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Algebraic Geometry: Moduli Spaces, Birational Geometry and Derived Aspects

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Abstract. The talks at the workshop and the research done during the week focused on aspects of algebraic geometry in the broad sense. Special emphasis was put on hyperkähler manifolds and derived categories.

Mathematics Subject Classification (2010): 14C, 14E, 14J, 14N.

Introduction by the Organizers

The meeting in 2020 was different. It was deemed worthwhile to run the workshop, although only one of the organizers could attend, only Europeans, mostly from Germany and France, could be present physically and the week had more the character of 'Research in several groups' than a proper workshop. The mostly young participants, the overwhelming majority for the first time at the MFO, enjoyed the experience but the workshop was lacking some of its usual energy and could not fulfill its role as a catalyst in the area of algebraic geometry. Many of the original invited participants (only a couple declined the invitation) expressed explicitly their regret about not being able to attend the workshop and their interest in a workshop on the original topics in the very near future.

The format of the workshop was naturally also somewhat different. We had only one talk in the morning, usually by a first time participant and not video recorded, and one talk in the afternoon that was streamed online (and watched by some of the external participants). The afternoon speakers were asked to make their talks accessible to a wide audience, which was very much appreciated by everybody.

There was a clear emphasis on hyperkähler geometry and derived categories, but we did arrange for one (after dinner) talk on foliations in the minimal model program, a new research direction that has been picking up speed recently. We tried to keep the workshop thematically broad but certain new active directions, e.g. moduli spaces of Fano varieties, could not be covered as none of the experts could come.

Despite the exceptional situation of this installment of the workshop series, the participants spent a productive and stimulating week at the MFO. People used the opportunity to discuss mathematics in person for the first time in months, pursued existing research projects and started new ones. Some of the originally invited participants used the opportunity to invite a collaborator along, which contributed to the success of the week.

Workshop: Algebraic Geometry: Moduli Spaces, Birational Geometry and Derived Aspects

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Abstracts

Elliptic quintics on cubic fourfolds, O'Grady 10 and Lagrangian fibrations

Laura Pertusi

(joint work with Chunyi Li, Xiaolei Zhao)

In this talk we study certain moduli spaces of semistable objects in the Kuznetsov component of a cubic fourfold. We show that they admit a symplectic resolution $\widetilde{M}$ which is a smooth projective hyperkähler manifold deformation equivalent to the 10-dimensional example constructed by O'Grady. As a first application, we construct a birational model of $\widetilde{M}$ which is a compactification of the twisted intermediate Jacobian of the cubic fourfold. Secondly, we show that $\widetilde{M}$ is the MRC quotient of the main component of the Hilbert scheme of elliptic quintic curves in the cubic fourfold, as conjectured by Castravet.

1. Introduction

Moduli spaces of stable sheaves on a K3 surface provide examples of projective hyperkähler manifolds, which are deformation equivalent to Hilbert schemes of points on a K3 surface, by the seminal work of Mukai [15] and the contribution of many other authors, including Beauville, Huybrechts, O'Grady, Yoshioka [19, 20]. In [16] O'Grady considered the case when the moduli space contains also strictly semistable sheaves. In particular, he constructed a symplectic resolution of the singular moduli space of semistable torsion-free sheaves on a K3 surface with rank 2, trivial first Chern class and second Chern class equal to 4. This construction provides a new example of a hyperkähler manifold of dimension 10, not deformation equivalent to the previous construction. O'Grady's result was generalized by Lehn and Sorger in [12] to moduli spaces of semistable sheaves on a K3 surface having Mukai vector of the form $v = 2v_0$ with $v_0^2 = 2$. In addition, they showed that the symplectic resolution of the moduli space can be obtained by blowing up the singular locus with the reduced scheme structure.

In the paper [14] under report we investigate the analogous situation of O'Grady's example, in the case of moduli spaces of semistable complexes in the noncommutative K3 surface associated to a smooth cubic fourfold. Indeed, by [9] the bounded derived category of a cubic fourfold Y has a semiorthogonal decomposition of the form

$$\mathrm{D}^{\mathrm{b}}(Y) = \langle \mathrm{Ku}(Y), \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2) \rangle,$$

where $\mathrm{Ku}(Y)$ is a triangulated subcategory of K3 type, in the sense that it has the same Serre functor and Hochschild homology as the derived category of a K3 surface. We call this category $\mathrm{Ku}(Y)$ the *Kuznetsov component* of Y.

2. Kuznetsov component of a cubic fourfold

One reason to study $\mathrm{Ku}(Y)$ is related to the birational geometry of Y. For instance, there is a folklore conjecture [9, Conjecture 1.1] saying that Y is rational if and only if $\mathrm{Ku}(Y)$ is equivalent to the derived category of a K3 surface.

Another interest in studying $\mathrm{Ku}(Y)$ is to generalize Mukai's construct to this noncommutative K3 surface. Bayer, Lahoz, Macrì and Stellari construct Bridgeland stability conditions on $\mathrm{Ku}(Y)$ in [3]. In a second paper [4], joint also with Nuer and Perry, they develop the deep theory of families of stability conditions, which allows studying the properties of moduli spaces of stable objects in $\mathrm{Ku}(Y)$ by deforming to cubic fourfolds whose Kuznetsov components are equivalent to the derived category of a K3 surface. As a consequence, they produced infinite series of unirational, locally complete families of smooth polarized hyperkähler manifolds, deformation equivalent to Hilbert schemes of points on a K3 surface. These hyperkähler manifolds are given as moduli spaces of stable objects in $\mathrm{Ku}(Y)$ of primitive Mukai vector.

It is worth to point out that the hyperkähler manifolds constructed from some Hilbert schemes of rational curves of low degree in Y can be interpreted as moduli spaces of stable objects in $\mathrm{Ku}(Y)$. Indeed, we gave in [13] a description of the Fano variety of lines in Y and, when Y does not contain a plane, of the hyperkähler 8-fold constructed in [11] using twisted cubic curves in Y, as moduli spaces of stable objects in $\mathrm{Ku}(Y)$ with primitive Mukai vector.

3. O'Grady spaces

The *algebraic Mukai lattice* $H^*_{\mathrm{alg}}(\mathrm{Ku}(Y),\mathbb{Z})$ of $\mathrm{Ku}(Y)$, introduced in [3, Proposition and Definition 9.5], consists of algebraic cohomology classes of Y which are orthogonal to the classes of $\mathcal{O}_Y$, $\mathcal{O}_Y(1)$, $\mathcal{O}_Y(2)$ with respect to the Euler pairing. On the other hand, we denote by $\mathrm{Stab}^\dagger(\mathrm{Ku}(Y))$ the connected component of the stability manifold containing the stability conditions constructed in [3].

Consider now a vector $v = 2v_0 \in H^*_{\mathrm{alg}}(\mathrm{Ku}(Y),\mathbb{Z})$ such that v_0 is primitive with $v_0^2 = 2$. Let τ be a stability condition in $\mathrm{Stab}^\dagger(\mathrm{Ku}(Y))$ which is generic with respect to v, in other words, the strictly τ-semistable objects with Mukai vector v are (S-equivalent to) direct sums of τ-stable objects with Mukai vector v_0. Let M be the moduli space of τ-semistable objects with Mukai vector v. Our first result is the following.

Theorem 3.1 ([14], Theorem 1.1)**.** *The moduli space M has a symplectic resolution $\widetilde{M}$, which is a* 10*-dimensional smooth projective hyperkähler manifold, deformation equivalent to the O'Grady's example constructed in* [16]*.*

Idea of proof. We apply the same argument used in [12] in the case of the moduli space of semistable sheaves over a polarized K3 surface. The strategy is to study the local structure of the moduli space at the worst singularity and prove that its normal cone is isomorphic to an affine model obtained as a nilpotent orbit in the symplectic Lie algebra $\mathfrak{sp}(4)$. It turns out that the singularity is formally isomorphic to its normal cone. Since the singularity at the generic point of the singular locus of M is of type A_1, one can conclude that the blow up $\widetilde{M}$ of M at its singular locus endowed with the reduced scheme structure is a symplectic resolution of M. The main difference with [12] is that in the case of moduli of sheaves, the moduli is constructed as a GIT quotient. To study the local structure, it is enough to take an étale slice. In our case, we instead use the deep result on étale slice of algebraic stacks [2]. The other properties of $\widetilde{M}$ (projectivity, deformation type) are obtained by degeneration to the locus of cubic fourfolds with Kuznetsov component equivalent to the bounded derived category of a K3 surface, as in [4]. □

4. Applications

We explain two main applications, which make a connection between the derived categorical viewpoint of Theorem 3.1 and the classical construction of hyperkähler manifolds from Y.

Recall that by [1], the algebraic Mukai lattice of $\mathrm{Ku}(Y)$ contains two classes λ_1 and λ_2 spanning an A_2-lattice. Motivated by classical geometric constructions (as it will be clear later), we consider the case $v_0 = \lambda_1+\lambda_2$, $v = 2v_0$ and we analyze the objects in $M := M_\sigma(v)$ where σ is a stability condition as constructed in [3]. It is not difficult to see that by [13] the strictly semistable locus of M is identified with the symmetric square of the Fano variety of lines in Y, up to a perturbation of the stability condition. On the other hand, stable objects are harder to describe. If X is a smooth hyperplane section of Y, in other words, X is a smooth cubic threefold, then the moduli space M_{inst} parametrizing rank 2 instanton sheaves on X have been described in [8]. In particular, stable sheaves in M_{inst} belong to one of the following classes: rank 2 stable vector bundles constructed from non-degenerate elliptic quintics in X, rank 2 stable torsion free sheaves associated to smooth conics in X. Moreover, the strictly semistable objects in M_{inst} are direct sums of two ideal sheaves of lines in X. By [5, 8] the moduli space M_{inst} is birational to the translate $J^2(X)$ of the intermediate Jacobian, which parameterizes 1-cycles of degree 2 on X.

A key result for our applications is the following theorem, which provides a description of an open subset of the stable locus of $M := M_\sigma(2\lambda_1 + 2\lambda_2)$.

Theorem 4.1 ([14], Theorem 1.2)**.** *Let X be a smooth hyperplane section of Y. Then the projection in $\mathrm{Ku}(Y)$ of the stable rank 2 instanton sheaves associated to non degenerate elliptic quintic curves and smooth conics in X are σ-stable objects with Mukai vector $2\lambda_1 + 2\lambda_2$.*

We apply Theorem 4.1 to show that, up to a perturbation of the stability condition σ in $\mathrm{Stab}^\dagger(\mathrm{Ku}(Y))$, the sympletic resolution $\widetilde{M}$, given by Theorem 3.1 has a deep connection to a classical construction of Jacobian fibration associated to Y. Consider the $(\mathbb{P}^5)^\vee$-family of cubic threefolds obtained as hyperplane sections of Y and let $\mathbb{P}_0$ be its smooth locus. Consider the twisted family of intermediate Jacobians $p : J \to \mathbb{P}_0$, whose fibers are the twisted intermediate Jacobians of the smooth cubic threefolds parametrized by $\mathbb{P}_0$. It is known that there exists a holomorphic symplectic form on J by [7]. However, it remained a long standing question whether J can be compactified to a hyperkähler manifold $\bar{J}$ and a Lagrangian fibration $\bar{J} \to (\mathbb{P}^5)^\vee$ extending p. This has been recently proved for very general cubic fourfolds in the beautiful works [10] for the untwisted family and [18] by Voisin for J. We mention that in the recent preprint [17], Saccà extended the result for the untwisted family in [10] to all smooth cubic fourfolds. The same argument applies to the twisted family and extends Voisin's result to all smooth cubic fourfolds (see [17, Remark 1.10]).

Our main result is the following modular construction of a hyperkähler compactification of J for every cubic fourfold Y, obtained combining Theorems 3.1, 4.1 and some techniques in birational geometry of hyperkähler varieties.

Theorem 4.2 ([14], Theorem 1.4). *There exists a hyperkähler manifold N birational to $\widetilde{M}$, which admits a Lagrangian fibration structure compactifying the twisted intermediate Jacobian family $J \to \mathbb{P}_0$.*

It is worth to note that N and $\widetilde{M}$ are birational, but not isomorphic if Y is very general [14, Example 6.8]. Also N is isomorphic to the compactification constructed by Voisin if Y is very general [14, Remark 6.9].

The next application arises from the following conjecture of Castravet.

Conjecture 4.3 ([6, Page 416]). *Let $\mathcal{C}$ be the connected component of the Hilbert scheme $\mathrm{Hilb}^{5m}(Y)$ containing elliptic quintics in Y. Then the maximally rationally connected quotient of $\mathcal{C}$ is birationally equivalent to the twisted intermediate Jacobian of Y.*

Using Theorems 3.1, 4.1 and 4.2 we are able to prove Conjecture 4.3.

Proposition 4.4 ([14], Proposition 1.5). *The projection functor in $\mathrm{Ku}(Y)$ induces a rational map $\mathcal{C} \dashrightarrow M$ which is the maximally rationally connected fibration of $\mathcal{C}$. The maximally rationally connected quotient of $\mathcal{C}$ is birational to the the twisted family J of intermediate Jacobians of Y.*

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How do semiorthogonal decompositions behave in families?

