FOURIER–MUKAI PARTNERS AND GENERALIZED KUMMER STRUCTURES ON GENERALIZED KUMMER SURFACES OF ORDER 3

XAVIER ROULLEAU, ALESSANDRA SARTI

ABSTRACT. A generalized Kummer surface X of order 3 is the minimal resolution of the quotient of an abelian surface A by an order 3 symplectic automorphism. We study a generalization of a problem of Shioda for classical Kummer surfaces, which is to understand how much X is determined by A and conversely. The surface X posses a big and nef divisor L_X such that $L_X^2 = 0$ or $2 \bmod 6$. We show that for surfaces with $L_X^2 = 6k$ with $k \neq 0, 6 \bmod 9$, the surface X determines the transcendental lattice $T(A)$ of A and the Hodge structure on $T(A)$. Conversely if A and B are Fourier-Mukai partners (i.e. if the Hodge structures of their transcendental lattices are isomorphic) and Y is the generalized Kummer surface which is the minimal resolution of the quotient of B by an order 3 symplectic automorphism, we obtain that X and Y are isomorphic. These results are also know to hold for surfaces with $L_X^2 = 2 \bmod 6$ from a previous work. When $k = 0$ or $6 \bmod 9$, we show that X determines $T(A)$ and its Hodge structure, but the converse does not hold in general.

1. Introduction

An (algebraic and complex) generalized Kummer surface $X = \mathrm{Km}_3(A)$ of order 3 is a K3 surface which is the minimal resolution of the quotient A/G_A of an abelian surface A by an order 3 symplectic automorphism group G_A. In [11, 6, 10] we study when there exists another Abelian surface B and order 3 automorphism group G_B such that (B, G_B) is not isomorphic to (A, G_A) but the generalized Kummer surfaces $\mathrm{Km}_3(A)$ and $\mathrm{Km}_3(B)$ are. That question is a natural extension to generalized Kummer surfaces of a problem of Shioda on classical Kummer surfaces.

Recall that the quotient surface A/G_A has 9 cups singularities, each of these being resolved by two (-2)-curves on X. For a very general abelian surface A, the K3 surface X has Picard number 19. The positive generator L_X of the orthogonal complement in the Néron-Severi group $\mathrm{NS}(X)$ of the 18 exceptional curves of the resolution $X \to A/G_A$ satisfies $L_X^2 = 0$ or $2 \bmod 6$.

Let us also recall that for an abelian surface (respectively a K3 surface) Y, a Fourier-Mukai partner of Y is an abelian surface (respectively K3 surface) Y' such that there is an isomorphism of Hodge structures

$$(T(Y), \mathbb{C}\omega_Y) \simeq (T(Y'), \mathbb{C}\omega_{Y'}).$$

Here the transcendental lattice $T(Y)$ is the orthogonal complement in $H^2(Y,\mathbb{Z})$ of the Néron-Severi lattice $\mathrm{NS}(Y) \subset H^2(Y,\mathbb{Z})$ and ω_Y is a generator of the space of holomorphic 2-forms.

Let (A, G_A) and (B, G_B) be two abelian surfaces with an order 3 symplectic automorphism group. Suppose that $X = \mathrm{Km}_3(A)$ is a generalized Kummer surface with Picard number 19. Consider the following two assertions:
(I) The surface B is a Fourier–Mukai partner of A.
(II) The surfaces $\mathrm{Km}_3(B)$ and $\mathrm{Km}_3(A)$ are isomorphic.
As shown in [11], for generalized Kummer surfaces X with $L_X^2 = 2 \bmod 6$, the assertions (I) and (II) are equivalent. The aim of this paper is to prove the following result:

Theorem 1. *Let X be a generalized Kummer surface such that $L_X^2 = 6k$, for $k \in \mathbb{N}^*$.*
a) Suppose that $k \neq 0$ and $6 \bmod 9$. Then (I) is equivalent to (II).
b) Suppose $k = 0$ or $6 \bmod 9$. Then (II) implies (I), but (I) does not implies (II) in general: there exist abelian surfaces with order 3 symplectic automorphisms that are Fourier-Mukai partners and such that the associated generalized Kummer surfaces are not isomorphic.

By [5], since $X = \mathrm{Km}_3(A)$ has Picard number $19 > 2 + \ell$ (here ℓ is the length of the discriminant group of $\mathrm{NS}(X)$, which is also the length of $T(X)$, and therefore is ≤ 3), if a K3 surface is a Fourier-Mukai partner of $\mathrm{Km}_3(A)$, then it is isomorphic to $\mathrm{Km}_3(A)$. The assertion (II) is therefore equivalent to :
(II') The surfaces $\mathrm{Km}_3(A)$ and $\mathrm{Km}_3(B)$ are Fourier-Mukai partners, i.e. there exists a Hodge isometry

$$(T(\mathrm{Km}_3(A)), \mathbb{C}\omega_{\mathrm{Km}_3(A)}) \simeq (T(\mathrm{Km}_3(B)), \mathbb{C}\omega_{\mathrm{Km}_3(B)}).$$

In case $L_X^2 = 2 \bmod 3$, the equivalence between (I) and (II') (and therefore with (II)) is much simpler. Indeed, it is easy to compare the Hodge structures, since in that case $T(\mathrm{Km}_3(A))$ is isometric to $T(A)(3)$, (where the 3 means that the quadratic intersection form of the lattice is multiplied by 3). In case $L_X^2 = 0 \bmod 6$, one only knows that $T(\mathrm{Km}_3(A))$ contains an index 3 sub-lattice isometric to $T(A)(3)$ and that subtlety makes the proof much more involved.

A generalized Kummer structure on the generalized Kummer surface X is an isomorphism class of pairs (B, G_B) such that the associated generalized Kummer surface $\mathrm{Km}_3(B)$ is isomorphic to $X = \mathrm{Km}_3(A)$. The generalized problem of Shioda can be rephrased as the problem to understand if there is a unique generalized Kummer structure. Theorem 1 shows that a generalized Kummer structure $\{(B, G_B)\}$ on X is such that B is a Fourier-Mukai partner of A, for X such that $L_X^2 \neq 0, 6 \bmod 9$. Theorem 1 is a key-result in order to compute the number of generalized Kummer structures in [10] according to the value of L_X^2, for $L_X^2 \neq 0, 6 \bmod 9$.

In [4, 5], Hosono, Lian, Oguiso and Yau study the analogous problem for the classical Kummer surfaces (and compute the number of Kummer structures of some Kummer surfaces). For these surfaces, one always has $T(\mathrm{Km}(A)) = T(A)(2)$, and the equivalence between the analog of (I) and (II) for Kummer surfaces is immediate. The fact that for generalized Kummer surfaces (I) does not imply (II) is all the more noteworthy.

The paper is structured as follows: In Section 2, we give preliminaries and notations on lattice theory. In Section 3, we explain how we proceed to prove Theorem 1. One wants in particular to know the over-lattices of $T(A)(3)$ which are isomorphic to $T(X)$. Sections 4 and 5 are devoted to compute these over-lattices. Section 6 is a proof of Theorem 1 when $k \neq 0, 6 \bmod 9$, and section 7 deals with the remaining cases.

Acknowledgements. The authors thank Simon Brandhorst and Igor Reider for useful conversations.

2. Preliminaries on lattices and notations

The following section gives preliminaries and notations on lattice theory, a standard reference is [8, Section 1]. The part on torsion quadratic modules of the form $(\frac{u}{v})$ is well-known, but we couldn't find a proper reference.

2.1. The discriminant group of a lattice as a torsion quadratic module.

Let L be an even lattice ; the intersection of two elements $v, v' \in L$ is denoted by $vv' \in \mathbb{Z}$. The bilinear pairing extends to $L \otimes \mathbb{Q}$, and the dual of L is

$$\check{L} := \{v \in L \otimes \mathbb{Q} \,|\, \forall w \in L,\, vw \in \mathbb{Z}\}.$$

The discriminant group of L is a torsion quadratic module, denoted by A_L: its subjacent group is the finite abelian group $\check{L}/L$, and its quadratic form $q_L : \check{L}/L \to \mathbb{Q}/2\mathbb{Z}$ is defined for $\frac{1}{n}w \in \check{L}$ (for $n \in \mathbb{Z}, n \neq 0$ and $w \in L$ such that $\frac{1}{n}w \in \check{L}$) by

$$q_L(\frac{1}{n}w) = \frac{1}{n^2}w^2 \bmod 2\mathbb{Z}.$$

If $v \in L$, one has

$$(\frac{1}{n}w + v)^2 = \frac{1}{n^2}w^2 + 2\frac{1}{n}wv + v^2.$$

Since $\frac{1}{n}w \in \check{L}$, $2\frac{1}{n}wv$ is an even integer and since L is even, one has $v^2 \in 2\mathbb{Z}$, thus the quadratic form q_L is a well defined function on the finite group $\check{L}/L$.

Let L be the lattice $\mathbb{Z}^n$ with Gram matrix G for the canonical basis. The columns $c_1, \dots, c_n$ of G^{-1} considered as elements of $\mathbb{Q}^n/\mathbb{Z}^n$ generate the group A_L, moreover the quadratic form q_L on these generators is given by the matrix $G^{-1} = ({}^t c_i G c_j)_{1 \leq i,j \leq n}$, where the diagonal entries of G^{-1} are taken modulo $2\mathbb{Z}$, and the other entries are taken modulo $\mathbb{Z}$ (and respecting the symmetry).

