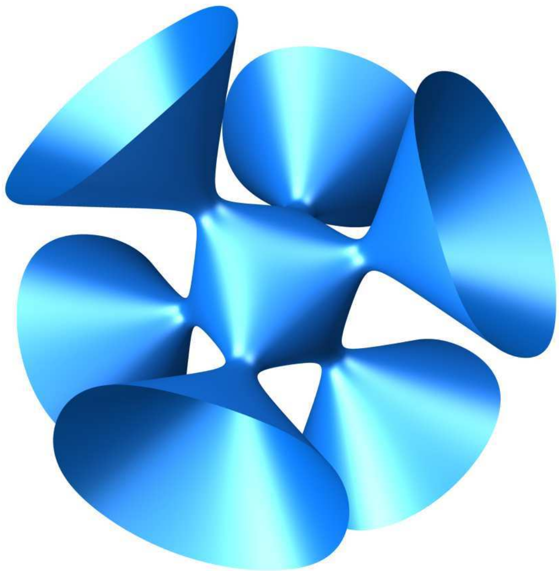
The geometry of some special algebraic varieties



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1. Zusammenfassung

Das Thema von meiner Habilitation ist *Die Geometrie einiger spezieller algebraischer Varietäten*, insbesondere untersuche ich die K3-Flächen. Die Arbeiten, die ich vorlege, sind die Artikeln:

(1) mit Wolf Barth, *Polyhedral Groups and Pencils of K3-Surfaces with maximal Picard Number*, **Asian J. of Math.** Vol. 7, No. 4, pp. 519–538, 2003.

(2) *Symmetric surfaces with many Singularities*, **Comm. in Algebra** Vol. 32, No. 10, pp. 3745-3770, 2004.

(3) *A geometrical construction for the generators of some reflection group*, **Serdica Math. J.**, 31, pp. 229-242, 2005.

(4) with Andreas Knutsen, Carla Novelli, *On Varieties that are uniruled by lines*, **Compositio Math.** 142, pp. 889–906, 2006.

(5) *Group Actions, cyclic coverings and families of K3 surfaces*, erscheint in **Canadian Math. Bull.**

(6) *Transcendental lattices of some K3 surfaces*, erscheint in **Math. Nachr.**

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(9) with Alice Garbagnati, *Symplectic automorphisms of prime order on K3 surfaces*, Preprint math.AG/0603742, eingereicht.

(10) with Samuel Boissière, *Counting lines on surfaces*, Preprint math.AG/0606100, eingereicht.

(11) with Alice Garbagnati, *Projective models of K3 surfaces with an even set*, Preprint math.AG/0611182, eingereicht.

Ich werde sie im Folgenden kurz beschreiben, man siehe die Literaturangaben für die verwendeten Abkürzungen.

1) Arbeiten über Flächen mit vielen Doppelpunkten: In meiner Promotion habe ich mich mit der Frage beschäftigt, wieviele gewöhnliche Doppelpunkte eine Fläche vom Grad d in $\mathbb{P}_3$ maximal haben kann, und ich habe drei neue eindimensionale Familien von Flächen in $\mathbb{P}_3$ beschrieben. Diese haben Grad 6,8 bzw. 12 und die Symmetrien der sogenannten bi-polyedrischen Tetraedergruppe (=G_6), Oktaedergruppe(=G_8) bzw. Ikosaedergruppe (=G_{12}), d.h. die Polynome, die die Familien definieren, sind invariant unten der Operation von $G_d \subset SO(4,\mathbb{R})$, $d=6$, 8 bzw. 12. Jede Familie enth"alt genau vier Fl"achen mit gew"ohnlichen Doppelpunkten. Insbesondere gibt es in der Familie vom Grad 12 eine Fläche, die 600 gewöhnliche Doppelpunkte hat (s. [Sa1]).
Die Gruppen G_6 und G_{12} sind Untergruppen der Spiegelungsgruppen $[3,4,3]$ und $[3,3,5]$. Mit Hilfe der G_6- bzw. $G_{12}-$invarianten Flächen in $\mathbb{P}_3$ konnte ich in der Arbeit [Sa3] eine

einfache geometrische Konstruktion für die Erzeugenden des Rings der invarianten Polynome vom Grad $2, 6, 8, 12$ bzw. $2, 12, 20, 30$ angeben (diese wurden auf eine andere Weise von Racah beschrieben). In der Arbeit [Sa2] betrachte ich weitere Untergruppen der $SO(4, \mathbb{R})$ und untersuche deren eindimensionalen Familien von invarianten Flächen in $\mathbb{P}_3$. Ich schränke meine Untersuchung auf die Gruppen ein, die die Heisenberggruppe enthalten. Zusammen mit den Grppen G_d ergeben die Gruppen aus [Sa2] eine vollständige Liste von Gruppen, die die Heisenberggruppe enthalten.

2) Flächen mit vielen (disjunkten) Geraden: In der Arbeit [BoSa2] mit Samuel Boissière (Universität Nizza) konstruieren wir Flächen in $\mathbb{P}_3$ mit vielen (disjunkten) Geraden. Für diese Anzahl gibt es Abschätzungen von Segre und Miyaoka. Es ist wohlbekannt, dass eine glatte Kubik 27 Geraden enthält; für Flächen vom Grad vier gibt es Arbeiten u.a. von Segre und Nikulin. Das Problem ist noch offen für den Grad $d \geq 5$. Klassische Beispiele sind die Fermatsche Flächen $x_0^d + x_1^d + x_2^d + x_3^d = 0$, die $3d^2$ Geraden enthalten. Andere Beispiele sind die Flächen der Art: $\phi(x, y) - \psi(z, t) = 0$ wobei ϕ, ψ homogene Polynome vom Grad d sind. In diesem Artikel beschreiben wir Flächen, die gegeben sind durch die Gleichung: $\phi(x, y) - \psi(z, t) = 0$ vollständig und wir geben alle möglichen Anzahlen von Geraden an. Wir studieren außerdem einige Flächen mit vielen Symmetrien und wir geben ein Beispiel einer Fläche vom Grad acht mit 352 Geraden an. Das verbessert ein Ergebnis von Caporaso-Harris-Mazur, die eine Fläche mit 256 Geraden konstruieren. Wir geben auch einige neue Beispiele von Flächen mit vielen disjunkten Geraden an, die ein vorheriges Ergebnis von Rams verbessern.

3) Arbeit über die Klassifikation von geregelten 3-Mannigfaltigkeiten: In [KNS] konnten wir das folgende Ergebnis zeigen: Sei X eine irreduzible Varietät vom Dimension $k \geq 3$, $\mathcal{H}$ ist ein global erzeugter und big Geradenbündel auf X mit $\mathcal{H}^k := d$, $n = dim\ H^0(X, \mathcal{H}) - 1$. Wenn $d < 2(n-k) - 4$, und $(k, d, n) \neq (3, 27, 19)$, dann ist X geregelt von Geraden. Im Fall von 3-Mannigfaltigkeiten ist diese Abschätzung optimal, denn für $d = 2n - 10$ haben wir Beispiele von 3-Mannigfaltigkeiten gefunden, die nicht geregelt von Geraden sind. Unser Ergebnis gilt insbesondere für Varietäten in $\mathbb{P}^n$. Im Fall von Flächen wurde eine solche Abschätzung von M. Reid und Xiao angegeben. Das bis jetzt beste Ergebnis für k-Mannigfaltigkeiten X in $\mathbb{P}^n$, X glatt war von Horowitz. Er zeigte: ist der Grad $d < 3/2(n - k - 1)$, dann ist X geregelt von Geraden. Unser Ergebnis verbessert dieses Resultat, außerdem gilt es für jede Varietät ohne Annahme über die Singularitäten von X. Um unser Ergebnis zu zeigen, verwenden wir die Mori-Theorie und das Minimal-Model-Programm, insbesondere benutzen wir einige Ergebnisse von Mella.

4) Arbeiten über K3-Flächen: Eine K3-Fläche S ist eine glatte, kompakte Fläche über $\mathbb{C}$, die einfach zusammenhängend ist und ein triviales kanonisches Bündel hat. Die K3-Flächen sind von besonderem Interesse wegen ihrer wichtigen Eigenschaften, z.B. sind sie alle zueinander diffeomorph, es gilt die Surjektivität der Periodenabbildung, und nach dem Theorem von Torelli kann man sie durch die Hodge-Struktur (also durch das transzendete Gitter und das Picard-Gitter) klassifizieren. Sie wurden in den letzten Jahren eingehend untersucht, z.B. wegen ihrer arithmetischen Eigenschaften und nicht zuletzt wegen ihrer Rolle in der Physik und insbesondere in der String-Theorie: Sie sind Calabi-Yau-Mannigfaltigkeiten der Dimension zwei und spielen eine wichtige Rolle in der Spiegel-Symmetrie. Mit diesen Flächen habe ich mich sehr intensiv in den letzten Jahren beschäftigt, und insbesondere habe ich mich mit den folgenden Themen befaßt:

K3-Flächen mit großer Picard-Zahl. In den Arbeiten [BaSa], [Sa4] und [Sa5] beschäftige ich mich mit Familien von K3-Flächen mit großer Picard-Zahl (das Maximum für eine K3-Fläche ist 20). Es ist schwierig, Beispiele von solchen Familien zu konstruieren und die Flächen

in der Familie zu identifizieren, die eine höhere Picard-Zahl haben. Damit verbunden ist das Problem, Büschel von K3-Flächen mit großer Picard-Zahl und minimaler Anzahl von singulären Fasern zu konstruieren. Einige Beispiele sind in Arbeiten von Beauville, Belcastro, Verrill-Yui, Narumiya-Shiga enthalten. In [BaSa] zusammen mit Wolf Barth beschreibe ich die Quotienten der Familien $\{X_\lambda\}_{\lambda\in\mathbb{P}_1}$ nach den Gruppen G_d (s. Arbeit [Sa1]): Diese sind Familien von K3-Flächen, bei denen die allgemeine Fläche Picard-Zahl 19 hat und es vier singuläre Fasern gibt, die Picard-Zahl 20 haben. Insgesamt enthält die Familie aber fünf singuläre Fasern. In dem Artikel berechnen wir das Picard-Gitter der Flächen explizit. In [Sa4] beschreibe ich weitere Familien von K3-Flächen mit großer Picard-Zahl und kleiner Anzahl von singulären Fasern. Hier betrachte ich spezielle Untergruppen G von G_d. Dann ist die G_d-invariante Familie $\{X_\lambda\}_{\lambda\in\mathbb{P}_1}$ auch G-invariant und unter einigen Bedingungen sind die Quotienten X_λ/G wieder K3-Flächen mit großer Picard-Zahl. Wenn außerdem G Normalteiler von G_d mit $[G:G_d]=2,3$ ist, kann man X_λ/G als 2- bzw. 3-zyklische Überlagerung von X_λ/G_d betrachten. Mit Hilfe dieser Überlagerung kann man das Picard-Gitter von X_λ/G in vielen Fällen genau identifizieren. Mit Hilfe der Gitter-Theorie und Ergebnissen über quadratische Formen kann man das transzendente Gitter der Flächen berechnen. Das wurde von Barth für die Flächen aus [BaSa] durchgeführt. In [Sa5] berechne ich es für die Flächen aus [Sa4]. Damit kann ich dann die K3-Flächen klassifizieren.
Ich beschäftige mich weiter mit diesen Flächen in der Arbeit in Vorbereitung [Sa6], in der ich projektive Modelle der Flächen untersuche.

Symplektische Automorphismen auf K3-Flächen. Im Rahmen meines DFG-Forschungsprojekts in Mailand *Die Geometrie einiger Familien von K3-Flächen und symplektische Automorphismen auf K3-Flächen* habe ich mich mit Automorphismen auf K3-Flächen beschäftigt, die die 2-holomorphe Form invariant lassen (*symplektische Automorphismen*). Solche Automorphismen der Ordnung zwei heißen *Nikulin-Involutionen*. In dem Artikel [GS], untersuche ich sie zusammen mit Bert van Geemen.

Nach einer Arbeit von Nikulin induzieren sie eine eindeutige Operation auf $H^2(X,\mathbb{Z})$. Wir studieren die Neron-Severi-Gruppe und das transzendente Gitter. Insbesondere zeigen wir, dass, wenn X eine Nikulin-Involution besitzt, die Picard-Zahl ≥ 9 ist und die Neron-Severi-Gruppe eine Kopie des Gitters $E_8(-2)$ enthält (das ist das Gitter E_8 mit der Bilinearform multipliziert mit -2). Im Fall $\rho=9$ bestimmen wir mit Hilfe der Gitter-Theorie vollständig die Struktur der Neron-Severi-Gruppe in Abhängigkeit von der Polarisierung der K3-Fläche. Wir geben an und untersuchen viele konkrete Beispiele, die die allgemeinen Sätze beleuchten u. a. doppelte Überlagerungen der Ebene, Quartiken in $\mathbb{P}_3$, vollständige Durchschnitte und insbesondere K3-Flächen mit elliptischer Faserung.

In der Arbeit [GaSa1] beschäftige ich mich zusammen mit Alice Garbagnati (Universität Mailand) mit symplektischen Automorphismen der Ordnung $3,5,7$. Nach einer Arbeit von Nikulin sind diese zusammen mit den Automorphismen der Ordnung zwei alle möglichen Primordnungen für solche Automorphismen. Mit Hilfe von elliptischen Faserungen auf K3-Flächen und der Gittertheorie konnten wir die Wirkung auf $H^2(X,\mathbb{Z})$ vollständig beschreiben. Wie in dem Fall der Ordnung zwei (nach einem Ergebnis von Nikulin) ist diese Wirkung eindeutig, d.h. unabhängig von der Wahl der K3-Fläche. Ich beschäftige mich mit ähnlichen Problemen (auch im Fall von nicht-symplektischen Automorphismen) in den Arbeiten in Vorbereitung: [AS] zusammen mit Michela Artebani (Universität Mailand) und [GaSa3] zusammen mit Alice Garbagnati.

