Autumn School

Plan

1) Motivation / Introduction (Lect 1)
2) IHS-manifolds: definitions & properties, lattice theory (Lect 1–2)
3) Automorphisms: properties, examples (Lect 2–3–4)
4) Automorphisms of IHS $\sim K3^{[2]}$, rk Pic $=2$ .... (Lect 4)

§ **Motivation / Introduction** Before go to hyperkähler (= irreducible holomorphic symplectic = IHS) manifolds let's recall what happen in low dimension.

1. Elliptic curves (dim $=1$)

$E = \dfrac{\mathbb{C}}{\Lambda_2}$, Λ_2 is a rank 2 lattice, $\Lambda_2 = \mathbb{Z} \oplus \tau\mathbb{Z}$, $\tau \in \mathbb{C}$, $\operatorname{Im}\tau > 0$

$\updownarrow$

Compact Riemann surface of genus 1

These are all projective $E \hookrightarrow \mathbb{P}^2(\mathbb{C})$ and have a concrete description:

$$y^2 = x(x-1)(x-\lambda); \quad \lambda \in \mathbb{C}$$ (double cover of $\mathbb{P}^1$ ramified in $0, 1, \lambda, \infty$)

• $\exists!$ (modulo constant) 1-form holomorphic global without zeros (locally dz) i.e. (so $K_E \sim 0$) and

$$\Omega_E = \mathcal{O}_E(K_E) = \mathcal{O}_E$$

cotangent bundle holomorphic of 1-forms $f(z)dz$ — canonical divisor

$$H^0(\Omega_E) = \mathbb{C}\cdot dz$$

• are not simply connected.

2. Dimension = 2 : surfaces not projective in general (2)

most simply connected

2-dimensional tori $\qquad\qquad$ K3 surfaces.

$$T = \mathbb{C}^2 / \Lambda_4 \quad , \ \mathrm{rk}\,\Lambda_4 = 4 \text{ cocompact}.$$

$$\Omega^2_T = \mathcal{O}_T(K_T) = \mathcal{O}_T$$

$K_T \sim 0$ trivial.

$\boxed{\text{K3 surfaces}}$: most easy example:

$$\{ \underbrace{f_4(x_0, x_1, x_2, x_3)}_{\text{homogeneous of deg } 4} = 0 \} \subset \mathbb{P}^3(\mathbb{C}) \quad \text{smooth is K3}$$

concrete eg: $\{ x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0 \}$ Fermat surface.

Def A K3 surface is a cpct. cplx smooth surface S ($\dim S = 2$)

s.t.
- $\pi_1(S) = \{1\}$
- $H^0(S, \Omega^2_S) = \mathbb{C} \cdot \omega_S$

(Name: A. Weil 1958 Kummer, Kähler, Kodaira and beautiful mountain K2 in Cachemire)

ω_S = global holomorphic 2 form without zeros

Consequences: $K_S \sim 0$ (Trivial canonical divisor).

Rm: ① K3 surfaces are the simply connected analogous of ell. curves

② If X is a cpct, cplx, (Kähler) vary. (smooth)

$$c_1(X) = c_1^{\mathbb{R}}(T_X) \in H^2(X, \mathbb{Z}) \otimes \mathbb{R} \quad \text{(first chern class)}.$$

$$c_1^{\mathbb{R}}(X) = 0 \iff \exists\, m \in \mathbb{N}^* \text{ s.t. } \left(\Omega_X^{\dim X}\right)^{\otimes m} = \mathcal{O}_X$$

(i.e. $mK_X \sim 0$)

So that c_1 gives informations on the differential forms on X.

In our examples $c_1(\text{Torus}) = c_1(K3) = 0$

Tori + K3 are "stones" to construct Kähler surfaces with $c_1 = 0$

Prop 1 X be a Kähler surface, $c_1(X) = 0$

Then X is an étale quotient of a Torus or of a K3-surface:

3

E.g. • $X = \frac{S}{\langle i \rangle}$, S K3, i a.s. invol. $i \subset X$ ($i^*\omega_S = -\omega_S$) $\mathrm{Fix}(i) = \phi$

$\Rightarrow X$ is an $\boxed{\text{Enriques Surface}}$ has $2K_X \sim 0$

• $\boxed{\text{bielliptic Surfaces}}$ $X = \frac{E \times F}{G}$; E, F elliptic curves

(G group fin. de Translat. of E st G acts on F by $\frac{F}{G} = \mathbb{P}_1$)

All classified by Bagnera - De Franchis beginning 1900, have $jK_X \sim 0$ $j \in \{2, 4, 3, 6\}$.

+ K3, Tori

Rm ① these are all the surfaces with Kodaira-dimension $K = 0$ in the Enriques - Kodaira classification.

② If one removes "Kähler" then prop 1 is no more true, $\exists$ surfaces (not Kähler) with $c_1 = 0$ not quotients of K3 & Tori (Kodaira primary & secondary)

A Theorem of Beauville-Bogomolov.

Recall that compact complex mnf X is Kähler if $\exists$ real (1,1)-form ~~[illegible]~~ that comes from an hermitian metric and it is closed.

Theorem (Bogomolov 1974 - Beauville 1984, Berger 1955 - Yau 1982)

X cpt, cplx Kähler unramified $c_1(X)^{\mathbb{R}} = 0$

there exists an étale finite covering $\tilde{X}$ of X s.t.

$$\tilde{X} = T \times \prod_i V_i \times \prod_j X_j \quad \text{where}$$

) T = Torus

• V_i is simply connected projective ($\Rightarrow$ Kähler) $\dim V_i \geq 3$

$$\Lambda^{\dim V_i} \Omega_{V_i} = \Omega^{\dim V_i}_{V_i} = \mathcal{O}_{V_i}(K_{V_i}) = \mathcal{O}_{V_i}$$

(i.e. $K_{V_i} \sim 0$ Trivial canonical bundle)

& $H^0(V_i, \Omega^m_{V_i}) = 0 \quad \forall\, 0 < m < \dim V_i$

(i.e. Hodge numbers: $h^{m,0} = 0$, $0 < m < \dim V_i$)

They are (simply connected) CY manifolds

(iii) X_j is simply connected cpct Kähler manf. admitting a global holo. 2-form φ_j everywhere non deg.

$\hookrightarrow$ they are IHS manifolds

(A big theorem the proof uses results of Berger of 1955 that classify the holonomy group of Riemannian mf. (diff geometry).)

Rmk . In dim = 2 we have: $CY = IHS = K3$

. In dim > 2 : $\exists$ IHS et $CY \neq IHS$.

can be not simply connected but always proj. ($H^{2,0} = 0$ if dim ≥ 3)

Irreducible holomorphic symplectic (IHS) = hyperkähler manifolds

Are the varieties in part (iii) of the theorem:

Def (these are described by Beauville in a paper of '83)

X cpct, cplx, mnf., Kähler is an IHS manifold:

* $\pi_1(X) = \{1\}$

* $\exists !$ global holomorphic 2-form everywhere non-deg s.t.

$$H^0(X, \Omega^2_X) = \mathbb{C}\cdot\varphi.$$

analytic topol.

Locally $\varphi = \sum a_{ij}(z_1, \dots, z_n)\, dz_i \wedge dz_j$, $(z_1, \dots, z_n)$ local coord at a pt $p \in X$.

holo

Rmk ① "non-deg" means as an alternating form on the holo tg bundle $T^{1,0}$:

$\dim X = n$

$$T^{1,0}_{X,p} \times T^{1,0}_{X,p} \longrightarrow \mathbb{C}$$
$$(u, v) \longmapsto \varphi(u, v)$$

is given by an alternating $n \times n$ matrix M
($\varphi(u,u) = 0$
$\varphi(u,v) = -\varphi(v,u)$)

So that

$$M = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & 0 & \\ & & \ddots & & \\ & & & 0 & 1 \\ & 0 & & -1 & 0 \end{pmatrix} \Rightarrow \dim X \text{ is } \underline{\text{even}}$$

($\dim T^{1,0}_{X,p}$ is even).