2.2. **Genus of lattice, discriminant group and over-lattices.** By definition, two lattices L_1, L_2 are in the same genus if $L_1 \otimes \mathbb{Q}_p \simeq L_2 \otimes \mathbb{Q}_p$ for all primes p and $L_1 \otimes \mathbb{R} \simeq L_2 \otimes \mathbb{R}$. One has

Theorem 2. *(*[8, Corollary 1.9.4]*) The lattices L_1, L_2 are in the same genus if and only if they have the same signature and have isometric discriminant groups.*

Any over-lattice L_H of L is (up to isometry) the pull-back $L_H = \pi^{-1}(H)$ by the quotient map $\pi : \check{L} \to \check{L}/L$ of an isotropic subgroup H contained in $\mathrm{A}_L = \check{L}/L$.

An isometry $g \in O(L)$ induces an isometry $\bar{g}$ of A_L. If $H_1 \subset \mathrm{A}_L$ is an isotropic sub-group, so is $H_2 = \bar{g}(H_1)$ and g induces an isometry between the over-lattices L_{H_1} and L_{H_2} preserving the lattice L. More generally, two over-lattices $\iota_1 : L \hookrightarrow L_1, \iota_2 : L \hookrightarrow L_2$ are said isomorphic if there exists an isometry g of L extending to an isometry $\tilde{g}$ between L_1 and L_2. Then the following diagram

$$\begin{array}{ccc} L & \overset{\iota_1}{\to} & L_1 \\ \downarrow g & & \downarrow \tilde{g} \\ L & \overset{\iota_2}{\to} & L_2 \end{array}$$

is commutative. Let H_1, H_2 be the isotropic sub-groups of A_L corresponding to over-lattices L_1, L_2. One has:

Proposition 3. *(*[8, Proposition 1.4.3]*). The over-lattices $\iota_1 : L \hookrightarrow L_1, \iota_2 : L \hookrightarrow L_2$ are isomorphic if and only if there exist an isometry g of L such $\bar{g}(H_1) = H_2$.*

Let M be a lattice containing L with finite index, let $H_1, \dots, H_n$ be the set of all isotropic sub-groups of the (finite) group A_L such that their associated over-lattices $L_1, \dots, L_n$ are isometric to M. Let $\iota_j : L \hookrightarrow L_j$ be the inclusion of L in L_j. Suppose that the over-lattices $\iota_j : L \hookrightarrow L_j$, $j = 1, \dots, n$ are isomorphic. Then:

Corollary 4. *Up to isometries of L and M, there exists a unique embedding of L in M.*

Proof. Let $\iota : L \hookrightarrow M$, $\iota' : L \hookrightarrow M$ be two embeddings, let H, H' be the two isotropic sub-groups of L corresponding to these over-lattices. There exists two indices $s, t \in \{1, \dots, n\}$ such that (up to isometries) $\iota : L \hookrightarrow M$ is isomorphic to $\iota_s : L \hookrightarrow L_s$ and $\iota' : L \hookrightarrow M$ is isomorphic to $\iota_t : L \hookrightarrow L_t$, the over-lattices L_s, L_t corresponding to H_s and H_t respectively. Since by hypothesis $\iota_s : L \hookrightarrow L_s$ and $\iota_t : L \hookrightarrow L_t$ are isomorphic, the over-lattices $\iota : L \hookrightarrow M$, $\iota' : L \hookrightarrow M$ are isomorphic, thus there exists an isometry $\tilde{g}$ of M such that $\iota \circ \tilde{g} = \iota'$. □

2.3. **Examples of quadratic torsion modules.** In this section, we give some examples of quadratic torsion modules which we will later use to compute the discriminant groups of various lattices.

Consider $\frac{u}{v} \in \mathbb{Q} \setminus \{0\}$ with u, v coprime such that either u or v is even. Let us denote by $(\frac{u}{v})$ the torsion quadratic module $(\mathbb{Z}/v\mathbb{Z}, q)$ with quadratic form $q : \mathbb{Z}/v\mathbb{Z} \to \mathbb{Q}/2\mathbb{Z}$ defined by

$$q(x) = \frac{ux^2}{v} \in (\frac{1}{v}\mathbb{Z})/2\mathbb{Z} \subset \mathbb{Q}/2\mathbb{Z}.$$

If $x' = x + tv$ with $t \in \mathbb{Z}$, one has

$$q(x') = \frac{u(x^2 + 2xtv + t^2v^2)}{v} = \frac{ux^2}{v} + u(2xt + t^2v) = \frac{ux^2}{v} = q(x),$$

thus q is well-defined; here we use that u or v is even at the third equality (if both were odd, the form q would be well-defined only modulo $\mathbb{Z}$). For u, u' coprime to v, one has

$$(\frac{u}{v}) = (\frac{u'}{v})$$

if and only if $u = u' \bmod 2v$.

Suppose that $v = ab$ with a, b coprime integers, and let $s, t \in \mathbb{Z}$ such that $as + bt = 1$. If s', t' is another Bézout pair, one has $s' = s + mb$ and $t' = t - ma$ for $m \in \mathbb{Z}$.

• Suppose that u is even, (thus a and b are odd). Then, since u is even, one has $us = us' \bmod 2b$, $ut = ut' \bmod 2a$ and the torsion quadratic modules

$$(\frac{us}{b}),\ (\frac{ut}{a})$$

are independent of the choice of the Bézout pair (s, t).

• Suppose that u is odd and a, say, is even, b is odd. The relation $as+bt = 1$ implies that t is odd. Moreover, since b is odd, up to changing s by $s+mb$ for $m \in \mathbb{Z}$, one can choose s to be even, and in fact one must do it in order for the quadratic form of $(\frac{us}{b})$ to have values in $\mathbb{Z}/2\mathbb{Z}$. That choice of s is unique modulo $2b$. The choice of t is then unique modulo $2a$, and in the notations: $(\frac{us}{b}), (\frac{ut}{a})$, it is always implicitly supposed that s is even.

In both cases u even or odd, the relation

$$\frac{u}{ab} = \frac{tu}{a} + \frac{su}{b}$$

implies that $(\frac{u}{v})$ decomposes as a direct sum

$$(\frac{u}{v}) = (\frac{tu}{a}) + (\frac{su}{b})$$

where we use the canonical isomorphism $x \bmod ab \to (x \bmod a, x \bmod b)$ between $\mathbb{Z}/ab\mathbb{Z}$ and $\mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z}$ (we recall that a, b are coprime integers). For $v = ab$ with ab coprime, we denote by $(\frac{u}{v})_a = (\frac{tu}{a})$ the restriction of $(\frac{u}{v})$ to $\mathbb{Z}/a\mathbb{Z}$. Since s is even, in $\mathbb{Z}/2a\mathbb{Z}$ one has $\bar{b}\bar{t} = \bar{1}$, thus we sometimes write $(\frac{u}{v})_a = (\frac{u/b}{a})$.

The decomposition $(\frac{u}{ab}) = (\frac{tu}{a}) + (\frac{su}{b})$ implies that if $v = p_1^{n_1} \dots p_k^{n_k}$ is the factorization of v into a product of distinct primes, the torsion quadratic module $(\frac{u}{v})$ is isometric to a direct sum

$$(\frac{u_1}{p_1^{n_1}})+\cdots+(\frac{u_k}{p_k^{n_k}})$$

for some integers u_i coprime to p_i and such that u_i or p_i is even. More generally, using that a finite abelian group is the direct sum of its p-sub-groups, a torsion quadratic module M is a direct sum $M=\oplus_{p\,\mathrm{prime}}M_p$ of p-sub-groups. For $p\geq 3$, M_p is a direct sum of torsion quadratic modules of the form $(\frac{u}{p^n})$ (for u even, coprime to p and $n\in\mathbb{N}^*$). For $p=2$, the situation is more complicated, see [3].

Let be $w\in\mathbb{Z}$ such that $\bar{w}\in\mathbb{Z}/v\mathbb{Z}$ is a unit. For any $n\in\mathbb{Z}$, one has

$$u(w+nv)^2=uw^2+2nuwv+un^2v^2=uw^2 \bmod 2v,$$

(where in the last equality we use that u or v is even), thus group of units $(\mathbb{Z}/v\mathbb{Z})^*$ acts on the set of torsion quadratic modules $(\frac{u}{v})$ by $\bar{w}.(\frac{u}{v})=(\frac{uw^2}{v})$. One has

$$\forall x\in\mathbb{Z}/v\mathbb{Z},\ \frac{uw^2}{v}x^2=\frac{u}{v}(\bar{w}x)^2,$$

therefore the automorphism of $\mathbb{Z}/v\mathbb{Z}$ defined by $x\to\bar{w}x$ is an isometry between the torsion quadratic modules $\bar{w}.(\frac{u}{v})$ and $(\frac{u}{v})$. Conversely, if there is an isometry between $(\frac{u}{v})$ and $(\frac{u'}{v})$, the subjacent automorphism ψ of $\mathbb{Z}/v\mathbb{Z}$ being of the form $x\to\bar{w}x$ for some $\bar{w}\in(\mathbb{Z}/v\mathbb{Z})^*$, one has $u'=uw^2 \bmod 2v$, thus $u'=uw^2 \bmod v$ and $u/u'\in(\mathbb{Z}/v\mathbb{Z})^*$ is a square.