Zwei-teilbaren Mengen von acht disjunkten rationalen Kurven. In der Arbeit [GaSa2] zusammen mit Alice Garbagnati (Universität Mailand) untersuchen wir K3-Flächen mit einer *2-teilbaren Menge* von acht (-2)-rationalen Kurven (das heißt die Summe der

Kurven ist äquivalent zu zweimal einem Divisor in der Picard Gruppe) und mit einer 2-teilbaren Menge von gewöhnlichen Doppelpunkten (das heißt, die (-2)-rationalen Kurven in der minimalen Auflösung sind eine 2-teilbare Menge). Solche K3-Flächen sind minimale Auflösungen und Quotienten einer K3-Fläche nach einer Nikulin-Involution, ihre Untersuchung vervollständigt die Ergebnisse der Artikel [GS] im folgenden Sinn: Gegeben sei eine K3-Fläche mit Automorphismus. Es ist natürlich zu fragen, was für eine Fläche der Quotient ist und mit welchen Eigenschaften (z.B. Singularitäten). In [GaSa2] studieren wir K3-Flächen mit gerader Menge von rationalen Kurven und mit kleinstmöglicher Picard-Zahl, die neun ist. Es werden die Flächen klassifiziert und es wird deren Modulraum beschrieben. Insbesondere beschreiben wir viele projektive Modelle, mit denen wir die Untersuchung erweitern und fortsetzen, die von Barth angefangen worden ist.

5) Arbeit über die McKay-Korrespondenz in Dimension zwei und drei: In der Arbeit [BoSa1] zusammen mit Samuel Boissière (Universität Nizza) geben wir eine Beziehung zwischen der McKay-Korrespondenz in Dimension zwei und in Dimension drei. Sei G eine endliche Untergruppe der $SO(3,\mathbb{R})$ und sei $\tilde{G} \subset SU(2)$ die binäre Gruppe zur Gruppe G. Die Gruppe $\widetilde{G}$ operiert auf $\mathbb{C}^2$ und der Quotient ist eine ADE-Flächensingularität. Ihre Auflösung besteht aus glatten, rationalen (-2)-Kurven mit einem ADE-Dynkin Diagramm als dualem Graph. Die McKay-Korrespondenz assoziert die Ecken des Graphs mit den irreduziblen Darstellungen ($\neq 1$) von $\widetilde{G}$. Die Auflösung der Quotienten $\mathbb{C}^2/\widetilde{G}$ und $\mathbb{C}^3/G$ sind Hilbert-Nakamura-Schemata und die exzeptionellen Kurven beider Auflösungen über dem Ursprung haben sehr ähnliche Eigenschaften. Wir zeigen, dass es einen Morphismus zwischen diesen beiden Auflösungen gibt, der bestimmte Kurven in der exzeptionellen Faser kontrahiert. Für den Beweis benutzen wir die McKay-Korrespondenz zwischen Darstellungen und exzeptionelle Kurven in Dimension zwei und drei, sowie die Theorie von Hilbert-Nakamura-Schemata.

2. Introduction

In the papers which I collect for my Habilitation at the University of Mainz, I concentrate my attention to algebraic varieties with some special geometric properties (for example with many singularities or many lines) with particular attention to K3 surfaces, which occupy an important place in the classification of algebraic surfaces. An example of such a surface is given on the cover: this is a surface of equation

$$1 + x^4 + y^4 + z^4 + a(x^2 + y^2 + z^2 + 1)^2 = 0, \quad a = -0,49.$$

in fact any quartic surface in $\mathbb{P}_3$ is an example of a K3 surface.
About varieties with special geometric property there are easy questions with difficult answers, for example which is the maximal number of lines a surface of degree d in $\mathbb{P}_3$ can have? Or which is the maximal number of nodes? The first question has an answer up to the degree four, in general there are bounds of Segre and Miyaoka. It is well known that a smooth cubic contains 27 lines; for surfaces of degree four this maximal number is also known: it is 64, 16 if we assume that the lines are skew; these are results of Segre and Nikulin. The problem is still open for degree $d \geq 5$. Classical examples are the Fermat surfaces $x^d + y^d + z^d + t^d = 0$, which contain $3d^2$ lines. Other examples are the surfaces of the kind $\varphi(x, y) - \psi(z, t) = 0$, where φ and ψ are homogeneous polynomials of degree d. There are more results given by Caporaso-Harris-Mazur in [CHM], who construct examples of surfaces with many lines in each degree and there are results of Rams in [Ram2] about skew lines. In the case of the question about the nodes there is an answer up to the degree six. Unfortunately there are no-standard methods to construct examples. A successful idea is to consider surfaces with many symmetries, this was used by Barth, Endraß and other people to construct examples of surfaces with many nodes. I used it to construct examples of surfaces with many nodes or with many lines (cf. [Sa2], [Sa3], [BoSa2]). Strictly connected to the problem of lines on algebraic varieties is the problem to determine when a variety is coverd by lines or more precisely when it is uniruled by lines. The answer to the problem in the case of surfaces is given independently by M. Reid in [Re] and by Xiao in [Xi]. They show that for $d < (4/3)(n-2)$ a surface $X \subset \mathbb{P}_n$ of degree d is uniruled by lines (except when $n = 9$ and $(X, \mathcal{O}_X(1)) = (\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2})$). For varieties of higher degree this bound was unknown, in [KNS] we give a bound for varieties of any degree and we show that this is optimal in the case of threefolds.
Another topic of my work are the K3 surfaces, these are smooth, compact complex surfaces which are simply connected and have trivial canonical bundle. The K3 surfaces are of particular interest because of their important properties, for example they are all diffeomerphic to eachother, the period map is surjective and due to a theorem of Torelli, they can be classified by their Hodge structure which involves the transcendental lattice and the Picard lattice. They have been very much studied in the last years for example for their arithmetic properties and for their importance in Physics and in particular in the String-Theory: they are Calabi-Yau manifold of dimension two and play an important role in the Mirror-Symmetry. I have worked very much with these surfaces in the last years in particular with the following topics: K3 surfaces with big Picard number and automorphisms on K3 surfaces. The maximal Picard number for a K3 surface is 20, however non-trivial families have at most Picard number 19. It is difficult to construct such families and to identify the surfaces with Picard number 20 (i.e. the *singular* K3 surfaces). Strict connected to this, there is the problem of constructing families of K3 surfaces with big Picard number and small number of singular (in the usual sense) K3 surfaces. Some examples are studied by Beauville, Belcastro, Verrill-Yui, Narumiya-Shiga.

During my DFG-researchproject *Die geometrie einiger Familien von K3-Flächen und symplektische Automorphismen auf K3-Flächen* in Milan by B. van Geemen, I studied symplectic automorphisms on K3 surfaces (i.e. those automorphisms leaving the holomorphic two form invariant). These were extensively studied in the recent past. In a famous paper from 1979 [N1], Nikulin described the finite automorphism groups of K3 surfaces, in particular he classified all abelian groups which act symplectically on a K3 surface, that is, they leave the holomorphic two form invariant. This classification was completed in 1988 by Mukai in [Muk]: he classified all isomorphism classes of finite groups acting symplectically on a K3 surface. The first case to study are the symplectic involutions (called *Nikulin involutions* by Morrison in [Mo]). These are also important for their relation with the Shioda-Inose structure introduced by Morrison in [Mo]. This structure relates K3 surfaces with large Picard number to abelian surfaces and was studied for example in [L], [NS], [vGT]. For my research also the results on elliptic fibrations are important. In fact, given a K3 surface with an elliptic fibration and a section, the group of sections of the fibrations is the Mordell-Weil group of the surface. A section of finite order defines a symplectic automorphism of the same order. The study of these automorphisms is often very useful for gaining an understanding of the general case. The literature on elliptic fibrations is vast, works of particular importance for my research are of Shioda, [Shio] and Shimada, [Shim]. The last paper classifies all fibres of type ADE in an elliptic fibration and also describes the torsion group of the Mordell-Weil group (that is the part generated by sections of finite order).

Nikulin showed that the action induced by these automorphisms on the second cohomology group with integer coefficients, $H^2(X,\mathbb{Z})$, is determined by its order and does not depend on the particular choice of the K3 surface X. In particular, we have a canonical decomposition into an invariant lattice and its perpendicular. In the case of Nikulin involutions, this decomposition was identified by Morrison in [Mo], he showed that the invariant part is isometric with $U\oplus U\oplus U\oplus E_8(-2)$ (where U is a copy of the unimodular even hyperbolic plane and $E_8(-2)$ is the lattice E_8 with the bilinear form multiplied by -2) and its perpendicular is $E_8(-2)$. The question about this decomposition is of course of interest for any other group in the classification of Nikulin. In [GaSa1] we give this decomposition in the case of prym order automorphisms.

Given an automorphism of a surface it is natural to ask about the properties of the quotient surface. In the case of symplectic automorphisms, the quotient has only ADE singularities and its minimal resolution is a K3 surface. The first case to study are the quotients by a Nikulin involutions. More in general it is interesting to study surfaces with an even set of eight rational (-2)-curves (that is, the sum of the curves is twice a divisor in the Picard group) or with even sets of nodes (that is, the (-2)-curves in the resolution are an even set). In [B2] Barth gives a description of some projective models of such surfaces. In [GaSa2] we continue this description.

It is also interesting to study non-symplectic automorphisms, examples of such finite order automorphisms are given in the papers [DGK], [K] and in the paper [A1], [A2], [A3], in the particular case of order four automorphisms. In this case the study of the K3 surfaces and of concrete examples is more complicated, for instance, there are no results on their classification.

3. Description of the scientific works

The works which I submit for my Habilitation are the papers:

(1) with Wolf Barth, *Polyhedral Groups and Pencils of K3-Surfaces with maximal Picard Number*, **Asian J. of Math.** Vol. 7, No. 4, pp. 519–538, 2003.

(2) *Symmetric surfaces with many Singularities*, **Comm. in Algebra** Vol. 32, No. 10, pp. 3745-3770, 2004.

(3) *A geometrical construction for the generators of some reflection group*, **Serdica Math. J.**, 31, pp. 229-242, 2005.

(4) with Andreas Knutsen, Carla Novelli, *On Varieties that are uniruled by lines*, **Compositio Math.** 142, pp. 889–906, 2006.

(5) *Group Actions, cyclic coverings and families of K3 surfaces*, to appear in **Canadian Math. Bull.**

(6) *Transcendental lattices of some K3 surfaces*, to appear in **Math. Nachr.**

(7) with Bert van Geemen, *Nikulin involutions on K3 surfaces*, to appear in **Math. Z.**

(8) with Samuel Boissière, *Contraction of excess fibres between the Mckay correspondence in dimensions two and three*, to apper in **Ann. Inst. Fourier**

(9) with Alice Garbagnati, *Symplectic automorphisms of prime order on K3 surfaces*, Preprint math.AG/0603742, submitted.

(10) with Samuel Boissière, *Counting lines on surfaces*, Preprint math.AG/0606100, submitted.

(11) with Alice Garbagnati, *Projective models of K3 surfaces with an even set*, Preprint math.AG/0611182, submitted.

these are the papers [BaSa], [Sa2], [Sa3], [KNS], [Sa4], [Sa5], [GS], [BoSa1], [GaSa1], [BoSa2], [GaSa2] from the reference list. First I give a quick overwiev of the contents and then I will explain more in details:
1) the papers [Sa2], [Sa3], [BoSa2] deal with surfaces with many nodes, lines and many symmetries,
2) the papers [BaSa], [Sa4], [Sa5], deal with some special families of K3 surfaces with Picard number 19,
3) the papers [GS],[GaSa1],[GaSa2] are about symplectic automorphisms of K3 surfaces,
4) the paper [KNS] give a criterion for varieties in any degree to be uniruled by lines,
5) finally the paper [BoSa1] is about a special case of the Mckay correspondence, and it is related to the works of 2).
1) Surfaces with many double points. In my PhD thesis I worked about the question: which is the maximal number of nodes a surface of degree d in $\mathbb{P}_3$ can have. I have found three new one dimensional families of surfaces $\{X^d_\lambda\}_{\lambda\in\mathbb{P}_1}$ in $\mathbb{P}_3$, they have degree $d=6,8$ resp. 12 and have the symmetries of the so called bipolyhedral tetrahedralgroup ($=G_6$), octahedralgroup ($=G_8$) resp. icosahedralgroup ($=G_{12}$), this means that the polynomials which define the families are invariant for the operation of $G_d\subset SO(4,\mathbb{R})$, $d=6,8,12$. Each family contains exactly four surfaces with nodes. In particular the family of degree 12 contains a

surface with 600 nodes (these results are contained in my paper [Sa1]).
The groups G_6 and G_{12} are subgroups of the reflection groups $[3,4,3]$ and $[3,3,5]$, which are the symmetry groups of some special four dimensional polyhedra. By using the G_6- resp. $G_{12}-$invariant surfaces in $\mathbb{P}_3$ I give an easy geometric construction for the generators of the rings of the invariant polynomials, these have degree $2,6,8,12$ resp. $2,12,20,30$ (these were described before in a different way by Racah in [Rac]). In the paper [Sa2] I consider other subgroups of $SO(4,\mathbb{R})$ and I study the one dimensional families of invariant surfaces in $\mathbb{P}_3$. I restrict my study to the subgroups which contain the Heisenberg group. These together with the groups G_d give a complete list of subgroups of $SO(4,\mathbb{R})$ which contain the Heisenberg group and have a one dimensional family of invariant surfaces.
2) Surfaces with many (disjoint) lines. In the paper [BoSa2] together with Samuel Boissière (University of Nizza) we construct surfaces in $\mathbb{P}_3$ with many (disjoint) lines. First we describe the surfaces $\varphi(x,y) - \psi(z,t) = 0$ completely and we give for any degree d all the possible numbers of lines. We study also surfaces with many symmetries and we give an example of a surface of degree eight with 352 lines. This result improves a preceding result of Caporaso-Harris-Mazur [CHM], who construct a surface with 256 lines. We give also some new examples of surfaces with many disjoint lines, which improve some previous result of Rams [Ram2].
3) Classification of uniruled varieties. In [KNS] together with Andreas Knutsen (University of Rom) and Carla Novelli (University of Genova) we show the following result: Let X be an irreducible variety of dimension $k \geq 3$, $\mathcal{H}$ a globally generated and big line bundle on X with $\mathcal{H}^k := d$, $n = dimH^0(X,\mathcal{H}) - 1$. If $d < 2(n-k) - 4$ and $(k,d,n) \neq (3,27,19)$ then X is uniruled by lines. In the case of threefolds this is an optimal bound, since for $d = 2n - 10$ there are examples of threefolds which are not uniruled by lines. Our result holds in particular for varieties in $\mathbb{P}_n$. In the case of surfaces a similar bound was given by M. Reid and Xiao. Until now the best result for smooth k-folds X in $\mathbb{P}_n$ was a result of Horowitz. He showed that for $d < (3/2)(n-k-1)$, then X is uniruled by lines. Our result improves the result of Horowitz and moreover it holds without any assumption on the singularities if X. To prove our result we use Mori-Theory and the Minimal-Model-program, in particular we use some previous results of Mella.
4) K3 surfaces with big Picard number. In the papers [BaSa], [Sa4], [Sa5] I work with families of K3 surfaces with Picard number 19. In [BaSa] together with Wolf Barth (University of Erlangen) I describe the quotients of the one dimensional families $\{X^d_\lambda\}_{\lambda\in\mathbb{P}_1}$ by the groups G_d (cf. **1** above and the paper [Sa1]): these are families of K3 surfaces, in which the general K3 surface has Picard number 19 and there are exactly five singular fibers: one is a degeneration and four have nodes, the latter have Picard number 20. In the paper we describe completely the Picard lattice. In [Sa4] I describe more families of K3-surfaces with big Picard number and small number of singular fibers, I consider some special subgroup G of G_d, then clearly the G_d-invariant family $\{X^d_\lambda\}_{\lambda\in\mathbb{P}_1}$ is also G-invariant and under some assumption on the groups G, the quotients X^d_λ/G are again K3 surfaces with big Picard number. Moreover if G is a normal subgroup with $[G : G_d] = 2,3$ then I describe X^d_λ/G as 2-cyclic, resp. 3-cyclic covering of X^d_λ/G_d. This description is very helpful to identify the Picard lattice of the covering surfaces. Then in [Sa5] with the help of lattice theory (cf. [N2]) and results on quadratic forms I describe the transcendental lattices of the surfaces described in [Sa4] and I classify them.
I still work on these families of surfaces, in fact I'm looking for projective models of them. This is the topic of the work in progress [Sa6].
5) Symplectic automorphisms on K3 surfaces. The symplectic automorphisms of order