Pieter Belmans

(joint work with Shinnosuke Okawa, Andrea T. Ricolfi)

In this talk I gave a brief history of semiorthogonal decompositions, explained how they can be studied using Fourier–Mukai transforms, and how they behave in families. What follows is an overview of the historical motivation with additional references, and a summary of the results on moduli spaces of semiorthogonal decompositions.

1. Semiorthogonal decompositions

For an introduction to semiorthogonal decompositions, and many more examples, one is referred to [13]. They are introduced to understand the structure of the derived category of coherent sheaves of a smooth projective variety X, from now on denoted $\mathbf{D}^{\mathrm{b}}(X)$.

A brief history of semiorthogonal decompositions: 3 examples.
In [2] Beĭlinson described the derived category of $\mathbb{P}^n$, starting the whole field.

Example 1.1. *Using the usual notation for exceptional collections we have that*

$$\mathbf{D}^{\mathrm{b}}(\mathbb{P}^n) = \langle \mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \ldots, \mathcal{O}_{\mathbb{P}^n}(n) \rangle \tag{1}$$

Here all the admissible subcategories are equivalent to $\mathbf{D}^{\mathrm{b}}(k)$*.*

The next step came with the introduction of semiorthogonal decompositions by Bondal–Orlov in [6].

Example 1.2. *The first example is given by the (smooth) intersection of two even-dimensional quadrics $X = Q_1 \cap Q_2$ in $\mathbb{P}^{2g+1}$, for which we have*

$$\mathbf{D}^{\mathrm{b}}(X) = \langle \mathbf{D}^{\mathrm{b}}(C), \mathcal{O}_X, \mathcal{O}_X(1), \ldots, \mathcal{O}_X(2g-3) \rangle \tag{2}$$

where C is a hyperelliptic curve of genus g. As we will discuss in the next example, $\mathbf{D}^{\mathrm{b}}(C)$ cannot be decomposed further, so this is a semiorthogonal decomposition into "atomic" components.

By now there exists a vast literature on exceptional collections and semiorthogonal decompositions. The final example we want to give discusses the *absence* of non-trivial semiorthogonal decompositions.

Example 1.3. *Let X be either a Calabi–Yau variety, or a curve of genus $g \geq 2$. Then by Bridgeland* [7] *resp. Okawa* [14, Theorem 1.1] *we have that $\mathbf{D}^{\mathrm{b}}(X)$ is indecomposable.*

Families of varieties. Once one understands a semiorthogonal decomposition of one fibre in a family, what can be said about semiorthogonal decompositions for other fibres?

An important guiding principle here is *Dubrovin's conjecture* [8]. It states that $\mathrm{BQH}^\bullet(X)$ is (generically) semisimple if and only if $\mathbf{D}^{\mathrm{b}}(X)$ admits a full exceptional collection. But $\mathrm{BQH}^\bullet(X)$ depends on the symplectic, not the complex structure of X. Hence conjecturally the existence of full exceptional collection is constant in nice families. A more general version of this conjecture for arbitrary semiorthogonal decompositions can be found in [17].

The behavior of exceptional collections in families is studied in [10] using Lieblich's deformation theory of perfect complexes. Full exceptional collections are shown to extend to étale neighbourhoods. In what follows we discuss how this deformation theory result generalises, by constructing an actual moduli space of semiorthogonal decompositions with arbitrary components.

2. The moduli space of semiorthogonal decompositions

The general procedure to make exceptional collections and semiorthogonal decompositions behave well in a family over a base S is that of S-linearity [12]. The main result is then following [4, Theorem A].

Theorem 2.1. *Let $f\colon \mathcal{X} \to S$ be a smooth projective morphism, where S is an excellent scheme. Then there exists an algebraic space $\mathrm{SOD}_f \to S$, such that*

(1) $\mathrm{SOD}_f \to S$ *is* étale*;*
(2) there exists a functorial bijection between $\mathrm{SOD}_f(T \to S)$ and the set of T-linear semiorthogonal decompositions (of length 2) of $\operatorname{Perf} \mathcal{X} \times_T S$.

The proof consists of checking Artin's axioms for étale algebraic spaces, in the form of Hall–Rydh [9, Theorem 11.3]. For this we use that S-linear semiorthogonal decompositions can be represented using (morphisms of) Fourier–Mukai kernels. The main technical ingredient is then a deformation theory of *morphisms* of complexes in a derived category (with a fixed lift of the codomain), generalising the deformation theory of complexes.

The main important geometric feature of this moduli space (which is rather strange in other respects, see §3) is that it is étale over S. This is consistent with the suggestion of Dubrovin's conjecture.

Application: indecomposability. We can use the moduli space of semiorthogonal decompositions to show that having an indecomposable derived category *specialises* in a family of smooth projective varieties. The details for this are contained in the joint work [1] with Francesco Bastianelli. We can obtain for example the following result.

Theorem 2.2. *Let C be a smooth projective curve of genus $g \geq 2$. Let $n = 1, \ldots, \lfloor \frac{g+3}{2} \rfloor - 1$. Then $\mathbf{D}^{\mathrm{b}}(\operatorname{Sym}^n C)$ is indecomposable.*

Its proof is obtained by bootstrapping from the indecomposability result [11, Theorem 1.4], analysing the relationship between the gonality of a curve and the base locus of the canonical linear system (see also [5]). This theorem settles (the easier) half of [3, Conjecture 2], which suggests the indecomposability of $\mathbf{D}^{\mathrm{b}}(\operatorname{Sym}^n C)$ for n up to $g-1$.

3. Examples, pathologies and amplifications

By describing SOD_f in 3 instances, we can see how this algebraic space has an interesting geometry, and what kind of variations we can moreover study.

Example 3.1. *Let $f\colon X \to \operatorname{Spec} k$ be an example from Example 1.3. Then SOD_f consists of two points, given by the trivial semiorthogonal decompositions $\langle \mathbf{D}^{\mathrm{b}}(X), \emptyset \rangle$ and $\langle \emptyset, \mathbf{D}^{\mathrm{b}}(X) \rangle$.*

To remedy this, one can study the open and closed algebraic subspace $\mathrm{ntSOD}_f \subset \mathrm{SOD}_f$, only parametrising non-trivial semiorthogonal decompositions.

More interestingly we can consider Beĭlinson's semiorthogonal decompositions from Example 1.1.

Example 3.2. *For $n = 1$ (folklore) and $n = 2$* [16] *there exists a classification of semiorthogonal decompositions of $\mathbf{D}^{\mathrm{b}}(\mathbb{P}^n)$. For $f\colon \mathbb{P}^1 \to \operatorname{Spec} k$ it shows that $\mathrm{ntSOD}_f = \bigcup_{i \in \mathbb{Z}} \operatorname{Spec} k$, indexing the decomposition $\langle \mathcal{O}_{\mathbb{P}^1}(i), \mathcal{O}_{\mathbb{P}^1}(i+1) \rangle$.*

This shows that SOD_f and ntSOD_f are usually *not quasicompact.* It also shows the necessity to extend the definition of the moduli space to incorporate semiorthogonal decompositions of length ℓ. This can be done, and yields moduli spaces SOD_f^ℓ and ntSOD_f^ℓ with similar properties.

One can show that SOD_f^ℓ and ntSOD_f^ℓ admit (commuting) actions by the group $\mathrm{Auteq}(f)$ of f-linear autoequivalences and the braid group Br_ℓ acting by mutations. The quotient by these groups might yield more tractable moduli spaces.

Finally, the most interesting behaviour is showcased by the following example.

Example 3.3. *Let $f\colon \mathcal{X} \to \mathbb{A}^1$ be the degeneration of $\mathbb{P}^1 \times \mathbb{P}^1$ into the second Hirzebruch surface $\mathbb{F}_2$ (at the point $0 \in \mathbb{A}^1$). By comparing the classification of exceptional objects for the quadric with the results of* [15]*, one can construct distinct families of exceptional objects, which agree on $\mathbb{A}^1 \setminus \{0\}$, but give different exceptional objects in $\mathbf{D}^{\mathrm{b}}(\mathbb{F}_2)$.*

This shows that SOD_f can in general be *non-separated.* This is an important feature of the behavior of semiorthogonal decompositions in families, and it would be interesting to understand this in more instances.

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Equivariant categories and fixed loci of holomorphic symplectic varieties

Thorsten Beckmann
(joint work with Georg Oberdieck)

In this talk I explained the close relationship between equivariant categories and fixed loci of finite groups acting on Bridgeland moduli spaces. The abstract theory was accompanied by examples of holomorphic symplectic varieties.

1. Equivariant categories

If a finite group G acts on a symplectic surface S and preserves the symplectic form, then the quotient variety S/G has isolated ADE singularities and admits a crepant resolution S' which is again symplectic. The derived McKay correspondence [4] provides a natural equivalence between the bounded derived category of G-equivariant sheaves on S and the bounded derived category $D^b(S')$ of coherent sheaves on S'.

One may ask what happens if we work more generally with an abstract action of a finite group G on the derived category $D^b(S)$. Is the equivariant category, assuming reasonable conditions, again equivalent to the derived category of a symplectic surface S'?

Our setup is the following: Let S be a non-singular complex projective surface which is symplectic, hence either a K3 surface or an abelian surface. Let G be a finite group acting symplectically and faithfully on $D^b(S)$ such that there exists a stability condition $\sigma \in \mathrm{Stab}^\dagger(S)$ which is fixed by the action of G.

Write $\Lambda = H^{2*}(S, \mathbb{Z})$ for the even cohomology lattice. The induced G-action on cohomology preserves the sublattice Λ_{alg}. We write Λ_{alg}^G for the invariant sublattice. Let $M_\sigma(v)$ be the moduli space of σ-semistable objects with Mukai vector v. If v is G-invariant, then we have an induced action of G on $M_\sigma(v)$. Let $G^\vee = \mathrm{Hom}(G, \mathbb{C}^*)$ be the group of characters of G.

Theorem 1.1. *Let $v \in \Lambda_{\mathrm{alg}}^G$ such that $M_\sigma(v)$ is a fine moduli space. If the fixed locus $M_\sigma(v)^G$ has a 2-dimensional G-linearizable connected component F, then there exists a connected étale cover $S' \to F$ of degree dividing the order of $G^\vee$ and an equivalence*

$$D^b(S') \xrightarrow{\cong} D^b(S)_G$$

induced by the restriction of the universal family to $S' \times S$.

We say here that a connected component of $M_\sigma(v)^G$ is *G-linearizable* if for some (or equivalently any) point on it the corresponding G-invariant object in $D^b(S)$ admits a G-linearization. For cyclic groups, the condition on F to be G-linearizable is automatically satisfied.

We state a version of Theorem 1.1 where we drop the condition on the moduli space to be fine. This is useful since not every group action on $D^b(S)$ induces an action on a fine moduli space.

Definition 1.2. *Let Z denote the central charge of σ. A vector $v \in \Lambda_{\mathrm{alg}}^G$ is* (G,σ)-generic *if it is primitive and for every splitting $v = v_0 + v_1$ with $v_0, v_1 \in \Lambda_{\mathrm{alg}}^G \setminus \mathbb{Z}v$ the values $Z(v_0)$ and $Z(v_1)$ have different slopes.*

Given any primitive vector $v \in \Lambda_{\mathrm{alg}}^G$, one can show that after a small deformation of σ along G-fixed stability conditions the class v becomes (G,σ)-generic.

Let also $\mathcal{M}_\sigma(v)$ denote the moduli stack of σ-semistable objects in class v.

Theorem 1.3. *Let $v \in \Lambda_{\mathrm{alg}}^G$ be (G,σ)-generic.*

(a) The fixed stack $\mathcal{M}_\sigma(v)^G$ has a good moduli space $\pi\colon \mathcal{M}_\sigma(v)^G \to N$ which is smooth, symplectic and proper. The map π is a $\mathbb{G}_m$-gerbe.

(b) If N has a 2-dimensional connected component S', then the restriction of the universal family induces an equivalence

$$D^b(S', \alpha) \xrightarrow{\cong} D^b(S)_G$$

where $\alpha \in \mathrm{Br}(S')$ is the Brauer class of the gerbe.

Here we let $D^b(S', \alpha)$ denote the derived category of α-twisted coherent sheaves on S'. The notion of a good moduli space was introduced in [1]. The fixed stack is taken in the categorical sense of Romagny [5].

For the proof we use Orlov's result on Fourier–Mukai functors to construct an action of G on the stack $\mathfrak{M}$ of universally gluable objects in $D^b(S)$ in the sense of Lieblich. The fixed stack $\mathfrak{M}^G$ is precisely the stack of objects in the equivariant category $D^b(S)_G$. By transferring geometric properties from $\mathfrak{M}$ to its fixed stack, this yields a well-behaved moduli theory for objects in the equivariant category. The restriction of the universal family of $\mathfrak{M}^G$ to components, which are 2-dimensional and parametrize stable objects, then leads to a Fourier–Mukai kernel which induces the desired equivalence.