Suppose that v is an odd prime. The set S_q of square elements in $(\mathbb{Z}/v\mathbb{Z})^*$ is a sub-group of index 2. Let $m_2:\mathbb{Z}/v\mathbb{Z}\to 2\mathbb{Z}/\mathbb{Z}2v\mathbb{Z}$ be the map defined by $x+v\mathbb{Z}\to 2x+2v\mathbb{Z}$; it is a bijection. Let E be the set

$$E=m_2((\mathbb{Z}/v\mathbb{Z})^*).$$

The set E is in bijection with the set of quadratic torsion modules by the map $u\in E\to(\frac{u}{v})$. There is a natural faithful action of S_q on E given by

$$(s,u)\in S_q\times E\to m_2(m_2^{-1}(u)s)\in E.$$

That action has two orbits, corresponding to the invertible square, and invertible non-square in S_q. Thus there are only two isometry classes among the torsion quadratic modules $(\frac{u}{v})$, $u\in E$. For example, the only torsion quadratic modules on $\mathbb{Z}/3^a\mathbb{Z}$ up to isometry are $(\frac{2}{3^a})$ and $(\frac{4}{3^a})$ (for $a\geq 1$).

3. Fourier-Mukai partners in case $L_X^2=0 \bmod 6$

Suppose that the generalized Kummer surface $X=\mathrm{Km}_3(A)$ satisfies $L_X^2=6k$. Then (see [10, Theorem 7]), there exists a basis of $T(A)$ with Gram matrix

$$\begin{pmatrix}-2k & 0 & 0\\ 0 & 2 & 3\\ 0 & 3 & 6\end{pmatrix},$$

therefore $T(A)(3)$ has a basis with Gram matrix

$$\begin{pmatrix} -6k & 0 & 0 \\ 0 & 6 & 9 \\ 0 & 9 & 18 \end{pmatrix}. \tag{3.1}$$

Let $T(X)$ be the transcendental lattice of X: this is the orthogonal complement of $\mathrm{NS}(X)$ in $H^2(X,\mathbb{Z})$.

Proposition 5. *The lattice $T(X)$ has a basis with Gram matrix*

$$\begin{pmatrix} -6k & 0 & 0 \\ 0 & 6 & 3 \\ 0 & 3 & 2 \end{pmatrix}.$$

It is unique in its genus. The discriminant group $A_{T(X)}$ is generated by $(\frac{1}{6k},0,0)$, $(0,\frac{2}{3},-1)$ and is isomorphic to $\mathbb{Z}/6k\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$.

Proof. See [10, Section 2.4] for the first two assertions; the last one follows from a direct computation. □

One has

Proposition 6. *There exists a morphism $\pi_{A*}: T(A)(3)\to T(X)$ such that π_{A*} is an isometry onto its image and $\pi_{A*}(T(A))$ has index 3 in $T(X)$.*

Proof. The morphism π_{A*} is defined in [11, Section 2.3], where it is shown that the lattice $\pi_{A*}(T(A))$ is isometric to $T(A)(3)$. For the assertion on the index, it is sufficient to compare the discriminant of the two lattices. □

Suppose that there exists in $T(A)(3)\otimes\mathbb{Q}$ a unique over-lattice T of $T(A)(3)$ such that T is isometric to $T(X)$, where $X=\mathrm{Km}_3(A)$. Let us suppose moreover that T contains a unique sub-lattice isometric to $T(A)(3)$. Let (B,G_B) be another abelian surface B with an order 3 symplectic automorphism group G_B.

Theorem 7. *Under the above hypothesis, there exists a Hodge isometry*

$$(T(A),\mathbb{C}\omega_A)\simeq(T(B),\mathbb{C}\omega_B) \tag{3.2}$$

if and only if there exists a Hodge isometry

$$(T(\mathrm{Km}_3(A)),\mathbb{C}\omega_{\mathrm{Km}_3(A)})\simeq(T(\mathrm{Km}_3(B)),\mathbb{C}\omega_{\mathrm{Km}_3(B)}).$$

Proof. By Proposition 5, the lattice $T(X)$ is uniquely determined by the lattice $T(A)$ and conversely, in particular, the lattice $T(X)$ is isometric to $T(X')$ (where $X'=\mathrm{Km}_3(B)$) if and only $T(A)$ is isometric to $T(B)$. Suppose that one has an isomorphism of Hodge structures $(T(A),\mathbb{C}\omega_A)\simeq(T(B),\mathbb{C}\omega_B)$, then $T(A)\simeq T(B)$ and from the hypothesis, there exists in $T(B)(3)\otimes\mathbb{Q}$ a unique over-lattice T' isomorphic to $T(X')$. The Hodge structure $(T(X),\mathbb{C}\omega_X)$ is then uniquely determined by $(T,\mathbb{C}\omega_A)$: necessarily $(T(X),\mathbb{C}\omega_X)\simeq(T,\mathbb{C}\omega_A)$, and also $(T(X'),\mathbb{C}\omega_{X'})\simeq(T',\mathbb{C}\omega_A)$, therefore the Hodge structures $(T(X),\mathbb{C}\omega_X)$ and $(T(X'),\mathbb{C}\omega_{X'})$ are isomorphic.

Conversely, suppose that there exists an isomorphism

$$\phi : (T(X), \mathbb{C}\omega_X) \to (T(X'), \mathbb{C}\omega_{X'})$$

of Hodge structures. Since T contains a unique sub-lattice isometric to $T(A)(3)$ and $T(X) \simeq T \simeq T(X')$, the sub-lattice $T(A)(3)$ is sent by ϕ to $T(B)(3)$ and there is a Hodge isomorphism $(T(A), \mathbb{C}\omega_A) \simeq (T(B), \mathbb{C}\omega_B)$. $\square$

Let us therefore study the over-lattices T of $T(A)(3)$ such that $T(A)(3)$ has index 3 in T. We can suppose that these lattices are contained in $T(A)(3)\otimes\mathbb{Q}$. The quadratic form on the discriminant group $\mathrm{A}_{T(A)(3)}$ is given by

$$Q = \begin{pmatrix} -\frac{1}{6k} & 0 & 0 \\ 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{2}{9} \end{pmatrix}.$$

In the following list of 13 elements of the form $(v; s_q) \in \mathrm{A}_{T(A)(3)} \times \mathbb{Q}/2\mathbb{Z}$:

$$\begin{aligned}
&(0,0,\frac{1}{3};0) \\
&(0,\frac{1}{3},0;\tfrac{2}{3}),(0,\frac{1}{3},\frac{1}{3};\tfrac{2}{3}),(0,\frac{1}{3},\frac{2}{3};\tfrac{14}{3}), \\
&(\frac{1}{3},0,0;-\tfrac{2}{3}k),(\frac{1}{3},0,\frac{1}{3};-\tfrac{2}{3}k),(\frac{1}{3},0,\frac{2}{3};-\tfrac{2}{3}k) \\
&(\frac{1}{3},\frac{1}{3},0;-\tfrac{2}{3}k+\tfrac{2}{3}),(\frac{1}{3},\frac{1}{3},\frac{1}{3};-\tfrac{2}{3}k+\tfrac{2}{3}),(\frac{1}{3},\frac{1}{3},\frac{2}{3};-\tfrac{2}{3}k+\tfrac{14}{3}), \\
&(\frac{1}{3},\frac{2}{3},0;-\tfrac{2}{3}k+\tfrac{8}{3}),(\frac{1}{3},\frac{2}{3},\frac{1}{3};-\tfrac{2}{3}k+\tfrac{2}{3}),(\frac{1}{3},\frac{2}{3},\frac{2}{3};-\tfrac{2}{3}k+\tfrac{8}{3}),
\end{aligned} \tag{3.3}$$

are the 13 generators

$$v = (a_1, a_2, a_3) \in \mathrm{A}_{T(A)(3)}$$

of the 13 order 3 sub-groups H_v of $\mathrm{A}_{T(A)(3)} \simeq \mathbb{Z}/6k\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/9\mathbb{Z}$, and their square $s_q = vQ\,{}^tv \in \mathbb{Q}/2\mathbb{Z}$. We denote by T_v the pull-back in $T(A)(3)\otimes\mathbb{Q}$ of the group $H_v \subset \mathrm{A}_{T(A)(3)}$. We remark that the groups generated by the elements

$$(0,\frac{1}{3},0;\tfrac{2}{3}),(0,\frac{1}{3},\frac{1}{3};\tfrac{2}{3}),(0,\frac{1}{3},\frac{2}{3};\tfrac{14}{3})$$

in list 3.3 are not isotropic sub-groups, thus their pull-back to $T(A)(3)\otimes\mathbb{Q}$ are not over-lattices of $T(A)(3)$. For any k, the case $v_0 = (0,0,\frac{1}{3})$ gives an index 3 over-lattice isometric to $T(X)$.