two are called *Nikulin involutions*. In the article [GS] I study them together with Bert van Geemen (University of Milan).
By a paper of Nikulin they induce a unique (up to isometry) action on $H^2(X,\mathbb{Z})$, this means that the operation is independent from the choice of the K3 surface. We study the Picard lattice and the transcendental lattice, in particular we show that if a K3 surface X has a Nikulin involution then the Picard number is $\rho \geq 9$ and the Picard lattice contain a copy of the lattice $E_8(-2)$. In the case of $\rho = 9$ with the help of lattice theory we describe completely the structure of the Picard lattice. We discuss also many concrete examples, which explains the general results, in particular double covers of the plane, quartics in $\mathbb{P}_3$, complete intersections and in particular elliptic fibrations.
In the paper [GaSa1] together with Alice Garbagnati (University of Milan) I study symplectic automorphisms of order $3, 5, 7$. Together with the order two these are all possible prime orders for such automorphisms (at least in characteristic zero). By using lattice theory and elliptic fibrations we identify completely the action on $H^2(X,\mathbb{Z})$. In the case of the order three automorphism we show that the orthogonal complement to the invariant sublattice of $H^2(X,\mathbb{Z})$ is the well known rank twelve Coxeter-Todd lattice.
I still work on similar questions in the paper in preparation [GaSa3] with Alice Garbagnati and in the case of non-symplectic automorphisms of order three in the work in progress [AS] with Michela Artebani (University of Milan).
6) Even sets of eight disjoint rational curves. In the submitted preprint [GaSa2] together with Alice Garbagnati, we study K3 surfaces with an even set of eight disjoint (-2)-rational curves or with an even set of eight nodes. Such K3 surfaces are minimal resolution and quotient of a K3 surface by a Nikulin involution, their study complete the results of the paper [GS] in the following meaning: we consider a K3 surface with a Nikulin involution, then it is natural to ask what is the quotient surface and which properties has, for example which kind of singularities. In [GaSa2] we study K3 surfaces with an even set of rational curves and with the smallest possible Picard number, which is nine. We classify the surfaces and we describe their moduli space. In particular we describe many projective models, which continue and complete the study started by Barth in [B2] of such surfaces.
7) The Mckay correspondence in dimension two and three. In the paper [BoSa1] together with Samuel Boissière (University of Nizza) we give a relation between the Mckay correspondence in dimension two and in dimension three. Let G be a finite subgroup of $SO(3,\mathbb{R})$ and let $\tilde{G} \subset SU(2)$ be the binary group associated to the group G. The group $\tilde{G}$ operates on $\mathbb{C}^2$ and the quotient is an ADE-surface singularity. Its resolution consists of smooth (-2)-rational curves with an ADE-Dynkin diagram as dual graph. The Mckay correspondence associates to the vertices of the graph the irreducible representations ($\neq 1$) of $\tilde{G}$. The resolutions of the quotients $\mathbb{C}^2/\tilde{G}$ and $\mathbb{C}^3/G$ are Hilbert-Nakamura-Schemes and the exceptional curves of the resolutions on the origin have very similar properties. We show that there exists a morphism between these two resolutions, which contracts some curves in the exceptional fiber. For the proof we use the Mckay correspondence in dimension two and three, and also the theory of Hilbert-Nakamura-Schemes. The study of these resolutions is related to the study of the resolutions of the singularities of the four special K3 surfaces of the families X^d_λ/G_d, which are fibrations of the singular space $\mathbb{P}_3/G_d$ (cf. **3**).

4. Short description of the other scientific works

1. In the paper [Sa1] from my PhD thesis I work on the question: which is the maximal number of nodes a surface of degree d in $\mathbb{P}_3$ can have. For the degree $d \leq 6$ this problem

is solved by results of Cayley, Kummer, Beauville and Barth. For $d \geq 7$ the exact number is unknown. There are bound of Varchenko and Miyaoka. In this paper I find a surface of degree 12 with 600 nodes, which improves the previous lower bound of 576 nodes of Kreiß for a surface in this degree.
2. In the paper [ES] together with Philippe Ellia (University of Ferrara) I prove the Hartshorne conjecture for codimension two subvarieties in the case of 2-arithmetic Buchsbaum varieties. The exactly formulation of the Hartshorne conjecture for varieties of codimension two is the following: each smooth variety of codimension two in $\mathbb{P}_n$, $n \geq 7$ is complete intersection.

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Symmetric surfaces with many singularities

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SYMMETRIC SURFACES WITH MANY SINGULARITIES

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Abstract. Let $G \subset SO(4)$ denote a finite subgroup containing the Heisenberg group. In these notes we classify all these groups, we find the dimension of the spaces of G-invariant polynomials and we give equations for the generators whenever the space has dimension two. Then we complete the study of the corresponding G-invariant pencils of surfaces in $\mathbb{P}_3$ which we started in [S]. It turns out that we have five more pencils, two of them containing surfaces with nodes.

0. Introduction

Consider the Klein four group $V \subseteq SO(3)$. Let $\tilde{V}$ denote its inverse image in $SU(2)$ under the universal covering $SU(2) \rightarrow SO(3)$. The image of the direct product $\tilde{V} \times \tilde{V}$ in $SO(4)$ under the double covering $SU(2) \times SU(2) \rightarrow SO(4)$ is the Heisenberg group H. In this note we classify all the subgroups G of $SO(4)$ which contain H. First we classify all the subgroups of $SU(2) \times SU(2)$ which contain $\tilde{V} \times \tilde{V}$, then their images in $SO(4)$ are the groups G (cf. proposition 1.1 and section 1.4). They operate in a natural way on $\mathbb{C}[x_0, x_1, x_2, x_3]$, the ring of polynomials in four variables with complex coefficients. In section 3 we give generators for the spaces $\mathbb{C}[x_0, x_1, x_2, x_3]_j^G$ of homogeneous G-invariant polynomials of degree j whenever this dimension is two. Since the groups G contain H, we have invariant polynomials only in even degree. When the dimension is two the generators are the multiple quadric $q^{j/2} = (x_0^2 + x_1^2 + x_2^2 + x_3^2)^{j/2}$ (trivial invariant) and another polynomial of degree j which we denote by f. The pencils

$$f + \lambda q^{j/2} = 0, \ \lambda \in \mathbb{P}_1,$$

of surfaces in the three dimensional complex projective space $\mathbb{P}_3$ have then a large symmetry group (this is the reason why we consider just subgroups of $SO(4)$ containig H). We describe them in section 4. In particular we find the singular surfaces contained in it. In [S] we considered the case of $G = TT$, OO, II which are the images in $SO(4)$ of the direct products $\tilde{T} \times \tilde{T}$, $\tilde{O} \times \tilde{O}$, $\tilde{I} \times \tilde{I}$ where $\tilde{T}$ denotes the binary tetrahedral group, $\tilde{O}$ the binary octahedral group, $\tilde{I}$ the binary icosahedral group in $SU(2)$. We denoted the groups there by G_6, G_8 and G_{12} and we called them bi-polyhedral groups. We found pencils containing surfaces with many nodes (=ordinary double points). In particular the degree twelve II-invariant pencil contains a surface with 600 nodes which improves the previous lower bound for the maximal number of nodes of a surface of degree twelve in $\mathbb{P}_3$ (cf. [C]). Here we describe the other G-invariant pencils and show that we have two more pencils which contain surfaces with nodes (the others do not contain surfaces with isolated singularities at all). We list the groups G and the degrees j below, as well as the number of nodes on the singular surfaces. In each pencil we have four of these singular surfaces and the nodes there form just one G-orbit. For the convenience of the reader we recall the results about the TT-, OO-, and II-invariant pencils too.

G	order	j	nodes			
$(OO)'$	192	4	4	12	16	8
TT	288	6	12	48	48	12
OO	1152	8	24	72	144	96
IO	2880	12	240	360	240	120
II	7200	12	300	600	360	60

The group IO is the image in $SO(4)$ of the direct product of the binary icosahedral group with the binary octahedral group in $SU(2)\times SU(2)$ and $(OO)'$ is a subgroup of OO which we describe in section 1.2 and 1.4. This is an index two subgroup of the reflection group [3,3,4] (cf. [Co2], p. 226) and has the same invariant polynomials. In all the cases but one ($G = IO$), the singular surfaces contain just isolated singularities (i.e. the nodes). The surfaces in the IO-invariant pencil contain two double lines in the base locus.
The [3,3,4]-invariant polynomials of degree two and four were already known by Coxeter in [Co1]. Here we show that in the pencil of degree four we have a surface with the maximal number possible of nodes (=16) which is a so called Kummer surface. Finally in section 6 we give a computer picture of the IO-invariant surface of degree 12 with 360 nodes.
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1. Symmetry groups

Denote by $H \subseteq SO(4)$ the Heisenberg group (with 32 elements). We want to collect in a systematic way all the finite subgroups of $SO(4)$ containing H, and their polynomial invariants of low degree. These are invariants of the Heisenberg group with extra symmetries.

1.1. **Ternary groups.** We specify the following matrices in $SO(3)$

$$A_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, A_2 := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, R_n := \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & -b \\ 0 & b & a \end{pmatrix},$$

$$S := \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}, U := \frac{1}{2}\begin{pmatrix} \tau-1 & -\tau & 1 \\ \tau & 1 & \tau-1 \\ -1 & \tau-1 & \tau \end{pmatrix},$$

where $\tau := \frac{1}{2}(1+\sqrt{5}) = 2\cdot\cos(\frac{\pi}{5})$, $a :=\cos\frac{2\cdot\pi}{n}$, $b :=\sin\frac{2\cdot\pi}{n}$. These matrices generate the following subgroups of $SO(3)$

	generators	order	group	name
V	A_1, A_2	4	$\mathbb{Z}_2 \times \mathbb{Z}_2$	Klein four
D_n	A_2, R_n	$2n$	D_n	dihedral
T	A_1, S	12	$Alt(4)$	tetrahedral
O	A_1, R_4, S	24	$Sym(4)$	octahedral
I	A_1, S, U	60	$Alt(5)$	icosahedral

Where $Alt(4)$ and $Alt(5)$ denote the group of even permutations of four and five elements and $Sym(4)$ denotes the permutation group of four elements. Whenever $n \in \mathbb{N}$, $n \neq 0$, is *even*, V is contained in each of the above groups. By the classification of the finite subgroups of $SO(3)$, these are all such subgroups.
The groups T, O and I are the rotation groups of tetrahedron, octahedron and icosahedron. An identification with the permutation groups is given in [Co2], p. 46-50 as well as an identification of D_n with the symmetry group of a regular polygon with n vertices (ibid. p. 46). Sometimes it is useful for the computations to identify the matrices above with the cycles of the permutation groups. Indeed the identification of T and O with subgroups of the permutation group $Sym(4)$ is obtained by letting them act on the four space diagonals of the unit cube. Let these lines and vectors generating them be

$$d_1 : (1,1,1),\ d_2 : (-1,1,1),\ d_3 : (1,-1,1),\ d_4 : (1,1,-1)$$

The matrices in O permute these lines by

$$A_1 : (12)(34),\ A_2 : (13)(24),\ R_4 : (1423),\ S : (123).$$

Using this correspondence with permutation groups, it is easy to write down their conjugacy classes. We write the conjugacy classes of the dihedral group D_n, $n = 2l$, $l \geq 2$ too. In the next table we characterize a conjugacy class by one of its elements. Under each representative we write the number of elements in the conjugacy class.

group	repr. of a conj. class and number of elements				
V	1	A_1	A_2	A_3	
	1	1	1	1	
D_n	1	R_n^k	R_n^l	A_2	A_2R_n
	1	2	1	l	l
T	1	A_1	S	S^2	
	1	3	4	4	
O	1	A_1	R_4A_2	R_4	S
	1	3	6	6	8

where $k = 1, \ldots, l-1$.
The symmetries of T obviously leave invariant the icosahedron (cf. [Co2], p. 52) with vertices

$$(\pm 1, \pm\tau, 0),\ (0, \pm 1, \pm\tau),\ (\pm\tau, 0, \pm 1)$$

The matrix U permutes these vertices as

$$\pm(0,1,\tau) \mapsto \pm(0,1,\tau),$$
$$\pm(0,1,-\tau) \mapsto \pm(-\tau,0,1) \mapsto \pm(-1,-\tau,0) \mapsto \pm(1,-\tau,0) \mapsto \pm(\tau,0,-1) \mapsto \pm(0,1,-\tau).$$

So together with the group T the symmetry U generates a group of order at least $12 \cdot 5 = 60$, contained in the symmetry group of the icosahedron specified. Therefore it coincides with the icosahedral group $I \cong A_5$. The action of I on the five cosets of I/T defines the map $I \mapsto A_5$. Using

$$A_1 \cdot U = U^4 \cdot A_1,\ A_1 \cdot U^2 = U^3 \cdot A_1,$$
$$S \cdot U = U^3 \cdot A_2,\ S \cdot U^2 = U \cdot A_2 S,\ S \cdot U^3 = U^2 \cdot S^2,\ S \cdot U^4 = U^4 \cdot S^2 A_2,$$

one finds that under this map

$$A_1 \mapsto (14)(23),\ S \mapsto (132),\ U \mapsto (12345).$$

Using this correspondence, one enumerates the conjugacy classes in I

group	repr. of a conj. class and number of elements				
I	1	A_1	S	U	U^2
	1	15	20	12	12

1.2. **Subgroups of products of ternary groups.** Here we classify *subdirect products* $G \subseteq G_1 \times G_2$, where G_1 and G_2 are finite ternary groups V, D_n (n even), T, O or I. We assume that G contains the subgroup $V \times V \subseteq G_1 \times G_2$. We are interested in these subgroups only up to interchanging the factors G_1 and G_2. So we assume that we are in one of the following cases

- $|G_1| \geq |G_2|$ and $G_1, G_2 \neq D_n$,
- $G_1 \neq D_n$, $G_2 = D_n$,
- $G_1 = D_n$ and $G_2 = D_m$.