We write $\dim X = 2m$ ($\neq$ CY that exist also in odd dim e.g. quintic 3-fold in $\mathbb{P}^4$).

(bil., alt., non deg) Locally: $dz_1 \wedge dy_1 + \dots + dz_m \wedge dy_m$

② φ is a symplectic form: it is also closed $d\varphi = 0$

(holo forms on a Kähler mnf. are closed).

(So IHS mnf has 2 symp. form $\to$ (1,1) real form, the Kähler form.

$\to$ (2,0) cplx. form.

More consequences:

$\dim X = 2m$

- φ is a (2,0) form non deg $\Leftrightarrow$ $\wedge^m \varphi$ has no zeros so $K_X \sim 0$ for an IHS.

Beauville using holonomy (geo. diff. arguments)

- $H^0(X, \Omega_X^p) = 0$ p odd.
- $H^0(X, \Omega_X^{2p}) = \mathbb{C} \cdot \varphi^p$ $0 \leq p \leq \frac{1}{2} \dim X$

$\varphi^p = \wedge^p \varphi$

$$\Rightarrow \chi(\mathcal{O}_X) = h^0 - h^1 + h^2 - h^3 + \dots + h^{\overbrace{2m}^{\dim X}} = m+1$$

Famous examples ($\neq$ K3)

Fujiki – Beauville

$S^{[2]}$ 1983

$S^{[n]} = \mathrm{Hilb}^n(S)$, S K3

= Hilbert scheme of n pts on a K3 surface

$\dim S^{[n]} = 2m$

$b_2 = 23$

+ defo. (FPK, Fano var. of lines..., LLSvS for $\mathrm{Hilb}^{[3]}(S)$)

not defo equiv. since $\neq$ Betti numbers

$K_n(A) =$ (A: 2-dim Torus.)

= generalized Kummer manifold.

$\dim K_n(A) = 2n$

$b_2 = 7$

+ no defo known! (Recent work in progress by O'Grady...)

+ $\exists$ 2 examples by O'Grady ~ 2000 in dim 6 & 10.
$b_2 = 8$ & $b_2 = 24$.

(Some moduli spaces of sheaves on K3 & abelian surfaces)

Recall the construction of $S^{[n]}$, S K3

S a K3 surface, $S^{[n]}$ = Hilbert scheme (Douady space if S not proj).
= space that parametrizes finite analytic subspaces Z,
supp$(Z) = \{$pts$\}$ (i.e. dim $Z = 0$) and $\ell g(\mathcal{O}_Z) = n$
(ex. n pts on S, or pts with tg direction)

Let $S^n = \underbrace{S \times \cdots \times S}_{n\text{-times}}$; $S^{(n)} = \dfrac{\overbrace{S \times \cdots \times S}^{n\text{-times}}}{\Sigma_n}$ & permutations of n pts.

locus smooth

Hilbert Chow morphism

$$\begin{tikzcd} & & S^n \supset \Delta \arrow[d, "\pi"] & \\ E \subset S^{[n]} \arrow[rr, "\varepsilon"] & & S^{(n)} \supset D \end{tikzcd}$$

quotient map.

$\boxed{\text{Fogarty-Iarrobino 1970}}$

$S^{(n)}$ is singular along a divisor D (locus of singularities) small resolutions.

Let $\Delta = \bigcup_{1 \leq i < j \leq n} \Delta_{ij}$

$\Delta_{ij} = \{(x_1, \ldots, x_n) \in S^n \mid \exists\, i \neq j$ s.t. $x_i = x_j\}$ $1 \leq i < j \leq n$.

so let $D = \pi(\Delta)$

10/10/2017

$S^{[n]}$ is a smooth, cpct, cplx, simply connected dim $S^{[n]} = 2n$.
Varouchas 1988: $S^{[n]}$ is Kähler (paper of Beauville 1984 (before Varouchas) $S^{[n]}$ is IHS only for S proj.)
Beauville 1984: $S^{[n]}$ is IHS of dim $2n$.

Construction of $K_n(A)$ (Beauville).

not IHS
not simply conn!

$$\begin{tikzcd} & A^{n+1} \arrow[d] & \\ A^{[n+1]} \arrow[r, "\varepsilon"] \arrow[rr, bend right, "p"'] & A^{(n+1)} \arrow[r] & A \end{tikzcd}$$

summation map.

(not proj)

$\pi^{-1}(0) \subset A^{[n+1]}$ has dim $2n$ and is $K_n(A)$

$x_0 + \cdots + x_n = 0$.

Back to $S^{[n]}$: We have an embedding:

$$i : H^2(S, \mathbb{C}) \hookrightarrow H^2(S^{[n]}, \mathbb{C})$$ (respecting the Hodge decomps.)

and (*) $H^2(S^{[n]}, \mathbb{C}) = i\,(H^2(\underset{\substack{\uparrow \\ \text{K3 surface}}}{S}, \mathbb{C})) \oplus \mathbb{C}\underset{\substack{\uparrow \\ \text{excep. divisor}}}{[E]}$

(Beauville uses this to prove unicity of the (2,0)-form).

We have then: $\dim H^{2,0}(S^{[n]}) = \dim H^{2,0}(S) = 1$

($[E]$ is alg. class so that it is in $H^{1,1}(S^{[n]}) \cap H^2(S^{[n]}, \mathbb{Z})$)

One can write (*) at the level of integral cohomology:

$$H^2(S^{[n]}, \mathbb{Z}) = i\,(H^2(S, \mathbb{Z})) \oplus \mathbb{Z}\delta$$

where $2\delta = [E]$ (for $n=2$ easy to see, $\exists$ double cover...).

The free $\mathbb{Z}$-module $H^2(S^{[n]}, \mathbb{Z})$ plays an important rôle for IHS-manifolds as $H^2(S, \mathbb{Z})$ for K3 surfaces!

Lattice theory

Def A lattice of rk r is a free $\mathbb{Z}$-module $\mathcal{L}$ of rk r together with a symmetric bilinear form:

$$b : \mathcal{L} \times \mathcal{L} \longrightarrow \mathbb{Z}$$
$$(x, y) \longmapsto b(x,y) := \langle x, y \rangle.$$

Signature of $\mathcal{L}$ = signature of $\overset{\langle , \rangle}{b}$ on $\mathcal{L} \otimes \mathbb{R}$

$\mathcal{L}$ non deg if b, non deg.

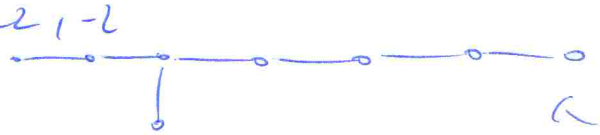
if $\mathcal{L}$ non deg at $\operatorname{sgn}(\mathcal{L}) = \underset{(t_+, 1)}{(1, t_-)} \Rightarrow \mathcal{L}$ is called hyperbolic

$\mathcal{L}$ is even if $\langle x, x \rangle \in 2\mathbb{Z}$ $\forall x \in \mathcal{L}$

$\mathcal{L}$ is called unimodular if $|\underset{\substack{\uparrow \\ \text{Gram matrix ass. to } \mathcal{L}}}{\det \mathcal{L}}| = 1$

example . $U=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ is the hyperbolic plane, hyperbolic lattice, even unimodular.

$$\mathbb{Z}^2\times\mathbb{Z}^2 \longrightarrow \mathbb{Z}$$
$$((a,b),(c,d)) \longmapsto (a,b)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}c\\d\end{pmatrix} = (b\ a)\begin{pmatrix}c\\d\end{pmatrix} = bc+ad$$

. $E_8(-1)$  is neg. def., even, unimodular lattice.