2. Fixed loci

After having seen how fixed loci determine the equivariant category, we describe how conversely the equivariant category controls the fixed loci of moduli spaces of stable objects.

Consider a symplectic action of a finite group G on $D^b(S)$. Assume that we have an equivalence

$$D^b(S', \alpha) \xrightarrow{\cong} D^b(S)_G.$$

The surface S' here is necessarily symplectic but can be disconnected. Let

$$P\colon H^{2*}(S',\mathbb{Z}) \to H^{2*}(S,\mathbb{Z})$$

be the map induced from the composition $D^b(S',\alpha) \to D^b(S)_G \to D^b(S)$ where the latter map is the forgetful functor. Given an element $v \in \Lambda^G_{\mathrm{alg}}$ we write

$$R_v = \{v' \in \Lambda_{(S',\alpha),\mathrm{alg}} \mid P(v') = v\}.$$

The G-invariant stability condition σ induces a stability condition, denoted σ_G, on $D^b(S)_G$ and hence on $D^b(S',\alpha)$. We write $M_{\sigma_G}(v')$ for the good moduli space of the stack $\mathcal{M}_{\sigma_G}(v')$.

Theorem 2.1. *Let $v \in \Lambda^G_{\mathrm{alg}}$ such that $M_\sigma(v)$ is a moduli space of stable objects. Then there exists a degree $|G^\vee|$ étale morphism*

$$\bigsqcup_{v' \in R_v} M_{\sigma_G}(v') \to M_\sigma(v)^G \tag{1}$$

whose image is the union of all G-linearizable connected components of $M_\sigma(v)^G$. If G is cyclic, then (1) *is surjective.*

We refer to [2, Thm. 3.20] for a more general version of Theorem 2.1 which applies to any variety with a suitable stability condition and where we do not require the equivariant category to be equivalent to the derived category of some variety.

If S' is a K3 surface and the equivalence is geometric, we can be more precise with our description of the fixed locus. Let

$$\overline{R}_v \subset \Lambda_{(S',\alpha),\mathrm{alg}}$$

be a set of representatives of the coset $R_v/G^\vee$.

Theorem 2.2. *Let $v \in \Lambda^G_{\mathrm{alg}}$ such that $M_\sigma(v)$ is a moduli space of stable objects. Suppose that G is cyclic and that we have an equivalence $D^b(S',\alpha) \to D^b(S)_G$ for a K3 surface S' which is induced from a universal family as in Theorem 1.3.*

Then the induced stability condition σ_G lies in $\mathrm{Stab}^\dagger(S')$ and we have an isomorphism

$$M_\sigma(v)^G \cong \bigsqcup_{v' \in \overline{R}_v} M_{\sigma_G}(v'). \tag{2}$$

We finally remark that symplectic actions of finite groups on moduli spaces of stable objects on K3 surfaces are always induced by actions on the derived category as considered above. Hence Theorems 2.1 and 2.2 in combination with Theorem 1.1 provide an effective method to determine the fixed locus of any such action.

Proposition 2.3. *Let S be a K3 surface and let $\sigma' \in \mathrm{Stab}^\dagger(S)$ be a stability condition. Let G be a finite cyclic group which acts faithfully and symplectically on a moduli space M of σ'-stable objects. Then there exists a faithful and symplectic action of G on $D^b(S)$ which induces the given G-action on M.*

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Stability conditions in families, without applications

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The construction of Bridgeland stability conditions in families [BLMNSP19] led to a number of applications, including an infinite sequence of locally complete unirational families of polarised Hyperkähler varieties of K3 type, or the complete classification of cubic fourfolds whose derived category contains the derived category of a K3 surface as its *Kuznetsov component*. This talk focussed instead on the foundational aspects, explaining the notion of a stability condition for a family of varieties precisely, along with some motivation for the definition.

Recall that a stability condition σ on a triangulated category $\mathcal{D}$ consists of

- a central charge, i.e. a group homomorphism $Z\colon K(\mathcal{D}) \to \mathbb{C}$, and
- a slicing, i.e. a list $\mathcal{P}(\phi)$ of semistable objects of phase ϕ for all $\phi \in \mathbb{R}$

satisfying a number of conditions. Now let $\pi\colon X \to S$ be a flat projective morphism. When does a collection of stability conditions σ_s on the derived categories of the fibers $D^b(X_s)$ form a *nice* family, that we can reasonably call a stability condition on X over S? Our goal is to answer this question.

1. Setting

Let $\pi\colon X \to S$ be a flat[1] projective[2] morphism, e.g. for S a scheme of finite type over a field.[3] To allow for stability conditions on semiorthogonal components of

[1] This is essential, e.g. as we frequently use base change arguments of the form $\mathrm{Hom}(E_s, F_s) = \left(\pi_* \mathcal{H}om(E,F)\right)_s$ for $E, F \in D^b(X)$ with E S-perfect.

[2] Unlike in the absolute case, we will need moduli spaces of semistable objects to be proper to get a satisfactory theory; thus the equivalent of working with compactly supported objects on a quasi-projective variety would require additional arguments.

[3] More precisely, we assume that S is Nagata, quasi-projective over a noetherian affine scheme, and X is noetherian affine scheme of finite Krull dimension; in particular, S can be a localisation of a finite type scheme over a field, or a Dedekind domain of mixed or infinite characteristic.

the fibers $D^b(X_s)$, we consider a semiorthogonal[4] component $\mathcal{D} \subset D^b(X)$ that is S-linear (i.e., preserved by tensoring with perfect objects on S).

Example 1.1. *If $\pi\colon X \to S$ is a family of smooth Fano varieties, we can consider*

$$\mathcal{D} = \{F \in D^b(X)\colon \pi_* \mathcal{H}om(O, F) = 0\}.$$

Here we can replace $\mathcal{O}$ by any relative exceptional object.

We write $\mathcal{D}_T$ and $\mathcal{D}_s$ for the base change of $\mathcal{D}$ for any morphism $T \to S$ or point $s \in S$, which exist by the S-linearity of $\mathcal{D}$. We write $\underline{\sigma} = (\sigma_s)_{s\in S}$ for a collection of stability conditions σ_s on $\mathcal{D}_s$ for all (closed or non-closed) points of S.

2. Base change for field extensions

We will frequently apply base change to such collections; this relies on base change of stability conditions for (not necessarily finite) field extensions.

If X is projective over a field, the *numerical K-group* $K_{\mathrm{num}}(\mathcal{D})$ of $\mathcal{D}$ is the quotient of $K(\mathcal{D})$ by the kernel of the Euler characteristic pairing $K(\mathrm{Perf}(\mathcal{D})) \times K(\mathcal{D}) \to \mathbb{Z}$, where $\mathrm{Perf}(\mathcal{D}) = \mathrm{Perf}(X) \cap \mathcal{D}$. A stability condition is numerical if its central charge factors via $K_{\mathrm{num}}(\mathcal{D})$.

Theorem 2.1 ([Sos12], [BLMNSP19, Theorem 12.17])**.** *Let $\sigma_k = (\mathcal{A}, Z)$ be a numerical stability condition on $\mathcal{D}$ over a field k, and let $k \subset \ell$ be any field extension. Then σ_k has a canonical base change to a stability condition $\sigma_\ell = (\mathcal{A}_\ell, Z_\ell)$ on $\mathcal{D}_\ell$.*

Here we recall that a stability condition can also be given via its associated *heart* $\mathcal{A} \subset D^b(X)$, the extension closure of $\bigcup_{\phi\in(0,1]} \mathcal{P}(\phi)$. Following Abramovich-Polishchuk [AP06, Pol07], $\mathcal{A}_\ell$ consists of objects $E \in \mathcal{D}_\ell$ whose associated object $E_k \in D_{\mathrm{qc}}(X)$ lies in the unbounded quasi-coherent version $\mathcal{A}_{\mathrm{qc}} \subset \mathcal{D}_{\mathrm{qc}}$ of the heart $\mathcal{A} \subset \mathcal{D}$. The central charge Z_ℓ is obtained by dualising the pull-back $K(\mathrm{Perf}(\mathcal{D}_k)) \to K(\mathrm{Perf}(\mathcal{D}_\ell))$ using $K_{\mathrm{num}}(\mathcal{D}_\ell)_{\mathbb{Q}} = K_{\mathrm{num}}(\mathrm{Perf}(\mathcal{D}))^\vee_{\mathbb{Q}}$.

3. Central charge in families

Consider two S-perfect objects $E, F \in D^b(X)$, from which we get objects $E_s, F_s \in \mathcal{D}_s$ for all $s \in S$. Whether E_s has bigger slope than F_s should not depend on the choice of s, and the same should hold after base change $T \to S$ for any connected scheme T. Therefore, the appropriate notion of a family of central charges is the following.

[4]We also assume that it is part of a semiorthogonal decomposition $D^b(X) = \langle \mathcal{D}_1, \mathcal{D}_1, \dots, \mathcal{D}_n\rangle$ where each inclusion $\mathcal{D}_i \to \mathcal{D}$ admits a right adjoint of finite cohomological amplitude.

Definition 3.1. *A collection $\underline{Z} = (Z_s)_{s \in S}$ of homomorphisms $Z_s \colon K_{\mathrm{num}}(\mathcal{D}_s) \to \mathbb{C}$ is a* central charge on $\mathcal{D}$ over S *if for every base change $T \to S$ and every T-perfect object $E \in \mathcal{D}_T$, the function*

$$t \mapsto Z_t(E_t)$$

is locally constant on T.

(Here Z_t is obtained as in Theorem 2.1 from Z_s for the image $s \in S$ of t.)

We will always fix a finite rank free abelian group Λ and group homomorphisms $v_t \colon K_{\mathrm{num}}(\mathcal{D}_t) \to \Lambda$ for all points t over S such that $v_t(E_t) \in \Lambda$ is locally constant in the above setting. Then we only consider central charges that factor via a common $Z \colon \Lambda \to \mathbb{C}$.

Example 3.2. *If $\pi \colon X \to S$ has a polarisation $\mathcal{O}_X(1)$, then any linear combination of the coefficients of the Hilbert polynomial with respect to $\mathcal{O}_X(1)$ gives a central charge on $\mathcal{D}$ over S. More generally, there is a pairing*

$$K_{S-\mathrm{Perf}}(\mathcal{D}) \times \bigoplus_{\overline{s} \in S} K(\mathcal{D}_{\overline{s}}) \to \mathbb{Z}$$

induced by $([E], [F_{\overline{s}}]) = \chi(E_{\overline{s}}, F_{\overline{s}})$ for $F_{\overline{s}} \in \mathcal{D}_{\overline{s}}$; we can set Λ to be the quotient of the right-hand side by the kernel of the pairing.

4. Openness

Recall that for slope- or Gieseker-stability, and a flat family of sheaves $E \in \mathrm{Coh}(X)$ over S, the set of fibers where E_s is *geometrically stable*[5] is open. In our situation, we instead use this property as a compatibility condition glueing stability conditions for different $s \in S$:

Definition 4.1. *A collection $\underline{\sigma}$ of stability conditions σ_s* universally satisfies openness of stability *if for every $T \to S$, and every T-perfect object $E \in \mathcal{D}_T$, the set*

$$\{t \in T \colon E_t \text{ is geometrically } \sigma_t\text{-stable}\} \subset T$$

is open.

Note that σ_t, and the notion of geometric σ_t-stability, are both given by Theorem 2.1. Due to a lack of intrinsic characterisation of σ_s-(semi)stable objects in the definition of stability conditions, Definition 4.1 seems the only reasonable compatibility condition between slicings on different fibers.

The existence of Harder-Narasimhan (HN) filtrations for the generic point, together with openness of stability, already implies the existence of *generic HN filtrations* for any S-perfect object $E \in \mathcal{D}$.

In the case of a product $X = X_0 \times S$, local constantness of central charges and openness of stability already guarantee that a collection of stability conditions is constant (under mild assumptions on S, see [BLMNSP19, Proposition 20.11]).

[5] stable after base change to the algebraic closure

5. Boundedness

We would like stability conditions $\underline{\sigma}$ on $\mathcal{D}$ over S to satisfy a version of Bridgeland's deformation theorem: given a small deformation $Z \rightsquigarrow Z'$ in $\operatorname{Hom}(\Lambda, \mathbb{C})$, there should be an associated deformation $\underline{\sigma} \rightsquigarrow \underline{\sigma'}$. Therefore, we need assumptions that ensure openness of stability in the sense of Definition 4.1 is preserved by such a deformation. Following [Bay19], it is sufficient to treat the case where the imaginary part of Z remains constant; then the heart $\mathcal{A}_s \subset \mathcal{D}_s$ of the stability condition σ_s remains constant for all $s \in S$, and a flat family E of objects $E_s \in \mathcal{A}_s$ is stable with respect to σ'_s at s if and only if E_s does not have a quotient in $\mathcal{A}_s$ that is destabilising with respect to Z'.