Additionally, passing to smaller subgroups $G_i' \subseteq G_i$ if necessary, we may assume that both projections

$$p_1 : G \to G_1',\ p_2 : G \to G_2'$$

are surjective, so that $G \subseteq G_1' \times G_2'$. Finally, we do not distinguish between groups conjugate in $SO(3) \times SO(3)$. In the table below we assume $n \neq m$, $n = 2l$, $m = 2l'$ and let $s :=\mathrm{lcm}(n, m)$.

Proposition 1.1. *The following list is a complete list of subgroups $G \subseteq G_1 \times G_2$ under the assumptions above:*

G_1	G_2	G	$\lvert G\rvert/16$	$G/V\times V$	G_1	G_2	G	$\lvert G\rvert/16$	$G/V\times V$
V	V	$V\times V$	1	1	V	D_n	$V\times D_n$	$n/2$	$\mathbb{Z}_l\times 1$
T	V	$T\times V$	3	$\mathbb{Z}_3\times 1$	T	D_n	$T\times D_n$	$3n/2$	$\mathbb{Z}_3\times\mathbb{Z}_l$
	T	$T\times T$	9	$\mathbb{Z}_3\times\mathbb{Z}_3$		$3\vert n$	$(T\times D_n)'$	$n/2$	$\mathbb{Z}_l$
		$(T\times T)'$	3	$\mathbb{Z}_3$	O	D_n	$O\times D_n$	$6n/2$	$D_3\times\mathbb{Z}_l$
O	V	$O\times V$	6	$D_3\times 1$		$4\vert n$	$(O\times D_n)'$	$3n/2$	$D_{3l/2}$
	T	$O\times T$	18	$D_3\times\mathbb{Z}_3$	I	D_n	$I\times D_n$	$15n/2$	
	O	$O\times O$	36	$D_3\times D_3$	D_n	D_n	$D_n\times D_n$	$n^2/4$	$\mathbb{Z}_l\times\mathbb{Z}_l$
	O	$O\times O$	36	$D_3\times D_3$			$(D_n\times D_n)'$	$n/2$	$\mathbb{Z}_l$
		$(O\times O)'$	6	D_3	D_n	D_m	$D_n\times D_m$	$nm/4$	$\mathbb{Z}_l\times\mathbb{Z}_{l'}$
		$(O\times O)''$	18	$\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_2$			$(D_n\times D_m)'$	$s/2$	$\mathbb{Z}_s$
I	V	$I\times V$	15						
	T	$I\times T$	45						
	O	$I\times O$	90						
	I	$I\times I$	225						

Proof. We discuss the cases one-by-one. But before that, we observe that the kernels $K_1\subseteq G_1\times 1$ and $K_2\subseteq 1\times G_2$ of both projections $p_i : G\to G_i$ are normal subgroups. This follows by conjugating component-wise from the surjectivity of both projections.
The groups $G_1\times V$ do not have proper subgroups containing $V\times V$ and mapping surjectively onto G_1, (for $G_1 = D_n$ too). So we do not need to consider the cases $G_2 = V$.
First consider the case of $G_1 = I$. Since I is simple, the kernel $K_1\subseteq I\times 1$ either coincides with $I\times 1$, or is trivial. The latter cannot be the case, because this kernel contains $V\times 1$. The only possibilities are the product cases $I\times V$, $I\times T$, $I\times O$, $I\times I$ and $I\times D_n$.
Let now $Q := G/V\times V$ and $Q_1 := G_1/V$, $Q_2 := G_2/V$. We consider Q a proper subdirect product of $Q_1\times Q_2$.
T,T: $Q\subseteq\mathbb{Z}_3\times\mathbb{Z}_3$ mapping surjectively onto both factors is either the diagonal or the anti-diagonal. The two corresponding subgroups are not conjugate in $T\times T$, but in $T\times O$ they are. The inverse image of the diagonal $\mathbb{Z}_3\subseteq\mathbb{Z}_3\times\mathbb{Z}_3$ in $T\times T$ is $(T\times T)'$.
O,T: $Q\subseteq D_3\times\mathbb{Z}_3$ mapping surjectively onto both factors would have order six and be isomorphic with D_3 under p_1. But there is no epimorphism of D_3 onto $\mathbb{Z}_3$. Such a group does not exist.
O,O: $Q\subseteq D_3\times D_3$ must have order six, twelve or 18. If it has order six, it is a graph of an isomorphism between both factors D_3. Then it is conjugate to the diagonal, and this leads to the subgroup $(O\times O)'\subseteq O\times O$. The case $\lvert Q\rvert=12$ cannot occur, because then the kernel $K_1\subseteq D_3\times 1$ would have order two, and could not be normal. If Q has order 18 both the kernels K_1 and K_2 have order three, and coincide with the unique proper normal subgroup of D_3. This implies that G contains $T\times T\subseteq O\times O$ and is the inverse image of a subgroup $\mathbb{Z}_2\subseteq\mathbb{Z}_2\times\mathbb{Z}_2= O\times O/T\times T$. By surjectivity of projections this can only be the diagonal. Its inverse image is $(O\times O)''$.
T,D_n: if $Q\subseteq\mathbb{Z}_3\times\mathbb{Z}_l$ maps surjectively onto both the components then three divides l. Let its inverse image be $(T\times D_n)'$.
O,D_n: if $Q\subseteq D_3\times\mathbb{Z}_l$ maps surjectively onto both factors then two divides l. We denote the inverse image by $(O\times D_n)'$.

D_n, D_n: $Q \subseteq \mathbb{Z}_l \times \mathbb{Z}_l$ mapping surjectively onto $\mathbb{Z}_l$ is conjugate to the diagonal. We call its inverse image $(D_n \times D_n)'$.
D_n, D_m: $Q \subseteq \mathbb{Z}_l \times \mathbb{Z}_{l'}$ mapping surjectively onto $\mathbb{Z}_l$ and $\mathbb{Z}_{l'}$ is generated by an element of order $s/2$. We call its inverse image $(D_n \times D_m)'$.
$\square$

1.3. **Binary groups.** We consider the standard double cover $SU(2) \to SO(3)$ (cf. [DV] p. 39–42). Let $\tilde{G}$ denotes the pre-image in $SU(2)$ of $G \subseteq SO(3)$. We specify the following matrices, $\tilde{M}$, which are in the pre-image of $M \in SO(3)$:

$$\tilde{A}_1 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \tilde{A}_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \tilde{A}_3 := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$
$$\tilde{S} := \tfrac{1}{2}\begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix}, \quad \tilde{U} := \tfrac{1}{2}\begin{pmatrix} \tau & \tau-1+i \\ 1-\tau+i & \tau \end{pmatrix}, \quad \tilde{R}_n := \begin{pmatrix} e^{\frac{i\pi}{n}} & 0 \\ 0 & e^{-\frac{i\pi}{n}} \end{pmatrix}.$$

since $\tilde{M}^{\mathrm{ord}(M)} = -1$, they have order $2{\cdot}\mathrm{ord}(M)$. By an argumentation as in [S] section 2, we can write the conjugacy classes in the binary groups:

group	repr. of a conj. class and number of elements							
$\tilde{V}$	1	-1	$\tilde{A}_1$	$\tilde{A}_2$	$\tilde{A}_3$			
	1	1	2	2	2			
$\tilde{D}_n$	1	-1	$\tilde{A}_2$	$\tilde{A}_2\tilde{R}_n$	$\tilde{R}_n^{\,l}$	$\tilde{R}_n^{\,k}$	$-\tilde{R}_n^{\,k}$	
	1	1	2l	2l	2	2	2	
$\tilde{T}$	1	-1	$\tilde{A}_1$	$\tilde{S}$	$-\tilde{S}$	$\tilde{S}^2$	$-\tilde{S}^2$	
	1	1	6	4	4	4	4	
$\tilde{O}$	1	-1	$\tilde{A}_1$	$\tilde{R}_4\tilde{A}_2$	$\tilde{R}_4$	$-\tilde{R}_4$	$\tilde{S}$	$-\tilde{S}$
	1	1	6	12	6	6	8	8
$\tilde{I}$	1	-1	$\tilde{A}_1$	$\tilde{S}$	$-\tilde{S}$	$\tilde{U}$	$-\tilde{U}$	$\tilde{U}^2$	$-\tilde{U}^2$
	1	1	30	20	20	12	12	12	12

where $k = 1, \ldots, l-1$.

1.4. **Quaternary groups.** Here we consider the images of the finite groups $\tilde{G}_1 \times \tilde{G}_2 \subseteq SU(2) \times SU(2)$ under the double covering map

$$SU(2) \times SU(2) \to SO(4), \quad (q, q') : p \mapsto q \cdot p \cdot q'^{-1}$$

(cf. [DV] p. 42–45), we abbreviate there G_1G_2. Since the corresponding subgroups of $SO(3) \times SO(3)$ contain the group $V \times V$, these contain the Heisenberg group $VV \subseteq SO(4)$, and by proposition 1.1 these are all such subgroups. We specify now the matrices:

$$(A_1,1) := \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad (1,A_1) := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$(A_2,1) := \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad (1,A_2) := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$(A_3,1) := \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad (1,A_3) := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$(R_n,1) := \begin{pmatrix} \alpha & -\beta & 0 & 0 \\ \beta & \alpha & 0 & 0 \\ 0 & 0 & \alpha & -\beta \\ 0 & 0 & \beta & \alpha \end{pmatrix}, \quad (1,R_n) := \begin{pmatrix} \alpha & \beta & 0 & 0 \\ -\beta & \alpha & 0 & 0 \\ 0 & 0 & \alpha & -\beta \\ 0 & 0 & \beta & \alpha \end{pmatrix},$$

$$(S,1) := \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad (1,S) := \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 & 1 \\ -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \end{pmatrix},$$

$$(U,1) := \frac{1}{2}\begin{pmatrix} \tau & 0 & 1-\tau & -1 \\ 0 & \tau & -1 & \tau-1 \\ \tau-1 & 1 & \tau & 0 \\ 1 & 1-\tau & 0 & \tau \end{pmatrix}, \quad (1,U) := \frac{1}{2}\begin{pmatrix} \tau & 0 & \tau-1 & 1 \\ 0 & \tau & -1 & \tau-1 \\ 1-\tau & 1 & \tau & 0 \\ -1 & 1-\tau & 0 & \tau \end{pmatrix},$$

where $\alpha := \cos\frac{\pi}{n}$, $\beta := \sin\frac{\pi}{n}$.
We can write the conjugacy classes of the groups $G_1G_2 \subseteq SO(4)$ in $SO(4)$ and their number of elements. These are the images of the conjugacy classes of $\tilde{G_1} \times \tilde{G_2}$ in $G_1G_2 \subseteq SO(4)$ under the double covering map. Observe that the matrices $(g_1, g_2) \in SO(4)$ with the same eigenvalues are conjugate (cf. (1.1) of [S]), this fact simplifies the computations considerably. In this section and in the next one we omit the groups D_nD_m and $(G_1G_2)'$ with $G_2 = D_n$. We return to those groups later. In the tables we use the following conventions:

- we omit the conjugacy classes of $+1, -1$ (these contain one element each)
- whenever the conjugacy classes (g_1, g_2) and its s-th power (g_1^s, g_2^s) are distinct we write them just one time since they have the same number of elements.

G_1G_2	order	$A_2,1$	A_2,A_2	$R_4,1$	R_4,A_2	R_4,R_4	$S,1$	S,A_2	S,R_4	S,S
VV	32	12	18	0	0	0	0	0	0	0
TV	96	12	18	0	0	0	8	48	0	0
TT	288	12	18	0	0	0	16	96	0	64
OV	192	24	54	6	36	0	8	48	0	0
OT	576	24	54	6	36	0	16	192	48	64
OO	1152	36	162	12	216	36	16	288	96	64
IV	480	36	90	0	0	0	20	120	0	0
IT	1440	36	90	0	0	0	28	360	0	160
IO	2880	48	270	6	180	0	28	600	120	160
II	7200	60	450	0	0	0	40	1200	0	400

G_1G_2	order	$A_2,1$	A_2,A_2	$R_4,1$	R_4,A_2	$S,1$	S,A_2
VD_n	$16n$	$8+4l$	$6(2l+1)$	0	0	0	0
TD_n	$48n$	$8+4l$	$6(2l+1)$	0	0	8	$16(2l+1)$
OD_n	$96n$	$20+4l$	$18(2l+1)$	6	$12(2l+1)$	8	$16(2l+1)$
ID_n	$240n$	$32+4l$	$30(2l+1)$	0	0	20	$40(2l+1)$

G_1G_2	$U,1$	U,A_2	U,R_4	U,S	U,U	U,U^2	U,R_n^k
IV	12	72	0	0	0	0	0
IT	12	72	0	96	0	0	0
IO	12	216	72	96	0	0	0
II	24	720	0	480	144	144	0
ID_n	12	$24(2l+1)$	0	0	0	0	24

G_1G_2	A_2,R_n^k	$1,R_n^k$	S,R_n^k	R_4,R_n^k
VD_n	12	2	0	0
TD_n	12	2	16	0
OD_n	36	2	16	12
ID_n	60	2	40	0

where $k = 1, \dots, l-1$. For the groups which are not products it is a little more complicated to write down the sizes of their conjugacy classes. But using the description from proposition 1.1 one finds

group	order	$A_2, 1$	A_2, A_2	R_4, A_2	R_4, R_4	$S, 1$	S, A_2	S, S
$(TT)'$	96	12	18	0	0	0	0	32
$(OO)'$	192	12	42	48	12	0	0	32
$(OO)''$	576	12	90	144	36	16	96	64