$\rightsquigarrow \begin{pmatrix}-2&1&&\\1&-2&&\\&&\ddots&\\&&&\end{pmatrix}$ Cartan matrix.

. $A_1(-1) = \langle -2\rangle$ $\quad \mathbb{Z}\times\mathbb{Z}\longrightarrow\mathbb{Z}$, $(a,b)\mapsto -2ab$, $A_n(-1)$ 

. ~~Homo~~ of lattices L, L' $\quad f: L\longrightarrow L'$ s.t. $\langle f(x),f(y)\rangle_{L'} = \langle x,y\rangle_L$

f inj. $\Rightarrow$ $L\hookrightarrow L'$ emb of lattices

if $L'=L$, f bij $\Rightarrow$ f is an ~~iso~~ isometry of L,

$O(L)$ = group of isometries.

Embedding of lattices is **primitive** : $f: L\hookrightarrow L'$ if $\dfrac{L'}{f(L)}$ is free

Example (primitive embeddings).

Take $U=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ the hyperbolic lattice

and $U(2)=\begin{pmatrix}0&2\\2&0\end{pmatrix}$ (in gen. $L(m)$ is the lattice L with the bil. form mult. by $m\in\mathbb{N}_{>0}$).

gen f_1,f_2 $\qquad$ gen e,f

$i: U(2)\hookrightarrow U$ $\qquad$ i inj + isometry

$f_1\longmapsto 2e = h_1$ $\quad$ does $h_1^2=h_2^2=0$, $h_1h_2 = 2ef = 2$

$f_2\longmapsto f = h_2$

but $\dfrac{U}{i(U(2))} = \dfrac{\mathbb{Z}}{2\mathbb{Z}}$ in fact $e\in U$ ~~and~~ $2e\in i(U(2))$ is torsion and the embedding can not be primitive

(on the other hand $rk(U(2)) = rk(U)$ so that $U(2)\hookrightarrow U$ can not be primitive since $U(2)\not\cong U$, $\nexists$ primitive emb $U(2)\hookrightarrow U$ iso!

$U(2)$ has index 2 in U.

Assume $\mathcal{L}$ non degenerate.

If $\mathcal{L}$ is even we can associate a quadratic form.

$$f : \mathcal{L} \to \mathbb{Z} \quad \text{s.t.} \quad 1)\ f(nx) = n^2 f(x)$$
$$2)\ \langle x,y\rangle = \frac{1}{2}\left(f(x+y) - f(x) - f(y)\right)$$

Recall the definition of dual lattice!

$$\mathcal{L}^\vee = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{L},\mathbb{Z}) = \{ v \in \mathcal{L}\otimes\mathbb{Q} \mid \langle v, c\rangle \in \mathbb{Z}\ \forall c \in \mathcal{L}\}$$

and $\mathcal{L} \hookrightarrow \mathcal{L}^\vee$, $x \mapsto \langle\ ,x\rangle$

discriminant group: $\frac{\mathcal{L}^\vee}{\mathcal{L}}$ is finite abelian group.

(assume $\mathcal{L}$ non deg!) and $\left|\frac{\mathcal{L}^\vee}{\mathcal{L}}\right| = |\det(\mathcal{L})|$

det of the matrix of $\mathcal{L}$.

If $\mathcal{L}$ is unimodular $\Rightarrow \frac{\mathcal{L}^\vee}{\mathcal{L}}$ is trivial.

$\mathcal{L}$ is said p-elementary, p prime if $\frac{\mathcal{L}^\vee}{\mathcal{L}} \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus e}$, $e \in \mathbb{Z}_{\geq 0}$

Ex. $A_1(-1)$ is 2-el.
$A_2(-1)$ is 3-el.

Lattice theory is a powerful tool in the study of hyperkähler manifolds!

$$\boxed{\text{An example with K3 surfaces}}$$

cup product $H^2(S,\mathbb{Z}) \times H^2(S,\mathbb{Z}) \to H^4(S,\mathbb{Z}) = \mathbb{Z}$
$(\alpha,\beta) \longmapsto \alpha\cup\beta$

S K3 surface, $H^2(S,\mathbb{Z}) \cong U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$ unimodular lattice of signature $(3,19)$

torsion free $\mathbb{Z}$-module

Consider the rk 1 lattice $\langle 2\rangle = A_1(-1)$

We can give a primitive embedding:

$$\langle 2\rangle \hookrightarrow U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \quad (\text{unique up to isometry})$$
$$g \longmapsto e+f \qquad e,f \text{ gen of a copy of } U.$$

By surjectivity of period map $\exists$ a K3 surface S with

$$NS(S) = \mathbb{Z}g, \quad g^2 = 2 \quad (NS(S) = \langle 2\rangle)$$

$H^{1,1}(S) \cap H^2(S,\mathbb{Z}) = H^{2,0}(S)^\perp \cap H^2(S,\mathbb{Z})$; $H^{2,0}(S) = \mathbb{C}\omega_S$ global hol 2 form

One can show that g is an ample class

By work of Saint-Donat (1974) we have a map (Study of proj. models!)

$$i \circlearrowright S \xrightarrow{2:1} \mathbb{P}_2$$
$$\cup$$
$$C_6 \text{ smooth sextic curve.}$$

So that S admits an involution $S \circlearrowleft i$ s.t. $\frac{S}{i} = \mathbb{P}_2$

i induces an action $i^* \circlearrowright H^2(S,\mathbb{Z})$ with:

$$H^2(S,\mathbb{Z})^{i^*} = \langle 2 \rangle$$ class coming from the class of a line on $\mathbb{P}_2$ ← only inv. class. fam.

$$\left(H^2(S,\mathbb{Z})^{i^*}\right)^\perp = U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2 \rangle$$
e.g. in U.

One can construct the involution directly on the lattice and then use Torelli:

$f: H^2(S,\mathbb{Z}) \longrightarrow H^2(S,\mathbb{Z})$ isom. then f effective Hodge isom. $\Rightarrow \exists!\, i : S \to S$ auto s.t. $i^* = f$.

preserves the Hodge decomp.

sends Kähler cone to Kähler cone (i.e. eg. ample class) to ample class

So if S is K3 with $NS(S) = \langle 2 \rangle = \mathbb{Z}h \Rightarrow h$ is ample (no -2 curves!)

Define the isometry as follows:

$$f: H^2(S,\mathbb{Z}) \longrightarrow H^2(S,\mathbb{Z}) \quad \text{is isom.} \quad f^2 = id.$$
$$v \longmapsto (v,h)h - v$$

. effective $\quad f^*(h) = 2h - h = h \quad$ ample class → ample class

$(\omega_S, h) = 0$ by def of $NS(S)$

. Hodge isom $\quad f(\omega_S) = -\omega_S$

$\Rightarrow \exists\, i \circlearrowright S$ s.t. $i^* = f$ and since

$$Aut(S) \longrightarrow O(S) \text{ inj.}$$

$\Rightarrow i^2 = Id$ is involution.

Rmk Torelli theorem works in the same way for $S^{[2]}$ and deformations! (see later)

Back to IHS manifolds

X IHS $\Rightarrow$ $\exists$ quadratic form non-degenerate, $\mathrm{Sgn}(f) = (3, b_2 - 3)$ (bilin. form assoc.)

$b_2 = \mathrm{rk}(H^2(X,\mathbb{Z}))$ called Beauville–Bogomolov–Fujiki quadratic form.