Let N^{over} be the number of over-lattices of $T(A)(3)$ in $T(A)(3)\otimes\mathbb{Q}$ that are isomorphic to $T(X)$. From the above discussion on case $v_0 = (0,0,\frac{1}{3})$, we get that $N^{over} \geq 1$. The aim of the next two Sections is to prove the following result:

Theorem 8. *We have*

	$k=1$ *or* $2 \bmod 3$	$k=0 \bmod 9$	$k=3 \bmod 9$	$k=6 \bmod 9$
N^{over}	1	3	1	2
w	$(0,0,\frac{1}{3})$	$(0,0,\frac{1}{3})$ $(\frac{1}{3},0,\frac{1}{3})$ $(\frac{1}{3},0,\frac{2}{3})$	$(0,0,\frac{1}{3})$	$(0,0,\frac{1}{3})$ $(\frac{1}{3},0,0)$

where in the last line are the elements w such that T_w is isometric to $T(X)$.

For $k = 2 \bmod 3$, we remark that the element $v = (0, 0, \frac{1}{3})$ in list (3.3) is the only possibility in order for T_v to be an over-lattice of $T(A)(3)$. Let us check the possibilities from the list (3.3) according to the remaining cases $k = 0$ or $1 \bmod 3$.

4. Case $k = 0 \bmod 3$

Let us prove Theorem 8 when $k = 0 \bmod 3$. In this case $k = 3k'$ for $k' \in \mathbb{Z}$, the possibilities different from $v_0 = (0, 0, \frac{1}{3}; 0)$ are

$$(\tfrac{1}{3}, 0, 0; -\tfrac{2}{3}k), (\tfrac{1}{3}, 0, \tfrac{1}{3}; -\tfrac{2}{3}k), (\tfrac{1}{3}, 0, \tfrac{2}{3}; -\tfrac{2}{3}k).$$

4.1. **Case $k = 3k'$ and $v = (\frac{1}{3}, 0, 0)$.** Let us study the case

$$v = (\frac{1}{3}, 0, 0) \in \mathrm{A}_{T(A)(3)}.$$

We have that $v^2 = -\frac{2}{3}k$, and this is 0 in $\mathbb{Q}/2\mathbb{Z}$ since $k = 3k'$ by assumption. Then the Gram matrix in some basis of the over-lattice T_v associated to H_v is

$$\begin{pmatrix} -2k' & 0 & 0 \\ 0 & 6 & 9 \\ 0 & 9 & 18 \end{pmatrix}.$$

The lattice T_v has discriminant group isomorphic to $\mathbb{Z}2k'\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}$, generated by the columns of the matrix

$$\begin{pmatrix} -\frac{1}{2k'} & 0 & 0 \\ 0 & \frac{2}{3} & \frac{-1}{3} \\ 0 & -\frac{1}{3} & \frac{2}{9} \end{pmatrix}.$$

Proposition 9. *The lattice T_v is isometric to $T(X)$ if and only if $k = 6 \bmod 9$.*

Proof. The elements

$$(\frac{1}{2k'}, 0, 0), (0, \tfrac{2}{3}, -\tfrac{1}{3}), (0, 0, \tfrac{1}{9})$$

generate $\mathrm{A}_{T_v} \simeq \mathbb{Z}/2k'\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/9\mathbb{Z}$ and have intersection matrix $\mathrm{Diag}(-\frac{1}{2k'}, \frac{2}{3}, \frac{2}{9})$.

The lattice $T(X)$ has discriminant group isomorphic to

$$\mathbb{Z}/6k\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} = \mathbb{Z}/18k'\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z},$$

thus if $3|k'$, T_v is not isometric to $T(X)$. If $3 \not| k'$, then

$$\mathrm{A}_{T(X)} \simeq \mathbb{Z}/2k'\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z} \simeq \mathrm{A}_{T_v}$$

and we must compare the quadratic forms. The quadratic form on $T(X)$ is $\mathrm{Diag}(-\frac{1}{18k'},\frac{2}{3})$. Since $3\nmid k'$, one has

$$(-\frac{1}{18k'})=(-\frac{u}{2k'})+(-\frac{v}{9}),$$

where $u\in(\mathbb{Z}/2k'\mathbb{Z})^*$ is such that $9u=1\bmod 2k'$ and $v\in(\mathbb{Z}/9\mathbb{Z})^*$ is such that $2kv=1\bmod 9$. Since $9u=1\bmod 2k'$, there exists $u'\in(\mathbb{Z}/2k'\mathbb{Z})^*$ such that $u'^2=u$, thus $(-\frac{u}{2k'})$ is isometric to $(-\frac{1}{2k'})$. Suppose that $k'=1\bmod 3$, then $-(2k')^{-1}=1,4\,\text{or}\,7\bmod 9$, and since $4=2^2\bmod 9$, $7=5^2\bmod 9$, we obtain that $(-\frac{v}{9})=(\frac{4}{9})$. Suppose that $k'=2\bmod 3$, then $-(2k')^{-1}=2,5\,\text{or}\,8\bmod 9$, and since $5=2\cdot 5^2\bmod 9$, $8=2\cdot 2^2\bmod 9$, we obtain that $(-\frac{v}{9})=(\frac{2}{9})$. Therefore, the quadratic form on $\mathrm{A}_{T(X)}$ is isometric to

$$\begin{array}{l}(-\frac{1}{2k'})+(\frac{4}{9})+(\frac{2}{3})\ \text{if}\,k'=1\bmod 3\\ (-\frac{1}{2k'})+(\frac{2}{9})+(\frac{2}{3})\ \text{if}\,k'=2\bmod 3.\end{array}\tag{4.1}$$

Thus $T(X)$ is isometric to T_v if and only if $k=6\bmod 9$. In that case $T(X)$ and T_v are in the same genus, but we know from proposition 5 that $T(X)$ is unique in its genus, therefore $T(X)$ and T_v are isometric. □

4.2. **Case $k=3k'$ and $v=(\frac{1}{3},0,\frac{1}{3})$.** Let T_v be the over-lattice associated to $v=(\frac{1}{3},0,\frac{1}{3})$. There is basis in which the Gram matrix of $T=T_v$ is

$$\begin{pmatrix}-2k'+2 & 0 & 3\\ 0 & 6 & 9\\ 3 & 9 & 18\end{pmatrix}.$$

The discriminant group A_T of T has order $54k'$; it is generated by the columns c_1,c_2,c_3 of

$$\begin{pmatrix}-\frac{1}{2k'} & 0 & \frac{1}{3k'}\\ -\frac{1}{2k'} & \frac{2}{3} & \frac{1}{3}(\frac{1}{k'}-1)\\ \frac{1}{3k'} & -\frac{1}{3} & \frac{2}{9}(1-\frac{1}{k'})\end{pmatrix}.$$

Making the substitution $c_2\to 2c_2+2k'c_1$, (which is possible since c_2 has order 3), gives the generators

$$\begin{pmatrix}-\frac{1}{2k'} & 0 & \frac{1}{3k'}\\ -\frac{1}{2k'} & \frac{1}{3} & \frac{1-k'}{3k'}\\ \frac{1}{3k'} & 0 & \frac{2}{9}(\frac{k'-1}{k'})\end{pmatrix}\tag{4.2}$$

with intersection matrix

$$\begin{pmatrix}\frac{-1}{2k'} & 0 & \frac{1}{3k'}\\ 0 & \frac{2}{3} & 0\\ \frac{1}{3k'} & 0 & \frac{2(k'-1)}{9k'}\end{pmatrix},$$

thus T is isometric to

$$\begin{pmatrix}\frac{2}{3}\end{pmatrix}+\begin{pmatrix}\frac{-1}{2k'} & \frac{1}{3k'}\\ \frac{1}{3k'} & \frac{2(k'-1)}{9k'}\end{pmatrix}.\tag{4.3}$$

We will also use the transformation $c_3 \to c_3 + c_2$ in Equation 4.2; then one gets the generators e_1, e_2, e_3

$$\left(\begin{array}{ccc} -\frac{1}{2k'} & 0 & \frac{1}{3k'} \\ -\frac{1}{2k'} & \frac{1}{3} & \frac{1}{3k'} \\ \frac{1}{3k'} & 0 & \frac{2}{9}(\frac{k'-1}{k'}) \end{array}\right) \tag{4.4}$$

with intersection matrix

$$\left(\begin{array}{ccc} -\frac{1}{2k'} & 0 & \frac{1}{3k'} \\ 0 & \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3k'} & \frac{2}{3} & \frac{2(4k'-1)}{9k'} \end{array}\right).$$

4.2.1. *Sub-case* $v = (\frac{1}{3}, 0, \frac{1}{3})$ *and* $k = 0 \bmod 9$. Suppose that $k = 0 \bmod 9$. Let us prove

Lemma 10. *The lattice T associated to $v = (\frac{1}{3}, 0, \frac{1}{3})$ is isometric to $T(X)$*