2. Poincaré series

In this section we want to find the dimension of the spaces of homogeneous invariant polynomials of a given degree. We consider the Poincaré series

$$p(t) := \sum_{j=0}^{\infty} \dim \mathbb{C}[x_0, x_1, x_2, x_3]_j^G \cdot t^j$$

where G is a group as in section 1.4. By a theorem of Molien ([B] p. 21) and an easy computation as in [S], (2.1), it can be written as

$$p(t) = \frac{1}{|G|} \sum \frac{n_g}{\det(g - 1 \cdot t)}$$

where the sum runs over all the conjugacy classes of G and n_g denote their number of elements. At the denominator we have the characteristic polynomials. Using the numbers of conjugacy classes (under $SO(4)$) given in section 1.4 and computing their characteristic polynomials, the power series package of MAPLE produces the following table of dimensions m_d of invariant polynomials in degree d of the groups $G \subseteq G_1 G_2$. Observe that since G contains the Heisenberg group we do not have invariant polynomials of odd degree. First we consider the case of $G_i \neq D_n$:

group	m_2	m_4	m_6	m_8	m_{10}	m_{12}
VV	1	5	6	15	19	35
TV	1	1	**2**	5	5	13
TT	1	1	**2**	3	3	7
$(TT)'$	1	3	4	7	9	15
OV	1	1	1	4	4	8
OT	1	1	1	**2**	**2**	4
OO	1	1	1	**2**	**2**	3
$(OO)'$	1	**2**	3	5	6	9
$(OO)''$	1	1	**2**	3	3	5
IV	1	1	1	1	1	5
IT	1	1	1	1	1	3
IO	1	1	1	1	1	**2**
II	1	1	1	1	1	**2**

Whenever $G_1 \neq D_n$ and $G_2 = D_n$, we can write the finite sums $p(t)$ for each n. Here we compute the first coefficients of the Poincaré series for $n = 4, 6, 8$.

group	m_2	m_4	m_6	m_8	m_{10}	m_{12}
VD_4	1	3	3	9	11	19
VD_6	1	3	3	6	6	14
VD_8	1	3	3	6	6	10
TD_4	1	1	1	3	3	7
TD_6	1	1	1	**2**	**2**	6
TD_8	1	1	1	**2**	**2**	4
OD_4	1	1	1	3	3	5
OD_6	1	1	1	**2**	**2**	4
OD_8	1	1	1	**2**	**2**	3
ID_4	1	1	1	1	1	3
ID_6	1	1	1	1	1	3
ID_8	1	1	1	1	1	**2**

Of course, whenever $m_d = 1$, the space of invariant polynomials is spanned by the $d/2$-th power of the invariant quadric (*trivial invariant*)

$$q := x_0^2 + x_1^2 + x_2^2 + x_3^2.$$

3. Invariants

In this section we want to compute a system of generators for the spaces $\mathbb{C}[x_0, x_1, x_2, x_3]_j^G$ whenever this space has dimension two and $G \subseteq \mathrm{SO}(4)$ is a finite subgroup containing the Heisenberg group. We distinguish two cases.

(3.1) *First case.* Assume that $G \subseteq G_1G_2$ with $G_i \neq D_n$, $i = 1, 2$. We do some remarks on the groups G which simplify the computations of the invariant polynomials.

- the group $TV \subseteq TT$.
- The group TT is contained in $(OO)''$. In fact the generators modulo VV are

group	TT	$(OO)''$
generators	$(1, S)$	$(1, S), (S, 1)$
	$(S, 1)$	(R_4A_2, R_4A_2)

with

$$(R_4A_2, R_4A_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Similarly to [S] section 3, $(OO)''$ is a subgroup of index two in the reflection group of the $\{3, 4, 3\}$-cell (cf. [Co2] p. 149 for the definition of this polytope), if we add the generator

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

we get the whole reflection group.

- Finally since $OT \subseteq OO$, they have the same two invariant polynomials in degree eight. Those of degree ten are obtained just by multiplication with the quadric q.

By these remarks and considering G as in the assumption, it follows that we have to compute the generators of six two-dimensional spaces $\mathbb{C}[x_0, x_1, x_2, x_3]_j^G$ for the following pairs (G, j):

$$((OO)', 4) \quad (TT, 6) \quad (OO, 8) \quad (IO, 12) \quad (II, 12).$$

The generators of the TT-, OO- and II-invariant spaces are given in [S] as well as a description of the corresponding pencils of surfaces in $\mathbb{P}_3$ (base locus and singular surfaces). Here we examine the remaining cases and in the next section we describe the corresponding pencils of surfaces in $\mathbb{P}_3$. The basic idea to find generators of the invariant spaces is the same as in [S]. For the computations here we use the matrix representation of the groups given in section 1.4 (see also [ST1] and [ST2]).
$(OO)'$-invariants. We start with the space of Heisenberg invariant quartics. It has dimension five, being spanned by

$$\begin{aligned} &f_0 := x_0^4 + x_1^4 + x_2^4 + x_3^4, \\ &f_1 := 2(x_0^2 x_1^2 + x_2^2 x_3^2), \qquad f_2 := 2(x_0^2 x_2^2 + x_1^2 x_3^2), \quad f_3 := 2(x_0^2 x_3^2 + x_1^2 x_2^2), \\ &f_4 := 4 x_0 x_1 x_2 x_3. \end{aligned}$$

In terms of these invariants

$$q^2 = f_0 + f_1 + f_2 + f_3.$$

Modulo VV, the group $(OO)'$ is generated by $(R_4 A_2, R_4 A_2)$ and (S, S). Tracing the action of these generators on $f_0, \dots, f_4$ one finds the invariants q^2 and f_0.
IO-invariants. The generators $(U, 1)$, $(1, S)$ and $(1, R_4)$ of IO operate on the space $\mathbb{C}[x_0, x_1, x_2, x_3]_{12}^{VV}$ which is 35-dimensional. A computation with MAPLE shows that it contains the non trivial IO-invariant polynomial

$$\begin{aligned} S_{IO} \quad := \quad & -\textstyle\sum_i f_i^3 + 11 \sum_{i,j} f_i^2 f_j + (f_0^2 - 14 f_4^2) \sum_i f_i + 30 f_0 f_4^2 - 2 f_0 \sum_{i,j} f_i f_j \\ & -30 f_1 f_2 f_3 + 3\sqrt{5} f_4 (2 f_0 \textstyle\sum_i f_i - \sum_{i,j} f_i f_j - f_0^2 - \sum_i f_i^2 + 4 f_4^2) \\ & +6\sqrt{5} F_a \end{aligned}$$

where the sums run over all the indices $i, j = 1, 2, 3$, $i \neq j$, and

$$F_a := f_1^2 f_2 + f_3^2 f_1 + f_2^2 f_3 - f_1 f_2^2 - f_3 f_1^2 - f_2 f_3^2$$

is the anti-symmetric part of S_{IO}.
(3.2)*Second case.* Assume that $G \subseteq G_1 G_2$ with $G_1 = D_n$ or $G_2 = D_m$, $n, m \geq 4$, even. With the help of the table in section 1.4 we discuss the following cases:

- D_nD_m: observe that VD_m is contained in D_nD_m. A direct computation shows that the generator $(1, R_n)$ leaves invariant a three-dimensional family of degree four VV-invariant polynomials. Generators in this case are $f_2 + f_3$, q^2, $f_2 - f_4$. Now the matrix $(R_n, 1)$ of D_nD_m operates on these three-dimensional space leaving invariant q^2 and $f_2 + f_3$.
- since $(D_nD_m)' \subseteq D_nD_m$ these groups have already an at least two-dimensional family of invariant polynomials of degree four, which is generated by q^2 and $f_2 + f_3$.
- TD_n, $n \geq 6$: we have a five-dimensional family of TV-invariant in degree eight. By the action of the generator $(1, R_n)$, $n \geq 6$ we have a two-dimensional family of invariants. Put

  $$K_i := x_0^2 + x_1^2 - x_2^2 - x_3^2 + (-1)^i 2x_0(x_2 + x_3) + 2x_1(x_2 - x_3),\ i = 0, 3$$
  $$K_i := x_0^2 + x_1^2 - x_2^2 - x_3^2 + (-1)^i 2x_0(x_2 + x_3) - 2x_1(x_2 - x_3),\ i = 1, 2$$

  then generators are

  $$q^4 \text{ and } P_8 := \prod_{i=0}^{3} K_i.$$

- Suppose three divides n. Since the groups $(TD_n)'$ are contained in TD_n, they have at least a two-dimensional family of invariant polynomials in degree eight. We consider now the action of the extra generators (S, R_n) of $(TD_n)'$ on the space of degree four, resp. six VV-invariant polynomials. A direct computation with MAPLE shows that we have no invariants other then the quadric q^2, resp. q^3.
- OD_n, $n \geq 6$: we have $TD_n \subseteq OD_n$ and the extra generator $(R_4, 1)$ leaves the previous polynomials invariant.
- Suppose now four divides n. Since the groups $(OD_n)'$ are contained in OD_n, they have at least a two-dimensional family of invariant polynomials in degree eight. Modulo VV the groups $(OD_n)'$ have generators $(S, 1)$ and (R_4, R_n), hence they contain the group TV. This has no non-trivial invariant polynomials of degree four and has a two-dimensional family of invariant polynomials in degree six, generated by q^3 and $S_6(x)$ (cf. [S] p. 437). For $n \geq 6$, $S_6(x)$ is not $(OD_n)'$-invariant, for $n = 4$ it is. In any case we do not get new invariants.
- ID_n, $n \geq 8$: we have a five-dimensional family of IV-invariant polynomials in degree twelve. The action of $(1, R_n)$, $n \geq 8$ on this space produces a two-dimensional family of invariant polynomials. Put

  $$\begin{aligned}
  K_0' &:= x_0^2 + x_1^2 - x_2^2 - x_3^2 + 2(1 - \tau)(x_0x_2 - x_1x_3) \\
  K_1' &:= x_0^2 + x_1^2 - x_2^2 - x_3^2 + 2\tau(x_0x_3 + x_1x_2) \\
  K_2' &:= x_0^2 + x_1^2 - x_2^2 - x_3^2 - 2(1 - \tau)(x_0x_2 - x_1x_3) \\
  K_3' &:= x_0^2 + x_1^2 - x_2^2 - x_3^2 - 2\tau(x_0x_3 + x_1x_2) \\
  K_4' &:= 2(1 - \tau)(x_0x_3 + x_1x_2) + 2(x_0x_2 - x_1x_3) \\
  K_5' &:= -2\tau(x_0x_3 + x_1x_2) + 2(x_0x_2 - x_1x_3)
  \end{aligned}$$

  then generators are

  $$q^6 \text{ and } P_{12} := \prod_{i=0}^{5} K_i'.$$

In conclusion, the dimension of $\mathbb{C}[x_0, x_1, x_2, x_3]^G_j$ is two, for (G, j) equal to

$$(D_nD_m, 4)\ n, m \geq 4, \quad (OD_n, 8)\ n \geq 6, \quad (ID_n, 12)\ n \geq 8.$$

Remark 3.3 Observe that the polynomials q^2 and $f_2 + f_3$ are D_nD_n-invariant even if n is an odd integer. As in the even case, generators of these groups are $(A_2, 1)$, $(1, A_2)$ and $(R_n, 1)$, $(1, R_n)$.

4. Invariant pencils

4.1. **The pencil of $(OO)'$-invariant quartics.** We take the generators q^2 and $f_0 := x_0^4 + x_1^4 + x_2^4 + x_3^4$. The base locus is a double curve of degree eight. Let C_8 denote the curve $\{q = 0,\ f_0 = 0\}$. The pencil is invariant under the matrix C too, hence by symmetry reasons the curve C_8 has bi-degree $(4, 4)$ on q (cf. [S], section 5). Moreover we have the following

Lemma 4.1. *The curve C_8 is smooth and irreducible.*

Proof. The Jacobian matrix of C_8 has rank two in each point, hence the curve is smooth. If $C_8 = C' \cup C''$, the curves C', C'' would meet in some point on q (observe that they cannot be lines of the same ruling), but this is impossible because C_8 is smooth. □

Now a singular point on a surface of the pencil ($\neq q^2$) is not contained on q (cf. (6.1) of [S]). Hence by this fact, lemma 4.1 and Bertini's theorem the general surface in the pencil is smooth. All the other surfaces but q^2 are irreducible and reduced and the singular ones have only isolated singularities.

The symmetry group of $\{3, 3, 4\}$. Consider the "cross polytope" β_4= $\{3, 3, 4\}$ in $\mathbb{R}^4$ with vertices the permutations of $(\pm 1, 0, 0, 0)$ as in [Co2] p. 156 and edge $\sqrt{2}$. The generators of $(OO)'$ permutes these points, hence $(OO)'$ is contained in the symmetry group $[3, 3, 4]$ of $\{3, 3, 4\}$, more precisely it is an index two subgroup. In fact the symmetry group of β_4 has order $2^4 \cdot 4! = 384$= $2 \cdot 192$ and by adding the generator C to $(OO)'$ we get the whole symmetry group $[3, 3, 4]$ (cf. [Co2] p. 226). In particular, observe that they have the same invariant polynomials. The polytope β_4 has $N_0 = 8$, $N_1 = 24$, $N_2 = 32$, $N_3 = 16$ the reciprocal "measure polytope", γ_4= $\{4, 3, 3\}$ has $N_0 = 16$, $N_1 = 32$, $N_2 = 24$, $N_3 = 8$. Hence we get four $[3, 4, 3]$-orbits of points: the vertices and the middle points of the edges of the β_4 and the vertices and the middle points of the edges of the reciprocal γ_4. These have coordinates the permutation of $(\pm 1, 0, 0, 0)$, $(\pm 1, \pm 1, 0, 0)$, resp. $(\pm 1, \pm 1, \pm 1, \pm 1)$, $(\pm 1, \pm 1, \pm 1, 0)$. As points of $\mathbb{P}_3$ these are singular on the surfaces $f_0 + \lambda q^2$ for $\lambda = -1, -\frac{1}{2}$ resp. $-\frac{1}{4}, -\frac{1}{3}$ and a direct computation shows that they are all ordinary double points. We do it for $\lambda = -1$ and $(1 : 0 : 0 : 0)$. In the affine chart $\{x_0 \neq 0\}$ the equation becomes

$$\begin{aligned} 0 &= 1 + x^4 + y^4 + z^4 - (1 + x^2 + y^2 + z^2)^2 \\ &= -2x^2 - 2y^2 - 2z^2 + \text{terms of degree} \geq 4, \end{aligned}$$

hence the rank of the Hessian matrix at $(0, 0, 0)$ is three. This shows that $(1 : 0 : 0 : 0)$ is an ordinary double point and so are all the points in its orbit.