So that $(H^2(X,\mathbb{Z}), f)$ is a lattice

Rmk ① ~~If~~ $H^{2,0}(X) = \mathbb{C}\,\omega_X$ one computes by extension to $H^2(X,\mathbb{R}) \otimes \mathbb{C}$

$\omega_X^2 := f(\omega_X) = 0$, $f(\omega_X + \bar\omega_X) > 0$ and $H^{1,1}(X) \perp_f (H^{2,0}(X) \oplus H^{0,2}(X))$ (as for K3 surfaces)

$\qquad \underset{(\omega_X+\bar\omega_X)^2}{\shortparallel}$

② if $X \underset{\text{defo}}{\sim} S^{[n]}$ (κ 4-fold) (i.e. X is equivalent by deformation!)

We have the cup product:

$$H^{2n}(X,\mathbb{Z}) \times H^{2n}(X,\mathbb{Z}) \longrightarrow \mathbb{Z} = H^{4n}(X,\mathbb{Z})$$

$$(\alpha, \beta) \longmapsto \int_X \alpha \cup \beta$$

$\exists$ a constant called Fujiki constant st. for $\alpha \in H^2(X,\mathbb{Z})$.

$$\int_X \alpha^{2n} = c_X f(\alpha)^n \qquad (\text{~~holds true in gen for $S^{[n]}$~~})$$

BBF form. (for $n=2$ ie $X \sim S^{[2]}$ then $c_X = 3$)

③ By def. BBF form one has:

$$H^2(S^{[n]}, \mathbb{Z}) \cong H^2(S,\mathbb{Z}) \oplus \mathbb{Z}\delta, \quad \delta \text{ st. } 2\delta = E \text{ (excep set)}$$

$$\underset{\text{isometric}}{\cong} \Lambda_{K3} \oplus \langle -2(n-1) \rangle$$

$\Lambda_{K3} = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$ the K3 lattice so that $\mathrm{Sgn}(H^2(S^{[n]},\mathbb{Z})) = (3, 20)$

Rmk ① Similar to K3 but no more unimodular!

② the moduli space for $S^{[n]}$ has one dimension more than for K3.

If K3 is projective: 19-dim moduli space but 20-dim mod. space for $S^{[n]}$.
the generic element in the moduli space is not of the form $S^{[n]}$!

6) Automorphisms

Assume that $G \curvearrowright X$ <u>finite</u> auto group (i.e. $g: X \longrightarrow X$ biholo. & bireg.)

We have a character (i.e. a group-hom.):

$$\alpha: G \longrightarrow \mathbb{C}^*$$
$$g \longmapsto \alpha(g) \quad \text{def. by } g^*\omega_X = \alpha(g)\,\omega_X$$

ω s.t. $H^0(\Omega^2_X) = \mathbb{C}\cdot\omega_X$ (g induces an action in cohom. that respects Hodge decomp.).

So that $\alpha(G) \subset \mathbb{C}^*$ is a subgroup: but all multiplicative subgroups of $\mathbb{C}^*$ are finite & cyclic. Let $\alpha(G) = \mu_m = \frac{\mathbb{Z}}{m\mathbb{Z}}$

group of m-roots of unity.

We have an exact sequence of groups:

$$1 \longrightarrow G_0 \hookrightarrow G \longrightarrow \mu_m \longrightarrow 1$$

clearly $\forall g \in G_0$ we have $g^*\omega_X = \omega_X$ (we say that g acts symplectically)

otherwise if $g \in G\setminus G_0$ we say that g acts non-symplectically i.e.

$$g^*\omega_X = \underset{\substack{\uparrow \\ \text{primitive } t\text{-root of unity}}}{\overset{\neq 1}{\zeta_t}}\,\omega_X \quad \text{with } t \,|\, o(g) \quad (\text{and } o(g) \,|\, |G|)$$

- if $o(g) = t \Rightarrow g$ acts purely non-symp. ~~[illegible]~~

in part $o(g)$ can be bigger and $t \,|\, o(g)$).

Rmk 1) If $g \in \mathrm{Aut}(X)$, $o(g) = p$ prime, g acts n.s. then $\boxed{\text{purely n.s.} \equiv \text{n.s.}}$

2) If $G \subset \mathrm{Aut}(X)$ and $\forall g \in G$, g acts purely n.s.

$\Rightarrow G_0 = \{id\}$ and G is cyclic

in part has finite order.

but $\exists\, G \subset \mathrm{Aut}(X)$ of infinite order; always sympl. of infinite order

e.g. if $X = K3$ take transl. by section of ∞ order on an elliptic K3

Examples ① If S is K3 with (non-)symp. auto,
i.e. $\sigma \circlearrowleft S$, $\sigma^* \omega_S = \omega_S$ (resp. $\sigma^* \omega_S = \zeta_4 \omega_S$), ζ_4^1
the σ induces an auto $\sigma^{[n]}$ of the same kind on
$S^{[n]} = \mathrm{Hilb}^{(n)}(S)$ (recall $H^2(S^{[n]}, \mathbb{C}) = H^2(S, \mathbb{Z}) \oplus \mathbb{C}[E]$)

Concretely: $S: \underbrace{x_0^4 + x_1^4 + x_2^4 + x_3^4}_{f(x_0 : \ldots : x_3)} = 0 \subset \mathbb{P}^3$ is the Fermat quartic K3 surface.

S has a lot of auto. ($|\mathrm{Aut}(S)| = \infty$)

$$\left.\begin{aligned} \sigma_1 &: (x_0 : x_1 : x_2 : x_3) \longmapsto (x_0 : x_1 : -x_2 : -x_3) \\ \sigma_2 &: (x_0 : x_1 : x_2 : x_3) \longmapsto (x_0 : x_1 : x_2 : -x_3) \end{aligned}\right\} \text{ involutions on } S.$$

2-form: $\omega_S = \dfrac{dx_2 \wedge dx_3}{\frac{\partial f}{\partial x_1}}$ $\quad (x_0 \neq 0, \frac{\partial f}{\partial x_1} \neq 0)$

$\sigma_1^* \omega_S = \omega_S$; $\sigma_2^* \omega_S = -\omega_S$

We get in a natural way $\sigma_i^{[n]} : S^{[n]} \longrightarrow S^{[n]}$

$\sigma_i^{[n]}(p_1, \ldots, p_n) = (\sigma_i(p_1), \ldots, \sigma_i(p_n))$ and

$\sigma_1^{[n]}$ symp. ; $\sigma_2^{[n]}$ non symp.

$\Big(\omega_{S^{[n]}} \overset{\text{essentially}}{=} p_1^* \omega_S + \ldots + p_n^* \omega_S \Big)$.
$\underset{\text{locally}}{=} dx_1 \wedge dy_1 + \ldots + dx_n \wedge dy_n$

$$\begin{tikzcd} & S \times \ldots \times S \arrow[dl, "p_1"'] \arrow[dr, "p_n"] & \\ S & & S \end{tikzcd}$$

Rmk ① if $f \in E \Rightarrow \sigma_i^{[n]}(f) \in E \Rightarrow \sigma_i^{[n]}$ leaves inv. E.

12/10/2017

② the auto of $S^{[n]}$ that come from auto of S are called "**natural**" and $f \circlearrowleft S^{[n]}$ is natural $\Leftrightarrow f(E) = E$ [BS] 2012

The fixed locus ④ ④

Another interesting example (Beauville)

$S \subset \mathbb{P}_3$ quartic K3 $\big/$ $\not\supset$ lines st. for generic quartic $NS(S) = \mathbb{Z}h$ $\big/$ let $p_1 \neq p_2 \in S$ and $\ell_{p_1 p_2} \cap S = \{p_1 \neq p_2\}$

$Z \in S^{[2]}$ st. $\mathrm{Supp}(Z) = \{p_1, p_2\}$, $Z' \in S^{[2]}$ st. $\mathrm{Supp}(Z') = \{p_1', p_2'\}$

(*)(*) The fixed locus. Let's look also at the fixed locus. 13

$\mathrm{Fix}_{\sigma_1}(S) = 8$ pts

$\mathrm{Fix}_{\sigma_2}(S) = \{x_3 = 0\} \cap S$ = plane curve of degree 4 = genus 3 curve $= C_3$

Let's look on $S^{[2]}$:

$\mathrm{Fix}_{\sigma_1}(S^{[2]}) = 28$ isolated fixed pts $+ 1$ K3 surface, Y

$\binom{8}{2} = 28$

comp. from pts of the form $(x, \sigma_1(x))$

$Y \xrightarrow{\text{bir.}} \dfrac{S}{\langle\sigma_1\rangle}$ & 8 A_1 singularities. 8 fixed pts (x, x)

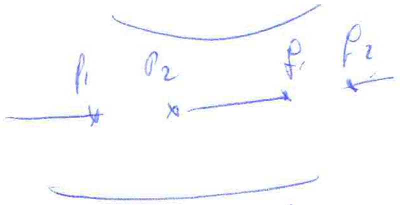
Camere & Mongardi 2010: Symplectic involutions on $S^{[2]}$, always have this fixed locus.