Proof. Let be $k' \in \mathbb{N}$ such that $k = 3k'$ and let us define k'' by $k' = 3^\alpha k''$ with $\alpha \geq 1$ and k'' prime to 3. Let us study the torsion quadratic module

$$Q = \left(\begin{array}{cc} \frac{-1}{2k'} & \frac{1}{3k'} \\ \frac{1}{3k'} & \frac{2(k'-1)}{9k'} \end{array}\right)$$

in Equation 4.3. Let $\mathrm{A}_T^{\perp 3}$ be the subgroup of elements that have prime to 3 order in the finite abelian group A_T. We can then define $v_2' = v_2 + \frac{2}{3}v_1$, where v_1, v_2 are the base vectors. Let $P = \left(\begin{array}{cc} 1 & \frac{2}{3} \\ 0 & 1 \end{array}\right)$ be the basis change matrix; one has

$${}^tPQP = \left(\begin{array}{cc} \frac{-1}{2k'} & 0 \\ 0 & \frac{2}{9} \end{array}\right) \simeq (\frac{-1}{2k'}) + (\frac{2}{9}),$$

where since we consider the sub-group $\mathrm{A}_T^{\perp 3}$, the form $(\frac{2}{9})$ is the zero form, and the quadratic form on $\mathrm{A}_T^{\perp 3}$ is therefore $(-\frac{1}{2k'})$. For the 3-torsion subgroup $\mathrm{A}_T(3)$, let us consider $v_1' = v_1 + \frac{3}{2-2k'}v_2$ (one has $2 - 2k' = 2 \bmod 3$). Under the basis change by $P' = \left(\begin{array}{cc} 1 & 0 \\ \frac{3}{2-2k'} & 1 \end{array}\right)$, the quadratic form on $\mathrm{A}_T(3)$ is isometric to

$$\left(\begin{array}{cc} \frac{-1}{2(k'-1)} & 0 \\ 0 & \frac{2}{9}(1-\frac{1}{k'}) \end{array}\right) \simeq \left(\frac{-1}{2(k'-1)}\right) + \left(\frac{2}{9}(1-\frac{1}{k'})\right)$$

and since we consider here the sub-group $\mathrm{A}_T(3)$ and $k' - 1 = -1 \bmod 3$, the form $\left(\frac{-1}{2(k'-1)}\right)$ is zero, and the quadratic form on $\mathrm{A}_T(3)$ is isometric to

$$\left(\frac{2}{9}(1-\frac{1}{k'})\right) = \left(\frac{2(k'-1)/k''}{3^{\alpha+2}}\right).$$

We thus conclude that the quadratic form on T is

$$(-\tfrac{1}{2k'}) \text{ for elements in } \mathrm{A}_T^{\perp 3},$$
$$\left(\tfrac{2(k'-1)}{9k'}\right) + (\tfrac{2}{3}) \text{ for elements in } \mathrm{A}_T(3).$$

Now the quadratic form on $T(X)$ is

$$(-\frac{1}{6k})+(\frac{2}{3}).$$

The two quadratic forms are equal if they are equal at any prime. At prime 3, they are equal if and only if

$$(-\frac{1/(2k'')}{3^{\alpha+2}})=(\frac{2(k'-1)/k''}{3^{\alpha+2}}),$$

this is the case if and only if

$$\frac{2(k'-1)/k''}{-1/(2k'')}=4(1-3^{\alpha}k'')$$

is a square modulo $3^{\alpha+2}$, this is equivalent to $1-3^{\alpha}k''\in 3^{\alpha+2}$ is a square. Let $v=1-3^{\alpha}k''/2\in\mathbb{Z}/3^{\alpha+2}\mathbb{Z}$, then $v^2=1-3^{\alpha}k''+3^{2\alpha}k''/4$ and therefore, when $\alpha\geq 2$, $v^2=1-3^{\alpha}k''$ is a square. One can check that this is also true for $\alpha=1$ by a direct computation. Thus we obtain that, at prime 3, the two quadratic forms are equal.

Now for the part coprime to 3, we must compare $(-\frac{1}{2k'})=(-\frac{1/3^{\alpha}}{2k''})$ with $(-\frac{1}{6k})=(-\frac{1/3^{\alpha+2}}{2k''})$: these forms are isometric since they differ by a square.

We thus proved that the two lattices T, $T(X)$ are in the same genus. Since the genus contains a unique element by Proposition 5, the two lattices are isomorphic. □

4.2.2. *Sub-case* $v=(\frac{1}{3},0,\frac{1}{3})$ *and* $k=3 \bmod 9$. Suppose that $k=3k'$ with $k'=1 \bmod 3$, and let us prove

Lemma 11. *The lattice* T *associated to* $v=(\frac{1}{3},0,\frac{1}{3})$ *is not isometric to* $T(X)$

Proof. Using Equation 4.2, the discriminant group is generated by the columns of

$$\begin{pmatrix} -\frac{1}{2k'} & 0 & \frac{1}{3k'} \\ -\frac{1}{2k'} & \frac{1}{3} & -\frac{k''}{k'} \\ \frac{1}{3k'} & 0 & \frac{2k''}{3k'} \end{pmatrix}$$

(where $k'=3k''+1$) and we remark that

$$c_3-2k''c_1=(\frac{1}{3},0,0)$$

thus the discriminant group A_T is generated by the columns of

$$\begin{pmatrix} -\frac{1}{2k'} & 0 & \frac{1}{3} \\ -\frac{1}{2k'} & \frac{1}{3} & 0 \\ \frac{1}{3k'} & 0 & 0 \end{pmatrix}$$

which shows that, when $k'=1 \bmod 3$, the discriminant group of T is

$$\mathrm{A}_T\simeq(\mathbb{Z}/3\mathbb{Z})^3\times\mathbb{Z}/2k'\mathbb{Z}.$$

It is not isomorphic to $\mathbb{Z}/18k'\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$, therefore T is not isometric to $T(X)$. □

4.2.3. *Sub-case $v=(\frac{1}{3},0,\frac{1}{3})$ and $k=6 \bmod 9$.* Suppose that $k=3k'$ with $k'=2 \bmod 3$. Let us prove

Lemma 12. *The lattice T associated to $v=(\frac{1}{3},0,\frac{1}{3})$ is not isometric to $T(X)$*

Proof. One can check that, as abstract groups, the discriminant groups $\mathrm{A}_{T(X)}$ and A_T are isomorphic. Let us study the 3-torsion part $\mathrm{A}_T(3)$ of A_T and the quadratic form restricted to that part. Since $3 \not| k'$, from 4.4, the generators of $\mathrm{A}_T(3)$ are the columns c_1, c_2, c_3 of the matrix

$$\left(\begin{array}{ccc} 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{4}{3} & 0 & \frac{2}{9}(k'-1) \end{array}\right).$$

Since $k' \neq 1 \bmod 3$, the element c_1 is a multiple of c_3 and the generators of $\mathrm{A}_T(3)$ are c_2, c_3, with intersection matrix

$$\left(\begin{array}{cc} \frac{2}{3} & \frac{2k'}{3} \\ \frac{2k'}{3} & \frac{2k'}{9}(4k'-1) \end{array}\right).$$

Then $c_3' = c_3 - kc_2$ is such that $c_3'c_2=0$ and $c_3'^2 = \frac{2}{9}k'(k'-1)$. We obtain that $\mathrm{A}_T(3)$ is isometric to $(\frac{2}{3})+(\frac{2k'(k'-1)}{9})$. Using that $k'=2,5$ or $8 \bmod 9$, it is easy to check that in any case $(\frac{2k'(k'-1)}{9}) \simeq (\frac{4}{9})$. We conclude that $\mathrm{A}_T(3)$ is isometric to $(\frac{2}{3})+(\frac{4}{9})$. But we know from Equation 4.1 that when $k'=2 \bmod 3$, the 3-torsion part of $\mathrm{A}_{T(X)}$ is $(\frac{2}{3})+(\frac{2}{9})$. Therefore T is not isometric to $T(X)$ when $k'=2 \bmod 3$. □

4.3. **Case $k=3k'$ and $v=(\frac{1}{3},0,\frac{2}{3})$.** Let $v=(\frac{1}{3},0,\frac{2}{3}) \in \mathrm{A}_{T(A)(3)}$, $v'=(\frac{1}{3},0,\frac{1}{3})$ and let $T_v, T_{v'}$ be the associated over-lattices.

Lemma 13. *The lattices T_v and $T_{v'}$ are isomorphic. The lattice T_v is isometric to $T(X)$ if and only if $k=0 \bmod 9$.*

Proof. The matrix

$$g=\left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & -3 \\ 0 & 1 & 2 \end{array}\right) \tag{4.5}$$

is an order 6 isometry of the lattice $T(A)(3)$ with Gram matrix

$$Q_{T(A)(3)}=\left(\begin{array}{ccc} -6k & 0 & 0 \\ 0 & 6 & 9 \\ 0 & 9 & 18 \end{array}\right)$$

(one has ${}^t g Q_{T(A)(3)} g = Q_{T(A)(3)}$) and $g(\frac{1}{3}(v_1+v_3)) = \frac{1}{3}(v_1+3v_2+2v_3)$. Its action on the discriminant group sends the group $H_{v'}$ generated by the class of $v'=\frac{1}{3}(v_1+v_3)$ in $\mathrm{A}_{T(A)(3)}$ to the group H_v generated by the class of $v=\frac{1}{3}(v_1+2v_3)$ in $\mathrm{A}_{T(A)(3)}$. Therefore, the element g induces an isometry between the over-lattices $T_{v'}, T_v$ which are the pull-back in $T(A)(3)\otimes\mathbb{Q}$ of $H_{v'}$ and H_v. The lattice T_v has therefore the same properties as the lattice $T_{v'}$ studied in Section 4.2. □

5. Case $k = 3k' + 1$

Let us prove Theorem 8 when $k = 1 \bmod 3$. When $k = 3k' + 1$, the cases different from $(0, 0, \frac{1}{3})$ are the six generators v in the following list

$$(\tfrac{1}{3}, \tfrac{1}{3}, 0), (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}), (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{2}{3}),$$
$$(\tfrac{1}{3}, \tfrac{2}{3}, 0), (\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{3}), (\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{2}{3}),$$

of the six order 3 isotropic groups H_v.