We collect the results on the singular surfaces in the following table. In the middle column we write just one point, but we mean all its permutations.

λ	nodes	number\description
-1	$(1:0:0:0)$	4
$-\frac{1}{2}$	$(\pm 1:\pm 1:0:0)$	12
$-\frac{1}{3}$	$(\pm 1:\pm 1:\pm 1:0)$	16, Kummer surface
$-\frac{1}{4}$	$(\pm 1:\pm 1:\pm 1:\pm 1)$	8
∞	$-$	double quadric

In the case of $\lambda = -\frac{1}{3}$, we get a surface with 16 nodes which is the maximal number possible for a surface of degree four. This is a Kummer surface. Observe that in this case too as in [S] the nodes are fix points under the action of some matrices in $[3, 3, 4]$, resp. in $(OO)'$ and they are contained on lines of fix points (see (5.2) and (6.3) of [S]). Moreover an estimation as in section 8 of [S] shows that whenever the lines of fix points do not meet the base locus, they contain exactly four nodes. It is natural to expect that these are all the singular surfaces in the $(OO)'$-invariant pencil (as in the case of the TT-, OO-, and II-invariant pencils). This is a direct consequence of the following

Proposition 4.1. *1. The conjugacy classes in $(OO)'$ (under $(OO)'$) with eigenvalues ± 1 are the following*

conj. class	(A_2, A_2)	(A_2, A_1)	(R_4A_2, R_4A_2)	(S, S)	(R_4, R_4)
number of elements	6	12	24	32	12
number of fix lines	6	12	24	16	6

2. The fix lines of the matrices in these conjugacy classes contain the maximal number possible of node.

Proof. Choosing a fix line for each of the representative above and intersecting with the singular surfaces we find

matrix and fix line	value of λ		int. points	
(A_2, A_2) :	-1	$-\frac{1}{2}$	$(1:0:0:0)$	$(\pm 1:0:1:0)$
$x_1 = x_3 = 0$			$(0:0:1:0)$	
(A_2, A_1) :	$-\frac{1}{2}$	$-\frac{1}{4}$	$(0:1:1:0)$	$(1:1:1:1)$
$x_0 = x_3, x_1 = x_2$			$(1:0:0:1)$	$(-1:1:1:-1)$
(R_4A_2, R_4A_2) :	-1	$-\frac{1}{2}$	$(1:0:0:0)$	$(0:0:1:1)$
$x_1 = 0, x_2 = x_3$	$-\frac{1}{3}$		$(\pm 1:0:1:1)$	
(S, S) :	-1	$-\frac{1}{3}$	$(1:0:0:0)$	$(0:1:1:1)$
$x_1 = x_2 = x_3$	$-\frac{1}{4}$		$(\pm 1:1:1:1)$	

where we do not write the matrix (R_4, R_4), since the fix lines of the matrices in its conjugacy class are the same as those of the matrices in the conjugacy class of (A_2, A_2). From this table follows that:

- the fix lines above meet different surfaces, hence the conjugacy classes of these matrices are in fact, all distinct (cf. also [S] (7.3)).

- The fix lines contain the maximal number possible (four), of nodes (this shows *2.*)

About the number of fix lines: observe that the conjugacy classes of (S,S) has order 32 and contains the elements (S^2,S^2) hence we have $\frac{32}{2}=16$ distinct fix lines. Finally the conjugacy class of (A_2,A_2) contains six elements, that of (A_2,A_1) contains twelve elements and that of (R_4A_2,R_4A_2) contains 24 elements. Since each element in these conjugacy classes has two fix lines and the elements with minus sign are in the same conjugacy class, the number of fix lines is the same as the number of matrices. □

We know that the singular points form $[3,4,3]$-orbits, but we can now show something more.

Lemma 4.2. *The nodes on each singular surface form one $(OO)'$-orbit:*

λ	-1	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{4}$
orbit	4	12	16	8
fix group mod. ± 1	S_4	D_4	D_3	A_4
order	24	8	6	12

Proof. The situation is easy for $\lambda=-1$. In fact the matrix $C\in[3,4,3]$ leaves each singular point fix, hence the group $(OO)'$ musts permute them. Consider $\lambda=-\frac{1}{2},-\frac{1}{3}$ resp. -$\frac{1}{4}$ and assume that the orbit's length of singular points is less or equal then 12, 16, resp. 8. Then the fix group in $(OO)'$ mod. ±1 has order bigger or equal then 8, 6 resp. 12. Checking in the table given in the previous page we see that in fact the only possibility is to have equality. □

Put now N_0 =number of nodes on a surface in the pencil, N_1 =number of fix lines of matrices in the same conjugacy class, n_0=number of nodes on a line, n_1=number of line through a point. Knowing the fix groups of the singular points and using the formula

$$N_0\cdot n_1=N_1\cdot n_0 \tag{1}$$

for a configuration of lines and points (cf. [S] section 11.), we can write the table:

Repr. of the conj. class	value of λ			Configuration		
(A_2,A_2)	-1	$-\frac{1}{2}$		$(4_3,6_2)$	$(12_1,6_2)$	
(A_2,A_1)	$-\frac{1}{2}$	$-\frac{1}{4}$		$(12_2,12_2)$	$(8_3,12_2)$	
(R_4A_2,R_4A_2)	-1	$-\frac{1}{2}$	$-\frac{1}{3}$	$(4_6,24_1)$	$(12_2,24_1)$	$(16_3,24_2)$
(S,S)	-1	$-\frac{1}{3}$	$-\frac{1}{4}$	$(4_4,16_1)$	$(16_1,16_1)$	$(8_4,16_2)$

where in the second column we mean the nodes of the surfaces with the given λ.

4.2. **The pencil of IO-invariant 12-ics.** We take the generators q^6 and S_{IO}. We compute first the base locus. Consider the groups $(1, O)$, resp. $(I, 1)$, which operate on the two rulings of the quadric q, each element there has two lines of fix points (cf. (5.4) of [S]). For the convenience of the reader we recall the table on the length of the orbits under the action of the octahedral group O, and of the icosahedral group I:

octahedron	icosahedron
24, 12, 8, 6	60, 30, 20, 12

Now we show:

Lemma 4.3. *The variety $q \cap S_{IO}$ consists of an $(I, 1)$-orbit of twelve lines and of an $(1, O)$-orbit of six singular lines of S_{IO}.*

Proof. We have deg($q \cap S_{IO}$)=24. By the table above it can only have bi-degree $(12, 12)$ or $(12, 6)$. An argumentation as in [S] (5.5), (b), shows that $q \cap S_{IO}$ splits into the union of lines of the two rulings of q. More precisely it contains the $(I, 1)$-orbit of twelve lines and the $(1, O)$-orbit of twelve or of six lines. In the last case the lines have multiplicity two in the intersection. So take the fix line $(\lambda : \mu : i\lambda : i\mu)$, $(\lambda : \mu) \in \mathbb{P}_1$, of the matrix $(1, A_2)$ in the orbit of length six. A direct computation (with MAPLE) shows that it is singular on S_{IO}, so we are done. □

Lemma 4.4. *Let p be a singular point on a surface of the pencil $S_{IO} + \lambda q^6$ and assume that p is on the quadric q. Then $p \in L_i$, $i = 1, \cdots, 6$, where the L_i's denote the singular lines of the base locus.*

Proof. As in the proof of (6.1) in [S], if p is singular on a surface in the pencil and $p \in q$, then p is a singular point of $q \cap S_{IO}$. Hence $p \in L_i$. □

By this fact and Bertini's theorem follows that:

Lemma 4.5. *1. The general surface in the pencil is smooth away from the lines $L_1, \ldots, L_6$. Moreover we have:*
2. The singular surfaces have only isolated singularities away from the singular lines $L_1, \ldots, L_6$. In particular they are irreducible and reduced.

Proof of 2. The surface q^6 is the only not reduced surface in the pencil. Indeed, assume that there is another not reduced surface $\{f_\alpha^m = 0\}$, $\alpha \cdot m = 12$, and deg$f_\alpha = \alpha$. Then the twelve lines of the base locus in the intersection of f_α^m and q would be singular too, which is not the case. Assume now that there is a surface S in the pencil which contains a singular curve C. The latter meets the quadric in at least one point, which by lemma 4.4 is on some L_i. This point is a "pinch point" of S (for the definition of "pinch point" cf. [SR] p. 423). We compute explicitly the pinch points along a line L_i. Because of symmetry reason it is enough to do it just for one line. We take L_1:$\{x_2 = ix_0, x_3 = ix_1\}$. First we consider the transformation $x_0 \mapsto y_0$, $x_1 \mapsto y_1$, $x_2 \mapsto y_2 + i \cdot y_0$, $x_3 \mapsto y_3 + i \cdot y_1$, which maps L_1 to $\{y_2 = y_3 = 0\}$. We call the transform L_1 again.

Let $F := F(y_0, y_1, y_2, y_3, \lambda)$ be the equation of the pencil after the transformation. We can write the Taylor expansion of F along L_1

$$F = A(y_0, y_1, \lambda) \cdot y_2^2 + B(y_0, y_1, \lambda) \cdot y_2 \cdot y_3 + C(y_0, y_1, \lambda) \cdot y_3^2$$
$$+\text{terms of order} \geq 3,$$

the pinch points are solution of

$$\det \begin{pmatrix} A & \frac{1}{2}B \\ \frac{1}{2}B & C \end{pmatrix} = 0$$

which splits into the product

$$\begin{array}{l}(x_1^4 + 2x_1^3x_0 + 2x_0^2x_1^2 - 2x_0^3x_1 + x_0^4)(x_1^4 - 2x_1^3x_0 + 2x_0^2x_1^2 + 2x_0^3x_1 + x_0^4) \\ (x_0^2 + 3x_0x_1 + \sqrt{5}x_0x_1 - x_1^2)(x_0^2 - 3x_0x_1 - \sqrt{5}x_0x_1 - x_1^2) \\ (3x_0^4 - 2\sqrt{5}x_0^2x_1^2 + 3x_1^4)(x_0^4 + 16x_0^2x_1^2 - 6\sqrt{5}x_0^2x_1^2 + x_1^4) = 0.\end{array}$$

this has twenty distinct solutions (independent from λ), therefore we have “simple” pinch points on L_i, hence no singular curves. □

In this case as in the case of the pencils of [S] the singular points are contained on lines of fix points (cf. (6.3) of [S]) and we get an estimate for the number of isolated singularities as follows: Denote by S a surface of the IO-invariant pencil then $\deg(S \cap \partial_i S) = 12 \cdot 11$ and $S \cdot \partial_i S = C + 2L_1 + \ldots + 2L_6$ where $\deg C = 12 \cdot 11 - 12$. Since C cannot be singular, there is a $j = 0, 1, 2, 3$ s. t. $C \not\subseteq \partial_j S$. Hence $\deg(C \cap \partial_j S) = (12 \cdot 11 - 12)(12 - 1) = 120 \cdot 11$. Since the singular points are computed two times in the intersection their number is $\leq \frac{120 \cdot 11}{2} = 660$. Denote by L a line of fix points and assume that it meets the base locus in two points z_1, z_2. Then for each surface $S \neq q^6$ $\text{mult}_{z_j}(L \cdot S) = 2$. In fact a line like L there can be only a fix line of a matrix in the conjugacy class of (A_2, A_2) and it meets the base locus at some line L_i. Moreover an argumentation as in (7.1) of [S] shows that L cannot be tangent to S at z_i. By this fact and by a computation as in section 8 of [S], we find that a fix line contains ≤ 8 singular points if it meets the base locus, and ≤ 12 singular points otherwise. Finally, before giving an exact description of the singular points on the fix lines (cf. table below) we remark that the fix lines of elements in the same conjugacy class form one orbit under the action of IO and the points where they meet are real (cf. (7.3) and (7.5) of [S]).
The singular surfaces. We proceed by a direct computation using the lines of fix points. The matrices of IO with fix lines containing singular points are in the conjugacy class of (A_2, A_2), (S, S) or (A_2, R_4A_2). The total number of distinct fix lines is 90, 80, resp. 180. In the table below we give the singular surfaces and the singular points on the fix lines. We choose a representative in each conjugacy class and a fix line of it, moreover we put $a' := \sqrt{2} - 1$, $a := \sqrt{2} + 1$,$\gamma_1 = 2\sqrt{2} - \sqrt{5}$, $\gamma_1' = 2\sqrt{2} + \sqrt{5}$, $\gamma_2 := 2 + \sqrt{2} + \sqrt{5} + \sqrt{10}$, $\gamma_2' := -2 + \sqrt{2} - \sqrt{5} + \sqrt{10}$, $\gamma_3 := -2 + \sqrt{2} + \sqrt{5} - \sqrt{10}$, $\gamma_3' := 2 + \sqrt{2} - \sqrt{5} - \sqrt{10}$, $\alpha_1 := \frac{1}{2}(-1 + \sqrt{2} - \sqrt{10})$, $\alpha_1' := \frac{1}{2}(1 + \sqrt{2} - \sqrt{10})$, $\beta_1 := \frac{1}{2}(4 - 3\sqrt{2} - 2\sqrt{5} + \sqrt{10})$, $\alpha_2 := \frac{1}{2}(1 + \sqrt{2} + \sqrt{10})$, $\alpha_2' := \frac{1}{2}(-1 + \sqrt{2} + \sqrt{10})$, $\alpha_3 := -2 + \frac{3}{2}\sqrt{2} - \sqrt{5} + \frac{1}{2}\sqrt{10}$,

$\alpha_3' := 2+\frac{3}{2}\sqrt{2}+\sqrt{5}+\frac{1}{2}\sqrt{10}$, $\beta_2 := \frac{1}{2}(-10+7\sqrt{2}-4\sqrt{5}+3\sqrt{10})$, $\beta_3 := \frac{1}{2}(\sqrt{2}-2\sqrt{5}+\sqrt{10})$, $\beta_4 := \frac{1}{2}(2+\sqrt{2}+\sqrt{10})$, $c_1 := -\frac{74}{972}+\frac{4}{243}\sqrt{10}$, $c_2 := -\frac{74}{972}-\frac{4}{243}\sqrt{10}$:

matrix and fix line	value of λ		int. points	
(A_2, A_2) :	0	$-\frac{1}{8}$	$(1:0:0:0)$	$(\pm a':0:1:0)$
$x_1 = x_3 = 0$			$(0:0:1:0)$	$(1:0:\pm a':0)$
			$(\pm 1:0:1:0)$	
(A_2, R_4A_2) :	$-\frac{1}{8}$		$(-a:1:0:0)$	
$x_0 = -ax_1, x_3 = a'x_2$			$(0:0:1:a')$	
			$(1:\pm a':1:a')$	
	c_1	c_2	$(\alpha_1:-\beta_1:1:a')$	$(\alpha_1':-\beta_3:1:a')$
			$(-\alpha_2:\beta_1:1:a')$	$(-\alpha_2':\beta_3:1:a')$
			$(\alpha_3:\beta_2:1:a')$	$(\alpha_3':\beta_4:1:a')$
			$(-\alpha_3:-\beta_2:1:a')$	$(-\alpha_3':-\beta_4:1:a')$
(S, S) :	0		$(1:0:0:0)$	
$x_1 = x_2 = x_3$			$(0:1:1:1)$	
			$(-1:1:1:1)$	
			$(-\sqrt{5}:1:1:1)$	
			$(\sqrt{5}\pm 2:1:1:1)$	
	c_1	c_2	$(\gamma_1:1:1:1)$	$(\gamma_1':1:1:1)$
			$(\gamma_2:1:1:1)$	$(\gamma_2':1:1:1)$
			$(\gamma_3:1:1:1)$	$(\gamma_3':1:1:1)$

Here the fix line of (A_2, A_2) contains two points of the base locus. Observe that the number of singular points is maximal on the lines, hence the four given λ's are the only values corresponding to singular surfaces.

Proposition 4.2. *In the pencil $S_{IO} + \lambda q^6$ we have the following IO-orbits of nodes*

λ	0	c_1	$-\frac{1}{8}$	c_2
orbit	120	240	360	240
fix group mod. ± 1	A_4	D_3	$\mathbb{Z}_2 \times \mathbb{Z}_2$	D_3
order	12	6	4	6

Proof. We analyze case by case the four different surface. Then a direct computation as in section 4.2, shows that the singular points are nodes.
I. $\underline{\lambda = 0}$. Observe that $(1:0:0:0) \in S_{IO}$ is contained in three fix lines of elements in the conjugacy class of (A_2, A_2). Moreover

(*) there is a matrix of order four $(1,\gamma) \in IO$, with $\gamma^2 = A_2$,

Hence $(1,\gamma)$ and (A_2, A_2) commute and the four points on the fix line of (A_2, A_2) form one orbit under the action of $(1,\gamma)$. By this fact follows that we have a configuration of lines and points. By the formula (1), we get $N_0 \cdot 3 = 90 \cdot 4$, hence $N_0 = 120$. The point $(1:0:0:0)$ is on the fix line of (S, S) too, more precisely it is contained on four such lines. The formula $120 \cdot 4 = 80 \cdot n_0$ shows $n_0 = 6$ so the six points on the fix line are in the orbit of length 120 too. In conclusion we have just one IO-orbit of singular double points.
II. $\underline{\lambda = -\frac{1}{16}}$. The four points on the fix line of (A_2, A_2) form one orbit by (*). The formula

(1) gives $N_0 \cdot n_1 = 360$. If $n_1 \geq 2$ then $N_0 \leq 180$ and the fix group in IO mod. ± 1 has order ≥ 8. Checking in the table on page 24 this is not possible. Hence $N_0 = 360$ and $n_1 = 1$. The fix group has order four and it is isomorphic with $\mathbb{Z}_2 \times \mathbb{Z}_2$. Each singular point is contained in two fix lines of matrices in the conjugacy class of (A_2, R_4A_2). Now the formula (1) with $N_0 = 360$, $n_1 = 2$ and $N_1 = 180$ gives $n_0 = 4$. So we conclude that the singular points on these fix lines are in the orbit of length 360 too.
III. $\underline{\lambda = c_i,\ i = 1, 2}$. The three points on the fix line of (S, S) form one orbit under the action of $(S, 1)$. The formula (1) gives $N_0 \cdot n_1 = 240$. If $n_1 \geq 2$ then $N_0 \leq 120$ and the fix group mod. ± 1 has order ≥ 12, which is not possible (check in the table on page 24 again). Hence $N_0 = 240$ and $n_1 = 1$. The fix group has order six and it is isomorphic with D_3, hence three fix lines of matrices in (A_2, R_4A_2) contain a singular points. The formula (1) in this case gives $n_0 = 4$, which shows that the singular points on these fix lines are in the previous orbit too. □

We give now the table of the configurations of lines and points:

Repr. of the conj. class	value of λ		Configuration	
(A_2, A_2)	0	$-\frac{1}{8}$	$(120_3, 90_4)$	$(360_1, 90_4)$
(A_2, R_4A_2)	$-\frac{1}{8}$	c_i	$(360_2, 180_4)$	$(240_3, 180_4)$
(S, S)	0	c_i	$(120_4, 80_6)$	$(240_1, 80_3)$

4.3. **The pencil of D_nD_m-invariant quartics.** We take the generators q^2 and $f_2 + f_3 = (x_0 - ix_1)(x_0 + ix_1)(x_2 - ix_3)(x_2 + ix_3)$. The latter is the union of four complex planes meeting each other at the four complex lines $\{x_0 \pm ix_1 = 0,\ x_2 \pm ix_3 = 0\}$ on q. The pencil contains the multiple quadric $q'^2 := q^2 - 2(f_2 + f_3) = (x_0^2 + x_1^2 - x_2^2 - x_3^2)^2$ too. Hence with this new generator the equation of the pencil becomes

$$q'^2 + \lambda q^2 = (q' + i\sqrt{\lambda}q)(q' - i\sqrt{\lambda}q).$$

A surface in the pencil is the union of two quadrics which meet each other at the four lines given above. An easy computation shows that none of the surface in the pencil contains isolated singularities.

4.4. **The pencil of OD_n-invariant octics ($n \geq 6$) and of ID_n-invariant 12-ics ($n \geq 8$).** We take generators q^4, q^6 and $P_8 = \prod_{i=0}^{3} K_i$, $P_{12} = \prod_{i=0}^{5} K_i'$ (cf. **(3.1.2)**) and we denote the pencils by $\Pi_m(\lambda) := P_m + \lambda q^{\frac{m}{2}}$, $m = 8, 12$. We have

Lemma 4.6. *The base locus of the pencil is $\frac{m}{2}$-times the intersection $q \cap P_m$, $m = 8, 12$. Where (as sets)*

$$\begin{aligned} q \cap P_8 &= \text{fix lines of } (1, R_n), \\ &\quad\ \text{orbit of eight lines under } (O, 1), \\ q \cap P_{12} &= \text{fix lines of } (1, R_n), \\ &\quad\ \text{orbit of twelve lines under } (I, 1). \end{aligned}$$

Proof. The element $(1, R_n)$ has fix lines $L_1 : \{x_0 = -ix_1,\ x_2 = ix_3\}$, $L_2 : \{x_0 = ix_1,\ x_2 = -ix_3\}$ for each n. These are contained in each quadric K_s, $s = 0, \dots, 3$, resp. K'_t, $t = 0, \dots, 5$, hence they have at least multiplicity four, resp. six in the intersection $q \cap P_m$, $m = 8, 12$. By this fact and an argumentation as in [S], (5.5), it follows that the lines of the length $m = 8, 12$-orbit under $(O, 1)$, resp. $(I, 1)$ are in the base locus too. Moreover the previous multiplicity of intersection are exactly four, resp. six. □

Lemma 4.7. *The pencil does not contain surfaces with isolated singular points.*

Proof. The groups have order $96n$, resp. $240n$, since an isolated singular point has orbit of finite length, it is fixed by $(1, R_n)$. This shows that it is on the lines of the base locus, hence not isolated. A contradiction to the assumption. □

Proposition 4.3. *In the pencil we have the following singular surfaces, with one orbit of double lines which are fix lines for the elements of some conjugacy class:*

value of λ	OD_n :	-1	$\frac{1}{3}$	0
	ID_n :	0		
orbit	OD_n :	6	8	12
	ID_n :	30		
fix lines in the conj. class of		$(A_3, R_n^{\frac{n}{2}})$	(S, R_6^2)	$(R, R_n^{\frac{n}{2}})$

Proof. It is a direct computation using the equations $x_0 = x_2, x_1 = -x_3$; $x_0 = -(1+\sqrt{3})x_3 + x_1, x_2 = x_3 + (1-\sqrt{3})x_1$ and $x_2 = (1-\sqrt{2})x_0, x_3 = (\sqrt{2}-1)x_1$ of a fix line of $(A_3, R_n^{\frac{n}{2}})$, (S, R_6^2), resp. $(R, R_n^{\frac{n}{2}})$. □

5. Final Remarks

1) By [Co2] p. 292 there are sixteen regular polytopes $\{p, q, r\}$ in four dimensions. These polytopes correspond to four distinct symmetry groups $[p, q, r]$ listed in the table below. Three of these symmetry groups can be obtained from groups which we described above by adding extra generators C and C', where C denotes the matrix given in section 3, C' is the matrix of [S] p. 433.

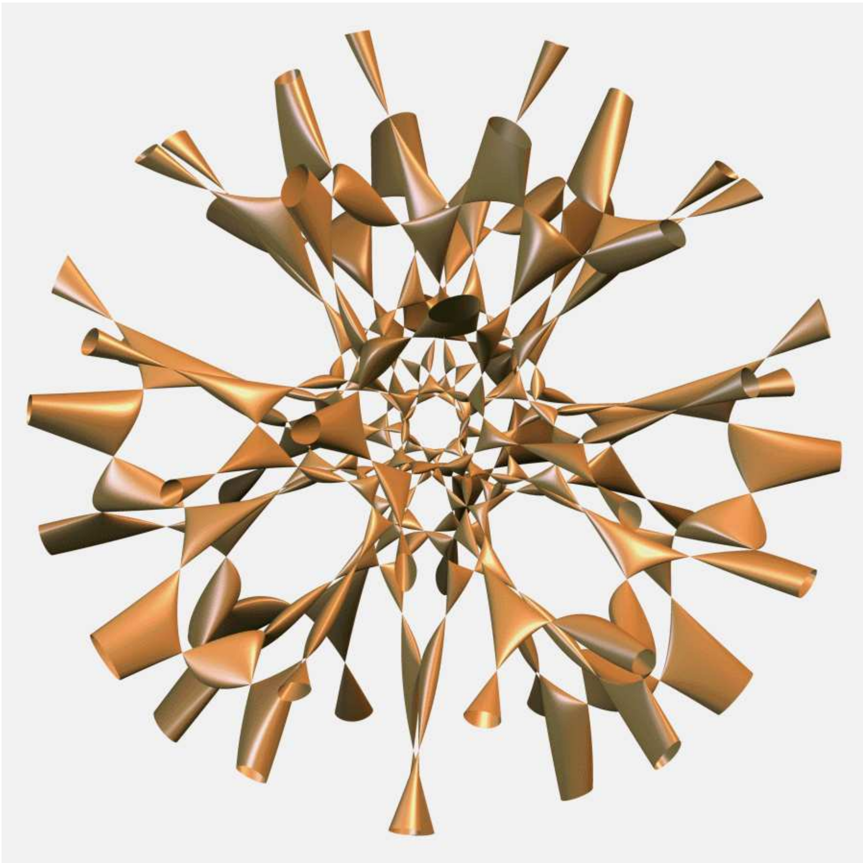
symmetry groups	$[3,3,3]$	$[3,3,4]$	$[3,4,3]$	$[3,3,5]$
our groups		$(OO)'$	TT	II
extra generators		C	C, C'	C

The group $[3,3,3] \cong Sym(5)$ does not contain the Heisenberg group H, in fact it has some invariant polynomials of odd degree (cf. [Co1] p. 780). In these notes we complete the description of the G-invariant pencils of surfaces whenever G is the symmetry group of a regular four dimensional polytope and G contains H.
2) In remark 3.3, with $n = 3$, we give the degree four invariant polynomials of the *bi-polyhedral dihedral group* $D_3D_3 \subseteq SO(4)$. By Mukai [M] the quotient $\mathbb{P}_3/D_3D_3$ is isomorphic with the Satake compactification of the moduli space of abelian surfaces with $(1, 2)$-polarization, hence this invariant polynomials should be related to modular forms.

3) In [BS] the quotients X/G, $G = TT, OO, II$ are described. It would be interesting to examine the quotients in the remaining cases.

6. Computer Picture

We exhibit a computer picture of the $I \times O$-symmetric surface of degree 12 with 360 nodes. This has been realized with the program SURF written by S. Endraß.



$I \times O$-symmetric surface of degree 12 with 360 nodes

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A geometrical construction for the polynomial invariants of some reflection groups

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A GEOMETRICAL CONSTRUCTION FOR THE POLYNOMIAL INVARIANTS OF SOME REFLECTION GROUPS

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Abstract. We construct invariant polynomials for the reflection groups $[3, 4, 3]$ and $[3, 3, 5]$ by using some special sets of lines on the quadric $\mathbb{P}_1 \times \mathbb{P}_1$ in $\mathbb{P}_3$. Then we give a simple proof of the well known fact that the ring of invariants are rationally generated in degree 2,6,8,12 and 2,12,20,30.

Key Words: Polynomial invariants, Reflection and Coxeter groups, Group actions on varieties.

Mathematics Subject Classification: Primary 20F55, 13F20; Secondary 14L30.

0. Introduction

There are four groups generated by reflections which operate on the four-dimensional Euclidean space. These are the symmetry groups of some regular four dimensional polytopes and are described in [C2, p. 145 and Table I p. 292-295]. With the notation there the polytopes, the groups and their orders are

Polytope	$5-\text{cell}$	$16-\text{cell}$	$24-\text{cell}$	$600-\text{cell}$
Group	$[3, 3, 3]$	$[3, 3, 4]$	$[3, 4, 3]$	$[3, 3, 5]$
Order	120	384	1152	14400

They operate in a natural way on the ring of polynomials $R = \mathbb{R}[x_0, x_1, x_2, x_3]$ and it is well known that the ring of invariants R^G (G one of the groups above) is algebraically generated by a set of four independent polynomials (cf. [B, p. 357]). Coxeter shows in [C1] that the rings R^G, $G = [3, 3, 3]$ or $[3, 3, 4]$ are generated in degree $2, 3, 4, 5$ resp. $2, 4, 6, 8$ and since the product of the degrees is equal to the order of the group, any other invariant polynomial is a combination with real coefficients of products of these invariants (i.e., in the terminology of [C1], the ring R^G is *rationally generated* by the polynomials). Coxeter also gives equations for the generators. In the case of the groups $[3, 4, 3]$ and $[3, 3, 5]$ he recalls a result of Racah,(cf. [R]), who shows with the help of the theory of Lie groups that the rings R^G are rationally generated in degree $2, 6, 8, 12$ resp. $2, 12, 20, 30$.
Equations for these generating polynomials can be found e.g. in [M], [Sm, p. 218], [CS, p. 203] and most recently in [IKM] (the groups are often denoted in the literature by F_4 and H_4). The method used by Metha in [M] is simple: He considers the equations of the reflecting hyperplanes and he finds a set of linear forms which are invariant under the action of the groups $[3, 4, 3]$ resp. $[3, 3, 5]$, then he uses these to give equations for the polynomial invariants (a similar method is used by Coxeter in the case of the groups $[3, 3, 3]$ and $[3, 3, 4]$). In [Sm, p. 218] Smith explains how to obtain equations for the invariant polynomials of the rings R^G, but he refers to [CS] for the explicit equations, however only in the case of the group $[3, 4, 3]$. In fact Conway and Sloane use coding theory to

construct the invariants of this group, but they do not consider the case of $[3,3,5]$. In [IKM] the authors find the invariants by solving a special system of partial differential equations. But as they say the method is quite elaborated and they need the support of computer-algebra.

In this paper we give a different construction, which should be interesting in particular from the point of view of algebraic geometry: We consider some special $[3,4,3]$-, resp. $[3,3,5]$-orbit of lines on the quadric $\mathbb{P}_1 \times \mathbb{P}_1$ in $\mathbb{P}_3$ and construct the invariant polynomials by using the action of the group and geometric considerations. We remark that in our construction of the polynomials we use very little computer-algebra, in fact only MAPLE for some computation in Proposition 2.1 and 3.2 (cf. Section 4). Otherwise everything is proved by hand and by geometric considerations. This construction seems to be interesting for the following reasons:

1. We can give a simple proof of Racah's result,
2. We establish relations between the invariants of the groups $[3,4,3]$ and $[3,3,5]$ and the invariants of some binary subgroups of $SU(2)$,
3. The construction may be helpful in the study of the geometry of the algebraic surfaces defined by the zero sets of the invariant polynomials. We have in fact families of surfaces with many symmetries and by the construction, for example it is possible to determine immediately the base locus of the families, which consists of sets of lines on $\mathbb{P}_1 \times \mathbb{P}_1$.

We explain now briefly our method and also the structure of the paper: Denote by T, O and I the rotations subgroups in $SO(3,\mathbb{R})$ of the platonic solids: tetrahedron, octahedron and icosahedron, it is well known that $SO(4,\mathbb{R})$ contains central extensions G_6 of $T \times T$, G_8 of $O \times O$ and G_{12} of $I \times I$ by ± 1. Then G_6 is an index four subgroup of $[3,4,3]$ and G_{12} is an index two subgroup of $[3,3,5]$ (cf. e.g. [Sa], Section 3). These two groups, and G_8 too, act on the three dimensional projective space $\mathbb{P}_3$, and in particular on the two ruling of the quadric $\mathbb{P}_1 \times \mathbb{P}_1$ (this action is studied in [Sa]). The quadric can be described as the zero set of the quadratic form:

$$x_0^2 + x_1^2 + x_2^2 + x_3^2$$

which is $[3,4,3]$- and $[3,3,5]$-invariant. By considering some special orbits of lines of $\mathbb{P}_1 \times \mathbb{P}_1$ under G_6, G_{12} and G_8, it is possible to construct explicitly $[3,4,3]$- and $[3,3,5]$-invariant polynomials, this is done and explained in details in Section 2. In Section 3 we show that our polynomials generate the rings of invariants R^G by showing some relations between them and the invariant forms of the binary tetrahedral group and of the binary icosahedral group in $SU(2)$. More precisely we define a surjective map between polynomials of degree d on $\mathbb{P}_3$ and polynomials of be-degree (d,d) on $\mathbb{P}_1 \times \mathbb{P}_1$. Then we show that the image of a G_n-invariant polynomial $n = 6, 12$ splits into the product of two invariant polynomials of the same degree under the action of the binary subgroup in $SU(2)$ corresponding to G_n (there are classical $2:1$ maps $SU(2) \longrightarrow SO(3)$, $SU(2) \times SU(2) \longrightarrow SO(4)$ which we recall in Section 1). This corresponds in some sense to the fact that G_n contains the product $G \times G$ (for $n = 6$ is $G = T$ and for $n = 12$ is $G = I$) and each copy $G \times 1$, $1 \times G$ operates on one ruling of $\mathbb{P}_1 \times \mathbb{P}_1$ and leaves the other ruling invariant. This relation is the main ingredient in our proof of the result of Racah (Corollary 2.1). It seems to be however interesting by itself. Finally Section 4 contains explicit computations and in Section 5 we present open problems and possible applications of the results of the paper. *It is a pleasure to thank W. Barth of the University of Erlangen for many helpful discussions and the referees for pointing me out some important bibliographical information.*

1. Notations and preliminaries

Denote by R the ring of polynomials in four variables with real coefficients $\mathbb{R}[x_0, x_1, x_2, x_3]$, by G a finite group of homogeneous linear substitutions, and by R^G the ring of invariant polynomials.
1. A set of polynomials $F_1, \ldots, F_n$ in R is called *algebraically dependent* if there is a non trivial relation

$$\sum \alpha_I (F_1^{i_1} \cdot \ldots \cdot F_n^{i_n}) = 0,$$

where $I = (i_1, \ldots, i_n) \in \mathbb{N}^n$, $\alpha_I \in \mathbb{R}$.
2. The polynomials are called *algebraically independent* if they are not dependent. For the ring R^G, there always exists a set of four algebraically independent polynomials (cf. [B], thm. I, p. 357).
3. We say that R^G is *algebraically generated* by a set of polynomials $F_1, \ldots, F_4$, if for any other polynomial $P \in R^G$ we have an algebraic relation

$$\sum \alpha_I (P^{i_0} \cdot F_1^{i_1} \cdot \ldots \cdot F_4^{i_4}) = 0.$$

4. We say that the ring R^G is *rationally generated* by a set of polynomials $F_1, \ldots, F_4$, if for any other polynomial $P \in R^G$ we have a relation

$$\sum \alpha_I (F_1^{i_1} \cdot \ldots \cdot F_4^{i_4}) = P, \ \alpha_I \in \mathbb{R}$$

5. The four polynomials of 3 are called a *basic set* if they have the smallest possible degree (cf. [C1]).
6. There are two classical $2:1$ coverings

$$\rho : SU(2) \to SO(3) \text{ and } \sigma : SU(2) \times SU(2) \to SO(4),$$

we denote by T, O, I the tetrahedral group, the octahedral group and the icosahedral group in $SO(3)$ and by $\tilde{T}$, $\tilde{O}$, $\tilde{I}$ the corresponding binary groups in $SU(2)$ via the map ρ. The σ-images of $\tilde{T} \times \tilde{T}$, $\tilde{O} \times \tilde{O}$ and $\tilde{I} \times \tilde{I}$ in $SO(4)$ are denoted by G_6, G_8 and G_{12}. By abuse of notation we write (p, q) for the image in $SO(4)$ of an element $(p, q) \in SU(2) \times SU(2)$. As showed in [Sa] (3.1) p. 436, the groups G_6 and G_{12} are subgroups of index four respectively two in the reflections groups $[3, 4, 3]$ and $[3, 3, 5]$.

2. Geometrical construction

Denote by $\tilde{G}$ one of the groups $\tilde{T}$, $\tilde{O}$ or $\tilde{I}$. Clearly, the subgroups $\tilde{G} \times 1$ and $1 \times \tilde{G}$ of $SO(4)$ are isomorphic to $\tilde{G}$. Moreover, each of them operates on one of the two rulings of the quadric $\mathbb{P}_1 \times \mathbb{P}_1$ and leaves invariant the other ruling (as shown in [Sa]). We recall the lengths of the orbits of points under the action of the groups T, O and I

group	T	O	I
lengths of the orbits	12, 6, 4	24, 12, 8, 6	60, 30, 20, 12

These lines are fixed by elements $(p, 1) \in \tilde{G} \times 1$ on one ruling, resp. $(1, p') \in 1 \times \tilde{G}$ on the other ruling of the quadric. Recall that these elements have two lines of fix points with eigenvalues α, $\bar{\alpha}$ which are in fact the eigenvalues of p and p'. We call two lines L, L' of $\mathbb{P}_1 \times \mathbb{P}_1$ a *couple* if L is fixed by $(p, 1)$ with eigenvalue α and L' is fixed by $(1, p)$ with the same eigenvalue.

2.1. **The invariant polynomials of G_6 and of G_{12}.** Consider the six couples of lines $L_1, L_1', \ldots, L_6, L_6'$ in $\mathbb{P}_1 \times \mathbb{P}_1$ which form one orbit under the action of $\tilde{T} \times 1$, resp. $1 \times \tilde{T}$, and denote by $f_{11}^{(6)}, \ldots, f_{66}^{(6)}$ the six planes generated by such a couple of lines (and by abuse of notation their equation, too). Now set

$$F_6 = \sum_{g \in \tilde{T} \times 1} g(f_{11}^{(6)} \cdot f_{22}^{(6)} \cdot \ldots \cdot f_{66}^{(6)}) = \sum_{g \in \tilde{T} \times 1} g(f_{11}^{(6)}) \cdot g(f_{22}^{(6)}) \cdot \ldots \cdot g(f_{66}^{(6)}).$$

Observe that an element $g \in \tilde{T} \times 1$ leaves each line of one ruling invariant and operates on the six lines of the other ruling. A similar action is given by an element of $1 \times \tilde{T}$. Since we sum over all the elements of $\tilde{T} \times 1$, the action of $1 \times \tilde{T}$ does not give anything new, hence F_6 is G_6-invariant. Furthermore observe that F_6 has real coefficients. In fact, in the above product, for each plane generated by the lines L_i, L_i' we also take the plane generated by the lines which consist of the conjugate points. The latter has equation $\bar{f_{ii}}^{(6)}$, i.e., we have an index $j \neq i$ with $f_{jj}^{(6)} = \bar{f_{ii}}^{(6)}$ and the products $f_{ii}^{(6)} \cdot \bar{f_{ii}}^{(6)}$ have real coefficients.
Consider now the orbits of lengths eight and twelve under the action of $\tilde{O} \times 1$ and $1 \times \tilde{O}$ and the planes $f_{ii}^{(8)}$, $f_{jj}^{(12)}$ generated by the eight, respectively by the twelve couples of lines. As before the polynomials

$$\begin{aligned} F_8 &= \sum_{g \in \tilde{T} \times 1} g(f_{11}^{(8)} \cdot \ldots \cdot f_{88}^{(8)}), \\ F_{12} &= \sum_{g \in \tilde{T} \times 1} g(f_{11}^{(12)} \cdot \ldots \cdot f_{1212}^{(12)}) \end{aligned}$$

are G_6-invariant and have real coefficients.
Finally we consider the lines of $\mathbb{P}_1 \times \mathbb{P}_1$ which form orbits of length $12, 20$ and 30 under the action of $\tilde{I} \times 1$ resp. $1 \times \tilde{I}$. The planes generated by the couples of lines produce the G_{12}-invariant real polynomials

$$\begin{aligned} \Gamma_{12} &= \sum_{g \in \tilde{I} \times 1} g(h_{11}^{(12)} \cdot \ldots \cdot h_{1212}^{(12)}), \\ \Gamma_{20} &= \sum_{g \in \tilde{I} \times 1} g(h_{11}^{(20)} \cdot \ldots \cdot h_{2020}^{(20)}), \\ \Gamma_{30} &= \sum_{g \in \tilde{I} \times 1} g(h_{11}^{(30)} \cdot \ldots \cdot h_{3030}^{(30)}). \end{aligned}$$

2.2. **The invariant polynomials of the reflection groups.** We consider the matrices

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad C' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

as in [Sa] (3.1) p. 436, the groups generated by G_6, C, C' and G_{12}, C are the reflections groups $[3, 4, 3]$ respectively $[3, 3, 5]$.

Proposition 2.1. *1. The polynomials F_6, F_8, F_{12}, Γ_{12}, Γ_{20}, Γ_{30} are C invariant.*
2. The polynomials F_6, F_8, F_{12} are C' invariant.

Proof. 1. The matrix C interchanges the two rulings of the quadric, hence the polynomials F_i and Γ_j are invariant by construction. We prove 2 by a direct computation in the last Section. □

From this fact we obtain

Corollary 2.1. *The polynomials q, F_6, F_8, F_{12} are $[3,4,3]$-invariant and the polynomials $q, \Gamma_{12}, \Gamma_{20}, \Gamma_{30}$ are $[3,3,5]$-invariant.*

Here we denote by q the quadric $\mathbb{P}_1 \times \mathbb{P}_1$.

3. The rings of invariant forms

Identify $\mathbb{P}_3$ with $\mathbb{P}M(2\times 2, \mathbb{C})$ by the map

$$(x_0 : x_1 : x_2 : x_3) \quad \mapsto \quad \begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix}. \tag{1}$$

Furthermore consider the map

$$\begin{array}{ccc} \mathbb{C}^2 \times \mathbb{C}^2 & \longrightarrow & M(2\times 2, \mathbb{C}) \\ ((z_0, z_1), (z_2, z_3)) & \longmapsto & \begin{pmatrix} z_0 z_2 & z_0 z_3 \\ z_1 z_2 & z_1 z_3 \end{pmatrix} = \mathcal{Z}. \end{array} \tag{2}$$

Then $\mathcal{Z}$ is a matrix of determinant $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0$ which is the equation of q. Now denote by $\mathcal{O}_{\mathbb{P}_3}(n)$ the sheaf of regular functions of degree n on $\mathbb{P}_3$ and by $\mathcal{O}_q(n,n)$ the sheaf of regular function of be-degree (n,n) on the quadric q. We obtain a surjective map between the global sections

$$\phi : \quad H^0(\mathcal{O}_{\mathbb{P}_3}(n)) \quad \longrightarrow \quad H^0(\mathcal{O}_q(n,n)) \tag{3}$$

by doing the substitution

$$x_0 = \frac{z_0 z_2 + z_1 z_3}{2}, \quad x_1 = \frac{z_0 z_2 - z_1 z_3}{2i},$$

$$x_2 = \frac{z_0 z_3 - z_1 z_2}{2}, \quad x_3 = \frac{z_0 z_3 + z_1 z_2}{2i}$$

in a polynomial $p(x_0, x_1, x_2, x_3) \in H^0(\mathcal{O}_{\mathbb{P}_3}(n))$. