $\varphi^* f := f \circ \varphi$ is a rational function on X. In conclusion, we get

$$\varphi : \mathscr{M}_Y(Y) \longrightarrow \mathscr{M}_X(X).$$

One can define an inverse correspondence.

Proposition 3.1.2. *Let X and Y be two projective varieties. The following are equivalent:*

1. *X and Y are birational;*
2. *there is a dense open subset U of X and a dense open subset V of Y which are isomorphic;*
3. *$\mathscr{M}_X(X) \simeq \mathscr{M}_Y(Y)$ as $\mathbb{C}-$algebras.*

3.1.2 The Zariski Main Theorem

In this subsection I birefly recall properties of birational maps and of fibers of morphism. Let me recall the following definition:

Definition 3.1.3. A projective variety X is *normal* if for every $x \in X$ the local ring $\mathscr{O}_{X,x}$ is a normal ring, i. e. it is integrally closed in its quotient field.

If $x \in X$ is a smooth point, then $\mathscr{O}_{X,x}$ is normal, so that any smooth projective variety is normal. If X is normal and if $x \in X$ is such that the local ring $\mathscr{O}_{X,x}$ has dimension 1, then $\mathscr{O}_{X,x}$ is regular. Hence if X is normal, the singular locus of X has codimension at least 2. In particular, every normal curve is smooth.

Definition 3.1.4. Let X be a projective variety. A *normalisation* of X is a pair (X', f) such that X' is a normal projective variety, and $f : X' \longrightarrow X$ is a morphism which is finite and birational.

This definition coincides with the usual definition of normalisation. Indeed, we have the following:

Proposition 3.1.3. *Let X be a projective variety, and let (X', f) be a normalisation of X. If (X'', g) is another normalization of X, then there is an isomorphism $h : X' \longrightarrow X''$ such that $g \circ h = f$.*

Proof. We start with X affine, and let $R(X) := \mathscr{O}_X$. It is a Noetherian ring whose integral closure in $k(X)$ is denoted R'. Then R' is a finitely generated $R(X)-$module, so that there is an ideal I of $\mathbb{C}[x_1, ...x_n]$ such that $R' \simeq \mathbb{C}[x_1, ..., x_n]/I$. The closed subvariety of $\mathbb{C}^n$ defined by I is denoted X': it is a normal variety admitting a finite morphism $f : X' \longrightarrow X$ defined by the inclusion $R(X) \subseteq R'$. Moreover, the quotient field of $R(X)$ and of R' are equal, hence by Proposition 3.1.2 f is birational. Now, X'' is normal if and only if $R(X'')$ is normal, and as X'' is a normalisation of X, then $R(X) \subseteq R(X'')$. Hence X'' and X' correspond to the integral closure of $R(X)$ in its quotient field, and we are done.

If X is projective, one produces a normalisation as in the previous case, and applies the statement to local pieces. $\square$

We can now proceed with the proof of the Zariski Main Theorem:

Theorem 3.1.4. ***(Zariski's Main Theorem)***. *Let Y be a normal projective variety, and let $f : X \longrightarrow Y$ be a birational morphism. Then the fibers of f are connected.*

Proof. The first step is to prove the following lemma:

Lemma 3.1.5. *Let X, Y be two projective varieties, and let $f : X \longrightarrow Y$ be a morphism.*

1. *If the natural map $\mathscr{O}_Y \longrightarrow f_*\mathscr{O}_X$ is an isomorphism, then the fibers of f are connected and non empty.*

2. *If Y is normal, f is surjective and its fibers are connected, then the natural map $\mathscr{O}_Y \longrightarrow f_*\mathscr{O}_X$ is an isomorphism.*

Proof. We can work in the euclidean topology (i. e. we consider X and Y to be complex varieties). Let $y \in Y$ be such that $f^{-1}(y) =: X_y$ is not connected. As f is a morphism between two projective varieties, then f is proper. Hence, there is an open subset V of Y (in the euclidean topology) such that $y \in V$ and $f^{-1}(V)$ is not connected. Indeed, if $F = A \cup B$ is a disconnected fiber whose connected components are A and B, we have that A and B are compact. Then, let U be a neighborhood of A and W a neighborhood of B in X which are disjoint. Again by compactness (following from the fact that f is proper) all points of X which are sufficiently close to F are in $U \cup W$. Now, simply take $V := f(W)$.

Now, let $y := f(F)$. The inverse image of a sufficiently small ball centered at y cannot be connected, and the canonical map $\mathscr{O}_{Y,y} \longrightarrow (f_*\mathscr{O}_X)_y$ cannot be surjective. Hence if the hypethesis of item 1 is verified, we need that every fiber has to be connected. For non-emptyness, just remark that if $X_y = \emptyset$, then the map $\mathscr{O}_{Y,y} \longrightarrow (f_*\mathscr{O}_X)_y$ is not injective, and we are done.

For the converse, let $y \in Y$ and let V be an open neighborhood of y in the complex topology. Let $g \in \mathscr{O}_X(f^{-1}(V))$ be a bounded holomorphic function. On every smooth fiber F of f the function g has the same value by definition, as F is connected. Hence, let V' be the open subset of V given by smooth points of V where f has maximal rank. Then there is a bounded continuous function h on V' such that $g_{|f^{-1}(V')} = h' \circ f$. As f has maximal rank at every point of $f^{-1}(V')$, for every such a point there is a neighborhood of the form $U \times V''$ for some V'' open subset of V', and f is simply the projection to V''. Hence on V'' the map h has to be holomorphic (as g is), hence $h \in \mathscr{O}_Y(V')$. As V is normal, we can extended h to a holomorphic function on V, and in this way we see that $\mathscr{O}_{Y,y} \longrightarrow (f_*\mathscr{O}_X)_y$ is an isomorphism. □

Now, let us suppose that $f : X \longrightarrow Y$ is a birational morphism, and that Y is normal. By Lemma 3.1.5, in order to prove that the fibers of f are connected, we just need to prove that $\mathscr{O}_Y \longrightarrow f_*\mathscr{O}_X$ is an isomophism. As the statement is local, we can suppose X and Y to be affine. As f is birational, $f_*\mathscr{O}_X$ is coherent, hence $H^0(Y, f_*\mathscr{O}_X)$ is a finitely generated $H^0(Y, \mathscr{O}_Y)-$module. But $H^0(Y, \mathscr{O}_Y)$ is normal as Y is normal, and $H^0(Y, f_*\mathscr{O}_X)$ is normal too. As f is birational, their fields of fractions are isomorphic, hence we need $H^0(Y, f_*\mathscr{O}_X) \simeq H^0(Y, f_*\mathscr{O}_X)$. But this implies that the morphism $\mathscr{O}_Y \longrightarrow f_*\mathscr{O}_X$ is an isomorphism, as X and Y are affine, and we are done. □

Let me first prove the following corollary:

Corollary 3.1.6. *Let X and Y be smooth projective surface. If $f : X \longrightarrow Y$, is a surjective and birational morphism, then $q(X) = q(Y)$ and $p_g(X) = p_g(Y)$. In particular, $\chi(\mathscr{O}_X) = \chi(\mathscr{O}_Y)$.*

Proof. Let $f : X \dashrightarrow Y$ be a birational morphism. By the Zariski Main Theorem the fibers of f are connected, and we can apply Lemma 3.1.5 to get $f_*\mathscr{O}_X \simeq \mathscr{O}_Y$. Now, let us use the projection formula:

$$h^i(Y, \mathscr{O}_Y) = h^i(Y, f_*f^*\mathscr{O}_Y) = h^i(X, f^*\mathscr{O}_Y) = h^i(X, \mathscr{O}_X),$$

as $\mathscr{O}_X \simeq f^*\mathscr{O}_Y$. Hence $h^{0,1}(X) = h^{0,1}(Y)$ and $h^{0,2}(X) = h^{0,2}(Y)$, and we are done. □

Another important corollary is the following: if $f : X \dashrightarrow Y$ is a rational map, we can consider $\Gamma_f \subseteq X \times Y$ to be the closure of the graph of f. We have then the two natural projections $p : \Gamma_f \longrightarrow X$ and $q : \Gamma_f \longrightarrow Y$.

Corollary 3.1.7. *Let X be a normal projective variety, and let Y be any projective variety. Let $f : X \dashrightarrow Y$ be a rational map. Then for every point $x \in X$ the set $f(x) := q(p^{-1}(x))$ is connected.*

Proof. The map p is a birational morphism, and we can apply to it the Zariski Main Theorem as X is normal. Hence $p^{-1}(x)$ is connected for every $x \in X$, so that $f(x)$ is connected. $\square$

Corollary 3.1.8. *Let X and Y be two projective varieties, and let (X', f) and (Y', g) be their normalisations. Every morphism $h : X \longrightarrow Y$ induces a morphism $k_h : X' \longrightarrow Y'$ such that $g \circ k_h = h \circ f$.*

Proof. The morphisms f and g are birational, so we have a rational inverse f^{-1} and g^{-1}. Let $k_h := g^{-1} \circ h \circ f : X' \longrightarrow Y'$. We need to show that k_h is a morphism: if $x \in X'$, we have $g^{-1}(h(f(x)))$ is a finite set. This is connected by the previous corollary as X' is normal, so it consists of only one point, and k_h is a morphism. $\square$

Another important application is the following:

Theorem 3.1.9. ***(Stein's factorization)***. *Let X, Y be projective varieties, and $f : X \longrightarrow Y$ a surjective morphism. Then there is projective variety Y', a map $f' : X \longrightarrow Y'$ with connected fibers and a finite morphism $g' : Y' \longrightarrow Y$, such that $f = g' \circ f'$. Moreover, if X is normal, then Y' is normal.*

Corollary 3.1.10. *Let $f : X \longrightarrow Y$ be a morphism of projective varieties. If Y is normal and the generic fiber of f is connected, then every fiber is connected.*

Proof. Let us take the Stein factorisation $g' : Y' \longrightarrow Y$. as the generic fiber of f is connected, the degree of g' is generically 1, so that g' is a birational map. By the Zariski Main Theorem the fibers of g' are connected, hence the consist of only 1 point, and g' is an isomorphism. Then $f = f'$ has connected fibers. $\square$

Corollary 3.1.11. *Let $f : X \dashrightarrow Y$ be a proper map, and suppose X to be normal. Moreover let $U \subseteq X$ be the maximal open subset over which f is defined. Then $codim_X(X \setminus U) \geq 2$.*

Proof. In the first case, the fibers of f are connected by the Zariski Main Theorem. If $y \in Y$ is a point such that $p^{-1}(y)$ consists of a single point, then we can find an open subset $V \subseteq Y$, $y \in V$, such that $f : f^{-1}(V) \longrightarrow V$ is finite. Hence $X \setminus U$ is exactly the locus where the fibers have positive dimension, so that $codim_X(X \setminus U) \geq 2$. $\square$

3.1.3 Blow-up of a point

In this subsection, let X be a complex surface, and let $q \in X$. Moreover, let U be an open neighborhood of x in X, and let (x, y) be local coordinates, so that the point q corresponds to $(0, 0)$. Define the following:

$$\widetilde{U} := \{((x, y), (z : w)) \in U \times \mathbb{P}^1 \mid xw = yz\}.$$

Clearly, we have a map $p_U : \widetilde{U} \longrightarrow U$, which is the projection onto the first factor. Notice that if $(x, y) \neq (0, 0)$, then $p_U^{-1}((x, y)) = \{((x, y), (x : y)\}$. Moreover, we have that $p_U^{-1}(q) = \{q\} \times \mathbb{P}^1$. Hence,

$$p_U : p_U^{-1}(U \setminus \{q\}) \longrightarrow U \setminus \{q\}$$

is an isomorphism, and $p_U^{-1}(q) \simeq \mathbb{P}^1$ is a curve contracted by p_U. Now, let us take the gluing of X and $\widetilde{U}$ along $X \setminus \{q\}$ and $\widetilde{U} \setminus \{q\} \simeq U \setminus \{q\}$. In this way we obtain a surface $\widetilde{S}$ together with a morphism $p : \widetilde{X} \longrightarrow X$. Notice that p gives an isomorphism between $X \setminus \{q\}$ and $\widetilde{X} \setminus p^{-1}(q)$, and contracts the curve $\mathbb{P}^1 \simeq p^{-1}(q)$ to the point q.

Definition 3.1.5. The morphism $p : \widetilde{X} \longrightarrow X$ is called *blow-up of* X along q. We often write $Bl_q(S) := \widetilde{S}$. The curve $E := p^{-1}(q) \simeq \mathbb{P}^1$ is called *exceptional curve* of the blow-up.

The first remark is that the blow-up of a surface X along a point q is a birational map. Hence $\widetilde{X}$ and X are birational equivalent surfaces. Moreover, the morphism p is surjective, hence by Corollary 3.1.6 we have $q(\widetilde{X}) = q(X)$, $p_g(\widetilde{X}) = p_g(X)$ and $\chi(\mathscr{O}_{\widetilde{X}}) = \chi(\mathscr{O}_X)$.

Remark 3.1.1. If X is projective, even $\widetilde{X}$ is projective. Indeed, we can define the blow-up of $\mathbb{P}^n$ at the point $(0 : .. : 0 : 1)$ as a subvariety $\widetilde{\mathbb{P}^n} \subseteq \mathbb{P}^n \times \mathbb{P}^{n-1}$ in a way completely analogue to the previous one (here instead of $\mathbb{P}^1$ we need $\mathbb{P}^{n-1}$): simply take the subvariety defined by equations $x_i y_j = x_j y_i$ for every $i, j = 0, ..., n-1$, where x_i are coordinates for $\mathbb{P}^n$ and y_j are coordinates for $\mathbb{P}^{n-1}$. Next take the Segre embedding of $\mathbb{P}^n \times \mathbb{P}^{n-1}$ in $\mathbb{P}^{n^2+n-1}$. Now, every subvariety of $\mathbb{P}^n \times \mathbb{P}^{n-1}$ of bihomogenous equation correspond to a projective subvariety of $\mathbb{P}^{n^2+n-1}$ via the Segre embedding. It is now easy to see that the blow up of a projective variety at one point is still projective.

Definition 3.1.6. Let C be a curve in X. The *strict transform of* C under he blow-up p of X along q is the closure $\widetilde{C}$ in $\widetilde{X}$ of $p^{-1}(C \setminus \{q\})$.

Notice that the strict transform of a curve is always a curve. If $q \notin C$, then $\widetilde{C} = p^{-1}(C)$, so that $\widetilde{C} = p^*C$. More generally, one has the following:

Lemma 3.1.12. *Let C be an irreducible curve passing through q with multiplicity m. Then*

$$p^*C = \widetilde{C} + mE.$$

Proof. Clearly we have $p^*C = \widetilde{C} + kE$ for some $k \in \mathbb{Z}$. Use local coordinates (x, y) around q (in the open subset U). On the open subset $\widetilde{U} \cap (U \times \{z \neq 0\})$ we can then consider coordinates $u = x$ and $v = w/z$. In these coordinates, we see that $p(u,v) = (u, uv)$. Let f be a local equation of C at q. As C has multiplicity m at q, we have

$$f = f_m(x,y) + O(m),$$

where $f_m(x,y)$ is homogenous of degree m. Hence

$$f \circ p(u,v) = f_m(u,uv) + O(m) = u^m(f_m(1,v) + O(m)).$$

Then the component E has multiplicity m. □

We now prove the following, which we will use frequently in the following:

Proposition 3.1.13. *Let X be a smooth projective surface, and $p : \widetilde{X} \longrightarrow X$ be the blow-up of X along $q \in X$, whose exceptional curve is denoted E.*

1. *The morphism of groups*

$$Pic(X) \oplus \mathbb{Z} \longrightarrow Pic(\widetilde{X}), \qquad (\mathscr{O}_X(D), n) \mapsto \mathscr{O}_{\widetilde{X}}(p^*D + nE)$$

is an isomorphism, and similarly for the Néron-Severi groups. In particular $h^{1,1}(\widetilde{X}) = h^{1,1}(X) + 1$ and $\rho(\widetilde{X}) = \rho(X) + 1$.

2. *For every two divisor D, D' on X we have*

$$p^*D \cdot p^*D' = D \cdot D', \qquad p^*D \cdot E = 0, \qquad E \cdot E = -1.$$

3. *We have $K_{\widetilde{X}} = p^*K_X + E$.*

Proof. Let us start from the second item: let C and C' be two curves on X such that $q \notin C \cap C'$. Then

$$C \cdot C' = p^*C \cdot p^*C' = \widetilde{C} \cdot \widetilde{C}',$$

where the first equality comes by Proposition 2.4.4 as p is finite of degree 1 outside q, and the second comes from Lemma 3.1.12. Moreover, it is clear that $\widetilde{C} \cdot E = 0$, by projection formula, as E is contracted to q. For $E \cdot E$ consider the following: let C be a curve of X passing through q with multiplicity 1. Hence $\widetilde{C}$ meets E transversally, so that $\widetilde{C} \cdot E = 1$. Now, by Lemma 3.1.12 we have $\widetilde{C} = p^*C - E$, so that $(p^*C - E) \cdot E = 1$. But now, by the projection formula one has $p^*C \cdot E = 0$ as E is contracted to q, so that $E^2 = -1$. The remaining cases follow from this.

Let us now show item 1: every curve on $\widetilde{X}$ is either a multiple of E, or it is not. If it is not, then it is not contracted to a point, and it is then the proper transform of a curve on X. The map $Pic(X) \oplus \mathbb{Z} \longrightarrow Pic(\widetilde{X})$ is surjective. To show injectivity, simply suppose that $p^*D + nE$ is linearily equivalent to 0. Then

$$0 = (p^*D + nE) \cdot E = -n,$$

by the previous part of the proof, so that $n = 0$. Hence $p^*D \sim 0$, so that $D \sim 0$ on X, and injectivity is shown.

To conclude, we have that $K_{\widetilde{X}} = p^*K_X + kE$ for some $k \in \mathbb{Z}$. Now, use the adjunction formula:

$$-2 = deg(K_E) = deg((K_{\widetilde{X}} + E)_{|E}) = K_{\widetilde{X}} \cdot E + E^2 = p^*K_X \cdot E + (k+1)E^2.$$

Now, $p^*K_X \cdot E = 0$ by the projection formula, and $E^2 = -1$ by item 2, so that $2 = k+1$. In conclusion, $k = 1$ and we are done. □

3.1.4 Indeterminacies

In this section we prove a fundamental result of surface theory: every rational map is composition of blow-ups. Let me first show the relationship between resolution of indeterminacies of a rational map, and blow-up. Let us consider $f : S \dashrightarrow \mathbb{P}^n$, and let Ind_f be the indeterminacy locus of f. Then let $f(S) := \overline{S \setminus Ind_f}$ be the Zariski closure of the image of f.

Proposition 3.1.14. *There is finite sequence*

$$S_n \xrightarrow{p_n} S_{n-1} \xrightarrow{p_{n-1}} \dots \overset{p_1}{S_0} := S,$$

where for each n the map p_n is the blow-up of S_{n-1} along a point, such that the map

$$f_n := f \circ p_1 \circ \dots \circ p_n : S_n \longrightarrow \mathbb{P}^n$$

is a morphism.

Proof. We can assume that $f(S)$ is not contained in a hyperplane of $\mathbb{P}^n$. If one considers a system of hyperplanes on $\mathbb{P}^n$, we then get a linear system $|D|$ on X associated to a divisor D, whose base locus is contained in Ind_f. If $Ind_f = \emptyset$, then we are done. So let us suppose $Ind_f \neq \emptyset$, and let $q \in Ind_f$. Let $S_1 := Bl_q(S)$, and let p_1 be the associated blow-up. Then $p_1^* = D_1 + mE_1$, where D_1 is the proper transform of D and $E_1 = exc(p_1)$, for some integer $m_1 \in \mathbb{Z}$. We can choose m_1 so that $E_1 \notin Bs(D_1)$. If $Bs(D_1) = \emptyset$, then we are done. If $Bs(D_1) \neq \emptyset$, let $q_1 \in Bs(D_1)$ and proceed as before. In this way we produce a surface $S_k = Bl_{q_{k-1}}(S_{k-1})$ and $p_k : S_k \longrightarrow S_{k-1}$, and on S_k we have D_k such that $p^*D_{k-1} = D_k + m_kE_k$, where $m_k \in \mathbb{Z}$ is such that $E_k := exc(p_k)$ is not in the base locus of D_k. Now, by Proposition 3.1.13 we have

$$0 \leq D_k^2 = D_{k-1}^2 - m_k^2 < D_{k-1}^2,$$

where the first inequality comes from the fact that D_k has no curve in its base locus (since the indeterminacy locus has codimension at least 2 by Corollary 3.1.11). Hence the sequence $\{D_k^2\}$ must stabilise for some $k \gg 0$. Hence for $k \gg 0$ we have that $|D_k|$ has no base locus, and we are done. □

Example 3.1.2. 2.1.2Let $X = \mathbb{P}^2$, and let $q \in X$. The projection from q gives a map

$$\pi_q : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1,$$

which is not defined only at q. Blowing-up q in X, we easily solve the indeterminacy.

Example 3.1.3. Using the same notations as in Example 3.1.1, we see that the map $Q \dashrightarrow \mathbb{P}^2$ is not defined at 0, and that the two lines passing through 0 contained in Q are projected to two distint points A and B. Then, blowing up Q at 0 we solve the indeterminacy, and we can show that Bl_0Q is the blow-up of $\mathbb{P}^2$ at A and B.

Now, the locus where a rational map is not defined has codimension at least 2. For a smooth (or normal) surface, this implies that the indeterminacy locus is given by a finite number of points. To solve the indeterminacies we have then to blow them up, getting a finite number of curves which are contracted.

Lemma 3.1.15. *Let X and Y be two projective surfaces, and suppose Y to be smooth. Let $f : X \longrightarrow Y$ be a birational map. If $p \in Y$ is such that the inverse of f is not defined at p, then $f^{-1}(p)$ is a curve.*

Proof. If the inverse map $g : Y \dashrightarrow X$ of f is not defined at p, this means that $f^{-1}(p)$ consists at least of 2 distinct points. By the Zariski Main Theorem the fibers of f are connected, hence $f^{-1}(p)$ is a curve. □

Corollary 3.1.16. *Let X, Y be two smooth projective surfaces, and let $f : X \dashrightarrow Y$ be a birational map. If f is not defined at $x \in X$, then the inverse map $g : Y \dashrightarrow X$ contracts a curve to x.*

Proof. Let us suppose that the maximal open subset of X over which f is defined is U. Let $\Gamma_f \subseteq X \times Y$ be the closure of the graph of f, and let $p : \Gamma_f \longrightarrow X$ and $q : \Gamma_f \longrightarrow Y$ be the two projections. If f is not defined at x, then the inverse of q (which is birational) is not defined at x. Hence $q^{-1}(x)$ is a curve C by Lemma 3.1.15, which is contracted to x by q. Now, notice that $g = q \circ (q')^{-1}$, hence C is contracted to x by g. □

Here we can finally prove the main theorem on blow-ups of surfaces.

Theorem 3.1.17. *Let X, Y be two smooth projective surfaces, and $f : X \longrightarrow Y$ a morphism which is birational. If $y \in Y$ is a point where the inverse g of f is not defined, then there is a morphism $h : X \longrightarrow Bl_yY$ which is birational and such that $f = p \circ h$, where $p : Bl_yY \longrightarrow Y$ is the blow-up morphism.*

Proof. Let p^{-1} be the inverse of p, and let $h := p^{-1} \circ f$. Moreover, let h^{-1} be the inverse of h. we want to show that h is a morphism, and to do so we assume the contrary. Hence, by Corollary 3.1.16 there is a curve C on Bl_yY

contracted by h^{-1} to a point $x \in X$. As $h^{-1} = f^{-1} \circ p$, and $h^{-1}(C) = x$, we have $f^{-1}(p(C)) = x$, so that $p(C) = f(x)$. Hence C is a curve contracted by p. As p contracts only its exceptional curve E, we have $E = C$ and $f(x) = y$. Let u be a local coordinate at y on Y. If f^*u is not a local coordinate at x on X, we would have $f^*u \in m_x^2$. Hence $p^*u = (h^{-1})^* f^*u \in m_e^2$ for every $e \in E$ where it is defined, i. e. at every $e \in E$ but a finite number. The blow-up has the property that any local coordinate at y on Y lifts to a local coordinate at every point of E, but one. Hence f^*u has to be a local coordinate at x on X. But this implies that h is defined at x, so that h is a morphism, and we are done. □

From this, one can prove the following:

Theorem 3.1.18. *Let $f : X \longrightarrow Y$ be a birational morphism between two smooth projective surfaces. Hence f is the composition of a finite number of blow-ups and isomorphisms.*

Proof. Let us suppose f is not an isomorphism. Then there is a point $y_1 \in Y$ over which the inverse map f^{-1} is not defined. By Theorem 3.1.17, there is a blow-up $p_1 : Y_1 := Bl_{y_1}Y \longrightarrow Y$ such that $f = p_1 \circ f_1$, for some birational morphism $f_1 : X \longrightarrow Y_1$. Notice that f contracts all those curves which f_1 contracts, and contracts a curve more, namely the curve which f_1 maps to $exc(p_1)$. Hence the number of curves that f_1 contracts is strictly smaller than the one of the curves contracted by f. If no curve is contracted by f_1, then f_1^{-1} is everywhere defined and f_1 is an isomorphism, and we are done. Otherwise there is a point $y_2 \in Y_1$ on which f_1^{-1} is not defined, and we can proceed as before. Notice that this procedure has to finish since the number of contracted curves strictly decreases. □

As a corollary we have the following:

Corollary 3.1.19. *Let X, Y be smooth projective surfaces, and $f : X \dashrightarrow Y$ a birational map. Hence there is a smooth projective surface Z and two morphisms $g : Z \longrightarrow X$ and $h : Z \longrightarrow Y$ which are composition of blow-ups and isomorphisms and such that $h = f \circ g$.*

Notice that from this it follows that if X and Y are two birational surfaces, then $q(X) = q(Y)$, $p_g(X) = p_g(Y)$ and $\chi(\mathscr{O}_X) = \chi(\mathscr{O}_Y)$.

3.1.5 Castelnuovo's contraction theorem

In this section we prove one of the main results in the classification theory of surfaces. As we have seen in the previous section, the only birational maps between smooth projective surfaces are isomorphisms and blow-ups along points. If we blow up a smooth projective surface X at a point $x \in X$, we obtain a smooth projective surface $\widetilde{X}$ on which there is a curve E, the exceptional curve of the blow-up, such that $E^2 = -1$. One of the most surprising properties of surfaces is that every curve C such that $C^2 = -1$ is the exceptional curve of some blow-up. Before proving this, let me introduce a definition:

Definition 3.1.7. A $(-1)-$*curve* on a surface X is a smooth rational curve C such that $C^2 = -1$.

Theorem 3.1.20. ***(Castelnuovo's contraction)***. *Let X be a smooth projective surface, and suppose there is $(-1)-$curve E. Then there is a smooth projective surface Y and a point $y \in Y$ such that $X \simeq Bl_yY$ and $E = p^{-1}(y)$, where $p : X \longrightarrow Y$ is the blow-up morphism.*

Proof. Let H be a very ample divisor on X, and suppose that $H^1(X, \mathscr{O}_X(H)) =$ (this is always possible up to changing H with mH for $m \gg 0$). Let $d := H \cdot E$, and $H' := H + dE$. Then $H' \cdot E = H \cdot E + dE^2 = 0$ as $E^2 = -1$. Let us consider the map $\varphi_{H'}$, which is defined on $X \setminus Bs(H')$. We want to show that $\varphi_{H'}$ is defined on the whole X. In order to do so, we define a good basis for $H^0(X, \mathscr{O}_X(H'))$: let $i = 1, ..., d$, and consider the exact sequence of sheaves

$$0 \longrightarrow \mathscr{O}_X(H + (i-1)E) \longrightarrow \mathscr{O}_X(H + iE) \longrightarrow \mathscr{O}_E(d-i) \longrightarrow 0.$$

This is exact as $E^2 = 1$ and

$$deg((H + iE)_{|E}) = (H + iE) \cdot E = H \cdot E + iE^2 = d - i.$$

Now, this exact sequence induces a long exact sequence in cohomology, and we have

$$H^1(X, \mathscr{O}_X(H + (i-1)E) \longrightarrow H^1(X, \mathscr{O}_X(H + iE) \longrightarrow H^1(E, \mathscr{O}_E(d-i)).$$

The last term of this exact sequence is trivial for every $i = 1, ..., d$ as $E \simeq \mathbb{P}^1$ and $d - i \geq 0$. Moreover, if $i = 1$, then the first element is $H^1(X, \mathscr{O}_X(H)) = 0$ by hypothesis, so that $H^1(X, \mathscr{O}_X(H + E)) = 0$. Using this process in successive steps, we finally get that $H^1(X, \mathscr{O}_X(H + iE)) = 0$ for every $i = 1, ..., d$. Hence the restriction map

$$H^0(X, \mathscr{O}_X(H + iE)) \longrightarrow H^0(E, \mathscr{O}_E(d-i))$$

is surjective for every $i = 1, ..., d$. Then

$$H^0(X, \mathscr{O}_X(H + dE)) \simeq H^0(X, \mathscr{O}_X(H)) \oplus H^0(E, \mathscr{O}_E(1)) \oplus ... \oplus H^0(E, \mathscr{O}_X(d)).$$

Hence to determine a basis of $H^0(X, \mathscr{O}_X(H'))$ we need to fix a basis $\{s_0, ..., s_n\}$ of $H^0(X, \mathscr{O}_X(H))$, and for every $i = 1, ..., d$ a set $\{s_{i,0}, ..., s_{i,d-i}\}$ of sections of $H^0(X, \mathscr{O}_X(H + iE))$ which restricts to a basis of $H^0(E, \mathscr{O}_E(d-i))$ (here

$h^0(\mathbb{P}^1, \mathscr{O}(d-i)) = d-i+1)$. Hence, if $s \in H^0(X, \mathscr{O}_X(E))$ (it exists as E is effective), we get the basis

$$\{s^d s_0, ..., s^d s_n, s^{d-1} s_{1,0}, ..., s^{d-1} s_{1,d-1}, ..., s s_{d-1,0}, s s_{d-1,1}, s_{d,0}\}$$

for $H^0(X, \mathscr{O}_X(H'))$. Now, let's look at the map $\varphi_{H'}$ associated to this basis: if $x \in E$, then $s(x) = 0$ and $s_{d,0}(x) = 1$, so that $\varphi_{H'}(E) = (0 : 0 : ... : 0 : 1) := P$, and E is contracted to a point. If $x \notin E$, then $s(x) \neq 0$, and $\varphi_{H'}$ is defined at x: indeed H has no base locus, so that there is $j = 1, ..., n$ such that $s_j(x) \neq 0$. Hence the map $\varphi_{H'}$ is everywhere well defined, and notice that if $x \notin E$, then $\varphi_{H'}(x) \neq P$. . Moreover, it is an isomorphism between $X \setminus E$ and $\varphi_{H'}(X \setminus E) = \varphi_{H'}(X) \setminus \{P\}$. Indeed, outside of E we have that $H = H'$, so that H' separates points outside E as H does it (more precisely, if $x, y \in X \setminus E$ are such that $\varphi_{H'}(x) = \varphi_{H'}(y)$, then $s_j(x) = s_j(y)$ for every $j = 0, ..., n$, as $s^d(x), s^d(y) \neq 0$; but as H is very ample, it separates points, so that this implies $x = y$).

Now, let us show that P is smooth in $\varphi_{H'}(X)$. As $s_{d,0}$ gives a basis for $H^0(E, \mathscr{O}_E)$, there is an open neighborhood U of E in X over which $s_{d,0} \neq 0$. The sheaf $\mathscr{O}_U(-E)$ has then two sections $t_0 := s_{d-1,0}/s_{d,0}$ and $t_1 := s_{d-1,1}/s_{d,0}$. Clearly they restrict to a basis of $H^0(E, \mathscr{O}_E(1))$. Up to shrinking U, we can suppose that t_0 and t_1 are not simultaneously trivial on U. Hence we get a map

$$z_2 : U \longrightarrow \mathbb{P}^1, \qquad z_2(x) := (t_0(x) : t_1(x)).$$

Moreover, let

$$z_1 : U \longrightarrow \mathbb{C}^2, \qquad z_1(x) := (s(x)t_0(x), s(x)t_1(x)),$$

which is trivial exactly on E. If one looks at the map

$$(z_1, z_2) : U \longrightarrow \mathbb{C}^2 \times \mathbb{P}^1, \qquad (z_1, z_2)(x) = (z_1(x), z_2(x)),$$

we easily see that it maps to the blow-up $\widetilde{\mathbb{C}^2}$ of $\mathbb{C}^2$ at the point $(0,0)$ inside $\mathbb{C}^2 \times \mathbb{P}^1$. Hence we get a map $z : U \longrightarrow \widetilde{\mathbb{C}^2}$. Let now $p : \widetilde{\mathbb{C}^2} \longrightarrow \mathbb{C}^2$ be the blow-up morphism, and let $\widetilde{z} : \varphi_{H'}(U) \longrightarrow \mathbb{C}^2$ be the map induced by z. Hence we have $p \circ z = \widetilde{z} \circ \varphi_{H'}$.

Let now $Y := \varphi_{H'}(X)$, and notice that $\varphi_{H'}(U) \cap Y \subseteq Y$ is an open subset. Observe the following: the map z maps E to the exceptional divisor F of p, and it is an isomorphism. Using the complex topology, we can see that z is an isomorphism locally around every point of E. Indeed, let (u, v) coordinates on $\mathbb{C}^2$, and let $(U : V)$ be coordinates on $\mathbb{P}^1$. The local equation of $\widetilde{\mathbb{C}^2}$ is $uV = Uv$. Let $q \in E$, and let suppose $z(q) = ((0,1), (0 : 1))$. Hence v and U/V are local coordinates on $\widetilde{\mathbb{C}^2}$ around $z(q)$, and we easily see that $z^*v = st_1$ and $z^*U/V = t_0/t_1$. Then z^*v vanishes with multiplicity 1 on E, while z^*U/V is a local coordinate around q. Hence, in a neighborhood of q where these two functions are defined, they give a coordinate system, and z is an isomorphism. Hence, as E is compact in X, there is an open neighborhood U of E which maps isomorphically to an open neighborhood W of F. Let now

$$\varphi_{H'} \circ z_{|W}^{-1} : W \longrightarrow \varphi_{H'}(U).$$

As this contracts the exceptional curve F in W, by Theorem 3.1.17 there must be a map $g : p(W) \longrightarrow \varphi_{H'}(U)$ such that $g \circ p = \varphi_{H'} \circ z_{|W}^{-1}$. Hence, by commutativity properties, one easily gets $g = \widetilde{z}^{-1}$. Hence, we see that $\varphi_{H'}(U)$ is isomorphic to $p(W)$, which is smooth, so that P is a smooth point in Y. It is now clear that $\varphi_{H'} : X \longrightarrow Y$ is exactly the blow-up of Y along P, and the exceptional locus is E. □

The main conclusion of this theorem is the following: whenever on a smooth surface X there is a $(-1)-$curve, we can contract it as it is the product of a blow-up. The obtained surface Y is again smooth and projective, but $\rho(Y) = \rho(X) - 1$ be Proposition 3.1.13. Let me define the following:

Definition 3.1.8. A surface X is called *minimal* if every birational morphism $f : X \longrightarrow Y$ is an isomorphism. A *minimal model* for a surface X is a surface X' which is minimal and birational to X.

Here is an important corollary of the Castelnuovo Contraction Theorem:

Corollary 3.1.21. *A surface is minimal if and only if it has no $(-1)-$curve. In particular, every surface has a minimal model.*

Proof. Let X be a smooth projective surface, and let us suppose it is minimal. If there is a $(-1)-$curve E on E, by the Castelnuovo Contraction Theorem there is a smooth projective surface Y and $y \in Y$ such that $X = Bl_yY$. In particular, there is a birational morphism $p : X \longrightarrow Y$ which is not an isomorphism, hence X cannot be minimal, and we get a contradiction. Hence every minimal smooth projective surface has no $(-1)-$curve.

For the converse, let us suppose that X has no $(-1)-$curve, but X is not minimal. Then there is a birational morphism $f : X \longrightarrow Y$ to a smooth projective surface Y which is not an isomorphism. Hence, by Theorem 3.1.18 f is the composition of isomorphisms and blow-ups, and there is at least one blow-up. As f is defined on X, then this introduces an exceptional curve on X, which is a $(-1)-$curve. This is a contradiction, and we are done.

Let us prove the remaining part. Let X be a smooth projective surface. If it is minimal, then it is a minimal model for itself. If it is not minimal, then there is a $(-1)-$curve E on X. By the Castelnuovo Contraction Theorem, E is the exceptional curve of a blow-up, so that contracting it we get a new smooth projective surface X_1, such that $\rho(X) = \rho(X_1) + 1$ by Proposition 3.1.13. If X_1

is minimal, we are done. Otherwise one can still contract exceptional curves, producing a sequence of blow-ups

$$X = X_0 \xrightarrow{p_0} X_1 \xrightarrow{p_1} ... \xrightarrow{p_{n-1}} X_n \xrightarrow{p_n} ...$$

Notice that as $\rho(X_n) = \rho(X) - n$, and $\rho(X_n) > 0$, this sequence has to terminate for some $n_0 < \rho(X)$. Hence X_{n_0} cannot contain $(-1)-$curves, being then minimal. Hence it is a minimal model for X. $\square$

This is a very important point: as every surface admits a minimal model, once a birational classification of minimal surfaces is completed, even the birational classification of surfaces is completed. The classification we are going to present is hence only a classification of minimal surfaces. There is anyway an important point to remark: every surfaces admits a minimal model. How many are them? Let us present a simple example:

Example 3.1.4. Let $X := \mathbb{P}^2$ and $Y := \mathbb{P}^1 \times \mathbb{P}^1$. In Example 3.1.1 we have shown that these two smooth projective surfaces are birational and not isomorphic. Anyway, they are minimal. Indeed, on X every irreducible curve C is such that $C^2 > 0$, hence there is no $(-1)-$curve. Similarily, every curve C on Y is such that C^2 is even, so there is no $(-1)-$curve on Y, and these two surfaces are minimal. Hence, if S is a smooth projective surface birational to $\mathbb{P}^2$ it can have at least two non-isomorphic birational models.

3.2 Kodaira dimension

In this section we introduce the notion of Kodaira dimension of a normal projective variety. In the case of surfaces, it respresents an important birational invariant, which is used to give a first rough classification. In the following, let X be a normal projective variety, and let D be an effective Cartier divisor on X. Consider the following field

$$Q(X, D) := \{s/t \,|\, s, t \in H^0(X, \mathscr{O}_X(kD)),\ t \neq 0,\ k \geq 0\},$$

which is the homogenous field of fractions of the ring

$$R(X, D) := \bigoplus_{k \geq 0} H^0(X, \mathscr{O}_X(kD)).$$

Notice that if D is a hyperplane section, then $H^0(X, \mathscr{O}_X(kD))$ is given by homogenous regular functions on X of degree k, so that $Q(X, D) = \mathscr{M}_X(X)$, the field of functions of X. More generally, we have the following:

Lemma 3.2.1. *Let X be a normal projective variety, and let D be an effective divisor. The field $Q(X, D)$ is algebraically closed in $\mathscr{M}_X(X)$. In particular, its transcendence degree d is finite and $0 \leq d \leq dim(X)$.*

Proof. Let f be a rational function on X, and suppose that there are $a_1, ...a_n \in Q(X, D)$ such that $f^n + a_1 f^{n-1} + ... + a_n = 0$. We can suppose $a_i = s_i/t$ for some $s_i, t \in H^0(X, \mathscr{O}_X(kD)$ for every $i = 1, ..., n$. Hence, $h := ft$ is a meromorphic section of $\mathscr{O}_X(kD)$ verifying

$$h^n + s_1 h^{n-1} + ... + s_n = 0,$$

where s_i are all regular. Let now $x \in X$: as X is normal, $\mathscr{O}_{X,x}$ is a normal ring, and $s_{i,x} \in \mathscr{O}_{X,x}$. Hence $h_x \in \mathscr{O}_{X,x}$ for every $x \in X$, so that h is a regular section of $\mathscr{O}_X(kD)$. Hence $f = h/t \in Q(X, D)$, and we are done. □

Notice that id D is not effective, we cannot define $Q(X, D)$ in the same way. If for every $k \in \mathbb{N}$ we have $h^0(X, \mathscr{O}_X(kD)) = 0$, then $R(X, D) = Q(X, D) = \mathbb{C}$. Otherwise, consider the set

$$\mathbb{N}(D) := \{k \in \mathbb{N} \,|\, h^0(X, \mathscr{O}_X(kD)) > 0\}.$$

Then we can define $Q(X, D)$ to be the homogenous quotient field of the ring $R(X, D) := \bigoplus_{k \in \mathbb{N}(D)} H^0(X, \mathscr{O}_X(D))$. Obviously, Lemma 3.2.1 holds in this case.

3.2.1 Definition and geometrical interpretation

We can then give the following definition:

Definition 3.2.1. Let X be a normal projective variety, and let D be a divisor. If $\mathbb{N}(D) \neq \emptyset$, we call *Kodaira dimension* of D the degree of transcendence $\kappa(D)$ of $Q(X, D)$. Otherwise we define $\kappa(D) := -\infty$. The *Kodaira dimension* of X is

$$\kappa(X) := \kappa(K_X) \in \{-\infty, 0, ..., dim(X)\}.$$

Remark 3.2.1. Notice that if X is a normal projective surface, and K_X is nef, then $\kappa(X) \geq 0$. Indeed, as K_X is nef, there is $m \gg 0$ such that mK_X is effective. Hence $h^0(X, \mathscr{O}_X(mK_X)) > 0$, and $\kappa(X) \geq 0$.

This abstract definition is strictly related to the geometry of X. Let us take $k \in \mathbb{N}(D)$, so that $h^0(X, \mathscr{O}_X(kD)) > 0$. Hence we can define the morphism

$$\varphi_{kD} : X \dashrightarrow \mathbb{P}^{N_{kD}},$$

where $N_{kD} := h^0(X, \mathscr{O}_X(kD)) - 1$. Let $U_{kD} := X \setminus Bs(kD)$, and let

$$W_{kD} := \overline{\varphi_{kD}(U_{kD})}.$$

This is a projective variety whose fields of functions is easily described in terms of sections of kD: indeed, if $\{s_0, ..., s_{N_{kD}}\}$ is a basis for $H^0(X, \mathscr{O}_X(kD))$, then it is easy to see that

$$\mathscr{M}(W_{kD}) = \mathbb{C}(s_1/s_0, ..., s_{N_{kD}}/s_0).$$

As a corollary of this, we see that $Q(X, D) = \bigcup_{k \in \mathbb{N}(D)} \mathscr{M}(W_{kD})$.

Proposition 3.2.2. *Let X be a normal projective variety, and let D be a divisor such that $\mathbb{N}(D) \neq \emptyset$. Then*

$$\kappa(D) = max\{dim(W_{kD}) \,|\, k \in \mathbb{N}(D)\}.$$

Proof. Let us suppose that $\mathbb{N}(D) = \mathbb{N}$, and let $k \in \mathbb{N}$. The natural injection $\mathscr{O}_X(kD) \longrightarrow \mathscr{O}_X((k+1)D)$ induces a natural inclusion $H^0(X, \mathscr{O}_X(kD)) \subseteq H^0(X, \mathscr{O}_X((k+1)D))$. By the previous description of $\mathscr{M}(W_{kD})$ we then get an inclusion $ik : \mathscr{M}(W_{kD}) \longrightarrow \mathscr{M}(W_{(k+1)D})$. Now, the quotient field $\mathscr{M}(X)$ of X is finitely generated over $\mathscr{C}$ (indeed, the transcendence degree of $\mathscr{M}(X)$ is the dimension of X), hence $Q(X, D)$ is finitely generated over $\mathbb{C}$. So the sequence of inclusions i_k must stabilize, hence there is $k_0 \in \mathbb{N}$ such that $Q(X, D) = \mathscr{M}(W_{k_0 D})$. Their transcendence degree is then equal, so that $\kappa(D) = dim(W_{k_0 D})$. Now, simply notice that as i_k is an inclusion, we have $dim(W_{kD}) \leq dim(W_{(k+1)D})$ for every $k \in \mathbb{N}$, so that we finally prove the proposition if $\mathbb{N}(D) = \mathbb{N}$. The remaining case is similar. □

Corollary 3.2.3. *If X is a normal projective variety such that K_X is ample, then $\kappa(X) = dim(X)$.*

Proof. If K_X is ample, there is $m \gg 0$ such that mK_X is very ample, and the map φ_{mK_X} is a closed embedding. Hence $dim(X) = dim(W_{mK_X})$, so that $\kappa(X) = dim(X)$. □

Another important result is the following:

Proposition 3.2.4. *Let X be a normal projective surface, and let D be a divisor such that $\mathbb{N}(D) \neq \emptyset$. Then the generic fiber of φ_{kD} is connected.*

Proof. Let me start by supposing that there is $k \in \mathbb{N}(D)$ such that φ_{kD} is everywhere defined on X, i. e. $Bs(kD) = \emptyset$. Let $f := \varphi_{kD} : X \longrightarrow W_{kD}$. We can consider the Stein factorization of f, so that there is a normal projective variety Y and two morphisms

$$X \xrightarrow{f'} Y \xrightarrow{g} W_{kD},$$

such that $g \circ f' = f$, g is finite and f' has connected fibers. We have then $\mathscr{M}(W_{kD}) \subseteq \mathscr{M}(Y) \subseteq \mathscr{M}(X)$. Moreover, notice that if kD has no base locus, the same is true for every $k'D$, with $k' \geq k$: hence we can suppose $k \gg 0$ so that $\mathscr{M}(W_{kD}) = Q(X, D)$. As $Q(X, D)$ is algebraically closed in $\mathscr{M}(X)$, we then have $\mathscr{M}(Y) = \mathscr{M}(W_{kD})$. The morphism g is then birational by Proposition 3.1.2. Generically, then, the fibers of f and f' are equal, so that the generic fiber of f is connected.