Lemma 14. *Let T be the over-lattice corresponding to the isotropic order 3 group $H_v = \langle v \rangle$, with v in the above list. The lattice T is not isometric to $T(X)$.*

Proof. The isometry g in Equation 4.5 acts on the discriminant group $\mathrm{A}_{T(A)(3)}$ and the above 6 elements form one orbit for that action. It is therefore enough to study the over-lattice T corresponding to one of these elements, say $v = (\frac{1}{3}, \frac{1}{3}, 0)$. The Gram matrix of the corresponding over-lattice in some basis is

$$Q_T = \begin{pmatrix} -2k' & 1 & 0 \\ 1 & 6 & 9 \\ 0 & 9 & 18 \end{pmatrix}$$

the discriminant group has order $54k' + 18 = 18k$ (here, $k = 3k' + 1$); it is generated by the columns c_1, c_2, c_3 of

$$Q_T^{-1} = \begin{pmatrix} -\frac{3}{2k} & \frac{1}{k} & -\frac{1}{2k} \\ \frac{1}{k} & \frac{2k'}{k} & -\frac{k'}{k} \\ -\frac{1}{2k} & -\frac{k'}{k} & \frac{4k-3}{18k} \end{pmatrix}.$$

In A_T, one has $c_2 = 2k'c_1$. Moreover, $c_1 - 3c_3 = (0, 0, -\frac{2}{3})$. We observe that the column $2kc_3$ is $(0, 0, \frac{4k-3}{9})$. Since $4k - 3$ is coprime to 9, the group generated by $2kc_3$ contains $(0, 0, -\frac{2}{3})$. We thus obtain that the discriminant group is cyclic, generated by c_3: T cannot be isometric to $T(X)$. □

6. Preservation of $T(A)(3)$ into $T(X)$ under $O(T(X))$, proof of the main Theorem in case $k \neq 0, 6 \bmod 9$

We recall that $T(X)$ is the lattice with Gram matrix

$$Q_1 = \begin{pmatrix} -6k & 0 & 0 \\ 0 & 6 & 3 \\ 0 & 3 & 2 \end{pmatrix}$$

in basis $\beta_1 = (e_1, e_2, e_3)$. Let $T_1 \simeq T(A)(3)$ be the lattice generated by $e_1, e_2, 3e_3$. Let us show that

Proposition 15. *The orthogonal group $O(T(X))$ preserves T_1.*

Proof. Let $g = (a_{ij})_{1 \leq i,j \leq 3} \in O(T(X))$; one has ${}^t g Q_1 g = Q_1$. The lattice T_1 is preserved by g if and only if $ge_1, ge_2, 3ge_3 \in T_1$. Since $T_1 = \langle e_1, e_2, 3e_3 \rangle$, this is the case if and only if the coefficients a_{31}, a_{32} are $0 \bmod 3$. The relation ${}^t g Q_1 g = Q_1$ implies that mod 3, one has

$$ {}^t g Q_1 g = \begin{pmatrix} * & * & 2a_{31}a_{33} \\ * & * & 2a_{32}a_{33} \\ * & * & 2a_{33}^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, $$

thus $a_{33} = 1$ or $2 \bmod 3$ and $a_{31} = a_{32} = 0 \bmod 3$, which implies the result. □

Let T_2 be a lattice with Gram matrix

$$ Q_2 = \begin{pmatrix} -6k & 0 & 0 \\ 0 & 6 & 9 \\ 0 & 9 & 18 \end{pmatrix} $$

in some basis v_1, v_2, v_3; it is isomorphic to T_1. Let $\iota : T_2 \hookrightarrow T(X)$ be an embedding of lattices. Let us identify T_2 with its image in $T(X)$ trough the embedding ι.

Proposition 16. *Suppose that $k \neq 0$ and $6 \bmod 9$. Then $T_1 = T_2$, in other words: $T(X)$ contains a unique sub-lattice isomorphic to $T(A)(3)$.*

Proof. By Theorem 8, the hypothesis on k implies that the over-lattice $T(X)$ of T_2 is obtained by $\frac{1}{3}v_3 \in T(X)$. Then $\beta_2 = (v_1, v_2, \frac{1}{3}v_3)$ is a basis of $T(X)$. These vectors have intersection matrix Q_2 equal to Q_1. Let P be the base-change matrix between these two basis, one has

$$ Q_1 = {}^t P Q_2 P $$

and since $Q_1 = Q_2$, the matrix P sending the base β_1 to the base β_2 defines an element of $O(T(X))$. Since $O(T(X))$ preserves T_1, the vectors v_1, v_2, v_3 are in T_1, thus $T_1 = T_2$. □

Corollary 17. *When $k \neq 0$ and $6 \bmod 9$, the hypothesis of Theorem 7 are satisfied, and therefore it proves Theorem 1 in these cases.*

7. Cases $L_X^2 = 6k$ with $k = 0$ or $6 \bmod 9$

7.1. **On the implication (II) ⇒ (I) in cases $k = 0$ or $6 \bmod 9$.** Suppose that $k = 0$ or $6 \bmod 9$. Let $v_1, v_2, \dots$ be the generators of the (distinct) isotropic groups $H_1, H_2, \dots$ of $\mathrm{A}_{T(A)(3)}$ such that the corresponding over-lattices $T_1, T_2, \ldots$ are isometric to $T(X)$. Let (B, G_B) be another abelian surface with an order 3 symplectic automorphism group.

Proposition 18. *Suppose that $k = 0$ or $6 \bmod 9$ and that the over-lattices $T_1, T_2, \dots$ of $T(A)(3)$ are isomorphic. If there exists an isomorphism of Hodge structures*

$$ \psi : (T(\mathrm{Km}_3(A)), \mathbb{C}\omega_{\mathrm{Km}_3(A)}) \to (T(\mathrm{Km}_3(B)), \mathbb{C}\omega_{\mathrm{Km}_3(B)}) $$

then there exists an isomorphism of Hodge structures

$$ (T(A), \mathbb{C}\omega_A) \simeq (T(B), \mathbb{C}\omega_B). $$

Proof. Since the over-lattices $T_1, T_2, \dots$ are isomorphic, by Corollary 4, there is a unique embedding of $T(A)(3)$ in $T(\mathrm{Km}_3(A))$ up to isometries. Moreover, by Proposition 15, any isometries of $T(\mathrm{Km}_3(A))$ preserves the image of $T(A)(3)$, thus we recover uniquely $(T(A), \mathbb{C}\omega_A)$ from $(T(\mathrm{Km}_3(A)), \mathbb{C}\omega_{\mathrm{Km}_3(A)})$. The isomorphism ψ must send $T(A)(3)$ to $T(B)(3)$, and therefore it induces an isomorphism of Hodge structures between $(T(A), \mathbb{C}\omega_A)$ and $(T(B), \mathbb{C}\omega_B)$. □

The next sub-section shows that the over-lattices $T_1, T_2, \dots$ of $T(A)(3)$ are indeed isomorphic, so that the hypothesis of Proposition 18 are satisfied, and that will prove the implication (II) ⇒ (I) of Theorem 1.

7.2. **On the orthogonal group of $T(A)(3)$ and isomorphic over-lattices.** Recall that $T(A)(3)$ has a basis with Gram matrix

$$\begin{pmatrix} -6k & 0 & 0 \\ 0 & 6 & 9 \\ 0 & 9 & 18 \end{pmatrix}. \tag{7.1}$$

Proposition 19. *The lattice $T(A)(3)$ is unique in its genus and the map*

$$O(T(A)(3)) \to O(A_{T(A)(3)})$$

is surjective.

Proof. The discriminant group $\mathrm{A}_{T(A)(3)}$ is (isomorphic to) $\mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/6k\mathbb{Z}$. For a prime p, the length ℓ_p of the p-torsion subgroup is $\ell_3 = 3$, $\ell_p = 1$ for $p \neq 3$ dividing k, otherwise $\ell_p = 0$.