$\mathrm{Fix}_{\sigma_2}(S^{[2]}) = C_3^{(2)} \cup \underbrace{\dfrac{S}{\langle\sigma_2\rangle}}_{\text{smooth rational surface}}$

$C_3^{(2)}$: surface of general type

Classification of the fixed locus of all n.s. invol. on $S^{[2]}$ by Beauville (2010)

Always a smooth surface (not nec. connected)



$i_{B,2}^2 = id$, $\quad i_B = i_{B,2}: S^{[2]} \dashrightarrow S^{[2]}$

$Z \longmapsto Z'$

Beauville: i_B is a non-symp. invol. (one can extend it everywhere).

More in general $S \subset \mathbb{P}^4$, $S = Q \cap C$ (quadric, cubic), i.e. $\deg S = 6$

3 pts on S, p_1, p_2, p_3 (general enough) span a $\mathbb{P}_2 = H_1 \cap H_2 =: P$, $H_1 \cong \mathbb{P}_3$, $H_2 \cong \mathbb{P}_3$

and $P \cap S = \{p_1, p_2, p_3\} \cup \{f_1, f_2, f_3\}$ (distinct in gen.) one gets a birat invol.

$$i_{B,3}: S^{[3]} \dashrightarrow S^{[3]}$$
$$\{p_1, p_2, p_3\} \longmapsto \{f_1, f_2, f_3\}$$

only birat invol. not def eg. if 3 pts are on a line!

One can get more birat invol:

$$S \subset \mathbb{P}^{n+1}, \quad \deg S = 2n$$

Take n pts (gen. enough), $\mathrm{Span}(n \text{ pts}) = \mathbb{P}^{n-1} =: P_{n-1}$ that cuts

$P_{n-1} \cap S = \{n \text{ pts}\} \cup \{n \text{ pts}\}$ one gets an invol.

$$i_{B,n}: S^{[n]} \dashrightarrow S^{[n]}$$

Beauville: never biregular if $n > 2$.

Rk ① the constr of $i_{B,n}$, $n > 2$ answers a question of Bogomolov, $\exists$ birat map between smooth IHS var. without being biregular? Answer is yes! (not possible for K3 surfaces).

2) $i_{B,2} = i_B$ is non-natural: one can show i_B does not leave E invariant (come back later on this fact...) $\left(H^2(S^{[2]}, \mathbb{Z})^{i_B} = \langle 2 \rangle\right)$

Important problem when studying autos: construct non-natural autos i.e. autos that does not come from K3 (Torelli...)

15

More properties of automorphisms (G finite).

Recall 2 important sublattices:

XIHS $\quad H^2(X,\mathbb{Z}) \supset NS(X) = H^{1,1}(X) \cap H^2(X,\mathbb{Z}) \underset{\text{for IHS}}{=} Pic(X)$.

$NS(X) = \frac{Pic(X)}{Pic_0(X)}$ Néron-Severi group.

$H^2(X,\mathbb{Z}) \supset T_X = NS(X)^{\perp} \cap H^2(X,\mathbb{Z})$ Transcendental lattice

Rmk $\quad \omega_X \in H^{1,1}(X)^{\perp} \cap H^2(X,\mathbb{C}) \Rightarrow \omega_X \in T_X \otimes \mathbb{C}$.

for $G \curvearrowright X$ IHS we have seen an exact sequence

$$1 \to G_0 \to G \to \mu_m \to 1$$

We want to find a bound for m and so give the possible orders of the cyclic group $\frac{G}{G_0}$ ($= G$ if $G_0 = \{1\}$ and G acts purely non-sympl., for simplicity.

Let us assume that G acts purely non-symplectically

i.e. $G = \langle \rho \rangle$ and $|G| = m$. satisfies

Prop ① the euler function $=\#\{$primitive m-roots of unity$\}$

$$\varphi(m) \le rk\, T_X$$

proof $\quad \rho^* \omega_X = \zeta_m \omega_X$, $\quad \zeta_m$ primitive m-root of unity

$\omega_X \in T_X \otimes \mathbb{C}$ $\quad$ i.e. ζ_m is e.v. of ρ^* acting on $T_X \otimes \mathbb{C}$

$\Rightarrow$ min. poly of ζ_m divides the char poly of ρ^* on $T_X \otimes \mathbb{C}$

Φ_m = cyclotomic poly of degree $\varphi(m)$

$\Rightarrow \quad \varphi(m) \le rk\, T_X$.

Rmk ① $\quad \varphi(m) \le rk\, T_X \le b_2 - rk\, NS(X)$.

Rmk if $X \sim S^{[n]}$ $\varphi(m) \leq \mathrm{rk}\, T_X \leq \underset{\shortparallel}{\leq} 23 - \mathrm{rk}\, NS(X)$.
$\mathrm{rk}\, H^2(X,\mathbb{Z})$

Prop ② $|G| = m$ $G = \langle g \rangle$ p.m. $g^* \curvearrowright T_X \otimes \mathbb{C}$ then g^* acts by primitive roots of unity.
$g^* \omega_X = \zeta_m \omega_X$

proof (sketch) $\boxed{\text{I. Step}}$

Assume $\exists$ $v \in H^2(X,\mathbb{Z})$ s.t. $g^* v = v$ we show $v \in NS(X)$:

By using BBF: $(g^* v, g^* \omega_X) = (v, \omega_X)$

$$\underset{\shortparallel}{(g^* v, g^* \omega_X)} \quad (v, \zeta \omega_X) \Rightarrow (v, \omega_X) = 0$$

$\Rightarrow v \in NS(X) = \omega_X^\perp \cap H^2(X,\mathbb{Z})$

So $T_X \not\ni$ fixed vectors for g^*

$\boxed{\text{II Step}}$ if $\exists$ v s.t. $g^* v = \zeta_t v$ $t | m$ t not primitive. $t \neq m$.

(Apply step I to g^t) $m = t \cdot t'$

$g^{*t} v = \zeta_t^t v = v$ but then $(g^{*t} v, g^{*t} \omega_X) = (v, \omega_X)$

$\shortparallel$ $(v, \zeta_m^t \omega_X) \Rightarrow v \in NS(X)$

$\neq$ 1 is $\zeta_{t'}$

(It is the same argument as in I!)

Corollary ③ $\varphi(m) \,|\, \mathrm{rk}\, T_X$ ($|\overset{\langle g \rangle}{\overset{\shortparallel}{G}}| = m$)
G acts p.m. $g^* \omega_X = \zeta_m \omega_X$

proof || the char poly of g^* on $T_X \otimes \mathbb{C}$ is a power of the min poly of g^*
since only primitive root of unity.
which is the cyclotomic polynomial. Φ_m

Rmk All primitive m-roots of unity have the same multiplicity in $T_X \otimes \mathbb{C}$.

Recall that if X is IHS and $G \ni \sigma$ purely n.s.