In general, the situation is more complicated. Let $k \in \mathbb{N}(D)$, and let $f := \varphi_{kD}$: by Corollary 3.1.11, as X is normal we have that $codim_X(Bs(kD)) \geq 2$, since $Bs(kD) = Ind_f$, so that $Bs(kD)$ is given by a finite number of points. By Proposition 3.1.14 we know that to solve the indeterminacies of f we just need to blow up X in $Bs(kD)$, and we get a projective surface $\widetilde{X}$ together with a morphism $p : \widetilde{X} \longrightarrow X$ such that the map

$$\widetilde{f} := f \circ p : \widetilde{X} \longrightarrow W_{kD}$$

is a morphism. As X is normal, the fibers of p are connected by the Zariski Main Theorem. Moreover, we have

$$p^* : H^0(X, \mathscr{O}_X(kD)) \longrightarrow H^0(\widetilde{X}, p^*\mathscr{O}_X(kD))$$

is an isomorphism. Up to replacing $\widetilde{X}$ with its normalisation, we can suppose $\widetilde{X}$ to be normal. Hence we can replace X with another smooth projective surface X', and the morphism f is replaced by f', which is defined by means of the same basis (up to the identification of global sections). We can apply the previous part and guarantee that the generic fiber of φ_{kD} is always connected, for every $k \in \mathbb{N}(D)$. □

We conclude with the following, which is a beautiful equivalent definition of the Kodaira dimension. Let me introduce the following:

Definition 3.2.2. Let X be a normal projective variety. The *$m-th$ plurigenus* of X is

$$P_m(X) := h^0(X, \mathscr{O}_X(mK_X)).$$

Proposition 3.2.5. *Let X be a normal projective variety, and let D be a divisor such that $\mathbb{N}(D) \neq \emptyset$. There are $\alpha, \beta \in \mathbb{R}_{>0}$ such that if $k \gg 0$ we have*

$$\alpha k^{\kappa(D)} \leq h^0(X, \mathscr{O}_X(kD)) \leq \beta k^{\kappa(D)}.$$

In particular, we have $P_m(X) \sim m^{\kappa(X)}$.

Proof. Up to replace D with some multiple and X with some blow-up, we can suppose that $f := \varphi_D$ is everywhere defined, and that $dim(W) = \kappa(D)$, where $W = W_D$. Let F be the fixed part of $|D|$, so that $D = F + f^*H$, where H is a hyperplane section of W. We have the following sequence of inequalities:

$$h^0(X, \mathscr{O}_X(kD)) \geq h^0(X, f^*\mathscr{O}_W(kH)) \geq h^0(W, \mathscr{O}_W(kH)) \geq \alpha k^{\kappa(D)},$$

by the Hilbert polynomial.

For the remaining inequality, let us first suppose that the fixed part F does not contain any component mapping surjectively on W. Then F is contained in the pull-back of some divisor G on W, so that

$$h^0(X, \mathscr{O}_X(kD) \leq h^0(X, f^*\mathscr{O}_W(kH + kG).$$

Up to adding a very ample divisor to G, we can suppose that $H+G$ is very ample. By the Hilbert polynomial we then get

$$h^0(X, \mathscr{O}_W(kH+kG)) \leq \beta k^{\kappa(D)},$$

and we are done. So now suppose that F contains a component which surjects onto W. Let us write $F = F' + F''$, where F' is the maximal divisor such that $f(F') = W$. Clearly we have $|D| = |D - F'| + F'$. If $|kD| = |kD - kF'| + kF'$, we can just replace D by $D - F'$ in the preceding part, and we are done. We then need to show that $|kD| = |kD - kF'| + kF'$ for every k. Clearly we have $|kD - kF'| + kF' \subseteq |kD|$, and suppose there is $E \in |kD|$, but $E \notin |kD - kF'| + kF'$. Let G be the maximal divisor such that $E \in |kD - G| + G$ and which does not contain kF'. Then we can write $E = E' + G$, where $E' \in |kD - G|$, and E' passes through some points of F', but not all. Now, $f(F') = W$, so the fibers of f meet F'. If $x \in E'$, but $x \notin F'$, we have that $f^{-1}(f(x))$ intersects F' in some point. If $y \in f^{-1}f(x) \cap F'$, we have then $\varphi_{kD}(x) \neq \varphi_{kD}(y)$.

The generic fiber of f and of φ_{kD} are connected by Proposition 3.2.4, and the seconds are contained in the firsts. as φ_{kD} is not constant on the generic fiber of f, then the generic fiber of φ_{kD} is a lower dimensional subvariety of the generic fiber of f. Now, since

$$dim(W_{kD}) + dim(\varphi_{kD}^{-1}(pt)) = dim(X) = dim(W) + dim(f^{-1}(pt)),$$

we get that $dim(W_{kD}) > dim(W)$, which is impossible as $dim(W) = \kappa(D)$ by hypothesis. Hence $|kD| = |kD - kF'| + kF'$ for every k, and we are done. $\square$

Here is a basic property of the Kodaira dimension:

Proposition 3.2.6. *If X and Y are two birational normal projective surfaces, then $P_m(X) = P_m(Y)$ for every $m > 0$, and $\kappa(X) = \kappa(Y)$.*

Proof. Let $f : X \dashrightarrow Y$ be a birational map. By Corollary 3.1.19 there is projective surface Z and two maps $g : Z \longrightarrow X$ and $h : Z \longrightarrow Y$ which are compositions of blow-ups and isomorphisms, and such that $f \circ g = h$. If we are able to show that the Kodaira dimension does not change under blow-up, we are then done. Now, let X be a normal projective surface, and let $p : \widetilde{X} \longrightarrow X$ be the blow-up of X along $x \in X$, and let E be the exceptional divisor. By Proposition 3.1.13 we have that $K_{\widetilde{X}} = p^*K_X + E$.

Let D be an effective divisor on $\widetilde{X}$ which is linear equivalent to $mK_{\widetilde{X}}$ for some $m \in \mathbb{N}$. Then we have

$$D \cdot E = mK_{\widetilde{X}} \cdot E = mp^*K_X \cdot E + mE^2 = -m.$$

As D is a divisor on $\widetilde{X}$, we can write it as $D = D' + kE$, where $k \geq 0$ and D' is an effective divisor on $\widetilde{X}$ which does not contain E. Hence $D' = D - kE$, and as D' and E are effective and E is not contained in D' we have

$$0 \leq D' \cdot E = D \cdot E - kE^2 = k - m.$$

Hence $k \geq m$, so that the divisor $D - mE$ is still effective: indeed $D - mE = D' + (k-m)E$, where D' is effective and $k \geq m$. As $D \sim mK_{\widetilde{X}}$, we get $D - mE \sim mK_{\widetilde{X}} - mE$. The map

$$|mK_{\widetilde{E}}| \longrightarrow |m(K_{\widetilde{X}} - E)|, \qquad D \mapsto D - mE$$

is then well-defined and it is clearly an isomorphism. Hence

$$h^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(mK_{\widetilde{X}})) = h^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(m(K_{\widetilde{X}} - E))) =$$

$$= h^0(\widetilde{X}, p^*\mathscr{O}_X(mK_X)) = h^0(X, \mathscr{O}_X(mK_X)).$$

Hence $\kappa(\widetilde{X}) = \kappa(X)$, and we are done. □

The final result we need is the following:

Proposition 3.2.7. *Let X and Y be to smooth projective varieties, ad suppose that $f : X \longrightarrow Y$ is an unramified covering. If D is a divisor on Y, then $\kappa(D) = \kappa(f^*D)$. In particular, $\kappa(X) = \kappa(Y)$.*

Proof. Has f is an unramified covering, one has $f^*K_Y = K_X$. Hence the second part is a consequence of the first one. Now, f is surjective, so that the induced map

$$f^* : H^0(Y, \mathscr{O}_Y(D)) \longrightarrow H^0(X, f^*\mathscr{O}_Y(D))$$

is injective. Hence $\kappa(D) \leq \kappa(f^*(D))$. We then just need to show the converse. As f is a covering, the map $f_* : \pi_1(Y) \longrightarrow \pi_1(X)$ is injective, and there is a normal subgroup $N \lhd \pi_1(X)$ such that $N \subseteq f_*(\pi_1(Y))$, and such that the quotient $\pi_1(X)/N$ is a finite group occurring as a group of desk transformations of a Galois covering $f' : X' \longrightarrow Y$ factoring f. One the reduces to the case where f is a Galois covering.

Clearly, if $\kappa(f^*(D)) = -\infty$, then $\kappa(D) = -\infty$, so we assume $\kappa(f^*(D)) \geq 0$. Let $k \gg 0$ such that $dim(W_{f^*(kD)}) = \kappa(f^*(D))$. If one fixes a basis $\{s_0, ..., s_n\}$ of $H^0(X, f^*\mathscr{O}_Y(kD))$, we have that $\kappa(f^*(D))$ is the transcendence degree of $L := \mathbb{C}(s_1/s_0, ..., s_n/s_0)$. As $f : X \longrightarrow Y$ is a Galois cover, let G be its Galois group, which acts on X, and hence on L, and consider the invariant subfield $K \subseteq L$. Consider the following:

$$\prod_{g \in G} (x - g^*(s_i/s_0)) = x^m + a_1(s_i/s_0)x^{m-1} + ... + a_m(s_i/s_0),$$

for every $i = 1, ..., n$. Hence the field K, which is $G-$invariant, is generated by

$$\{a_1(s_1/s_0), ..., a_1(s_n/s_0), ..., a_m(s_1/s_0), ..., a_m(s_n/s_0)\}.$$

Notice that $t_0 := \prod_{g \in G} g^*(s_0)$ is a $G-$invariant section of $f^*\mathscr{O}_Y(kmD)$, and similarily for $t_0 a_j(s_i/s_0)$. Hence, they define sections of $\mathscr{O}_W(km)$, where $W = im(\varphi_{kD})$. As $a_j(s_i/s_0)$ is a quotient of two $G-$invariant global sections of $f^*\mathscr{O}_Y(kmD)$, they define sections of $\mathscr{O}_{W_{kmD}}$, so that there is finally a canonical inclusion of K inside the field Q of functions of W_{kmD}. Hence the transcendence degree of K is at most the transcendence degree of Q, which is $\kappa(D)$. As the transcendence degree of L is $\kappa(f^*(D))$, and it is at most the transcendence degree of K, we are done. □

To conclude the section, we give some examples:

Example 3.2.1. The first example I describe is the case of a Riemann surface X. The degree of the canonical divisor is very easy to compute, as we have

$$deg(K_X) = 2g(X) - 2,$$

where $g(X)$ is the genus of X. Using Riemann-Roch for curves and Serre's duality, this is easy to prove: indeed, $h^i(X, \mathscr{O}_X) = h^{1-i}(X, \mathscr{O}_X(K_X))$, so that $\chi(\mathscr{O}_X) = -\chi(K_K)$. Hence

$$deg(K_X) - g(X) + 1 = \chi(K_X) = -\chi(\mathscr{O}_X) = -g(X) + 1,$$

so that $deg(K_X) = 2g(X) - 2$.

If $g(X) = 0$, i. e. $X \simeq \mathbb{P}^1$, then $deg(mK_X) = -2m < 0$, so that $P_m(X) = 0$ for every $m > 0$. This implies that $\kappa(\mathbb{P}^1) = -\infty$.

If $g(X) = 1$, i. e. X is an elliptic curve, then $deg(mK_X) = 0$ for every $m > 0$. One can actually show more, namely that $\mathscr{O}_X(K_X) \simeq \mathscr{O}_X$, so that $P_m(X) = 1$ for every $m > 0$. Hence $\kappa(X) = 0$.

If $g(X) \geq 2$, then $deg(K_X) > 0$. Hence K_X is ample, so that $\kappa(X) = 1$.

Example 3.2.2. The Kodaira dimension of $\mathbb{P}^2$ is $\kappa(\mathbb{P}^2) = -\infty$. Indeed, we have $\mathscr{O}_{\mathbb{P}^2}(mK_{\mathbb{P}^2}) = \mathscr{O}(-3m)$, so that $P_m(\mathbb{P}^2) = 0$ for every $m > 0$. As the Kodaira dimension is a birational invariant, we see that $\kappa(\mathbb{P}^1 \times \mathbb{P}^1) = -\infty$.

Example 3.2.3. Let X be a generic hypersurface of degree 4 in $\mathbb{P}^3$, i. e. $X \in |\mathscr{O}(4)|$. As it is generic, by Bertini's Theorem X is smooth, hence normal. Moreover, we can use the adjunction formula to calculate the canonical bundle of X: we have $\mathscr{O}_X(K_X) \simeq \mathscr{O}_{\mathbb{P}^3}(K_{\mathbb{P}^3} + X)_{|X}$. But now notice that $\mathscr{O}_{\mathbb{P}^3}(K_{\mathbb{P}^3}) = \mathscr{O}(-4)$, so that $K_X = 0$. Hence $P_m(X) = 1$ for every $m > 0$. By Proposition 3.2.5, we then get $\kappa(X) = 0$.

3.2.2 Kodaira dimension and nefness

We want to give a precise connection between the Kodaira dimension and the nefness of the canonical bundle. we have already shown that if X is a smooth projective surface and K_X is nef, then $\kappa(X) \geq 0$, and that if K_X is ample, then $\kappa(X) = 2$. It then remains to look at the cases where K_X is not nef. The main result is the following:

Proposition 3.2.8. *Let X be a smooth projective surface.*

1. *If there is an irreducible curve C on X such that $K_X \cdot C < 0$ and $C^2 \geq 0$, then $P_m(X) = 0$ for every $m > 0$.*

2. *If there is $m > 0$ such that $P_m(X) \neq 0$, and there is an irreducible curve C on X such that $K_X \cdot C < 0$, then C is a $(-1)-$curve.*

Proof. For the first item, let me suppose there is $m > 0$ such that $P_m(X) \neq 0$. Then $|mK_X| \neq \emptyset$, and let $D \in |mK_X|$. Clearly, one can write $D = nC + R$, where $k \in \mathbb{Z}$ and R is an effective divisor which does not contain C. Since $m > 0$ and $K_X \cdot C < 0$, we get $D \cdot C < 0$, so that $k \neq 0$ (otherwise $D \cdot C = 0$). Moreover, as D is effective, we need $k \geq 0$, so that finally $k \geq 1$. But now

$$0 > mK_X \cdot C = D \cdot C = kC^2 + R \cdot C \geq kC^2 \geq 0,$$

and we get a contradiction. Hence $P_m(X) = 0$ for every $m > 0$.

For the second item, we can have only two possibilities, namely $K_X \cdot C \leq -2$ or $K_X \cdot C = -1$. In the first case, use the genus formula:

$$0 \leq g(C) = 1 + \frac{1}{2}(K_X \cdot C + C^2) \leq C^2,$$

so that one can apply item 1 to get that $P_m(X) = 0$ for every $m > 0$. This is in contradiction with the hypothesis, so that we need $K_X \cdot C = -1$. By the first item, one needs $C^2 < 0$, and by the genus formula we have

$$0 \leq g(C) = 1 + \frac{1}{2}(C^2 - 1),$$

so that $C^2 \geq -1$. Hence $C^2 = -1$, and the genus formula gives $g(C) = 0$, so that C is a $(-1)-$curve. $\square$

We can use this result to prove the following:

Proposition 3.2.9. *Let X be a smooth projective surface, and suppose that K_X is not nef. Hence either X is not minimal, or $\kappa(X) = -\infty$.*

Proof. As K_X is not nef, there must be an irreducible curve C on X such that $K_X \cdot C < 0$. Let us suppose that $\kappa(X) \geq 0$. Then there must be $m > 0$ such that $P_m(X) \neq 0$, and we can apply Proposition 3.2.8 to get that C is a $(-1)-$curve. Hence X is not minimal. $\square$

Corollary 3.2.10. *Let X be a minimal surface. Then $\kappa(X) = -\infty$ if and only if K_X is not nef.*

Proof. If K_X is not nef and X is minimal, then $\kappa(X) = -\infty$ by Proposition 3.2.9. If $\kappa(X) = -\infty$, then K_X cannot be nef: by Remark 3.2.1, if K_X is nef we have $\kappa(X) \geq 0$. $\square$

There is another important result we need to show, and as a corollary of it we prove that every smooth projective surface whose Kodaira dimension is non-negative has a unique minimal model.

Proposition 3.2.11. *Let X and Y be two smooth projective surfaces, and let $f : X \dashrightarrow Y$ be a birational map.*

1. *If K_X is nef, then $f : X \longrightarrow Y$ is a morphism.*
2. *If K_X and K_Y are nef, then $f : X \longrightarrow Y$ is an isomorphism.*

Proof. Before, we need to prove the following:

Lemma 3.2.12. *Let S be a surface, and let $f : S' \longrightarrow S$ be a birational morphism. If K_S is nef and C' is a curve on S' which is not contracted by f, then $K_{S'} \cdot C' \geq 0$.*

Proof. Since every birational morphism f is a composition of blow-ups and isomorphisms, we just need to look at the case where f s the blow-up of S at some point q. Let E be the exceptional divisor on S', and let us suppose that C' is a curve such that $C := f(C')$ is a curve on X. Hence we have

$$K_{S'} \cdot C' = (f^*K_S + E) \cdot (f^*C - mE) = K_S \cdot C + m \geq 0,$$

as m is the multiplicity of C at q (hence $m \geq 0$). $\square$

Now, let $f : X \dashrightarrow Y$ be a birational map. Let us suppose that K_X is nef, but f is not a morphism: hence we blow up X along at least one point in order to solve the indeterminacies of f. Let E be an exceptional curve on the blow-up $\widetilde{X}$ of X: then $K_{\widetilde{X}} \cdot E = -1$. Indeed, by the genus formula and the fact that E is a $(-1)-$curve we have

$$0 = g(E) = 1 + \frac{1}{2}(K_{\widetilde{X}} \cdot E + E^2 = \frac{1}{2}(K_{\widetilde{X}} \cdot E + 1).$$

Now, notice that if $p : \widetilde{X} \longrightarrow X$ is obtained by the minimal number of blow-ups so that $f \circ p : \widetilde{X} \longrightarrow Y$ is a morphism, the exceptional divisor of the last blow-up is not contracted by f, so that by Lemma 3.2.12 we have a contradiction. In conclusion, f is a morphism

For the remaining item, we just apply item 1 to $f^{-1} : Y \dashrightarrow X$: as K_Y is nef, it is a morphism, which is inverse to f, so that f is an isomorphism. $\square$

Corollary 3.2.13. *Let X be a smooth projective surface such that $\kappa(X) \geq 0$. Then X has a unique (up to isomorphism) minimal model.*

Proof. Let X' and X'' be two minimal models for X. In particular, X' and X'' are birational to X, and hence there is a birational map $f : X' \dashrightarrow X''$. As the Kodaira dimension is a birational invariant, we have that $\kappa(X') = \kappa(X'') = \kappa(X) \geq 0$. Moreover, as X' and X'' are minimal, the fact that $\kappa(X') = \kappa(X'') \geq 0$ implies that $K_{X'}$ and $K_{X''}$ are nef: indeed, if $K_{X'}$ is not nef, as X' is minimal this implies that $\kappa(X') = -\infty$ by Proposition 3.2.9. Hence we can apply Proposition 3.2.11, and f is an isomorphism. $\square$

3.3 Albanese tori

This section is devoted to introduce a very general tool in algebraic geometry, which is the Albanese torus of every complex manifold. This will be used later in the proof of the Enriques-Castelnuovo birational classification of smooth projective surfaces. Let me start with the following: let V be a complex vector space of dimension d, and let Γ be a maximal rank lattice in V. Let $T := V/\Gamma$, which has the structure of compact complex manifold, and which is called complex torus of dimension d.

Lemma 3.3.1. *Let $\gamma \in \Gamma$, and let $t_T(\gamma)$ be the algebraic $1-$cycle on T defined by the line $[0, \gamma] \subseteq V$. Then the map*

$$t_T : \Gamma \longrightarrow H_1(T, \mathbb{Z}), \qquad \gamma \mapsto t_T(\gamma)$$

is an isomorphism. The map

$$\tau_T : V^* \longrightarrow H^0(T, \Omega_T), \qquad \tau_T(f) := df$$

is an isomorphism. Moreover, for every $\gamma \in \Gamma$ and every $f \in V^*$ we have $\int_{t_T(\gamma)} \tau_T(f) = f(\gamma)$.

Using this lemma, one can easily produce a morphism

$$j : H_1(X, \mathbb{Z}) \longrightarrow H^0(X, \Omega_X)^*.$$

Indeed, if $\alpha \in H_1(X, \mathbb{Z})$, by Lemma 3.3.1 there is $\gamma \in \Gamma$ such that $\alpha = t_T(\gamma)$. Now simply define

$$j(\alpha) : H^0(X, \Omega_X) \longrightarrow \mathbb{C}, \qquad j(\alpha)(\omega) := \int_\gamma \omega.$$

The following definition hence makes sense:

Definition 3.3.1. The *Albanese torus* of a compact Kähler manifold X is

$$Alb(X) := H^0(X, \Omega_X)^*/im(H_1(X, \mathbb{Z})).$$

Proposition 3.3.2. *If X is a compact Kähler manifold, then $Alb(X)$ is a complex torus of dimension $q(X)$.*

Proof. One just needs to verify that $im(H_1(X,\mathbb{Z}))$ is a maximal rank lattice in $H^0(X,\Omega_X)^*$. To do this, first of all notice that as X is compact and Kähler, then the Hodge decomposition shows that $b_1(X) = 2q(X)$. Let $\{\omega_1,...,\omega_{q(X)}\}$ be a basis for $H^0(X,\Omega_X)$, and let $\{\alpha_1,...,\alpha_{2q(X)}\}$ be a basis for $H_1(X,\mathbb{Z})$. For every $i = 1,...,2q(X)$, let $v_i \in \mathbb{C}^{q(X)}$ be the complex vector whose $j-$th component is $\int_{t_T^{-1}(\alpha_i)} \omega_j$. If these vectors are linearily dependent over $\mathbb{R}$, then there are non-trivial $\lambda_1,...,\lambda_{2q(X)} \in \mathbb{R}$ such that for every $j = 1,...,q(X)$ we have

$$\sum_{i=1}^{2q(X)} \lambda_i \int_{t_T^{-1}(\alpha_i)} \omega_j = 0.$$

Hence for every $\omega \in H^0(X,\Omega_X)$ we have

$$\sum_{i=1}^{2q(X)} \lambda_i \int_{t_T^{-1}(\alpha_i)} \omega = 0.$$

As this is over $\mathbb{R}$, this equality holds for $\overline{\omega}$ too. By the Hodge decomposition, the linear functional $\sum_{i=1}^{2q(X)} \lambda_i \int_{t_T^{-1}(\alpha_i)}$ is then trivial on $H^1(X,\mathbb{C})$. As this is the dual of $\sum_{i=1}^{2q(X)} \lambda_i\alpha_i$, we finally get $\lambda_i = 0$ for every $i = 1,...,2q(X)$, so that all the vectors v_i are linearily independent over $\mathbb{R}$. But clearly this means that $im(H_1(X,\mathbb{Z}))$ is a maximal rank lattice of $H^0(X,\Omega_X)^*$, and we are done. $\square$

The main property of the Albanese torus of X is that one can define the following map: fix a point $x_0 \in X$. For every point $x \in X$ we can consider any path $\gamma(x_0,x)$ from x_0 to x, and we can define

$$a_X : X \longrightarrow Alb(X), \qquad a_X(x) := \int_{\gamma(x_0,x)}.$$

This is a well defined map (it does not depend on $\gamma(x_0,x)$), and moreover it is holomorphic.

Definition 3.3.2. The map a_X is the *Albanese map* of X.

Proposition 3.3.3. *The map $a_X^* : H^0(Alb(X),\Omega_{Alb(X)}) \longrightarrow H^0(X,\Omega_X)$ induced by a_X is an isomorphism. Moreover, we have that $a_X^* = \tau_{Alb(X)}^{-1}$.*

Proof. As $\tau_{Alb(X)}$ is an isomorphism, we just need to show that $a_X^* = \tau_{Alb(X)}^{-1}$. Let $\omega \in H^0(X,\Omega_X)$: we show that $a_X^*(\tau_{Alb(X)}(\omega)) = \omega$. As $a_X^*(\tau_{Alb(X)}(\omega))$ is a holomorphic $1-$form on X, we can evaluate it at every point $x \in X$. Let U be an open neighborhood of x in X, and let $q : H^0(X,\Omega_X)^* \longrightarrow Alb(X)$ be the projection. Hence

$$a_X^*(\tau_{Alb(X)}(\omega))(x) = a_X^*(q^*(\tau_{Alb(X)}(\omega)))(x) = a_X^*(d((\omega,.)))(x) =$$

$$= d((\omega, a_X(x))) = d\left(\int_{\gamma(x_0,x)} \omega \right) = \omega(x).$$

□

Example 3.3.1. Let X be a Riemann surface: then $Alb(X) = Jac(X)$. Indeed

$$Alb(X) = H^0(X, \mathscr{O}_X(K_X))^*/im(H_1(X, \mathbb{Z})),$$

and

$$Jac(X) = H^1(X, \mathscr{O}_X)/im(H^1(X, \mathbb{Z})).$$

But now using Serre's duality one sees that $H^0(X, \mathscr{O}_X(K_X))^* \simeq H^1(X, \mathscr{O}_X)$. Moreover, this respects the duality between $H_1(X, \mathbb{Z})$ and $H^1(X, \mathbb{Z})$, and we are done. Moreover, one has that the Albanese map is the Abel-Jacobi map for X.

The Albanese variety and the Albanese map satisfy a universal property as follows:

Theorem 3.3.4. *(**Universal property of the Albanese**). Let X be a compact Kähler manifold, T a complex torus and $f : X \longrightarrow T$ a morphism. Then there is a (unique) map $g : Alb(X) \longrightarrow T$ such that $f = g \circ a_X$.*

Proof. The uniqueness of g is given by the commutativity. Let us prove the following:

Lemma 3.3.5. *Let V and V' be two complex vector spaces, and let $\Gamma \subseteq V$ and $\Gamma' \subseteq V'$ be two maximal rank lattices. Moreover, let $T := V/\Gamma$ and $T' := V'/\Gamma'$.*

1. *Every linear map $H : V \longrightarrow V'$ such that $H(\Gamma) \subseteq \Gamma'$ induces a morphism $h : T \longrightarrow T'$.*
2. *Every morphism $h : T \longrightarrow T'$ is induced by the composition of a translation and of a linear map $H : V \longrightarrow V'$ such that $H(\Gamma) \subseteq \Gamma'$. If H^t is the transpose of H, then we have $\tau_{T'} \circ H^t = h^* \circ \tau_T$.*

Proof. Item 1 is clear. For item 2: by the universal property of liftings, the map h lifts to a holomorphic map $\widetilde{h} : V \longrightarrow V'$ such that for every $v \in V$ and every $\gamma \in \Gamma$ we have $\widetilde{h}(v + \gamma) - \widetilde{h}(v) \in \Gamma'$. As $\widetilde{h}$ is holomorphic and Γ' is discrete, the value of $\widetilde{h}(v + \gamma) - \widetilde{h}(v)$ does not depend on v, hence all partial derivatives of $\widetilde{h}$ are invariant under translation via an element in Γ. They define hence a holomorphic function on T. As T is compact, they have to be constant, so that $\widetilde{h}(v) = H(v) + k$, where k is a constant vector and $H : V \longrightarrow V'$ is a linear map such that $H(\Gamma) \subseteq \Gamma'$. The rest is clear. □

Let us now consider $f : X \longrightarrow T$, where $T = V/\Gamma$ for some complex vector space V and some maximal rank lattice Γ in V. Consider the linear map

$$G := \tau_{Alb(X)} \circ f^* : H^0(T, \Omega_T) \simeq V \longrightarrow H^0(Alb(X), \Omega_{Alb(X)}).$$

By Proposition 3.3.3 we have $\tau_{Alb(X)}^{-1} = a_X^*$, so that we finally get $f^* = a_X^* \circ G$. Moreover, let $\widetilde{F} : H^0(X, \Omega_X)^* \longrightarrow V$ be a map such that $\tau_{Alb(X)}^{-1} \circ \widetilde{F}^t = G \circ \tau_T$. By construction hence $\widetilde{F}^t = f^* \circ \tau_T$.

Let me prove that $\widetilde{F}$ preserves the lattices: let $\gamma \in H_1(X, \mathbb{Z})$, and let $h \in V^*$. Hence

$$h\left(\widetilde{F} \int_\gamma\right) = \int_\gamma \widetilde{F}^t(h) = \int_\gamma (f^* \circ \tau_T)(h) = \int_{f_*\gamma} \tau_T(h).$$

But now by Lemma 3.3.1 this last is $h(\tau_T^{-1} f_*\gamma)$. As this is true for every $\gamma \in H_1(X, \mathbb{Z})$ and every $h \in V^*$, this simply means that $\tau_T^{-1} \circ f_* = \widetilde{F}$. Hence $f_* = \tau_T \circ \widetilde{F}$, and the lattice are hence preserved.

Now, by Lemma 3.3.5 the map $\widetilde{F}$ induces a map $g : Alb(X) \longrightarrow T$, which is unique up to translation. But $g(0) = f(x_0)$, so that g is unique. □

Remark 3.3.1. If $q(X) = 0$, then $Alb(X)$ is simply a point. The universal property of the Albanese variety then implies that every map $f : X \longrightarrow T$, where T is a complex torus, is constant.

Remark 3.3.2. The Albanese variety $Alb(X)$ is the smallest subtorus of $Alb(X)$ generated by the image of a_X. This follows easily by the universal property of the Albanese.

Remark 3.3.3. The Albanese variety is functorial: this follows immediately from the universal property. In particular, if $f : X \longrightarrow Y$ is a morphism, it induces a morphism $Alb(f) : Alb(X) \longrightarrow Alb(Y)$. If f is surjective, by the previous remark one gets that $Alb(f)$ is surjective.

Let me now prove the last result we need on Albanese varieties: in general the Albanese map is not surjective, but one can say something in the case where the image is a curve.

Proposition 3.3.6. *If the image of a_X is a curve C in $Alb(X)$, then a_X has connected fibers. Moreover, C is smooth and $g(C) = q(X)$.*

Proof. We have $a_X : X \longrightarrow C$, and consider its Stein factorisation, i. e. a manifold Y together with two morphisms $f : X \longrightarrow Y$ with connected fibers, and $g : Y \longrightarrow C$ which is finite. The manifold X is normal, hence Y is normal. Moreover, it has to be a curve, hence Y is a smooth curve. The morphism f induces a morphism $Alb(f) : Alb(X) \longrightarrow Alb(Y)$, and we know that $Alb(Y) = Jac(Y)$. Moreover, the map g induces a morphism $Alb(g) : Jac(Y) \longrightarrow Alb(X)$. Moreover, as f is surjective, we have $Alb(f)$ surjective. By functoriality we have

$$Alb(g) \circ Alb(f) \circ a_X = g \circ f = a_X.$$

By the universal property of the Albanese, we then get $Alb(g) \circ Alb(f) = id_{Alb(X)}$. Now, $Alb(f)$ is surjective, hence it has to be an isomorphism with inverse $Alb(g)$. But now the map a_Y is injective (as Y is a smooth curve), so that $g = Alb(g) \circ a_Y$ is injective, hence g is an isomorphism. In conclusion the fibers of a_X are the fibers of f, hence they are connected. Moreover C is smooth. Finally, simply notice that $g(C) = g(Y)$, and $g(Y) = dim(Jac(Y))$. As $Alb(f)$ is an isomorphism, then

$$g(C) = dim(Alb(X)) = q(X),$$

and we are done. $\square$

Chapter 4

The Enriques-Castelnuovo Classification

This chapter is devoted to the proof of the Enriques-Castelnuovo birational classification of smooth projective surfaces. As we have seen in the previous chapter, if X is a smooth projective surface, one can perform a very natural transformation on X, which is the blow-up of X along a point x. The product of such a transformation is a smooth projective surface $\widetilde{X}$ together with a proper morphism $p : \widetilde{X} \longrightarrow X$. This map is not an isomorphism: its restriction to $\widetilde{X} \setminus p^{-1}(x)$ gives an isomorphism with $X \setminus \{x\}$, but is contracts a the smooth rational curve $p^{-1}(x)$ to the point x. Hence p is not an isomorphism. Moreover, on $\widetilde{X}$ there is a curve more than on X, so that $\rho(\widetilde{X}) = \rho(X) + 1$, so that $\widetilde{X}$ and X cannot be isomorphic.

It is then unreasonable to expect a classification of smooth projective surfaces up to isomorphism: one can consider $p_1, ..., p_n$ different points on X, and consider $\widetilde{X}_i := Bl_{p_1,...,p_i}(X)$: we obtain $n+1$ surfaces which are not isomorphic, but which are strictly related. They are, indeed, birational. Hence, it seems reasonable to expect that it is possible to perform a birational classification of smooth projective surfaces: this was the main idea Enriques and Castelnuovo had, and it is the starting point of the classification. Indeed, from Theorem 3.1.18, we know that every birational map between two smooth projective surfaces is composition of isomorphisms and blow-ups. Moreover, from Theorem 3.1.20 we know that a surface is the blow-up of another one if and only if it carries a $(-1)-$curve. A surface which does not contain $(-1)-$curves is minimal, and we have seen that every smooth projective surface admits a minimal model, which is again a smooth projective surface. The birational classification of smooth projective surfaces is then the classification, up to isomorphism, of minimal smooth projective surfaces.

A first attempt to classify minimal smooth projective surfaces is to use their

Kodaira dimension: it is a birational invariant, hence two minimal surfaces with different Kodaira dimension cannot be isomorphic. As seen in the previous chapter, there is a first distinction: surfaces with Kodaira dimension $-\infty$, and surfaces of non-negative Kodaira dimension. For minimal surfaces, this can be read in terms of the nefness of their canonical bundle: those surfaces whose canonical bundle is not nef are in the first family, otherwise they are in the second one.

Anyway, a classification by means of Kodaira dimension is too rough: there are several examples of surfaces having completely different characteristics, but having the same Kodaira dimension. One could then try to give a finer classification, for example using topological and analytic invariants like the Betti numbers, the irregularity and the plurigenera. The Betti numbers are not birational invariants, hence they cannot e used for our purposes; anyway, irregularity and plurigenera are important birational invariants, as we are going to show. Our goal il to show the following:

Theorem 4.0.7. ***(Enriques-Castelnuovo's birational classification)****. Let X be a smooth projective surface. A minimal model of X is then one of the following:*

1. *Kodaira dimension $-\infty$:*
    - *rational surface;*
    - *ruled surface;*
2. *Kodaira dimension 0:*
    - *abelian surface;*
    - *K3 surface;*
    - *Enriques surface;*
    - *bielliptic surface;*
3. *Kodaira dimension 1: proper elliptic surface;*
4. *Kodaira dimension 2: surface of general type.*

Clearly, there are still several things to define in the statement of the previous Theorem, and this will be one of the major goals of the following sections. We start with the classification of minimal surfaces of Kodaira dimension $-\infty$, and we will define rational and ruled surfaces. We will next prove the remaining part of the classification, together with definitions.

4.1 The Kawamata-Mori Rationality Theorem

The Kawamata-Mori Rationality Theorem is one of the most important results in Mori Theory, and was proven at the beginning of the 80s. It came hence much later then the Enriques-Castelnuovo Classification: it was however soon realized that this Theorem, which holds in greater generality than the one we are going to state, gives a serious shortcut in the proof of the classification. As this result is basic in the context of the MMP, I decided to prove and use it (following the modern fashion of view the original birational classification of surfaces as the toy case for the MMP).

Before stating and proving the theorem, I need to recall some basic facts we have seen in Chapter 2. If X is a smooth projective surface, on the real vector space $N^1_{\mathbb{R}}(X)$ we have a non-degenerate intersection form, and one may view a divisor D either as an element of $N^1_{\mathbb{R}}(X)$, or as the hyperplane

$$D^{\perp} := \{\alpha \in N^1_{\mathbb{R}}(X) \,|\, \alpha \cdot D = 0\}.$$

Moreover, in $N^1_{\mathbb{R}}(X)$ we have the important cone $Eff(X)$, whose dual cone is $Nef(X)$: a divisor D is nef if and only if $D \cdot \alpha \geq 0$ for every $\alpha \in Eff(X)$. If one uses the notation

$$N^1_{\mathbb{R}}(X)_{D\geq 0} := \{\alpha \in N^1_{\mathbb{R}}(X) \,|\, \alpha \cdot D \geq 0\},$$

and similar for $N^1_{\mathbb{R}}(X)_{D\geq 0}$, $N^1_{\mathbb{R}}(X)_{D>0}$ and $N^1_{\mathbb{R}}(X)_{D<0}$. Using this notation, we easily see that D is nef if and only if $Eff(X)$ is contained in $N^1_{\mathbb{R}}(X)_{D\geq 0}$, and that D is ample if and only if $Eff(X) \setminus \{0\}$ is contained in $N^1_{\mathbb{R}}(X)_{D>0}$.

If D is not nef, then there is a part of $Eff(X)$ lying in $N^1_{\mathbb{R}}(X)_{D<0}$. Moreover, if $s \in \mathbb{R}$ and H is ample, we know that $H + sD$ is ample for $0 < |s| \ll 0$, but for $s \gg 0$ we have that $H + sD$ is not nef, as D is not nef. The hyperplane $(H+sD)^{\perp}$ moves in $N^1_{\mathbb{R}}(X)$ as s changes: for small values of s the cone $Eff(X)$ is contained in the positive hyperspace it defines, but for high values of s there will be a part of $Eff(X)$ contained in the negative hyperspace. Hence, there is $s_0(H, D) \in \mathbb{R}$ such that $H + sD$ is nef for every $s \leq s_0(H, D)$, and $H + sD$ is not nef for every $s > s_0(H, D)$, i. e.

$$s_0(H, D) := sup\{s \in \mathbb{R} \,|\, H + sD \in Nef(X)\}.$$

Example 4.1.1. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$, and consider $D = K_X$. Using the isomorphism $N^1_{\mathbb{R}}(X) \simeq \mathbb{R}^2$ which sends h_1, h_2 to the canonical basis of $\mathbb{R}^2$, we have that $D = (-2, -2)$. As we have seen in Example 2.5.2, D is not nef, and

$$Eff(X) = Nef(X) = \{(x, y) \in \mathbb{R}^2 \,|\, x, y \geq 0\}.$$

Let us consider $H = (1, 2)$, which is an ample divisor, so that $H + sD = (1 - 2s, 2 - 2s)$. The line $(H + sD)^{\perp}$ is then

$$(H + sD)^{\perp} = \{(x, y) \in \mathbb{R}^2 \,|\, (2 - 2s)y + (1 - 2s)x = 0\}.$$

Now, notice that if $s = 0$, then the line $H^{\perp}$ has equation $y = -x/2$, and for $0 \leq s \leq 1/2$ the effective cone is always in the positive half-plane defined by $H + sD$. For $s > 1/2$, this is no longer true, so that $s_0(H, K_X) = 1/2$.

In this example we have seen that $s_0(H, K_X) \in \mathbb{Q}$: this is the general phenomenon described by the Kawamata-Mori Rationality Theorem:

Theorem 4.1.1. ***(Kawamata-Mori's Rationality Theorem)****. Let X be a smooth projective surface, H an ample divisor on X. Suppose that K_X is not nef. Then $s_0(H, K_X) \in \mathbb{Q}$.*

Proof. For the proof we need the following result in number theory, which I won't prove:

Lemma 4.1.2. ***(Hardy-Wright)****. Let $\alpha \in \mathbb{R}$, $\alpha \neq 0$, and let $\epsilon \in \mathbb{Q}_{>0}$. Moreover, let*

$$W_{\alpha,\epsilon} := \{(x, y) \in \mathbb{R}^2 \mid \alpha x < y < \alpha x + \epsilon\}.$$

If $\alpha \notin \mathbb{Q}$, then $W_{\alpha,\epsilon} \cap (\mathbb{N} \times \mathbb{N})$ is infinite.

Now, let $u, v \in \mathbb{N}$, and let $D_{u,v} := uH + vK_X$, and let $W := W_{s_0(H,K_X),1}$ as in the previous lemma. If $(u, v) \in W$, then $D_{u,v} - K_X$ is ample: we have $D_{u,v} - K_X = uH + (v-1)K_X$, and $s_0(H, K_X) < v/u < s_0(H, K_X) + 1/u$; hence $(v-1)/u = v/u - 1/u < s_0(H, K_X)$, and by definition of $s_0(H, K_X$ we have that $D_{u,v} - K_X$ is ample. By the Kodaira Vanishing Theorem we can then conclude that $h^i(X, \mathscr{O}_X(D_{u,v})) = 0$ for $i = 1, 2$ and for every $(u, v) \in W$. Let now

$$P(u, v) := \chi(uH + vK_X) =$$

$$= \frac{1}{2}(H^2u^2 + K_X^2v^2 + 2H \cdot K_Xuv + H \cdot K_Xu + K_X^2v) + \chi(\mathscr{O}_X),$$

which is a quadratic polynomial in u, v. Notice that the coefficient of u^2 is $H^2 > 0$, as H is ample, so that $P(u, v)$ is not identically 0.

Let us suppose that $s_0(H, K_X) \notin \mathbb{Q}$. By Lemma 4.1.2 there is in infinite number of $u, v \in \mathbb{N}$ such that $(u, v) \in W$. There are two possibilities: either $P(u, v) = 0$ for every $(u, v) \in W$, or there is $(u, v) \in W$ such taht $P(u, v) \neq 0$. Let us show that the first possibility can be excluded.

Indeed, since $s_0(H, K_X) \notin \mathbb{Q}$, then there are arbitrary large $p, q \in \mathbb{N}$ such that $s_0(H, K_X) < p/q < s_0(H, K_X) + 1/3q$. Hence, if $k = 1, 2, 3$ then the couple $(kq, kp) \in W$ for arbitrary large integers p, q. If $P(u, v) = 0$ for every $(u, v) \in W$, then $P(kq, kp) = 0$ for $k = 1, 2, 3$, so that $P(u, v)$ restricted to the line $\mathbb{R}_{>0}(p, q)$ vanishes at three different points. As it is a quadratic polynomial, it has to vanish on the whole line. As one can consider infinite many lines, then $P(u, v)$ would be identically zero, which is not possible.

In conclusion, there is $(u, v) \in W$ such that $P(u, v) \neq 0$. Now, recall that for this choice of (u, v) we have $h^0(X\mathscr{O}_X(D_{u,v})) = P(u, v)$, so that there must be an effective divisor $L \sim uH + vK_X$: we can then suppose $D_{u,v}$ to be effective. But as $(u,v) \in W$, we have $v/u > s_0(H,K_X)$, so that $D_{u,v}$ is not nef. There is then an irreducible curve C such that $D_{u,v} \cdot C < 0$. As $D_{u,v}$ is effective, this forces C to be one of the components of $D_{u,v}$. Let these be $D_1, ..., D_r$: without loss of generality, we can suppose $C = D_1$. Resuming, we have

$$\left(H + \frac{v}{u}K_X\right) \cdot D_1 < 0, \qquad (H + s_0(H,K_X)K_X) \cdot D_1 \geq 0.$$

We can even say more: as $H \cdot K_X, K_X^2 \in \mathbb{Z}$ and $s_0(H,K_X) \notin \mathbb{Q}$, then $(H + s_0(H,K_X)K_X) \cdot D_1 > 0$. Then, there must be some $\rho_1 \in \mathbb{Q}_{>0}$ such that $(H + \rho_1 K_X) \cdot D_1 \geq 0$ and $s_0(H,K_X) < \rho_1 < v/u$. We can repeat the same argument for every irreducible component of $D_{u,v}$, getting for each $i = 1, ..., r$ a positive rational number ρ_i such that $s_0(H,K_X) < \rho_i < v/u$, and $(H + \rho_i K_X) \cdot D_i > 0$. Let now

$$\rho := min\{\rho_1, ..., \rho_r\}.$$

Then $s_0(H,K_X) < \rho < v/u$ and $(H + \rho K_X) \cdot D_i \geq 0$ for every $i = 1, ..., r$.

Now, the divisor $H + \rho K_X$ is nef. Let C be an irreducible curve different from $D_1, ..., D_r$; we need to check that $(H + \rho K_X) \cdot C \geq 0$. We have

$$(H+\rho K_X)\cdot C = \rho\left(\frac{1}{\rho}H + K_X\right)\cdot C = \rho\left(\left(\frac{u}{v}H + K_X\right)\cdot C + \left(\frac{1}{\rho} - \frac{u}{v}\right)H\cdot C\right) =$$

$$= \rho\left(\frac{1}{v}D_{u,v}\cdot C + \left(\frac{1}{\rho} - \frac{u}{v}\right)H \cdot C\right) > 0,$$

as $\rho > 0$, $D_{u,v}$ is effective and C is not one of its components, $\rho < v/u$ and H is ample.

Hence $H + \rho K_X$ is nef, so that $\rho \leq s_0(H,K_X)$, which is a contradiction. In conclusion, we need $s_0(H,K_X) \in \mathbb{Q}$. □

This result is appearently innocent, but it is really far from that, as we are going to see in the next section.

4.2 Kodaira dimension $-\infty$

Using the Kawamata-Mori Rationality Theorem we can proceed to the classification of smooth projective surfaces having Kodaira dimension $-\infty$. We first introduce some examples, in particular rational and geometrically ruled surfaces. We will then proceed to a classification.

4.2.1 Ruled surfaces

Here we introduce three main definitions: rational surfaces, ruled surfaces and geometrically ruled surfaces. In this section we show some of their important features, in particular which of them are minimal, and their numerical birational invariants.

Definition 4.2.1. A smooth projective surface X is *ruled* if it is birational to a product $C\times\mathbb{P}^1$, where C is a smooth curve. If C is $\mathbb{P}^1$, then X is called *rational.*

Notice that every rational surface is then birational to $\mathbb{P}^2$, as $\mathbb{P}^1\times\mathbb{P}^1$ is birational to $\mathbb{P}^2$. Conversely, every smooth projective surface which is birational to $\mathbb{P}^2$ is rational.

Definition 4.2.2. A smooth projective surface X is *geometrically ruled* if there is a surjective morphism $p: X \longrightarrow C$ of maximal rank, where C is a smooth curve, and the fibers of p are all isomorphic to $\mathbb{P}^1$.

A first important remark is that it is by no means clear which is the realtion between ruled and geometrically ruled surfaces. Here is the first property:

Proposition 4.2.1. *Let X be a surface, C a curve and $p: X \longrightarrow C$ a surjective morphism. If $c\in C$ is a regular value for p such that $p^{-1}(c)\simeq\mathbb{P}^1$, then there is a Zariski-open neighborhood U of c in C such that $p^{-1}(U)\simeq U\times\mathbb{P}^1$ in a fiber preserving manner. In particular, every geometrically ruled surface is ruled.*

Proof. The proof is rather long, so it will be divided in several steps.

