ON THE CONE CONJECTURE FOR ENRIQUES MANIFOLDS

GIANLUCA PACIENZA AND ALESSANDRA SARTI

À Claire, avec gratitude et admiration

Abstract. Enriques manifolds are non simply connected manifolds whose universal cover is irreducible holomorphic symplectic, and as such they are natural generalizations of Enriques surfaces. The goal of this note is to prove the Morrison-Kawamata cone conjecture for such manifolds when the degree of the cover is prime using the analogous result (established by Amerik-Verbitsky) for their universal cover. We also verify the conjecture for the known examples having non-prime degree.

1. Introduction

An Enriques manifold is a connected complex manifold X which is not simply connected and whose universal covering $\widetilde{X}$ is an irreducible holomorphic symplectic (IHS) manifold. Enriques manifolds were simultaneously and independently introduced in [BNWS11] and [OS11a] as a natural generalization of Enriques surfaces. The fundamental group of an Enriques manifold is a cyclic and finite group $G = \langle g \rangle$ and its order is called the *index* of X, which is also the order of the torsion canonical class $K_X \in \mathrm{Pic}(X)$. From the definition it follows that an Enriques manifold is even dimensional. Moreover, as $X = \widetilde{X}/G$, it is compact and, since $h^{2,0}(X) = h^{0,2}(X) = 0$, it turns out that an Enriques manifold is always projective (cf [BNWS11, Proposition 2.1, (4)] and [OS11a, Corollary 2.7]) and so is its universal cover. In those papers examples of Enriques manifolds of index 2, 3 and 4 are constructed, while their periods are studied in [OS11b].

The Morrison-Kawamata cone conjecture (see [Mor93, Ka97]) concerns the action of the automorphism group of manifolds (and more generally pairs) with numerically trivial canonical class on the cone of nef classes which are effective and predicts the existence of a rational, polyhedral fundamental domain for such action. More precisely, and more generally, we have the following.

Conjecture 1.1 (The Morrison-Kawamata Cone conjecture). *Let $X \to S$ be a K-trivial fiber space, that is a proper surjective morphism $f : X \to S$ with connected fibers between normal varieties such that X has $\mathbb{Q}$-factorial and terminal singularities and K_X is zero in $N^1(X/S)$.*

1. *There exists a rational polyhedral cone Π which is a fundamental domain for the action of $\mathrm{Aut}(X/S)$ on $\mathrm{Nef}^e(X/S) := \mathrm{Nef}(X/S) \cap \mathrm{Eff}(X/S)$ in the sense that*
   (a) $\mathrm{Nef}^e(X/S) = \cup_{g \in \mathrm{Aut}(X/S)} g^*\Pi$.
   (b) *$int(\Pi) \cap int(g^*\Pi) = \emptyset$, unless $g^* = id$ in $\mathrm{GL}(N^1(X/S))$.*
2. *There exists a rational polyhedral cone Π' which is a fundamental domain in the sense above for the action of $\mathrm{Bir}(X/S)$ on $\overline{\mathrm{Mov}}^e(X/S) := \overline{\mathrm{Mov}}(X/S) \cap \mathrm{Eff}(X/S)$.*

Item (2) above is also known as the birational Cone conjecture.

Date: March 18, 2023.

1991 *Mathematics Subject Classification.* 14C99, 14J28, 14J35, 14J40.

Key words and phrases. holomorphic symplectic varieties, ample cone, automorphisms, Enriques manifolds.

The conjecture has been proved in dimension 2 by Sterk-Looijenga, Namikawa, Kawamata, and Totaro (see [Ste85, Na85, Ka97, Tot10]), by Prendergast-Smith [PS12] for abelian varieties, by Amerik-Verbitsky [AV17, AV20] for IHS manifolds, building upon [Mar11], see also Markman-Yoshioka [MY15] and by Lehn-Mongardi-Pacienza [LMP22] for singular IHS varieties. For a recent extension to a not necessarily closed field of characteristic 0 see [Ta21, Theorem 1.0.5]. Notice that for IHS varieties $\widetilde{X}$ in the conjecture $\mathrm{Nef}^e(\widetilde{X})$ is replaced by the convex hull $\mathrm{Nef}^+(\widetilde{X})$ of $\mathrm{Nef}(\widetilde{X})\cap \mathrm{Pic}(\widetilde{X})_{\mathbb{Q}}$ inside $\mathrm{Pic}(\widetilde{X})_{\mathbb{R}}$ (same for $\mathrm{Mov}^e(X)$ which is replaced by the convex hull $\mathrm{Mov}^+(X)$ of $\mathrm{Mov}(X)\cap \mathrm{Pic}(\widetilde{X})_{\mathbb{Q}}$ inside $\mathrm{Pic}(\widetilde{X})_{\mathbb{R}}$). One can see that $\mathrm{Nef}^e(\widetilde{X}) \subset \mathrm{Nef}^+(\widetilde{X})$ (cf. [MY15, Remark 1.4]). This statement is analogous to the classical one for the 4 known deformations type (and equivalent in general modulo the SYZ-conjecture). For Calabi-Yau varietes the conjecture is open in general. For a recent result in this direction, see [GLW22] and the reference therein. For recent results in the relative case, establishing in particular the conjecture for families of $K3$ surfaces, see [LZ22]. We refer the reader to [Tot12, LOP18] for nice introductions to this topic. The conjecture is deeply related with birational geometry. Item (1) of the conjecture yields the finiteness, up to automorphisms, of birational contractions and fiber space structures of the initial variety, while item (2) implies, modulo standard conjectures of the MMP, the finiteness of minimal models, up to isomorphisms, of any $\mathbb{Q}$-factorial and terminal variety with non-negative Kodaira dimension (cf. [CL14, Theorem 2.4]).

From now on we will restrict ourselves to the absolute and smooth case. By the Beauville-Bogomolov decomposition theorem (see [Be83a]) we know that any K-trivial variety V admits a finite étale cover $\widetilde{V} \to V$, where $\widetilde{V}$ is a product of Calabi-Yau manifolds, IHS manifolds and an abelian variety. A general question is: suppose we know Conjecture 1.1 for $\widetilde{V}$, can we deduce for V? Our main result provides a positive answer when we have only one factor of IHS type.

Theorem 1.2. *The Morrison-Kawamata cone conjecture (respectively the birational Morrison-Kawamata cone conjecture) holds for the* $\mathrm{Aut}(X)$*-action on the* Nef^+*-cone (respectively for the* $\mathrm{Bir}(X)$*-action on* $\overline{\mathrm{Mov}}^+(X)$*) on any Enriques manifold* X *of prime index* p*.*

The index 4 Enriques manifolds are therefore left out by our result. Nevertheless, the proof of Theorem 1.2 and the explicit computation of the action of G on the cohomology of the IHS cover allow us to verify the conjecture in the index 4 known cases, see Section 4.1 for more details. Of course it would be nice to have a general argument valid for any index.