$\Rightarrow X$ is projective so that $\mathrm{rk}\, NS(X) \geq 1$

$\mathrm{rk}\, T_X \leq b_2 - 1$

Theorem $X \sim S^{[n]}$, $n \geq 2$, $X \circlearrowleft \sigma$, σ purely n.s., $o(\sigma) = p$ prime.

$\sigma^* \omega_X = \zeta_p \omega_X$.

$\Rightarrow \quad p \in \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$

Proof

$p - 1 = \varphi(p) \mid 23 - 1 = 22 \Rightarrow p$ as in the list.

Rmk ① all cases are possible for $n = 2$.

② for $X = S$ K3 $\quad p \in \{2, 3, 5, 7, 11, 13, 17, 19\}$

One constructs examples (natural) on $S^{[n]}$.

③ $p = 23$ is special. Since not possible for K3. OK on $S^{[2]}$, Work in progress for $S^{[n]}$, $n \geq 3$ 2017 (Cattaneo - Camere)

For $p = 23$ an important ingredient is the $\boxed{\text{Torelli theorem}}$ by Markman - Verbitsky:

Theorem (Torelli theorem for $X \sim S^{[2]}$)

$O^+(H^2(X, \mathbb{Z})) =$ isometries of $H^2(X, \mathbb{Z})$ that preserve the positive cone $=$ conn. comp. of $\{\alpha \in H^{1,1}_{\mathbb{R}}(X) \mid f(\alpha) > 0\}$ $\cup$ Kähler cone.

Let $\varphi \in O^+(H^2(X, \mathbb{Z}))$ s.t. φ respect the Hodge decomposition on $H^2(X, \mathbb{C}) = H^2(X, \mathbb{Z}) \otimes \mathbb{C}$ $\Rightarrow \exists f \in \mathrm{Aut}(X)$ s.t. $f^* = \varphi$ iff φ preserves a Kähler (e.g. ample of X proj) class.

Rmk ② for $S^{[2]}$: Torelli $=$ Torelli for K3!

② For $p = 23$ one uses Torelli + surjectivity of period map. To find $X \sim S^{[2]}$, $X \circlearrowleft \sigma$, $o(\sigma) = 23$ n.s. and we have $\mathrm{rk}\, T_X = 22$; $\mathrm{rk}\, NS(X) = 1$, $NS(X) = \mathbb{Z}L$, $L^2 = 46$, this X is unique but σ may not

$f(L)$

$\mathbb{X}$ (open) which is the fundamental ... of ... .

For some $\cancel{X} \neq S^{[2]}$ (since rk $NS(S^{[2]}) \geq 2$).

(for $S^{[n]}$, $n \geq 3$ similar methods, numerical cond. depends on n!)

13/10/2017

Results from joint work with: Boissière, Camere, Cattaneo, Nieper-Wißkirchen.

More examples: Fano variety of lines of a cubic fourfold.

$V \subset \mathbb{P}^5$ smooth cubic hypersurface, $V: f_3(x_0, .., x_5) = 0$

$F(V) = \{ \ell \in Gr(1,5) \mid \ell \subset V \}$, this has $\dim F(V) = 4$

Grassmannian of lines of $\mathbb{P}^5$

Beauville - Donagi (1985): $F(V) \overset{\text{defo}}{\sim} S^{[2]}$

(they find special cubics s.t. $F(V) = S^{[2]}$ for some K3 S).

One can find automorphisms of $F(V)$ as follows:

Let $\sigma : \mathbb{P}_5 \longrightarrow \mathbb{P}_5$ auto and take V a σ-inv. cubic i.e.

$\sigma(V) = V \Rightarrow \sigma$ induces an auto on $F(V)$. Concrete examples

$$\sigma : (x_0 : \cdots : x_5) \longmapsto (x_0 : \cdots : x_4 : \zeta x_5), \quad \zeta = e^{\frac{2\pi i}{3}}$$

and take $V: f_3(x_0, .., x_4) + x_5^3 = 0$ (this is the whole family of σ-inv. cubics)

Matsumura - Monsky 1964: all autos of V are induced by autos of $\mathbb{P}^5$

González - Aguilera - Liendo 2010: classification of families of smooth cubic 4-folds V in $\mathbb{P}^5$ with auto. So we get $\sigma \curvearrowright F(V)$ one can see essentially that since $\det \sigma \neq 1 \Rightarrow \sigma$ acts non-symp. on $F(V)$

Interesting object when study auto is the fixed locus:

$$\mathrm{Fix}_\sigma(\mathbb{P}^5) = \{(0:0:0:1)\} \cup \mathbb{P}^4_{x_0 : \cdots : x_4}.$$

$$V^\sigma = \underset{\{x_5=0\}}{\mathbb{P}^4} \cap V = \mathcal{C} : \{ f_3(x_0, .., x_4) = 0 \} \quad \text{smooth cubic 3-fold.}$$

(In fact $\overset{\sigma}{\circlearrowright} V \xrightarrow{3:1} \mathbb{P}^4 \supset \mathcal{C} \leftarrow$ ramif. locus).

$F(V)^\sigma = \{$ lines on V that are preserved by $\sigma\}$

Let $\ell \subset V$ s.t. $\sigma(\ell) = \ell$ i.e. σ is auto of $\ell \cong \mathbb{P}_1 \Rightarrow$

$\Rightarrow$ ① ℓ is pointwise fixed. $\Rightarrow \ell \subset \mathcal{C}$

② ℓ cont 2 (isol.) fixed pts $\quad \ell \cap \{x_5 = 0\}$ are 2 pts

Claim: Case ② is not possible: $\ell \cap \underset{\{x_5=0\}}{\mathcal{C}} = 2$ pts $\Rightarrow \ell \subset \{x_5 = 0\} \cap \underset{\ell \subset V}{V}$

$\Rightarrow \ell \subset \mathcal{C} \Rightarrow \ell$ fixed pointwise.

So that $F(V)^\sigma = F(\mathcal{C})$ is the Fano surface of a cubic 3-fold (is a surface of general type, studied by Fano 1904).

~~Remark~~ Remarks

① One can compute $H^2(F(V),\mathbb{Z})^\sigma = \langle 6 \rangle$, and

$(H^2(F(V),\mathbb{Z})^\sigma)^\perp = U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus A_2(-1)$

② One can construct a moduli space for IHS $\sim S^{[n]}$ with n.s. auto of prime order $p \geq 3$ $\Rightarrow$ Ball quotients (follows a construction of Dolgachev - van Geemen - Kondo for K3-surfaces 2003)

In our case we get a 10-dim ball quotient. $\cong$ (BCS, 2017) moduli space of smooth cubic 3-folds described by Allcock - Carlson - Toledo (2011) [see thm. at the end of lectures]

③ the family cont an IHS $F(V) \cong S^{[2]}$ for some S ($S \subset \mathbb{P}^d$, deg $S = 14$) so we produce in this way a n.s. auto of order 3 non natural on $S^{[2]}$. (Work in progress to construct auto geometrically!)

Thm (BCS) X very "general" (i.e. rk Pic $= 2$ + away from special subfamilies)

$X \sim S^{[2]}$, $NS(X) \neq \mathbb{Z}L$, $L^2 = 2 \Rightarrow \mathrm{Aut}(X) = \{\mathrm{id}\}$

Proof (sketch)

1 step $\mathrm{Aut}(X)$ is finite (discrete subgroup of a compact Lie group)

2 step Study auto of finite order i.e. of prime order.

~~[illegible]~~

If $\sigma \in \mathrm{Aut}(X)$ sympl. $\Rightarrow$ rk $NS(X) \geq 8$ (Boissière–Camere, similar res. as for K3)

$\Rightarrow \nexists$ sympl. auto.