Step 1. We show that $H^2(X,\mathscr{O}_X)=0$. This is easy: indeed, by Serre's Duality we have $H^2(X,\mathscr{O}_X)=0$ if and only if $H^0(X,\mathscr{O}_X(K_X))=0$. Let us suppose that this last is not verified, so that there is an effective divisor K_X. Let F be the fiber over c, which is a smooth rational curve. Since $F^2=0$, by the genus formula we get

$$0=1+\frac{1}{2}K_X\cdot F,$$

so that $K_X\cdot F=-2$. However, K_X is effective, so that $K_X=nF+G$, where $n\in\mathbb{N}$ and G is effective and not containing F. Hence

$$-2=K_X\cdot F=(nF+G)\cdot F)=G\cdot F\geq 0,$$

which is not possible. In conclusion $p_g(X)=0$.

Step 2. As $p_g(X)=0$, the first Chern class morphism is surjective. We want to produce a divisor H such that $H\cdot F=1$. To do that, it is then sufficient to find a class $h\in H^2(X,\mathbb{Z})$ such that $h\cdot f=1$, where $f=c_1(\mathscr{O}_X(F))$. Let $\alpha\in H^2(X,\mathbb{Z})/tors$, and let us consider $\alpha\cdot f\in\mathbb{Z}$: the set

$$I_f:=\{\alpha\cdot f\in\mathbb{Z}\,|\,\alpha\in H^2(X,\mathbb{Z})/tors\}$$

is an ideal of $\mathbb{Z}$, hence there must be $d\in\mathbb{Z}$ such that $I_f=(d)$. Let us consider the linear functional

$$\delta: H^2(X,\mathbb{Z})/tors\longrightarrow\mathbb{Z},\qquad \delta(\alpha):=\frac{\alpha\cdot f}{d}.$$

This is then an element in $(H^2(X,\mathbb{Z})/tors)^*$, so that by Poincaré duality there must be $f' \in H^2(X,\mathbb{Z})/tors$ such that $\delta(\alpha) = \alpha \cdot f'$ for every $\alpha \in H^2(X,\mathbb{Z})/tors$. Hence we have $f \cdot \alpha = df' \cdot \alpha$ for every $\alpha \in H^2(X,\mathbb{Z})/tors$, so that $f = df'$.

If we show that $f = f'$, i. e. $d = 1$, we are done: we would have $I_f = \mathbb{Z}$, so there is a divisor H such that $H \cdot F = 1$. To do so, let $k = c_1(\mathscr{O}_X(K_X))$. As $f \cdot k = -2$ by Step 1, we need $f' \cdot k = -2/d$, so that $d = 1, 2$. Now, let α be the class of an irreducible smooth curve: then we can use the genus formula to get that $\alpha^2 + \alpha \cdot k$ is even. By linearity, this is then true on the whole $H^2(X,\mathbb{Z})/tors$. Moreover, notice that $f' \cdot k + (f')^2 = f' \cdot k$, which has then to be even, so that $-2/d$ is even. In conclusion $d = 1$.

Step 3. Consider now the divisor H such that $H \cdot F = 1$, and consider the exact sequence

$$0 \longrightarrow \mathscr{O}_X(H + (r-1)F) \longrightarrow \mathscr{O}_X(H + rF) \longrightarrow \mathscr{O}_F(H + rF) \longrightarrow 0.$$

Notice that

$$deg(\mathscr{O}_F(H + rF)) = (H + rF) \cdot F = H \cdot F = 1,$$

so that $\mathscr{O}_F(H + rF) = \mathscr{O}(1)$. As $H^2(F, \mathscr{O}(1)) = 0$, the map

$$b_r : H^1(X, \mathscr{O}_X(H + (r-1)F)) \longrightarrow H^1(X, \mathscr{O}_X(H + rF))$$

is surjective for every $r \in \mathbb{N}$. The sequence decreases for r increasing, so that there is $r_0 \in \mathbb{N}$ such that for every $r > r_0$ the map b_r is an isomorphism. Hence, the map

$$a_r : H^0(X, \mathscr{O}_X(H + rF)) \longrightarrow H^0(F, \mathscr{O}(1))$$

is surjective for $r \gg 0$.

Consider a plane V of $H^0(X, \mathscr{O}_X(H + rF))$ which is mapped isomorphically by a_r onto $H^0(F, \mathscr{O}(1))$, and consider the corresponding pencil $P = \mathbb{P}(V)$. As $\mathscr{O}(1)$ is very ample and $V \simeq H^0(F, \mathscr{O}(1))$ via a_r, we have that P separates points of F. The possible fixed locus Z of P is then given by points lying in fibers distinct from F, or of curves in fibers disjoint from F. Let us consider $U := C \setminus p(Z)$, which is a Zariski-open subset of C containing c. Moreover, let $P' := P_{|p^{-1}(U)}$. The generic member C_t of the moving part of P meets F in only one point (as $C_t \sim H + rF$, so that $C_t \cdot F = H \cdot F = 1$), so that if it is reducible is must contain some fibers. If it was the case, another C_s in the moving part of P, with $s \neq t$, would meet C_t in the intersection points of these fibers, and hence they would be base points. Hence P', which is the moving part of P, consists exactly of sections of $p_{|p^{-1}(U)}$. It then defines a map

$$g : p^{-1}(U) \longrightarrow \mathbb{P}^1,$$

sending a point $x \in p^{-1}(U)$ to the intersection point of C_t with F, where C_t is the only element of P' passing through x.

Notice that the fibers of g meet the fibers of p at exactly one point: the morphism

$$(p, g) : p^{-1}(U) \longrightarrow U \times \mathbb{P}^1$$

is then an isomorphism preserving the fibers. $\square$

Remark 4.2.1. In the proof we have shown that if X is a geometrically ruled surface, then $p_g(X) = 0$. Moreover, we have shown that there is always a divisor H such that $H \cdot F = 1$, where F is the class of a fiber.

We have a lot more informations about the structure of geometrically ruled surfaces: we know indeed how to produce them all.

Theorem 4.2.2. *Let $p : X \longrightarrow C$ be a geometrically ruled surface. Then there is a rank 2 vector bundle E on C and an isomorphism $f : X \longrightarrow \mathbb{P}(E)$ such that $\pi \circ f = p$, where $\pi : \mathbb{P}(E) \longrightarrow C$ is the canonical projection. Moreover, two geometrically ruled surfaces $\mathbb{P}(E)$ and $\mathbb{P}(E')$ over C are isomorphic if and only if there is a line bundle $\mathscr{L}$ on C such that $E \simeq E' \otimes \mathscr{L}$.*

Proof. Consider the sheaf $GL(2, \mathscr{O}_C)$ defined in the following way: for every open subset $U \subseteq C$, let

$$GL(2, \mathscr{O}_C)(U) := GL(2, \mathscr{O}_C(U)),$$

with the obvious restriction morphisms. The quotient of $GL(2, \mathscr{O}_C)$ under the natural action of $\mathbb{C}^*$ defines another sheaf, denoted $PGL(2, \mathscr{O}_C)$. It is easy to show that the isomorphism classes of rank 2 vector bundles on C are classified by $H^1(C, GL(2, \mathscr{O}_C))$, and that the isomorphism classes of $\mathbb{P}^1-$bundles on C are classified by $H^1(C, PGL(2, \mathscr{O}_C))$. We have an evident exact sequence of sheaves

$$1 \longrightarrow \mathscr{O}_C^* \longrightarrow GL(2, \mathscr{O}_C) \longrightarrow PGL(2, \mathscr{O}_C) \longrightarrow 1,$$

which in cohomology gives the exact sequence

$$Pic(C) \xrightarrow{a} H^1(C, GL(2, \mathscr{O}_C)) \xrightarrow{p} H^1(C, PGL(2, \mathscr{O}_C)) \longrightarrow H^2(C, \mathscr{O}_C^*).$$

First of all, notice that $p(e) = p(e')$ if and only if there is $\mathscr{L} \in Pic(X)$ such that $e = e' \cdot a(\mathscr{L})$, and the second part is shown. For the first part, one just needs to prove that $H^2(C, \mathscr{O}_C) = 0$, which follows immediately from the exponential sequence of C, as C is a curve. $\square$

4.2.2 Invariants of geometrically ruled surfaces

Let me calculate some important invariants of geometrically ruled surfaces. By Theorem 4.2.2, every geometrically ruled surface X on a smooth curve C is the projective bundle associated to a rank 2 vector bundle E on C. By Remark 4.2.1 there is a divisor H such that $H \cdot F = 1$, and we denote the corresponding line bundle as $\mathscr{O}_X(1)$. Let $h := c_1(\mathscr{O}_X(1))$. I recall how one can define this line bundle: we have $p : X = \mathbb{P}(E) \longrightarrow X$, and every point $x \in X$ corresponds hence to a line $D_{p(x)}$ in the $2-$dimensional vector space $E_{p(x)}$. Consider the sub-line bundle $N \subseteq p^*E$ such that for every $x \in X$ we have $N_x = D_{p(x)}$. Then $\mathscr{O}_X(1)$ is the quotient of p^*E by N. First of all, let me describe the Picard group and the intersection form on a geometrically ruled surface.

Proposition 4.2.3. *Let $p : X = \mathbb{P}(E) \longrightarrow C$ be a geometrically ruled surface.*

1. *The Picard group of X is $Pic(X) \simeq p^*Pic(C) \oplus \mathbb{Z} \cdot \mathscr{O}_X(1)$. In particular, $H^2(X, \mathbb{Z})$ is generated by the class f of a fiber and by h.*
2. *Let $d := deg(E)$. Then $h^2 = d$.*
3. *We have $c_1(K_X) = -2h+(2g(C)-2+d)f$. In particular $K_X^2 = 8(1-g(C))$.*

Proof. We clearly have a map

$$p^*Pic(C) \oplus \mathbb{Z} \cdot \mathscr{O}_X(1) \longrightarrow Pic(X), \qquad (p^*\mathscr{L}, n) \mapsto p^*\mathscr{L} \otimes \mathscr{O}_X(n),$$

where $\mathscr{O}_X(n) := \mathscr{O}_X(1)^{\otimes n}$. This map is injective: if $p^*\mathscr{L} \otimes \mathscr{O}_X(n) \simeq p^*\mathscr{M} \otimes \mathscr{O}_X(m)$ we have $p^*\mathscr{L} \simeq p^*\mathscr{M} \otimes \mathscr{O}_X(m-n)$. As $\mathscr{O}_X(1)$ is not contracted to a point by p, we then need $n = m$. Hence $p^*\mathscr{L} \simeq p^*\mathscr{M}$.

We then need to show the surjectivity: let D be an effective divisor on X, and write $D = nH + G$, where $n \in \mathbb{N}$ and G is effective and does not contain H. Then we need to show that G is the pull-back of a line bundle on C. If $G \cdot F \neq 0$, then by construction we need $G = kH$ for some integer k. Hence $G \cdot F = 0$, so that G has components lying on fibers, and G is the pull-back of some divisor on C. For non effective divisors D, it is more complicated: the idea is to show that for $n \gg 0$ we have $D_n := D + nF$ effective. Then item 1 follows from what we have just shown. Now, consider D_n. As $D \cdot F = 0$, we have $D^2 = D_n^2$, and

$$D_n \cdot K_X = D \cdot K_X + nF \cdot K_X = D \cdot K_X + ndeg(K_F) = D \cdot K_X - 2n.$$

By Riemann-Roch we then get

$$h^0(X, \mathscr{O}_X(D_n)) + h^0(X, \mathscr{O}_X(K_X - D_n)) \geq \chi(\mathscr{O}_X) + n + \frac{1}{2}(D^2 - D \cdot K_X).$$

But if $n \gg 0$ we have $h^0(X, \mathscr{O}_X(K_X - D_n)) = 0$: if A is a very ample divisor on X, we have that $(K_X - D_n) \cdot A = K_X \cdot A - D \cdot A - nF \cdot A < 0$ for $n \gg 0$, so that $K_X - D_n$ is not effective for $n \gg 0$. In conclusion, we get $h^0(X, \mathscr{O}_X(D_n)) \sim n$ for $n \gg 0$, and we are done.

For the $H^2(X, \mathbb{Z})$, simply use Remark 4.2.1: we have $H^2(X, \mathscr{O}_X) = 0$, so that c_1 is surjective. Hence $H^2(X, \mathbb{Z})$ is generated by f and h.

For the second item, consider the exact sequence defining $\mathscr{O}_X(1)$:

$$0 \longrightarrow N \longrightarrow p^*E \longrightarrow \mathscr{O}_X(1) \longrightarrow 0.$$

Hence $N \otimes \mathscr{O}_X(1) \simeq det(E)$, so that $h = p^*e - c_1(N)$, where $e = c_1(det(E))$. Hence $h^2 = h \cdot p^*e - h \cdot c_1(N)$. Now, for every exact sequence

$$0 \longrightarrow L \longrightarrow F \longrightarrow M \longrightarrow 0,$$

where L, M are line bundles on a surface, and F is a rank 2 vector bundle, we have

$$L \cdot M = L^{-1} \cdot M^{-1} = \chi(\mathscr{O}_X) - \chi(L) - \chi(M) + \chi(L \otimes M) =$$
$$= \chi(\mathscr{O}_X) - \chi(F) + \chi(det(F)),$$

which depends only on F. In our case we have then

$$c_1(N) \cdot h = \chi(\mathscr{O}_X) - \chi(p^*(E)) - \chi(det(E)).$$

As E is a rank 2 vector bundle on a curve C, there are two line bundles L, M on C and an exact sequence

$$0 \longrightarrow L \longrightarrow E \longrightarrow M \longrightarrow 0,$$

(see Lemma 5.1.1 below), so that

$$c_1(N) \cdot h = p^*L \cdot p^*M = 0,$$

hence $h^2 = h \cdot p^*e = deg(det(E)) = deg(E)$, and we are done with item 2.

For the third item, we have $c_1(K_X) = ah + bf$. Moreover

$$-2 = c_1(K_X) \cdot f = ah + bf = a,$$

so that $c_1(K_X) = -2h + bf$. Then $K_X \cdot H = -2d + b$. The genus formula gives

$$2g(X) - 2 = K_X \cdot H + H^2 = -2d + b + d = b - d,$$

so that $b = 2g(X) + d - 2$. Finally

$$K_X^2 = (-2h + (2g(X) + d - 2)f)^2 = 4d - 8g(X) - 4d + 8 = 8(1 - g(X)).$$

□

Remark 4.2.2. Let X be a geometrically ruled surface. Then K_X is not nef: indeed we have that f is an effective class, and that $K_X \cdot f = -2$.

To conclude, we show the following:

Proposition 4.2.4. *Let X be a ruled surface birational to $C \times \mathbb{P}^1$. Then $p_g(X) = P_m(X) = 0$ for every $m > 0$, and $q(X) = g(C)$. In particular $\kappa(X) = -\infty$.*

Proof. As $p_g(X)$, $P_m(X)$ and $q(X)$ are birational invariants, we can use $C \times \mathbb{P}^1$ to calculate them. Here notice that

$$\Omega_{C\times\mathbb{P}^1} \simeq p^*\Omega_C \oplus q^*\Omega_{\mathbb{P}^1},$$

where $q : C\times\mathbb{P}^1 \longrightarrow \mathbb{P}^1$ is the projection. Hence $K_{C\times\mathbb{P}^1} \simeq p^*K_C + q^*K_{\mathbb{P}^1}$. Then

$$q(X) = h^0(C \times \mathbb{P}^1, \Omega_{C\times\mathbb{P}^1}) = h^0(C, \mathscr{O}_C(K_C)) + h^0(\mathbb{P}^1, \mathscr{O}(-2)) = g(C),$$

$$p_g(X) = h^0(C \times \mathbb{P}^1, \mathscr{O}_{C\times\mathbb{P}^1}(K_{C\times\mathbb{P}^1})) = h^0(C, \mathscr{O}_C(K_C)) \cdot h^0(\mathbb{P}^1, \mathscr{O}(-2)) = 0.$$

Similar calculations show that $P_m(X) = 0$ for every $m > 0$. $\square$

4.2.3 Rational surfaces

A rational surface X is a ruled surface which is birational to $\mathbb{P}^1 \times \mathbb{P}^1$. Hence, a rational surface is birational to $\mathbb{P}^2$ and one can calculate its numerical invariants from this (without using Proposition 4.2.4). Namely, if X is rational, then $q(X) = p_g(X) = 0$ and $P_m(X) = 0$ for every $m > 0$.

If a rational surface is geometrically ruled, then by Theorem 4.2.2 it is isomorphic to a projective bundle associated to a rank 2 vector bundle on $\mathbb{P}^1$. One can be more explicit in the classification of geometrically ruled rational surfaces. Here is the main result of the section:

Theorem 4.2.5. *Let E be a vector bundle of rank n on $\mathbb{P}^1$. Then there are $a_1, ..., a_n \in \mathbb{Z}$ such that $E \simeq \mathscr{O}(a_1) \oplus ... \oplus \mathscr{O}(a_n)$.*

Proof. Use the following lemma:

Lemma 4.2.6. *Let U, V, W be three vector bundles on a complex manifold X and consider the exact sequence*

$$0 \longrightarrow U \xrightarrow{a} V \xrightarrow{b} W \longrightarrow 0.$$

It splits if and only if $H^1(X, \mathscr{H}om(W, U)) = 0$.

Proof. Apply the functor $\mathscr{H}om(W, .)$ to the exact sequence: As V, W, U are locally free, one gets an exact sequence

$$0 \longrightarrow \mathscr{H}om(W,U) \xrightarrow{a\circ} \mathscr{H}om(W,V) \xrightarrow{b\circ} \mathscr{H}om(W,W) \longrightarrow 0. \tag{4.1}$$

The exact sequence of the lemma splits if and only if there is a morphism $h : W \longrightarrow V$ such that $b \circ h = id_W$. Looking at the long exact sequence induced by the exact sequence (4.1), we have

$$Hom(W,V) \xrightarrow{b\circ} Hom(W,W) \longrightarrow H^1(X, \mathscr{H}om(W,U)).$$

The splitting of the exact sequence of the lemma is then equivalent to the vanishing of $H^1(X, \mathscr{H}om(W,U))$. $\square$

Now, let E be a vector bundle on $\mathbb{P}^1$, and for every $p \in \mathbb{Z}$ let $E(p) := E \otimes \mathscr{O}(p)$. Now, fix $k \in \mathbb{Z}$ such that $H^0(\mathbb{P}^1, E(k)) \neq 0$, but for every $h < k$ we have $H^0(\mathbb{P}^1, E(h)) = 0$. Let $s \in H^0(\mathbb{P}^1, E(k))$: we have $Z(s) = \emptyset$. Indeed, if $Z(s) \neq \emptyset$, it would define an effective line bundle $\mathscr{O}(p)$, $p > 0$, so that $E(k-p)$ has a section. As we supposed that for every $h < 0$ the vector bundle $E(h)$ has no section, this is not possible, so that $Z(s) = \emptyset$. Then s defines the trivial line bundle $\mathscr{O}_{\mathbb{P}^1}$, and an injection

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^1} \longrightarrow E(k) \longrightarrow F \longrightarrow 0.$$

Twisting by $\mathscr{O}(-1)$ we then get

$$0 \longrightarrow \mathscr{O}(-1) \longrightarrow E(k-1) \longrightarrow F(-1) \longrightarrow 0.$$

Recall that $H^0(\mathbb{P}^1, E(k-1)) = 0$. Moreover, we have $H^1(\mathbb{P}^1, \mathscr{O}(-1)) = 0$, so that $H^0(\mathbb{P}^1, F(-1)) = 0$, and $H^0(\mathbb{P}^1, F(-2)) = 0$. Using Serre's Duality we then get $H^1(\mathbb{P}^1, F^*) = 0$. But this means $H^1(\mathbb{P}^1, \mathscr{H}om(F, \mathscr{O}_{\mathbb{P}^1})) = 0$. Apply Lemma 4.2.6 to get hence that $E \simeq \mathscr{O}(-k) \oplus F$.

But now F has rank $n-1$, so that one can apply induction to see that there are $a_2, ..., a_n \in \mathbb{Z}$ such that $F \simeq \mathscr{O}(a_2) \oplus ... \oplus \mathscr{O}(a_n)$. Now simply let $a_1 := -k$, and we are done. □

Theorem 4.2.2 together with Theorem 4.2.5 implies that every geometrically ruled rational surface X is of the form $X \simeq \mathbb{P}(\mathscr{O}(a_1) \oplus \mathscr{O}(a_2))$. Notice that tensoring with $\mathscr{O}(-a_1)$ does not change X, so that one can always suppose that $X \simeq \mathbb{P}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(n))$ for some $n \geq 0$. Now, let me introduce the following notation: for every $n \geq 0$, let

$$\mathbb{F}_n := \mathbb{P}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(n)).$$

Definition 4.2.3. The surface $\mathbb{F}_n$ is called *Hirzebruch surface*.

Resuming, we have shown that every geometrically ruled rational surface is a Hirzebruch surface.

4.2.4 Minimal ruled surfaces

In this section we show some important results about minimal ruled surfaces. Up to now we have seen that every geometrically ruled surface is ruled, and that every ruled surface is birational to a geometrically ruled one. It is by no means clear that a ruled surface is geometrically ruled: this is, indeed, not true in general.

Proposition 4.2.7. ***(Noether-Enriques)****. Let X be a minimal surface, and let us suppose that there is a surjective morphism $p : X \longrightarrow C$ to a smooth curve such that the generic fiber is a smooth rational curve. Then p gives X the structure of a geometrically ruled surface.*

Proof. Let $c \in C$ be a generic point and consider $F := p^{-1}(c)$. Then $F \simeq \mathbb{P}^1$, and $F^2 = 0$. Then

$$F \cdot K_X = 2g(F) - 2 = -2.$$

Following the same argument as in Step 2 of Proposition 4.2.1, we have that F is primitive, i. e. if divisor F' is such that $F = nF'$ for some $n \in \mathbb{Z}$, then $n = \pm 1$. Now, let us consider G to be any fiber (no more only generic). We have $G^2 = 0$ and $G \cdot K_X = -2$, so that $g(G) = 0$ for every fiber.

We now show that there are no reducible fibers. Let us suppose that $F = \sum_i n_i C_i$ is a reducible fiber. We have

$$n_i C_i^2 = C_i \cdot (F - \sum_{j \neq i} n_j C_j) = -\sum_{j \neq i} n_j C_i \cdot C_j < 0$$

as $C_i \cdot F = 0$ and there is at least one $j \neq i$ such that $C_i \cdot C_j \neq 0$ (as the fibers are connected: this is true for the generic fiber, hence is everywhere true by easy topological argument). Hence $n_j > 0$ and $C_i \cdot C_j \geq 0$, for every $j \neq i$, and there is $j \neq i$ such that $C_i \cdot C_j > 0$. In conclusion $C_i^2 < 0$. By the genus formula we have

$$-2 = 2g(C_i) - 2 = C_i^{-2} + C_i \cdot K_X,$$

so that $C_i \cdot K_X \geq -1$, and $C_i \cdot K_X = -1$ if and only if $C_i^2 = -1$. In this last case C_i would be a $(-1)-$curve, but as X is minimal, by Corollary 3.1.21 the surface X does not contain any $(-1)-$curve. Hence $C_i \cdot K_X \geq 0$. As this is true for every i, we then get

$$-2 = K_X \cdot F = K_X \cdot \sum_i n_i C_i = \sum_i n_i C_i \cdot K_X \geq 0,$$

which is impossible. In conclusion, there are no reducible fibers.

Now, every point $c \in C$ is a regular value for p. By Proposition 4.2.1 there is a Zariski-open neighborhood U of c in C such that $p^{-1}(U) \simeq U \times \mathbb{P}^1$ in a fiber preserving manner. But this implies that $p : X \longrightarrow C$ is a geometrically ruled surface. □

The main importance of this is the following:

Theorem 4.2.8. *Let X be a ruled surface which is not rational. Then X is geometrically ruled if and only if it is minimal.*

Proof. As X is ruled, there is a smooth curve C such that X is birational to $C \times \mathbb{P}^1$. We have then a birational map $X \dashrightarrow C \times \mathbb{P}^1$, and composing with the projection to C we get a rational map $p : X \dashrightarrow C$. If this map is not a morphism, we need to blow-up X along the indeterminacies of p in order to get a morphism $\widetilde{p} : \widetilde{X} \longrightarrow C$. On $\widetilde{X}$ there will then be $(-1)-$curves: all these are then mapped to a point of C (otherwise the image would be a rational curve, but C is not rational as X is not rational). But this is not possible if we are blowing-up indeterminacies, and p is then a morphism. Now, we have a surjective map $p : X \longrightarrow C$ whose generic fiber is a smooth rational curve. As X is minimal, we can apply Proposition 4.2.7 to get that X has the structure of a geometrically ruled surface.

To conclude we just need to show that every geometrically ruled non-rational surface is minimal. We have $p : X \longrightarrow C$, where C is not rational. If X is not minimal, then there is a $(-1)-$curve E on X. As $E^2 = -1$, E is not contained in a fiber, hence $p(E) = C$. But this implies C is rational, which is not the case, and we are done. $\square$

In conclusion, the minimal models of ruled surfaces which are not rational are geometrically ruled surfaces. Moreover, one sees from this that minimal models of ruled non-rational surfaces are of the form $\mathbb{P}(E)$, where E is a rank 2 vector bundle on a smooth curve C. This implies two things: first of all, there is no uniqueness of the minimal model; second, one can in principle classify all minimal models of non-rational ruled surfaces simply knowing a classification of rank 2 vector bundles on curves.

Let us then look at rational surfaces. A first remark is the following:

Remark 4.2.3. There is an example of minimal rational surface which is not geometrically ruled: it is simply $\mathbb{P}^2$. Indeed, if $\mathbb{P}^2$ was geometrically ruled, then there would be some $n \geq 0$ such that $\mathbb{P}^2 \simeq \mathbb{F}_n$. But recall that $Pic(\mathbb{P}^2) \simeq \mathbb{Z}$, while $Pic(\mathbb{F}_n) \simeq Pic(\mathbb{P}^1) \oplus \mathbb{Z} \simeq \mathbb{Z}^2$ by Proposition 4.2.3. Hence $\mathbb{P}^2$ cannot be isomorphic to any $\mathbb{F}_n$, so it is not geometrically ruled.

Anyway, one can give a complete list of minimal rational surfaces. First of all one has the following:

Proposition 4.2.9. *The Hirzebruch surface $\mathbb{F}_1$ is isomorphic to the blow-up of $\mathbb{P}^2$ along a point. In particular, $\mathbb{F}_1$ is not minimal.*

Proof. Let x_0, x_1, x_2 be homogenous coordinates on $\mathbb{P}^2$, and let y_0, y_1 be homogenous coordinates on $\mathbb{P}^1$. Consider the point $P = (1 : 0 : 0) \in \mathbb{P}^2$. The blow-up of $\mathbb{P}^2$ along P is then a smooth projective surface $V \subseteq \mathbb{P}^2 \times \mathbb{P}^1$ of equation $x_0 y_1 = x_1 y_0$. Now, $\mathbb{F}_1 = \mathbb{P}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(1))$. The vector bundle $E := \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(1)$ is generated by its global sections 1 of $\mathscr{O}_{\mathbb{P}^1}$ and y_0, y_1 of $\mathscr{O}(1)$.

The algebraic variety $\mathbb{F}_1$ is then $Proj(S(E))$, where $S(E)$ is the symmetric algebra of E. we have a map of graded rings

$$\mathbb{C}[x_0, x_1, x_2] \otimes \mathbb{C}[y_0, y_1] \longrightarrow S(E),$$

which sends x_0 to y_0, x_1 to y_1, x_2 to 1 and y_o and y_1 to themself. This map is clearly surjective, and it kernel is the ideal generated by $x_0 y_1 - x_1 y_0$. Hence $\mathbb{F}_1$ is the subvariety of $\mathbb{P}^2 \times \mathbb{P}^1$ of equation $x_0 y_1 = y_1 x_0$. In conclusion we have $\mathbb{F}_1 \simeq Bl_P \mathbb{P}^2$, and we are done. $\square$

All the other Hirzebruch surfaces are minimal:

Theorem 4.2.10. *Let X be a minimal rational surface. If it is geometrically ruled, then $X \simeq \mathbb{F}_n$ for some $n \neq 1$.*

Proof. First of all, $\mathbb{P}^2$ is ruled. If $n = 0$, then $\mathbb{F}_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$, as easily seen, so this one is minimal too. Let now consider $n > 1$, and let $p_n : \mathbb{F}_n \longrightarrow \mathbb{P}^1$ be the canonical projection. On $\mathbb{F}_n$ there are two natural type of sections of p_n one can describe: one is

$$\widehat{S} : \mathbb{P}^1 \longrightarrow \mathbb{F}_n, \qquad \widehat{S}(x) := (0,1) \in \mathscr{O}^2_{\mathbb{P}^1,x}.$$

The other one is obtained considering any homogenous polynomial of degree n in two variables, and considering

$$S_p : \mathbb{P}^1 \longrightarrow \mathbb{F}_n, \qquad S_p(x) := (1, p(x)) \in \mathscr{O}^2_{\mathbb{P}^1,x}.$$

Notice that $\widehat{S}$ and S_p define curves in $\mathbb{F}_n$, and that $S_p \sim S_q$ for every two homogenous polynomials p, q of degree n, so that one can define $S := S_p$ for some homogenous polynomial p of degree n. It is easy to see that $S \cdot \widehat{S} = 0$, and one has $S \cdot S = n$ (as $S_p \cap S_q$ coincides with the set of common zeroes of p and q: as p and q can be chosen to have no common factors, and they have degree n, we have that this set is given by n points).

By Proposition 4.2.3, there must be some $a \in \mathbb{N}_0$ such that $\widehat{S} = S + aF$. Hence

$$0 = \widehat{S} \cdot S = S^2 + aS \cdot F = a + n,$$

hence $\widehat{S} = S - nF$. Moreover, we have

$$\widehat{S} \cdot \widehat{S} = (S - nF) \cdot (S - NF) = S^2 - 2nS \cdot F + n^2 F^2 = -n.$$

As $n > 1$, we see that $\widehat{S}$ is not a $(-1)-$curve. Now, let us consider an irreducible effective curve $C = aS + bF$, and let us suppose that C is not linearly equivalent to $\widehat{S}$. Hence

$$C \cdot \widehat{S} = (aS + bF) \cdot (S - nF) = aS^2 + (b - na)S \cdot F - nbF^2 = b \geq 0.$$

Moreover, we have $C \cdot F = (aS + bF) \cdot F = a \geq 0$. Finally, let us consider

$$C^2 = (aS + bF)^2 = a^2 n + 2ab \geq 0,$$

so that the only irreducible curve of negative self-intersection is $\widehat{S}$. In conclusion, $\mathbb{F}_n$ is minimal for every $n \neq 0$. $\square$

To conclude, let us first notice that $\mathbb{P}^2$ is not isomorphic to any $\mathbb{F}_n$ by Remark 4.2.3, and that if $n \neq m$, then $\mathbb{F}_n$ is not isomorphic to $\mathbb{F}_m$: on $\mathbb{F}_n$ there is a curve of self-intersection $-n$, and on $\mathbb{F}_m$ there is no such a curve. This implies that even for rational surfaces there is no unique minimal model. Anyway, notice that we have not proved yet that $\mathbb{P}^2$ and the Hirzebruch surfaces are the only minimal rational surfaces. This will be proven in the next section.

4.2.5 Classification for Kodaira dimension $-\infty$

As far as now, we introduced several examples of surfaces having Kodaira dimension $-\infty$. We have the following, which completes the classification of surfaces of Kodaira dimension $-\infty$.

Theorem 4.2.11. *Let X be a minimal surface such that $\kappa(X) = -\infty$. Then X is either geometrically ruled or $X \simeq \mathbb{P}^2$.*

Proof. The proof uses the Kawamata-Mori Rationality Theorem. First of all, by Corollary 3.2.10 as X is minimal and $\kappa(X) = -\infty$ we have that K_X is not nef. Consider the positive half ray $R := \mathbb{Q}_{>0} \cdot (-K_X)$ in $N^1_{\mathbb{Q}}(X)$. We have only two possibilites: either $Pic(X) \subseteq R$, i. e. $-K_X$ is ample and $\rho(X) = 1$; or there is $H \in Pic(X)$ which is not a positive multiple of $-K_X$.

Let us begin with the first case, so we suppose that $-K_X$ is ample and that $\rho(X) = 1$. As $-K_X$ is ample, by the Kodaira Vanishing Theorem we have that $h^i(X, \mathscr{O}_X) = 0$ for every $i > 0$, i. e. $p_g(X) = q(X) = 0$. Hence, by Example 2.1.3 the first Chern class is an isomorphism

$$Pic(X) \xrightarrow{c_1} H^2(X, \mathbb{Z}).$$

Hence $b_2(X) = 1$. As $q(X) = 0$ and X is Kähler we get $b_1(X) = 0$. In conclusion we get $e(X) = 3$. Apply Noether's Formula:

$$1 = 1 - q(X) + p_g(X) = \chi(\mathscr{O}_X) = \frac{1}{12}(K_X^2 + e(X)),$$

so that $K_X^2 = 9$.

Now, as $Pic(X)$ as rank 1, one can consider an ample generator H. As $-K_X$ is ample, $H - K_X$ is ample too, so that by the Kodaira Vanishing Theorem one gets $h^i(X, \mathscr{O}_X(H)) = 0$ for $i = 1, 2$. In conclusion we have

$$h^0(X, \mathscr{O}_X(H)) = \chi(H) = 1 - q(X) + p_g(X) + \frac{1}{2}(H^2 - H \cdot K_X).$$

Moreover, as $H - K_X$ is ample, we need $H - K_X = nH$ for some $n \in \mathbb{N}$, hence $K_X = (n + 1)H$, where $n + 1 > 1$. As $K_X^2 = 9$ we get

$$9 = K_X^2 = (n + 1)^2 H^2,$$

so that H^2 properly divises 9. In conclusion $H^2 = 1$ and $K_X = -3H$. Using this, we finally get $h^0(X, \mathscr{O}_X(H)) = 3$. As H is ample, it defines a dominant rational map

$$f : X \dashrightarrow \mathbb{P}^2,$$

mapping H to a line. Now, we have $H^2 = 1$, hence $|H|$ has no base points, so that $f : X \longrightarrow \mathbb{P}^2$ is everywhere defined. Now, f cannot contract curves, as $\rho(X) = \rho(\mathbb{P}^2) = 1$, hence f is finite of degree d. Recall now that

$$1 = H^2 = (f^*\mathscr{O}(1))^2 = d\mathscr{O}_1^2 = d,$$

so that $d = 1$, and the map f is birational. If f^{-1} is not defined at some points, then f contracts a curve to that point, but as f does not contract curves, f^{-1} is everywhere defined. In conclusion $X \simeq \mathbb{P}^2$.

Let us consider the remaining case, namely there is an ample divisor H such that $H \notin R$. Let $s := s_0(H, K_X)$, which is a rational number by the Kawamata-Mori Rationality Theorem, and let $L := uH + vK_X$ for $u, v \in \mathbb{N}$ such that $v/u = s$. Then L is a nef divisor, so that $L^2 \geq 0$. Moreover, for every $m \geq 1$ the $\mathbb{Q}-$divisor $L - \frac{1}{m}K_X$ is ample: indeed

$$L - \frac{1}{m}K_X = uH + \left(v - \frac{1}{m}\right)K_X,$$

and $(v - (1/m))/u < v/u = s$. Multiplying by m we get the ample divisor $mL - K_X$, for every $m \geq 1$. Using the Kodaira Vanishing Theorem we then have $h^i(X, \mathscr{O}_X(mL)) = 0$ for $i = 1, 2$. By Riemann-Roch we finally get

$$h^0(X, \mathscr{O}_X(mL)) = \chi(\mathscr{O}_X) + \frac{1}{2}(m^2L^2 - mL \cdot K_X).$$

Recall that $L^2 \geq 0$, so that we have only two possibilities: either $L^2 > 0$, or $L^2 = 0$.

Let us begin with the case $L^2 > 0$. As L is nef, for every effective divisor D we have $L \cdot D \geq 0$. If D is such that $L \cdot D = 0$, then

$$0 = L \cdot D = (uH + vK_X) \cdot D = uH \cdot D + vK_X \cdot D.$$

As $u, v \in \mathbb{N}$ and $H \cdot D > 0$ as H is ample, we need $K_X \cdot D < 0$. Moreover, as $L^2 > 0$ and $L \cdot D = 0$, by the Hodge Index Theorem we get $D^2 < 0$. Now, use the genus formula:

$$g(D) = 1 + \frac{1}{2}(K_X \cdot D + D^2) \leq 0,$$

so that $g(D) = 0$. Moreover, by the same formula we get

$$K_X \cdot D + D^2 = -2.$$

If $D^2 < -1$, then we would have $K_X \cdot D \geq 0$, which is not possible, so that $D^2 = -1$. In conclusion D is an exceptional curve. As X is minimal, such a D cannot exist on X, so that for every irreducible effective divisor we have $L \cdot D > 0$. By the Nakai-Moishezon Criterion, we can finally conclude that L is ample: but L is on the border of $Nef(X)$, hence it cannot be ample, so we must have $L^2 = 0$.

So, we consider $L^2 = 0$. As L is nef we need $L \cdot H \geq 0$. If $L \cdot H = 0$, the Hodge Index Theorem implies $L^2 \equiv 0$. In conclusion, we have $uH \equiv -vK_X$, i. e. $H \in R$, which is not possible. In conclusion, we have $L \cdot H > 0$. Now

$$0 = \frac{1}{u}L^2 = L \cdot (H + sK_X) = L \cdot H + sL \cdot K_X.$$

As $s > 0$ and $L \cdot H > 0$, we get $L \cdot K_X < 0$. As we have seen before we have

$$h^0(X, \mathscr{O}_X(mL)) \geq -\frac{1}{2}L \cdot K_X m + \chi(\mathscr{O}_X),$$

so that $h^0(X, \mathscr{O}_X(mL)) \sim m$. Up to replacing L with mL we can then suppose $dim|L| > 1$.

Let us write $L = L' + M$, where M is the fixed part, and L' the moving part. As L' moves in a linear system, it is clear that L' is nef. Hence $(L')^2 \geq 0$. Moreover M is effective, so $L' \cdot M \geq 0$ as L' is nef. Then

$$0 = L^2 = L \cdot L' + L \cdot M \geq 0,$$

so that we need $L \cdot L' = L \cdot M = 0$. Hence

$$0 = L \cdot L' = (L')^2 + L' \cdot M \geq 0,$$

so that $(L')^2 = L' \cdot M = 0$. Now, let D be an irreducible component of a member of $|L'|$. As $L \cdot L' = 0$, we have $L \cdot D = 0$. Similarily, as $M \cdot L' = 0$ we have $M \cdot D = 0$. Hence

$$L' \cdot D = (L - M) \cdot D = L \cdot D - M \cdot D = 0,$$

which implies $D^2 \leq 0$. But now, remark that

$$0 = L \cdot D = uH \cdot D + vK_X \cdot D,$$

so that $D \cdot K_X < 0$, as H is ample and $u, v > 0$. As shown previously, this implies that $g(D) = 0$ and that $D^2 = 0$.

What we have just done applies to every linear system of L containing no fixed part. Consider then a $1-$dimensional $V \subseteq |L|$, and take out its fixed part, so that we get a pencil $\mathbb{P}$ of curves F such that $F^2 = 0$. Every irreducible component of a member of $|F|$ is a smooth rational curve. As $F^2 = 0$, there can be no fixed points, hence $|F|$ gives a morphism

$$f : X \longrightarrow \mathbb{P}^1,$$

as $dim|F| = 1$. Consider now the Stein factorization of f: there is a smooth curve C and a map $f' : S \longrightarrow C$ whose fibers are connected, and are the components of the members of F. Hence this is a $\mathbb{P}^1-$fibration, and X has the structure of a geometrically ruled surface. □

Notice that this theorem implies that the only smooth projective surfaces of Kodaira dimension $-\infty$ are the ruled surfaces. If X it is not rational, then we had already seen that its minimal model is geometrically ruled. If X is rational we can now conclude:

Corollary 4.2.12. *Let X be a minimal rational surface. Then X is either $\mathbb{P}^2$ or $\mathbb{F}_n$ for some $n \neq 1$.*

Proof. As X is a minimal rational surface, it is a minimal surface such that $\kappa(X) = -\infty$. Hence by Theorem 4.2.11 X is either geometrically ruled or $\mathbb{P}^2$. But if X is geometrically ruled, by Theorem 4.2.10 we have that X is a Hirzebruch surface $\mathbb{F}_n$ for $n \neq 1$ and we are done. □

4.2.6 The Castelnuovo Rationality Criterion

In this subsection I want to prove an important corollary of the Kawamata-Mori Rationality Theorem that will be useful in the following, in the proof of the Castelnuovo-Enriques Classification.

Proposition 4.2.13. *Let X be a smooth projective surface, and let us suppose that K_X is nef. Then one of the following is possible:*

1. $K_X^2 > 0$. *In this case we have even $P_2(X) > 0$ and*

$$P_m(X) \geq \frac{1}{2}m(m-1)K_X^2 + 1 - q(X) + p_q(X)$$

for every $m \geq 2$.

2. $K_X^2 = 0$ *and $q(X) = 0$. In this case $P_2(X) > 0$.*

3. $K_X^2 = 0$, *$p_g(X) > 0$ and $q(X) > 0$.*

4. $K_X^2 = 0$, *$p_q(X) = 0$, $q(X) = 1$ and $b_2(X) = 2$.*

Proof. As K_X is nef, then $K_X^2 \geq 0$: then we can have only $K_X^2 > 0$ or $K_X^2 = 0$. First of all, notice that if $p_g(X) > 0$, then $P_m(X) > 0$ for every $m > 0$. In order to show that $P_2 > 0$ we can then only look at the case where $p_g(X) = 0$.

So let us suppose that $p_g(X) = 0$. Use the Noether Formula to get

$$12(1 - q(X)) = K_X^2 + 2 - 4q(X) + b_2(X).$$

If $K_X^2 = 0$ and $q(X) = 1$, then one gets $b_2(X) = 2$, and the last case is shown. If $q(X) > 1$, one gets

$$b_2(X) = 10 - 8q(X) < 0,$$

which is not possibile. Hence, if $K_X^2 = 0$ and $p_g(X) = 0$, we need $q(X) = 1$ and $b_2(X) = 2$, or $q(X) = 0$ (so that $b_2(X) = 10$).

Now, use the following easy lemma:

Lemma 4.2.14. *Let X be a smooth projective surface, and let $\mathscr{L}$ be a nef line bundle on X. If $h^0(X, \mathscr{L}^{-1}) \neq 0$, then $\mathscr{L} \simeq \mathscr{O}_X$.Recall that a minimal surface X is such that $\kappa(X) = -\infty$ if and only if K_X is not nef.*

Proof. Suppose $\mathscr{L}$ is not trivial, and let C be an effective curve transversal to a divisor defined by $\mathscr{L}^{-1}$: then

$$0 \leq \mathscr{L} \cdot \mathscr{O}_X(C) = -\mathscr{L}^{-1} \cdot \mathscr{O}_X(C) < 0,$$

and we are done. □

As K_X is nef, we need $h^0(X, \mathscr{O}_X(rK_X)) = 0$ for every $r < 0$ by Lemma 4.2.14. Hence if $m > 1$ we have $h^0(X, \mathscr{O}_X(-(m-1)K_X))$, and

$$P_m(X) = h^0(X, \mathscr{O}_X(mK_X)) = h^0(X, \mathscr{O}_X(mK_X)) + h^0(X, \mathscr{O}_X(-(m-1)K_X)) \geq$$

$$\geq \frac{1}{2}m(m-1)K_X^2 + 1 - q(X) + p_g(X).$$

If $K_X^2 > 0$ and $m = 2$ we have then

$$P_2(X) \geq K_X^2 + 1 - q(X) + p_g(X) > 0.$$

If $K_X^2 = 0$ and $q(X) = 0$, then $P_2(X) \geq 1 + p_g(X) > 0$. □

The reason why I decided to put this Proposition here is that it as an easy corollary on rational surfaces. This Theorem, which here appears as an easy corollary of previously shown results was one of the main points in the birational classification of smooth projective surfaces given by Enriques and Castelnuovo. Without the Kawamata-Mori Rationality Theorem, its proof is hard and tricky.

Theorem 4.2.15. ***(Castelnuovo's Rationality Theorem)****. Let X be a minimal smooth projective surface. Then X is rational if and only if $q(X) = 0$ and $P_2(X) = 0$.*

Proof. Let us suppose X rational. Then by Proposition 4.2.4 we have $q(X) = P_2(X) = 0$.

Let us show the converse. If X is smooth projective surface such that $q(X) = P_2(X) = 0$, then by Proposition 4.2.13 K_X is not nef (if X is nef, we have $P_2(X) > 0$ or $q(X) > 0$). Then, as X is minimal, we have $\kappa(X) = -\infty$, so that X can only be either geometrically ruled or $\mathbb{P}^2$. In any case, as $q(X) = 0$, we need X to be rational. □

Remark 4.2.4. Notice that if $P_2(X) = 0$, then even $p_g(X) = 0$ by Serre's Duality. Hence it seems quite reasonble to expect that the Castelnuovo Rationality Theorem works even with the hypothesis $p_g(X) = q(X) = 0$. This is not the case, as we are going to see with Enriques surfaces.

An interesting corollary of the Castelnuovo Rationality Theorem concerns unirational surfaces.

Definition 4.2.4. A smooth projective variety X is *unirational* if there is a dominant rational map $\mathbb{P}^n \dashrightarrow X$, where $n = dim(X)$.

It is a well known result that every unirational curve is rational: indeed if C is a smooth curve which is unirational then there is a surjective morphism $f : \mathbb{P}^1 \longrightarrow C$. If ω is a holomorphic form on C, then $f^*\omega$ is a holomorphic form on $\mathbb{P}^1$. But as $h^1(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(K_{\mathbb{P}^1})) = 0$, we have $f^*\omega = 0$, so that $\omega = 0$. Hence $g(C) = 0$, and C is rational. This result is known as the *Lüroth Theorem*, and one would like to generalize it to any dimension.

Corollary 4.2.16. *Let X be a smooth projective surface which is unirational. Then X is rational.*

Proof. As X is unirational, there is a dominant map $\mathbb{P}^2 \dashrightarrow X$. Resolving the indeterminacies we get a dominant morphism $R \longrightarrow X$, where R is a rational surface. Hence $q(R) = P_2(R) = 0$ by the Castelnuovo Rationality Theorem, so that $q(X) = P_2(X) = 0$. Again by the Castelnuovo Rationality Theorem, one then concludes that X is rational. $\square$

This result is not true in higher dimensions.

4.3 Non-negative Kodaira dimension

In this section we continue with the classification of surfaces. As for Kodaira dimension $-\infty$ we are done, we need to look at those smooth projective surfaces of non-negative Kodaira dimension.

Thus, we can have Kodaira dimension 0, 1 or 2. We start by giving some definitions and important examples, introducing the notion of elliptic surface, K3 surface, abelian surface, Enriques surface, bielliptic surface, surface of general type. The classification proceeds using important properties of fibrations and of Abelian tori.

4.3.1 Definitions and examples

The first definition we need id the following, which generalizes the notion of geometrically ruled surface.

Definition 4.3.1. Let X be a surface and C be a smooth curve. A morphism $f : X \longrightarrow C$ is called *fibration* if it is surjective and has connected fibers. If the generic fiber of f is a smooth curve of genus g, then f is called *$g-$fibration.*

Notice that if all the fibers of f have genus 0, then f gives X the structure of a geometrically ruled surface. If X is minimal and f is a $0-$fibration, then X is a geometrically ruled surface. A particular and very important case is the following:

Definition 4.3.2. A $1-$fibration is called *elliptic fibration.* If a surface X admits an elliptic fibration, then X is called *elliptic surface.*

Let me give some easy example:

Example 4.3.1. Let C, F be two smooth curves, and consider $X := C \times F$. We have then two fibrations on X: one is $p_C : X \longrightarrow C$, the projection onto C, which is a $g(F)-$fibration; the other is $p_F : X \longrightarrow F$, the projection onto F, which is a $g(C)-$fibration. For this kind of surfaces is very easy to calculate invariants. Indeed, we have $\Omega_X \simeq p_C^*\Omega_C \oplus p_F^*\Omega_F$, so that

$$q(X) = h^0(X, \Omega_X) = h^0(C, \Omega_C) + h^0(F, \Omega_F) = g(C) + g(F).$$

Moreover, we have $\mathscr{O}_X(K_X) \simeq p_C^*\mathscr{O}_C(K_C) \otimes p_F^*\mathscr{O}_F(K_F)$, so that

$$p_g(X) = h^0(X, \mathscr{O}_X(K_X)) = h^0(C, \mathscr{O}_C(K_C)) \cdot h^0(F, \mathscr{O}_F(K_F)) = g(C) \cdot g(F),$$

and similarly we get

$$P_m(X) = P_m(C) \cdot P_m(F).$$

Hence, we have the following cases:

1. one between C and F is a smooth rational curve, and without loss of generality we can suppose that it is F. Then $P_m(X) = 0$, so that $\kappa(X) = -\infty$. Moreover $q(X) = g(C)$ and $p_g(X) = 0$. In fact, X is a geometrically ruled surface.

2. the two curves C and F are elliptic. Hence $q(X) = 1$, $p_g(X) = 1$ and $\kappa(X) = 0$.

3. one of the two curves is elliptic and the other has higher genus. Then $q(X) \geq 3$, $p_g(X) \geq 2$ and $\kappa(X) = 1$.

4. the two curves are not elliptic nor rational. Hence $q(X) \geq 4$, $p_g(X) \geq 4$ and $\kappa(X) = 2$.

This is then an important family of examples, as in this way we are able to produce examples of smooth projective surfaces of any possible Kodaira dimension.

Example 4.3.2. Let C and F be two smooth curves, and consider the group $Aut(F)$ of automorphisms of F. Consider an open cover $\{U_i\}_{i \in I}$ of C, and for every $i \in I$ let $X_i := U_i \times F$. For every $i, j \in I$ such that $U_{ij} \neq \emptyset$, consider a transition function $\varphi_{ij} : U_{ij} \longrightarrow Aut(F)$. Then we can glue together X_i and X_j using φ_{ij}. This gives a surface X with the structure of a $g(F)-$fibration.

For the next example we need the following lemma:

Lemma 4.3.1. *Let D and D' be two smooth curves, and suppose that D' is the quotient of D under the action of a finite group G. Then we have*

$$\mathscr{O}_D(mK_D) \simeq f^*\mathscr{O}_{D'}\bigg(mK_{D'} + \sum_{j=1}^{m}\Big(1 - \frac{1}{r_j}\Big)Q_j\bigg),$$

for every $m > 0$.

Proof. For this, let us consider the following: let D and D' be two smooth curves, and let $f : D \longrightarrow D'$ be a morphism. Consider the ramification points $P_1, ..., P_n \in D$, and let $r_i \in \mathbb{N}$ be the ramification index at P_i. The *ramification divisor* is

$$R := \sum_{i=1}^{n}(r_i - 1)P_i,$$

and the Hurwitz formula gives $\mathscr{O}_D(K_D) = f^*\mathscr{O}_{D'}(K_{D'}) \otimes \mathscr{O}_D(R)$. A special case is when $D \longrightarrow D'$ is the quotient under the action of a group G: then the ramification index is the same for every point of a fiber. If $Q_1, ..., Q_m \in D'$ are points over which we have ramification, we have a ramification index r_j for every $j = 1, ..., m$ for every point R_k lying over Q_j, and

$$R = \sum_{j=1}^{m}(r_j - 1)f^{-1}(Q_j).$$

Now, recall that $f^*Q_j = e_j f^{-1}(Q_j)$, so that

$$R = \sum_{j=1}^{m}\left(1 - \frac{1}{r_j}\right)f^*Q_j.$$

As a consequence, we get a formula for the pluricanonical bundle of D:

$$\mathscr{O}_D(mK_D) \simeq f^*\mathscr{O}_{D'}\left(mK_{D'} + \sum_{j=1}^{m}\left(1 - \frac{1}{r_j}\right)Q_j\right).$$

□

Example 4.3.3. Let $a, b \in \mathbb{Z}$, and consider $G := (\mathbb{Z}/\mathbb{Z}a) \oplus (\mathbb{Z}/\mathbb{Z}b)$. Let F be an elliptic curve such that G is a subgroup of the group ot translations of F, i. e. G acts on F as translations. Moreover, let $C \longrightarrow \mathbb{P}^1$ be a Galois cover of $\mathbb{P}^1$ ramified along three different points, and whose covering group is G. Hence G acts on $C \times F$, and we can consider $X := (C \times F)/G$. It has two fibrations: one is $p : X \longrightarrow \mathbb{P}^1$ whose fiber is F, and one is $q : X \longrightarrow F/G$ whose fiber is C. We have the following equalities:

$$H^0(X, \Omega_X) = H^0(C \times F, \Omega_{C\times F})^G = H^0(C, \Omega_C)^G \oplus H^0(F, \Omega_F)^G,$$

and similar for

$$H^0(X, \mathscr{O}_X(mK_X)) = H^0(C, \mathscr{O}_C(mK_C))^G \otimes H^0(F, \mathscr{O}_F(mK_F))^G.$$

By Lemma 4.3.1 we have

$$h^0(C, \mathscr{O}_C(mK_C))^G = h^0\left(\mathbb{P}^1, \mathscr{O}\left(m\left(-2 + \sum_j\left(1 - \frac{1}{r_j}\right)\right)\right)\right),$$

and

$$h^0(F, \mathscr{O}_F(mK_F))^G = h^0\left(F/G, \mathscr{O}_{F/G}\left(mK_{F/G} + \sum_j \left(1 - \frac{1}{s_j}\right)Q_j\right)\right),$$

where s_j is the ramification index for F and G. As a conclusion, we get

$$q(X) = g(\mathbb{P}^1) + g(F/G) = g(F/G), \qquad p_g(X) = g(\mathbb{P}^1)g(F/G) = 0,$$

$$P_m(X) = h^0(\mathbb{P}^1, \mathscr{O}(-2m + R_m(C,G))) \cdot h^0(F/G, \mathscr{O}_{F/G}(mK_{F/G} + R_m(F,G))),$$

where $R_m(D,G) := \sum_j (1 - \frac{1}{r_j})Q_j$ is the ramification divisor on $D' = D/G$. More generally, on can consider C to be the Galois cover of C' with Galois group G, and one gets similar formulas for the invariants:

$$q(X) = g(C/G) + g(F/G), \qquad p_g(X) = g(C/G) \cdot g(F/G),$$

$$P_m(X) = h^0(C/G, \mathscr{O}(mK_{C/G} + R_m(C,G))) \cdot h^0(F/G, \mathscr{O}(mK_{F/G} + R_m(F,G))).$$

As a particular case of this example, we get an important definition:

Definition 4.3.3. Let E and F be two elliptic curves, and let G be a group of translations of F acting on E. The surface $X := (E \times F)/G$ is a *bielliptic surface* if $p_g(X) = 0$.

By Example 4.3.3 the surface X is bielliptic if and only if one of the two quotients between E/G and F/G is a smooth rational curve. We now introduce the remaining definitions and examples we will need in the following.

Definition 4.3.4. A surface X is a *K3 surface* if $q(X) = 0$ and $K_X = 0$.

A surface X is an *Enriques surface* if $q(X) = 0$, $p_g(X) = 0$ and $2K_X = 0$.

A surface X is an *abelian surface* if it is a $2-$dimensional complex torus admitting an embedding in a projective space.

A surface X is a *surface of general type* if $\kappa(X) = 2$.

Let me just finish this section with some examples:

Example 4.3.4. An example of abelian surface is the following: let E and F be two elliptic curves. Then $X := E \times F$ is an abelian surface. Indeed, we have $E = \mathbb{C}/\Gamma$ and $F = \mathbb{C}/\Gamma'$ for some lattices Γ, Γ', and as they are curves they admit an embedding in some projective spaces $j_E : E \longrightarrow \mathbb{P}^n$ and $j_F : F \longrightarrow \mathbb{P}^m$. Hence X admits an embedding in a projective space

$$X \xrightarrow{(j_E, j_F)} \mathbb{P}^n \times \mathbb{P}^m \xrightarrow{s} \mathbb{P}^N,$$

where s is the Segre embedding. But now notice that $X \simeq \mathbb{C}^2/L$, where $L = \Gamma \times \Gamma'$. Hence X is a complex torus. In conclusion, X is an abelian surface.

Example 4.3.5. We already met an example of K3 surface: in $\mathbb{P}^3$ consider a generic quadric surface X. As we have seen in Example 3.2.3, we have that X is smooth and $K_X = 0$. To show that it is indeed a K3 surface, we need to prove that $q(X) = 0$. This follows from a Theorem of Lefschetz: as $\mathbb{P}^3$ is simply connected, a generic hypersurface is simply connected too, so that $b_1(X) = 0$. Then $q(X) = 0$, and X is a K3 surface. More generally, we have the following example: let $n \in \mathbb{N}$ and consider $d_1, ..., d_n \in \mathbb{N}$. In $\mathbb{P}^{n+2}$ let X be the complete intersection of multidegree $(d_1, ..., d_n)$, i. e. $X = H_1 \cap ... \cap H_n$, where H_j is a hypersurface of $\mathbb{P}^{n+2}$ of degree d_j. Then we have

$$\mathscr{O}_X(K_X) = \mathscr{O}_{\mathbb{P}^{n+2}}(-n - 3 + d_1 + ... + d_n)_{|X}.$$

If $d_1 + ... + d_n = n + 3$, then we have that $K_X = 0$. Again by the Lefschetz Theorem we have that $q(X) = 0$, so that X is a K3 surface. Notice that if $n = 1$, then we need $d_1 = 4$, so that X is the generic quartic in $\mathbb{P}^3$ we saw before. If $n = 2$, then we need $d_1 + d_2 = 5$, so that $d_1 = 2$ and $d_2 = 3$; if $n = 3$ we have $d_1 + d_2 + d_3 = 6$, so that $d_1 = d_2 = d_3 = 2$, and so on.

Example 4.3.6. We now describe an example of Enriques surface. If X is a K3 surface with an involution $i : X \longrightarrow X$, i. e. $i \circ i = id_X$, which has no fixed points, we can consider the quotient $\widetilde{X} := X/i$. We have $p : X \longrightarrow \widetilde{X}$ which is a degree 2 cover. If C is a curve on $\widetilde{X}$, it is easy to see that $p_*p^*C = 2C$, hence by linearity we get $p_*p^*D = 2D$ for every divisor D. In particular we have $p_*p^*K_{\widetilde{X}} = 2K_{\widetilde{X}}$. But now notice that $p^*K_{\widetilde{X}} = K_X$, hence it is trivial since X is a K3 surface. Then $2K_{\widetilde{X}} = 0$. Moreover, we have $\chi(\mathscr{O}_X) = 2\chi(\mathscr{O}_{\widetilde{X}})$. Notice that as X is a K3 surface, we have $q(X) = 0$ and $p_g(X) = 1$, so that $\chi(\mathscr{O}_X) = 2$. Hence $\chi(\mathscr{O}_{\widetilde{X}}) = 1$, which implies $q(\widetilde{X}) = p_g(\widetilde{X})$. Now, as p is a double cover, we easily get that $q(\widetilde{X}) = q(X) = 0$, so that $p_g(\widetilde{X}) = 0$. Notice that since $h^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(K_{\widetilde{X}})) = 0$, we need to have $K_{\widetilde{X}} \neq 0$. Hence X is an Enriques surface.

The only problem now is to find an example of K3 surface having an involution without fixed points. In $\mathbb{P}^5$ consider the surface X obtained as complete intersection of three quadrics. If $x_0, ..., x_5$ are homogenous coordinates on $\mathbb{P}^5$, we can consider the quadrics of the form

$$P_i(x_0, x_1, x_2) + Q_i(x_3, x_4, x_5) = 0,$$

where $i = 1, 2, 3$ and P_i, Q_i are homogenous polynomials of degree 2 for every $i = 1, 2, 3$. For generic choices of P_i and Q_j, the surface X is smooth (by Bertini), and it has a natural involution

$$i : X \longrightarrow X, \qquad i(x_0 : x_1 : x_2 : x_3 : x_4 : x_5) := (x_0 : x_1 : x_2 : -x_3 : -x_4 : -x_5).$$

This involution has no fixed points if and only if the three quadrics P_1, P_2, P_3 (resp. Q_1, Q_2, Q_3) have no common points. Hence, choosing them generic we have an involution without fixed points. In conclusion X/i is an Enriques surface.

4.3.2 Fibrations

Before starting the proof of the Enriques-Castelnuovo Classification for non-negative Kodaira dimension, we need to collect some basic facts about fibrations. Let me start with the following.

Lemma 4.3.2. ***(Zariski's Lemma).*** *Let X be a surface and X a smooth curve. Let $f : X \longrightarrow C$ be a fibration, and let $F = \sum_i m_i C_i$ be a fiber, where $m_i \in \mathbb{N}$ and C_i are the irreducible components of F. Moreover, let $D := \sum_i r_i C_i$, with $r_i \in \mathbb{Q}$. Then $D^2 \leq 0$, and $D^2 = 0$ if and only if $D = rF$ for some $r \in \mathbb{Q}$.*

Proof. Let $F_i := m_i C_i$, so that we can write $F := \sum_i F_i$. Let $s_i := r_i/m_i$, so that $D = \sum_i s_i F_i$. We have

$$D^2 = \sum_i s_i^2 F_i^2 + 2\sum_{i<j} s_i s_j F_i \cdot F_j =$$

$$= \sum_i s_i^2 F_i \cdot F - \sum_{i<j}(s_i^2 + s_j^2 - 2s_i s_j) F_i \cdot F_j = -\sum_{i<j}(s_i - s_j)^2 F_i F_j \leq 0,$$

as $F_i \cdot F = 0$ and $F_i \cdot F_j \geq 0$ as F is connected (f is a fibration). Notice that in order to have $D^2 = 0$ we need $s_i = s_j$ for every i, j (as there is at least one pair i, j such that $F_i \cdot F_j > 0$. Hence let $r := s_i$, so that $D = \sum_i rF_i = rF$, and we are done. □

Let me introduce the following:

Definition 4.3.5. Let X be a surface, C a smooth curve and $f : X \longrightarrow C$ a fibration. Let $F = \sum_i m_i C_i$ be a fiber, where $m_i \in \mathbb{N}$ and C_i are the irreducible components of F. The *multiplicity* of F is $m(F) = gcd(m_i)$. The fiber F is a *multiple fiber* if and only if $m(F) > 1$.

Let now $f : X \longrightarrow C$ be a fibration on the surface X, and let $c \in C$. Consider $F = f^{-1}(c)$ the fiber over c, and suppose it is multiple, i. e. $F = mF'$ where F' is a fiber and $m > 1$. Consider an open neighborhood U of $c \in C$, and let z be a local coordinate around c on U. Then, the function $z \circ f : f^{-1}(U) \longrightarrow \mathbb{C}$ vanishes with order m along F'. As the fibers are connected, every holomorphic function g over $f^{-1}(U)$ is of the form $h \circ f$, where h is a holomorphic function on U. Hence every $g \in \mathscr{O}_X(f^{-1}(U))$ vanishes with order at least m along F'.

Now, consider the line bundle $\mathscr{O}_{f^{-1}(U)}(mF')$. By the previous discussion, this sheaf is trivial, hence every function in $\mathscr{O}_{f^{-1}}(F')$ has order at most m. If a function $g \in \mathscr{O}_{f^{-1}(U)}(F')$ vanishes with order smaller than m, then we would get a function on $f^{-1}(U)$ vanishing with order smaller than m along F', which is not possible. Hence, the line bundle $\mathscr{O}_{f^{-1}(U)}(F')$ is torsion of order m in $Pic(f^{-1}(U))$. we have the following:

Proposition 4.3.3. *Let $f : X \longrightarrow C$ be a fibration on the surface X, and let $F = mF'$ be a fiber, $m \in \mathbb{N}$. The line bundle $\mathscr{O}_{F'}(F')$ is torsion of order m in $Pic(F')$.*

Proof. For the proof of this proposition we need the following lemma:

Lemma 4.3.4. *Let X be a surface, C an effective divisor on X. Consider $r_C : \mathscr{O}_X \longrightarrow \mathscr{O}_C$ the restcriction morphism, and its analogue $r^*_C : \mathscr{O}^*_X \longrightarrow \mathscr{O}^*_C$. Then we have $exp \circ r_C = r^*_C \circ exp$. In particular, the exponential sequence of X remains exact when restricted to C.*

Proof. We can assume that C is connected. Let $g \in \mathscr{I}_C$, so that $r_C(g) = 0$ and $exp(r_C(g)) = 1$. We need to show that $r^*_C(exp(g)) = 1$. Notice that $exp(g) = 1 + \sum_{m\geq 1}(2\pi i g^m)/(m!)$, so that $exp(g) - 1 \in \mathscr{I}_C$. Hence $r^*_C(exp(g) - 1) = 0$, so that $r^*_C(exp(g)) = 1$, and we are done.

For the remaining part, it is clear that the exponential is the surjective, and we only need to prove the following: let $f \in \mathscr{O}_X$ such that $r^*_C(exp(f)) = 1$. Then $f_{|C} = n$ for some $n \in \mathbb{Z}$. Indeed, we have $exp(f) - 1 \in \mathscr{I}_C$. Let $C = \sum_i n_i C_i$, where $n_i \in \mathbb{N}$ and C_i is an irreducible curve. We have that $f_{|C_i} = m_i$ by the classical exponential sequence, and by connectedness of C we need m_i to be constant, i. e. $f_{|C_i} = n$ for some integer n. Hence $f - n$ vanishes with some order m_i along every component m_i. Then even $exp(f) - 1$ vanishes with order m_i along every C_i, but as $exp(g) - 1 \in \mathscr{I}_C$ we need $m_i = n_i$. In conclusion we have $f - n \in \mathscr{I}_C$, so that $f_{|C} = n$. □

It is clear that F' is torsion in $Pic(F')$ and that its period divises m. By topological facts, the fiber F' is deformation retract of some open neighborhood, and we can suppose it to be of the form $V := f^{-1}(U)$ for some open subset U of C. The restriction map

$$H^p(V, \mathbb{Z}) \longrightarrow H^p(U, \mathbb{Z})$$

is then an isomorphism for every p. By the previous lemma we then get a commutative diagram

$$\begin{array}{ccccccc}
H^0(V,\mathbb{Z}) & \xrightarrow{a} & H^0(\mathscr{O}_V) & \xrightarrow{b} & Pic(V) & \xrightarrow{c} & H^2(V,\mathbb{Z}) \\
\| & & d\downarrow & & e\downarrow & & \| \\
H^0(F',\mathbb{Z}) & \xrightarrow{a'} & H^0(\mathscr{O}_{F'}) & \xrightarrow{b'} & Pic(F') & \xrightarrow{c'} & H^2(F',\mathbb{Z}).
\end{array}$$

Now, let us suppose that $\mathscr{O}_{F'}(F')$ is torsion of period p, and consider $pF' \in Pic(V)$. As $c(pF') = c'(pF') = 0$, there is $\alpha \in H^1(V, \mathscr{O}_V)$ such that $b(\alpha)pF'$. By commutativity we have $e(b(\alpha)) = b'(d(\alpha)) = 0$, so that there is $\beta \in H^1(F', \mathbb{Z})$ such that $d(\alpha) = a'(\beta)$. Hence $\alpha = d(\beta)$, so that $pF' = 0$ in $Pic(V)$. But F' as period m in $Pic(V)$, so that p is a multiple of m. Hence $p = m$, and we are done. □

There are still two results we will need on fibrations, namely a topological one and something about isotrivial fibrations. The first one is the following, for which we need the following lemma:

Lemma 4.3.5. *Let $C = \sum_i C_i$ be a curve on a surface X, where C_i are the irreducible components of C. Then $e(C) \geq \chi(\mathscr{O}_C)$, and $e(C) = 2\chi(\mathscr{O}_C)$ if and only if C is smooth.*

Proof. Consider the normalization $\nu : \widetilde{C} \longrightarrow C$ of the curve C. We have $\mathbb{C}_C \subseteq \nu_*\mathbb{C}_{\widetilde{C}}$, and let δ be the quotient. Similarily, we have $\mathscr{O}_C \subseteq \nu_*\mathscr{O}_{\widetilde{C}}$, and let Δ be the quotient. Hence we have $e(\widetilde{C}) = e(C) + h^0(C, \delta)$ and $\chi(\mathscr{O}_{\widetilde{C}}) = \chi(\mathscr{O}_C) + h^0(C, \Delta)$. Moreover, we have $e(\widetilde{C}) = 2\chi(\mathscr{O}_{\widetilde{C}})$, since $\chi(\mathscr{O}_{\widetilde{C}}) = 1 - g(\widetilde{C})$ and $e(\widetilde{C}) = 2 - b_1(\widetilde{X})$, where $b_1(\widetilde{C}) = 2g(\widetilde{C})$. Hence we have

$$e(C) = 2\chi(\mathscr{O}_C) + 2h^0(C, \Delta) - 2h^0(C, \delta).$$

The natural map $\delta \longrightarrow \Delta$ is an injection, so that $h^0(C, \delta) \leq h^0(C, \Delta)$, so that $e(C) \geq 2\chi(\mathscr{O}_C)$, and equality holds if and only if C is smooth. $\square$

Using this one proves the following:

Proposition 4.3.6. *Let $f : X \longrightarrow C$ be a fibration, and suppose that X is a surface with K_X nef. Moreover, let $\delta(f)$ be the set of critical values of f, i. e. $\delta(f) := \{c \in C \,|\, df(x) = 0 \ for\ every\ x \in f^{-1}(c)\}$. Let F be a smooth fiber of f, and for every $c \in \delta(f)$ let F_c be the fiber over c. Then*

$$e(X) = e(C)e(F) + \sum_{c \in \delta(f)} (e(F_c) - e(F)).$$

Moreover, we have $e(F_c) \geq e(F)$ for every $c \in \delta(f)$, and $e(F_c) = e(F)$ if and only if F_c supports a smooth elliptic curve.

Proof. As f is locally a trivial fibration over the set $C \setminus \delta(f)$, we can show that

$$e(X) = e(X \setminus f^{-1}(\delta(f))) + e(f^{-1}(\delta(f))) = e(C \setminus \delta(f))e(F) + \sum_{c \in \delta(f)} e(F_c).$$

Notice that $e(C \setminus \delta(f)) = e(C) - \sum_{c \in \delta(f)} 1$, so that we finally get the first part of the statement. Now, consider a singular fiber $F_c = \sum_i m_i C_i$, where $m_i \in \mathbb{N}$ and C_i are the irreducible components of F_c. Let $F'_c := \sum_i C_i$. By Lemma 4.3.5 we have $e(F'_c) \geq \chi(\mathscr{O}_{F'_c}$ with equality if and only if F'_c supports a smooth elliptic curve. By the adjunction formula we have

$$e(F'_c) \geq 2\chi(\mathscr{O}_{F'_c}) = -(F'_c)^2 - F'_c \cdot K_X.$$

By Lemma 4.3.2 we have $(F'_c)^2 \leq 0$, and

$$-\sum_i C_i \cdot K_X \geq -\sum_i m_i C_i \cdot K_X = -F'_c \cdot K_X = -F \cdot K_X = e(F).$$

The first inequality follows indeed from the fact that K_X is nef, and the last equality follows from andjunction formula, the fact that $F^2 = 0$ and Lemma 4.3.5. In conclusion we get $e(F_c) \geq e(F)$. Equality happens if and only if $(F_c')^2 = 0$, i. e. by the Zariski Lemma if and only if $F_c = nF_c'$ for some positive integer n. We then get $e(F_c') = e(F)/n$. Notice that as K_X is nef the generic fiber is not rational, so that $e(F) \leq 0$. Hence $e(F_c') = e(F) = 0$, and we are done. □

The conclusion of this section deals with isotrivial fibrations.

Definition 4.3.6. Let X and Y be two manifolds, and $f : X \longrightarrow Y$ be a fibration. Then f is *isotrivial* if there is a finite unramified cover $g : Y' \longrightarrow Y$ such that the pull-back $f' : X \times_Y Y' \longrightarrow Y'$ is isomorphic to a trivial fibration $Y' \times F \longrightarrow Y'$ for some projective manifold F.

Let me give a first example of an isotrivial fibration.

Example 4.3.7. Let Y be a manifold, and let G be a finite group which is a quotient of $\pi_1(Y)$ and which acts on a projective manifold F. Consider $g : Y' \longrightarrow Y$ be the cover defined by G, and let $X := (Y' \times F)/G$. Then X admits a natural isotrivial fibration on Y. Even a converse is true: if $X \longrightarrow Y$ is a fiber bundle whose fiber F has a finite automorphism group is an isotrivial fibration arising in this way.

The following proposition will be used in the proof of the classification.

Proposition 4.3.7. *Let $f : X \longrightarrow C$ be a fibration of a surface onto a curve of genus 0 or 1. Then f is isotrivial.*

Proof. The proof uses the theory of periods of curves, so we are just going to sketch it. As $f : X \longrightarrow C$ is a fibration, it is proper and surjective, hence it is differentiably locally trivial (this is Ehresman's Theorem). This implies that the family $\cup_{c\in C} H_1(F_c, \mathbb{Z})$ is locally trivial in the euclidean topology of C. As F_c is a curve, $H_1(F_c, \mathbb{Z})$ is a rank $2g(F_c)$ lattice. If one considers the sheaf $f_*\mathscr{O}_X(K_X)$, it is coherent as f is proper, and as $H^0(F_c, \mathscr{O}_{F_c}(K_{F_c}))$ is locally of constant dimension, hence $f_*\mathscr{O}_X(K_X)$ is a locally free sheaf whose rank is $g(F)$. For every point $y \in C$ there is an open neighborhood U over which $H_1(F_c, \mathbb{Z})$ is constant on U and $f_*\mathscr{O}_X(K_X)$ is free on U. This means that we can find a basis $\gamma_1, ..., \gamma_{2g(F)}$ of $H_1(F_y, \mathbb{Z})$ over U, and a basis $\omega_1(y), ..., \omega_{g(F)}(y)$ which depends holomorphically on y, so that the matrix $\Omega := [\int_{\gamma_i} \omega_j(y)]$ is normalized, i. e. it is symmetric and has positive definite imaginary part. Let now

$$\mathbb{H}_g := \{\Omega \in \mathbb{C}^{g,g} \,|\, \Omega = \Omega^T,\ Im(\Omega) > 0\},$$

over which the symplectic group Γ_g acts. We then get a holomorphic map

$$U \longrightarrow \mathbb{H}_g,$$

which does not extend to C, but to $\widetilde{C}$, the universal cover of C, getting the period map $p : \widetilde{C} \longrightarrow \mathbb{H}_g$.

Now, as $g(C) = 0, 1$, the universal cover of C is either $\mathbb{P}^1$ or $\mathbb{C}$. As $g(F) = 0, 1$, we have $\mathbb{H}_1$ bounded, so that p is constant. Then all the fibers have the same period, so that by Torelli's Theorem on curves we need them to be isomorphic. The isotriviality follows from Example 4.3.7 if $g(F) \geq 2$, as in this case the fiber has a finite group of automorphism. The case of genus 1 fiber is more involved: the action of $\pi_1(C)$ on the group of $n-$torsion points of the fiber can be trivialized up to passing to a finite unramified cover. Hence, globally one gets a zero section: but then the group of automorphism of an elliptic curve which preserves the origin is finite, and one can use again Example 4.3.7. □

4.3.3 Elliptic fibrations

Let me now come to elliptic fibrations. The first result we need is the following:

Lemma 4.3.8. *Let $f : X \longrightarrow C$ be an elliptic fibration, and let $F = mF'$ be a multiple fiber. Then F' is either an elliptic curve, a rational curve with one ordinary double point or a cycle of smooth rational curves.*

Proof. If F is a smooth fiber, we have $K_X \cdot F = 0$: indeed a smooth fiber is an elliptic curve, so that $g(F) = 1$. Moreover F^2, hence the genus formula gives $K_X \cdot F = 0$. Hence $K_X \cdot F' = 0$, so that $\chi(\mathscr{O}_{F'}) = 0$. If F is irreducible, then F' is either an elliptic curve or a rational curve with an ordinary double point. If F' is reducible and C_i is a component of F', we have $C_i^2 < 0$ by Lemma 4.3.2. Now, $C_i^2 \neq -1$ as X is minimal, and $C_i \cdot K_X = 0$. Hence by the genus formula we get $2g(C_i) - 2 = C_i^2 < 0$. Hence $g(C_i) = 0$ and $C_i^2 = -2$. Now, let us consider two different components C_i and C_j: then $C_i \cdot C_j = 0, 1, 2$. Indeed as F is connected we have $C_i \cdot C_j \geq 0$. Moreover by Lemma 4.3.2 we have

$$0 \geq (C_1 + C_2)^2 = 2C_1 \cdot C_2 - 4,$$

so that $C_1 \cdot C_2 \leq 2$.

Now, F' is not simply connected. Indeed, in $Pic(F')$ there is a non-trivial torsion line bundle by Proposition 4.3.3, hence $H^1(F', \mathscr{O}_{F'}) \neq 0$, so that $b_1(F')$ is not 0. Now, let $F' = \sum_k n_k C_k$, where $n_i \in \mathbb{N}$. If there are $i \neq j$ such that $C_i \cdot C_j = 2$, then we have

$$0 = C_i \cdot F' = \sum_k n_k C_i \cdot C_k = -2n_i + \sum_{k \neq i} n_k C_i \cdot C_k.$$

If $n_i \leq n_j$ we get

$$2n_i = 2n_j + \sum_{k \neq i,j} n_k C_i \cdot C_k,$$

then we need $n_k = 1$ for every $k \neq i, j$, and we have only two components, which hence form a cycle. If $C_i \cdot C_j = 1$, and three pairwise distinct components meet in a point, one uses a similar argument to show that there are no more components. But this implies that F' is simply connected, which is not the case, so we need that the components of F' form a cycle of smooth rational curves. $\square$

As a corollary we have the following:

Corollary 4.3.9. *Let $f : X \longrightarrow C$ be an elliptic fibration, and let F be a primitive fiber. Then the dualising sheaf $\omega_F := \mathscr{O}_F(K_X + F)$ is trivial.*

Proof. If F is elliptic, it is smooth and $\omega_F = \mathscr{O}_F(K_F)$. Hence $\omega_F = \mathscr{O}_F$ and we are done in this case. If F is not smooth, then by Lemma 4.3.8 F can only be either a rational curve with an ordinary double point, or a cycle of smooth rational curves. Hence we have either F is $\mathbb{P}^1$ or its components are $\mathbb{P}^1$. There is a global meromorphic $2-$form that has two poles with opposite residues, and we get a trivializing section of ω_F. $\square$

The final result of this section is the following:

Theorem 4.3.10. *Let $f : X \longrightarrow C$ be an elliptic fibration, and suppose that K_X is nef. Let $F_1, ..., F_m$ be the multiple fibers of f, and let $n_i := m(F_i)$, so that $F_i = n_i F_i'$. Then there is a divisor D on C such that $deg(D) = \chi(\mathscr{O}_X) - \chi(\mathscr{O}_C)$ and*

$$K_X = f^* D + \sum_{i=1}^{m} (n_i - 1) F_i'.$$

Proof. The proof is divided in two steps: first of all we show the formula for the canonical bundle, and the we calculate the degree of D.

Consider N smooth fibers $G_1, ..., G_N$ of f, and consider the exact sequence

$$0 \longrightarrow \mathscr{O}_X(K_X) \longrightarrow \mathscr{O}_X\left(K_X + \sum_{i=1}^{N} G_i\right) \longrightarrow \bigoplus_{i=1}^{N} \mathscr{O}_{G_i}(K_X + G_i) \longrightarrow 0.$$

By adjunction formula we have $\mathscr{O}_{G_i}(K_X + G_i) \simeq \mathscr{O}_{G_i}$, as G_i is an elliptic curve. That the long exact sequence induced by this gives

$$h^0\left(X, \mathscr{O}_X\left(K_X + \sum_{i=1}^{N} G_i\right)\right) \geq p_g(X) - q(X) + (N - 1).$$

If $N \gg 0$ then there is an effective divisor $L \in |K_X + \sum_i G_i|$. Let now F be any fiber, so that $F \cdot G_i = 0$ and $F \cdot K_X = 0$ by the genus formula. Hence $F \cdot L = 0$, so that L consists of components L_i of fibers, for $i = 1, ..., M$. As L is effective, we have

$$L \cdot L_i = K_X \cdot L_i + \sum_j G_j \cdot L_i = K_X \cdot L_i \geq 0,$$

as K_X is nef. Now, let me prove the following:

Lemma 4.3.11. *Let $f : X \longrightarrow C$ be an elliptic fibration with K_X nef. Then $K_X^2 = 0$.*

Proof. As K_X is nef, we have $K_X^2 \geq 0$. Moreover, we have $\kappa(X) \geq 0$, so there is $n \in \ltimes$ such that $|nK_X|$ is not empty, and take $D \in |nK_X|$. As $D \cdot F = 0$, the components of D are components of fibers. Then $n^2 K_X^2 = D^2 \leq 0$ by Lemma 4.3.2, so that $K_X^2 = 0$. □

Using this, we get

$$0 = K_X^2 = K_X \cdot (L - \sum_i G_i) = K_X \cdot L,$$

so that $K_X \cdot L_i = L \cdot L_i = 0$. In particular, we get $L^2 = 0$. Now, let $L = \sum_i L_i'$, where L_i' is the part supported on a fiber. Hence $(L_i')^2 = 0$, so that L_i' is a rational multiple of a fiber by Lemma 4.3.2. Any part supported on a multiple fiber F_i can then be written in the form $k_i F_i' + r_i F_i$, where $0 < k_i < n_i$ and $r_i \in \mathbb{Z}$. The part supported on a non-multiple fiber is of the form $r_j G_j$ for some $r_j \in \mathbb{Z}$. In conclusion we get

$$K_X = f^* D + \sum_i k_i F_i'.$$

Now, notice that

$$\mathscr{O}_{F_i'}((k_i + 1)F_i') \simeq \mathscr{O}_{F_i'}(K_X + F_i') \simeq \mathscr{O}_{F_i'},$$

where the last isomorphism comes from Coroloary 4.3.9. By Proposition 4.3.3 we then have that $k_i + 1$ is a multiple of n_i. As $k_i + 1 \leq n_i$ we finally get $n_i - 1 = k_i$, and the first step is proved.

Let us now calculate the degree of D. We compute $h^0(X, \mathscr{O}_X(K_X + \sum_j G_j))$ in two ways. The first one is the following: the divisor $\sum_i (n_i - 1)F_i'$ is a fixed component of the linear system $|K_X + \sum_j G_j|$. Writing $c_j := f(G_j)$ we then get

$$h^0(X, \mathscr{O}_X(K_X + \sum_j G_j)) = h^0(X, \mathscr{O}_X(f^* D + \sum_j G_j)) =$$

$$= h^0(C, \mathscr{O}_C(D + \sum_j c_j)) = deg(D) + N + 1 - g(C),$$

for $N \gg 0$, by Riemann-Roch. However, using the previous part we have

$$h^0(X, \mathscr{O}_X(K_X + \sum_j G_j)) = \chi(\mathscr{O}_X) + N - 1 + d,$$

where $d := dim(im(H^1(X, \mathscr{O}_X(K_X)) \longrightarrow H^1(X, \mathscr{O}_X(K_X + \sum_j G_j))))$. Hence

$$deg(D) = \chi(\mathscr{O}_X) + g(C) - 2 + d.$$

We then only need to calculate d. By Serre's Duality this is simply

$$d = dim(im(H^1(X, \mathscr{O}_X(-\sum_j G_J)) \longrightarrow H^1(X, \mathscr{O}_X(K_X)))) =$$

$$= dim(\ker(H^1(X, \mathscr{O}_X) \longrightarrow \bigoplus_j H^1(G_j, \mathscr{O}_{G_j}))).$$

Use now the Hodge decomposition: we have $H^1(X, \mathscr{O}_X) \simeq \overline{H^0(X, \Omega_X)}$, and that $H^1(G_j, \mathscr{O}_{G_j}) \simeq \overline{H^0(G_j, \Omega_{G_j})}$, hence

$$d = dim(\ker(H^0(X, \Omega_X) \longrightarrow \oplus_j H^0(G_J, \Omega_{G_j}))).$$

This kernel is given by those holomorphic $1-$forms which are pull-back of holomorphic $1-$forms on C, so that $d = g(C)$. In conclusion

$$deg(D) = \chi(\mathscr{O}_X) - 2 + 2g(C) = \chi(\mathscr{O}_X) - \chi(\mathscr{O}_C),$$

and we are done. □

As a corollary of this, we have the following:

Corollary 4.3.12. *Let $f : X \longrightarrow C$ be an elliptic fibration. Then $\kappa(X) \leq 1$. In particular, $\kappa(X) = 1$ if and only if $2g(C) - 2 + \chi(\mathscr{O}_X) + \sum_i (n_i - 1)/n_i \neq 0$.*

Proof. The formula for the canonical bundle of X can be written an $K_X = f^*L$, where L is a divisor of degree $d = 2g(C) - 2 + \chi(\mathscr{O}_X) + \sum_i \frac{n_i - 1}{n_i}$. Hence $mK_X = f^*L_m$, where L_m is a line bundle of degree md on C. Hence

$$P_m(X) = h^0(C, \mathscr{O}_C(L_m)) \sim m\bigg(2g(C) - 2 + \chi(\mathscr{O}_X) + \sum_i \frac{n_i - 1}{n_i}\bigg).$$

In conclusion $\kappa(X) \leq 1$, and the remaining part is clear. □

4.3.4 Classification for Kodaira dimension 0 and 1

We are finally able to conclude the classification for Kodaira dimension 0 or 1. The first result we need is the following:

Lemma 4.3.13. *Let X be a smooth minimal projective surface with $\kappa(X) = 0$. Then $K_X^2 = 0$ and $P_m(X) = 0, 1$ for every $m > 0$, and there is $m > 0$ such that $P_m(X) = 1$.*

Proof. Let us first show the first item. As $\kappa(X) = 0$, there is $m > 0$ such that $|mK_X| \neq \emptyset$. Let $D \in |mK_X|$, so that $D \cdot mK_X \geq 0$ implies $K_X^2 \geq 0$. Now, suppose $K_X > 0$. Then by Riemann-Roch we get

$$h^0(X, \mathscr{O}_X(nK_X)) + h^0(X, \mathscr{O}_X((1-n)K_X)) \geq \chi(\mathscr{O}_X) + \binom{n}{2} K_X^2.$$

For $n > 1$ we have $|(1-n)K_X| = \emptyset$, otherwise there would be $E \sim (1-n)K_X$ such that $E \cdot K_X \geq 0$, getting $K_X^2 \leq 0$. In conclusion $P_n(X) \sim n$, so that $\kappa(X) > 0$, hence $K_X^2 = 0$.

The remaining part is easier: suppose that $m > 0$ is such that $P_m(X) > 1$, i. e. there are at least two linearly independent $\sigma, \tau \in H^0(X, \mathscr{O}_X(mK_X))$. Hence in for every $n > 0$ the elements $\sigma^n, \sigma^{n-1}\tau, ..., \tau^n$ are linearly independent in $H^0(X, \mathscr{O}_X(nmK_X))$, i. e. $P_{nm}(X) \geq n+1$, so that $\kappa(X) \geq 1$. But since $\kappa(X) = 0$, this is not possibile, so that $P_m(X) = 0, 1$. Moreover, if for every $m > 0$ we have $P_m(X) = 0$, then $\kappa(X) = -\infty$, so that in order to have $\kappa(X) = 0$ there must be $m > 0$ such that $P_m(X) = 1$. □

Here is the first point of the classification.

Proposition 4.3.14. *Let X be a smooth minimal projective surface, and suppose that K_X is nef. If $K_X^2 = 0$, $q(X) = 1$ and $p_g(X) = 0$, then $\kappa(X) = 0, 1$. Moreover, in this case $\kappa(X) = 0$ if and only if X is a bielliptic surface.*

Proof. As $q(X) = 1$, the Albanese torus $Alb(X)$ associated to X is a curve. As it has to be abelian, $Alb(X)$ is an elliptic curve. Recall that

$$a_X : X \longrightarrow Alb(X)$$

has connected fibers by Proposition 3.3.6. Now, as K_X is nef, $K_X^2 = 0$, $q(X) = 1$ and $p_g(X) = 0$, by Proposition 4.2.13 we need that $b_2(X) = 2$. Moreover, as $q(X) = 1$ we have $b_1(X) = 2$, so that $e(X) = 0$. By Proposition 4.3.6 we have then

$$e(Alb(X))e(F) = -\sum_{c \in \delta(a_X)} (e(F_c) - e(F)) \leq 0.$$

By Lemma 4.3.5 we have for every smooth curve C that $e(C) = 2 - 2g(C) \leq 2$. As K_X is nef, the generic fiber F cannot be rational, so that $e(F) \leq 0$. We have then only two possibilities: either $e(F) < 0$, so that $g(F) \geq 2$; or $e(F) = 0$, so that the generic fiber is elliptic. But then for every $c \in \delta(a_X)$ we have $e(F_c) = e(F)$, which by Proposition 4.3.6 happens if and only if F_c supports on an elliptic curve.

Let us start with the first case, i. e. $g(F) \geq 2$ for the generic fiber, so that $e(F) < 0$. As $e(Alb(X)) \cdot e(F) \leq 0$, we can only have $e(Alb(X)) = 0, 1$. Hence we can apply Proposition 4.3.7 to get that a_X is the isotrivial. Hence, there is a finite unramified cover $\widetilde{X} \longrightarrow X$ such that $\widetilde{X} = Alb(X) \times F$. Recall that the $\kappa(\widetilde{X}) = \kappa(X)$ by Proposition 3.2.7, hence $\kappa(X) = \kappa(Alb(X) \times F)$. Now, notice that $g(F) > 1$, and $g(Alb(X)) = 0, 1$. If $g(Alb(X)) = 0$, then $e(F_c) = e(F)$ for every $c \in \delta(a_X)$, so that by Proposition 4.3.6 the fiber F is elliptic. This is not possibile, so that $Alb(X)$ is an elliptic curve. By Example 4.3.1 we then get that $\kappa(X) = 1$. Notice that in this case X is not bielliptic, as every bielliptic surface has Kodaira dimension 0.

Let us then suppose that $g(F) = 1$, so that every fiber is elliptic. If there are multiple fibers, then one can use Corollary 4.3.12 to get that $\kappa(X) = 1$: indeed $\chi(\mathscr{O}_X) = 0$, so that $\kappa(X) = 1$ if and only if $2g(Alb(X)) - 2 + \sum_i (n_i - 1)/n_i \neq 0$. If $g(Alb(X)) > 0$ and there are multiple fibers, this is automathic. Let us then suppose that $g(Alb(X)) = 0$, so that a_X is an elliptic fibration over $\mathbb{P}^1$. If $\kappa(X) = 0$ then $\chi(\mathscr{O}_X) - 2 + k - \sum_{i=1}^k 1/n_i = 0$ by Corollary 4.3.12. As $\chi(\mathscr{O}_X) \geq 0$, we then get $2 - k + \sum_{i=1}^k 1/n_i \geq 0$. Hence there are only the following possibilites:

$$\chi(\mathscr{O}_X) = 1, \quad k = 2, \quad n_1 = n_2 = 2,$$

$$\chi(\mathscr{O}_X) = 2, \quad k = 3, \quad n_1 = n_2 = n_3 = 3,$$

$$\chi(\mathscr{O}_X) = 2, \quad k = 3, \quad n_1 = 2,\ n_2 = n_3 = 4,$$

$$\chi(\mathscr{O}_X) = 2, \quad k = 3, \quad n_1 = 2,\ n_2 = 3,\ n_3 = 6,$$

$$\chi(\mathscr{O}_X) = 2, \quad k = 4, \quad n_1 = n_2 = n_3 = n_4 = 4.$$

In conclusion, we get that $\kappa(X) = 0$ if $\chi(\mathscr{O}_X) = 1, 2$, but we have $\chi(\mathscr{O}_X) = 0$, so that $\kappa(X) = 1$. We can then suppose that there are no multiple fibers.

We have then a surface with an elliptic fibration a_X which is everywhere of maximal rank. Then by Proposition 4.3.6 we get that $e(Alb(X)) = 0$, so that $g(Alb(X)) = 1$, and one can then apply Proposition 4.3.7 to get that a_X is an isotrivial fibration. In conclusion there are two elliptic curves E and F and a group G of automorphisms of F acting as translations on E such that $X = (E \times F)/G$. As by hypothesis we have $p_g(X) = 0$, then X is a bielliptic surface. □

Let me first conclude the classification for Kodaira dimension 1.