As for the proof, if $\pi : \widetilde{X} \to X = \widetilde{X}/G$ is the covering map, by Amerik-Verbitsky [AV17, AV20] there exists a rational polyhedral convex domain $\widetilde{D}$ which is a fundamental domain for the action of $\mathrm{Aut}(\widetilde{X})$ on $\mathrm{Nef}^+(\widetilde{X})$. We set

$$D := \widetilde{D} \cap \pi^* N^1(X).$$

Where recall that $N^1(X)$ denotes the Néron-Severi group of X which coincides with the Picard group of X. The proof of Theorem 1.2 then consists in showing that $\pi_*(D)$ is a fundamental domain for the action of $\mathrm{Aut}(X)$ on $\mathrm{Nef}^+(X)$: the rationality and polyhedrality of D are implied by those of $\widetilde{D}$. The main point is to show that if $\xi \in \pi^* N^1(X)$ and φ is an automorphism of $\widetilde{X}$ such that $\varphi^*(\xi) \in \widetilde{D}$, then φ "descends" to an automorphism of X, i.e. φ commutes with G. The idea to show this commutativity is to check it only on cohomology and then use Global Torelli to conclude. Some care has to be taken, as it is well known that there are non-trivial automorphisms acting trivially on cohomology for the 2 deformation classes coming from abelian surfaces (see Remark 2.2 for details). Let us finally point out that the commutativity is shown first on the general point in the moduli space of Enriques manifolds, for which we check (under the hypothesis that the index is prime) that $\mathrm{Pic}(\widetilde{X}) = H^2(\widetilde{X}, \mathbb{Z})^G$ and then, via a limiting process, we extend the result to special points.

Acknowledgments. We thank A. Chiodo for his kind invitation to Jussieu, where this project started and L. Kamenova, G. Mongardi and A. Oblomkov, the organizers of the workshop "Hyperkähler quotients, singularities and quivers" at the Simons Center for Geometry and Physics, where it was completed. We thank S. Boissière, C. Bonnafé, E. Floris, A. Garbagnati and E. Markman for very useful exchanges. G.P. is partially supported by the CNRS International Emerging Actions project "Birational and arithmetic aspects of orbifolds". A.S. is partially supported by the ANR project No. ANR-20-CE40-0026-01 (SMAGP).

2. Preliminaries

2.1. **Basic facts on IHS manifolds.** An irreducible holomorphic symplectic manifold is a compact Kähler manifold $\widetilde{X}$ which is simply connected and carries a holomorphic symplectic 2–form σ everywhere non–degenerate, such that $H^0(\widetilde{X},\Omega^2_{\widetilde{X}}) = \mathbb{C}\cdot\sigma$. For a general introduction of the subject, we refer to [Huy99].

Let $\widetilde{X}$ be an irreducible holomorphic symplectic manifold of dimension $2n \geq 2$. Let $\sigma \in H^0(\widetilde{X},\Omega^2_{\widetilde{X}})$ such that $\int_{\widetilde{X}} \sigma^n\bar{\sigma}^n = 1$. Then, following [Be83a], the second cohomology group $H^2(\widetilde{X},\mathbb{C})$ is endowed with a quadratic form $q = q_{\widetilde{X}}$ defined as follows

$$q(a) := \frac{n}{2}\int_{\widetilde{X}}(\sigma\bar{\sigma})^{n-1}a^2 + (1-n)\left(\int_{\widetilde{X}}\sigma^n\bar{\sigma}^{n-1}a\right)\cdot\left(\int_{\widetilde{X}}\sigma^{n-1}\bar{\sigma}^n a\right),\ a \in H^2(\widetilde{X},\mathbb{C}),$$

which is non-degenerate and, up to a positive multiple, is induced by an integral nondivisible quadratic form on $H^2(\widetilde{X},\mathbb{Z})$ of signature $(3, b_2(\widetilde{X})-3)$. The form q is called the Beauville-Bogomolov-Fujiki quadratic form of $\widetilde{X}$. By Fujiki [Fuj87], there exists a positive rational number $c = c_{\widetilde{X}}$ (the Fujiki constant of $\widetilde{X}$) such that

$$c\cdot q^n(\alpha) = \int_{\widetilde{X}}\alpha^{2n},\ \forall\alpha \in H^2(\widetilde{X},\mathbb{Z}).$$

We recall also the following Hodge theoretic version of Torelli theorem.

Theorem 2.1. [Mar11, Theorem 1.3] *Let Z and Y be two irreducible holomorphic symplectic manifolds deformation equivalent one to each other. Then:*

1. *Z and Y are bimeromorphic if and only if there exists a parallel transport operator $f : H^2(Z,\mathbb{Z}) \longrightarrow H^2(Y,\mathbb{Z})$ that is an isomorphism of Hodge structures;*
2. *if this is the case, there exists an isomorphism $\bar{f} : Y \longrightarrow Z$ inducing f if and only if f preserves a Kähler class.*

Remark 2.2. As usual $\widetilde{X}$ denotes an IHS manifold. Recall that $\nu : \mathrm{Aut}(\widetilde{X}) \longrightarrow \mathrm{Aut}(H^2(\widetilde{X},\mathbb{Z}))$ is finite, by [Huy99, Proposition 9.1]. Notice moreover that by a result of Hassett and Tschinkel, [HT13, Theorem 2.1] the kernel of the homomorphism ν is invariant under smooth deformations of the manifold $\widetilde{X}$. This allows to compute it for all 4 known deformation types of IHS manifolds, thanks to [Be83a, Proposition 10], [BNWS11, Corollary 3.3] and [MW17, Theorems 3.1 and 5.2]. We have that ν is injective for deformations of punctual Hilbert schemes of $K3$ surfaces and OG10 manifolds, while the kernel is generated by the group of translations by points of order n on an abelian surface and by $-\mathrm{id}$ (respectively equal to $(\mathbb{Z}/2\mathbb{Z})^{\times 8}$) if $\widetilde{X}$ is a deformation of a generalized Kummer of dimension $2n$ (respectively of an OG6 manifold).

2.2. **Basic facts on Enriques manifolds.** Consider now the quotient $\pi : \widetilde{X} \to X = \widetilde{X}/G$, where $|G| = d$. Recall that the automorphisms of X identify to those of $\widetilde{X}$ commuting with g, i.e.

$$\mathrm{Aut}(X) = \{\widetilde{\tau} \in \mathrm{Aut}(\widetilde{X}) : \widetilde{\tau}\circ g = g\circ\widetilde{\tau}\}/G. \tag{1}$$

Moreover the action of g on $\widetilde{X}$ cannot be symplectic, i.e. we cannot have $g^*\sigma_{\widetilde{X}} = \sigma_{\widetilde{X}}$, otherwise X would have $h^{2,0}(X) \neq 0$. Therefore there exists a d-th root of unity $\lambda \neq 1$ such that $g^*\sigma_{\widetilde{X}} = \lambda\sigma_{\widetilde{X}}$. The action must be moreover purely non–symplectic, i.e. λ is a primitive root of unity of the same order of g, otherwise, if G contains symplectic automorphisms, there are points with a non trivial stabilizer and the quotient would be singular. Observe that if G is of prime order then a non–symplectic action is the same as a purely non–symplectic one.