· If $\sigma \in \mathrm{Aut}(X)$ n.s. $o(\sigma) = $ prime $\Rightarrow p \in \{2, 3, \ldots, 23\}$ and we have seen $(p-1) \mid \mathrm{rk}\, T_X = 22 = \mathrm{rk}\, H^2(X,\mathbb{Z}) - \mathrm{rk}\, NS(X)$ ① $\Rightarrow p \in \{2, 3, 23\}$!

· $\underline{p = 23}$: then $\exists X \sim S^{[2]}$ with n.s. auto, but X is unique. (mod space = 0-dim)

· $\underline{p = 3}$: $\exists X \sim S^{[2]}$ with $NS(S^{[2]}) = \langle 6 \rangle$ but again the family is 10-dim (& take pt out side)

$\underline{p = 2}$: then if $X \sim S^{[2]}$ and rk $NS(S^{[2]}) = 1$

with n.s. invol. $\Rightarrow NS(S^{[2]}) = \langle 2 \rangle$ ∃ always n.s. invol. (see later) ☐

classific.

$\underline{\text{Rmk}}$: If $p = 2 \Rightarrow X$ is defo of double EPW sextic. $\boxed{\text{O'Grady}}$

Eisenbud–Popescu–Walter.

their construction generalizes that of K3 surf.

$$S \xrightarrow{2:1} \mathbb{P}^2 \supset C_6 \qquad C_6 = \text{smooth sextic curve.}$$

Double EPW:

$$X \xrightarrow{2:1} Y \subset \mathbb{P}^5$$

ramific. is a surface $= \mathrm{Sing}(Y)$.

Y: special 4-fold of degree 6 hyp.

We have more in general the following

$\underline{\text{Thm ①}}$ (BCS) 2017 $X \underset{\text{defo}}{\sim} S^{[2]}$, $NS(X) \ni h$, ample class $h^2 = 2$ (X is of type $\langle 2 \rangle$-pol)

$\Rightarrow X$ admits a n.s. invol. i s.t. i^* acts on $H^2(X,\mathbb{Z})$ as a reflection in the class $h \in H^2(X,\mathbb{Z})$!

$\underline{\text{Proof}}$ (by Torelli)

$$\varphi: H^2(X,\mathbb{Z}) \longrightarrow H^2(X,\mathbb{Z})$$
$$v \longmapsto (v,h)h - v$$

$\underline{\text{to Torelli}}$

· φ is isometry (ok)

- φ is Hodge isom: $NS(X) = \omega_X^{\perp} \cap H^2(X,\mathbb{Z})$

$\Rightarrow \omega_X \in \ell^{\perp} \Rightarrow \varphi(\omega_X) = -\omega_X$

$\Rightarrow$ Hodge decomp preserved!

φ effective: $\varphi(h) = h \Rightarrow \varphi$ preserves the Kähler cone.

$\Rightarrow \exists\, i \in Aut(X)$ s.t. $i^* = \varphi$ and $(i^*)^2 = id_{|H^2(X,\mathbb{Z})}$

recall: now the map
$Aut(X) \longrightarrow O(H^2(X,\mathbb{Z}))$ is inj. (Beauville)

$\Rightarrow i$ is (n.s.) invol. on X.

□

Rmk • Here we have that $H^2(X,\mathbb{Z})^{i^*} = \mathbb{Z}h = \langle 2 \rangle$

• Same result as for K3 surfaces! (See example from Spring!)

We have also a converse:

Thm ② $X \sim S^{[2]}$ with n.s. invol. i s.t. rk $H^2(X,\mathbb{Z})^{i^*} = 1$

$\Rightarrow H^2(X,\mathbb{Z})^{i^*} = \mathbb{Z}D$, $D^2 = 2$, D ample.

Proof (Beauville) $\Rightarrow$ X is proj $\Rightarrow \exists$ ample invariant class. i.e.
n.s. auto. $\exists L \in NS(X)$ L ample + $i^* L = L$

$\Rightarrow L \in H^2(X,\mathbb{Z})^{i^*} = \mathbb{Z}D$ which is gen by a class $D \Rightarrow D$ is also ample (inv. ample ~~cone~~ cone is gen by D!).

By using classific of n.s. invol on $X \sim S^{[2]}$
$\Rightarrow H^2(X,\mathbb{Z})^{i^*} = \langle 2 \rangle$ i.e. $D^2 = 2$
(BCS) 2016

Rmk: again same holds for K3.

Q How to describe these involutions? One can do it up to deformations:

Prop (X,i), $X \sim S^{[2]}$, $i \curvearrowright X$ n.s. invol. rk $H^2(X,\mathbb{Z})^{i^*} = 1$ $= \langle 2 \rangle$

$\Leftrightarrow (X,i) \underset{defo}{\sim} (S^{[2]}, i_B) \left(\underset{defo}{\sim} (X_{EPW}, j_{EPW})\right)$

F on S m.

$$H^2(S^{[2]}, i_B^*)^{\perp} = \mathbb{Z}h - \delta \quad \text{where} \quad NS(S^{[2]}) = \mathbb{Z}h \oplus \mathbb{Z}\delta$$
$$(h-\delta)^2 = 2 \qquad h^2 = 4$$

and all $H^2(X, i)$ + type does not change by deformations.

$\Rightarrow$ uses a description of families of moduli spaces of IHS $\sim S^{[2]}$ with n.s. invol.

Since here inv. lattice small $\Rightarrow$ only one family of deformations.

Rmk our prop proves in a $\neq$ way that $(S^{[2]}, i_B) \underset{\text{defo}}{\sim} (X_{EPW}, j_{EPW})$ already shown by [Ferretti 2008]

Classifying: A problem in the study of automorphisms is to classify them. i.e. describe moduli spaces, fixed locus ...

~~[illegible]~~ [EL]

- K3 surfaces (a lot is known). • finite abelian symplectic • all symplectic finite groups • (partly) n.s. of prime order (2, 3, .., 19). (Nikulin, + Artebani + Garbagnati, S., many people!)

IHS? One knows that an auto σ induces an action on

$H^2(X, \mathbb{Z})$ and we have the two sublattices.

$$T := H^2(X,\mathbb{Z})^{\sigma} = \{x \in H^2(X,\mathbb{Z}) \mid \sigma(x) = x\} \quad \text{inv. lattice}$$
$$S := ((H^2(X,\mathbb{Z}))^{\sigma})^{\perp} \cap H^2(X,\mathbb{Z}) \quad \text{orthog. complement}.$$

The starting pt in classifying is study possibilities for S & T.

Let's assume σ p.n.s. of prime order. $\sigma \curvearrowright X \sim S^{[2]}$

Recall that here $T = H^2(X,\mathbb{Z}) \subset NS(X)$

$$\text{Transcendental lattice} = T_X \subset S.$$

and $H^2(X, \mathbb{Z}) = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2 \rangle$

Properties of S & T:

Thm(1) $X \sim S^{[2]}$

1) $\mathrm{rk}\, S = (p-1)m$, $m \in \mathbb{Z}_{\geq 1}$

$\mathrm{sgn}\, S = (2, (p-1)m - 2)$, $\mathrm{sgn}\, T = (1, 22 - (p-1)m)$

2) $\frac{H^2(X,\mathbb{Z})}{S \oplus T} \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)$ discriminant groups $\frac{T^\vee}{T}$ & $\frac{S^\vee}{S}$

3) $A_T = \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a}$, $A_S \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a}$ i.e. S is a p-elementary lattice.