For any proof of such a statement, we first need to bound the set of objects that could destabilise E_s, independent of $s \in S$. This is done via two requirements. The first is a straightforward generalisation of the support property in the absolute case; here we fix Λ as in Section 3 and assume each Z_s is given by $Z \circ v_s$ for a common central charge $Z\colon \Lambda \to \mathbb{C}$.

Definition 5.1. *A collection $\underline{\sigma}$ of stability conditions σ_s on $\mathcal{D}_s$ satisfies the support property with respect to Λ if there exists a quadratic form Q on $\Lambda \otimes \mathbb{R}$ such that*

- *Q is negative definite on the kernel of Z, and*
- *for every s, and every σ_s-semistable object $E \in \mathcal{D}_s$, we have $Q(v_s(E)) \geq 0$.*

By completely elementary linear algebra, this guarantees that for a bounded subset $K \subset \mathbb{C}$, there are only finitely many classes $v \in \Lambda$ with $Z(v) \in K$ for which there exists $s \in S$ and a σ_s-semistable object $E \in \mathcal{D}_s$ with $v_s(E_s) = v$. Similarly, given a σ_s-semistable object E_s and a small deformation $Z \rightsquigarrow Z'$ as considered above, there exists a finite set of classes in Λ (depending only on Q, Z and Z') that could support a quotient $E_s \twoheadrightarrow Q$ destabilising E_s with respect to Z'.

Next, we need to bound the set of semistable objects of fixed class $v \in \Lambda$:

Definition 5.2. *A collection $\underline{\sigma}$ satisfies* boundedness *if for every $v \in \Lambda$ the stack $\mathcal{M}^{st}_{\underline{\sigma}}(v)$ of stable objects of class v is bounded: there exists a scheme B of finite type over S and a B-perfect object $E \in \mathcal{D}_B$ such that*

- *E_b is geometrically σ_b-stable with $v_b(E_b) = v$ for all $b \in B$, and*
- *for every geometric point $\overline{s} \to S$ and every $\sigma_{\overline{s}}$-stable object F with $v_{\overline{s}}(F) = v$, there is a factorisation $\overline{s} \to B \to S$ such that $E_{\overline{s}} = F$.*

Combining Definitions 5.1 and 5.2 with the existence of quot schemes, one can show that after a deformation $Z \rightsquigarrow Z'$, the set of geometrically Z'-stable objects is constructible in a flat family: the unstable locus is the image of a map from a finite union of finite type schemes.

6. Toward valuative criteria: base change to Dedekind schemes

To get from "constructible" to "open", we need to show that the unstable locus is closed under specialisation. For this, we need stronger assumptions after base change to a DVR;[6] slightly more generally, we make them after such a base change to an integral regular one-dimensional scheme, or *Dedekind scheme*, C.

Definition 6.1. *Assume that $S = C$ is a Dedekind scheme. For $\phi \in \mathbb{R}$, let $\mathcal{P}_C(\phi) \subset \mathcal{D}$ be the extension-closure of all $(i_c)_*\mathcal{P}_c(\phi)$ for closed points $c \in C$, and of objects $E \in \mathcal{D}$ such that $E_c \in \mathcal{P}_c(\phi)$ for all $c \in C$. We say that a $\underline{\sigma}$* integrates to a HN structure on $\mathcal{D}$ *if for every $E \in \mathcal{D}$ there exists a sequence of maps*

$$0 = E_0 \xrightarrow{f_1} E_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} E_n = E$$

such that the cone of f_i is in $\mathcal{P}_C(\phi_i)$ for some real numbers $\phi_1 > \phi_2 > \cdots > \phi_n$.

If $\underline{\sigma}$ satisfies all previous definitions, this is equivalent to the existence of a heart $\mathcal{A}_C \subset \mathcal{D}$ integrating the hearts $\mathcal{A}_c \subset \mathcal{D}_c$ associated to σ_c on the fibers, in the sense that $\mathcal{A}_c = (i_c)_*^{-1}(\mathcal{A})$ for all $c \in C$.

A HN structure strengthens the notion of generic HN filtrations, formally combining them with the existence of *semistable reduction*.

In particular, one can show that moduli stacks of semistable objects satisfy the existence part of the valuative criterion. Namely, if R is a DVR with fraction field K and residue field k, and $E_K \in \mathcal{P}_K(\phi) \subset \mathcal{D}_{\mathrm{Spec}K}$ is a σ_K-semistable object, we first consider an arbitrary extension $E_R \in \mathcal{D}_{\mathrm{Spec}R}$; using its HN filtration in the sense of Definition 6.1, we may assume $E_R \in \mathcal{P}_{\mathrm{Spec}R}(\phi)$; taking the quotient by the maximal R-torsion subobject we may assume that E_R is also flat over R; this is only possible if $(E_R)_k$ is σ_k-semistable, and so E_R induces the desired map from $\mathrm{Spec}R$ to the stack of semistable objects.

Combined with the existence of quot schemes, this also implies that in the situation of a small deformation $Z \rightsquigarrow Z'$ discussed in the previous section, the unstable locus is closed under specialisation.

7. Final Definition

Definition 7.1. *A stability condition on $\mathcal{D}$ over S is a collection $\underline{\sigma} = (\sigma_s)_{s\in S}$ of numerical stability conditions σ_s on $\mathcal{D}_s$ such that*

(1) the central charges Z_s are universally locally constant (Definition 3.1),
(2) $\underline{\sigma}$ universally satisfies openness of stability (Definition 4.1),
(3) satisfies the support property with respect to some Λ (Definition 5.1),
(4) satisfies boundedness (Definition 5.2), and
(5) after every base change $C \to S$ essentially of finite type to a Dedekind scheme C, the collection $\underline{\sigma}_C$ of fiberwise stability conditions on $\mathcal{D}_C$ (induced by Theorem 2.1) integrates to a HN structure on $\mathcal{D}_C$.

[6] This is where we use the assumption that S is Nagata: it is enough to verify valuative criteria after a base change to a DVR that is given by the localisation of a finite type morphism.

8. Results

Combining [Lie06, Tod08, AHLH18] we immediately obtain the following:

Theorem 8.1 ([BLMNSP19, Theorem 21.24]). *Let $\underline{\sigma}$ be a stability condition on $\mathcal{D}$ over S, and let $v \in \Lambda$. The moduli stack $\mathcal{M}_{\underline{\sigma}}(\mathcal{D}, v)$ of semistable objects of class v exists as an Artin stack of finite type over S. Moreover:*

- *In characteristic 0, it has a* good *moduli space $M_{\underline{\sigma}}(\mathcal{D}, v)$, an algebraic space proper over S.*
- *In arbitrary characteristic, when semistability and stability coincide, it has a coarse moduli space $M_{\underline{\sigma}}(\mathcal{D}, v)$, proper over S.*

The existence as an Artin stack follows from openness of stability and boundedness; the existence and properness of moduli spaces follows from HN structures over DVRs combined with the results and arguments in [AHLH18].

As sketched over Sections 5 and 6, openness of is preserved under deformations, which is the key ingredient for the analogue of the main result of [Bri07]:

Theorem 8.2 ([BLMNSP19, Theorem 22.2]). *The space* $\mathrm{Stab}_\Lambda(\mathcal{D}/S)$ *of stability conditions on $\mathcal{D}$ over S is a complex manifold, and the forgetful morphism* $\mathrm{Stab}_\Lambda(\mathcal{D}/S) \to \mathrm{Hom}(\Lambda, \mathbb{C})$*, given by $\underline{\sigma} \mapsto Z$, is a local isomorphism.*

9. Existence

While we don't have a general existence result, in practice we can construct stability conditions over S whenever we know a construction of stability conditions on the fibers: for families of surfaces (also in characteristic p [Kos20c]), Fano threefolds [Sch14, Li19a, BMSZ17], abelian threefolds [MP16, BMS16], threefolds with nef tangent bundles [Kos20a], quintic threefolds [Li19b], Calabi-Yau double/triple solids [Kos20b]. Similarly, it applies to Kuznetsov components of Fano threefolds or cubic fourfolds [BLMS17], or Gushel-Mukai fourfolds [PPZ19].

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Quasimaps to moduli spaces of sheaves

Denis Nesterov

1. Introduction

Let $(X, \mathfrak{X})$ be a pair, such that $\mathfrak{X}$ is an Artin stack locally of finite type and $X \subseteq \mathfrak{X}$ as an open substack of finite type. By $\mathcal{H}^2(\mathfrak{X})$ we denote some cohomology- like group associated to $\mathfrak{X}$, which is a subject to a choice, a reader can safely imagine it to be the second cohomology with integer coefficients $\mathrm{H}^2(\mathfrak{X}, \mathbb{Z})$. We comment on the choice of this group later in the remark.

Definition 1.1. *A map $f : C \to \mathfrak{X}$ is a quasimap to $(X, \mathfrak{X})$ of genus g and of degree $\beta \in \mathrm{Hom}(\mathcal{H}^2(\mathfrak{X}), \mathbb{Z})$ if*

- *C is a nodal connected curve of genus g;*
- *$f^* = \beta$;*
- *$|\{t \in C| \ f(t) \in \mathfrak{X}/X\}| < \infty$.*

A quasimap f is stable if

- *$\{t \in C| \ f(t) \in \mathfrak{X}/X\} \cap \{\text{nodes}\} = \emptyset$;*
- *C doesn't have rational tails;*
- *$|\mathrm{Aut}(f)| < \infty$.*

Let $QM_g(X, \mathfrak{X}, \beta)$ be the moduli stack of stable quasimaps of genus g and degree β to a pair $(X, \mathfrak{X})$. By $QM_C(X, \mathfrak{X}, \beta)$ we denote the moduli of quasimaps of degree β from a fixed curve C satisfying the first condition of the stability. We will suppress $\mathfrak{X}$ from the notation, when it is clear what pair is considered.

In [1] the moduli spaces of GIT quasimaps $QM_g(V/\!\!/G, [V/G], \beta)$ were defined, and the essential properties of the moduli spaces were proven, namely properness and existence of perfect obstruction theory. In [2], [3] statements of wall-crossing between Gromov-Witten invariants and quasimaps invariants for certain GIT quotients were established. In particular, one of the most striking applications of theory was the genus 0 wall-crossing for a quantic threefold, viewed as a quotient of its affine cone by $\mathbb{G}_m$-action, which coincided with the wall-crossing between Gromov-Witten invariants and B-model invariants of the quintic mirror. Hence GIT quasimaps provided a conceptual mathematical interpretation of an instance of Mirror Symmetry.

2. Quasimaps to moduli spaces of sheaves

Let's now consider the next most familiar pair, namely $(\mathcal{M}_v, \mathcal{C}oh_v(S))$, where $\mathcal{C}oh_v(S)$ is a moduli stack of coherent sheaves in a class v on a surface S, and $\mathcal{M}_v$ is a locus of Gieseker-stable sheaves for some ample line bundle. The stack $\mathcal{M}_v$ is a $\mathbb{G}_m$-gerb over a quasi-projective scheme M_v, which is given by quotienting out $\mathbb{G}_m$-automorphisms coming from the multiplication by scalars. The same can be done for $\mathcal{C}oh_v(S)$, the resulting *rigidified* stack is denoted by $\mathcal{C}oh_v(S)/\!\!/\mathbb{G}_m$. The moduli space M_v naturally embeds into $\mathcal{C}oh_v(S)/\!\!/\mathbb{G}_m$, giving rise to the following square

$$\begin{tikzcd} \mathcal{M}_v \arrow[r, hook] \arrow[d, "\mathbb{G}_m\text{-gerb}"] & \mathcal{C}oh_v(S) \arrow[d, "\mathbb{G}_m\text{-gerb}"] \\ M_v \arrow[r, hook] & \mathcal{C}oh_v(S)/\!\!/\mathbb{G}_m \end{tikzcd}$$

We are interested in the theory of quisimaps to a pair $(M_v, \mathcal{C}oh_v(S)/\!\!/\mathbb{G}_m)$, but quasimaps to $(\mathcal{M}_v, \mathcal{C}oh_v(S))$ are more accessible. We start with the latter.