Lemma 2.3. *For any Enriques manifold X we have $H^2(X,\mathbb{Z}) = \operatorname{Pic}(X)$. Moreover we have $\pi^*H^2(X,\mathbb{Z}) = H^2(\widetilde{X},\mathbb{Z})^G \subset \operatorname{Pic}(\widetilde{X})$. In particular*

$$\dim_{\mathbb{R}} \pi^* N^1(X) = \dim_{\mathbb{R}} \pi^* \operatorname{Pic}(X) = \operatorname{rk} H^2(\widetilde{X},\mathbb{Z})^G.$$

Proof. Consider $\xi \in H^2(X,\mathbb{Z})$, we have

$$q(\pi^*\xi,\sigma) = q(g^*\pi^*\xi, g^*\sigma) = q(\pi^*\xi,\lambda\sigma) = \lambda q(\pi^*\xi,\sigma)$$

with $\lambda \neq 1$ as g is not a symplectic automorphism. Then $q(\pi^*\xi,\sigma) = 0$, which implies that $\pi^*\xi \in \operatorname{Pic}(\widetilde{X})$, as $\operatorname{Pic}(\widetilde{X}) = \sigma^{\perp_q} \cap H^2(\widetilde{X},\mathbb{Z})$ so that $\xi \in \operatorname{Pic}(X)$. We have shown that $H^2(X,\mathbb{Z}) \subset \operatorname{Pic}(X) \subset H^2(X,\mathbb{Z})$ so we get the first equality (notice that this equality also follows immediately from the fact that $H^{2,0}(X) = 0$, but we will need the argument below). By construction $\pi^*H^2(X,\mathbb{Z}) \subset H^2(\widetilde{X},\mathbb{Z})^G$ with finite index and moreover by [OS11a, proof of Proposition 2.8, and proof of Proposition 5.1] we get in fact equality. Take now a G–invariant class η in $H^2(\widetilde{X},\mathbb{Z})$. Since $\pi^*\operatorname{Pic}(X) \subset \operatorname{Pic}(\widetilde{X})$ the previous equality implies that $H^2(\widetilde{X},\mathbb{Z})^G \subset \operatorname{Pic}(\widetilde{X})$ (which is in fact a more general fact for non–symplectic automorphisms acting on IHS manifold, but this gives an easy proof). □

2.3. **Examples.** We recall the following examples from [BNWS11] and [OS11a] which to our knowledge are the only known examples so far.

2.3.1. *From $K3^{[n]}$ manifolds.* Consider a K3 surface S with an Enriques involution ι. On the Hilbert scheme $S^{[n]}$ consider the natural involution $\iota^{[n]}$. Notice that $\iota^{[n]}$ has no fixed points if n is odd and $S^{[n]}/\langle\iota^{[n]}\rangle$ is an Enriques manifold of dimension $2n$ and index 2. Moreover in this case we know the invariant sublattice for the action of $\iota^{[n]}$. In fact $H^2(S,\mathbb{Z})^\iota = U(2) \oplus E_8(2)$ where U is the hyperbolic plane with bilinear form multiplied by 2 and $E_8(2)$ is the negative definite lattice associated to the corresponding Dynkin diagram and bilinear form multiplied by 2. Since $\iota^{[n]}$ is a natural automorphism, we have

$$H^2(S^{[n]},\mathbb{Z})^{\iota^{[n]}} = H^2(S,\mathbb{Z})^\iota \oplus \langle -2(n-1)\rangle = U(2) \oplus E_8(2) \oplus \langle -2(n-1)\rangle.$$

Notice that the punctual Hilbert scheme $(S/\iota)^{[n]}$ of the Enriques surface *is not* an Enriques manifold, in the sense of our definition, as its universal cover is a Calabi-Yau manifold, see [OS11a, Theorem 3.1]

2.3.2. *From Kum_n manifolds.* These examples arise from a bielliptic surface, i.e. a surface S with torsion canonical class of order $d \in \{2,3,4\}$ admitting a finite étale covering by an abelian surface $A \to S$, and specifically, under certain conditions on d and n, from a free action of a finite group on $A^{[n+1]}$ which preserves $\mathrm{Kum}_n(A) \subset A^{[n+1]}$. We refer the reader to [OS11a, Section 6] and [BNWS11, Section 4.2] for further details. We recall in detail the example for $d = 4$ in Section 4.1.

2.4. **Moduli spaces of marked Enriques manifolds.** We follow the presentation in [BCS16]. See also [OS11b] for another equivalent approach and for the study of the period map.

Let $\widetilde{X}$ be an IHS manifold. Recall that a marking for $\widetilde{X}$ is an isometry to a given lattice $H^2(\widetilde{X},\mathbb{Z}) \to L$.

Let M be an even non-degenerate lattice of rank $\rho \geq 1$ and signature $(1, \rho-1)$. An *M-polarized* IHS manifold is a pair $(\widetilde{X}, j)$ where $\widetilde{X}$ is a projective IHS manifold and j is a primitive embedding of lattices $j : M \hookrightarrow \mathrm{NS}(\widetilde{X})$. Two M-polarized IHS manifolds $(\widetilde{X}_1, j_1)$ and $(\widetilde{X}_2, j_2)$ are called *equivalent* if there exists an isomorphism $f : \widetilde{X}_1 \to \widetilde{X}_2$ such that $j_1 = f^* \circ j_2$. As in [DK07, Section 10] and [D96] one can construct a moduli space of *marked* M-polarized IHS manifolds as follows. We fix a primitive embedding of M in L and we identify M with its image in L. A *marking* of $(\widetilde{X}, j)$ is an isomorphism of lattices $\eta : L \to H^2(\widetilde{X},\mathbb{Z})$ such that $\eta_{|M} = j$. As observed in [D96, p.11], if the embedding of M in L is unique up to an isometry of L then every M-polarization admits a compatible marking. Two M-polarized marked IHS manifolds $(\widetilde{X}_1, j_1, \eta_1)$ and $(\widetilde{X}_2, j_2, \eta_2)$ are called *equivalent* if there exists an isomorphism $f : \widetilde{X}_1 \to \widetilde{X}_2$ such that $\eta_1 = f^* \circ \eta_2$ (this clearly implies that $j_1 = f^* \circ j_2$). Let $T := M^\perp \cap L$ be the orthogonal complement of M in L and set

$$\Omega_M := \{x \in \mathbb{P}(T \otimes \mathbb{C}) \,|\, q(x) = 0,\, q(x + \bar{x}) > 0\}.$$

Since T has signature $(2, \mathrm{rk}(T) - \rho)$ the period domain Ω_M is a disjoint union of two connected components of dimension $\mathrm{rk}(T) - \rho$. For each M-polarized marked IHS manifolds (X, j, η), since $\eta(M) \subset \mathrm{NS}(\widetilde{X})$, we have $\eta^{-1}(H^{2,0}(\widetilde{X})) \in \Omega_M$. On the other hand, by the surjectivity of the period map [Huy99, Theorem 8.1] restricted to any connected component $\mathfrak{M}_L^0$ of $\mathfrak{M}_L$ (the moduli space of IHS manifolds with second integral cohomology isometric to a given lattice L) we can associate to each point $\omega \in \Omega_M$ an M-polarized IHS manifold as follows: there exists a marked pair $(\widetilde{X}, \eta) \in \mathfrak{M}_L^0$ such that $\eta^{-1}(H^{2,0}(\widetilde{X})) = \omega \in \mathbb{P}(T \otimes \mathbb{C})$ so $M = T^\perp \subset \omega^\perp \cap L$, hence $\eta(M) \subset H^{2,0}(\widetilde{X})^\perp \cap H^{1,1}_{\mathbb{Z}}(\widetilde{X}) = \mathrm{NS}(\widetilde{X})$ and we take $(\widetilde{X}, \eta_{|M}, \eta)$. Recall that for a generic point in the moduli space we have $\mathrm{NS}(\widetilde{X}) = \eta(M)$.

By the Local Torelli Theorem for IHS manifolds, an M-polarized IHS manifold $(\widetilde{X}, j)$ has a local deformation space $\mathrm{Def}_M(\widetilde{X})$ that is contractible, smooth of dimension $\mathrm{rk}(T) - \rho$, such that the period map $P\colon \mathrm{Def}_M(\widetilde{X}) \to \Omega_M$ is a local isomorphism (see [D96]). By gluing all these local deformation spaces one obtains a moduli space K_M of marked M-polarized IHS manifolds that is a non-separated analytic space, with a period map $P\colon K_M \to \Omega_M$.

To construct a period domain for Enriques manifolds we have to take the non–symplectic action into account.