Theorem 4.3.15. *Let X be a smooth minimal projective surface such that K_X is nef and $K_X^2 = 0$. Then $\kappa(X) = 0, 1$. Moreover, every smooth minimal projective surface of Kodaira dimension 1 admits an elliptic fibration.*

Proof. As K_X is nef and X is minimal, we have $\kappa(X) \geq 0$. Let us suppose tha $\kappa(X) \geq 1$. Hence, for $n \gg 0$ the linear system $|nK_X|$ has at least dimension 1. Let D_f be the fixed part of this linear system, and let D be a moving divisor in it. Then

$$0 = n^2 K_X = K_X \cdot (D + D_f) = K_X \cdot D + K_X \cdot D_f \geq 0,$$

as K_X is nef. Hence $K_X \cdot D = K_X \cdot D_f = 0$. Now

$$0 = nD \cdot K_X = D^2 + D \cdot D_f \geq 0,$$

as D is moving, so that $D^2 = D \cdot D_f = 0$.

In particular, this implies that the map $Sf := \varphi_{nK_X} : X \longrightarrow |nK_X|$ maps every divisor in $|D|$ to a point, hence the image of this map is a curve. As this is true for every $n \gg 0$, we then get that $|nK_X|$ has dimension at most 1, so that $\kappa(X) = 1$.

Let us then suppose that $\kappa(X) = 1$. By the genus formula, as $F \cdot K_X = 0$ for every smooth fiber F, we get that $g(F) = 1$, hence the generic fiber is an elliptic curve, and X admits an elliptic fibration. □

Definition 4.3.7. An elliptic surface of Kodaira dimension 1 is called *proper elliptic surface.*

Using this terminology, the second part of the previous statement can be restated by saying that every surface of Kodaira dimension 1 is proper elliptic. The final step in the classification is the following:

Theorem 4.3.16. *Let X be a smooth minimal projective surface with $\kappa(X) = 0$. Then X can only be one of the following:*

1. *a K3 surface;*
2. *an abelian surface;*
3. *a bielliptic surface;*
4. *an Enriques surface.*

Proof. As $\kappa(X) = 0$, we have that $K_X^2 = 0$ and K_X is nef. Hence, by Proposition 4.2.13 we have only three possibilites. The first one is $q(X) = 1$ and $p_g(X) = 0$, and in this case X is bielliptic by Proposition 4.3.14.

For the remaining cases, notice that we have only two possibilities: either $p_g(X) = 0$, or $p_g(X) > 0$. In the first case, we need $q(X) = 0$ (otherwise $q(X) = 1$ by Proposition 4.2.13, and hence X is bielliptic), so that $P_2(X) = 1$.

Let us consider $p_g(X) = 0$, and let us suppose that $P_3(X) \neq 0$. Then $P_3(X) = 1$ as $\kappa(X) = 0$. As $P_2(X) = P_3(X) = 1$, then let $D_2 \in |2K_X|$ and $D_3 \in |3K_X|$, so that $3D_2, 2D_3 \in |6K_X|$. Since $P_6(X) \leq 1$ as $\kappa(X) = 0$, then we need $3D_2 = 2D_3$, and there must be an effective divisor D such that $2D = D_2$ and $3D = D_3$. But then $D \in |K_X|$, but as $p_g(X) = 0$ then this is not possible, hence $P_3(X) = 0$. Hence $h^0(X, \mathscr{O}_X(3K_X)) = 0$, so that Riemann-Roch we get $h^0(X, \mathscr{O}_X(-2K_X)) \geq 1$. But as $P_2(X) = h^0(X, \mathscr{O}_X(2K_X)) = 1$, this is possible if and only if $2K_X = 0$. Hence in this case X is an Enriques surface.

We are then left with the case $p_g(X) > 0$. This implies that $p_g(X) = 1$, and consider the Noether Formula

$$12(2 - q(X)) = e(X) = 2 - 4q(X) + b_2(X).$$

Then $b_2(X) = 22 - 8q(X)$, which clearly implies that $q(X) = 0, 1, 2$.

If $q(X)=0$, then by the Riemann-Roch inequality we get

$$h^0(X,\mathscr{O}_X(2K_X))+h^0(X,\mathscr{O}_X(-K_X))\geq 2.$$

As $h^0(X,\mathscr{O}_X(2K_X))=0,1$ we then get $h^0(X,\mathscr{O}_X(-K_S))\geq 1$. As $p_g(X)=1$, this implies that K_X has to be trivial. In conclusion, in this case X is a K3 surface.

We can now suppose $p_g(X)=1$ and $q(X)>0$, so that $q(X)=1,2$. As $q(X)>0$, we have $Pic^0(X)\neq 0$, so that there is a non-tivial line bundle $\mathscr{O}_X(\tau)$ such that $\mathscr{O}_X(2\tau)\simeq\mathscr{O}_X$. If $q(X)=1$, then use the Riemann-Roch inequality to get that

$$h^0(X,\mathscr{O}_X(\tau))+h^0(X,\mathscr{O}_X(K_X-\tau))\geq 1.$$

Then $h^0(X,\mathscr{O}_X(K_X-\tau))\geq 1$, so that there is $D\in|K_X-\tau|$. Then $2D=2K_X$, which implies $D=K_X$, so that τ is trivial. But this is not possible, hence $q(X)\neq 1$.

In conclusion we need $q(X)=2$, consider the canonical bundle $K_X=\sum_i m_iC_i$, where C_i are the components of K_X. As K_X is nef we get

$$0=K_X^2=\sum_i m_iC_i\cdot K_X,$$

so that $C_i\cdot K_X=$ for every i. Similarily we have

$$0=C_i\cdot K_X=m_iC_i^2+\sum_{j\neq i}m_jC_i\cdot C_j\geq m_iC_i^2\geq 0.$$

In conclusion $C_i^2\leq 0$ and we have only two possibilities: as

$$2g(C_i)-2=C_i^2\leq 0,$$

we need either that $C_i^2=-2$ and $g(C_i)=0$, or $C_i^2=0$ and $g(C_i)=1$. In this last case we get $C_i\cdot C_j=0$ fpor every i,j, hence we can divide the components C_i of K_X in two families: one is given by smooth rational curves, the other by smooth elliptic curves or rational curves with an ordinary double point.

Consider the Albanese map $a_X:X\longrightarrow Alb(X)$. As $q(X)=2$, then $Alb(X)$ is an abelian surface, and the image of a_X is either a curve C or the whole $Alb(X)$. In the first case, as $q(X)=2$, the curve C has genus 2 by Proposition 3.3.6, and we have a fibration $a_X:X\longrightarrow C$. As the components of the canonical bundle are either rational or elliptic, they cannot be mapped via a_X on C, so they must be contracted to points. By the Zariski Lemma, this implies that K_X is a multiple of a fiber. Let F be such a fiber, and consider $D=(a/b)F$, where $a,b\in\mathbb{N}$. Then $bD=a_X^*(aP)$ for some point $P\in C$: hence $h^0(X,\mathscr{O}_X(nbK_X))$ and $h^0(X,\mathscr{O}_X(nbD))$ grow indefinitely with n. As $\kappa(X)=0$ this is not possible, so K_X has to be a trivial multiple of a fiber, i. e. $K_X=0$. Now, consider a cover $C'\longrightarrow C$ which is unramified and of degree $d\geq 2$. Pulling-back the fibration we get a surface X' together with a fibration $f' : X' \longrightarrow C'$ and an unramified cover $X' \longrightarrow X$ of degree d. Hence $K_{X'} = 0$, $\chi(\mathscr{O}_{X'}) = d\chi(\mathscr{O}_X) = 0$, so that $q(X') = 2$. But $q(X') \geq g(C') \geq 3$, so that the possibility of having $im(a_X)$ of dimension 1 is not possible.

Hence a_X has 2-dimensional image, which is the torus $Alb(X)$. As every 2−cycle on $Alb(X)$ is homologous to a 2−cycle which lifts to X, the map

$$a_X^* : H^2(Alb(X), \mathbb{C}) \longrightarrow H^2(X, \mathbb{C})$$

is injective. Now, notice that $b_2(X) = 22 - 8q(X) = 6$, and $b_2(Alb(X)) = 6$ as it is a torus, so that a_X^* is an isomorphism: in particular, no fundamental cohomology class of a curve is mapped to zero. Hence a_X does not contract any curve, and it has then to be a finite morphism.

Consider D to be a component of K_X. As $a_X(D)$ is not a point, it has to be an elliptic curve mapping to an elliptic curve E of $Alb(X)$. Consider $E' := Alb(X)/E$, which is an elliptic curve, and consider the surjective morphism $X \longrightarrow E'$. The Stein factorization of this gives an elliptic fibration on X, and D is one of the fibers. As $D^2 = 0$, the Zariski Lemma implies that D is a multiple of a fiber. Following arguments as before, one gets $\kappa(X) = 1$, so that this is not possible. In conclusion $K_X = 0$. By the formula of the canonical divisor under coverings we conclude that a_X is an unramified covering. In conclusion, X is a torus, and we are done. □

The previous Theorem then concludes the proof of the Enriques-Castelnuovo birational classification of smooth projective surfaces. We can now to present all the invariants of surfaces of Kodaira dimension 0:

1. Let X be a K3 surface: then $K_X = 0$, $P_m(X) = 1$ for every $m > 0$,

$$q(X) = 0, \quad p_g = 1, \quad \chi(\mathscr{O}_X) = 2,$$

$$e(X) = 24, \quad b_1(X) = 0 \quad b_2(X) = 22.$$

2. Let X be an abelian surface: then $K_X = 0$, $P_m(X) = 1$ for every $m > 0$,

$$q(X) = 2, \quad p_g(X) = 1, \quad \chi(\mathscr{O}_X) = 0,$$

$$e(X) = 0, \quad b_1(X) = 4, \quad b_2(X) = 6.$$

3. Let X be an Enriques surface: then $K_X \neq 0$ but $2K_X = 0$, $P_m(X) = 0$ if m is odd and $P_m(X) = 1$ if m is even,

$$q(X) = 0, \quad p_g(X) = 0, \quad \chi(\mathscr{O}_X) = 1,$$

$$e(X) = 12, \quad b_1(X) = 0, \quad b_2(X) = 10.$$

4. Let X be a bielliptic surface: then $K_X \neq 0$ but torsion of period 4 or 6 (we will see this),

$$q(X) = 1, \quad p_g(X) = 0, \quad \chi(\mathscr{O}_X) = 0,$$

$$e(X) = 0, \qquad b_1(X) = 2, \qquad b_2(X) = 2.$$

Chapter 5

Further properties and classifications

In the previous chapter we completed the proof of the Enriques-Castelnuovo birational classification of smooth projective surfaces. If X is such a surface, then it can be ot two types only: either X is minimal, or it is not. The difference between these two cases is on the existence of $(-1)-$curves by the Castelnuovo Contraction Theorem. As every smooth projective surface is birational to a minimal one, the classification proceeds by classifying only minimal surfaces. Then, the classification divides in 4 families, by means of the Kodaira dimension, which can be $-\infty$, 0, 1 or 2.

For surfaces of Kodaira dimension 2, the Enriques-Castelnuovo Theorem doesn't say anything: we only know that they exist (see Example 4.3.1), and that they do not admit elliptic or rational fibrations. For surfaces of Kodaira dimension 1, the Enriques-Castelnuovo Classification says only that they are all proper elliptic, and that they cannot have $p_g = 0$ and $q(X) = 1$.

A more complete classification is given for Kodaira dimension 0: these surfaces can only be K3, Enriques, abelian or bielliptic, and we have seen how to chaacterize them in terms of self-intersection of the canonical bundle and of birational invariants like q and p_g. Finally, for Kodaira dimension $-\infty$ we have shown that minimal surfaces of this type can only be either $\mathbb{P}^2$ or geometrically ruled. Moreover, we have a complete list of rational geometrically ruled surfaces, namely the Hirzebruch surfaces $\mathbb{F}_n$.

It is then natural to ask if it is possibile to give a complete birational classification of smooth projective surfaces, going beyond the Enriques-Castelnuovo Theorem. The desired result would be a complete list of minimal surfaces. This is the aim of this chapter, i. e. to study further properties and characteristics of surfaces aiming to a finer classification. In the following we will then introduce some important features, examples and properties of ruled surface, rational surfaces, K3 surfacess, Enriques surfaces, abelian surface, bielliptic surfaces, elliptic surfaces ans surfaces of general type.

5.1 Ruled surfaces

In this section we start by the first family we encounter in the birational classification by Enriques and Castelnuovo. A ruled surface X over a smooth curve C has $\kappa(X) = -\infty$, so that $P_m(X) = 0$ for every $m > 0$, $q(X) \geq g(C)$ and $p_g(X) = 0$. Moreover, if X is not ruled, i. e. $q(X) > 0$, if is minimal if and only if is geometrically ruled, i. e. there is rank 2 vector bundle E on C such that $X \simeq \mathbb{P}(E)$. If X is ruled, i. e. $q(X) = 0$, then using properties of rank 2 vector bundles on $\mathbb{P}^1$ we have been able to present a complete list of the possibilities of X: if X is not geometrically ruled, then it can be only $\mathbb{P}^2$; if it is geometrically ruled it can only be one of the Hirzebruch surfaces $\mathbb{F}_n$, for $n \neq 1$.

In this section we analyze the situation for non-rational minimal ruled surfaces, i. e. geometrically ruled surfaces with $q > 0$. Moreover, we will present some criterion to distinguish between non-rational ruled surface and bielliptic surface (which have the same irregularity and geometrical genus).

5.1.1 Classification of non-rational ruled surfaces

If X is a non-rational ruled surface, by Theorem 4.2.8 X is minimal if and only if it is geometrically ruled, and by Theorem 4.2.2 this is the case if and only if it is a projective bundle over a genus $g > 0$ curve. Hence, a birational classification of non-rational ruled surfaces is equivalent to a classification of rank 2 vector bundles on curves up to a tensor with a line bundle.

So, let X be a ruled surface birational tu $C \times \mathbb{P}^1$, where C is a smooth curve of genus $g > 0$. Let E be a rank 2 vector bundle on C, and $L \in Pic(C)$. Notice that

$$deg(E \times L) = deg(E) + 2deg(L),$$

so that we can always suppose E to be of degree 0 (or af any desired degree).

Lemma 5.1.1. *Let C be a smooth curve of genus g, and suppose E to be a rank 2 vector bundle on C.*

1. *There are $L, M \in Pic(X)$ such that there is an exact sequence*

$$0 \longrightarrow L \longrightarrow E \longrightarrow M \longrightarrow 0,$$

   i. e. E defines an element in $Ext^1(M, L)$.

2. *If $h^0(C, E) > 0$, then we can choose $L \simeq \mathscr{O}_C(D)$, where D is a divisor on C which can be either trivial or effective.*

3. *If $h^0(C,E) > 1$, then we can choose D to be effective.*

Proof. Up to changing E with $E \times H$ for some ample line bundle, we can always suppose $h^0(C,E) > 0$ and $deg(E) > 0$.

There is then $s \in H^0(C,E)$ which is not zero, so it defines a morphism $s^* : E^* \longrightarrow \mathscr{O}_C$. The image is clearly an ideal of $\mathscr{O}_C$, i. e. it is of the form $\mathscr{O}_C(-D)$ where D is a the divisor $D = Z(s)$. We have then a surjective map $s^* : E^* \longrightarrow \mathscr{O}_C(-D)$, and the kernel of s^* is clearly a line bundle. Taking duals, this implies item 1 and 2.

For item 3, we just need to show that we can choose s to to vanish at some points. As $h^0(C,E) > 1$ there are at least two linearily independent sections $s,t \in H^0(C,E)$. As we supposed $deg(E) > 0$, we have $deg(det(E)) > 0$, so that the section $s \wedge t \in H^0(C, det(E))$ has to vanish at some points. Let $p \in Z(s \wedge t)$. Hence there are $\mu, \lambda \in \mathbb{C} \setminus \{0\}$ such that $\mu s(p) + \lambda t(p) = 0$. In conclusion $\mu s + \lambda t$ vanishes at p, so that there is global section of E which vanishes at some points, and we are done. □

Corollary 5.1.2. *(**Riemann-Roch's Theorem for rank 2 vector bundles**). Let C be a smooth curve of genus g and E a vector bundle of rank 2 on C. Then*

$$\chi(E) = deg(E) - 2g + 2.$$

Proof. By Lemma 5.1.1 there are two line bundles $L, M \in Pic(C)$ such that $\chi(E) = \chi(L) + \chi(M)$. By the Riemann-Roch Theorem for curves we then get

$$\chi(E) = deg(L) + deg(M) - 2g + 2.$$

But notice that

$$deg(E) = deg(det(E)) = deg(L \times M) = deg(L) + deg(M),$$

and we are done. □

As every rank two vector bundle E on C is the extension of a line bundle M by another line bundle L on C, it seems natural to ask when the sequence splits, i. e. $E \simeq L \oplus M$. The sequence can be rewritten as

$$0 \longrightarrow L \otimes M^{-1} \longrightarrow E \otimes M^{-1} \longrightarrow \mathscr{O}_C \longrightarrow 0,$$

so that $E \simeq L \oplus M$ if and only if $H^1(C, L \otimes M^{-1}) = 0$. Here is one of the two main results of this section:

Theorem 5.1.3. *Let C be an elliptic curve. A rank 2 vector bundle E on C can be one of the following:*

1. *there are two line bundles $L, M \in Pic(C)$ such that $E \simeq L \oplus M$;*

2. *there is a line bundle $L \in Pic(C)$ such that E is isomorphic to the non-trivial extension of L by L.*

3. *there is a line bundle $L \in Pic(C)$ and a point $p \in C$ such that E is isomorphic to the non-trivial extension of $L \otimes \mathscr{O}_C(p)$ by L.*

Proof. We can assume E to have degree $d = 1, 2$, so that $h^0(C, E) > 0$ by Corollary 5.1.2. By Lemma 5.1.1 there are then a line bundle L_k and a line bundle M_{d-k} of degrees $k \geq 0$ and $d-k$ respectively, such that there is an exact sequence

$$0 \longrightarrow L_k \longrightarrow E \longrightarrow M_{d-k} \longrightarrow 0.$$

If $d = 2$, we can even suppose $k > 0$ by Lemma 5.1.1. If $k = 0$, we can suppose $L_0 \simeq \mathscr{O}_C$. Then E defines an element in $H^1(C, L_k \otimes M_{d-k}^{-1})$, where $deg(L_k \otimes M_{d-k}^{-1}) = 2k - d$. If $2k > d$, then, this is 0, so that E is split. We then need to analyze the case $2k \leq d$. As $d = 1, 2$, this is $2k \leq 1$ or $2k \leq 2$, and as $k \geq 0$ the first is the case $k = 0$, the second is $k = 1$ (as if $d = 2$ we assume $k > 0$).

In the first case we have then $deg(E) = 1$, and E is an extension of a line bundle of degree 1 by $\mathscr{O}_C$. Notice that in this case $h^1(C, \mathscr{O}_C(p) = h^0(C, \mathscr{O}_C(-p)) = 0$ by Serre's duality, as C is elliptic, so that

$$h^1(C, \mathscr{O}_C(-p)) = h^0(C, \mathscr{O}_C(p)) = deg(\mathscr{O}_C(p)) = 1,$$

by Riemann-Roch. Hence E is a non-trivial extension, and we get item 3.

In the second case we have $deg(E) = 2$, and E is an extension of a line bundle of degree 1 by a line bundle of degree 1. Up to torsion, we have $L_1 \otimes M_1^{-1} \simeq \mathscr{O}_C$, so that $L_1 \simeq M_1$. Again, notice that

$$h^1(C, L_1 \otimes M_1^{-1}) = h^1(C, \mathscr{O}_C) = g(C) = 1,$$

as C is elliptic, so that E is a non-trivial extension. We finally get item 2, and we are done. □

This gives a complete classification of geometrically ruled surfaces of with $q = 1$. They are all of the form $\mathbb{P}(\mathscr{O}_C \oplus L)$ for every $L \in Pic(X)$, or of the form $\mathbb{P}(E)$ where E is the non-trivial extension of $\mathscr{O}_C$ via $\mathscr{O}_C$, or the non-trivial extension of $\mathscr{O}_C(p)$ via $\mathscr{O}_C$ (for any point $p \in C$). The first case is the same one we have for rational geometrically ruled surfaces (as $Pic(\mathbb{P}^1) \simeq \mathbb{Z}$, it is even more precise).

It then remains to study the last case, i. e. whene $g > 1$. Here the result is much less precise, and we have the following:

Theorem 5.1.4. *Let C be a smooth curve of genus $g \geq 2$. For every line bundle $L \in Pic(C)$ there is a quasi-projective variety S_L of dimension $dim(S_L) \geq 2g-3$ parameterizing indecomposable rank 2 vector bundles E such that $det(E) \simeq L$.*

Proof. Let $L \in Pic(C)$ be a line bundle of degree $g-1>0$, and suppose that $h^0(C,L)=0$. Then $h^0(C,L^{-1})=0$, so that

$$h^1(C,L^{-1}) = 2g-2 \geq 2.$$

Consider an affine hyperplane S_L in $H^1(C,L^{-1})$ which does not pass through the origin. Notice that S_L is a quasi-projective variety of dimension $2g-3$. Moreover, every point $s \in S_L$ corresponds to an exact sequence

$$0 \longrightarrow \mathscr{O}_C \longrightarrow E_s \longrightarrow L \longrightarrow 0.$$

Notice that $h^0(C,E_s)=0$, so that s determines a unique extension which is indecomposable, and $deg(E_s)=deg(L)$. □

5.1.2 Ruled and bielliptic surfaces

According to the Enriques-Castelnuovo Classification, there are two types of surfaces having $p_g=0$ and $q>0$: namely, non-rational ruled surfaces and bielliptic surfaces. Here we want to give criteria to characterize those surfaces in the first or in the second class. The first result is the following:

Proposition 5.1.5. *Let X be a smooth projective surface.*

1. *If $p_g(X)=0$ and $q(X)\geq 1$, then $K_X^2 \leq 0$, and $K_X^2=0$ if and only if $q(X)=1$ and $b_2(X)=2$.*
2. *If $K_X^2<0$ and X is minimal, then $p_g(X)=0$ and $q(X)\geq 1$.*

Proof. For the first item, as that $p_g(X)=0$ Noether's Formula gives

$$12-12q(X) = K_X^2+2-4q(X)+b_2(X),$$

so that $K_X^2 = 10-8q(X)-b_2(X)$. As $q(X)\geq 1$, we have that $K_X^2\leq 0$ if and only if $b_2(X)\geq 2$ when $q(X)=1$. To prove this consider the Albanese map $a_X : X \longrightarrow Alb(X)$. As $q(X)=1$, the Albanese torus is an elliptic curve, and the generic fiber F is connected. Since $F^2=0$ and $H\cdot F>0$ for every hyperplane section H, we see that F and H are linearly independent, so that $b_2(X)\geq 2$.

For the second item, as X is minimal and $K_X^2<0$, then K_X is not nef. Hence X is either geometrically ruled or $\mathbb{P}^2$, so that $p_g(X)=0$. If X is $\mathbb{P}^2$, then $K_X^2=9$, so that X has to be geometrically ruled. If $q(X)=0$, then X is rational. But in this case $K_X^2=8$ by Proposition 4.2.3, so that $q(X)>0$, □

Notice then that one can distinguish between non-rational ruled surfaces and bielliptic surfaces by means of the self-intersection of the canonical bundle: X is bielliptic if and only if $K_X^2=0$. Moreover, we have shown that the only minimal surfaces whose canonical bundle has negative self-intersection are non-rational geometrically ruled surfaces. To conclude this section, let me resume all the invariants and numbers of a non-rational minimal ruled surface X over a curve C of genus $g > 0$: we have

$$K_X^2 = 8 - 8g, \qquad P_m(X) = 0, \qquad \kappa(X) = -\infty,$$
$$q(X) = g, \qquad p_g(X) = 0, \qquad \chi(\mathscr{O}_X) = 1 - g,$$
$$e(X) = 4 - 4g, \qquad b_1(X) = 2g, \qquad b_2(X) = 2.$$

Moreover, as $b_2(X) = 2p_g(X) + h^{1,1}(X)$, we have $h^{1,1}(X) = 2$, so that $\rho(X) = 2$.

5.2 Rational surfaces

In this section we want to present several examples of ruled surfaces. Recall that a ruled surface is minimal if and only if it is $\mathbb{P}^2$ or $\mathbb{F}_n$ where $n = 0$ or $n > 1$. Hence we hava complete classification of ruled surfaces. Let me resume the invariants of a rational surface X: if $X \simeq \mathbb{P}^2$ then

$$K_X^2 = 9, \qquad P_m(X) = 0, \qquad \kappa(X) = -\infty,$$
$$q(X) = 0, \qquad p_g(X) = 0, \qquad \chi(\mathscr{O}_X) = 1,$$
$$e(X) = 3, \qquad b_1(X) = 0, \qquad b_2(X) = 1.$$

Moreover, as $b_2(X) = 2p_g(X) + h^{1,1}(X)$, we have $h^{1,1}(X) = 1$, so that $\rho(X) = 1$. If X is $\mathbb{F}_n$ for $n \neq 1$, we have

$$K_X^2 = 8, \qquad P_m(X) = 0, \qquad \kappa(X) = -\infty,$$
$$q(X) = 0, \qquad p_g(X) = 0, \qquad \chi(\mathscr{O}_X) = 1,$$
$$e(X) = 4, \qquad b_1(X) = 0, \qquad b_2(X) = 2.$$

Moreover, $h^{1,1}(X) = 2$, so that $\rho(X) = 2$. Notice that every rational surface X is birational to $\mathbb{P}^2$, so that every rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^N$ gives a rational map $X \dashrightarrow \mathbb{P}^N$. Then we just need to study rational maps from $\mathbb{P}^2$ to some projective space. The advantage is that such rational maps correspond to linear systems associated to some line bundle $\mathscr{O}(n)$ on $\mathbb{P}^2$, which have no fixed components. We need the following steps:

First of all, we need to determine N, which is $N = \binom{n+2}{n}$. Then we need to determine if the map associated to $\mathscr{O}(n)$ is everywhere defined, i. e. if there are base points. In general the linear system $|\mathscr{O}(n)|$ has base points, so that to have a well-defined map we need to blow up $\mathbb{P}^2$ in the base locus of $|\mathscr{O}(n)|$. This base locus is given by a finite set of points $\{p_1, ..., p_r\}$, so we have to consider the blow-up $P_r := Bl_{p_1,...,p_r}\mathbb{P}^2$, and we get a map

$$f_{r,n} : P_r \longrightarrow \mathbb{P}^{\binom{n+2}{n}}.$$

It corresponds to a linear system V on P_r: if $H = f^*\mathscr{O}(1)$, m_i is the minimum of the multiplicities of the members of $|\mathscr{O}(n)|$ at the point p_i, and we let E_i be the exceptional divisor obtained blowing-up p_i, then we have

$$V \subseteq |nH - \sum_{i=1}^{r} m_i E_i|.$$

we are particularily interested in the case where $f_{r,n}$ is an embedding. If this is the case, $f_{r,n}(P_r) \subseteq \mathbb{P}^{\binom{n+2}{n}}$ is a rational surface in a projective space. The aim of this section is to study the geometry of $f_{r,n}(P_r)$.

Notice that the Picard group of $f_{r,n}(P_r)$ has an orthogonal basis given by $H, E_1, ..., E_r$, where $H^2 = 1$ and $E_i^2 = -1$. Moreover, every hyperplane section L is linearily equivalent to $nH + \sum_{i=1}^{r} m_i E_i$, so that

$$L^2 = n^2 - \sum_{i=1}^{r} m_i.$$

This is the degree of $f_{r,n}(P_r)$ in $\mathbb{P}^N$. Other important points are the number of lines on $f_{r,n}(P_r)$, i. e. of curves on it having intersection 1 with a hyperplane section, an possible equations of $f_{r,n}(P_r)$.

There is again another point which is interesting for us: if $H \in |\mathscr{O}(n)|$ is a hyperplane section, it corresponds to a point $h \in |\mathscr{O}(n)|^*$. The linear system $|H|$ then defines a rational map

$$\mathbb{P}^2 \dashrightarrow |H|^*,$$

which is easily seen to be the composition of φ_n with the projection of $|\mathscr{O}(n)|$ onto $|H|$ away from h. If $x \in P_r$, then the rational surface obtained from the system of curves of $|\mathscr{O}(n)|$ passing through x is exactly the projection of $f_{r,n}(P_r)$ away from $f_{r,n}(x)$. Then it is natural to consider projections of $f_{r,n}(P_r)$ into projective spaces of dimension smaller than $\binom{n+2}{n}$. We will then need two results on this kind of projections:

Lemma 5.2.1. *Let X be a surface in $\mathbb{P}^N$, and $p \in \mathbb{P}^N \setminus X$ (resp. $p \in X$). Let $f_p : X \longrightarrow \mathbb{P}^{N-1}$ be the projection from p (resp. $f_p : Bl_p(X) \longrightarrow \mathbb{P}^{N-1}$ is the projection from p). Then f_p is an embedding if and only if there is no line L of $\mathbb{P}^N$ passing through p such that $L \cap X$ consists of at least 2 (resp. 3) points counted with multiplicities.*

Proof. The proof is immediate. □

As the set of bisecants (resp. tangents) to X is parameterized by the complement of the diagonal in $X \times X$ (resp. by $\mathbb{P}(T_X)$), it follows that the union of bisecants (resp. tangents) to X lies in a subvariety of $\mathbb{P}^N$ of dimension d, where $d \leq 5$ (resp. $d \leq 4$).

Lemma 5.2.2. *Every surface is isomorphic, via generic projection, to a smooth surface in $\mathbb{P}^5$.*

The cases we want to study are more precisely when $n = 2$ and $n = 3$.

5.2.1 Linear systems of conics

Consider the linear system $|\mathscr{O}(2)|$, i. e. the complete linear system of conics. The map defined by it is

$$j : \mathbb{P}^2 \longrightarrow \mathbb{P}^5, \qquad j(x_0 : x_1 : x_2) := (x_0^2 : x_0x_1 : x_0x_2 : x_1^2 : x_1x_2 : x_2^2).$$

It is the easy to see that j is an embedding, which is called the *Veronese embedding*. The surface $V := f_{0,2}(\mathbb{P}^2)$ is called the *Veronese surface*, and its degree is $(2H)^2 = 4$, where H is an hyperplane section of $\mathbb{P}^5$.

Proposition 5.2.3. *The Veronese surface does not contain any line, but contains a $2-$dimensional linear system of conics.*

Proof. If L is a line on V, then we would have

$$1 = L \cdot H = \mathscr{O}(1) \cdot \mathscr{O}(2) = 2,$$

which is clearly not possible. If L is a line in $\mathbb{P}^2$, then $j(L)$ is a curve in $\mathbb{P}^5$, and $j(L) \cdot H = 2$, so that $j(L)$ is a conic. But notice that $h^0(\mathbb{P}^2, \mathscr{O}(1)) = 2$, so V contains a $2-$dimensional linear system of conics. $\square$

Corollary 5.2.4. *Let p be a generic point of $\mathbb{P}^5$, and consider $f_p : V \longrightarrow \mathbb{P}^4$ to be the projection from V. Then f_p gives an isomorphism between V and $f_p(V) \subseteq \mathbb{P}^4$.*

Proof. Let L be a line in $\mathbb{P}^2$, and consider the plane $P(L)$ defined by the conic $j(L)$. Moreover, let $X := \cup_{L \in |\mathscr{O}(1)|} P(L)$. Then we have $dim(X) \leq 4$. Consider now

$$Z := \{(L, x) \in |\mathscr{O}(1)| \times \mathbb{P}^5 \,|\, x \in P(L)\},$$

which is a $\mathbb{P}^2-$bundle over $|\mathscr{O}(1)|$. Notice that Xs the projection to $\mathbb{P}^5$ of Z. Let now $x, y \in V$, where $x \neq y$. The line $L(x, y)$ defined by x and y is contained in the plane $P(L)$, where L is the line of $\mathbb{P}^2$ defined by $j^{-1}(x)$ and $j^{-1}(y)$. Hence every bisecant of V lieas in X, and the result follows. $\square$

The generic projection of $f_p(V)$ to $\mathbb{P}^3$ is called *Steiner surface*, which is a singular surface: it has 3 double lines meeting in a point, which is triple.

Let us now look at the projection of V from a point $p \in V$. As V contains no lines, there is a well defined morphism

$$f_p : V \setminus \{p\} \longrightarrow S,$$

where S is a cubic surface in $\mathbb{P}^4$. The map f_p is 1 to 1 where it is defined, so that to define it every where we need to blow-up the point p. As a consequence we get an embedding

$$j : \mathbb{F}_1 \longrightarrow \mathbb{P}^4$$

corresponding to the linear system $|H + F|$, where H and F are the two generators of the Picard group of $\mathbb{F}_1$. Using this, we can determine lines on S.

Proposition 5.2.5. *The cubic surface S containes two types of lines: a linear system $\{L_t\}_{t\in\mathbb{P}^1}$ such that $L_t \cap L_{t'} = \emptyset$ for every $t \neq t'$, and a line L such that $L \cdot L_t = 1$ for every $t \in \mathbb{P}^1$.*

Proof. As $F \cdot (H + F) = 1$, we have that $j(F)$ is a line on S, so that on S we have a family of lines $\{L_t\}_{t\in\mathbb{P}^1}$, with the property that $L_t \cap L_{t'} = \emptyset$ for $t \neq t'$.

Now, let E be the exceptional curve on $\mathbb{F}_1$. Then $E = H - F$, so that

$$E \cdot (H + F) = (H - F) \cdot (H + F) = H^2 = 1$$

by Proposition 4.2.3, so that $j(E)$ is a line on S which meets every line L_t: indeed

$$j(E) \cdot L_t = E \cdot F = H \cdot F = 1.$$

If C is a curve on $\mathbb{F}_1$, then $C = aH + bF$ with $a, b \geq 0$, so that

$$C \cdot (H + F) = (aH + bF) \cdot (H + F) = 2a + b,$$

which is 1 if and only if $a = 0$ and $b = 1$, i. e. $C = F$. In conclusion, these are the only lines on S. $\square$

To conclude, we study projections of S from points of $\mathbb{P}^4$. We begin with the following

Lemma 5.2.6. *The cubic S is contained in a $2-$dimensional linear system of quadrics in $\mathbb{P}^4$, of which it is the intersection. For every pencil $\{\lambda_1 Q_1 + \lambda_2 Q_2\}$ of quadrics in $\mathbb{P}^4$ we have $Q_1 \cap Q_2 = S \cup P$, where P is a plane and $S \cap P$ is a conic. Conversely, for every conic on S lying in a plane P, we have that $S \cup P$ is the intersection of two quadrics.*

Proof. A quadric on $\mathbb{P}^4$ cuts on S the strict transform of a quartic on $\mathbb{P}^2$ passing through 0 with multiplicity 2. Now, the linear systems $|\mathscr{O}_{\mathbb{P}^4}(2)|$ and $|\mathscr{O}_{\mathbb{P}^2}(4)|$ have the same dimension, and as passing through a point with multiplicity 2 gives three conditions on a system of plane curves, there must be at least 3 linearily independent quadrics of $\mathbb{P}^4$ containing S. Any two of them is irreducible, and their intersection is a degree 4 surface containing S. As S has degree 3, we need that the intersection of the two quadrics is $S \cup P$, where P is a plane. If P has equations $L = M = 0$, then two quadrics containing S have equations $LA_i + MB_i = 0$, where $i = 1, 2$. The determinant $A_1B_2 - A_2B_1$ vanishes at every point of $S \setminus (S \cap P)$, hence S lies in the quadric of equation $A_1B_2 - A_2B_1 = 0$. Then S is the intersection of these three quadrics, so that the linear system of quadrics containing S has dimension 2. The intersectionn$S \cap P$ is easily seen to be a conic. If conversely C is a conic in $P \cap S$, then any quadric contains P if and only if if contains a point of $P \setminus C$, so that there is a pencil of quadrics containing $S \cap P$, and the intersection of the members of this pencil is $S \cap P$. □

Corollary 5.2.7. *The projection of S from a point $p \in \mathbb{P}^4 \setminus S$ is a cubic surface in $\mathbb{P}^3$ whose singularities are a double line. The projection of S from a point $p \in S$ is a quadric in $\mathbb{P}^3$, which is smooth if and only if $p \notin L$.*

Proof. By Lemma 5.2.6 there are two quadrics Q_1 and Q_2 containing S and p. Then $Q_1 \cap Q_2 = S \cup P$, where P is a plane passing through p. Every bisecant of S through p cuts Q_i in three points, thus it lies in $Q_1 \cap Q_2$, hence in P. Hence f_p is an isomorphism outside $C := S \cap P$, which is a conic. The restriction to C is 2 to 1, so that $f_p(S)$ has a double line of singularities.

If $p \in S$, then every line through p cuts S in a single point if $p \notin L$. In this case, the line L is projected to a singular point. □

5.2.2 Linear systems of cubics

Consider $p_1, ..., p_r \in \mathbb{P}^2$ be r distinct points in general position (i. e. none of then are collinear, nor 6 of them lie on a quadric), where $r \leq 6$, and take V to be the linear system of cubics passing through $p_1, ..., p_r$. Then V defines a rational map

$$\mathbb{P}^2 \dashrightarrow \mathbb{P}^d,$$

and it is easy to calculate d: indeed $h^0(\mathbb{P}^2, \mathscr{O}(3)) = 10$, so that $|\mathscr{O}(3)|$ has dimension 9. As we are taking cubics passing through $p_1, ..., p_r$ in general position, we have r independent conditions, so that $d = 9 - r$. Consider now P_r, the blow up of $\mathbb{P}^2$ along $p_1, ..., p_r$, and consider the morphism

$$j : P_r \longrightarrow \mathbb{P}^d.$$

Notice that j is the morphism associated to $|3H - \sum_{i=1}^{r} E_i|$, where H is a line on $\mathbb{P}^2$ and E_i is the exceptional divisor over p_i.

Definition 5.2.1. The image $S_d := j(P_r)$ is called *Del Pezzo surface of degree d*.

Proposition 5.2.8. *The map j is an embedding, and the Del Pezzo surface S_d is a surface of degree d in $\mathbb{P}^d$.*

Proof. In order to show that j is an embedding, we need to show that the linear system V separates points and tangent directions. In order to do this, we just need to look at the case $r = 6$, the others follow. Consider $i < j \in \{1, ..., 6\}$, and let $x \in P_6$. Moreover, let $\pi : P_6 \longrightarrow \mathbb{P}^2$ be the blow-up. Suppose that $x \notin L(p_i, p_j)$, where $L(p_i, p_j)$ is the line in $\mathbb{P}^2$ passing through p_i and p_j.

As the points are in general position, there is only one conic Q_{ij}^x passing through $\pi(x)$ and p_k for every $k \neq i, j$: indeed if $x \notin E_i \cup E_j$ (where $E_i = \pi^{-1}(p_i)$), then $\pi(x) \neq p_i, p_j$, hence the conic Q_{ij}^x is defined by 5 independent conditions, so it is unique. If $x \in E_i$, then there is a 1−dimensional pencil of conics passing through p_k with $k \neq i, j$, and to pass through x means that this conic has to pass through p_i with tangent direction given by x. Hence we have again 5 independent conditions, and there is only one conic Q_{ij}^x.

Similarily, there is only one conic Q_i through the points p_j with $j \neq i$. If $\widetilde{Q}_i$ is its proper transform, we have that $\widetilde{Q}_i \cap \widetilde{Q}_j = \emptyset$ for $i \neq j$.

Now, consider $x, y \in P_6$, with $x \neq y$, and let i be such that $p_i \neq \pi(x), \pi(y)$ and $x \notin \widetilde{Q}_i$. Then $\widetilde{Q}_{ij}^x \cap \widetilde{Q}_{ik}^x = \{x\}$ for every $j \neq k$, with $j, k \neq i$ and $p_k \neq \pi(x)$. Hence $y \in \widetilde{Q}_{ij}^x$ for at most one value of $j \neq i$. Hence there is at most one $j \neq i$ such that the cubic $\widetilde{Q}_{ij}^x \cup \widetilde{L}(p_i, p_j)$ passes through x but not through j. Hence the morphism j separates points.

Let $x \in \mathbb{P}^2 \setminus \{p_1, ..., p_6\}$, and consider the cubics $Q_i \cup L(p_i, x)$ for $i = 1, ..., 6$. They have different tangents at x, so that j is an immersion on $P_6 \setminus \cup_{i=1}^6 E_i$. Let now $x \in E_1$: the conics Q_{23}^x and Q_{24}^x intersect at x with multiplicity 2. The cubics $\widetilde{Q}_{23}^x \cup \widetilde{L}(p_2, p_3)$ and $\widetilde{Q}_{34}^x \cup L(p_2, p_4)$ then have different tangent directions at x, so that j is an embedding on E_1. Similar calculations show that the same is true for $i = 2, ..., 6$, so that j is an embedding.

To calculate the degree of S_d, simply notice that

$$deg(S_d) = \left(3H - \sum_{i=1}^{r} E_i\right)^2 = 9 - r = d,$$

and we are done. $\square$

The next topic is to calculate how many lines we have on S_d.

Theorem 5.2.9. *On the Del Pezzo surface of degree d there are finitely many lines, which are the images under $j : P_r \longrightarrow S_d$ of one of the following curves on P_r:*

1. *the exceptional curves E_i, for $i = 1, ..., r$;*
2. *the curves $\widetilde{L}(p_i, p_j)$ for $i, j = 1, ..., r$, $i < j$;*
3. *the conics $\widetilde{Q}_i$, for $i = 1, ..., r$.*

Proof. By the formula of the canonical bundle of the blow up, we get that

$$K_{P_r} = \pi^*(-3H) + \sum_{i=1}^{r} E_i.$$

As $j^*\mathscr{O}(1) = \mathscr{O}_{P_r}(3H - \sum_{i=1}^r E_i)$, we get that the very ample divisor defining j is $-K_{P_r}$. Now, if E_i is an exceptional divisor, we have

$$-E_i \cdot K_X = 3E_i \cdot \pi^* H - \sum_{j=1}^r E_i \cdot E_j = 1,$$

so that $j(E_i)$ is a line on S_d.

Now, let L be a line on S_d which is not of the form $j(E_i)$ for any $i = 1, ..., r$. Then

$$1 = -L \cdot K_X = 3L \cdot p^* H - \sum_{j=1}^r L \cdot E_i.$$

We have $L \cdot E_i \geq 0$ and $L \cdot p^* H \geq 0$ as L is effective. Notice that $L = m\pi^* H + \sum_{i=1}^r m_i E_i$ for some $m, m_1, ..., m_r \in \mathbb{N}_0$, so that

$$1 = 3m - \sum_{i=1}^r m_i.$$

As L is not one of the E_i, we need $m_i = 0, 1$, and we need $m > 0$. Hence we have the following possible solutions: if $m = 1$, then we need

$$\sum_{i=1}^r m_i = 2,$$

so that the only possible case is $r = 2$ and $m_1 = m_2 = 1$. Hence

$$L = \pi^* H + E_i + E_j$$

for some $i \neq j$, so that L is the proper transform of a line passing through p_i and p_j. But we have other possible cases: if $m = 2$, then we have

$$\sum_{i=1}^r m_i = 5,$$

so that there are only the following possibility, namely $r = 5$ and $m_i = 1$ for $i = 1, ..., 5$, i. e. L is the proper transform of a cubic passing through 5 of the p_i's. If $m \geq 3$, we have

$$6 \geq r \sum_{i=1}^r m_i = 3m - 1 \geq 8,$$

which is not possible, hence we are done. $\square$

We can then give a complete description of lines on S_d for every $d = 3, ..., 8$.

1. If $d = 3$, i. e. $r = 6$, then we have 6 exceptional divisors, 15 lines and 6 conics, so that on S_3 we have 27 lines.

2. If $d = 4$, i. e. $r = 5$, then we have 5 exceptional divisors, 10 lines and 1 conic, so that on S_4 we have 16 lines.

3. If $d = 5$, i. e. $r = 4$, then we have 4 exceptional divisors, 6 lines and no conics, so that on S_5 we have 10 lines.

4. If $d = 6$, i. e. $r = 3$, then we have 3 exceptional divisors, 3 lines and no conics, so that on S_6 we have 6 lines.

5. If $d = 7$, i. e. $r = 2$, then we have 2 exceptional divisors, 1 line and no conics, so that on S_7 we have 3 lines.

6. If $d = 8$, i. e. $r = 1$, then we have 1 exceptional divisor, no lines and no conics, so that on S_8 we have 1 line.

To conclude this section, we want to prove some results about cubics an other surfaces. Let me start with two lemmas:

Lemma 5.2.10. *Let $S \subseteq \mathbb{P}^3$ be a smooth cubic surface. Then S contains a line.*

Proof. Let $V := |\mathscr{O}(3)|$ be the linear system of cubics in $\mathbb{P}^3$, and let G be the Grassmannian of lines in $\mathbb{P}^3$. Consider

$$Z := \{(L, C) \in G \times V \,|\, L \subseteq G\},$$

which has two projections $p : Z \longrightarrow G$ and $q : Z \longrightarrow V$. A cubic in $\mathbb{P}^3$ contains the line $x_2 = x_3 = 0$ if and only if the coefficients of x_0^3, $x_0^2x_1$, $x_0x_1^2$ and x_1^3 vanish, so that the fiber of p over $x_2 = x_3 = 0$ has dimension equal to $dim(V) - 4$. As $dim(G) = 4$, then $dim(Z) = dim(V)$. If there is a cubic S such that no line is contained in S, then $q(Z) \subseteq V$ has codimension at least 1, and the fiber $q^{-1}(S)$ is either empty or positive dimensional. As S_3 is a cubic containing finitely many lines, we see that this is not possible so that q is surjective, and we are done. □

Lemma 5.2.11. *Let $S \subseteq \mathbb{P}^3$ be a smooth cubic surface, and let L be a line on S. Then there are exactly 10 distinct lines which are distinct from L and meeting L. Moreover, these 10 lines fall into 5 disjoint pairs of concurrent lines. In particular, S contains two disjoint lines.*

Proof. Consider the pencil $\{P_t\}_{t \in \mathbb{P}^1}$ of planes through L. Then $S \cap P_t = L \cup C_t$, where C_t is a conic. Notice that C_t does not contain L, and cannot be a double line: indeed, if the plane $x = 0$ cuts S in the line $y = 0$ counted twice, then S has equation $xQ + yZ^2 = 0$, where Q is a conic and Z is a line. But then S would have singularities, which is not possibile. Hence C_t has to be either smooth or singular, and in this second case it is the union of two cuncurrent lines distinct from L but meeting it.

Conversely, if L' is a line meeting L and distinct from L, they define a plane P which meets S along L. Hence $P \cap S = L \cup L' \cup M$, and L' and M are two cuncurrent lines meeting L. In conclusion, all the lines meeting L are of this form. Let us now suppose that L is given by equation $x_2 = x_3 = 0$ in $\mathbb{P}^3$. The equation of S is then

$$ax_0^2 + 2bx_0x_1 + cx_1^2 + 2dx_0 + 2ex_1 + f = 0,$$

where a, b, c, d, e, f are homogenous polynomials in x_2 and x_3. The equation of C_t is obtained setting $x_2 = tx_3$, so that C_t is singular if and only if its determinant $\Delta(x_2, x_3)$ vanishes. This is a homogenous quintic in x_2, x_3 which has no multiple root: if $x_2 = 0$ is a root, let $x \in C_0$ be the singular point. If $x \notin L$ we can suppose C_0 to be described by equation $x_0x_1 = 0$. Then every coefficient of $\Delta(x_2, x_3)$ is divisible by x_2, except b, so that x is smooth in S and f is not divisible by x_2^2. In conclusion $\Delta(x_2, x_3)$ is not divisible by x_2^2. If $x \in L$, we can write C_0 as $x_0^2 - x_1^2$, and repeating the argument as before we get that $\Delta(x_2, x_3)$ is not divisible by multiple roots.

In conclusion, $\{C_t\}_{t\in\mathbb{P}^1}$ contains 5 singular conics, givins the 10 lines of the statement. If three of these lines meet at a point $x \in S$, they are coplanar (they lie in the tangent plane of S at p). If C_0 and C_1 are singular conics in the pencil, suppose $C_i = D_i \cup D_i'$. Since D_0, L and D_1 are not coplanar, we have $D_0 \cap D_1 = \emptyset$, and we are done. $\square$

These two lemmas are used to show the following important result:

Theorem 5.2.12. *We have the two following properties:*

1. *A smooth surface $S \subseteq \mathbb{P}^3$ is a cubic if and only if it is S_3.*
2. *A smooth surface $S \subseteq \mathbb{P}^4$ is a complete intersection of two quadrics if and only if it is S_4.*

In particular, every smooth cubic surface in $\mathbb{P}^3$ contains 27 lines.

Proof. We only prove the first item. The Del Pezzo surface S_3 is a cubic by Propositon 5.2.8, so that we need to show the converse. Let S be a smooth cubic surface in $\mathbb{P}^3$. By Lemmas 5.2.10 and 5.2.11 there are two disjoint lines L, L' on S, and consider the rational map

$$\phi : L \times L' \dashrightarrow S$$

defined as follows: if (p, p') is generic in $L \times L'$, the line through p and p' meets S at a third point p'', so that $\phi(p, p') := p''$. Consider the rational map

$$\psi : S \dashrightarrow L \times L'$$

defined as follows: if $x \in S \setminus (L \cup L')$, let $P_{x,L}$ be the plane passing through L and x, and let $p := L \cap P_{x,L'}$ and $p' := L' \cap P_{x,L}$. Then put $\psi(x) := (p, p')$. It is easy to see that ϕ and ψ are inverse to each other, and we can define ψ

on the whole S: if $x \in L$, simply define $p := L \cap T_x(S)$ and $p' = L' \cap P_{x,L}$, and similarily on for L'. Then we have a birational map which contracts curves meeting L and L'.

These lines come in pairs D_i, D'_i for $i = 1, ..., 5$ defining a plane P_i by Lemma 5.2.11. The plane P_i meets L' in a point, lying either on D_i or on D'_i, so that one line in each pair meets L'. Hence the morphism ψ contracts 5 disjoint lines. Hence S is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ with 5 points blown-up. As $\mathbb{P}^1 \times \mathbb{P}^1$ is isomorphic to $\mathbb{P}^2$ blown-up in a point, so that S is isomorphic to $\mathbb{P}^2$ blow-up in 6 points. But S is embeddend in $\mathbb{P}^3$ via the anticanonical embedding, so that $S \simeq S_3$. $\square$

5.3 K3 surfaces

Starting from this section, we collect properties and informations about the geometry of smooth surfaces of non-negative Kodaira dimension. A first class one meets in the Enriques-Castelnuovo Classification is the one of K3 surfaces. This is a very interesting class of surfaces, whose geometry is very rich. It has been investigated since 1950's, and we are going to collect here some of the main results. First of all, let me recall all the invariants. If X is a K3 surface, then $K_X = 0$, so that

$$K_X^2 = 0, \qquad P_m(X) = 1, \qquad \kappa(X) = 0,$$

$$q(X) = 0, \qquad p_g(X) = 1, \qquad \chi(\mathscr{O}_X) = 2,$$

$$e(X) = 24, \qquad b_1(X) = 0, \qquad b_2(X) = 22.$$

Notice that the first Chern class $c_1 : Pic(X) \longrightarrow H^2(X, \mathbb{Z})$ is injective, so that $0 < \rho(X) \leq h^{1,1}(X)$, and as $p_g(X) = 1$ we have $h^{1,1}(X) = 20$. In conclusion we have different types of K3 surfaces according to $\rho(X)$, which can be any integer between 1 and 20.

5.3.1 The Kummer surface

We have already seen several examples of K3 surfaces: in Example 4.3.5 we have shown that the generic complete intersection X of hypersurfaces $H_1, ..., H_r$ of degrees $d_1, ..., d_r$ in $\mathbb{P}^{r+2}$ is a K3 surface if and only if $\sum_{i=1}^{r} d_i = r + 3$. In this first section I introduce a new example. Let A be an abelian surface, on which we have a natural involution

$$i : A \longrightarrow A, \qquad i(x) = -x,$$

(recall that $A = \mathbb{C}^2/\Gamma$ for some maximal rank lattice Γ, so that A inherits a group structure). The group A is isomorphic to $(\mathbb{R}/\mathbb{Z})^4$, so the involution i has

16 fixed points, which are $p_1, ..., p_{16}$. Consider

$$\pi : \widetilde{X} \longrightarrow A$$

to be the blow-up of A along $p_1, ..., p_{16}$, on which we have 16 exceptional curves $E_1, ..., E_{16}$. The involution i on A extends to an involution $\widetilde{i}$ on $\widetilde{A}$. Let $Kum(A) := \widetilde{A}/\widetilde{i}$ and $p : \widetilde{A} \longrightarrow Kum(A)$ be the quotient map.

Definition 5.3.1. The surface $Kum(A)$ is called *Kummer surface associated to A*.

Proposition 5.3.1. *Let A be an abelian surface. Then $Kum(A)$ is a K3 surface.*

Proof. First of all, we need to show that $Kum(A)$ is smooth. The involution i has no fixed points on $A \backslash \{p_1, ..., p_{16}\}$, so that $\widetilde{i}$ has no fixed points on $\widetilde{A} \backslash \cup_{i=1}^{16} E_i$. Hence p is étale outside $\cup_{i=1}^{16} E_i$, and $Kum(A) \setminus p(\cup_{i=1}^{16} E_i)$ is smooth. Let then $q \in E_i$ for some i. Writing $A = \mathbb{C}^2/\Gamma$ we have local coordinates x, y around p_i, with the property that $i^*x = -x$ and $i^*y = -y$. Let $x' := \pi^*x$ and $y' := \pi^*y$. On $\widetilde{A}$ we can then consider local coordinates around q which are x' and $t = y'/x'$. The involution $\widetilde{i}$ then acts as follows: $\widetilde{i}^*x' = -x'$ and $\widetilde{i}^*t = t$. Hence t and $u := (x')^2$ form local coordinates on $Kum(A)$ around $p(q)$, so that $Kum(A)$ is smooth.

In order to show that X is a K3 surface, we need $K_{Kum(A)} = 0$ and $q(Kum(A)) = 0$. Let us start with the canonical divisor. The abelian surface A has a holomorphic $2-$form which is nowhere zero. Writing $A = \mathbb{C}^2/\Gamma$, this is symple the form ω on A induced by $dx \wedge dy$. Notice that $i^*\omega = \omega$, and the two form $\pi^*\omega$ is invariant under the action of $\widetilde{i}$: hence $\pi^*\omega$ descends to a holomorphic $2-$form $\widetilde{\omega}$ on $Kum(A)$. To show that $K_{Kum(A)}$ is trivial, we simply need to show that $\widetilde{\omega}$ is nowhere zero on $Kum(A)$. Let D be the zero divisor of $\widetilde{\omega}$. As $Z(\omega) = 0$, we have that $Z(\pi^*\omega)$ is concentrated on the exceptional curves. Let $C_i := p(E_i)$, giving 16 curves on $Kum(X)$. As $p^*\widetilde{\omega} = \pi^*\omega$, we have that D is concentrated on the C_i's. Let now $q \in E_i$. Then, in local coordinates as before, we have

$$\pi^*\omega = dx' \wedge dy' = dx' \wedge d(tx') = x'dx' \wedge dt = \frac{1}{2}du \wedge dt.$$

Then the form $\widetilde{\omega}$ is not zero at q, so that $D = 0$. In conclusion $K_{Kum(A)} = 0$.

Now, let us suppose we have a holomorphic $1-$form on $Kum(A)$, so that $q(Kum(A)) > 0$. This defines a holomorphic $1-$form on $\widetilde{A}$ which is invariant under $\widetilde{i}$. As the map

$$\pi^* : H^0(A, \Omega_A) \longrightarrow H^0(\widetilde{A}, \Omega_{\widetilde{A}})$$

is an isomorphism, we get that there exist a holomorphic $1-$form on A which is invariant under i. But $H^0(A, \Omega_A) = \mathbb{C}^2$ spanned by dx and dy, and these forms are such that $i^*dx = -dx$ and $i^*dy = -dy$. Hence for every holomorphic $1-$form $\alpha \in H^0(A, \Omega_A)$ we have $i^*\alpha = -\alpha$, so that the only possible holomorphic $1-$form invariant under i is $\alpha = 0$. But this implies that $q(Kum(A)) = 0$, so that $Kum(A)$ is a K3 surface. $\square$

The Kummer surfaces are special among the class of K3 surfaces: the curves C_i in the proof are independent, and $C_i^2 = -2$. Hence $\rho(Kum(A)) \geq 16$. Moreover, the hyperplane section H of A induces an hyperplane section on $Kum(A)$, so that $\rho(X) \geq 17$. We have even the following easy property: first of all, we define a $(-2)-$*curve* on a surface X to be a curve C on X which is smooth and rational, and $C^2 = -2$. Now, if X is a K3 surface and C is a curve on X, then by the genus formula we have that C^2 is even and that $C^2 \geq -2$, and it is -2 if and only if $g(C) = 0$. Then every smooth curve C on a K3 surface of negative self-intersection is a $(-2)-$curve. Using this, there is a beautiful characterization of Kummer surfaces among K3 surfaces:

Proposition 5.3.2. *Let X be a K3 surface containing 16 disjoint irreducible curves $C_1, ..., C_{16}$ such that C_i is a $(-2)-$curve and the divisor $\sum_{i=1}^{16} C_i$ is divisible by 2 in $Pic(X)$. Then X is a Kummer surface.*

Proof. As $\sum_{i=1}^{16} C_i$ is divisible by 2 in $Pic(X)$, there is a double cover $p : Z \longrightarrow X$ whose branching locus is exactly $\sum_{i=1}^{16} C_i$. The curve $D_i := p^{-1}(C_i)$ are 16 disjoint curves such that $D_i^2 = C_i^2/2 = -1$, and $g(D_i) = g(C_i)/2 = 0$, so they are all $(-1)-$curves. Contracting them we get a map $\pi : Z \longrightarrow Y$, where Y is a smooth surface.

We have that $K_Z = \pi^*K_Y + \sum_{i=1}^{16} D_i$, and $p^*K_X = K_Z - \sum_{i=1}^{16} D_i$, so that $\pi^*K_Y = p^*K_X = 0$. In conclusion we need $K_Y = 0$. Moreover, we have $e(Y) = e(Z) - 16$, and $e(Z) = 2(e(X) - 16)$. As X is a K3 surface, we have $e(X) = 24$, so that $e(Z) = 16$ and $e(Y) = 0$. Then $q(Y) = p_g(Y) + 1$, and $p_g(Y) = 1$ as $K_Y = 0$, so that $q(Y) = 2$. By the Enriques-Castelnuovo Classification, we have then that Y is an abelian surface.

To show that $X = Kum(Y)$ we still need to look at the involution on Y. The involution on Z given by exchanging sheets descends to an involution i on Y which has excatly 16 fixed points, namely the images of the D_i's. Now, the i-invariant part of $H^1(Y, \mathbb{Q})$ is isomorphic to $H^1(X, \mathbb{Q}) = 0$, so that i acts as $-id$ on $H^1(Y, \mathbb{Q})$, and $X = Kum(Y)$. $\square$

5.3.2 Projectivity

Let us now study maps from a K3 surfaces to some projective spaces. The main result we need is the following:

Theorem 5.3.3. *Let X be a K3 surface, and let $C \subseteq X$ be a smooth curve of genus g.*

1. *We have* $C^2 = 2g - 2$ *and* $h^0(X, \mathscr{O}_X(C)) = g + 1$.

2. *If* $g \geq 1$, *then* $|C|$ *has no base points, and it defines a morphism*

$$\phi_g : X \longrightarrow \mathbb{P}^g.$$

The map $\phi_{g|C} : C \longrightarrow \mathbb{P}^{g-1}$ *is the map associated to the canonical linear system of* C.

3. *If* $g = 1$, *then* ϕ_g *is an elliptic fibration over* $\mathbb{P}^1$ *whose generic fiber is* C.

4. *If* $g = 2$, *then* $\phi_g : X \longrightarrow \mathbb{P}^2$ *is a double cover of* $\mathbb{P}^2$ *ramified along a sextic.*

5. *If* $g \geq 3$, *then* ϕ_g *is either a birational morphism to its image and the generic curve in* $|C|$ *is non-hyperelliptic; or it is double cover of a rational surface (possibly singular) of degree* $g - 1$ *in* $\mathbb{P}^g$, *and the generic curve in* $|C|$ *is hyperelliptic.*

6. *If* $g = 2$, *then the map* φ_{3C} *is birational. If* $g \geq 3$, *then the map* φ_{2C} *is birational*

Proof. Let us start with the first item. As $g(C) = g$ and $K_X =$ as X is a K3 surface, then we get by the genus formula that $C^2 = 2g - 2$. Now, consider the exact sequence

$$0 \longrightarrow \mathscr{O}_X(-C) \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_C \longrightarrow 0.$$

As $H^0(X, \mathscr{O}_X) \simeq H^0(C, \mathscr{O}_C \simeq \mathbb{C}$, and as $H^1(X, \mathscr{O}_X) = 0$ as X is a K3 surface, we then get from this that $H^1(X, \mathscr{O}_X(-C)) = 0$ and $H^0(X, \mathscr{O}_X(-C)) = 0$. As $K_X = 0$, by Serre's Duality this gives $h^2(X, \mathscr{O}_X(C)) = h^1(X, \mathscr{O}_X(C)) = 0$. Then

$$h^0(X, \mathscr{O}_X(C)) = \chi(\mathscr{O}_X(C)) = \chi(\mathscr{O}_X) + g - 1 = g + 1,$$

and we are done.

For the second item, as $K_X = 0$ we have $\mathscr{O}_C(K_C) = \mathscr{O}_C(C)$, by the adjunction formula, so that we have an exact sequence

$$0 \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_X(C) \longrightarrow \mathscr{O}_C(C) \longrightarrow 0.$$

As $H^1(X, \mathscr{O}_X) = 0$, the map $H^0(X, \mathscr{O}_X(C)) \longrightarrow H^0(C, \mathscr{O}_C(C))$ is surjective, hence the complete linear system $|C|$ cuts on C a complete linear system, which is the canonical linear system on C. As $g \geq 1$, $deg(K_C) = 2g - 2 \geq 0$, so that it has no base points on C. In conclusion, the linear system $|C|$ has no base points on X. Then ϕ is a morphism, and the rest is clear.

For the third item, as $g = 1$ we have $C^2 = 0$, and $h^0(X, \mathscr{O}_X(C)) = 2$ by item 1. Then we have a $1-$dimensional pencil defined by $|C|$, every curve of which is contracted by ϕ: hence we have a fibration whose generic memeber is a smooth curve of genus 1, hence we have an elliptic fibration.

For the third item, as $g = 2$, we have $C^2 = 2$ and $h^0(X, \mathscr{O}_X(C)) = 3$. As $\mathscr{O}_X(C) = \phi^*\mathscr{O}(1)$, we have that ϕ has degree 2. Let $\Delta \subseteq \mathbb{P}^2$ be the branche curve, and let L be a line in $\mathbb{P}^2$. Then $L \cdot \Delta = deg(\Delta)$. As $\phi^{-1}(L)$ is a double cover of L ramified along $deg(\Delta)$ points of $L \cap \Delta$. But notice that $C = \phi^{-1}$ and $g(C) = 2$, so that the number of branching points is 6, and we are done.

For the fourth item, if C is non-hyperelliptic, then $\phi_{|C}$ is an embedding, hence $C = \phi^{-1}\phi(C)$, so that ϕ is birational. If ϕ is not birational, every smooth curve in $|C|$ has to be hyperelliptic. Then for a geniric point $x \in X$ the set $\phi^{-1}\phi(x)$ consists of 2 points, hence ϕ has degree 2. The image is the a surface S of degree $g-1$ in $\mathbb{P}^g$, possibly singular, whose hyperplane sections are the rational curves $\phi(C)$. Then S is rational.

The fifth item is easy: the restriction of ϕ_{3C} (resp. ϕ_{2C}) is the $3-$canonical (resp. $2-$canonical) map, hence it is an embedding, so that ϕ_{3C} (resp. ϕ_{2C}) is birational. $\square$

Let us now have a look to some particular cases, and let us suppose that X is a K3 surface and C is a smooth curve of genus $g \geq 3$ such that ϕ is birational. If $g = 3$, then the map $\phi : X \longrightarrow \mathbb{P}^3$ gives a surface $\phi(X) \subseteq \mathbb{P}^3$. The it has to be a quartic surface, i. e. K3 surfaces of degree 4 in $\mathbb{P}^3$ are parameterized by an open subset of the projective space $|\mathscr{O}_{\mathbb{P}^3}(4)|$.

If $g = 4$, the situation is more complicated: we have that $\phi(X)$ is a surface of degree 6 in $\mathbb{P}^4$. As $h^0(X, \mathscr{O}_X(2C)) = 14$ by Riemann-Roch, and $h^0(\mathbb{P}^4, \mathscr{O}(2)) = 15$. Hence there is a quadric containing $\phi(X)$. In a similar way, we have $h^0(X, \mathscr{O}_X(3C)) = 29$, and $h^0(\mathbb{P}^4, \mathscr{O}(3)) = 35$, so there is a cubic containing $\phi(C)$. Hence $\phi(X)$ is complete intersection of a quadric and a cubic.

The case $g = 5$ is similar to the preceding one: it is a surface in $\mathbb{P}^5$ of degree 8, and the generic one is complete intersection of 3 linearily independent quadrics. Moreover, we have the following: in any of these cases, there is a quasi-projective variety T_g parameterizing K3 surfaces of degree $2g-2$ in $\mathbb{P}^g$. As instance, if $g = 3$, this is an open subvariety in $|\mathscr{O}_{\mathbb{P}^3}(4)| \simeq \mathbb{P}^{34}$. It is then quaite natural to ask about the dimension of T_g, i. e. to determine how many K3 surfaces of degree $2g-2$ are in $\mathbb{P}^g$. More precisely, on T_g acts the group $PGL(g+1)$, and two K3 surfaces are isomorphic if they are in the orbit of this group. One can prove that the stabilizer is finite, For $g = 3$, this group has dimension 15, and hence the dimension of the moduli space of K3 surfaces of degree 4 in $\mathbb{P}^3$ is $34 - 15 = 19$.

For $g = 2$, we have that every K3 surface of degree 2 is double cover of $\mathbb{P}^2$ ramified along a sextic, so that T_g is contained in $|\mathscr{O}_{\mathbb{P}^2}(6)| \simeq \mathbb{P}^{27}$, and the dimension of $PGL(3)$ is 8. Hence the dimension of the moduli space of K3 surfaces of degree 2 is $27 - 8 = 19$. In a similar way one can see that the moduli spaces of K3 surfaces of degree 6 in $\mathbb{P}^4$ and of degree 8 in $\mathbb{P}^5$ have dimension 19.

For higher g the situation is more complicated.

Proposition 5.3.4. *For every $g \geq 3$ there are K3 surfaces $S_{2g-2} \subseteq \mathbb{P}^g$ of degree $2g-2$.*

Proof. By Theorem 5.3.3 we just need to produce a K3 surface admitting a very ample divisor of self-intersection $2g-2$. Actually, one can use the following fact: if H is a hyperplane section on X and $|E|$ is a linear system without base points, then $H+E$ is very ample. We have three cases:

Case 1: $g = 3k$. Consider a quartic $S \subseteq \mathbb{P}^3$, and suppose it contains a line. Consider the pencil $|H-L|$, which is a pencil of elliptic curves. This pencil has no base points, and if $E \in |H-L|$ we have then that $D_k := H + (k-1)E$ is very ample. Moreover

$$D_k^2 = H^2 + (k-1)^2E^2 + 2(k-1)H \cdot E = 4 + 6(k-1) = 2g-2.$$

Case 2: $g = 3k+1$. Let $Q \subseteq \mathbb{P}^4$ be a qaudric with a double point, and let $V \subseteq \mathbb{P}^4$ be a cubic such that $S = Q \cap V$ is a smooth surface. We have two pencil of planes on Q: one of them cuts on V a pencil $|E|$ of elliptic curves. Again it has no base points and $D_k := H + (k-1)E$ is a very ample divisor such that $D_k^2 = 2g-2$.

Case 3: $g = 3k+2$. Let $S \subseteq \mathbb{P}^3$ be a smooth quartic containing a line L and a twisted cubic T disjoint from L. Set $E := H - L$ and $H' := 2H - T$. Again one can show that $D_k := H' + kE$ is very ample and that $D_k^2 = 2g-2$. □

Even in this case, one can prove that there is a manifold T_g parameterizing K3 surfaces of degree $2g-2$ in $\mathbb{P}^g$, and that $PGL(g+1)$ acts on it. The moduli space is again of dimension 19. Now, let me introduce the following definition:

Definition 5.3.2. A complex surface is called *K3 surface* if $K_X = 0$ and $q(X) = 0$.

If X is smooth and projective, then X is the familiar K3 surface on which we have been talking up to now. As this definition makes sense for general complex surfaces, one can ask if there are K3 surfaces which are not projective. To talk of this, we need to introduce an importna notion, which is very general

Definition 5.3.3. Let X be any complex manifold. A *deformation* of X is the datum of a complex manifold $\mathscr{X}$ together with a smooth morphism $f : \mathscr{X} \longrightarrow S$ for some complex manifold S, such that there is $s_0 \in S$ such that $f^{-1}(s_0) = X$. The *Kuranishi family* for X is the universal deformation of X.

One of the most important results in deformation theory of surfaces is the following:

Theorem 5.3.5. *Let X be a complex surface. The minimal model of any deformation of X lies in the same class in the Enriques-Castelnuovo classification as the minimal model of X.*

In particular, any deformation of a K3 surface is again a K3 surface. One can prove that the Kuranishi family exists for every K3 surface X, and let it be $f : \mathscr{X} \longrightarrow S$, where S is called the deformation space of X. By general results on deformations of complex manifolds, one has that $dim(S) = h^1(X, T_X)$, and that S is smooth if and only if $h^2(X, T_X) = 0$. If X is a K3 surface, then we have $T_X \simeq \Omega_X^*$, so that Serre's Duality gives $h^1(X, T_X) = h^1(X, \Omega_X) = h^{1,1}(X) = 20$, and $h^2(X, T_X) = h^0(X, \Omega_X) = q(X) = 0$. Hence S is smooth and has dimension 20. Now if one deforms a projective K3 surface, and the deformation is still projective, the deformation space would be of dimension 19 by the previous part, hence this means that there are K3 surfaces which are not projective! Here we have a first example of non-projective complex surfaces entering in the Enriques-Castelnuovo Classification. There is another important result that I want to state:

Theorem 5.3.6. ***(Siu)****. Every K3 surface is Kähler.*

In conclusion, K3 surfaces are all Kähler surfaces which are not-necessarily projective.

5.3.3 The Torelli Theorem for K3 surfaces

The last section on K3 surfaces is a digression on the Torelli Theorem, given an important creterion to establish if two K3 surfaces are isomorphic, and hence virtually completing the classification of K3 surface. First, let me introduce a definition:

Definition 5.3.4. The *K3 lattice* is the lattice

$$\Lambda_{K3} := E_8(-1) \oplus E_8(-1) \oplus U \oplus U \oplus U,$$

where $E_8(-1)$ is the rank 8 lattice whose intersection form is

$$\begin{vmatrix} -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{vmatrix},$$

and U is the rank 2 lattice whose interesection form is

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}.$$

The first property we need is the following:

Proposition 5.3.7. *Let X be a K3 surface.*

1. *We have $H^1(X,\mathbb{Z}) = H^3(X,\mathbb{Z}) = 0$.*
2. *The $\mathbb{Z}-$module $H^2(X,\mathbb{Z})$ is free of rank 22, and as a lattice with respect to the cup produc it is isometric to Λ_{K3}.*

Proof. To prove the first item and the fact that $H^2(X,\mathbb{Z})$ is free of rank 22 it is sufficient to prove that $H_1(X,\mathbb{Z})$ has no torsion. If there is an element of $n-$torsion in $H_1(X,\mathbb{Z})$, this would determine an unramified cover $Y \longrightarrow X$ of degree n. As $K_X = 0$, this implies that $K_Y = 0$, so that $p_g(Y) = 1$. By the Noether Formula we then get

$$2 - 2q(Y) = \chi(Y) = n\chi(X) = 2n,$$

so that $n = 1$ and $q(Y) = 0$.

Now, the cup product on $H^2(X,\mathbb{Z})$ is even, and by Poincaré duality it is unimodular. Moreover, the index is easily calculated to be -16. Then we have two unimodular even lattices of the same index, Λ_{K3} and $H^2(X,\mathbb{Z})$, hence they are isometric. $\square$

Definition 5.3.5. A *marking* on a K3 surface X is the choice of an isometry between $H^2(X,\mathbb{Z})$ and Λ_{K3}. A marked K3 surface is the pair given by a K3 surface and a marking. Two marked K3 surfaces (X,ϕ) and (Y,ψ) are isomorphic if there is an isomorphism $f : X \longrightarrow Y$ such that $\phi \circ f^* = \psi$.

Let now $\Lambda_{\mathbb{C}} := \Lambda_{K3} \otimes_{\mathbb{Z}} \mathbb{C}$, which is a complex vector space of dimension 22, and let $\mathbb{P} := \mathbb{P}(\Lambda_{\mathbb{C}})$, which is a projective space of dimension 21. Inside of this, we can consider

$$\Omega := \{[\omega] \in \mathbb{P} \,|\, (\omega,\omega) = 0,\ (\omega,\overline{\omega}) > 0\}.$$

Now, let X be a marked K3 surface, and let ϕ be the marking. As X is Kähler by Siu's Theorem, we have a Hodge decomposition

$$H^2(X,\mathbb{C}) \simeq H^2(X,\mathscr{O}_X) \oplus H^{1,1}(X) \oplus \overline{H^2(X,\mathscr{O}_X)}.$$

As X is a K3 surface, we have that $p_g(X) = 1$, so that $H^2(X,\mathscr{O}_X) = \mathbb{C}\cdot\sigma$. The marking ϕ gives an isometry

$$\phi : H^2(X,\mathbb{C}) \longrightarrow \Lambda_{\mathbb{C}}.$$

Then the Hodge decomposition gives us a line $\mathbb{C}\cdot\phi(\sigma)\subseteq\Lambda_{\mathbb{C}}$, i. e. a point $[\sigma]\in\mathbb{P}$. Now, As σ is a nowhere vanishing form we have $(\sigma,\sigma)=0$ and $(\sigma,\overline{\sigma})>0$, so that $[\sigma]\in\Omega$. In conclusion, we get a map

$$\pi : S \longrightarrow \Omega, \qquad \pi(X) := [\sigma],$$

where S is the deformation space, which is called *period map*. It is a rather important question the to understant which periods K3 surfaces can have, and which informations on a K3 surface the period gathers.

Theorem 5.3.8. ***(Local Torelli Theorem)***. *Two marked K3 surfaces X and Y are isomorphic if and only if their periods are equal. Moreover, every $\omega\in\Omega$ is the period of a marked K3 surface.*

Hence we can classify marked K3 surfaces by means of the period, and Ω is the deformation space of K3 surfaces. But we have more:

Theorem 5.3.9. ***(Global Torelli Theorem)***. *Two K3 surfaces X and Y are isomorphic if and only if there is a Hodge isometry $f : H^2(X,\mathbb{Z}) \longrightarrow H^2(Y,\mathbb{Z})$.*

In conclusion, K3 surfaces can be classified by means of the lattice structure of $H^2(X,\mathbb{Z})$. Other important results on K3 surfaces are the following:

Theorem 5.3.10. *Every two K3 surfaces are diffeomorphic. In particular every K3 surface is simply connected.*

5.4 Enriques surfaces

A second class of surfaces of Kodaira dimension 0 is the one of Enriques surfaces, which are strictly related to K3 surfaces. Before starting with properties of Enriques surfaces, let me recall their invariants. If X is an Enriques surfaces, then $K_X\neq 0$ but $2K_X=0$, so that

$$K_X^2=0, \qquad P_m(X)\equiv_2 m-1, \qquad \kappa(X)=0,$$

$$q(X)=0, \qquad p_g(X)=0, \qquad \chi(\mathscr{O}_X)=1,$$

$$e(X)=12, \qquad b_1(X)=0, \qquad b_2(X)=10.$$

As $p_g(X)=0$, we then conclude that $h^{1,1}(X)=10$. Notice that as $q(X)=0$ and $p_g(X)=0$ the first Chern class is an isomorphism, so that $H^2(X,\mathbb{Z})\simeq Pic(X)$ and $\rho(X)=10$.

5.4.1 Construction and projectivity

In Chapter 4 we have seen how one can produce examples of Enriques surfaces: if $\widetilde{X}$ is a K3 surface admitting an involution i without fixed points, then $\widetilde{X}/i$ is an Enriques surface. One of the main results of this section is that every Enriques surface arises in this way.

Theorem 5.4.1. *Let X be an Enriques surface. Then there is a K3 surface $\widetilde{X}$ admitting an involution i without fixed points, such that $\widetilde{X}/i \simeq X$.*

Proof. As X is an Enriques surface, then K_X is not trivial, but $2K_X = 0$, i. e. K_X has torsion 2 in $Pic(X)$. Now, recall that the line bundle $\mathscr{O}_X(K_X)$ can be viewed as a rank 1 vector bundle $\mathscr{K}_X \longrightarrow X$, where $\mathscr{K}_X$ is a complex manifold of dimension 3. There is an isomorphism of line bundles $\alpha : \mathscr{O}_X(K_X) \longrightarrow \mathscr{O}_X$, and a point $s \in \mathscr{K}_X$ corresponds to a global section s of $\mathscr{O}_X(K_X)$: let then

$$\widetilde{X} := \{s \in \mathscr{K}_X \mid \alpha(s^{\otimes 2}) = 1\}.$$

Then $\widetilde{X}$ comes with a natural map $\pi : \widetilde{X} \longrightarrow X$, which is an unramified Galois cover of degree 2, i. e. on $\widetilde{X}$ there is an involution i without fixed points such that $\widetilde{X}/i = X$.

It remains only to show that X is a K3 surface. First of all, as π is an unramified cover of degree 2, we have $\chi(\mathscr{O}_{\widetilde{X}}) = 2\chi(\mathscr{O}_X) = 2$, as X is an Enriques surface, so that $q(\widetilde{X}) = p_g(\widetilde{X}) + 1$. Moreover, we have that $\pi^*K_X = K_{\widetilde{X}}$, but π^*K_X has a nowhere vanishing section: indeed, we have that $\pi^*K_X = \widetilde{X} \times_X \mathscr{K}_X$, and a section of π^*K_X is obtained simply defining a map $\widetilde{X} \longrightarrow \widetilde{X} \times_X \mathscr{K}_X$. Consider then the map

$$\sigma : \widetilde{X} \longrightarrow \widetilde{X} \times_X \mathscr{K}_X, \qquad \sigma(x) := (x, x),$$

which is everywhere non-vanishing. Hence $\pi^*K_X = 0$, so that $K_{\widetilde{X}} = 0$. This then implies that $p_g(\widetilde{X}) = 1$, so that $q(\widetilde{X}) = 0$, and $\widetilde{X}$ is in conclusion a K3 surface. □

Corollary 5.4.2. *Let X be an Enriques surface. Then $H_1(X, \mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$.*

Proof. By Theorem 5.4.1, there is a K3 surface $\widetilde{X}$ admitting an involution i such that $X = \widetilde{X}/i$. By Proposition 5.3.10, a K3 surface is simply connected, so that $\widetilde{X}$ is the universal cover of X, and $\pi_1(X)$ is the Galois group of the cover. As this is an unramified double cover, we get $H_1(X, \mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$. □

In conclusion, Enriques surfaces are strictly related to K3 surfaces, and we know hoy to produce them all. As for K3 surfaces, the definition of Enriques surface can be extended to complex surfaces as follows:

Definition 5.4.1. A complex surface X is called *Enriques surface* if $q(X) = p_g(X) = 0$ and $2K_X = 0$.

As there are K3 surfaces which are not projective, it is then quite natural to ask if the same is true for Enriques surfaces, as these are quotients of K3 surfaces. The result is the following:

Proposition 5.4.3. *Every Enriques surface is projective.*

Proof. As X in an Enriques surface, then $q(X) = p_g(X) = 0$, so that $c_1 : Pic(X) \longrightarrow H^2(X,\mathbb{Z})$ is an isomorphism. Hence for every class $c \in H^2(X,\mathbb{Z})$ there is a unique (up to isomorphism) line bundle $\mathscr{L}$ such that $c_1(\mathscr{L}) = c$. Now, recall the Hodge Index Theorem: the cup product on $H^2(X,\mathbb{Z})$ has signature $(1,9)$, so there is a class $c \in H^2(X,\mathbb{Z})$ such that $c^2 > 0$. Notice that $c \in H^{1,1}(X)$, so it is a positive integral $(1,1)-$form on X: by Kodaira Projectivity Criterion, we finally get that X is projective. $\square$

As every Enriques surface is projective, and it is the quotient of a K3 surface admitting an involution without fixed points, by the Kodaira Projectivity Criterion it follows that if a K3 surface admits an involution without fixed points, then it has to be projective. There are still interesting results on Enriques surfaces that we won't have the time to prove, like the following:

Theorem 5.4.4. *Let X be an Enriques surface. Then it admits an elliptic fibration.*

5.4.2 The Torelli Theorem for Enriques surfaces

As for K3 surfaces, one can try to study the structure of the lattice $H^2(X,\mathbb{Z})$. For K3 surfaces, this $\mathbb{Z}-$module is free of rank 22, but for Enriques surfaces the situation is different: its rank is 10, and it has torsion (as $2K_X = 0$). The first result we need is the following:

Proposition 5.4.5. *Let X be an Enriques surface.*

1. *There is an isomorphism $Pic(X) \simeq \mathbb{Z}^{10} \oplus \mathbb{Z}/2\mathbb{Z} \simeq H^2(X,\mathbb{Z})$*
2. *The free $\mathbb{Z}-$module $H^2(X,\mathbb{Z})_{free}$ has rank 10 and it is isometric, as a lattice, to $\Lambda_{En} := E_8(-1) \oplus U$.*

Proof. The first item follows immediately from $\rho(X) = 10$, the Universal Coefficients Theorem, and the fact that $H^1(X,\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$, so that $H^2(X,\mathbb{Z}) \simeq H^2(X,\mathbb{Z})_{free} \oplus \mathbb{Z}/2\mathbb{Z}$. As the first Chern class is an isomorphism, this completes the proof of item 1.

For the second item, we have that $H^2(X,\mathbb{Z})_{free}$ and Λ_{En} are two rank 10 unimodular even lattices with signature $(1,9)$ and the same index -8. Hence the have to be isometric. $\square$

For K3 surfaces, we have seen that the lattice $H^2(X,\mathbb{Z})$ is isometric to $\Lambda_{K3} = E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$. This lattice has a natural involution, as follows:

$$\sigma : \Lambda_{K3} \longrightarrow \Lambda_{K3}, \qquad \sigma(x,y,a,b,c) := (y,x,-a,c,b).$$

Let us then define

$$\Lambda^+_{K3} := \{l \in \Lambda_{K3} \,|, \sigma(l) = l\}, \qquad \Lambda^-_{K3} := \{l \in \Lambda_{K3} \,|\, \sigma(l) = -l\}.$$

The lattice Λ^+_{K3} is the isometric to $E_8(-2) \oplus U(2)$, and $l \in \Lambda^+_{K3}$ if and only if $(l,l) \equiv_4 0$. We have then an isomoetry

$$\frac{1}{2}\Lambda^+_{K3} \simeq \Lambda_{En}.$$

The first result we need is:

Proposition 5.4.6. *Let X be an Enriques surface, and let $p : \widetilde{X} \longrightarrow X$ be its universal cover, where $\widetilde{X}$ is a K3 surface admitting an involution i. Then there is an isometry*

$$\phi : H^2(\widetilde{X},\mathbb{Z}) \longrightarrow \Lambda_{K3}$$

such that $\phi \circ i^* = \sigma \circ \phi$.

Using this, the following definition makes sense:

Definition 5.4.2. A *marking* on an Enriques surface X is a marking ϕ on the associated K3 surface $\widetilde{X}$, such that $\phi \circ i^* = \sigma \circ \phi$. A *marked Enriques surface* is a pair given by an Enriques surface and a marking.

Let us now define the following: we defined the moduli space Ω, which is locally isomorphic to the deformation space S of marked K3 surfaces via the period map π. If (X,ϕ) is a marked Enriques surface, we can define its period simply as the period of $(\widetilde{X},\phi)$. As $\phi\circ i^* = \sigma\circ\phi$, the period $\pi(\widetilde{X},\phi)$ satisfies some more conditions. Indeed, if ω is a holomorphic $2-$form on $\widetilde{X}$ which is nowhere vanishing, then $i^*\omega = -\omega$ (otherwise it would descend to a holomorphic $2-$form on X, but $p_g(X) = 0$). Hence

$$\sigma_{\mathbb{C}}(\phi(\omega)) = \phi_{\mathbb{C}}(i^*(\omega)) = -\phi_{\mathbb{C}}(\omega),$$

so that $\phi_{\mathbb{C}}(\omega) \in \Lambda^-_{\mathbb{C}} := \Lambda^-_{K3} \otimes_{\mathbb{Z}} \mathbb{C}$. Let us now define

$$\Omega^- := \Omega \cap \mathbb{P}(\Lambda^-_{\mathbb{C}}),$$

so that the period $\pi(Y,\phi) \in \Omega^{-1}$.

Theorem 5.4.7. ***(Torelli's Theorem for Enriques surfaces)***. *Two Enriques surfaces are isomorphic if and only if their periods are equal.*

Proof. We give here an idea of the proof: let X, Y be two Enriques surfaces and let $p_X : \widetilde{X} \longrightarrow X$ and $p_Y : \widetilde{Y} \longrightarrow Y$ be the associated double covering K3. We can choose markings ϕ_X and ϕ_Y such that $\phi_X \circ i_X^* = \sigma \circ \phi_X$ and similar for ϕ_Y. The periods of (X, ϕ_X) and (Y, ϕ_Y) are equal if and only if the periods of $(\widetilde{X}, \phi_X)$ and $(\widetilde{Y}, \phi_Y)$ are equal, hence by the Local and Global Torelli Theorems for K3 surfaces, if and only if $\psi := \phi_Y \circ \phi_X^{-1} : H^2(\widetilde{Y}, \mathbb{Z}) \longrightarrow H^2(\widetilde{X}, \mathbb{Z})$ is a Hodge isometry, i. e. if and only if there is an isomorphism $g : \widetilde{X} \longrightarrow \widetilde{X}$ such that $g^* = \phi_Y \circ \phi_X^{-1}$. Notice that hence $g^* \circ i_Y^* = i_X^* \circ g^*$, so that $g^* \circ i_Y^* \circ (g^{-1})^* \circ i_X^* = id_{H^2(\widetilde{X}, \mathbb{Z})}$. By the Global Torelli Theorem this is true if and only if $g \circ i_Y \circ g^{-1} \circ i_X = id_{\widetilde{X}}$, so that $g \circ i_Y = i_X \circ g$, i. e. g descends to an isomorphism between X and Y. □

Another important result about Enriques surfaces is the following:

Theorem 5.4.8. *Any two Enriques surfaces are equivalent under deformation.*

5.5 Bielliptic surfaces

The next family of surfaces of Kodaira dimension 0 is given by bielliptic surfaces. The aim of this section is the give a complete list of the possible bielliptic surfaces. Let me recall the invariants: if X is a bielliptic surface, then $K_X \neq 0$ (and as we will see we have $4K_X = 0$ or $6K_X = 0$), and

$$K_X^2 = 0, \qquad P_m(X) = 0, 1, \qquad \kappa(X) = 0,$$

$$q(X) = 1, \qquad p_g(X) = 0, \qquad \chi(\mathscr{O}_X) = 0,$$

$$e(X) = 0, \qquad b_1(X) = 2, \qquad b_2(X) = 2.$$

Notice that as $p_g(X) = 0$, we get that $h^{1,1}(X) = 2$. The first Chern class is surjective, so that $\rho(X) = 2$. Moreover, notice that every bielliptic surface carries a natural elliptic fibration, and they are all projective.

The first result we need to show is the following:

Lemma 5.5.1. *Let E be an elliptic curve. Then every automorphism of E is the composition of a translation and a group automorphism. The non-trivial group automorphisms are the symmetry $x \mapsto -x$ and:*

1. *if $E = E_i := \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}i)$, the automorphism $x \mapsto \pm ix$;*
2. *if $E = E_\rho := \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\rho$, the automorphisms $x \mapsto \pm\rho x$ and $x \mapsto \pm\rho^2 x$, where $\rho^3 = 1, \rho \neq 1$.*

Proof. By Lemma 3.3.5, every automorphism of E is induced by an automorphism of $\mathbb{C}$ sending the lattice Γ (here $E = \mathbb{C}/\Gamma$)itself. Hence every automorphism is the composition of a group translation with a group automorphism, which are only those written above. □

Using this lemma, we can prove the following:

Proposition 5.5.2. *Let $X = (B \times C)/G$ be a minimal surface which is not ruled, where B and C are smooth curves and G is a finite group acting faithfully on B and C. Suppose that $p_g(X) = 0$ and $q(X) \geq 1$. Then the curve B and C are not rational, B/G is elliptic, C/G is rational and one of the following possibilities is verified:*

1. *the curve B is elliptic and the group G is a group of translations on B;*

2. *the curve C is elliptic and the group G acts freely on $C \times B$.*

Moreover, the following properties are verified:

- *we have $P_4(X) \neq 0$ or $P_6(X) \neq 0$. In particular $P_{12}(X) = 1$.*

- *If one between C and B is not elliptic, then $\kappa(X) = 1$; if B and C are elliptic, then $4K_X = 0$ or $6K_X = 0$, so that $12K_X = 0$.*

Proof. By Proposition 5.1.5, we know that if a surface X is minimal and has $p_g = 0$ and $q(X) \geq 1$, then $K_X^2 \leq 0$, and it is 0 if and only if $q(X) = 1$ and $b_2(X) = 2$. As X is not ruled, then we need $K_X^2 \geq 0$, so that finally $K_X^2 = 0$ and $q(X) = 1$. Recall that as $X = (B \times C)/G$, we have $q(X) = g(B/G) + g(C/G)$. As $q(X) = 1$, we then need that one between the two quotients is rational, and the other is elliptic. Let us suppose that B/G is elliptic, so that C/G is rational. If $C = \mathbb{P}^1$, then C/G is no longer rational, so that C has to be irrational. Similarily, B cannot be rational. Hence we have only two possibilites: either B is elliptic, or C is elliptic. In any case, the action of G is free on $B \times C$. In the case where B is elliptic, the finite group G has to be a group of translantions on B, so the first part is shown.

For the remaining part, notice that the second part of the second item follows from the first item: as B and C are elliptic, then $K_{B\times C} = 0$. If $D \in |4K_X|$, we have $\pi^* D = 0$ if $P_4(X) = 0$, where $\pi : B \times C \longrightarrow X$ is the étale cover. Similarily, if $P_6(X) = 0$, then $6K_X = 0$.

Let us first suppose that B is elliptic. We have

$$H^0(\Omega_X^2)^{\otimes k} \simeq (H^0(B \times C, \Omega^2_{B\times C})^{\otimes k})^G \simeq (H^0(B, \Omega_B)^{\otimes k})^G \otimes (H^0(C, \Omega_C)^{\otimes k})^G.$$

As G acts as translations on B by the first part, we have that $(H^0(B, \Omega_B))^G = H^0(B, \Omega_B)$, so that

$$P_m(X) = h^0(C, \mathscr{O}_C(mK_C))^G.$$

Now, let $\mathscr{L}_m := \mathscr{O}_{\mathbb{P}^1}(-2m + \sum_{p\in\mathbb{P}^1} m(1 - (1/e_p)))$, where e_p is the ramification index at p of the action og G on C. Hence

$$P_m(X) = h^0(\mathbb{P}^1, \mathscr{L}_m),$$

But now by Riemann-Roch

$$P_m(X) \geq -2m + \sum_{p \in \mathbb{P}^1} \left(m \left(1 - \frac{1}{e_p} \right) - 1 \right) = m \frac{2g(C) - 2}{n} - r,$$

where n is the order of G, r is the number of ramification points, by the Riemann-Hurwitz Formula. If $g(C) \geq 2$, then $\kappa(X) = 1$. Let us now suppose that C is elliptic. As $\sum_p (1 - (1/e_p)) \geq 2$ by the Riemann-Hurwitz Formula, we have to show that $deg(\mathscr{L}_m) \geq 0$ for some k. We have the following possibilites:

1. $r \geq 4$. Hence $2(1 - (1/e_i)) \geq 1$, we have that $deg(\mathscr{L}_2) \geq 0$.
2. $r = 3$. Then $(1/e_1)+(1/e_2)+(1/e_3) \leq 1$. If $e_1 \geq 3$, then $3(1-(1/e_1)) \geq 2$, so that $deg(\mathscr{L}_3) \geq 0$, and we can suppose $e_1 = 2$, i. e. $(1/e_2) + (1/e_3) \leq 1/2$. If $e_2 \geq 4$, then again $deg(\mathscr{L}_4) \geq 0$. If $e_2 = 3$, then $e_3 \geq 6$, so that $deg(\mathscr{L}_6) \geq 0$.