Quasimaps to $\mathcal{H}ilb^n(S)$. For simplicity let $v = (1, 0, -n)$. We also assume that C is smooth and $h^{1,0}(S) = 0$. There is a natural open embedding defined by associating to a quasimap the corresponding family of sheaves,

$$QM_C(\mathcal{H}ilb^n(S), \beta) \hookrightarrow \mathcal{C}oh(S \times C)$$

$$f \mapsto \mathcal{F}_f.$$

It is not difficult to show that $\mathcal{F}_f$ is torsion-free of rank 1, hence stable. Moreover, the degree β determines the Chern character of $\mathcal{F}_f$ by pulling back the determinant line bundles on the stack $\mathcal{C}oh_v(S)$ and applying Hirzebruch-Riemann-Roch theorem in an appropriate way. We denote the corresponding Chern character by v_β. Conversely, if $\mathcal{F}$ is a torsion-free sheaf of rank 1 on $S \times C$ in the class v_β, then it defines a quasimap to $\mathcal{H}ilb^n(S)$ of degree β. We therefore obtain the following identification

$$QM_C(\mathcal{H}ilb^n(S), \beta) \cong \mathcal{M}_{v_\beta}(S \times C). \tag{1}$$

Remark 2.1. *At this point the above conclusions depend on what one means by $\mathcal{H}^2(\mathcal{C}oh_v(S))$. In this case we define it to be a subgroup generated by classes of determinant line bundles in* $\lim\limits_{U \subset \mathcal{C}oh_v(S)} \mathrm{H}^2(U, \mathbb{Z})$*, where the limit is taken over substacks of finite type $U \subset \mathcal{C}oh_v(S)$. This forces the Chern character of $\mathcal{F}_f$ to determine the degree β. In fact, the determinant line bundle construction is conjectured to be surjective on algebraic classes of a moduli space, for example, it is true for* $\mathrm{Hilb}^n(S)$*. Something similar in a certain sense can be expected for an entire stack of coherent sheaves.*

Quasimaps to $\mathrm{Hilb}^n(S)$**.** The moduli $QM_C(\mathrm{Hilb}^n(S), \beta)$ is related to the moduli $QM_C(\mathcal{H}ilb^n(S), \beta)$ via the following property of $\mathcal{C}oh_v(S) /\!\!/ \mathbb{G}_m$,

$$\pi_0(\mathcal{C}oh_v(S) /\!\!/ \mathbb{G}_m(B)) \cong \pi_0(\mathcal{C}oh_v(S)(B))/\mathrm{Pic}(B),$$

for any scheme of finite type B, where $\pi_0(\dots)$ stands for isomorphism classes of objects of the corresponding groupoids. The quotient

$$\pi_0(\mathcal{C}oh_v(S)(B)) \to \pi_0(\mathcal{C}oh_v(S)(B))/\mathrm{Pic}(B)$$

admits a canonical section given by setting the determinant of a sheaf to be trivial. In fact, this can be upgraded to a section $\mathcal{C}oh_v(S) /\!\!/ \mathbb{G}_m \to \mathcal{C}oh_v(S)$, which allows us to lift the quasimaps. Combining this observation with (1) we get the following identification

$$QM_C(\mathrm{Hilb}^n(S), \beta) \cong \mathrm{Hilb}_{v_\beta}(S \times C). \tag{2}$$

To treat the moduli $QM_g(\mathrm{Hilb}^n(S), \beta)$ in the similar fashion as above we define $\mathrm{Hilb}^s(S \times C) \subseteq \mathrm{Hilb}(S \times C)$ to be the locus of ideal sheaves, which are perfect and $|\mathrm{Aut}_C(\mathcal{I})| < \infty$, where $\mathrm{Aut}_C(\mathcal{I})$ are automorphisms of C that fix $\mathcal{I}$. Let $\mathcal{M}_g^{ss}$ be the moduli of semistable curves (nodal connected curves with no rational tails) with universal curve $\mathcal{C}_g$. Then (2) extends in the following form.

Theorem 2.2. *The moduli* $QM_g(\mathrm{Hilb}^n(S), \beta)$ *is a proper Deligne-Mumford stack with a perfect obstruction theory. Moreover,*

$$QM_g(\mathrm{Hilb}^n(S), \beta) \cong \mathrm{Hilb}^s_{v_\beta}(S \times \mathcal{C}_g / \mathcal{M}_g^{ss}).$$

Some parts of the above story generalise to higher ranks and to moduli spaces of sheaves on varieties of higher dimensions, we, however, restrain from spelling it out, as it would require some definitions.

3. Application

One of potential applications of the theory sketched above is *Igusa cusp form conjecture* proposed in [4]. For the statement of the conjecture we now assume that S is a K3 surface. We also fix a genus 1 curve E. Let F_{GW} be the generating series associated to Gromov-Witten invariants of $\mathrm{Hilb}^n(S)$ for the curve E in the classes with a primitive non-exceptional component for all $n > 0$. Let F_{PT} be the generating series associated to stable pairs invariants of $S \times E$ in the classes primitive in S. We refer the reader to [4] for more precise definitions of these generating series. Then Igusa cusp form conjecture is the following two equalities.

Conjecture 3.1.

$$-\frac{1}{\chi_{10}} \overset{1}{=} F_{PT} \overset{2}{=} F_{GW} + F,$$

where χ_{10} is Igusa cusp form, and F is explicitly defined.

The first equality was proven in the series of articles [5], [6]. The second equality remains open, but the theory of quasimaps sheds some light on it. Firstly, we can safely substitute stable pairs theory with Donaldson-Thomas theory for the second equality after changing the term F. Then in this setting quasimaps allow us to treat Gromov-Witten theory and Donaldson-Thomas theory on equal footing, since

$$\mathrm{Hilb}_{v_\beta}(S \times E) \cong QM_E(\mathrm{Hilb}^n(S), \beta),$$

which also holds on the level of obstruction theories in an appropriate sense. In particular, the second equality of the conjecture can now be restated as an assertion of existence of an explicit wall-crossing between Gromov-Witten invariants and quasimaps invariants. We expect to obtain this wall-crossing by adjusting the approach of [2], [3] to our needs. Such wall-crossing will, in particular, determine the fixed curve genus 1 Gromov-Witten theory of $\mathrm{Hilb}^n(S)$.

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Recent progress on the birational geometry of foliations on threefolds

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(joint work with Paolo Cascini, Calum Spicer)

We will always work over $\mathbb{C}$.

A foliation on a normal variety X is a coherent subsheaf $\mathcal{F} \subset T_X$ such that

(1) $\mathcal{F}$ is saturated, i.e. $T_X/\mathcal{F}$ is torsion free, and
(2) $\mathcal{F}$ is closed under Lie bracket.

The rank of $\mathcal{F}$ is its rank as a sheaf. Its co-rank is its co-rank as a subsheaf of T_X. The canonical divisor of $\mathcal{F}$ is any Weil divisor $K_{\mathcal{F}}$ such that $\mathcal{O}(K_{\mathcal{F}}) \cong \det(\mathcal{F})$.

In analogy with the classical case of a normal projective variety X where it is expected that the birational geometry X is governed by the positivity properties of the canonical bundle $\mathcal{O}_X(K_X)$, a similar principle holds for foliations. Indeed, for a pair $(X, \mathcal{F})$ of a normal projective X and a foliation $\mathcal{F} \subset T_X$, one would like to construct a birational model X' of X where the geometry of the strict transform of $\mathcal{F}$ becomes particularly simple. As in the classical case, the way to construct such "simpler" birational models of the pair $(X, \mathcal{F})$ should rely on a careful analysis of the positivity properties of the canonical bundle of the foliation $\mathcal{O}_X(K_{\mathcal{F}})$.

In low dimension, the birational classification of foliations has seen many important advancements in recent year:

- for surfaces, there now is a very exhaustive and effective picture of the classification of rank 1 foliations, [6, 2, 7, 8];
- in dimension three, several foundational steps have been established towards a full classification both in the case of rank 1, [1, 5], as well as rank 2 foliations on algebraic threefolds, [10, 4, 11].

In dimension greater than 3, the analogue problem is still quite obscure for several concurring reasons, e.g., the lack of an analogue of resolution of singularities in this context.

The aim of this report is to focus on the new advancements in the birational classification of rank 2 foliations on threefolds.

1. The foliated minimal model program

We will be working with a slightly more comprehensive framework than the one just introduced: namely, we will consider a co-rank 1 foliation $\mathcal{F}$ on a normal algebraic threefold X, and effective divisor Δ on X with coefficients in $\mathbb{R}_{\geq 0}$, such that $K_{\mathcal{F}} + \Delta$ is $\mathbb{R}$-Cartier. The latter condition is necessary in order to be able to discuss intersection numbers for $K_{\mathcal{F}} + \Delta$.

Given a triple $(X, \mathcal{F}, \Delta)$ as above, one would like to construct suitable birational models where the geometry of the triple is as simple as possible. The guiding light in this quest should be the positivity of $K_{\mathcal{F}} + \Delta$ which is measured in terms of the positivity of the intersections of $K_{\mathcal{F}} + \Delta$ with complete curves contained in X.

In analogy with the classical Minimal Model Program (in short, MMP), we expect 2 different types of outcomes. Given a triple $(X, \mathcal{F}, \Delta)$, where X projective, and the singularities of the triple are mild – see below for more on singularities – we would like to algorithmically construct a triple $(X', \mathcal{F}', \Delta')$, and a birational contraction $\pi\colon X \dashrightarrow X'$ such that $\mathcal{F}'$ (resp. Δ') is the strict transform of $\mathcal{F}$ (resp. Δ) under π and:

(1) either $K_{\mathcal{F}'} + \Delta'$ is nef on X', that is, $(K_{\mathcal{F}'} + \Delta') \cdot C \geq 0$ for any complete curve $C \subset X'$; or
(2) X' is covered by rational curves that have negative intersection with $K_{\mathcal{F}'} + \Delta'$.

In contructing X' (and thus, π), we would like to preserve the geometric data encoded in the triple $(X, \mathcal{F}, \Delta)$: in particular, we do not want to alter the linear systems $|m(K_{\mathcal{F}} + \Delta)|$, as those carry many important geometric information about $\mathcal{F}$. In view of this, it follows that case (1) in the above dichotomy should correspond to the case where $K_{\mathcal{F}} + \Delta$ is pseudoeffective, while case (2) corresponds to the non-pseudoeffective case.

A triple $(X', \mathcal{F}', \Delta')$ corresponding to an outcome described in (1) above is called a *minimal model* of $(X, \mathcal{F}, \Delta)$, while it is called a *Mori fibre space* when it corresponds to an outcome described in (2).

The classic starting point in the birational classification of higher dimensional algebraic varieties is the quest for a smooth representative in every birational equivalence class. For foliations, it is not hard to see that this question already has a negative answer for rank 1 foliations on surfaces, cf. [2]. For the purpose of the birational classification, the class of *simple singularities* is the correct analogue of a smooth model in the classical case of the birational classification of algebraic varieties, cf. [4, Definition 2.7] for the precise definition. In dimension 2 and 3, it is proven that for any foliated pair $(X, \mathcal{F})$, where $\mathcal{F}$ has co-rank 1, there always exists a birational model where the strict transform of $\mathcal{F}$ has simple singularities, see [9, 3].

On the other hand, the class of simple singularities is not stable under any meaningful class of birational transformation; hence, it is important to identify a suitable class of singularities that works well for our own purpose. The right class of foliated singularities to consider is that of foliated divisorial log terminal (in short, F-dlt) singularities, an analogue in the category of foliated spaces of that of divisorial log terminal singularities in the MMP, cf. [4, § 3] for the precise definition and more details. F-dlt singularities can be nicely characterized in terms of discrepancy of their log canonical divisor; they contain simple singularities and it can be shown that they are stable under the type of birational transformations that are used in the foliated version of the MMP, cf. next section. Hence, form this point of view, they are the most natural class of singularities that we should consider if we want to work with foliated spaces with simple singularities and classify those.

2. MMP for rank 2 foliations on threefolds

To simplify the notation for triples $(X, \mathcal{F}, \Delta)$, we will omit X and just write $(\mathcal{F}, \Delta)$. We will assume that our pairs $(\mathcal{F}, \Delta)$ have F-dlt singularities and explain how to proceed algorithmically to produce a triple $(X', \mathcal{F}', \Delta')$ which is either a minimal model or a Mori fibre space for $(\mathcal{F}, \Delta)$.

The starting input of our algorithmic construction is an F-dlt pair $(\mathcal{F}, \Delta)$ – for example, we could start with a foliation $\mathcal{F}$ with simple singularities on X.

If $K_{\mathcal{F}} + \Delta$ is nef, then our algorithm can stop immediately, as $(\mathcal{F}, \Delta)$ is its own minimal model. On the other hand, if $K_{\mathcal{F}} + \Delta$ is not nef, the following result provides a way forward in our quest for minimal models or Mori fibre spaces.

Theorem 2.1. [10, 4] *Let X be a normal projective threefold and let $\mathcal{F}$ be a co-rank one foliation. Suppose that (X, D) is klt for some $D \geq 0$. Let $(\mathcal{F}, \Delta)$ be a F-dlt pair and let H be an ample $\mathbb{Q}$-divisor.*
Then there exist countable many curves $C_1, C_2, \ldots$ such that

$$\overline{NE(X)} = \overline{NE(X)}_{K_{\mathcal{F}}+\Delta\geq 0} + \sum \mathbb{R}_+[C_i].$$

Furthermore, for each i, C_i is a rational curve tangent to $\mathcal{F}$ such that $(K_{\mathcal{F}} + \Delta) \cdot C_i \geq -6$, and if $C \subset X$ is a curve such that $[C] \in \mathbb{R}_+[C_i]$ then C is tangent to $\mathcal{F}$. In particular, there exist only finitely many $(K_{\mathcal{F}} + \Delta + H)$-negative extremal rays.