Let $(\widetilde{X}, j)$ be an M-polarized IHS manifold and $G = \langle g \rangle$ a cyclic group of prime order $p \geq 2$ acting non-symplectically on $\widetilde{X}$. Assume that the action of G on $j(M)$ is the identity and that there exists a group homomorphism $\rho : G \longrightarrow O(L)$ such that

$$M = L^\rho := \{x \in L \,|\, \rho(g)(x) = x, \forall g \in G\}.$$

We define a *(ρ, M)-polarization* of $(\widetilde{X}, j)$ as a marking $\eta\colon L \to H^2(\widetilde{X},\mathbb{Z})$ such that $\eta_{|M} = j$ and $g^* = \eta \circ \rho(g) \circ \eta^{-1}$.

Two (ρ, M)-polarized IHS manifolds $(\widetilde{X}_1, j_1)$ and $(\widetilde{X}_2, j_2)$ are isomorphic if there are markings, $\eta_1\colon L \to H^2(\widetilde{X}_1,\mathbb{Z})$ and $\eta_2\colon L \longrightarrow H^2(\widetilde{X}_2,\mathbb{Z})$ such that $\eta_{i|M} = j_i$, and an isomorphism $f\colon \widetilde{X}_1 \to \widetilde{X}_2$ such that $\eta_1 = f^* \circ \eta_2$.

Recall that by construction $\mathbb{C}\sigma_{\widetilde{X}}$ is identified to the line in $L \otimes \mathbb{C}$ defined by $\eta^{-1}(H^{2,0}(\widetilde{X}))$. Let $\lambda \in \mathbb{C}^*$ such that $\rho(g)(\sigma_{\widetilde{X}}) = \lambda\sigma_{\widetilde{X}}$. Observe that $\lambda \neq 1$ since the action is non-symplectic and it is a primitive p-th root of unity since p is prime. By construction $\sigma_{\widetilde{X}}$ belongs to the eigenspace of $T \otimes \mathbb{C}$ relative to the eigenvalue λ, where $T = M^\perp \cap L$. We denote it by $T(\lambda)$ (if $p = 2$, we have $\lambda = -1$ and we denote $T(\lambda) = T_{\mathbb{R}}(\lambda) \otimes \mathbb{C}$, where $T_{\mathbb{R}}(\lambda)$ is the real eigenspace relative to $\lambda = -1$).

Assume that $\lambda \neq -1$, then the period belongs to the space

$$\Omega_M^{\rho,\lambda} := \{x \in \mathbb{P}(T(\lambda)) \,|\, q(x + \bar{x}) > 0\}$$

of dimension $\dim T(\lambda) - 1$ which is a complex ball if $\dim T(\lambda) \geq 2$. By using the fact that $\lambda \neq -1$ it is easy to check that every point $x \in \Omega_M^{\rho,\lambda}$ satisfies automatically the condition $q(x) = 0$.

If $\lambda = -1$ we set $\Omega_M^{\rho,\lambda} := \{x \in \mathbb{P}(T(\lambda)) \,|\, q(x) = 0,\, q(x + \bar{x}) > 0\}$. It has dimension $\dim T(\lambda) - 2$. Clearly in both cases $\Omega_M^{\rho,\lambda} \subset \Omega_M$.

Let X be a *marked* Enriques manifold of index d, that is the data of a (ρ, M)-*polarization* of $(\widetilde{X}, j)$ with a marking $\eta\colon L \to H^2(X, \mathbb{Z})$ such that $\eta_{|M} = j$ and $g^* = \eta \circ \rho(g) \circ \eta^{-1}$, so that $X = \widetilde{X}/G$. Recall that by the Bogomolov-Tian-Todorov theorem the Kuranishi space $\mathrm{Def}(X)$ of an Enriques manifold X of index d is smooth. Moreover Oguiso and Schröer verified in [OS11b, Proposition 1.2] that, after possibly shrinking $\mathrm{Def}(X)$, every point parametrizes an index d Enriques manifold.

If $\mathscr{X} \to B$ is a flat family of marked Enriques manifolds, then, as remarked in [OS11b, Section 2] the universal covering $\widetilde{\mathscr{X}} \to \mathscr{X}$ of the family is also the fiberwise universal covering. The period map of the family $\mathscr{X} \to B$ is then defined as

$$\mathfrak{p}_B : B \to \Omega_M^{\rho,\lambda},\ b \mapsto \eta^{-1}(H^{2,0}(\widetilde{X}_b)). \tag{2}$$

By [OS11b, Theorem 2.4] the local Torelli theorem holds, namely the period map $\mathfrak{p}_B$ is a local isomorphism. Hence, as for IHS manifolds, we can patch together the Kuranishi spaces via the Oguiso-Schröer local Torelli theorem, to construct a (non-separated) moduli space of (marked) Enriques manifolds in terms of (ρ, M)-polarized IHS manifolds.

Let us now briefly discuss the case when the order of G is not necessarily a prime, but the action of G is purely non–symplectic (recall that it means that g acts on the holomorphic two-form by multiplication by a primitive root of unity of the same order as g). So let $|G| = d$, where d is not necessarily a prime number and fix the action as $g^*\sigma_{\widetilde{X}} = \lambda\sigma_{\widetilde{X}}$ where λ is a primitive d–root of unity. Then take $T(\lambda)$ the eigenspace in $T \otimes \mathbb{C}$ relative to the eigenvalue λ , where we use here the same notations as before, i.e. M is the invariant sublattice in L for the action of $\rho(G)$ and T its orthogonal complement. As before the period $\sigma_{\widetilde{X}}$ of $\widetilde{X}$ belongs to $\Omega_M^{\rho,\lambda}$, but here the Néron–Severi group of a generic $\widetilde{X}$ in the moduli space just contains M and it is not equal (as it is the case for the prime order, see Lemma 3.2). This means concretely that all eigenspaces corresponding to non–primitive d–roots of unity for the actions of G on $H^2(\widetilde{X}, \mathbb{Z}) \otimes \mathbb{C}$ are contained in $\mathrm{Pic}(\widetilde{X}) \otimes \mathbb{C}$ and the eigenspaces for the primitive roots are contained in $T_{\widetilde{X}} \otimes \mathbb{C}$.

3. Proof of the cone conjecture

First we prove the following.

Lemma 3.1. $H^2(\widetilde{X}, \mathbb{Q}) = \pi^*[H^2(X, \mathbb{Q})] \oplus [\pi^*(H^2(X, \mathbb{Q}))]^{\perp_q}$

Proof. Let us take a non-zero class $\beta \in [\pi^*(H^2(X,\mathbb{Q}))]^{\perp_q}$. Suppose by contradiction that

$$\beta \in \pi^* H^2(X,\mathbb{Q}). \tag{3}$$

Then $q(\beta,\beta) = 0$. Let $A \in \mathrm{Pic}(\widetilde{X})$ an ample line bundle. Without loss of generality we may assume that $A = \pi^* B$ with B ample (so that A is G-invariant). By the choice of β and the G-invariance of A we have that $q(\beta, A) = 0$ which, by (3), contradicts the Hodge-index theorem for q. □

We start making the proof in the case that $|G|$ is cyclic of prime order and we take X generic in the moduli space, so that $\mathrm{Pic}(\widetilde{X}) = H^2(\widetilde{X},\mathbb{Z})^G$ as shown in Lemma 3.2.

3.1. **The proof when the order is prime at the general point.**

Lemma 3.2. *Under the hypothesis $|G| = p$ is a prime number for a general Enriques manifold $\widetilde{X} \to X$ of index p we have that* $\mathrm{Pic}(\widetilde{X}) = H^2(\widetilde{X},\mathbb{Z})^G$.

Proof. We use the notation introduced in Section 2.4. Set $M = H^2(\widetilde{X},\mathbb{Z})^G$ and $T = M^\perp \cap L$. We first claim that, if $t \in T \setminus \{0\}$, then $t^\perp$ can not contain $\Omega_M^{\rho,\lambda}$. To see this, first notice that $t^\perp \cap L$ is endowed with a weight 2 Hodge structure. If $t^\perp \supset \Omega_M^{\rho,\lambda}$ then it must contain the full projectivized eigenspace $\mathbb{P}T(\lambda)$. Therefore $t^\perp$ must contain the smallest weight 2 Hodge sub-structure of $H^2(\widetilde{X},\mathbb{Z})$ containing $T(\lambda)$, which is T since the natural representation $G \to \mathrm{GL}(T \otimes \mathbb{Q})$ is irreducible (since the eigenvectors of the representation of G in $\mathbb{C}^*$ are primitive roots of unity by the hypothesis $d = p$). Then $t \in T^\perp$, which is absurd, and the claim is proved.

Then for each $t \in T \setminus \{0\}$ consider the restriction H_t of the hyperplane $t^\perp$ to $\Omega_M^{\rho,\lambda}$ and let $\mathscr{H} := \bigcup_{t \in T\setminus\{0\}} H_t$. Each subset $\Omega_M^{\rho,\lambda} \setminus H_t$ is open and dense in $\Omega_M^{\rho,\lambda}$ hence by Baire's Theorem the subset $\Omega^0 := \Omega_M^{\rho,\lambda} \setminus \mathscr{H}$ is dense in $\Omega_M^{\rho,\lambda}$ since $\mathscr{H}$ is a countable union of complex closed subspaces. If $x = \mathfrak{p}(X,\eta) \in \Omega^0$ then $\mathrm{Pic}(\widetilde{X}) = \eta(L^{1,1}(x)) = \eta(M)$, where $L^{1,1}(x) = \{\ell \in L : x \cdot \ell = 0\}$. Hence we conclude by local Torelli [OS11b, Theorem 2.4]. □

Consider $\xi \in \mathrm{Nef}^+(X)$. Since the Cone conjecture holds on $\widetilde{X}$ we have the existence of $\widetilde{\tau} \in \mathrm{Aut}(\widetilde{X})$ and $\widetilde{\delta} \in \widetilde{D}$ such that

$$(\widetilde{\tau})^* \pi^* \xi = \widetilde{\delta}. \tag{4}$$

We will actually not need condition (4) and prove more generally that, when X is general $\widetilde{\tau}^*$ commutes with g^*, for any $\widetilde{\tau} \in \mathrm{Aut}(\widetilde{X})$.

Lemma 3.3. *Any automorphism $\widetilde{\tau} \in \mathrm{Aut}(\widetilde{X})$ commutes with g on* $H^2(\widetilde{X},\mathbb{Z})^G = \pi^* H^2(X,\mathbb{Z})$.

Proof. Consider $\eta \in H^2(\widetilde{X},\mathbb{Z})^G$ and set $e := (\widetilde{\tau})^*(\eta)$. First notice that $e \in H^2(\widetilde{X},\mathbb{Z})^G$. Indeed by Lemma 2.3 the class η lies in $\mathrm{Pic}(\widetilde{X})$ and automorphisms preserve the Picard group, hence $e = (\widetilde{\tau})^*(\eta) \in \mathrm{Pic}(\widetilde{X}) = H^2(\widetilde{X},\mathbb{Z})^G$. Therefore $g^*(e) = e$ from which we deduce

$$g^*((\widetilde{\tau})^*(\eta)) := g^*(e) = e = (\widetilde{\tau})^*(\eta) = (\widetilde{\tau})^*(g^*(\eta))$$

where the last equality holds since $\eta \in H^2(\widetilde{X},\mathbb{Z})^G$. □

We then check the commutativity also on $T(X) := (\pi^* H^2(X,\mathbb{Z}))^{\perp_q} \cap H^2(\widetilde{X},\mathbb{Z})$.

Lemma 3.4. *Any automorphism $\widetilde{\tau} \in \mathrm{Aut}(\widetilde{X})$ commutes with g on $T(X)$.*

Proof. Notice that $T(X)$ is the transcendental part of a weight two Hodge structure of $K3$ type. Hence by [Huy16, Corollary 3.4, Chapter 3] the group of all Hodge isometries of $T(X)$ is a finite cyclic group. The automorphisms preserve $\mathrm{Pic}(\widetilde{X})$, which, by Lemma 3.2, equals $H^2(\widetilde{X},\mathbb{Z})^G$. Therefore any element of $\mathrm{Aut}(\widetilde{X})$ preserves $T(X)$ and induces a Hodge isometry of $T(X)$. Then the restrictions to $T(X)$ of $(\widetilde{\tau})^*$ and g^* are elements of a finite cyclic group and as such commute. $\square$

By Lemma 3.1, Lemma 3.3 and Lemma 3.4 the automorphism $\widetilde{\tau}$ commutes with g on $H^2(\widetilde{X},\mathbb{Z})$. By Torelli Theorem 2.1 if the map $\mathrm{Aut}(\widetilde{X}) \longrightarrow O(H^2(\widetilde{X},\mathbb{Z}))$ is injective, then they commute also on $\widetilde{X}$ which by (1) implies that $\widetilde{\tau}$ descends to an automorphism τ on X and therefore $\widetilde{\delta} = (\pi)^*\delta \in \widetilde{D} \cap (\pi)^* N^1(X)$ for some class $\delta \in \mathrm{Nef}^+(X)$.

If the map $\mathrm{Aut}(\widetilde{X}) \longrightarrow O(H^2(\widetilde{X},\mathbb{Z}))$ is not injective, then by [Huy99, Proposition 9.1, item (v)] the kernel K is finite. Consider

$$\Gamma := \{\varphi \in \mathrm{Aut}(\widetilde{X}) \,|\, \varphi g^{-1}\varphi^{-1} g \in K\}$$

and observe that Γ is in fact equal to $\mathrm{Aut}(\widetilde{X})$ as we have shown that each automorphism induces an isometry that commutes on $H^2(\widetilde{X},\mathbb{Z})$ with g. Consider

$$\Gamma_0 := \{\varphi \in \mathrm{Aut}(\widetilde{X}) \,|\, \varphi g^{-1}\varphi^{-1} g = id\}$$

and notice that Γ_0 is a subgroup of $\mathrm{Aut}(\widetilde{X})$ which is not necessarily normal. Let us show that it has finite index in $\mathrm{Aut}(\widetilde{X})$. Consider the map

$$\beta : \mathrm{Aut}(\widetilde{X}) \longrightarrow K, \ \varphi \mapsto \varphi g^{-1}\varphi^{-1} g.$$

Then one easily checks that $\beta(\varphi_1) = \beta(\varphi_2)$ if and only if $\varphi_2^{-1}\varphi_1 \in \Gamma_0$. Therefore if we take the set $\mathrm{Aut}(\widetilde{X})/\Gamma_0$ of left cosets we have an injective map $\mathrm{Aut}(\widetilde{X})/\Gamma_0 \longrightarrow K$ showing that Γ_0 has finite index in $\mathrm{Aut}(\widetilde{X})$. Let now $\bar{\gamma}_1, \dots, \bar{\gamma}_r$ be all the classes in $\mathrm{Aut}(\widetilde{X})/\Gamma_0$ and φ be the automorphism such that $\varphi^*(\pi^*\xi) \in \widetilde{D}$. By considering the class $\bar{\varphi} \in \mathrm{Aut}(\widetilde{X})/\Gamma_0$ then $\bar{\varphi} = \bar{\gamma}_j$ for a certain $j = 1, \dots, r$ so that $\gamma_j^{-1}\varphi \in \Gamma_0$. We then modify the fundamental domain $\widetilde{D}$ by taking

$$\widetilde{D}' := \widetilde{D} \cup \bigcup_{i=1}^{r} (\gamma_i^{-1})^*(\widetilde{D}).$$

Observe that $\widetilde{D}'$ is obviously still rational and polyhedral and $(\gamma_j^{-1})^*\varphi^*(\pi^*\xi) \in \widetilde{D}'$. Moreover $\gamma_j^{-1}\varphi$ descends to an automorphism on X and we conclude as before. The theorem is then proved in the prime order case under the genericity assumption.