In any case, hence $P_4(X) \neq 0$ or $P_6(X) \neq 0$, so that $P_{12}(X) \neq 0$.

Now, let us suppose that B is not elliptic, hence C has to be elliptic. Assume that $C \neq E_i, E_\rho$. Hence the only automorphisms on C can be translations on C or $-id$, so that if $\omega \in H^0(C, \Omega_C)$, then $\omega^2 \in H^0(C, \Omega_C)^G$. Hence we have

$$P_{2m}(X) = h^0(B/G, \mathscr{O}_{B/G}(\sum_p (m(1 - (1/e_p)))p)),$$

as B/G is an elliptic curve. If $g(B) \geq 2$, then $\kappa(X) = 1$. If $g(B) = 1$, then this expression is not zero, so that $P_{2m}(X) \neq 0$. Similarily, if $C = E_i$, then we need to consider $\omega^{\otimes 4}$, so that $P_{4m}(X) \neq 0$ and the expression is the same. If $C = E_\rho$, we need $\omega^{\otimes 6}$, so that $P_6(X) \neq 0$. □

As a corollary we have the following:

Corollary 5.5.3. *A minimal surface X is ruled if and only if $P_{12}(X) = 0$.*

Proof. If X is ruled, then clearly we have $P_{12}(X) = 0$. If $P_{12}(X) = 0$, then $p_g(X) = P_2(X) = P_4(X) = P_6(X) = P_3(X) = 0$. Hence if $q(X) = 0$, then X is rational by the Castelnuovo Rationality Criterion. If $q(X) \geq 1$, we need $K_X^2 < 0$, so that K_X is not nef, and $\kappa(X) = -\infty$. In conclusion, X is ruled. □

Another corollary is the following

Corollary 5.5.4. *Every minimal surface X such that $\kappa(X) = 0$ has $P_{12}(X) = 1$ and $12K_X = 0$.*

Proof. If X is a K3 or abelian surface, then $P_{12}(X) = 1$ and $K_X(X) = 0$, so that $12K_X = 0$. If X is an Enriques surface, then $2K_X = 0$, so that $12K_X = 0$ and $P_{12}(X) = 0$. If X is bielliptic, then $4K_X = 0$ or $6K_X = 0$ by Proposition 5.5.2. Hence $12K_X = 0$, so that $P_{12}(X) = 1$. By The Enriques-Kodaira Classification, this is all. □

We can now give the complete list of bielliptic surfaces:

Theorem 5.5.5. ***(Bagnera-De Franchis)***. *Let $X = (E \times F)/G$ be a bielliptic surface. Suppose that G is a group of translations on E and acting of F. Then E, F and G can be one and only one of the following types:*

1. *E and F are any two elliptic curves, $G = \mathbb{Z}/2\mathbb{Z}$ acting on F by symmetry. In this case we have $2K_X = 0$;*

2. *E and F are any two elliptic curves, $G = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ acting on F by $x \mapsto -x$, $x \mapsto x + \epsilon$, where ϵ is a $2-$torsion point of F. In this case we have $2K_X = 0$;*

3. *E is any elliptic curve, $F = E_i$, $G = \mathbb{Z}/4\mathbb{Z}$ acting on F by $x \mapsto ix$. In this case we have $4K_X = 0$;*

4. *E is any elliptic curve, $F = E_i$, $G = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ acting on F by $x \mapsto ix$, $x \mapsto x + (\frac{1+i}{2})$. In this case we have $4K_X = 0$;*

5. *E is any elliptic curve, $F = E_\rho$, $G = \mathbb{Z}/3\mathbb{Z}$ acting on F by $x \mapsto \rho x$. In this case we have $6K_X = 0$;*

6. *E is any elliptic curve, $F = E_\rho$, $G = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ acting on F as $x \mapsto \rho x$, $x \mapsto x + (\frac{1-\rho}{3})$. In this case we have $6K_X = 0$;*

7. *E is any elliptic curve, $F = E_\rho$, $G = \mathbb{Z}/6\mathbb{Z}$ acting on F by $x \mapsto \rho^2 x$. In this case we have $6K_X = 0$.*

Proof. The group G is a subgroup of $Aut(F)$, so it has to be of the form $G = T \rtimes A$, where T is a group of translations on F, and A is a group of group automorphisms of F. Since $F/G \simeq \mathbb{P}^1$, the group A is not trivial. By Lemma 5.5.1, A can be only of the form $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/6\mathbb{Z}$. Moreover, G is s group of translations on E, so that $G = T \oplus A$, i. e. every element of T is $A-$invariant. Now, the fixed points of A can be the following:

1. If A acts as $x \mapsto -x$, the points of order 2.

2. If A acts as $x \mapsto ix$, i. e. $F = E_i$, the points 0 and $\frac{1+i}{2}$.

3. If A acts as $x \mapsto \rho x$, i. e. $F = E_\rho$ the points 0, $\frac{1-\rho}{3}$ and $-\frac{1+\rho}{3}$.

4. If A acts as $x \mapsto \rho^2 x$, i. e. $F = E_\rho$, the point 0.

Now, simply notice that as G is a group of translations on E, it can be generated by two elements, so that $G \neq F[2] \times A$ where $F[2]$ is the subgroup of $2-$torsion points. The last remaining possibilities are those listed in the statement. $\square$

5.6 Abelian surfaces

The last remaining class of surfaces of Kodaira dimension 0 is the one of abelian surfaces. As the theory of abelian surfaces is almost the theory of abelian varieties of any dimension, it is out of our goals to list their properties. Anyway, I would like to collect some basic facts about the Torelli Theorem for complex tori and projectivity criteria. I will also collect some basic facts about the cohomology of line bundles on abelian surfaces. Before, let me recall the invariants: if X is an abelian surface, then $K_X = 0$ and

$$K_X^2 = 0, \qquad P_m(X) = 1, \qquad \kappa(X) = 0,$$

$$q(X) = 2, \qquad p_g(X) = 1, \qquad \chi(\mathscr{O}_X) = 0,$$

$$e(X) = 0, \qquad b_1(X) = 4, \qquad b_2(X) = 6.$$

We have then $h^{1,1}(X) = 4$, so that $0 \leq \rho(X) \leq 4$. By definition, notice that every complex torus is Kähler. One of the main question is then when a complex torus is an abelian surface.

5.6.1 Properties of abelian surfaces

Recall that an abelian surface is defined as a complex torus of dimension 2 admitting an embedding into some projective space. Hence, any abelian surface X is of the form $X = \mathbb{C}^2/\Gamma$, where Γ is a maximal rank lattice (i. e. of rank 4) in $\mathbb{C}^2$. There are two stupid remarks to do immediately: as $\mathbb{C}^2$ is simply connected, it is then the universal cover of X, so that it is obvious that $\pi_1(X) \simeq \Gamma$. Moreover, as $\Gamma \simeq \mathbb{Z}^4$ is an abelian group, we have $H_1(X, \mathbb{Z}) \simeq \Gamma$. This gives an intrinsec description of the first singular cohomology group of an abelian surface (actually, of every complex torus), which is then a free $\mathbb{Z}-$module of rank 4. Let me first introduce the following definition: if one fixes a basis e_1, e_2 of $\mathbb{C}^2$ and a basis $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ of Γ, we can write every λ_i in terms of e_1, e_2, getting $\lambda_i = \lambda_{i1}e_1 + \lambda_{i2}e_2$, where $\lambda_{ij} \in \mathbb{C}$ for every $i = 1, ..., 4$ and $j = 1, 2$. We then get a matrix $\Pi_X := [\lambda_{ij}]_{i,j}$, depending on the chosen basis.

Definition 5.6.1. The matrix $\Pi_X \in M(2 \times 4, \mathbb{C})$ is called *period matrix* of X (with respect to the chosen basis).

Notice that Π_X determines the torus completely, but different matrices do not give necessarily different abelian surfaces. Moreover, not every matrices defines an abelian surface. We have the following:

Proposition 5.6.1. *Let $M \in M(2 \times 4, \mathbb{C})$, and let $\overline{M}$ be its complex conjugate matrix. Then there is a complex torus X such that $\Pi_X = M$ if and only if the matrix $P := \binom{M}{\overline{M}} \in M(4 \times 4, \mathbb{C})$ is invertible.*

Proof. The the matrix M defines a lattice $\Gamma \subseteq \mathbb{C}^2$, which defines a complex torus if and only if it is of maximal rank, i. e. if and only if the columns are linearily independent on $\mathbb{R}$. Now, if M does not define a complex torus, two columns have to be dependent, and let us suppose that thesare the first two columns. Then there is $x \in \mathbb{R}^4$, $x \neq 0$ such that $Mx = 0$, so that $Px = 0$. Hence $det(P) = 0$, and P is not invertible. Conversely, if P is not invertible, then there are $x, y \in \mathbb{R}^4$, $x, y \neq 0$, such that $P(x+iy) = 0$. Then $M(x+iy) = 0$ and $M(x - iy) = \overline{M(x + iy)} = 0$, so that $Mx = My = 0$, hence two columns of M are linearily dependent on $\mathbb{R}$, i. e. the lattice Γ has not maximal rank. $\square$

The period matrix is important even because it gives us informations on the projectivity of the associated complex torus. Before looking at this problem, let me discuss a little bit of singular cohomology of abelian surfaces. As $H_1(X, \mathbb{Z})$ is a free $\mathbb{Z}-$module, the universal coefficient theorem tells us that $H^1(X, \mathbb{Z}) \simeq Hom(H_1(X, \mathbb{Z}), \mathbb{Z})$ is free of rank 4, and $H^2(X, \mathbb{Z})$ is free of rank 6. But we have a more intrinsec description of $H^2(X, \mathbb{Z})$:

Proposition 5.6.2. *The canonical map*

$$\bigwedge^2 H^1(X, \mathbb{Z}) \longrightarrow H^2(X, \mathbb{Z})$$

induced by the cup product is an isomorphism.

Proof. This is just the Künneth formula. $\square$

As $H^1(X, \mathbb{Z}) = Hom(\Gamma, \mathbb{Z})$, we have hence

$$H^2(X, \mathbb{Z}) \simeq Alt^2(\Gamma, \mathbb{Z}) := \bigwedge^2 Hom(\Gamma, \mathbb{Z}),$$

which are alternating integral $2-$forms on Γ. In particular, we have

$$H^2(X, \mathbb{C}) \simeq Alt^2_{\mathbb{R}}(V, \mathbb{C}),$$

where $V := \Gamma \otimes_{\mathbb{Z}} \mathbb{C}$ is a complex vector space of dimension 4, and $Alt^2_{\mathbb{R}}(V, \mathbb{C})$ is the group of $\mathbb{R}-$linear alternating $2-$forms on V.

Notice that on $H^1(X, \mathbb{C})$ we have a Hodge structure of weight 1, i. e. a decomposition $H^1(X, \mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X)$, where $H^{i,j}(X)$ is the vector space of global $(i, j)-$forms on X. We have that $H^1(X, \mathbb{C})$ is canonically isomorphic to the complex vector space of $1-$forms on $\mathbb{C}^2$ invariant under Γ, and similarily we have that $H^{i,j}(X)$ is the vector space of $\Gamma-$invariant $(i, j)-$forms on $\mathbb{C}^2$. By Proposition 5.6.2 we have then that $H^2(X, \mathbb{C})$ is the vector space of $\Gamma-$invariant $2-$forms on $\mathbb{C}^2$, and the Hodge decomposition has the same property.

Now, let $\mathscr{L} \in Pic(X)$, so that $c_1(\mathscr{L}) \in H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$. Hence $c_1(\mathscr{L})$ can be seen either as a $\Gamma-$invariant $(1, 1)-$form on $\mathbb{C}^2$, or as an integral alternating $2-$form on Γ. It is interesting to understant which alternating $2-$forms on Γ are corresponding to the first Chern class of a line bundle:

Proposition 5.6.3. *Let $E \in Alt^2(V, \mathbb{R})$. The following are equivalent:*

1. *there is a line bundle $\mathscr{L} \in Pic(X)$ such that E represents $c_1(\mathscr{L})$;*
2. *$E(\Gamma, \Gamma) \subseteq \mathbb{Z}$ and $E(v, w) = E(iv, iw)$ for every $v, w \in V$.*

Proof. Let $\Omega := Hom_{\mathbb{C}}(V, \mathbb{C})$ and $\overline{\Omega} = Hom_{\overline{\mathbb{C}}}(V, \mathbb{C})$. Then it is easy to show that

$$Alt^2_{\mathbb{R}}(V, \mathbb{C}) \simeq \bigwedge^2 \Omega \oplus (\Omega \times \overline{\Omega}) \oplus \bigwedge^2 \overline{\Omega}.$$

Hence we have natural identifications $H^2(X, \mathbb{C}) \simeq \bigwedge^2 \Omega \oplus (\Omega \times \overline{\Omega}) \oplus \bigwedge^2 \overline{\Omega}$ and $H^{0,2}(X) \simeq \bigwedge^2 \overline{\Omega}$. If $\mathscr{L} \in Pic(X)$, then $c_1(\mathscr{L})$ can be represented by an element $E \in Alt^2_{\mathbb{R}}(V, \mathbb{C})$ that we can write as $E = E_1 + E_2 + E_3$ according to the previous decomposition. As E has values in $\mathbb{R}$ we need $E_1 = \overline{E_3}$, and $E_3 = 0$ as $c_1(\mathscr{L})$ is a $(1,1)-$form. Hence $E = E_2$ and it is easy to see that it verifies the two conditions in the statement. The converse is similar. □

There is another way to represent the first Chern class of a line bundle:

Proposition 5.6.4. *There is a 1 to 1 correspondence between alternating real forms E on V such that $E(\Gamma, \Gamma) \subseteq \mathbb{Z}$ and $E(v, w) = E(iv, iw)$ for every $v, w \in V$, and Hermitian forms H on V, i. e. $\mathbb{C}-$linear forms on V such that $H(v, w) = \overline{H(w, v)}$ for every $v, w \in V$.*

Proof. Let $E : V \times V \longrightarrow \mathbb{R}$ be alternating and such that $E(\Gamma, \Gamma) \subseteq \mathbb{Z}$ and $E(v, w) = E(iv, iw)$ for every $v, w \in V$. Define

$$H_E : V \times V \longrightarrow \mathbb{C}, \qquad H_E(v, w) := E(iv, w) + iE(v, w).$$

Then H is $\mathbb{C}-$linear and it is easy to verify that it is Hermitian. For the converse, let H be Hermitian on V, and define

$$E_H : V \times V \longrightarrow \mathbb{R}, \qquad E_H(v, w) := Im(H(v, w)),$$

and it is easy to verify that E_H is $\mathbb{R}-$linear and satisfies the properties of the statement. □

In conclusion, the first Chern class of any line bundle corresponds to an Hermitian form on V. Using this, we can finally prove the following:

Theorem 5.6.5. ***(Riemann's Projectivity Criterion)**. Let $X = V/\Gamma$ be a $2-$dimensional complex torus. Then X is an abelian surface if and only if there is a non-degenerate alternating matrix $A \in M(4 \times 4, \mathbb{Z})$ such that the two following relations are verified:*

1. *$\Pi_X \cdot A^{-1} \cdot \Pi_X^T = 0$,*
2. *$i\Pi_X \cdot A^{-1} \cdot \overline{\Pi_X}^T > 0$.*

Proof. Consider an arbitrary non-degenerate alternating $2-$form E on Γ (denote by E even its real extension to V), and consider

$$H : V \times V \longrightarrow \mathbb{C}, \qquad H(v,w) := E(iv,w) + iE(v,w).$$

Let A be the matrix representing E with respect to the basis $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ of Γ, which is the basis with respect to which the period matrix of X is Π_X. By Propositions 5.6.4 and 5.6.3, H is Hermitian if and only if there is a line bundle $\mathscr{L} \in Pic(X)$ such that $c_1(\mathscr{L})$ is represented by E. Moreover, X is projective if and only if $c_1(\mathscr{L})$ is a positive definite $(1,1)-$form by the Kodaira Projectivity Criterion. Notice that as $c_1(\mathscr{L})$ is represented by E, the positivity of $c_1(\mathscr{L})$ is equivalent to the fact that H is positive definite. Hence the projectivity of X is equivalent to the existence of a positive definite Hermitian form H on V. Le us now read these two properties in terms of the matrix A.

Step 1. The form H is Hermitian if and only if $\Pi_X \cdot A^{-1} \cdot \Pi_X^T = 0$. Indeed, the form H is Hermitian if and only if $E(v,w) = E(iv,iw)$ for every $v, w \in V$. Consider the matrix

$$M := \begin{bmatrix} \Pi_X \\ \overline{\Pi_X} \end{bmatrix}^{-1} \cdot \begin{bmatrix} iI_2 & 0 \\ 0 & iI_2 \end{bmatrix} \cdot \begin{bmatrix} \Pi_X \\ \overline{\Pi_X} \end{bmatrix}.$$

It is easy to see that $i\Pi_X = \Pi_X \cdot M$. Moreover, we have

$$E(\Pi_X x, \Pi_X y) = x^T A y,$$

for every $x, y \in \mathbb{R}^4$, so that H is Hermitian if and only if $A = M^T \cdot A \cdot M$. Equivalently, if and only if the following relation is verified:

$$\begin{bmatrix} iI_2 & 0 \\ 0 & -iI_2 \end{bmatrix} \cdot \left(\begin{bmatrix} \Pi_X \\ \overline{\Pi_X} \end{bmatrix} \cdot A^{-1} \cdot \begin{bmatrix} \Pi_X^T & \overline{\Pi_X}^T \end{bmatrix} \right)^{-1} \cdot \begin{bmatrix} iI_2 & 0 \\ 0 & -iI_2 \end{bmatrix} =$$

$$= \left(\begin{bmatrix} \Pi_X \\ \overline{\Pi_X} \end{bmatrix} \cdot A^{-1} \cdot \begin{bmatrix} \Pi_X^T & \overline{\Pi_X}^T \end{bmatrix} \right)^{-1}.$$

Comparing the 2×2 blocks on each side, we finish step 1.

Step 2. The Hermitian form H is positive definite if and only if $i\Pi_X \cdot A^{-1} \cdot \overline{\Pi_X}^T > 0$. Indeed, let $x, y \in \mathbb{R}^4$, and let $U := \Pi_X x$ and $v := \Pi_X y$. As H is Hermitian, by Step 1 we know that $\Pi_X \cdot A^{-1} \overline{\Pi_X}^T = 0$, so that we get

$$E(iu, v) = x^T M^T A y =$$

$$= [u \ \ \overline{u}] \cdot \begin{bmatrix} iI_2 & 0 \\ 0 & -iI_2 \end{bmatrix} \cdot \left(\begin{bmatrix} \Pi_X \\ \overline{\Pi_X} \end{bmatrix} \cdot A^{-1} \cdot \begin{bmatrix} \Pi_X^T & \overline{\Pi_X}^T \end{bmatrix} \right)^{-1} \cdot \begin{bmatrix} v \\ \overline{v} \end{bmatrix} =$$

$$= [u \ \ \overline{u}] \cdot \begin{bmatrix} 0 & i(\overline{\Pi_X} \cdot A^{-1} \cdot \Pi_X^T)^{-1} \\ -i(\overline{\Pi_X} \cdot A^{-1} \cdot \Pi_X^T)^{-1} & 0 \end{bmatrix} \cdot \begin{bmatrix} v \\ \overline{v} \end{bmatrix} =$$

$$= iu^T(\overline{\Pi_X} \cdot A^{-1} \cdot \Pi_X^T)^{-1}\overline{v} - i\overline{u}^T(\overline{\Pi_X} \cdot A^{-1} \cdot \Pi_X^T)^{-1}v.$$

Similarily one computes

$$E(u,v) = u^T(\overline{\Pi_X} \cdot A^{-1} \cdot \Pi_X^T)^{-1}\overline{v} + \overline{u}^T(\overline{\Pi_X} \cdot A^{-1} \cdot \Pi_X^T)^{-1}v,$$

so that

$$H(u,v) = 2iu^T(\overline{\Pi_X} \cdot A^{-1} \cdot \Pi_X^T)^{-1}\overline{v}.$$

Hence $H = 2i(\overline{\Pi_X} \cdot A^{-1} \cdot \Pi_X^T)^{-1}$, so that it is positive definite if and only if $i\Pi_X \cdot A^{-1} \cdot \overline{\Pi_X}^T > 0$. □

Now, let $\mathscr{L} \in Pic(X)$, where $X = V/\Gamma$ is any complex torus. We know that $c_1(\mathscr{L})$ is represented by an Hermitian form H on V, whose alternating real form $E = Im(H)$ is integer valued on Γ. It is a basic result in matrix theory that hence there is a basis $\lambda_1, \lambda_2, \mu_1, \mu_2$ of Γ with respect to which the form E is represented by a matrix

$$M_E := \begin{bmatrix} 0 & D \\ -D & 0 \end{bmatrix},$$

where $D = diag(d_1, d_2)$ is a diagonal matrix of eigenvalues $d_1, d_2 \in \mathbb{N}_0$, with $d_1 | d_2$. These elements are called *elementary divisor*, and are completely determined by E, hence by $\mathscr{L}$.

Definition 5.6.2. The *type* of $\mathscr{L}$ is the pair (d_1, d_2).

Notice that E is non-degenerate if and only if $d_1, d_2 > 0$. If $\mathscr{L}$ is ample, i. e. $c_1(\mathscr{L})$ is positive define, then $\mathscr{L}$ is called a *polarization*. If the type of $\mathscr{L}$ is $(1,1)$, then $\mathscr{L}$ is called a *principal polarization*. An abelian surface is then a $2-$dimensional complex torus admitting a polarization. An abelian surface is *principally polarized* if it admits a principal polarization. An interesting result is the following:

Theorem 5.6.6. *Let X be an abelian surface, and let $\mathscr{L}$ be a polarization of type (d_1, d_2) on X. If $d_1 \geq 3$, the map $\varphi_{\mathscr{L}} : X \longrightarrow \mathbb{P}^N$ is an embedding.*

5.6.2 The Torelli Theorem for abelian surfaces

Abelian surface, and more generally complex tori, are very special for several reasons. First of all, the theory of abelian surfaces and $2-$dimensional complex torus can almost completely be generalized to higher dimension. Moreover, as for K3 and Enriques surfaces we have a Torelli Theorem for $2-$dimensional complex tori: the main difference is that it does not involve $H^2(X, \mathbb{Z})$ but $H^1(X, \mathbb{Z})$.

Theorem 5.6.7. ***(Torelli's Theorem for complex tori)***. *Let X and X' be two complex tori of dimension 2. A Hodge morphism*

$$\phi : H^1(X', \mathbb{Z}) \longrightarrow H^1(X, \mathbb{Z})$$

is a Hodge isomorphism if and only if there is an isomorphism $f : X \longrightarrow X'$ such that $f^* = \phi$.

Proof. The proof uses properties of the Albanese map. Recall that $a_X : X \longrightarrow Alb(X)$ is an isomorphism by the universal property of the Albanese torus, hence X and X' are isomorphic if and only if $Alb(X)$ and $Alb(X')$ are isomorphic. Recall now the definition of $Alb(X)$: we have

$$Alb(X) := H^0(X, \Omega_X)^*/im(H_1(X, \mathbb{Z})).$$

The morphism ϕ induces a morphism

$$\phi^{1,0} : H^0(X', \Omega_{X'}) \longrightarrow H^0(X, \Omega_X)$$

such that $\phi^{1,0}(im(H_1(X', \mathbb{Z}))) \subseteq im(H_1(X, \mathbb{Z}))$. Hence ϕ is a Hodge isomorphism if and only if there is an isomorphism $h : Alb(X') \longrightarrow Alb(X)$. Hence we get an isomorphism

$$f := a_{X'}^{-1} \circ h^{-1} \circ a_X : X \longrightarrow X',$$

and it is easy to verify that $f^* = \phi$. $\square$

This provides a complete classification of 2−dimensional complex tori. One can read this in terms of $H^2(X, \mathbb{Z})$ and its Hodge structure, as $H^2(X, \mathbb{Z}) \simeq \bigwedge^2 H^1(X, \mathbb{Z})$. Indeed, one has the following:

Theorem 5.6.8. *Let X and X' be two 2−dimensional complex tori. A Hodge morphism respecting the intersection forms*

$$\phi : H^2(X', \mathbb{Z}) \longrightarrow H^2(X, \mathbb{Z})$$

is a Hodge isometry if and only if there is an isomorphism $f : X \longrightarrow X'$ such that $f^* = \phi$.

Proof. The morphism ϕ is of the form $\phi = \psi \wedge \psi$ as $H^2(X, \mathbb{Z}) \simeq \bigwedge^2 H^1(X, \mathbb{Z})$ for every complex torus X. One can show (using lattice theory) that ϕ is a Hodge isometry if and only if ψ is a Hodge isomorphism. But using the Torelli Theorem for complex tori, we have the statement. $\square$

5.7 Elliptic surfaces

The next class of surfaces we want to study is the one of elliptic surfaces. Recall that an elliptic surface is simply a surface X together with a holomorphic map $\pi : X \longrightarrow C$, where C is a smooth curve, and the generic fiber of π is a smooth elliptic curve. By the Enriques-Castelnuovo classification of surfaces, we have seen that if a surface X is elliptic, then $\kappa(X) \leq 1$, and we have seen examples of elliptic surfaces of Kodaira dimension $-\infty$, 0 and 1. It is then a natural question if it is possible to classify ellitpic surfaces up to birationality. In order to do so, let me introduce the following definition

Definition 5.7.1. An elliptic surface X is *relatively minimal* if it has no $(-1)-$curves in the fibers. Two elliptic surfaces $\pi_1 : X_1 \longrightarrow C$ and $\pi_2 : X_2 \longrightarrow C$ are *birational as elliptic surfaces* if there is a birational map $f : X_1 \dashrightarrow X_2$ such that $\pi_2 \circ f = \pi_1$.

In general, as the generic fiber of π is an elliptic surface, we suppose that there is a section $s : C \longrightarrow X$ of π: simply send a point $p \in C$ over which the fiber is smooth to $s(p) := O$, where O is the zero point of $\pi^{-1}(p)$. More generally one can consider X without sections, if there is no choice of origin of the fibers.

5.7.1 Classification of singular fibers

In this first section I want to present the classification of singular fibers of an elliptic fibration given by Kodaira. To prove this, I need to recall some basic facts about graphs and Dynkin diagrams. First of all, let me give the list of the Dynkin diagrams. These are A_n for $n > 0$ which is of the form is a cycle with $n + 1$ vertices; D_n, for $n \geq 4$; E_6, E_7 and E_8.

Let me recall some basic facts about graph theory: let G be a graph with possible loops and multiple edges, and consider the vector space V_G whose basis is in bijection with the vertices of G. We can define a symmetryc bilinear form $(.,.)$ on V_G as follows: $(v, v) = -2 + 2l(v)$, where $l(v)$ is the number of loops of v, for every vertex v; and $(v, w) = e(v, w)$ for every $v \neq w$, where $e(v, w)$ is the number of edges joining v and w. The vector space V_G together with the form $((.,.)$ is called the *associated form* to G.

Lemma 5.7.1. *We have the following:*

1. *The associated form to any Dynkin diagram is negative semi-definite with a $1-$dimensional kernel spanned by the vector of multiplicites.*

2. *Every graph contains or is contained in some Dynking diagram; every connected graph without loops or multiple edges cointains or is contained in some Dynkin diagram without loops or multiple edges.*

3. *If a graph has associated form which is negative semi-definite and has* $1-$*dimensional kernel, then it is a Dynkin diagram.*

Using this Lemma, we are ablo to show the following:

Theorem 5.7.2. ***(Kodaira's classification of singular fibers)****. Let* $\pi : X \longrightarrow C$ *be a minimal elliptic surface. Then the only possible fibers of* π *can be:*

1. *Type* I_0*: a smooth elliptic curve.*

2. *Type* I_1*: a nodal rational curve.*

3. *Type* I_n *for* $n \geq 3$*:* n *smooth rational curves meeting in a cycle, i. e. with dual graph* A_n*.*

4. *Type* I_n^* *for* $n \geq 0$*:* $n+5$ *smooth rational curves meeting with dual graph* D_{n+4}*.*

5. *Type* II*: a cuspidal rational curve.*

6. *Type* III*: two smooth rational curves meeting at one point with order 2.*

7. *Type* IV*: three smooth rational curves meeting at one point.*

8. *Type* $IV*$*: seven smooth rational curves meeting with dual graph* E_6*.*

9. *Type* $III*$*: eight smooth rational curves meeting with dual graph* E_7*.*

10. *Type* $II*$*: nine smooth rational curves meeting with dual graph* E_8*.*

11. *Type* ${}_mI_n$*: for* $n \geq 0$ *and* $m \geq 2$*: topologically* I_n*, but with multiplicity* m*.*

Moreover, every component of a fiber has self-intersection -2*.*

Proof. Let $\pi : X \longrightarrow C$ have a singular fiber over $0 \in C$, and let it be X_0. It can be written in the form $X_0 = \sum_i n_i C_i$, where $n_i \in \mathbb{N}$ and C_i is irreducible. Let $m := mult(X_0)$ be the multiplicity of the fiber. Write then $X_0 = mF$, where $F = \sum_i r_i C_i$, where $n_i = mr_i$.

If F is irreducible, then $g(F) = 1$ by the genus formula, so it can either be a smooth elliptic curve (i. e. type I_0), a nodal rational curve (i. e. type I_1) or a cuspidal rational curve (i. e. type II). Assume now that F is reducible, and let E be the generic fiber of π. As $K_X \cdot E = 0$, we have $K_X \cdot F = 0$, so that

$$0 = K_X \cdot F = \sum_i r_i C_i \cdot K_X = \sum_i r_i (2g(C_i) - 2 - C_i^2).$$

If $2g(C_i) - 2 - C_i^2 < 0$ for some i, then we get

$$-2 \leq 2g(C_i) - 2 < C_i^2 < 0,$$

where the last inequality follows from the Zariski Lemma. Hence we have $g(C_i) = 0$, so that C_i is a smooth rational curve, and $C_i^2 = -1$, which is not possible as X is relatively minimal. Hence we need $2g(C_i) - 2 \geq C_i^2$ for every i. As $r_i > 0$, we then need $2g(C_i) - 2 - C_i^2 = 0$ for every i. Now $C_i^2 < 0$, so that $2g(C_i) - 2 < 0$, hence $g(C_i) = 0$ and C_i is a smooth irreducible curve such that $C_i^2 = -2$.

Now, consider the dual graph G of F: for every component C_i we have a vertex, for every i, j we have $C_i \cdot C_j$ vertices joining v_i to v_j and multiplicity C_i^2 to the vertex v_i. As the intersection form is negative semi-definite, we need the graph G to be one of the Dynkin diagrams by Lemma 5.7.1. If $G = A_1$, then F is either I_2 if the two components meet in two different points, or III if the two components meet at one point. If $G = A_3$, then F is either I_3 if the three components meet in a cycle, or IV is they meet in a single point. In any other case we obtain the other possible diagrams.

Now, let us complete the analysis for $m > 1$. In this case F must be simply connected because of Corollary 4.3.9, so that the topological type of X_0 is I_n, and we are done. $\square$

One can actually give examples of elliptic surfaces carrying at least one of these fibers.

Example 5.7.1. Let us consider C_1 a smooth cubic in $\mathbb{P}^2$, and C_2 be another cubic. The pencil generated by C_1 and C_2 has 9 base points $p_1, ..., p_9$ counted with multiplicity. Consider $X := Bl_{p_1,...,p_9}\mathbb{P}^2$, which has a natural map

$$\pi : X \longrightarrow \mathbb{P}^1,$$

whose fiber is an element in the pencil generated by C_1 and C_2. As the generic element is a smooth cubic in $\mathbb{P}^2$, it is an elliptic curve. Notice that there are no $(-1)-$curves on X, so X is minimal.

Now, let us suppose that C_2 is a reduced curve meeting C_1 transversally, i. e. in 9 distinct points. The fiber of π corresponding to C_2 is then C_2 itself. Hence if C_2 is a nodal cubic we obtain a fiber of type I_1, if C_2 is a cuspidal cubic we obtain a fiber of type II. If $C_2 = Q + L$ is a conic plus a line, then the fiber is of type I_2 if L is not tangent to Q, of type III if the line is tangent. If C_2 is given by three distinct lines, the fiber is of type III if they are not cuncurrent, or of type IV is they are cuncurrent. If we chose $C_2 = 2L + M$, for L and M distinct such that C_2 meets transversally M and L in three points each, the fiber is of type I_0^* (here the proper transform of L occurs with multiplicity 2 in the fiber). If C_1 is tangent to L, we obtain a fiber of type I_1^*. If $C_2 = 3L$ and C_1 meets L in three distinct points, then we obtain a fiber of type IV^*. If C_1 is tangent to L, then we get a fiber of type III^*. If C_1 has a flex on L, then the fiber is of type II^*.

To get multiple fibers, we consider C a smooth cubic, $p_1, ..., p_9 \in C$ so that $\sum_i p_i$ is not a divisor in $3H$ (where H is the hyperplane section of C), but $2\sum_i p_i$ is a divisor in $6H$. Consider the set of sextics of $\mathbb{P}^2$ having 9 double points at the p_i's. There is at least one of such sextics, for example $2C$. One can show that there are 2 such sextics, one is $2C$ and the other is S. The generic member of this pencil has genus 1 because a smooth sextic has arithmetic genus 10. Blowing up $\mathbb{P}^2$ at $p_1, ..., p_9$ be obtain a smooth minimal surface having an elliptic fibration over $\mathbb{P}^1$, whose singular fiber is the proper transform of $2C$, which is of type ${}_2I_0$. If one considers curves of degree $3m$ having 9 $m-$fold points as base locus, and cutting out a divisor D on a smooth cubic sucht that mD is a divisor in $3mH$, we get an elliptic surface with singular fiber ${}_mI_0$.

5.7.2 Classification of elliptic surfaces

In this section we want to give a complete classification of relatively minimal elliptic surfaces according to the Enriques-Castelnuovo classification. The first property we need to prove is the following:

Proposition 5.7.3. *Let $\pi_1 : X_1 \longrightarrow C$ and $\pi_2 : X_2 \longrightarrow C$ be two minimal elliptic surfaces which are birational as elliptic surfaces aver C. Then they are isomorphic as elliptic surfaces over C. In particular, every elliptic surface over C has a unique minimal model as elliptic surface over C*

Proof. Let $f : X_1 \dashrightarrow X_2$ be a birational map such that $\pi_2 \circ f = \pi_1$. As f preserves the elliptic fibrations, it can be resolved only by blowing-up points and blowing-down curves in the fibers of π_i: hence we can find a surface X together with two morphisms $g_1 : X \longrightarrow X_1$ and $g_2 : X \longrightarrow X_2$ such that $f \circ g_1 = g_2$. Consider it to be such that the number of blow-ups to resolve f is minimal. Hence g_2 is an isomorphism: it if is not, we can factor it by a sequence of blow-ups, and let E be the first exceptional curve which g_2 contracts. If E is exceptional for g_1, then contracting it by g_1 we get a new surface X' resolving f, which is not possible as X is obtained by the minimal number of blow-ups resolving f, so that E is not exceptional for g_1.

Hence E is the proper transform of a component of a fiber of π_1. The image $\widetilde{E}$ of E in X_1 is either a smooth rational curve with self-intersection -2, a nodal or cuspidal rational curve with self-intersection 0, or an elliptic curve. As E is smooth and rational, $\widetilde{E}$ cannot be an elliptic curve. Moreover, $E^2 = -1$ as E is exceptional for g_2, so that $\widetilde{E}^2 \neq -2$. Hence $\widetilde{E}$ is a rational curve with a singular double point with self intersection 0. In the sequence of blow-ups to obtain X, this double point has to be blown-up. But in this case the self-intersection drops by 4, so $E^2 \leq -4$, which is not possible. Hence such an E cannot exists, so that g_2 is an isomorphism. Similarily one proves that g_1 is an isomorphism, so f is an isomorphism and we are done. $\square$

Now, let me describe an interesting property of elliptic surfaces. Let C be a smooth curve, $\pi : X \longrightarrow C$ an elliptic surface. The generic fiber of π is an elliptic curve, so that if can be written in its Weierstrass form: there are $A, B \in \mathbb{C}$ such that the equation of E in $\mathbb{P}^2$ is

$$zy^2 = x^3 + Axz^2 + Bz^3.$$

Moreover, the pair (A, B) is unique up to the action of $\mathbb{C}^*$ defined as $\lambda(A, B) = (\lambda^4 A, \lambda^6 B)$.

Definition 5.7.2. The *discriminant* of the Weierstrass equation of E is

$$\Delta := 4A^3 + 27B^2.$$

The *Weierstrass coefficients* are A and B.

It is an important property of the Weierstrass equation that E is singular if and only if $\Delta = 0$.

One can consider the inverse: let C be a smooth curve, $A, B \in k(C)$ be two non-trivial functions on C, and let $U \subseteq C$ be the open subset of C where A and B are defined. Then we can consider

$$X := \overline{\{((x : y : z), t) \in \mathbb{P}^2 \times U \,|\, zy^2 = x^3 + A(t)xz^2 + B(t)z^3\}},$$

which has a ntural projection $\pi : X \longrightarrow C$. The fiber $\pi^{-1}(t_0)$ on $t_0 \in C$ is the the curve

$$zy^2 = x^3 + A(t_0)xz^2 + B(t_0)z^3,$$

which is a smooth elliptic curve for the generic choice of t_0. Hence $\pi : X \longrightarrow C$is an elliptic surface. Notice that π has a natural section

$$s : C \longrightarrow X, \qquad s(t) := ((0 : 1 : 0), t) \in X,$$

and the singular fibers are only over those $t \in C$ such that $\Delta(t) = 0$. This can only be then of the form I_1 or II, as they are only nodal or cuspidal rational curves. Let me define the following:

Definition 5.7.3. Let $\pi : X \longrightarrow C$ be an elliptic surface. We say that it is a *Weierstrass fibration* if π has a section S and every fiber of π is either of type I_0, I_1 or II.

If π has a section S, then $S \cdot F = 1$ for every fiber F. Notice that every singular primitive fiber has at least one component of multiplicity 1, so that the section has to intersect it on that component, and only in it. Moreover, the existence of a section forces the elliptic surface to have no multiple fibers. Moreover, starting from every elliptic surface with a section, one can produce a Weierstrass fibration simply contracting every component of the fibers which do not intersect the section. Contracting curves one may get singularities. An important result is to show that every Weierstrass fibration has a Weierstrass equation. To do that, consider the following: let $\pi : X \longrightarrow C$ be a Weierstrass fibration with section S, and consider the exact sequence

$$0 \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_X(S) \longrightarrow N_{S/X} \longrightarrow 0.$$

As $N_{S/X}$ is supported on S and the fibers of π restricted to S are of dimension 0, we get that $\mathbb{R}^1\pi_* N_{S/X} = 0$. Moreover $(\pi_*\mathscr{O}_X(nS))_p \simeq H^0(X_p, \mathscr{O}_{X_p}(nS))$ and $(\mathbb{R}^1\pi_*\mathscr{O}_X(nS))_p \simeq H^1(X_p, \mathscr{O}_{X_p}(nS))$, where $X_p := \pi^{-1}(p)$. By Riemann-Roch on an elliptic curve, we see that $\pi_*\mathscr{O}_X \simeq \mathscr{O}_C$ and $\mathbb{R}^1\pi_*\mathscr{O}_X$ are line bundle; moreover, for every $n \geq 1$ the sheaf $\pi_*\mathscr{O}_X(nS)$ is locally free of rank n and $\mathbb{R}^1\pi_*\mathscr{O}_X(nS) = 0$. Hence we get that the natural map

$$\pi_* N_{S/X} \longrightarrow \mathbb{R}^1\pi_*\mathscr{O}_X$$

is an isomorphism.

Definition 5.7.4. The line bundle $\mathbb{L} \in Pic(C)$ defined as

$$\mathbb{L} := (\pi_* N_{S/X})^{-1} \simeq (\mathbb{R}^1\pi_*\mathscr{O}_X)^{-1}$$

is called the *fundamental line bundle* of the elliptic surface $\pi : X \longrightarrow C$.

Now let me define

Definition 5.7.5. A triple $(\mathscr{L}, A, B)$ gives *Weierstrass data* over C if $\mathscr{L} \in Pic(X)$, $A \in H^0(C, \mathscr{L}^4)$, $B \in H^0(C, \mathscr{L}^6)$ and the discriminant section $\Delta := 4A^3 + 27B^3 \in H^0(C, \mathscr{L}^{12})$ is not identically 0.

Theorem 5.7.4. *We have the following:*

1. *Let $\pi : X \longrightarrow C$ be a Weierstrass fibration. Then there are $A \in H^0(C, \mathbb{L}^4)$ and $B \in H^0(C, \mathbb{L}^6)$ such that $(\mathbb{L}, A, B)$ are Weierstrass data over C.*

2. *Let C be a smooth curve and $(\mathscr{L}, A, B)$ be Weierstrass data over C. Then they define a Weierstrass fibration $\pi : X \longrightarrow C$ such that $\mathscr{L} = \mathbb{L}$.*

3. *Any Weierstrass fibration can be seen as a section of a $\mathbb{P}^2-$bundle $\mathbb{P}$ over C, and $K_X = \pi^*(K_C \times \mathbb{L})$.*

Proof. We begin by proving the following:

Lemma 5.7.5. *Le $\pi : X \longrightarrow C$ be a Weierstrass fibration with a section S, and let $n \in \mathbb{N}$, $n \geq 2$. Then there is an exact sequence*

$$0 \longrightarrow \pi_*\mathscr{O}_X((n-1)S) \longrightarrow \pi_*\mathscr{O}_X(nS) \longrightarrow \mathbb{L}^{-n} \longrightarrow 0.$$

Moreover, we have a splitting

$$\pi_*\mathscr{O}_X(nS) = \mathscr{O}_C \oplus \mathbb{L}^{-2} \oplus ... \oplus \mathbb{L}^{-n}.$$

Proof. We have the natural exact sequence

$$0 \longrightarrow \mathscr{O}_X((n-1)S) \longrightarrow \mathscr{O}_X(nS) \longrightarrow \mathscr{O}_S(nS) \longrightarrow 0.$$

If $n \geq 2$ we have $\mathbb{R}^1\pi_*\mathscr{O}_X((n-1)S) = 0$, so that applying the functor π_* to this exact sequence we get the statement, as $\mathbb{L}^{-1} \simeq \pi_* N_{S/X}$, and $N_{S/X} \simeq \mathscr{O}_S(S)$.

The last part of the statement we need to prove that the sequence is split, and then apply induction. For the splitting of the sequence, this comes from properties of elliptic curves. □

Now, let us begin with the first item, so let $\pi : X \longrightarrow C$ be a Weierstrass fibration. Fix a sufficiently fine open cover $\{U_i\}_{i \in I}$ of X trivializing $\mathbb{L}$, and fix a basis section e_i of $\mathbb{L}_{|U_i}$ for every $i \in I$. Then e_i^{-n} is a basis for $\mathbb{L}^{-n}_{|U_i}$. Moreover, let α_{ij} be transition functions, i. e. $e_i = \alpha_{ij} e_j$.

Choose $f_i \in \pi_*\mathscr{O}_X(2S)(U_i)$ for every $i \in I$, such that f_i projects onto $e_i^{-2} \in \mathbb{L}^{-2}(U_i)$, and choose $g_i \in \pi_*\mathscr{O}_X(3S)(U_i)$ which projects onto $e_i^{-3} \in \mathbb{L}^{-3}(U_i)$. As the generic fiber of π is an elliptic curve, there are $a_1, ..., a_6 \in \mathscr{O}_C(U_i)$ such that

$$g_i^2 = a_6 f_i^3 + a_5 f_i g_i + a_4 f_i^2 + a_3 g_i + a_2 f_i + a_1,$$

which is an equality of section of $\pi_*\mathscr{O}_X(6S)(U_i)$.Taking the projections of the two members to $\mathbb{L}^{-6}(U_i)$ we get a relation

$$e_i^{-6} = a_6 e_i^{-6} + a_5 e_i^{-5} + a_4 e_i^{-4} + a_3 e_i^{-3} + a_2 e_i^{-2} + a_1,$$

so that we need $a_6 = 1$. To get a Weierstrass equation for the generic fiber we need to complete the square in g_i, getting a section $y_i \in \pi_*\mathscr{O}_X(3S)$, and the completing the cube in f_i, getting a section $x_i \in \pi_*\mathscr{O}_X(2S)$ such that

$$y_i^2 = x_i^3 + A_i x_i + B_i,$$

where $A_i, B_i \in \mathscr{O}_C(U_i)$. Moreover, we have $x_i = \alpha_{ij}^{-2} x_j$ and $y_i = \alpha_{ij}^{-3} y_j$, so that we finally get $A_i = \alpha_{ij}^4 A_j$ and $B_i = \alpha_{ij}^6 B_j$, so that $A_i e_i^4$ patch together to form a global section of $\mathbb{L}^4$, and $B_i e_i^6$ patch together to form a global section of $\mathbb{L}^6$. Notice that a point of $x \in X$ is singular if and only if $\Delta(\pi(x)) = 0$, and we get that $(\mathbb{L}, A, B)$ are Weiestrass data.

The second item is easier: given any $(\mathscr{L}, A, B)$ Weierstrass data over C, the Weierstrass fibration is simply built patching together local surfaces determined by local Weierstrass equations

$$y_i^2 = x_i^3 + A_i x_i + B_i,$$

where A_i and B_i are local representant of A and B over some open cover of C.

The third item goes as follows. Let $\pi : X \longrightarrow C$ be a Weierstrass fibration with section S. By Lemma 5.7.5 there is a splitting $\pi_*\mathscr{O}_X(3S) \simeq \mathscr{O}_C \oplus \mathbb{L}^{-2} \oplus \mathbb{L}^3$, which is a rank 3 vector bundle on C. Let $\mathbb{P} := \mathbb{P}(\pi_*\mathscr{O}_X(3S))$, which has a structure of $\mathbb{P}^{-2}$ vector bundle $p\mathbb{P} \longrightarrow C$ over C. Clearly, we have a map $f : X \longrightarrow \mathbb{P}$ such that $p \circ f = \pi$, induced by the fact that $\mathscr{O}_X(3S)$ is a line bundle on X. Moreover, the splitting gives generators of $\pi_*\mathscr{O}_X(3S)$ as 1, x and y satisfying a Weierstrass equation: as they generate the homogenous ring of a fiber of π, f is an embedding. An equation of X inside $\mathbb{P}$ is the Weierstrass equation with Weierstrass coefficients A and B.

Now, X is a divisor in $\mathbb{P}$, and its class in $\mathbb{P}$ is $(p^*\mathbb{L}^6)(3)$, as the individual terms of the equation are formally sections of $\mathbb{L}^6$, and the equation is of degree 3. Recall that the canonical class of $\mathbb{P}$ is $p^*(\mathscr{O}_C(K_C) \otimes \mathbb{L}^{-5})(-3)$. Hence

$$\mathscr{O}_X(K_X) = \mathscr{O}_X(K_{\mathbb{P}} + X) = p^*(\mathscr{O}_C(K_C) \otimes \mathbb{L}),$$

and we are done. □

As a consequence we have the following:

Lemma 5.7.6. *Let $\pi : X \longrightarrow C$ be a Weierstrass fibration, and let $\mathbb{L}$ be its fundamental line bundle. Then $deg(\mathbb{L}) \geq 0$. If $deg(\mathbb{L}) = 0$, then either $\mathbb{L}^4 = 0$ or $\mathbb{L}^6 = 0$, so that $\mathbb{L}$ is torsion of period 1,2,3,4 or 6 in $Pic(C)$. Moreover, X is a product of C with an elliptic curve if and only if $\mathbb{L} \simeq \mathscr{O}_C$. In particular if X is a product, then $H^0(C, \mathbb{L}^{-1}) = 1$, and if X is not a product then $H^0(C, \mathbb{L}^{-1}) = 0$.*

Proof. As we have a Weierstrass fibration, we have Weierstrass data $(\mathbb{L}, A, B)$. Hence Δ is a non vanishing global section of $\mathbb{L}^{12}$, so that $h^0(C, \mathbb{L}) > 0$. But this implies that $deg(\mathbb{L}) \geq 0$. If $deg(\mathbb{L}) = 0$, the at least one between A and B is non-zero, hence one between $\mathbb{L}^4$ and $\mathbb{L}^6$ is trivial.

If $L \simeq \mathscr{O}_C$, then $\mathbb{P} \simeq \mathbb{P}^2 \times C$, and A, B have to be constant. Hence X is a product. For the converse, let $X = C \times E$, where E is an elliptic curve. As X is a Weierstrass fibration, we have a section $s : C \longrightarrow C \times E$, i. e. a morphism $f : C \longrightarrow E$. Consider the automorphism

$$\sigma_f : X \longrightarrow X, \qquad \sigma_f(c, e) := (c, e - f(c)).$$

Then the section S is carried into the section (c, e_0), which has trivial normal bundle as it is a fiber of the projection to E. Hence $\mathbb{L} \simeq \mathscr{O}_C$.

The remaining part is clear if X is a product, otherwise it follows from the fact that $deg(\mathbb{L}) \leq 0$. □

We can finally pass to the classification of elliptic surfaces.

Lemma 5.7.7. *Let $\pi : X \longrightarrow C$ be a Weierstrass fibration. Then we have:*

1. *if X is a product, then $q(X) = g(C) + 1$;*

2. *if X is not a product, then $q(X) = g(C)$.*

Proof. We need to use the Leray spectral sequence: let $E_2^{p,q} := H^p(C, \mathbb{R}^q\pi_*\mathscr{O}_X)$. The Leray spectral sequence gives an exact sequence

$$0 \longrightarrow E_2^{1,0} \longrightarrow H^1(X, \mathscr{O}_X) \longrightarrow E_2^{0,1} \longrightarrow E_2^{2,0}.$$

Since $\pi_*\mathscr{O}_X \simeq \mathscr{O}_C$ we have $E_2^{2,0} = 0$ and $dim(E_2^{1,0}) = g(C)$. Hence the exact sequence gives

$$q(X) = g(C) + h^0(C, \mathbb{L}^{-1}),$$

so the result follows from Lemma 5.7.6. □

Lemma 5.7.8. *Let $\pi : X \longrightarrow C$ be a Weierstrass fibration. Then we have:*

1. *if X is a product, then $p_g(X) = g(C) + deg(\mathbb{L})$;*
2. *if X is not a product, then $p_g(X) = g(C) + deg(\mathbb{L}) - 1$.*

Proof. We have the following chain of equalities:

$$p_g(X) = h^2(X, \mathscr{O}_X) = h^0(X, \mathscr{O}_X(K_X)) = h^0(X, \pi^*\mathscr{O}_C(K_C) \otimes \mathbb{L}) =$$

$$= h^0(C, \mathscr{O}_C(K_C \otimes \mathbb{L})) = h^1(C, \mathbb{L}^{-1}) = h^0(C, \mathbb{L}^{-1}) - \chi(\mathbb{L}^{-1}) =$$

$$h^0(C, \mathbb{L}^{-1}) + deg(\mathbb{L}) - 1 + g(C),$$

and we are done by Lemma 5.7.6. □

As a consequence we have

$$\chi(\mathscr{O}_X) = deg(\mathbb{L}), \qquad e(X) = 12deg(\mathbb{L}).$$

Lemma 5.7.9. *Let $\pi : X \longrightarrow C$ be a Weierstrass fibration.*

1. *If $g(C) = 0$ and $deg(\mathbb{L}) \leq 1$, then $P_m(X) = 0$.*
2. *If $g(C) = 0$ and $deg(\mathbb{L}) > 1$, then $P_m(X) = 1 + m(deg(\mathbb{L}) - 2)$.*
3. *If $g(C) = 1$ and $deg(\mathbb{L}) \geq 1$, then $P_m(X) = mdeg(\mathbb{L})$.*
4. *If $g(C) = 1$ and $deg(\mathbb{L}) = 0$, then $P_m(X) = 1$ if the period of $\mathbb{L}$ divides m, and $P_m(X) = 0$ otherwise.*
5. *If $g(C) \geq 2$, then $P_m(X) = m(2g(C) - 2 + deg(\mathbb{L})) + 1 - g(C)$.*

Proof. We have the following chain of equalities

$$P_m(X) = h^0(X, \mathscr{O}_X(mK_X)) = h^0(C, \mathscr{O}_C(mK_C) \otimes \mathbb{L}^m) =$$

$$= \chi(\mathscr{O}_C(mK_C) \otimes \mathbb{L}^m) + h^1(C, \mathscr{O}_C(mK_C) \otimes \mathbb{L}^m) =$$

$$m(2g(C) - 2 + deg(\mathbb{L})) + 1 - g(C) + h^0(C, \mathscr{O}_C((1-m)K_C) \otimes \mathbb{L}^{-m}).$$

One can then conclude with the statement. □

Theorem 5.7.10. ***(Classification of elliptic surfaces)****. Let $\pi : X \longrightarrow C$ be an elliptic surface.*

1. *If $g(C) = 0$ and $deg(\mathbb{L}) = 0$, then $X \simeq E \times \mathbb{P}^1$, i. e. X is a ruled surface with $q = 1$.*
2. *If $g(C) = 0$ and $deg(\mathbb{L}) = 1$, then X is a rational surface.*
3. *If $g(C) = 0$ and $deg(\mathbb{L}) = 2$, then X is a K3 surface.*
4. *If $g(C) = 0$ and $deg(\mathbb{L}) > 2$, then X is a proper elliptic surface, i. e. $\kappa(X) = 1$.*
5. *If $g(C) = 1$ and $\mathbb{L} \simeq \mathscr{O}_C$, then X is an abelian surface.*
6. *If $g(C) = 1$ and $\mathbb{L}$ is torsion of period 2, 3, 4 or 6, then X is a bielliptic surface, and K_X has the same order of $\mathbb{L}$.*
7. *If $g(C) = 1$ and $deg(\mathbb{L}) \geq 1$, then X is a proper elliptic surface, i. e. $\kappa(X) = 1$.*
8. *If $g(C) \geq 2$, then X is a proper elliptic surface, i. e. $\kappa(X) = 1$.*

Proof. This is simply an application of the previous Lemmas. □

As a consequence of this, we end up with the following:

Theorem 5.7.11. *Let X be a smooth minimal projective surface. Then*

1. *$\kappa(X) = -\infty$ if and only if $P_{12} = 0$;*
2. *$\kappa(X) = 0$ if and only if $P_{12} = 1$;*
3. *$\kappa(X) = 1$ if and only if $P_{12} > 1$ and $K_X^2 = 0$;*
4. *$\kappa(X) = 2$ if and only if $P_{12} > 1$ and $K_X^2 > 0$.*

Proof. The proof is very easy: we have already proven that $\kappa(X) = -\infty$ if and only if $P_{12}(X) = 0$. If $\kappa(X) = 1$, then we know that X admits an elliptic fibration, so that by the previous Theorem we have that $P_{12}(X) > 1$. Moreover, we know that $K_X^2 = 0$. If $\kappa(X) = 0$, then we have already shown that $P_{12}(X) = 1$. If $P_{12}(X) = 1$, then $\kappa(X) \geq 0$, and $\kappa(X) \leq 1$ by Proposition 4.2.13, so that $\kappa(X) = 0, 1$. But if $\kappa(X) = 1$ we have $P_{12}(X) > 1$, hence $\kappa(X) = 0$. Finally if $\kappa(X) = 2$ we have $P_{12}(X) > 1$ and $K_X^2 > 0$ by Proposition 4.2.13. If $P_{12}(X) > 1$, then $\kappa(X) \geq 1$, but if $\kappa(X) = 1$ then $K_X^2 = 0$, so that $\kappa(X) = 2$. □

To conclude with this brief account on elliptic surfaces, I just want to mention that the Weierstrass equation of an elliptic surface is a very important tool for studying and classifying it. Recall that any Weierstrass fibration $\pi_X \longrightarrow C$ can be seen as a divisor in some $\mathbb{P}^{-2}-$bundle on a smooth curve C, whose equation is

$$Y^2Z = X^3 + AXZ^2 + BZ^3,$$

where A and B for Weierstrass data over C, together with $\mathbb{L}$. If one considers a sufficiently fine open cover $\{U_i\}_{i\in I}$ of C, one can see it as an equation $y = x^3 + Ax + B$, where A, B are functions on some open disc. A point $x \in X$ is singular if and only if the discriminant $\Delta(\pi(x)) = 0$, and the type of the singularity is completely determined by A, B and Δ. In order to resolve the singularity, we need to blow-up. If x is not singular for X, then we do not need to blow-up, and the fiber can either be a nodal rational curve (i. e. a fiber of type I_1) or a cuspidal rational curve (i. e. a fiber of type II). If x is singular, we need to blow-up, and the resulting fiber is completely determined by the singularity, hence by A, B and Δ. As instance, if the fiber has a nodal singularity which we can solve by blowing up only once, the fiber is of type I_2.

Let me now introduce some notations: let $c := \pi(x)$, and let a be the order of vanishing of A at c, b the order of vanishing of B at c, δ the order of vanishing of Δ at c. Moreover, let e be the topological Euler characteristic of $\pi^{-1}(c)$, r be the number of the components of $\pi^{-1}(c)$ not meeting the section and d be the number of the components of $\pi^{-1}(c)$ with multiplicity 1. The following is just an application of blow-ups, and the proof is then omitted:

Theorem 5.7.12. ***(Tate's Algorithm).*** *Let $\pi : X \longrightarrow C$, and consider the notations introduced above for some point $c \in C$. Then we have only the following possibilites:*

1. *The fiber is of type I_0 if and only if $\delta = 0$ and either $a \geq 0$ and $b = 0$, or $a = 0$ and $b \geq 0$. In this case $e = 0$, $r = 0$ and $d = 1$.*

2. *The fiber is of type I_n for $n > 0$ if and only if $a = b = 0$ and $\delta = n$. In this case $e = n$, $r = n - 1$ and $d = n$.*

3. *The fiber is of type I_0^* if and only if $\delta = 6$ and either $a = 2$ and $b \geq 3$, or $a \geq 3$ and $b = 3$. In this case $e = 6$, $r = 4$ and $d = 4$.*

4. *The fiber is of type I_n^* for $n > 0$ if and only if $a = 2$, $b = 3$ and $\delta = n+6$. In this case $e = n + 6$, $r = n + 4$ and $d = 4$.*

5. *The fiber is of type II if and only if $a \geq 1$, $b = 1$ and $\delta = 2$. In this case $e = 2$, $r = 0$ and $d = 1$.*

6. *The fiber is of type III if and only if $a = 1$, $b \geq 2$ and $\delta = 3$. In this case $e = 3$, $r = 1$ and $d = 2$.*

7. *The fiber is of type IV if and only if $a \geq 2$, $b = 2$ and $\delta = 4$. In this case $e = 4$, $r = 2$ and $d = 3$.*

8. *The fiber is of type IV^* if and only if $a \geq 3$, $b = 4$ and $\delta = 8$. In this case $e = 8$, $r = 6$ and $d = 3$.*

9. *The fiber is of type III^* if and only if $a = 3$, $b \geq 5$ and $\delta = 9$. In this case $e = 9$, $r = 7$ and $d = 2$.*

10. *The fiber is of type II^* if and only if $a \geq 4$, $b = 5$ and $\delta = 10$. In this case $e = 10$, $r = 8$ and $d = 1$.*

In particular, in every case we have $e = \delta$ and $0 \leq e - r \leq 2$, where $e = r$ if and only if the fiber is of type I_0, $e = r + 1$ if and only if the fiber is of type I_n for $n > 0$, and $e = r + 2$ in all other cases.

An important consequence is the following: let W be the set of points of C such that $\pi^{-1}(c)$ is singular. It is a finite set, possibly empty, and we have

$$e(X) = e(\pi^{-1}(C \setminus W)) + \sum_{c \in W} e(\pi^{-1}(c)) =$$

$$= e(F)e(C \setminus W) + \sum_{c \in W} e(\pi^{-1}(c)) = \sum_{c \in W} e(\pi^{-1}(c)),$$

where F is the fiber over a point of $C \setminus W$, which is an elliptic curve, and hence $e(F) = 0$. Hence $e(X)$ (and then $deg(\mathbb{L})$) is completely determined by the type of the fibers, i. e. by a, b and δ by the Tate's algorithm. Using now Theorem 5.7.10, we are able to classify an elliptic surface by knowing its singular fibers.

5.8 Surfaces of general type

The last remaining class of surfaces in the Enriques-Kodaira classification is the one of surfaces of general type. These form a very wild famil which is still to investigate, even if there are several interesting results and examples. The variety of examples renders quite impossible a further classification of surfaces of general type, but still we have different methods of investigation. A first one is to determine which possible invariants surfaces of general type can have, and in this directions we have several results; another way is to study pluricanonical maps and canonical models of surfaces of general type.

A first result is the following:

Proposition 5.8.1. *Let X be a minimal surface of general type. Then $K_X^2 > 0$ and $e(X) > 0$.*

Proof. Let H be a hyperplane section on X, and consider the exact sequence

$$0 \longrightarrow \mathscr{O}_X(nK_X - H) \longrightarrow \mathscr{O}_X(nK_X) \longrightarrow \mathscr{O}_H(nK_H) \longrightarrow 0.$$

As $\kappa(X) = 2$, tehere is $\alpha \in \mathbb{R}_{>0}$ such that $h^0(X, \mathscr{O}_X(nK_X)) > cn^2$ for $n \gg 0$, whereas $h^0(H, \mathscr{O}_H(nK_X))$ grows linearily with n at most. Hence there is $n_0 \gg 0$

such that $H^0(X, \mathscr{O}_X(n_0K_X - H)) \neq \emptyset$, so that there is an effective divisor D linearily equivalent to $n_0K_X - H$. Then $K_X \cdot D \geq 0$ as K_X is nef, so that

$$n_0K_X^2 = n_0K_X \cdot (D+H) = n_0K_X \cdot D + n_0K_X \cdot H \geq n_0K_X \cdot H^2 + H \cdot D \geq H^2 > 0.$$

The proof of the remaining part is omitted. □

Another important result is the following:

Theorem 5.8.2. ***(Noether's Inequality).*** *Let X be a minimal surface of general type. Then*

$$p_g(X) \leq \frac{K_X^2}{2} + 2.$$

Proof. As X is a minimal surface of general type, by Proposition 5.8.1 we have $K_X^2 \geq 1$, so that $p_g(X) \geq 3$. Let $|K_X| = |C| + V$, where V is the fixed part, and $|C|$ is the moving part. Then either $|C|$ is composed with a pencil of curves, either its general member is irreducible.

Let us suppose we have a pencil, i. e. there is a $1-$dimensional family of curves whose generic memeber F is irreducible, such that there is $n \in \mathbb{N}$ such that $K_X = nF + V$, and we have $p_g(X) = dim|C| + 1 \leq n + 1$. If $K_X \cdot F \leq 1$, then

$$1 \geq K_X \cdot F = (nF + V) \cdot F = nF^2 + F \cdot V,$$

so that $F^2 \leq 1$. As $F^2 \geq 0$ and $K_X \cdot F \equiv_2 F^2$, we have only $F^2 = K_X \cdot F = 1$ or $F^2 = K_X \cdot F = 0$. In the first case we get $n = 1$, so that $p_g \leq$, whereas we assumed $p_g(X) \geq 3$, so that we need $F^2 = K_X \cdot F = 0$. It is easy to see that this implies that F is a $(-2)-$curve, which is not possible, so that $|C|$ cannot have a pencil.

Hence $C^2 > 0$, and take an irreducible member C. The exact sequence

$$0 \longrightarrow \mathscr{O}_X(K_X) \longrightarrow \mathscr{O}_X(K_X + C) \longrightarrow \mathscr{O}_C(K_X + C) \longrightarrow 0$$

gives

$$h^0(C, \mathscr{O}_C(K_C)) = h^0(X, \mathscr{O}_X(K_X + C)) - p_g(X) + q(X) =$$
$$= 1 + \frac{1}{2}(C^2 + C \cdot K_X) \leq \frac{1}{2}K_X^2 + 1.$$

Then

$$h^0(C, \mathscr{O}_X(K_X + C)) = h^0(C, \mathscr{O}_X(2C + V)) \geq h^0(C, \mathscr{O}_X(2C)) \geq$$
$$\geq 2h^0(X, \mathscr{O}_X(C)) + 1 \geq 2h^0(X, \mathscr{O}_X(C)) - 3 =$$
$$= 2h^0(X, \mathscr{O}_X(C + V)) - 3 = 2p_g(X) - 3.$$

In conclusion we get the statement. □

As a corollary we have the following:

Corollary 5.8.3. *Let X be a minimal surface of general type.*

1. *If K_X^2 is even, then $5K_X^2 \geq e(X) - 36$, and if the equality holds we have $q(X) = 0$.*

2. *If K_X^2 is odd, then $5K_X^2 \geq e(X) - 30$, and if the equality holds we have $q(X) = 0$.*

Proof. Suppose K_X^2 to be even. Then

$$\chi(\mathscr{O}_X) \leq p_g(X) + 1 \leq \frac{1}{2}K_X^2 + 3.$$

By Noether's Formuls we then get the statement. Similarily for K_X^2 odd. □

Another important result, whose proof is omitted, is the following:

Theorem 5.8.4. ***(Bogomolov-Miyaoka)****. Let X be a minimal surface of general type. Then*

$$K_X^2 \leq 3e(X).$$

Let us now look at pluricanonical maps, and consider the following:

Definition 5.8.1. Let X be a surface, and $\mathscr{L} \in Pic(X)$. The rational morphism $\varphi_{\mathscr{L}}$ is a *C-isomorphism* if

1. the linear system $|\mathscr{L}|$ has no fixed components;

2. $\varphi_{\mathscr{L}}(X)$ is a normal surface with at most a finite number of rational double points as singularities.

3. the curves C_i contracted by $\varphi_{\mathscr{L}}$ are the $(-2)-$curves on X.

4. the morphism $\varphi_{\mathscr{L}}$ is an isomorphism outside the C_i's.

We have the following basic result:

Theorem 5.8.5. *Let X be a minimal surface of general type, and let $f_n := \varphi_{nK_X}$. Then f_n is a $C-$isomorphism if $n \geq 5$. In particular, the canonical ring $R(X) := \bigoplus_n H^0(X, \mathscr{O}_X(nK_X))$ is a finitely generated Noetherian ring.*

Notice then that every surface of general type has to be projective. The most important consequence of this fact is that one can define the canonical model of every minimal surface of general type X, which is

$$X_{can} := Proj(R(X)),$$

which is a normal projective surface having at most a finite number of singular points, all of which are rational double points.

Let me now describe some example of surfaces of general type:

Example 5.8.1. In $\mathbb{P}^{n+2}$ take $H_{d_1}, ..., H_{d_n}$ general hypersurfaces of whose degrees are $d_1, ..., d_n$. Let X be the complete intersection of the H_{d_i}'s. If $\sum_{i=1}^n d_i > n+3$, then X is a surface of general type.

Example 5.8.2. Let C_1 and C_2 be two smooth curves, $g(C_i) \geq 2$. Then $X = C_1 \times C_2$ is of general type. If G is a finite group of order $(g(C_1) - 1)(g(C_2) - 1)$ acting freely on C_1 and C_2, the quotient $Y := X/G$ is a surface of general type. This is called *Beauville surface.*

Example 5.8.3. If X is a surface of general type and $f : Y \longrightarrow X$ is surjective, where Y is a surface, then Y is of general type.

Example 5.8.4. Let $S \subseteq \mathbb{P}^3$ be the quintic of equation $X^5 + Y^5 + Z^5 + T^5 = 0$. Let ζ be a primitive root of 1, $\zeta \neq 1$, and consider

$$\sigma : S \longrightarrow S, \qquad \sigma(X : Y : Z : T) := (X : \zeta Y : \zeta^2 Z : \zeta^3 T),$$

which is an automorphism on S of order 5. The group G generated by σ acts freely on S, and the quotient $X := S/G$ is smooth. As $q(S) = 0$ and $K_S = H$, we have $p_g(S) = 4$ and $K_S^2 = 5$. Then $q(X) = 0$,

$$\chi(\mathscr{O}_X) = \frac{1}{5}\chi(\mathscr{O}_S) = 1,$$

so that $p_g(X) = 0$. Moreover we have

$$K_X^2 = \frac{1}{5}K_S^2 = 1,$$

hence X is a surface of general type with $p_g(X) = q(X) = 0$. This is called *Godeaux surface.*

Chapter 6

Generalizations of the Enriques-Castelnuovo Classification

This last chapter can be viewed more as an appendix than a true chapter. In the previous chapters we have proved the existence of a beautiful classification of smooth projective complex surfaces, which is called the Enriques-Castelnuovo Birational Classification. The main problem is if it is possible to generalize this classification theorem, i. e. to remove hypothesis on the objects we are classifying.

The first generalization is to consider projective complex surfaces, without any informations about singularities. Next, we want to classify complex surfaces, i. e. we get rid of the hypothesis of projectivity: this has been done by Kodaira, getting the so-called Enriques-Kodaira Birational Classification of surfaces. Another important step is to consider surfaces, i. e. not necessarily defined over the field $\mathbb{C}$: this has been done by Mumford and Kodaira, who provided a classification of surfaces defined over an arbitrary algebraicailly closed field k.

A final step is to consider a classification of higher dimensional analogues of surfaces, i. e. varieties or manifolds. This starts the nowadays called MMP, or Minimal Model Program, which is still open at several steps.

In this brief chapter I just want to give the basic results without any proof.

6.1 Singular projective surfaces

The first hypothesis we need to get rid of, is the smoothness of our surfaces. The result we need is the following, which was first proven by Hironaka in complete generality:

Theorem 6.1.1. ***(Hironaka)***. *Let X be a projective variety defined over an algebraically closed field of characteristic 0. Then there exists a smooth projective variety $\widetilde{X}$ together with a birational proper surjective morphism*

$$\pi : \widetilde{X} \longrightarrow X,$$

which is an isomorphism over $X \setminus X_{sing}$.

In the case of surface, this is quite explicit: if X is a projective surface, consider X_0 its normalization, and $\nu : X_0 \longrightarrow X$, the normalization map, which is birational. Now, as X_0 is normal, its singular locus is given by a finite number of points. Hence we can blow them up, getting a new surface $\pi_1 : X_1 \longrightarrow X_0$. If X_1 is smooth, then we are done. If X_1 is not smooth, we can blow it up in its singular points. We then get a sequence

$$... \xrightarrow{\pi_{n+1}} X_n \xrightarrow{\pi_n} X_{n-1} \xrightarrow{\pi_{n-1}} ... \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 \xrightarrow{\nu} X.$$

Zariski showed that this sequence must end, i. e. there is some n_0 in this sequence such that X_{n_0} is smooth. If for every $n < n_0$ the surface X_n is still singular, we put $\widetilde{X} := X_{n_0}$. In conclusion, every projective surface is birational to a smooth projective one, and as our classification is up to birationality, we are done.

Moreover, blowing up singular points one adds a finite number of smooth rational curves, none of which can be a $(-1)-$curve: the Castelnuovo Contraction Theorem shows that any $(-1)-$curve is contracted to a smooth point. If one supposes that $K_{\widetilde{X}} = 0$, then by the genus formula we get that every curve of this sort is a $(-2)-$curve.

6.2 The Enriques-Kodaira Classification

The next topic is to extend the classification of projective surfaces to all complex surfaces. We have already seen two examples: there are surfaces which are not Kähler, as the Hopf surface; and there surfaces which are Kähler, but not projective, as several K3 surfaces and $2-$dimensional complex tori. In the Enriques-Castelnuovo classification appear several other types of surfaces, which are all projective: ruled surfaces are easily seen to be all projective. Bielliptic surfaces are all projective by construction, and Enriques surfaces are all projective by Proposition 5.4.3. Finally, every surface of general type is projective.

Kodaira extended the Enriques-Castelnuovo classification to all complex surfaces, getting what is nowaday known as the Enriques-Kodaira Classification. One of the main difficulties is that for a generic complex surface there is no Hodge decomposition. We can still define the main objects we need, like line bundles, intersection product, Kodaira dimension, irregularity, plurigenera, birational morphisms (which is more convenient to call bimeromorphic). But it is no longer true, for example, that $b_1(X) = 2q(X)$: we have two main cases. Either $b_1(X)$ is even, or it is odd.

Definition 6.2.1. Let X be a complex surface. We define $q(X) := h^1(X, \mathscr{O}_X)$ and $h(X) := h^0(X, \Omega_X)$. We define $b^+(X)$ to be the number of positive eigenvalues of the intersection form on $H^2(X, \mathbb{R})$, and $b^-(X)$ to be the number of negative eigenvalues.

The first result we need is the following:

Proposition 6.2.1. *Let X be a compact complex surface. Then one of the two possibilities is verified:*

1. *we have that $b_1(X)$ is even, and in this case $q(X) = h(X) = \frac{1}{2}b_1(X)$ and $b^+(X) = 2p_g(X) + 1$;*
2. *we have that $b_1(X)$ is odd, and in this case $b_1(X) = 2q(X) - 1$, $h(X) = q(X) - 1$ and $b^+(X) = 2p_g(X)$.*

Proof. We just give a sketch for of the proof. One can prove that we always have

$$b^+(X) - b^-(X) = \frac{1}{3}(K_X^2 - 2e(X)).$$

Comparing with Noether's Formula, which is true for every complex surface, we have

$$(4 - 4q(X) + 4p_g(X)) + (b^+(X) - b^-(X)) = e(X) = 2 - 2b_1(X) + b_2(X).$$

As $b_2(X) = b^+(X) + b^-(X)$, we then get

$$(b^+(X) - 2p_g(X)) + (2q(X) - b_1(X)) = 1.$$

The two parenthesis of the first member are positive: indeed, for the first we have that $H^0(X, \mathscr{O}_X(K_X)) \oplus \overline{H^0(X, \mathscr{O}_X(K_X))}$, which has dimension $2p_g(X)$, injects by the de Rham cohomology, into $H^2(X, \mathbb{C})$, which has dimension $b_2(X)$. But one easily verifies that the restriction of the intersectionn form to the real part of this injection is positive, so that $b^+(X) \geq p_g(X)$. For the second part, we have an exact sequence of sheaves

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathscr{O}_X \longrightarrow Z\Omega_X \longrightarrow 0,$$

where $Z\Omega_X$ is the sheaf of closed holomorphic $1-$forms. As every global $1-$form is closed, we get an exact sequence

$$0 \longrightarrow H^0(X, \Omega_X) \longrightarrow H^1(X, \mathbb{C}) \longrightarrow H^1(X, \mathscr{O}_X),$$

so that $b_1(X) \leq q(X) + h(X)$. Again, $H^0(X, \Omega_X) \oplus \overline{H^0(X, \Omega_X)}$ injects into $H^1(X, \mathbb{C})$, so that $2h(X) \leq b_1(X)$. Hence $h(X) \leq q(X)$ and

$$2h(X) \leq b_1(X) \leq h(X) + q(X) \leq 2q(X),$$

which implies that $2q(X) - b_1(X) \geq 0$.

We have now only two possibilities: either $b^+(X) = 2p_g(X)$, so that $b_1(X) = 2q(X) - 1$, getting then $h(X) = q(X) - 1$; or $2q(X) = b_1(X)$, so that $h(X) = q(X)$ and $b^+(X) = 2p_g(X) + 1$. □

For the first case, we have this important result:

Theorem 6.2.2. *Let X be a complex surface. If $b_1(X)$ is even, then X is Kähler.*

Hence we can apply Hodge Theory, and almost every step in the classification goes through to this generalization. Hence we have a classification of Kähler surfaces, which is the same as the Enriques-Castelnuovo one.

The remaining case is then when $b_1(X)$ is odd. Let me start with the following definition:

Definition 6.2.2. A complex surface X is called *surface of Kodaira type VII* if $b_1(X) = 1$ and $\kappa(X) = -\infty$. A minimial surface of Kodaira type VII is sometimes called *surface of Kodaira type VII_0*.

Here is one of the main steps in the Enriques-Kodaira classification:

Theorem 6.2.3. ***(Kodaira)****. Let X be a minimal comple surface of Kodaira dimension $-\infty$. Then either X is projective, or it is a surface of Kodaira type VII_0.*

We have several examples of surfaces of Kodaira type VII. The first one is given by the Hopf surfaces. We have already seen an example of Hopf surface: consider the surface Y obtained as the quotient of $\mathbb{C}^2$ under the action of the homothety $z \mapsto 2z$. One can generalize this contruction taking the quotient not by $z \mapsto 2z$ but by the map $(z_1, z_2) \mapsto (\alpha_1 z_1, \alpha_2 z_2)$, where $0 < |\alpha_1| \leq |\alpha_2| < 1$. Such a surface is denoted H_{α_1,α_2}, and the original Hopf surface is $H_{1/2,1/2}$. The more general definition is the following:

Definition 6.2.3. A *Hopf surface* is a compact complex surface X whose universal cover is $\mathbb{C}^2 \setminus \{(0,0)\}$.

If X is a Hopf surface, then one can show that $b_1(X) = 1$, so that $q(X) = 1$; moreover $p_g(X) = 0$ and $b_2(X) = 0$. In general, a Hopf surface is not of the form H_{α_1,α_2}. Finally, every Hopf surface has Kodaira dimension $-\infty$. Moreover, one can prove that on every Hopf surface there is always a curve.

Another family of examples is the following. Let $M = [m_{ij}] \in SL(3,\mathbb{Z})$ be a matrix having one real positive eigenvalue $\alpha \in \mathbb{R}_{>0}$, and two complex conjugate eigenvalues $\beta, \overline{\beta}$ such that $\beta \neq \overline{\beta}$. A typical example is the matrix

$$M_n := \begin{bmatrix} 0 & 1 & 0 \\ n & 0 & 1 \\ 1 & 1-n & 0 \end{bmatrix},$$

where $n \in \mathbb{Z}$. Let now (a_1, a_2, a_3) be a real eigenvectors corresponding to α, and (b_1, b_2, b_3) be an eigenvector corresponding to β. As (a_1, a_2, a_3), (b_1, b_2, b_3) and $(\overline{b_1}, \overline{b_2}, \overline{b_3})$ are linearly independent over $\mathbb{C}$, the vectors $\binom{a_1}{b_1}$, $\binom{a_2}{b_2}$ and $\binom{a_3}{b_3}$ are linearly independent over $\mathbb{R}$. Now, let

$$\mathbb{H} := \{z \in \mathbb{C} \,|\, Im(z) > 0\},$$

and let

$$g_0 : \mathbb{H} \times \mathbb{C} \longrightarrow \mathbb{H} \times \mathbb{C}, \qquad g_0(z, w) := (\alpha z, \beta w),$$

and for every $i = 1, 2, 3$ let

$$g_i : \mathbb{H} \times \mathbb{C} \longrightarrow \mathbb{H} \times \mathbb{C}, \qquad g_0(z, w) := (z + a_i, w + b_i).$$

Let G_M be the group of automorphism of $\mathbb{H} \times \mathbb{C}$ generated by g_0, g_1, g_2, g_3. One can show that the action of G_M on $\mathbb{H} \times \mathbb{C}$ is free, proper and discontinuous, and the quotient $X_M := \mathbb{H} \times \mathbb{C}/G_M$ is a compact complex surface. We can even consider the group generated by the same g_i, where instead of β and b_i one uses $\overline{\beta}$ and $\overline{b_i}$.

Definition 6.2.4. The surface X_M is called *Inoue surface* associated to M.

If X is an Inoue surface, then one can show that $b_1(X) = 1$, so that $q(X) = 1$; moreover $p_g(X) = 0$ and $b_2(X) = 0$. Any Inoue surface has Kodaira dimension $-\infty$, and one can show that on an Inoue surface there are no curves.

Theorem 6.2.4. ***(Bogomolov).*** *Let X be a minimal surface of type VII_0. If $b_2(X) = 0$, then X is either a Hopf surface or an Inoue surface.*

For $b_2(X) > 0$, the situation is much less clear: we have several examples, like Inoue-Hirzebruch, Enoki surfaces and Kato surfaces. This last class of surfaces covers even Inoue-Hirzebruch surfaces and Enoki surfaces, and we have the following conjecture:

Conjecture 6.2.5. *Let X be a minimal surface of type VII_0. If $b_2(X) > 0$, then X is a Kato surface.*

The proof of this conjecture would conclude the classification of comples surfaces of Kodaira dimension $-\infty$. For non-negative Kodaira dimension, the situation is the following: we have already seen that if X is a complex surface such that $\kappa(X) = 2$, then X is projective, and if $\kappa(X) = 1$ then X carries an elliptic fibration. It then remains to study the case of Kodaira dimension 0.

Definition 6.2.5. A complex surface X is called *surface of primary Kodaira type* if $b_1(X) = 3$, and X amits an elliptic fibration over an elliptic curve. It is called *surface of secondary Kodaira type* if it admits a surface of primary Kodaira type as a proper unramified cover.

Every surface X of primary Kodaira type has trivial canonical bundle. The invariants are

$$q(X) = 2 \quad h(X) = 1, \quad p_g(X) = 1, \quad \chi(\mathscr{O}_X) = 0,$$

$$e(X) = 0, \quad b_1(X) = 3, \quad b_2(X) = 4.$$

Every surface X of secondary Kodaira type is the quotient of a primary Kodaira type surface by the action of a finite group, and it is possible to show that the possible orders of such a group are 2, 3, 4 or 6. In any case, the canonical bundle of X is not trivial, but has period 2, 3, 4 or 6. Moreover, X admits an elliptic fibration over $\mathbb{P}^1$. The invariants are

$$q(X) = 1 \quad h(X) = 0, \quad p_g(X) = 0, \quad \chi(\mathscr{O}_X) = 0,$$

$$e(X) = 0, \quad b_1(X) = 1, \quad b_2(X) = 0.$$

We have the following result:

Theorem 6.2.6. ***(Kodaira)***. *Let X be a minimal surface of Kodaira dimension 0. Then either X is Kähler, or it is of primary or secondary Kodaira type.*

Finally, the complete statement of the classification is:

Theorem 6.2.7. ***(Enriques-Kodaira Classification)***. *Let X be a minimal complex surface. Then X is one of the following:*

1. *Kodaira dimension $-\infty$:*
    - *rational surface (all projective);*
    - *non-rational geometrically ruled surface (all projective);*
    - *surface of Kodaira type VII_0 (all non-Kähler);*
2. *Kodaira dimension 0:*
    - *Enriques surface (all projective);*
    - *bielliptic surface (all projective);*
    - *K3 surface (all Kähler);*
    - $2-$*dimensional complex torus (all Kähler);*

- *surface of primary Kodaira type (all non-Kähler)*
- *surface of secondary Kodaira type (all non-Kähler)*

3. *Kodaira dimension 1: proper elliptic surface;*

4. *Kodaira dimension 2: surface of general type (all projective).*

6.3 Characteristic p

Zariski was the first to study surfaces defined over algebraically closed fields of any characteristic. Anyway, the first attempt in the classification were made by Mumford, at the end of the 1960s. The classification was then completed by Mumford and Bombieri in the late 1970s. To accomplish such a classification one needs all the machinery of scheme theory: let k be an algebraically closed field.

Definition 6.3.1. A $k-$*surface* is a a $2-$dimensional smooth projective scheme defined over the field k.

All the machinery we used in the Enriques-Castelnuovo classification can be extended to this more general setting, and all the definitions remains: hence we can talk about Kodaira dimension, ruled surfaces, elliptic fibrations, K3 surfaces and so on.

If X is a $k-$surface of Kodaira dimension $-\infty$, one can show that all the results we had in characteristic 0 are true even in characteristic $p > 0$. Hence the classification of $k-$surfaces of Kodaira dimension $-\infty$ is equal to the corresponding one over $\mathbb{C}$.

Anyway, things are different once we move to Kodaira dimension 0. Again, one has K3 surfaces, abelian surfaces, Enriques surfaces and bielliptic surfacea, but there are some differences.

For K3 surfaces, every property of K3 surfaces which is true over $\mathbb{C}$ is true even over any k algebraically closed. Moreover, Deligne shows that every K3 surface defined over an algebraically closed field k of characteristic p is obtained from a K3 surface defined over a field of characteristic 0 via reduction modulo p. Even for abelian surfaces the situation is the same as the one we have over $\mathbb{C}$.

For Enriques surfaces, the situation is more complicated: if $char(k) \neq 2$, these are all quotient of a K3 surface by an involution without fixed points, and they all carry an elliptic fibration. If $char(k) = 2$, this is no longer true: the canonical double cover is inseparable, so it is no longer a K3 surface. Moreover, they are not necessarily elliptic:

Definition 6.3.2. A $k-$surface X is *quasi-elliptic* if it admits a proper surjective morphism $\pi : X \longrightarrow C$, where C is a smooth $k-$curve, and the generic fiber of π is a cuspidal cubic.

One can show that every Enriques surface defined over an algebraically closed field k of characteristic 2 is either elliptic or quasi-elliptic.

For bielliptic surfaces, the situation is the same as over $\mathbb{C}$ if $char(k) \neq 2, 3$. If $char(k) = 3$, then we obtain all the possibilities of Theorem 5.5.5 but number 6 (i. e. $G = (\mathbb{Z}/3\mathbb{Z})^2$). If $char(k) = 2$, we obtain all the possibilites of Theorem 5.5.5, but number 4 (i. e. $G = (\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$). Moreover, for number 2 (i. e. $G = (\mathbb{Z}/2\mathbb{Z})^2$) we have to write $G = (\mathbb{Z}/2\mathbb{Z}) \oplus \mu_2$, where μ_2 is the group of square roots of 1 in k.

If $char(k) \neq 2, 3$, then this is all. Otherwise we have other types of surfaces:

Definition 6.3.3. A *quasi-bielliptic surface* is a surface $X = (E \times C)/G$, where E is an elliptic curve, C is a cuspidal cubic and G is a group of translations on E acting freely on C.

Any quasi-bielliptic surface has Kodaira dimension 0, and carries a quasi-elliptic fibration. Moreover, we can classify them like in Theorem 5.5.5, getting in particular that $K_X \neq 0$, but it is torsion of period 2, 3, 4 or 6. These surfaces appear only if $char(k) = 2, 3$.

Definition 6.3.4. A *non-classical Enriques surface* is a surface X such that $K_X = 0$ and $q(X) = 1$.

Non-classical Enriques surfaces appear only if $char(k) = 2$, and they are of two types: *singular non-classical Enriques surfaces*, for which the Frobenius map on $H^1(X, \mathscr{O}_X)$ is bijective; and *supersingular non-classical Enriques surfaces*, for which the Frobenius morphism on $H^1(X, \mathscr{O}_X)$ is zero. The singular non-classical Enriques surfaces are all quotient of a K3 surface by an involution without fixed points. The others have a canonical inseparable cover with covering group μ_2. Moreover, they are all elliptic or quasi-elliptic.

For surfaces of Kodaira dimension 1, it is possible to show that if $char(k) \neq 2, 3$, then they are all elliptic surface. If $char(k) = 2, 3$, they are either elliptic or quasi-elliptic.

Hence, we have the complete classification:

Theorem 6.3.1. ***(Mumford-Bombieri Classification)****. Let k be an algebraically closed field of any characteristic, and let X be a minimal $k-$surface. Then X is one of the following:*

1. *Kodaira dimension $-\infty$:*
    - *rational surface;*
    - *non-rational geometrically ruled surface;*

2. *Kodaira dimension 0:*
    - *Enriques surface;*
    - *non-classical Enriques surface (only if $char(k) = 2$);*
    - *bielliptic surface;*
    - *quasi-bielliptic surface (only if $char(k) = 2, 3$);*
    - *K3 surface;*
    - $2-$*dimensional torus;*
3. *Kodaira dimension 1:*
    - *proper elliptic surfaces;*
    - *proper quasi-elliptic surfaces (only if $char(k) = 2, 3$);*
4. *Kodaira dimension 2: surface of general type.*

6.4 Higher dimensional classification

The final step is to pass to a classification of complex projective varieties of any dimension. In this case one can still use line bundles, curves, Kodaira dimension an the machinery we introduced before, but there are some important differencies. First of all, we cannot define an interesection product on the Picard group: anyway, one can still define an intersection product between line bundles and curves. If X is our complex projective variety of dimension n, then we can define

$$H^2(X, \mathbb{Z}) \times H_2(X, \mathbb{Z}) \longrightarrow \mathbb{Z},$$

and we can define ample line bundles, nef line bundles and so on. The second, and more important difference, is that the Castelnuovo Contraction Criterion is no longer true: blow-up of points are not the only birational transformation we have to consider.

We have defined a minimal surface as a smooth projective surface having no $(-1)-$curves. If the Kodaira dimension is non-negative, this is equivalent as to ask that the canonical bundle is nef. Hence, a minimal projective variety is defined as a projective variety whose canonical bundle is nef. Notice that we didn't specify anything about smoothness: performing birational transformations like contraction of curves in this higher dimensional situation, we can produce singularities, hence we need to take care of singular projective varieties.

Anyway, if X is a projective variety such that K_X is not nef, then there is an effective curve C such that $K_X \cdot C < 0$. One of the main results of the MMP, which is shown using the Kawamata-Mori Rationality Theorem, is the following: consider the effective cone $Eff(X)$, which is defined as for surfaces.

For every line bundle $\mathscr{L} \in Pic(X)$ we can define $Eff(X)_{\mathscr{L}\geq 0}$ as the subcone of effective curves intersecting non-negatively $\mathscr{L}$. Then one can show that there are a finite number of curves $C_1, ..., C_m$ such that

$$Eff(X) = Eff(X)_{K_X \geq 0} + \sum_{i=1}^{m} \mathbb{R}_{\geq 0}[C_i].$$

This result is known as the Cone Theorem. Hence to produce a nef line bundle the idea is to contract these finitely many negative curves. Is exactly at this point that one can produce singularities even starting from smooth varieties. It is then one of the major efforts in the MMP to find the good type of singularities to allow in order to let this process run.

If one contracts an extremal curve C on X, there are three possible conditions that the obtained variety Y verifies:

Case 1: $dim(Y) < dim(X)$. In this case we get what is called a *Mori-Fano fiber space*. In the case of surfaces, this typically happens if $\kappa(X) = -\infty$, so that we have a $\mathbb{P}^1-$fibration over a smooth curve.

Case 2: $dim(Y) = dim(X)$, and there is a divisor of X which is contracted to a point. This is called a *divisorial contraction*. In this case one produces a nuew projective varieties with the same type of singularities of X, but whose Picard number is strictly smaller that the one of X. One can then continue with the MMP: if K_Y is nef, then Y is a minimal model for X, otherwise we start again.

Case 3: $dim(Y) = dim(X)$, and the exceptional locus has codimension at least 2. This is called a *small contraction*, and the situation is much more complicated. In this case, the canonical Weil divisor K_Y is no longer Cartier, nor even $\mathbb{Q}-$Cartier: indeed, we have $K_X \cdot C < 0$, where C is the contracted curve, and if K_Y is Cartier, then K_X is the pull-back of K_Y (as we have a small contraction, we have no exceptional divisors), so that $K_X \cdot C = 0$. The solution is the to replace Y by a birational projective variety Y^+ which has the same type of singularities as those of Y, but which has K_{Y^+} Cartier (or more generally $\mathbb{Q}-$Cartier). The existence of such a Y^+, which is called a *flip*, is granted by a deep theorem of Birkar-Cascini-Hacon-McKernan, Then, if K_{Y^+} is nef, we get a minimal model for X, otherwise we need to start again.

Notice that the first case is completely separated from the others. If one meets only divisorial contractions, as the Picard number always decreases, we need to stop at some point, so that we are able to produce a minimal model for X. If one meets small contractions, the Picard number does not change, and a priori we can continue without any stop. The main conjecture of the MMP is to show that we terminate the flip procedure, i. e. that we get a minimal model.

Notice that even in the case this conjecture would be proven true, we are not yet (an maybe we will never be) in the position to classify projective varieties of dimension n in a way similar to the Enriques-Castelnuovo Classification.