Given a $K_{\mathcal{F}} + \Delta$-negative extremal ray $R \subset \overline{NE}(X)$, we can look at the set $\mathrm{loc}(R)$ of all points $x \in X$ such that there exists a curve C with $x \in C$ and $[C] \in R$. We have the following three distinct possibilities:

- if $\mathrm{loc}(R) = X$, then [10, Theorem 8.9] implies that R is K_X-negative and so there exists a contraction $f\colon X \to Y$ with $\dim X > \dim Y$. Thus, X is covered by rational curves that have negative intersection with $K_{\mathcal{F}} + \Delta$, and it is a Mori fibre space; in this case we can stop our algorithm at this stage.
- If $\mathrm{loc}(R) = D$ is a divisor, then it is shown in [10, 4] that D can be contracted by means of a birational contraction $f\colon X \to Y$. Such a morphism f is called a divisorial contraction; moreover, f preserve the linear systems $|m(K_{\mathcal{F}} + \Delta)| = |m(K_{\mathcal{F}_Y} + \Delta_Y)|$. In this case, we substitute $(X, \mathcal{F}, \Delta)$ with $(Y, \mathcal{F}_Y, \Delta_Y := f_*\Delta)$, where $\mathcal{F}_Y$ is the strict transform of $\mathcal{F}$, and repeat our algorithm starting with this new triple.
- If $\mathrm{loc}(R) = C$ is a curve, then there exists a birational contraction $f\colon X \to Y$ whose exceptional locus is Y. Such an f is called a flipping contraction. In this case, $f_*(K_{\mathcal{F}} + \Delta)$ ceases to be $\mathbb{R}$-Cartier, as f is a small contraction but $K_{\mathcal{F}} + \Delta$ intersects C negatively. Hence, we cannot just substitute X with Y and $(\mathcal{F}, \Delta)$ with their strict transforms, as we would not be able to measure the positivity of $K_{\mathcal{F}} + \Delta$ anymore on Y.

In all the of cases above, by Theorem 2.1, the morphism f contracts curves tangent to the foliation, thus, f is equivariant with respect to the foliation.

In the last case above, that of a so-called flipping contraction, can be remedied by means of the so-called flip of $f\colon X \to Y$. A flip is nothing more than the following diagram of birational maps

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr, "f"'] & & X^+ \arrow[dl, "f^+"] \\ & Y & \end{tikzcd}$$

where $f^+ \colon X^+ \to Y$ is also a birational contraction whose exceptional locus has dimension 1 and $K_{\mathcal{F}^+} + \Delta^+$ is $\mathbb{R}$-Cartier and it has positive intersection with all curves contracted by f^+, where $\mathcal{F}^+$ (resp. Δ^+) is the strict transform of $\mathcal{F}$ (resp. Δ).

Theorem 2.2. [4] *Let $\mathcal{F}$ be a co-rank one foliation on a $\mathbb{Q}$-factorial projective threefold X. Let $(\mathcal{F}, \Delta)$ be a F-dlt pair on X.*
Let $\phi \colon X \to Y$ be a $(K_{\mathcal{F}} + \Delta)$-flipping contraction. Then the $(K_{\mathcal{F}} + \Delta)$-flip exists.

As $K_{\mathcal{F}^+} + \Delta^+$ is $\mathbb{R}$-Cartier, we can still discuss its intersection properties. Moreover, it is possible to prove that when making a flip we have the equality of linear systems $|m(K_{\mathcal{F}} + \Delta)| = |m(K_{\mathcal{F}^+} + \Delta^+)|$ and so we can substitute $(X, \mathcal{F}, \Delta)$ with $(X^+, \mathcal{F}^+, \Delta^+)$ and restart our analysis as we did above.

Theorem 2.2 is a delicate and fundamental result which relies on a careful analysis of the singularities of $\mathcal{F}$, together with an ingenious argument based on Artin's approximation theorem that is used to produce algebraic approximations to the (possibly formal/trascendental) separatrices[1] of $\mathcal{F}$ around $\mathrm{loc}(R)$.

As for a divisorial contraction $f \colon X \to Y$, the rank of the Picard group of X is strictly greater than that of Y, it follows that when running our algorithm, we can just produce a finite number of divisorial contractions. The same type of result is not a priori clear for the case of flipping contractions and flips. Using the Special Termination for foliated pairs proved in [4] and extending the Bott connection to the case of foliated pairs with terminal singularities, it has been proven in [11] that there cannot be infinite sequences of flips.

Theorem 2.3. [11] *Let X be a $\mathbb{Q}$-factorial quasi-projective threefold. Let $(\mathcal{F}, \Delta)$ be an F-dlt pair. Then starting at $(\mathcal{F}, \Delta)$ there is no infinite sequence of flips.*

Thus, all of the results contained in Theorem 1-3 can be summarized in the following final theorem, which can be summarized by saying that "the Minimal Model Programme terminates for co-rank 1 F-dlt foliated pairs on projective threefolds".

Theorem 2.4 (MMP for rank 2 foliations on threefolds)**.** *Let X be a $\mathbb{Q}$-factorial quasi-projective threefold. Let $(\mathcal{F}, \Delta)$ be an F-dlt pair. Then there exists a $(K_{\mathcal{F}} + \Delta)$-negative birational contraction $\pi \colon X \to X'$ and an F-dlt pair $(\mathcal{F}', \Delta')$ on X' such that:*

(1) *either, $K_{\mathcal{F}'} + \Delta'$ is nef; or,*
(2) *there exists a contraction $f' \colon X' \to Y$, with $\dim X' > \dim Y$ and $K_{\mathcal{F}'} + \Delta'$ is ample along the fibers of f'.*

[1]A separatrix is an invariant hypersurface for the foliation F that contains a singular point of the foliation.

In [11], a large suite of applications of the existence of the MMP is shown in the guise of an analysis of local and global properties of foliations on threefolds. The authors study foliation singularities proving the existence of first integrals for isolated canonical foliation singularities, an extension of Malgrange's theorem to the singular case, and derive a complete classification of terminal foliated threefolds singularities. They show the existence of separatrices for log canonical singularities. They also prove some hyperbolicity properties of foliations, showing that the failure of the canonical bundle to be nef implies the existence of entire holomorphic curves contained in the open strata of a natural stratification of the singular locus of the foliation.

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A survey on automorphisms of irreducible holomorphic symplectic manifolds I

Alessandra Sarti

The aim of the talk is to give an overview of recent results on the automorphisms group and on the birational transformations group of irreducible holomorphic symplectic manifolds and to formulate open questions.

1. Basic Facts/Questions

Let X be an irreducible holomorphic symplectic (IHS) manifold, i.e. a compact complex manifold which is simply connected and which carries a unique (up to scalar) global holomorphic 2-form ω_X which is everywhere non–degenerate. We study here $\mathrm{Aut}(X)$ the group of biregular transformations of X. We will also study some properties of $\mathrm{Bir}(X)$ the group of birational transformations of X. More precisely consider $G \subset \mathrm{Aut}(X)$ a finite subgroup. Then we have a group homomorphism

$$\alpha : G \longrightarrow \mathbb{C}^*,\ g \mapsto \alpha(g)$$

defined by $g^*\omega_X = \alpha(g)\omega_X$ with $\alpha(g) \in \mathbb{C}^*$. Since G is finite, $\alpha(G)$ is a finite subgroup of $\mathbb{C}^*$ hence it is cyclic. This means that it exists $m \in \mathbb{Z}_{>0}$ such that

$$\alpha(G) \cong \mathbb{Z}/m\mathbb{Z} \cong \mu_m$$

where μ_m denotes the group of m–th roots of the unity. We get an exact sequence:

$$1 \longrightarrow G_0 \longrightarrow G \xrightarrow{\alpha} \mu_m \longrightarrow 1$$

and for each $g \in G_0$ we have $g^*\omega_X = \omega_X$. This motivates the following

Definition 1.1. *We call $g \in G$ a* symplectic *automorphism if $g^*\omega_X = \omega_X$, we call g a* non–symplectic *automorphism otherwise, i.e. if there exists $\zeta_m \in \mathbb{C}^*$ an* m*–root of unity such that $g^*\omega_X = \zeta_m\omega_X$. If the order $o(g)$ of g is equal to m we say that g is a* purely non–symplectic *automorphism.*

Remark 1.2. Obviously if $o(g) = p$ a prime number then a non–symplectic automorphism is the same as a purely non–symplectic automorphism.

The study of automorphisms of irreducible holomorphic symplectic manifolds was started in the 80's by Nikulin in [16] for K3 surfaces and then generalized by Beauville in [2] to the higher dimension. The first natural question one wants to answer is the following

Question: How big can be a finite group G acting on X? More precisely one wants to describe groups G of maximum order acting on an IHS manifold X.

The complete answer was given only recently for K3 surfaces and it remains unknown for IHS manifolds of higher dimension. A classic result of Kondo in 1999 [14] shows that if G acts on a K3 surface then $|G| \leq 3840$ and there exists a unique pair (X_{Ko}, G_{Ko}) with $|G_{Ko}| = 3840$ and $X_{Ko} = \mathrm{Km}(E_i \times E_i)$ the Kummer K3 surface associated to the product of the elliptic curve E_i with itself, where E_i is the elliptic curve $\mathbb{C}/\mathbb{Z} \oplus i\mathbb{Z}$. Observe that here G contains the Mathieu group M_{20}, which is the group of maximum order acting symplectically on a K3 surface as shown by Mukai in [15]. In that paper Mukai gives a list of the 11 maximum groups of maximum order acting symplectically on a K3 surface (projective or not). Observe that $|G_{Ko}| = 4 \cdot 960$ where recall that $|M_{20}| = 960$. The classification of groups of maximum order acting on a K3 surface was completed only recently and it turns out that these groups all contain M_{20} acting symplectically. The following theorem was shown by C. Bonnafé and the author in [8] and independently by S. Brandhorst and K. Hashimoto in [9]:

Theorem 1.3. *Let X be a K3 surface and let G be a finite group acting on it. Assume moreover that G contains M_{20} acting symplectically then there are exactly* *three possibilities for* (G, X)*:*

$$(G_{Ko}, X_{Ko}),\ (G_{Mu}, X_{Mu}),\ (G_{BH}, X_{BH}),$$

where $|G_{Mu}| = |G_{BH}| = 1920 = 2 \cdot 960$ *and* X_{Mu} *and* X_{BH} *are Kummer surfaces.*

We do not give here a precise description of (G_{Mu}, X_{Mu}) and (G_{BH}, X_{BH}) but we send the reader back to the paper [8], where the authors describe projective models of the K3 surfaces and give the matrices which generate the groups. Observe that in [9] the authors classify more in general all the finite groups acting on a K3 surface and containing one of the 11 maximal groups described by Mukai in [15].

Before to mention further problems and the state of the art, recall that the Markmann–Verbitsky Torelli theorem, allows to describe automorphisms of IHS manifolds by using the properties of lattices, in fact $H^2(X, \mathbb{Z})$ is a lattice, i.e. a free $\mathbb{Z}$–module of signature $(3, b_2(X) - 2)$ with the Beauville–Bogomolov–Fujiki intersection form. We mention now two main problems that were studied these last years about automorphisms of IHS manifolds:

(1) Classify IHS manifolds with an action by $G \subset \mathrm{Aut}(X)$ a finite group: describe the fixed locus, describe the invariant sublattice $H^2(X, \mathbb{Z})^G$ and its orthogonal complement $(H^2(X, \mathbb{Z})^G)^\perp$, describe the moduli spaces, etc. Here one main ingredient is *lattice theory*.
(2) Study $\mathrm{Aut}(X)$, $\mathrm{Bir}(X)$ when the rank of the Picard group is small. Here one main ingredient is the knowledge of the cones $\mathrm{Mov}(X)$, $\mathrm{Amp}(X)$ and $\mathrm{Nef}(X)$.

Of course there is not a wall between these two problems and lattice theory remains an important tool also to attack the second problem. Moreover in each case one interesting task is to produce geometric examples of IHS manifolds with an automorphism of finite order. This remains in general a difficult goal since Torelli theorem can provide the existence of the automorphism, but it does not give any information on the geometric realization.

One starts the study of the two problems by dividing the IHS manifolds in the four deformation types: $K3^{[n]}$, $Kum_n(A)$, OG_{10} and OG_6, which denote respectively the Hilbert scheme of n points on a K3 surface, the generalized Kummer manifold of an abelian surface, the O'Grady's manifolds. Most of the results are obtained for $K3^{[n]}$ and its deformations, as we will see in the sequel. I will start with the second problem, even if it was considered only recently, some time after the study of the first one began.

2. The groups $\mathrm{Aut}(X)$ and $\mathrm{Bir}(X)$ when the rank of $\mathrm{Pic}(X)$ is small

The first easy question is the following:

Question: Assume the rank of $\mathrm{Pic}(X)$ is one, i.e. $\mathrm{Pic}(X) = \mathbb{Z}H$ with $H^2 > 0$, can we compute $\mathrm{Aut}(X)$ and $\mathrm{Bir}(X)$?

We recall first some general facts: since $0 = h^0(X, T_X) = \dim \mathrm{Aut}(X)$ we get that $\mathrm{Aut}(X)$ is a discrete group. Moreover since we assume here $\mathrm{Pic}(X) = \mathbb{Z}H$ with

$H^2 > 0$ then H is ample and necessarily an automorphism $\sigma \in \mathrm{Aut}(X)$ is such that $\sigma(H) = H$. Hence $\mathrm{Aut}(X)$ is a subgroup of some compact Lie group, this means that it is finite. Moreover an element $f \in \mathrm{Bir}(X)$ is also forced to preserve the ample class H hence by [13, Section 27.3] it is an automorphism. This means that $\mathrm{Aut}(X) = \mathrm{Bir}(X)$ is finite. In [6, Theorem 3.1] it is shown the following result in the case that X is deformation equivalent to $K3^{[2]}$ (we say that X is of type $K3^{[2]}$):

Theorem 2.1. *Let X be a very general IHS manifold of type $K3^{[2]}$ and assume that* $\mathrm{Pic}(X) \neq \mathbb{Z}H$ *with* $H^2 = 2$, *then* $\mathrm{Aut}(X) = \{id\}$.

In fact if $\mathrm{Pic}(X) = \mathbb{Z}H$ with $H^2 = 2$ (we just write $\mathrm{Pic}(X) = \langle 2 \rangle$) then $\mathrm{Aut}(X) = \mathbb{Z}/2\mathbb{Z}$ and X is a double EPW sextic. This is an IHS manifold introduced by O'Grady, which is the double cover of an EPW sextic, a special sextic hypersurface in $\mathbb{P}^5$. Hence in this case X admits in a natural way an involution which is non–symplectic. The other exceptions are for $\mathrm{Pic}(X) = \langle 46 \rangle$, then in [4] it is shown that there exists a unique IHS manifold X with a non–symplectic automorphism of order 23. If $\mathrm{Pic}(X) = \langle 6 \rangle$ then in [5] it is shown that it exists a 10–dimensional family of Fano varieties of lines of cubic fourfolds which admit a non–symplectic automorphism of order three.

If X is of type $K3^{[n]}$ with $n \geq 3$, the Theorem 2.1 was generalized recently by O. Debarre in [11, Proposition 4.3], but to my knowledge a similar result is not known for the others deformation types of IHS manifolds.

We consider now the case of the rank of $\mathrm{Pic}(X)$ equal to 2. Again one has several results for the case where X is of type $K3^{[n]}$, more precisely for $X = S^{[n]}$ where S is a K3 surface. Let $\mathrm{Pic}(S) = \mathbb{Z}H$ with $H^2 = 2t$ by abuse of notation we denote again by H the class induced on $S^{[n]}$ and we have

$$\mathrm{Pic}(X) = \mathbb{Z}H \oplus \mathbb{Z}\delta$$

where $\delta^2 = -2(n-1)$ and δ is the exceptional divisor coming from the Hilbert–Chow morphism. Again we ask:

Question: Can we describe $\mathrm{Aut}(S^{[n]})$, $\mathrm{Bir}(S^{[n]})$ when the rank of the Picard group is the smallest possible?

The answer for $\mathrm{Aut}(S^{[2]})$ is described in the following Theorem of [7], we keep the same notations as before:

Theorem 2.2.
1. *If $t = 1$ then* $\mathrm{Aut}(S^{[2]}) = \mathbb{Z}/2\mathbb{Z} = \langle \iota^{[2]} \rangle$. *Where $\iota^{[2]}$ is the involution induced on $S^{[2]}$ by the involution ι on S, which is called a* natural involution.
2. *If $t > 1$ then the following are equivalent*
   - *(a)* $\mathrm{Aut}(S^{[2]}) = \mathbb{Z}/2\mathbb{Z}$
   - *(b) there exists an ample class $D \in$* $\mathrm{Pic}(S^{[2]})$, *such that $D^2 = 2$.*
   - *(c) t is not a square and Pell's equation $P_t(-1) : x^2 - ty^2 = -1$ has a solution and Pell's equation $P_{4t}(5) : x^2 - 4ty^2 = 5$ has no solution.*

The last condition on $P_{4t}(5)$ is related to the study of $\mathrm{Amp}(S^{[2]})$ by A. Bayer and E. Macrì in [1]. Then in 2019 O. Debarre and E. Macri have shown in [12] that in the cases of Theorem 2.2 we have $\mathrm{Aut}(S^{[2]}) = \mathrm{Bir}(S^{[2]})$, moreover they show that there are values of t such that $\mathrm{Aut}(S^{[2]}) = \{id\}$ but $\mathrm{Bir}(S^{[2]}) = \mathbb{Z}/2\mathbb{Z}$ and they describe precisely such t's by using Pell's equations. Recently Al. Cattaneo in [10] has formulated a similar result as Theorem 2.2 for $S^{[n]}$, $n \geq 3$. In particular he has shown

Theorem 2.3. *(1) If* $t = 1$ *then* $\mathrm{Aut}(S^{[n]}) = \mathbb{Z}/2\mathbb{Z} = \langle \iota^{[n]} \rangle$. *Where* $\iota^{[n]}$ *is the natural involution.*
(2) If $2 \leq t \leq 2n - 3$ *then* $\mathrm{Aut}(S^{[n]}) = \{id\}$ *for* $n \geq 3$.

Finally in [3] P. Beri and Al. Cattaneo generalize the description of O. Debarre and E. Macrì to $\mathrm{Bir}(S^{[n]})$, $n \geq 3$. The interesting fact here is that there are cases where $\mathrm{Bir}(S^{[n]}) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ but $\mathrm{Aut}(S^{[n]}) = \mathbb{Z}/2\mathbb{Z}$. This situation is new and not possible if $n = 2$.

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A survey on automorphisms of irreducible holomorphic symplectic manifolds II

Samuel Boissière

1. Classification of automorphisms on IHS manifolds

Starting from an irreducible holomorphic symplectic (IHS) manifold X with a biregular automorphism σ, we associate an isometry of an abstract lattice as follows. The second cohomology space with integer coefficients $\mathrm{H}^2(X,\mathbb{Z})$ is a nondegenerate lattice for the Beauville–Bogomolov–Fujiki integral quadratic form. Denote by L a representative of the isometry class of this lattice and fix an isometry between $\mathrm{H}^2(X,\mathbb{Z})$ and L. The isometry σ^* induced by σ on $\mathrm{H}^2(X,\mathbb{Z})$ defines an isometry $\rho \in \mathrm{O}(L)$ which is well-defined up to conjugation. By the global Torelli theorem of Markman–Verbitsky [7], we can reconstruct σ from ρ, but not always uniquely: for this, we need the faithfulness of the representation $\mathrm{Aut}(X) \to \mathrm{O}\left(\mathrm{H}^2(X,\mathbb{Z})\right)$, $\sigma \mapsto (\sigma^*)^{-1}$.

However, in practice the classification process of pairs (X,σ), using lattice theory and Smith theory, provides less than ρ but rather the data of two primitive sublattices of L: the invariant lattice $T := L^\rho$ and its orthogonal complement $S := T^\perp$. It may happen, although not so often, that the data (T,S) corresponds to several possible automorphisms. The challenge is then to understand their relationship.

In this talk, I fix the following setup: X is an IHS manifold deformation equivalent to the Hilbert square of a K3 surface, or equivalently to the Fano variety of lines on a smooth cubic fourfold; σ is an order 3 automorphism acting nonsymplectically. Denoting by ω_X the holomorphic 2-form of X, we have $\sigma^*\omega_X = \zeta\omega_X$, with $\zeta := \mathrm{e}^{2\mathrm{i}\pi/3}$.

In the classification tables given in [4], I extract three most stimulating cases whose geometrical properties are sketched in [4, 5]:

(1) $T = \langle 6\rangle$, $S = U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus A_2(-1)$;
(2) $T = U \oplus A_2(-1)^{\oplus 5} \oplus \langle -2\rangle$, $S = U \oplus U(3) \oplus A_2(-1)^{\oplus 3}$;
(3) $T = U(3) \oplus E_6^\vee(-3) \oplus \langle -2\rangle$, $S = U \oplus U(3) \oplus A_2(-1)^{\oplus 5}$.

Case (2) is part of a work in progress with Davide Veniani and Annalisa Grossi. In this talk, I present some recent results on case (1), obtained in collaboration with Chiara Camere and Alessandra Sarti.

2. Moduli and periods

I consider a couple (X, σ) as above, with $\mathrm{H}^2(X, \mathbb{Z})^{\sigma^*} \cong \langle 6 \rangle = T$ primitively embedded in L. Since σ acts nonsymplectically, the Hodge structure of X is characterized by the line $\mathrm{H}^{2,0}(X)$, which lives in the ζ-eigenspace of the complexification of the lattice S, denoted $S_{\mathbb{C}}(\zeta)$. This gives a point in a period domain which is the 10-dimensional complex ball: $\{\omega \in \mathbb{P}(S_{\mathbb{C}}(\zeta)) \mid \langle \omega, \bar{\omega} \rangle > 0\} \cong \mathbb{B}^{10}$.

As a special case of a construction given in [5], I define a moduli space $\mathcal{N}^{\rho}_{\langle 6 \rangle}$ parametrizing pairs (X, σ) realizing the data (T, ρ, S) and a holomorphic map:

$$\mathcal{P} \colon \mathcal{N}^{\rho}_{\langle 6 \rangle} \longrightarrow \frac{\mathbb{B}^{10} \setminus \mathcal{H}}{\Gamma},$$

where Γ is the group of monodromies of X fixing the sublattice T and $\mathcal{H}$ is the union of the hyperplanes $\delta^{\perp}$ where $\delta \in S$ is an MBM class in the sense of Amerik–Verbistky [2].

3. What makes the case $T = \langle 6 \rangle$ so interesting?

I realize this action geometrically by taking a smooth cubic threefold $C \subset \mathbb{P}^4$ and the triple covering $Y \subset \mathbb{P}^5$ of $\mathbb{P}^4$ branched along C. Denote by X the Fano variety of lines of Y and by σ the automorphism induced on it by the covering automorphism. We have $(X, \sigma) \in \mathcal{N}^{\rho}_{\langle 6 \rangle}$. I will explain the proof of the following result, which is deeply connected to a work of Allcock–Carlson–Toledo [1].

Theorem 3.1. [6] *Every element in $\mathcal{N}^{\rho}_{\langle 6 \rangle}$ is a Fano variety of lines on a cyclic cubic fourfold with automorphism induced by the covering one.*

4. Degenerations and limit points

When the period point ω goes to a wall $\delta^{\perp}$, in particular in the case of a nodal degeneration, the representation ρ cannot be realized by an automorphism of an IHS manifold anymore. In order to define a limit pair (X_0, σ_0) in the moduli space of $\langle 6 \rangle$-polarized IHS manifolds, under a nodal degeneration C_0 of the cubic threefold C, we observe that generically, the Fano variety of lines on its cyclic covering Y_0 is singular along a smooth K3 surface Σ. This relates to a family of K3 surfaces with an order three automorphism τ studied in [3]. I will explain in which sense the definition $(X_0, \sigma_0) := (\Sigma^{[2]}, \tau^{[2]})$ is a holomorphic extension of the period map $\mathcal{P}$ to the nodal walls.

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Special surfaces in special cubic fourfolds

Emanuele Macrì
(joint work with Arend Bayer, Aaron Bertram, Alexander Perry)

In this talk, I reported on work in progress on a possible characterization of Hassett divisors on the moduli space of cubic fourfolds by the property of containing special surfaces. I sketched the construction of such special surfaces for infinitely many divisors and the relation with the work of Russo and Staglianò on rationality of such cubics in low discriminant.

1. The Main Theorem

Let $Y \subset \mathbb{P}^5$ denote a complex smooth cubic fourfold and let $h := [\mathcal{O}_Y(1)]$ be the class of a hyperplane section. By following [7], we say that Y is *special* of discriminant d, and use the notation $Y \in \mathcal{C}_d$, if there exists a surface $\Sigma \subset Y$ not homologous to a complete intersection such that the determinant of the intersection matrix

$$\begin{pmatrix} h^2 & h.\Sigma \\ h.\Sigma & \Sigma^2 \end{pmatrix}$$

is equal to d. The locus $\mathcal{C}_d$ is non-empty if and only if $d \equiv 0, 2 \pmod 6$ and $d > 6$; moreover, in such a case, $\mathcal{C}_d$ is an irreducible divisor in the moduli space of cubic fourfolds, which can also be described purely in terms of Hodge theory and periods (by the Global Torelli Theorem [17] and the surjectivity of the period map [10, 13]; we refer to the book in progress [8] for the general theory of cubic fourfolds).

The main result gives an actual surface defining the divisor $\mathcal{C}_d$, for special values of d.

Theorem 1.1. *Let $a \geq 1$ be an integer and let $d := 6a^2 + 6a + 2$. Let Y be a general cubic fourfold in $\mathcal{C}_d$. Then there exists a surface $\Sigma \subset Y$ such that*

- $\deg(\Sigma) := h.\Sigma = 1 + \frac{3}{2}a(a+1)$ *and* $\Sigma^2 = \frac{d+\deg(\Sigma)^2}{3}$;
- $H^*(Y, \mathcal{I}_\Sigma(a-j)) = 0$, *for all* $j = 0, 1, 2$.

For discriminant d as in Theorem 1.1, by [1, 2] there is a polarized K3 surface S of degree d associated to each cubic fourfold in $\mathcal{C}_d$. To be precise the surface Σ is not unique but it is a family of surfaces in Y, parameterized by the K3 surface

S. Moreover, if the cubic fourfold deforms in the divisor $\mathcal{C}_d$, the K3 surface S and the family of surfaces Σ deform along as well.

Example 1.2. *Let Σ be the surface in Theorem 1.1.*

(1) Let $a = 1$, and so $d = 14$. Then the surface Σ is a smooth quartic scroll, whose existence was observed in [5, 6, 16]*; explicitly, in the general case, this is $\mathbb{P}^1 \times \mathbb{P}^1$ embedded in $\mathbb{P}^5$ by the linear system $|\mathcal{O}(1,2)|$.*

(2) Let $a = 2$, and so $d = 38$. Then the surface Σ is a smooth "generalized" Coble surface, whose existence was observed in [14]*; explicitly, this is the blow-up of $\mathbb{P}^2$ in 10 general points embedded in $\mathbb{P}^5$ by the linear system $|10L - 3(E_1 + \ldots + E_{10})|$.*

A geometric description for the surfaces Σ is not known when $a \geq 3$. In particular, we currently do not know whether the surface is smooth or even integral. If it is smooth, all numerical invariants can be computed; in particular, it will not be rational for any $a \geq 3$. On the positive side, the construction does conjecturally generalize to any $d \equiv 2 \pmod 6$. In particular, it works in general for small discriminant (e.g., $d \leq 44$) and recovers well known surfaces (e.g., the ones in [14]). Moreover, it does provide as well many rational morphisms from the cubic fourfold, which can be described and studied by using the associated K3 surface S and [4]. For example, in the case $a = 2$, this recovers completely the picture described in [15].

The key insight in our construction comes from derived categories; in particular, the construction of Σ arises from understanding the *Kuznetsov component* $\mathrm{Ku}(Y)$ of Y ([9]) and moduli spaces therein ([3, 2]). For d as in Theorem 1.1, the K3 surface S associated to Y has indeed the property that $\mathrm{Ku}(Y) \cong \mathrm{D^b}(S)$ and the surface Σ arises from a Brill–Noether locus in a special moduli space of stable objects in $\mathrm{Ku}(Y)$. The second property in the statement of the theorem can in fact be rephrased by saying that the ideal sheaf $\mathcal{I}_\Sigma(a)$ belongs to $\mathrm{Ku}(Y)$.

In what follows, we will give an outline of the construction of Σ in Section 2 and how to induce rational morphisms from Y in Section 3.

2. K3 categories and Brill–Noether loci

Let Y be a cubic fourfold and let $\mathrm{D^b}(Y)$ denote the bounded derived category of coherent sheaves on Y. The Kuznetsov component $\mathrm{Ku}(Y)$ of Y is defined as the right orthogonal

$$\mathrm{Ku}(Y) := \langle \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2) \rangle^\perp \subset \mathrm{D^b}(Y).$$

We denote by i_* the inclusion functor $\mathrm{Ku}(Y) \to \mathrm{D^b}(Y)$ and by i^* its left adjoint $\mathrm{D^b}(Y) \to \mathrm{Ku}(Y)$.

The basic properties of the Kuznetsov component are the following.

- $\mathrm{Ku}(Y)$ is a *K3 category*, i.e., it is smooth, proper triangulated category over $\mathbb{C}$, with Serre functor given by [2], the shift by 2 functor ([9]).
- There is a cohomology lattice $(H(\mathrm{Ku}(Y), \mathbb{Z}), (-,-))$ naturally associated to $\mathrm{Ku}(Y)$, given by topological K-theory

$$H(\mathrm{Ku}(Y), \mathbb{Z}) := K_{\mathrm{top}}(\mathrm{Ku}(Y)) := \langle [\mathcal{O}_Y], [\mathcal{O}_Y(1)], [\mathcal{O}_Y(2)] \rangle^\perp \subset K_{\mathrm{top}}(Y),$$

where $(-,-)$ denotes the Mukai pairing. It has a Hodge structure of weight 2, given by Hochschild homology, and the Mukai vector gives a morphism $K(\mathrm{Ku}(Y)) \to H_{\mathrm{alg}}(\mathrm{Ku}(Y), \mathbb{Z})$ ([1]).

- The classes

$$\lambda_1 := i^*[\mathcal{O}_{\mathrm{line}}(1)] \qquad \lambda_2 := i^*[\mathcal{O}_{\mathrm{line}}(2)]$$

define a sublattice $A_2 := \langle \lambda_1, \lambda_2 \rangle \subset H_{\mathrm{alg}}(\mathrm{Ku}(Y), \mathbb{Z})$ ([7, 1]).
- There is a "canonical" (orbit of) stability condition σ_0 which deforms over all cubics; we denote by $\mathrm{Stab}(\mathrm{Ku}(Y))$ the connected component of the space of Bridgeland stability conditions containing σ_0 ([3]).
- Given a Mukai vector $v \in H_{\mathrm{alg}}(\mathrm{Ku}(Y), \mathbb{Z})$ and $\sigma \in \mathrm{Stab}(\mathrm{Ku}(Y))$, the moduli space $M_\sigma(v)$ behaves "as nice as" a moduli space of semistable sheaves on a K3 surface. In particular, if v is primitive and σ generic with respect to v, then $M_\sigma(v) \neq \emptyset$ if and only if $v^2 + 2 \geq 0$; in such a case, $M_\sigma(v)$ is a projective irreducible holomorphic symplectic manifold of dimension $v^2 + 2$, deformation equivalent to a Hilbert scheme of points on a K3 surface ([2]).
- If Y does not contain a plane, then for all $y \in Y$, the projection of the skyscraper sheaf $i^*k(y)$ is σ_0-stable of Mukai vector $\lambda_2 - \lambda_1$; in particular, we obtain an embedding $Y \hookrightarrow M_{\sigma_0}(\lambda_2 - \lambda_1)$ ([12]; the geometric construction is in [11]).

The last property is the starting point for our construction. Indeed, let us fix $v := \lambda_2 - \lambda_1$. If we could find another Mukai vector $u \in H_{\mathrm{alg}}(\mathrm{Ku}(Y), \mathbb{Z})$ such that $(u, v) = -1$, $u^2 + 2 \geq 0$, and the slope of u with respect to σ_0 is larger than the slope of v, then for $F \in M_{\sigma_0}(u)$, the Brill–Noether locus

$$\mathrm{BN}_F := \left\{ E \in M_{\sigma_0}(v) \, : \, \min(\hom(E, F), \mathrm{ext}^1(E, F)) \geq 1 \right\} \subset M_{\sigma_0}(v)$$

has expected codimension 2; in particular, the intersection $Y \cap \mathrm{BN}_F$ has expected codimension 2 as well, and so it could define a surface Σ_F (parameterized by $M_{\sigma_0}(u)$).

To make this argument work, we need to prove the existence of such u and study the non-triviality of BN_F and its intersection with Y. The existence of u is a straightforward computation: u exists if and only if $d \equiv 2 \pmod 6$.

This Brill–Noether locus can be studied directly in low discriminant; in general, we have to assume that $d = 6a^2 + 6a + 2$, $a \geq 1$. In such a case, we can choose u such that $u^2 = 0$. Let $S := M_{\sigma_0}(u)$. Then (up to in case slightly deform σ_0) S is a smooth projective K3 surface and the universal family $\mathcal{U}$ (which exists) gives a Fourier–Mukai equivalence

$$\Phi_{\mathcal{U}} \colon \mathrm{D^b}(S) \xrightarrow{\cong} \mathrm{Ku}(Y).$$

Lemma 2.1. *Let $a \geq 2$. Then, for all $E \in M_{\sigma_0}(v)$, we have $\Phi_{\mathcal{U}}^{-1}(E) \cong \mathcal{I}_\Gamma$, where $\Gamma \subset S$ is a 0-dimensional closed subscheme of length 4.*

In particular, by Lemma 2.1, we can identify $M_{\sigma_0}(v)$ with the Hilbert scheme $S^{[4]}$ (in the case $a = 1$ this is not true; this case has to be studied separately). We can use this to show the following.

Lemma 2.2. *Let $F \in M_{\sigma_0}(u)$ general. Then (up to shift and taking derived dual) we have*

$$i_* F \cong \mathcal{I}_{\Sigma_F}(a),$$

where $\Sigma_F \subset Y$ is a surface.

Theorem 1.1 follows then directly from Lemma 2.2.

3. Morphisms

We keep the notation as in the previous section, with $v = \lambda_2 - \lambda_1$, u, and $F \in M_{\sigma_0}(u)$. Then in "optimal situations" by taking extensions with F gives a well-defined rational map

$$g = g_F \colon M_{\sigma_0}(v) \dashrightarrow M_{\sigma_0}(v-u)$$

which induces a diagram

$$\begin{tikzcd} \mathrm{Bl}_{\Sigma_F} Y \arrow[r] \arrow[d, "\sigma"] & \mathrm{Bl}_{\mathrm{BN}_F} M_{\sigma_0}(v) \arrow[d] \arrow[dr] & \\ Y \arrow[r] & M_{\sigma_0}(v) \arrow[r, dashed, "g"] & M_{\sigma_0}(v-u) \end{tikzcd}$$

and so a closed embedding $f \colon \mathrm{Bl}_{\Sigma_F} Y \hookrightarrow M_{\sigma_0}(v-u)$.

If this is the case, and we let Δ denote the exceptional divisor of σ, we have the following result.

Lemma 3.1. *For a divisor classe $D \in \mathrm{NS}(M_{\sigma_0}(v-u)) \cong (v-u)^{\perp} \subset H_{\mathrm{alg}}(\mathrm{Ku}(Y), \mathbb{Z})$, we have*

$$f^* D = -\frac{(D + (D,u)\, u, \lambda_1 + \lambda_2)}{2}\, \sigma^* h + (D,u)\, \Delta.$$

If $d = 6a^2 + 6a + 2$ as in Theorem 1.1, then g is indeed well-defined: F corresponds to a skyscraper sheaf at a point $p \in S$, and the morphism g is nothing but the map

$$S^{[4]} \dashrightarrow S^{[5]}, \qquad \Gamma \mapsto p + \Gamma.$$

Moreover, by fixing p, the morphism f gives a closed embedding $\mathrm{Bl}_{\Sigma} Y \hookrightarrow S^{[5]}$.

To get rational maps from Y, we can study morphisms from $S^{[5]}$ and these can be studied by simply looking at the base locus decomposition of the movable cone $\mathrm{Mov}(S^{[5]})$, which has been completely described in [4].

In the example when $a = 2$ (and so, $d = 38$), we have the following diagram:

$$\begin{tikzcd} S^{[5]} \arrow[rr, dashed, leftrightarrow, "\cong"] \arrow[dr] & & M_1 \arrow[rr, dashed, leftrightarrow, "\cong"] \arrow[dl] \arrow[dr] & & M_2 \arrow[rr, dashed, leftrightarrow, "\cong"] \arrow[dl] \arrow[drr] & & \dots \\ & \overline{M}_1 & & \overline{M}_2 & & & \dots \end{tikzcd}$$

where the leftmost diagram is a Mukai flop at a $\mathbb{P}^3$-bundle over the Fano variety of lines $F(Y)$ of Y, and the next diagram is a Mukai flop at a $\mathbb{P}^2$-bundle over the product $S \times F(Y)$.

By taking the restriction of the above sequence of morphisms to Y, we find exactly the “trisecant flop” description in [15]

$$\begin{tikzcd}
& S^{[5]} \supset \mathrm{Bl}_\Sigma Y \arrow[dl, "\sigma"'] \arrow[dr] \arrow[rr, dashed, leftrightarrow, "\cong"] & & Y_1 \subset M_1 \arrow[dl] \arrow[dr] & \\
Y & & \overline{Y}_1 \subset \overline{M}_1 & & \overline{Y}_2 \cong \mathbb{P}^4 \subset \overline{M}_2 .
\end{tikzcd}$$

The divisors associated to the two birational maps from Y to $\overline{Y}_1$, respectively $\overline{Y}_2$, correspond, by using Lemma 3.1, to the linear systems $|\mathcal{I}_\Sigma(3)|$, respectively $|\mathcal{I}^2_\Sigma(5)|$ on Y. Concretely, trisecant lines and 5-secant conics in Y.

In [15] this was used to show the rationality of Y in $\mathcal{C}_{38}$. Conjecturally ([9]) all cubic fourfolds in $\mathcal{C}_d$, where $d = 6a^2+6a+2$, should be rational. The corresponding picture already in the case $a = 3$ ($d = 74$) is not understood: $S^{[5]}$ has only one interesting morphism, which corresponds to the linear system $|\mathcal{I}^3_\Sigma(10)|$ on Y, and the rationality of Y in $\mathcal{C}_{74}$ is not known.

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