3.2. **The cone conjecture for prime order for all points in moduli space.** In the previous section we have seen that Theorem 1.2 holds at the general point of the moduli space. Via a limiting procedure we show here how to deduce the same conclusion everywhere.

Theorem 3.5. *Let X_0 be a marked Enriques manifold and let $\mathfrak{M}_0 \subset \mathfrak{M}_L$ be an irreducible component of the moduli space containing $[X_0]$. Assume that the Kawamata-Morrison cone conjecture holds on manifolds X corresponding to the general point of $\mathfrak{M}_0$. Then the Kawamata-Morrison cone conjecture holds for X_0.*

Proof. Let $\widetilde{X}_0 \to X_0$ be the IHS universal cover. Consider a smooth integral curve $B \subset \mathfrak{M}_0$ with $0 = [X_0] \in B$ and such that for the general $b \in B \setminus \{0\}$ the Kawamata-Morrison cone conjecture holds on the corresponding manifold X_b. Let $\mathscr{X} \to B$ the associated family.

Consider a class $\xi_0 \in \mathrm{Nef}^+(X_0)$. Notice that by Lemma 2.3 there exists $\{\xi_b \in \mathrm{Nef}^+(X_b)\}_{b\in B}$ a family of nef effective classes with $\lim_{b\to 0} \xi_b = \xi_0$. Recall that the relative group of automorphisms $\mathrm{Aut}(\mathscr{X}/B)$ is an open subscheme of $\mathrm{Hilb}(\mathscr{X} \times_B \mathscr{X})$ (cf. [Kol96, Theorem I.1.10]). By hypothesis, for the general $b \in B \setminus \{0\}$ there exists a rational polyhedral fundamental domain D_b and an automorphism $\tau_b \in \mathrm{Aut}(X_b)$ such that $\tau^*\xi_b = \delta_b \in D_b$. We look at the τ_b's as automorphisms $\widetilde{\tau}_b$ on the universal covers $\widetilde{X}_b$ commuting with the G-action. By the regularity of the action

$$\mathrm{Aut}(\mathscr{X}/B) \times \mathrm{Nef}(\mathscr{X}/B) \to \mathrm{Nef}(\mathscr{X}/B)$$

the automorphisms $\tau_b \in \mathrm{Aut}(X_b)$ form a family over $B \setminus \{0\}$ i.e. there exists $\mathscr{T} \in \mathrm{Aut}(\mathscr{X}/(B \setminus \{0\}))$ such that $\mathscr{T}_{|X_b} = \tau_b$.

Consider the induced Hodge isometries $\widetilde{\tau}_b^* : H^2(\widetilde{X}_0, \mathbb{Z}) \cong H^2(\widetilde{X}_b, \mathbb{Z}) \to H^2(\widetilde{X}_b, \mathbb{Z}) \cong H^2(\widetilde{X}_0, \mathbb{Z})$.

We set $\widetilde{t}_0 := \lim_{b \to 0} \widetilde{\tau}_b^*$.

Notice that $\widetilde{t}_0$ is a Hodge isometry of $H^2(\widetilde{X}_0, \mathbb{Z})$. Moreover $\widetilde{t}_0$ is a monodromy operator as the $\widetilde{\tau}_b^*$'s are. Finally remark that $\widetilde{t}_0$ preserves an ample class again as the $\widetilde{\tau}_b^*$'s do. Therefore, by the Hodge-theoretic global Torelli theorem [Mar11, Theorem 1.3] there exists an automorphism $\widetilde{\tau}_0 : \widetilde{X} \to \widetilde{X}$ such that $\widetilde{\tau}_0^* = \widetilde{t}_0$. The condition of commuting with G being a closed condition we have that $\widetilde{\tau}_0^*$ descends to an automorphism $\tau_0 \in \mathrm{Aut}(X_0)$ which, by construction, is the limit in zero of the τ_b's . □

Proof of Theorem 1.2. The Morrison-Kawamata cone conjecture for the $\mathrm{Aut}(X)$-action on the $\mathrm{Nef}^+(X)$-cone for an Enriques manifold X of prime index p follows for the combination of Section 3.1 (where we use that the index is prime) and of Theorem 3.5 (which holds in general).

To obtain the birational Morrison-Kawamata cone conjecture for the $\mathrm{Bir}(X)$-action on $\overline{\mathrm{Mov}}^+(X)$ on any Enriques manifold X of prime index p the same argument works, as by [Huy99, Proposition 9.1, item (iv)] the kernel of the group homomorphism $\mathrm{Bir}(\widetilde{X}) \to O(H^2(\widetilde{X}, \mathbb{Z}))$ equals the kernel of $\mathrm{Aut}(\widetilde{X}) \to O(H^2(\widetilde{X}, \mathbb{Z}))$. □

4. Further results

Recall that the Lefschetz number of an IHS manifold $\widetilde{X}$ of dimension $2n$ with a purely non–symplectic action by an automorphism g of finite order is given by:

$$L(g) = \sum_{j=0}^{2n} (-1)^j tr(g^*_{|H^j(\widetilde{X}, \mathscr{O}_{\widetilde{X}})}) = 1 + \bar{\lambda} + \bar{\lambda}^2 + \ldots + \bar{\lambda}^n$$

where we fix the action $g^*\sigma_{\widetilde{X}} = \lambda\sigma_{\widetilde{X}}$ and we have used that $H^j(\mathscr{O}_{\widetilde{X}}) = \overline{H^0(\Omega^j_{\widetilde{X}})}$. Moreover we know that for $j = 2t$ then $\overline{H^0(\Omega^j_{\widetilde{X}})}$ is generated by $\overline{\omega^t_{\widetilde{X}}}$, whence the cohomology groups for odd j are zero. Since the fixed locus is empty we must have $L(g) = 0$. This means that λ is at most a primitive $n+1$ root of unity and so we have that $|G| = d \leq n+1$, more precisely d divides $n+1$. By looking at the Lefschetz number, we can a priori have an Enriques manifold of any index, provided that the dimension of the IHS manifold is high enough. However until now we know existence of Enriques manifolds only of index $2, 3, 4$ (in the case of the index 2 we have Enriques surfaces, a construction involving $K3^{[n]}$-type manifolds for n odd and a construction involving Kum_n manifolds of odd dimension), see [BNWS11, OS11a]. In general the proof of Theorem 1.2 yields the following two results.