By [7, Chapter VIII, Lemma 7.7(1)], the quadratic form Q is 2-regular, by [7, Chapter VIII, Lemma 7.6(3)] Q is 3-pseudoregular, and by [7, Chapter VIII, Lemma 7.6(1)] it is p-regular for any prime $p \geq 5$. One can therefore apply [7, Chapter VIII, Theorem 7.5 (4)] to conclude that the genus of $T(A)(3)$ is $\{T(A)(3)\}$ and that the natural map $O(T(A)(3)) \to O(\mathrm{A}_{T(A)(3)})$ is surjective. □

If L is a torsion quadratic module, there is a decomposition $L = \oplus_p L_p$ into p-torsion elements, and

$$O(L) = \prod_p O(L_p),$$

where O is the orthogonal group i.e. the group preserving the quadratic form. The discriminant group $\mathrm{A}_{T(A)(3)}$ has quadratic form

$$Q = \begin{pmatrix} -\frac{1}{6k} & 0 & 0 \\ 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{2}{9} \end{pmatrix}.$$

Let us prove the following result:

Proposition 20. *Suppose that $k = 6 \bmod 9$ and consider $v_0 = (0, 0, \frac{1}{3})$ and $v_1 = (\frac{1}{3}, 0, 0)$. The associated over-lattices T_{v_0}, T_{v_1} are isomorphic.*

Proof. In order to prove that the over-lattices T_{v_0}, T_{v_1} are isomorphic, one must show that there exists an element $O(T(A)(3))$ that acts on $\mathrm{A}_{T(A)(3)}$ by sending v_0 to v_1. But by Proposition 19, the map $O(T(A)(3)) \to O(\mathrm{A}_{T(A)(3)})$ is surjective, therefore it is sufficient to prove that there exists $\tau \in O(\mathrm{A}_{T(A)(3)})$ sending v_0 to v_1. Since we supposed that $k = 6 \bmod 9$, one has $\left(-\frac{1}{6k}\right)_3 = \left(\frac{2}{9}\right)$, thus on the 3-torsion part, the quadratic form is

$$Q_3 = \begin{pmatrix} \frac{2}{9} & 0 & 0 \\ 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{2}{9} \end{pmatrix}.$$

The matrix

$$\tau = \begin{pmatrix} 0 & -6 & 1 \\ 0 & 1 & 0 \\ 1 & 6 & 0 \end{pmatrix}$$

defines an automorphism of the (3-torsion part of the) group $\mathrm{A}_{T(A)(3)} = \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/6k\mathbb{Z}$ and satisfies ${}^t\tau Q_3 \tau - Q_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 12 & -1 \\ 0 & -1 & 0 \end{pmatrix}$, thus this is an element of the orthogonal group of $\mathrm{A}_{T(A)(3)}$. The transformation exchanges $(\frac{1}{9}, 0, 0)$ and $(0, 0, \frac{1}{9})$, thus it also exchanges $v_1 = (\frac{1}{3}, 0, 0)$ and $v_0 = (0, 0, \frac{1}{3})$. □

Proposition 21. *Suppose that $k = 0 \bmod 9$. Let be $v_0 = (0, 0, \frac{1}{3})$, $w_1 = (\frac{1}{3}, 0, \frac{1}{3})$ and $w_2 = (\frac{1}{3}, 0, \frac{2}{3})$, the elements of $A_{T(A)(3)}$. The associated over-lattices $T_{v_0}, T_{w_1}, T_{w_2}$ are isomorphic.*

Proof. The isometry in Equation (4.5) of $T(A)(3)$ sends w_1 to w_2. By Proposition 19, the map $O(T(A)(3)) \to O(\mathrm{A}_{T(A)(3)})$ is onto, therefore, in order to prove that the over-lattices $T_{v_0}, T_{w_1}, T_{w_2}$ are isomorphic, it is sufficient to find an element $\tau \in O(\mathrm{A}_{T(A)(3)})$ sending v_0 to an element in $\{w_1, 2w_1, w_2, 2w_2\}$. The group $\mathrm{A}_{T(A)(3)}$ is isomorphic to

$$\mathbb{Z}/6k\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}$$

through the map $(\frac{a}{6k}, \frac{b}{3}, \frac{c}{9}) \to (a, b, c)$ and we will often identify $(\frac{a}{6k}, \frac{b}{3}, \frac{c}{9})$ with (a, b, c). The discriminant group $\mathrm{A}_{T(A)(3)}$ has quadratic form

$$Q = \begin{pmatrix} -\frac{1}{6k} & 0 & 0 \\ 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{2}{9} \end{pmatrix}.$$

At prime 3, the quadratic form is

$$Q_3 = \begin{pmatrix} \frac{u}{3^{a+1}} & 0 & 0 \\ 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{2}{9} \end{pmatrix},$$

where $k = 3^a t$, for $a \geq 2$, t coprime to 3 and u is even such that $u(-2t) = 1 \bmod 3^{a+1}$. The torsion quadratic module $\left(\frac{u}{3^{a+1}}\right)$ is isometric to $\left(\frac{2}{3^{a+1}}\right)$ or $\left(\frac{4}{3^{a+1}}\right)$, and the isometry sends the order 3 element element $(3^a, 0, 0)$ to

$(w3^a,0,0)$ with $w\in\{1,2\}$. It remains therefore to study the following two possibilities:

• Suppose that $u=2$ and $a\geq 3$. The matrix

$$\tau_a=\left(\begin{array}{ccc} 2\cdot 3^{a-1}-1 & -14\cdot 3^a & 8\cdot 3^{a-1}\\ 0 & -4 & 1\\ 7 & -48 & 11\end{array}\right).$$

define an endomorphism of the group

$$\mathrm{A}_{T(A)(3)}=\mathbb{Z}/3^{a+1}\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/9\mathbb{Z}$$

(for example, the $(3,2)$ entry is the well-defined map $\mathbb{Z}/3\mathbb{Z}\to\mathbb{Z}/9\mathbb{Z}$, $\bar{x}\to\overline{-48x}$). The matrix

$$\left(\begin{array}{ccc} 2\cdot 3^{a-1}-1 & -14\cdot 3^a & 7\cdot 3^{a-1}\\ 3 & -10 & 2\\ 20 & -87 & 17\end{array}\right)$$

also define an endomorphism and one can check that its product with τ_a acts by the identity on $\mathrm{A}_{T(A)(3)}$, thus τ_a is an automorphism of $\mathrm{A}_{T(A)(3)}$. One computes that

$${}^t\tau_a Q_3\tau_a - Q_3=\left(\begin{array}{ccc} 10+8\cdot 3^{a-3} & -56\cdot(3^{a-2}+1) & 32\cdot 3^{a-3}+13\\ -56\cdot(3^{a-2}+1) & 392(3^{a-1}+1)+2 & -224\cdot 3^{a-2}-89\\ 32\cdot 3^{a-3}+13 & -224\cdot 3^{a-2}-89 & 128\cdot 3^{a-3}+20\end{array}\right)$$

therefore if $a\geq 3$, the entries of the above matrix are integers and the diagonal entries are even numbers, so that τ_a preserves the quadratic form of $\mathrm{A}_{T(A)(3)}$: it is an element of $O(\mathrm{A}_{T(A)(3)})$. One has $\tau_a(0,0,3)=(8\cdot 3^a,3,33)=(2\cdot 3^a,0,6)$, therefore the image by τ_a of the isotropic group generated by $v_0=(0,0,3)$ is the isotropic group generated by $w_1=(3^a,0,3)$: that implies that the two associated over-lattices are isomorphic.

In case $a=2$, it is not difficult to check that the matrix $\tau_2=\left(\begin{array}{ccc} 5 & -18 & 3\\ 2 & -10 & 2\\ 17 & -69 & 14\end{array}\right)$ gives an isometry of $\mathrm{A}_{T(A)(3)}$ with the same properties as above when $a\geq 3$.

• Suppose that $u=4$ and $a\geq 3$. The matrix

$$\theta_a=\left(\begin{array}{ccc} 4\cdot 3^{a-1}-1 & -14\cdot 3^a & 8\cdot 3^{a-1}\\ 1 & -10 & 2\\ 13 & -78 & 16\end{array}\right).$$

define an automorphism of the group $\mathrm{A}_{T(A)(3)}$. One has

$${}^t\theta_a Q_3\theta_a - Q_3=\left(\begin{array}{ccc} 26+64\cdot 3^{a-3} & -224\cdot 3^{a-2}-144 & 128\cdot 3^{a-3}+30\\ -224\cdot 3^{a-2}-144 & 784\cdot 3^{a-1}+898 & -448\cdot 3^{a-2}-185\\ 128\cdot 3^{a-3}+30 & -448\cdot 3^{a-2}-185 & 38+256\cdot 3^{a-3}\end{array}\right),$$

so that θ_a is an element of $O(\mathrm{A}_{T(A)(3)})$ when $a\geq 3$. Since

$$\theta_a(0,0,3)=(8\cdot 3^a,6,48)=(2\cdot 3^a,0,3),$$

the image by θ_a of the isotropic group generated by $v_0=(0,0,3)$ is the isotropic group generated by $2w_2=(2\cdot 3^a,0,3)$: that implies that the two associated over-lattices are isomorphic.

When $a = 2$, it is not difficult to check that the matrix $\theta_2 = \begin{pmatrix} 2 & -126 & 24 \\ 0 & -4 & 1 \\ 5 & -66 & 14 \end{pmatrix}$ gives an isometry with the same properties as above. □

Corollary 22. *Suppose that $k = 3k'$, with $k' = 0$ or $2 \bmod 3$. Suppose that there is an isomorphism of Hodge structures*

$$(T(\mathrm{Km}_3(A)), \mathbb{C}\omega_{\mathrm{Km}_3(A)}) \simeq (T(\mathrm{Km}_3(B)), \mathbb{C}\omega_{\mathrm{Km}_3(B)}).$$

Then there is an isomorphism of Hodge structures

$$(T(A), \mathbb{C}\omega_A) \simeq (T(B), \mathbb{C}\omega_B).$$

Proof. In this cases, all the over-lattices that are isometric to $T(X)$ are isomorphic, thus we can apply Proposition 18. □

7.3. **About the implication (I) ⇒ (II).** Suppose that $k = 0$ or $6 \bmod 9$. Let $v_1, v_2, \dots$ be generators of the isotropic groups $H_1, H_2, \dots$ of $\mathrm{A}_{T(A)(3)}$ such that the corresponding over-lattices $T_1, T_2, ...$ of $T(A)(3)$ are isometric to $T(X)$. We know from Propositions 20 and 21 that these lattices $T_1, T_2, \dots$ are isomorphic.

Proposition 23. *Let be $i \neq j$ and let ω defining a Hodge structure on $T_i \otimes \mathbb{Q} = T_j \otimes \mathbb{Q}$. The Hodge structures $(T_i, \mathbb{C}\omega)$ and $(T_j, \mathbb{C}\omega)$ are not isomorphic for a general period ω.*

Proof. Let us recall some facts about integral Hodge structures of K3 type, for which a reference is [12, Section 7.2.3]. A Hodge structure on a rank 3 lattice T of signature $(2, 1)$ is the data of a point $\omega \in \mathbb{P}(T \otimes \mathbb{C})$ such that $w^2 = 0$ and $w\bar{w} > 0$, for $w \in \omega = \mathbb{C}w$. Then the space $T \otimes \mathbb{C}$ decomposes as

$$T \otimes \mathbb{C} = \mathbb{C}w \oplus \mathbb{C}t \oplus \mathbb{C}\bar{w},$$

where $\mathbb{C}t$ is the orthogonal complement of the space $\mathbb{C}w \oplus \mathbb{C}\bar{w}$; this is a real subspace: $\overline{\mathbb{C}t} = \mathbb{C}t$. The set of Hodge structures is an (euclidian) open subset in the smooth quadric $Q_T \simeq \mathbb{P}^1$ defined by $\omega^2 = 0$. In fact, this is the complement of the real axis in $\mathbb{C} \subset \mathbb{P}^1 = Q_T$, in particular, it is biholomorphic to $\mathcal{H} \cup \overline{\mathcal{H}}$, where $\mathcal{H}$ is the complex upper-plane (see [1, Section 2.3]). Two Hodge structures ω, ω' on T are isomorphic if and only if there exists an isometry $g \in O(T)$ such that

$$g_{\mathbb{C}}(\omega) = \omega',$$

where $g_{\mathbb{C}}$ is the complexification of g (an isometry of T acts as an homography on $\mathcal{H} \cup \overline{\mathcal{H}} \subset Q_T$).

The fixed point set of a projective automorphism g acting on $\mathbb{P}(T \otimes \mathbb{C})$ is a union of linear subspaces, thus the stabilizer group (in the projective automorphism group) of the smooth quadric Q_T acts faithfully on Q_T.

Since we suppose that the over-lattices T_i and T_j are isomorphic, there exist isometries $h_| \in O(T(A)(3))$ and $h : T_i \to T_j$ such that the following diagram is commutative

$$\begin{array}{ccc} T(A)(3) & \hookrightarrow & T_i \\ \downarrow h_{|} & & \downarrow h \\ T(A)(3) & \hookrightarrow & T_j \end{array}.$$

We consider T_i and T_j contained in $T(A)(3)\otimes\mathbb{C}$. The complexification $h_{\mathbb{C}}$ of the lattice isometry h preserves the quadric $Q=\{\omega^2=0\}\subset\mathbb{P}(T(A)(3)\otimes\mathbb{C})$ containing the Hodge structures. That quadric is the same for $T(A)(3), T_i$ and T_j. For $\omega\in Q$, the isometry h induces an isomorphism between the Hodge structures (T_i,ω) and $(T_j, h_{\mathbb{C}}(\omega))$.

Suppose that for a general period $\omega\in Q$, the Hodge structures (T_i,ω) and (T_j,ω) are isomorphic. Then (T_j,ω) and $(T_j,h_{\mathbb{C}}(\omega))$ are isomorphic: there exists an isometry $g\in O(T_j)$ such that $g_{\mathbb{C}}(\omega)=h_{\mathbb{C}}(\omega)$. Since $O(T_j)$ is a countable set, there exists a $g\in O(T_j)$ such that for an infinite number of ω, one has $g_{\mathbb{C}}(\omega)=h_{\mathbb{C}}(\omega)$, and therefore, in fact, $g_{\mathbb{C}}(\omega)=h_{\mathbb{C}}(\omega)$ for all ω, thus $\forall\omega\in Q,\ h_{\mathbb{C}}^{-1}g_{\mathbb{C}}(\omega)=\omega$. Since the projective automorphism group preserving Q acts faithfully on Q , this implies that $h=\pm g$, and h is an isometry of T_j. This is a contradiction, therefore for general $\omega\in Q$, the Hodge structures (T_j,ω) and $(T_j,h_{\mathbb{C}}(\omega))$ are not isomorphic, and we conclude that (T_j,ω), (T_i,ω) are not isomorphic. □

Recall that for $k=0$ or $6 \bmod 9$, we denote by $T_1, T_2, ...$ the over-lattices of $T(A)(3)$ that are isometric to $T(X)$. Let us fix a period ω. The Hodge structure $(T(A)(3),\omega)$ induces Hodge structures

$$(T_j,\omega), j=1,2,...$$

and by Proposition 23, these Hodge structures are not isomorphic for general ω. One has

Corollary 24. *By the subjectivity of the Period map, there exist generalized Kummer surfaces surfaces $X_1, X_2,\dots$ such that $(T(X_s),\mathbb{C}\omega_{X_s})\simeq(T_s,\mathbb{C}\omega)$ for $s=1,2,...,$ with $X_s=\mathrm{Km}(A_s,G_s)$, such that $A_1,A_2,...$ are Fourier-Mukai partners but such that $X_1,X_2,\dots$ are not isomorphic.*

Remark 25. We remark that one has an isomorphism of $\mathbb{Q}$-Hodge structures $(T(X_i)\otimes\mathbb{Q},\mathbb{C}\omega)\simeq(T(X_j)\otimes\mathbb{Q},\mathbb{C}\omega)$, and according to [9] it is algebraic, i.e. is induced by a correspondence between X_i and X_j.

References

[1] Barth W., On the classification of K3 surfaces with nine cusps, Geom. Dedicata 72 (1998), no. 2, 171-178.

[2] Cartwright D., Roulleau X., A refinement of Bézout's Lemma, and order 3 elements in some quaternion algebras over $\mathbb{Q}$, arXiv

[3] Conway J. H., Sloane N. J. A., Sphere packings, lattices and groups. Third edition. Grun. der math. Wiss., 290. Springer-Verlag, New York, 1999. lxxiv+703

[4] Hosono S., Lian B.H., Oguiso K., Yau S.-T., Kummer structures on K3 surface: an old question of T. Shioda. Duke Math. J. 120 (2003), no. 3, 635–647.

[5] Hosono S., Lian B.H., Oguiso K., Yau S.-T., Fourier-Mukai number of a K3 surface, in Algebraic structures and moduli spaces, 177192, CRM Proc. Lecture Notes, 38, Amer. Math. Soc., Providence, RI, 2004.

[6] Kohel D., Roulleau X., Sarti A., A special configuration of 12 conics and generalized Kummer surfaces, to appear in Manus. Math.
[7] Miranda R., Morrison D. R., Embeddings of Integral Quadratic Forms, preprint
[8] V. V. Nikulin, Integral symmetric bilinear forms and some of their applications, Izv. Akad. Nauk SSSR Ser. Mat., 1979, Volume 43, Issue 1, Pages 111–177.
[9] V. V. Nikulin, On correspondences between surfaces of K3 type, Izv. Akad. Nauk SSSR Ser. Mat. 51 (1987), no. 2, 402–411, 448; translation in Math. USSR-Izv. 30 (1988), no. 2, 375–383
[10] Roulleau X., Number of Kummer structures and Moduli spaces of generalized Kummer surfaces, preprint arXiv
[11] Roulleau X., Sarti A., Constructions of Kummer structures on generalized Kummer surfaces, preprint arXiv
[12] Voisin C., Théorie de Hodge et géométrie algébrique complexe, Cours Spécialisés, 10. SMF, Paris, 2002. viii+595 pp.

Xavier Roulleau
Université d'Angers,
CNRS, LAREMA, SFR MATHSTIC,
F-49000 Angers, France
xavier.roulleau@univ-angers.fr

Alessandra Sarti
Université de Poitiers
Laboratoire de Mathématiques et Applications,
UMR 7348 du CNRS,
TSA 61125
11 bd Marie et Pierre Curie,
86073 Poitiers Cedex 9, France
Alessandra.Sarti@math.univ-poitiers.fr
http://www-math.sp2mi.univ-poitiers.fr/~sarti/