(for $p=2$ they may be interchanged!) **STOP HERE!!**

To: fere list of possible S combine the prop with formulas on topology of fixed locus e.g. Topological Lefschetz formula

$$\chi(X^{\langle\sigma\rangle}) = \sum_{i\geq 0}^{8} (-1)^i \operatorname{Tr}(\sigma^*|H^i(X,\mathbb{R}))$$

$$\underset{\substack{X \sim S^{[2]} \\ + \\ \text{Borcea sub.B.}}}{=} 2 + 2\operatorname{Tr}(\sigma^*|H^2(X,\mathbb{R})) + \operatorname{Tr}(\sigma^*|H^4(X,\mathbb{R}))$$

Recall that for $S^{[2]}$: $H^4(X,\mathbb{R}) \overset{\text{Verbitsky}}{=} \operatorname{Sym}^2(H^2(X,\mathbb{R}))$ one gets

* $\underline{p=2}$: one uses results of Beauville (classified fixed locus of n.s. invol. and one can find the lattice)

Assume $p \geq 3$ then:

Lemma (1) $\chi(X^G) = 324 - \frac{51}{2} m \cdot p + \frac{1}{2} m^2 p^2$

(2) $a \leq m$

Example (easy case!) Assume $\underline{p=3}$ if $\operatorname{rk} T = 1 \Rightarrow \operatorname{rk} S = 22 = \overset{p-1}{2}\cdot(11) \Rightarrow m = 11$

Here. $A_T = \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{3\mathbb{Z}} = \frac{\mathbb{Z}}{6\mathbb{Z}}$ (can not have more copies of $\frac{\mathbb{Z}}{3\mathbb{Z}}$ since A_T has only 1 generator!

$\Rightarrow a = 1$

$\operatorname{Sgn}(S) = (2,20)$ ($\operatorname{sg}(T) = (1,0)$)

use lattice theory:

$\underline{\text{thm}}$ (Nikulin). S even indef. lattice of signature (t_+, t_-) then

If $t_+ + t_- \geq 3 + l(A_S) \Rightarrow S \cong U \oplus W$, W even lattice

($l(A_S) = \#$ gen.)

We can apply in our case:

$22 \geq 3+1 = 4 \Rightarrow S \cong \underset{(1,1)}{U} \oplus \underset{(1,19)}{S'}$ ~~apply again to S': $19 \geq 3+1 = 4$.~~

$\Rightarrow S \cong U \oplus S'' \hookrightarrow U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2 \rangle$

We now thm (Rudakov - Shafarevich) [8] $p \neq 2$

Any p-elementary even hyperbolic lattice S' of rank ≥ 2 is uniquely det by signature and a, when $A_{S'} \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a}$ (+ exist. cond. on r & a)

In our case $\underline{a=1}$ so S' is 3-elementary with $a=1$:

$\Rightarrow S' \cong U \oplus E_8(-1)^{\oplus 2} \oplus A_2(-1)$ so that

$S = U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus A_2(-1)$

$T = \langle 6 \rangle$

And we find again: $F(V) \circlearrowleft \sigma$ with

$V: x_5^3 + f_3(x_0, \ldots, x_4) = 0$, σ induced by $x_5 \longmapsto \zeta x_5^3$. !

In general one can find a list of around 40 lattices S & T, ($p \geq 3$)

$\Rightarrow$ Pb find geometric realizations (all possible!)

I) Use natural auto.: several cases have $T = \overline{T} \oplus \langle -2 \rangle$, so one can find:

SK3, $S \circlearrowleft \eta$ with $H^2(S, \mathbb{Z})^{\eta} = \overline{T} \Rightarrow$

$H^2(S^{[2]}, \mathbb{Z})^{\eta^{[2]}} = \overline{T} \oplus \langle -2 \rangle$
exceptional set is fixed.

II) Some cases are not as before i.e. T does not admit a splitting $\overline{T} \oplus \langle -2 \rangle$ of which $T = \langle 6 \rangle$ so one case use the Fano varieties of lines of cubic 4-folds.

Rmk for $\underline{p=2}$ more than 100 cases, also $\neq$ embeddings i.e. same T but $\neq S$.

several realizations (e.g. EPW).

Moduli spaces.

Let $X \sim S^{[2]}$ We have a surjective period map.

$$\rho : \mathcal{M}^{0}_{\Lambda} \xrightarrow{\text{Connected comp.}} \mathcal{L}\Omega_{\Lambda}$$

$$(X, \eta) \mapsto \eta[H^{2,0}(X)]$$

$\eta : H^2(X, \mathbb{Z}) \longrightarrow \Lambda$ isometry, $\Lambda = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2 \rangle$

$\mathcal{L}\Omega_{\Lambda} = \{[\omega] \in \mathbb{P}(\Lambda \otimes \mathbb{C}) \mid q(\omega + \bar\omega) > 0,\ q(\omega) = 0\}$ 21-dim.
$\quad\ \mathbb{P}^{22}$

Let (X, σ) $X \sim S^{[2]}$, σ n.s. auto $o(\sigma) = p \geq 3$.

$\Rightarrow$ $T \subset NS(X)$ and $T_X \subset S$ $\Rightarrow$
invariance

$$\omega_X \in \{[\omega] \in \mathbb{P}(S \otimes \mathbb{C}) \mid \ldots\}$$

but $\sigma^* \omega_X = \zeta_p \omega_X$ $\Rightarrow$ $\omega_X \in S_{\zeta_p} \subset S \otimes \mathbb{C}$
ζ_p R eigenspace w.r.t. action of σ

$$\Rightarrow \omega_X \in \{[\omega] \in \mathbb{P}(S_{\zeta_p}) \mid (\omega, \omega) = 0, (\omega, \bar\omega) > 0\}$$
$\mathbb{P}(S_{\zeta_p}) = \mathbb{P}^{m-1}$, $(\omega,\omega) = q(\omega)$

$rk(S) = (p-1)m - 2$ (recall $rk S = m(p-1)$ each e.v. has same mult.)

but since σ isometry: $(\omega, \omega) = (\sigma\omega, \sigma\omega) =$
$= (\zeta_p \omega, \zeta_p \omega) \Rightarrow \zeta_p^2 = 1$ but $p \neq 2$

$\Rightarrow$ $(\omega, \omega) = 0$ is trivially verified.

$$\Rightarrow \omega_X \in \{[\omega] \in \mathbb{P}(\underbrace{S_{\zeta_p}}_{m-1 \text{ dim}}) \mid (\omega, \bar\omega) > 0\}$$

with the BBF quadratic form given on basis of type $(1, (m-1))$ ($\omega_X \in S_{\zeta_p}$ and $\omega\bar\omega > 0$)

$$\Rightarrow \omega_X \in \{[\omega] \in \mathbb{P}^{m-1} \mid |y_0|^2 - |y_1|^2 - \cdots - |y_{m-1}|^2 > 0\}$$
$[\omega] = (y_0 : \cdots : y_{m-1})$, $y_0 \neq 0$ put $= 1$

$$\omega_X \in \{[\omega] \in \mathbb{C}^{m-1} \mid |y_1|^2 + \cdots + |y_{m-1}|^2 < 1\}$$

This is a $(m-1)$-dimensional ball, ...

$$\frac{\mathbb{B}_{m-1} \setminus \mathcal{H}}{\Gamma}$$ ← complement of hyperplanes!

Γ ← subgroups to get rid of the unity.

as period domain for $X \sim S^{[2]}$ with an auto of "fixed" type.

<u>EX</u> In the case $T = \langle 6 \rangle$, $S = U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2 \rangle$ we find a Ball

$$\frac{\mathbb{B}_{10} \setminus \mathcal{H}}{\Gamma}$$ 10-dim. = moduli space of $X \sim S^{[2]}$ with a n.s. auto of the type of σ.

<u>Thm</u> (BCS) 2017

$X \sim S^{[2]}$, $\langle 6 \rangle \hookrightarrow NS(X)$ is an ample polarisation.

X admits a n.s. auto of order 3 with invariant lattice $T = \langle 6 \rangle$

$\Leftrightarrow$ $X \cong F(V)$, $V \xrightarrow{3:1} \mathbb{P}^4 \supset C$ and auto is induced by covering auto.

(V: smooth 3-fold)