Proposition 4.1. *Let $\widetilde{X} \to X = \widetilde{X}/G$ be an Enriques manifold. Assume the action of $G = \langle g \rangle$ on* $\mathrm{Pic}(\widetilde{X})$ *is the identity, i.e.* $\mathrm{Pic}(\widetilde{X}) = H^2(\widetilde{X}, \mathbb{Z})^G$. *Then the cone conjecture holds for the Enriques manifold X and all its deformations.*

Proof. We consider a small variant of the proof of Theorem 1.2. We consider the decomposition

$$H^2(\widetilde{X},\mathbb{Q}) = \mathrm{Pic}(\widetilde{X})_{\mathbb{Q}} \oplus T(\widetilde{X})_{\mathbb{Q}},\ T(\widetilde{X}) := \mathrm{Pic}(\widetilde{X})^{\perp_{q_{\widetilde{X}}}}.$$

Then we obviously have commutativity on $\mathrm{Pic}(\widetilde{X})$, while on $T(\widetilde{X})$, which is preserved by $\mathrm{Aut}(\widetilde{X})$, we have commutativity by applying [Huy16, Corollary 3.4, Chapter 3] as in the proof of Lemma 3.4. The conclusion now follows by Global Torelli, with the same care to take into account automorphisms acting trivially on cohomology. Now we can argue exactly as in the proof of Theorem 1.2 and use the limiting procedure as in the proof of Theorem 3.5. □

4.1. **The cone conjecture for the index 4 examples.** As a first application of Proposition 4.1 we show that the cone conjecture holds for the known example of index four constructed by using generalized Kummer manifolds. First of all we recall the construction (see [BNWS11, Section 4.2] and [OS11a, Section 6]). Take integers m, n such that $4m = (n + 1)$. Let E_i be the elliptic curve with complex multiplication by i and F is another elliptic curve. Set $A := E_i \times F$. For F a generic elliptic curve we have $\mathrm{rk}\,\mathrm{Pic}(E_i \times F) = 2$ (indeed $\mathrm{Pic}(E_i \times F) = \mathrm{Pic}(X) \times \mathrm{Pic}(F) \times Hom(Jac(E_i), Jac(F))$, and for the genericity of F we have $Hom(Jac(E_i), Jac(F)) = 0$. Consider now $a := (a_1, a_2)$ where $a_2 \in A$ is a 4-torsion point and $a_1 \in E_i[n + 1]$ is such that $2ma_1 \notin \mathbb{Z}(1 + i)/2$. Let t_a be the translation by the point a on $E_i \times F$ and $h = \mathrm{diag}(i\,1)$ is the multiplication by i on the first component. Set $\psi := t_a \circ h$ then ψ does not have fixed points on $\mathrm{Kum}_n(E_i \times F)$ as shown in [BNWS11, Section 4.2] and [OS11a, Section 6]. For $(z, w) \in E_i \times F$ we have $t_a(h(z, w)) = (iz + a_1, w + a_2)$. Let us now compute the action in cohomology. Recall that $H^2(A,\mathbb{Z}) = U \oplus U \oplus U$ and consider $H^2(A,\mathbb{R})$. If we write $z = z_1 + iz_2$ for the coordinate on E_i and $w = w_1 + iw_2$ for the coordinate on F then $H^2(A,\mathbb{R})$ is generated by the 2-forms

$$dz_1 \wedge dw_1,\ dz_1 \wedge dw_2,\ dz_2 \wedge dw_1,\ dz_2 \wedge dw_2,\ dz_1 \wedge dz_2,\ dw_1 \wedge dw_2.$$

A translation acts trivially in cohomology and the action of the multiplication by i acts sending $z_1 + iz_2$ to $-z_2 + iz_1$ so that the image under ψ^* of the previous basis is

$$\psi^*(dz_1 \wedge dw_1) = -dz_2 \wedge dw_1,\ \psi^*(dz_1 \wedge dw_2) = -dz_2 \wedge dw_2,\ \psi^*(dz_2 \wedge dw_1) = dz_1 \wedge dw_1,$$

$$\psi^*(dz_2 \wedge dw_2) = dz_1 \wedge dw_2,\ \psi^*(dz_1 \wedge dz_2) = dz_1 \wedge dz_2,\ \psi^*(dw_1 \wedge dw_2) = dw_1 \wedge dw_2.$$

Hence the matrix of ψ^* in this basis is

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The eigenvalues of this matrix are i and $-i$ with multiplicity 2 and then 1 with multiplicity 2. This means that the invariant part for the action of ψ on the cohomology of A is precisely $\mathrm{Pic}(E_i \times F)$. Consider the induced natural automorphism $\psi^{[n]}$ on $\mathrm{Kum}_n(E_i \times F)$. By the choice of a_1 and a_2 the induced automorphism $\psi^{[n]}$ acts freely on $\mathrm{Kum}_n(E_i \times F)$ (see [OS11a, Theorem 6.4]). From the above description we have that the invariant part for the action on the cohomology of $\mathrm{Kum}_n(E_i \times F)$ of the induced automorphism $\psi^{[n]}$ is exactly the Picard group (since the automorphism $\psi^{[n]}$ is natural it acts as the identity on the exceptional divisor). Therefore we can invoke Proposition 4.1 and conclude that Conjecture 1.1 holds in this case.

4.2. **The cone conjecture for other possible indices.** We now discuss Enriques manifolds that can arise from all the known examples of IHS manifolds. The two examples of O'Grady OG_6 and OG_{10} are six and 10–dimensional so that to produce Enriques manifolds we have to make the quotient by a fixed point free automorphism of order 2 or 4, respectively order 2, 3 or 6. It is an interesting open question to understand whether such automorphisms exist.

Consider now an Enriques manifold $\widetilde{X} \to X = \widetilde{X}/G$ such that $\widetilde{X}$ is a deformation of $K3^{[n]}$ or of Kum_n. In this case the group $G = \langle g \rangle$ is forced to have order d that divides $n+1$. Recall that the respective second Betti numbers are 23 and 7, so that, since a primitive root is an eigenvalue of the action of g^* on $H^2(\widetilde{X}, \mathbb{Z}) \otimes \mathbb{C}$, we must have for the Euler totient function $\varphi(d) \leq 22$ respectively ≤ 6 (in fact in general we have $\varphi(d) \leq b_2(\widetilde{X}) - 1$), see [OS11a, Proposition 2.9] for the details. It is easy then to find the list of possible index d and we can formulate:

Proposition 4.2. *Let $\widetilde{X} \to X = \widetilde{X}/G$ be an Enriques manifold quotient of an IHS manifold $\widetilde{X}$.*

a) *If $\widetilde{X}$ is of $K3^{[n]}$-type, then, see* [OS11a, Proposition 2.9], *$d := |G| \leq 66$ and*
$$d \in \{2, \ldots, 28, 30, 32, 33, 34, 36, 38, 40, 42, 44, 46, 48, 50, 54, 60, 66\}.$$
In these cases the cone conjecture holds for all $d \in \{2, 5, 7, 11, 13, 17, 19, 23, 46\}$.

b) *If $\widetilde{X}$ of* Kum_n*-type, then, see* [OS11a, Proposition 2.9], *$d := |G| \leq 18$ and*
$$d \in \{2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 18\}.$$
In these cases the cone conjecture holds for all $d \in \{2, 3, 5, 7, 9, 14, 18\}$.

Proof. For the list of possible d see [OS11a, Proposition 2.9] even if the cases $d = 48, 60$ are missing in that list in the first case and in the second case the case $d = 24$ is included, but this is not possible since $\varphi(24) = 8$ which is bigger then $b_2(\widetilde{X}) = 7$ for $\widetilde{X}$ of Kum_n-type. For prime d then one applies Theorem 1.2. In the case of $d = 3$ and $\widetilde{X}$ of $K3^{[n]}$–type one uses the results of [BCS16, Table 1] to show that a non–symplectic automorphism of order 3 on a $K3^{[2]}$–type manifold has never empty fixed locus. For the orders 46, 9, 14 and 18, we have $\varphi(46) = 22$, $\varphi(9) = 6$, $\varphi(14) = 6$ and $\varphi(18) = 6$ and we remark that the rank of the Picard group is forced to be equal to one. In fact in these cases the transcendental lattice has exactly rank equal to $\varphi(d)$ (see [Be83b, Proposition 6]) and each primitive root of unity appears with multiplicity one. Now we know that $\widetilde{X}$ always contains an ample invariant class so that the action of G is trivial on $\mathrm{Pic}(\widetilde{X})$ and one concludes with Proposition 4.1. □

References

[AV17] K. Amerik and M. Verbitsky, *Morrison-Kawamata cone conjecture for hyperkähler manifolds.* Ann. Sci. Éc. Norm. Supér.(4) **50** (2017), 973–993.

[AV20] K. Amerik and M. Verbitsky, *Collections of parabolic orbits in homogeneous spaces, homogeneous dynamics and hyperkähler geometry.* IMRN, Volume 2020, Issue 1, 25–38.

[Be83a] A. Beauville, *Variétés Kähleriennes dont la première classe de Chern est nulle.* J. Differential Geom. **18** (1983), no. 4, 755–782.

[Be83b] A. Beauville, *Some remarks on Kähler manifolds with $c_1 = 0$.*. Classification of algebraic and analytic manifolds, PM 39, 1–26; Birkhäuser (1983).

[BCS16] S. Boissière, C. Camere and A. Sarti, *Classification of automorphisms on a deformation family of hyperkähler fourfolds by p-elementary lattices.* Kyoto J. of Mathematics **56** (2016), no. 3, 465–499.

[BNWS11] S. Boissière, M. Nieper-Wisskirchen and A. Sarti, *Higher dimensional Enriques varieties and automorphisms of generalized Kummer varieties.* J. Math. Pures et Appli. **95** (2011), 553–563.

[CL14] P. Cascini, V. Lazić, *On the number of minimal models of a log smooth threefold*, J. Math. Pures Appl. **102** (2014), 597–616.

[DK07] I. V. Dolgachev and S. Kondo, Moduli of K3 surfaces and complex ball quotients, Arithmetic and geometry around hypergeometric functions, Progr. Math., vol. 260, Birkhäuser, Basel, 2007, pp. 43–100.

[D96] I.V. Dolgachev, Mirror symmetry for lattice polarized K3 surfaces, J. Math. Sci. **81** (1996), no. 3, 2599–2630.

[Ka97] Y. Kawamata, *On the cone of divisors of Calabi-Yau fiber spaces.* Internat. J. Math. **8** (1997), 665–687.

[Fuj87] A. Fujiki *On the de Rham cohomology group of a compact Kähler symplectic manifold.* Algebraic geometry, Sendai, 1985, volume 10 of Adv. Stud. Pure Math., p. 105–165. North-Holland, Amsterdam, 1987.

[GLW22] C. Gachet, H.-Y. Lin, L. Wang, *Nef cones of fiber products and an application to the Cone Conjecture*, preprint `arXiv:2210.02779`.

[HT13] B. Hassett, Yu. Tschinkel, *Hodge theory and Lagrangian planes on generalized Kummer fourfolds.* Moscow Mathematical Journal **13** (2013), no. 1, 33–56.

[Hor76] E. Horikawa, *On deformations of holomorphic maps. III.* Math. Ann. **222** (1976), no. 3., 275–282.

[Huy99] D. Huybrechts, *Compact hyperkähler manifolds: basic results.* Invent. Math. **135** (1999) 63–113.

[Huy11] D. Huybrechts, A global Torelli theorem for hyperkähler manifolds [after M. Verbitsky]. Séminaire Bourbaki: Vol. 2010/2011. Exposés 1027–1042. Astérisque No. 348 (2012), Exp. No. 1040, x, 375–403.

[Huy16] D. Huybrechts, *Lectures on K3 surfaces.* Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2016. doi:10.1017/CBO9781316594193.

[Kol96] J. Kollár, Rational Curves on Algebraic Varieties. Ergebnisse der Mathematik und ihrer Grenzgebiete 32, Springer Verlag, Berlin, 1996.

[LOP18] V. Lazić, K. Oguiso, Th. Peternell, *The Morrison–Kawamata Cone Conjecture and Abundance on Ricci flat manifolds*, Uniformization, Riemann-Hilbert Correspondence, Calabi-Yau manifolds and Picard-Fuchs Equations (L. Ji, S.-T. Yau, eds.), Advanced Lectures in Mathematics, vol. 42, International Press, 2018, pp. 157–185.

[LMP22] Ch. Lehn, G. Mongardi and G. Pacienza, The Morrison-Kawamata cone conjecture for singular symplectic varieties, preprint `arXiv:2207.14754`.

[LZ22] Z. Li, H. Zhao *On the relative Morrison-Kawamata cone conjecture*, preprint `arXiv:2206.13701v3`.

[Mar11] E. Markman, *A survey of Torelli and monodromy results for holomorphic-symplectic varieties. Complex and differential geometry*, 257–322, Springer Proc. Math., **8**, Springer, Heidelberg, 2011.

[MY15] E. Markman and K. Yoshioka *A proof of the Kawamata-Morrison Cone Conjecture for holomorphic symplectic varieties of K3 or generalized Kummer deformation type.* IMRN Volume 2015, Issue 24, 13563–13574.

[MW17] G. Mongardi and M. Wandel, *Automorphisms of O'Grady's manifolds acting trivially on cohomology*, Algebraic Geometry **4** (1) (2017) 104–119.

[Mor93] D. Morrison, *Compactifications of moduli spaces inspired by mirror symmetry.* Journées de géométrie algébrique d'Orsay (Orsay, 1992), Astérisque **218** (1993), 243–271.

[Na85] Y. Namikawa, *Periods of Enriques surfaces.* Math. Ann. **270** (1985), 201–222.

[OS11a] K. Oguiso and S. Schöer, *Enriques manifolds.* Journal für die reine und angewandte Mathematik, vol. 2011, no. 661, (2011), 215–235.

[OS11b] K. Oguiso and S. Schöer, *Periods of Enriques manifolds.* Pure and Applied Mathematics Quarterly Volume 7, Number 4 (Special Issue: In memory of Eckart Viehweg), (2011), 1631–1656.

[PS12] A. Prendergast-Smith, *The cone conjecture for abelian varieties.* J. Math. Sci. Univ. Tokyo **19** (2012), 243–261.

[Ste85] H. Sterk, *Finiteness results for algebraic K3 surfaces.* Math. Z. **189** (1985), 507–513.

[Ta21] T. Takamatsu, *On the finiteness of twists of irreducible symplectic varieties*, preprint `arXiv:2106.11651v2`.

[Tot10] B. Totaro, *The cone conjecture for Calabi–Yau pairs in dimension two.* Duke Math. J. **154** (2010), 241–263.

[Tot12] B. Totaro, *Algebraic surfaces and hyperbolic geometry.* Current Developments in Algebraic Geometry, ed. L. Caporaso, J. McKernan, M. Mustaţă, and M. Popa, MSRI Publications **59**, Cambridge (2012), 405–426.

[Ver13] M. Verbitsky, A global Torelli theorem for hyperkahler manifolds, Duke Math. J. **162** (15) (2013) 2929–2986.

Institut Elie Cartan de Lorraine, Université de Lorraine et CNRS, 54000 Nancy, France

Email address: `pacienza@univ-lorraine.fr`

UMR 7348 du CNRS, Université de Poitiers, Bât. H3 - Site du Futuroscope, TSA 61125, 11 bd Marie et Pierre Curie, 86073 POITIERS Cedex 9, France

Email address: `Alessandra.Sarti@math.univ-poitiers.fr`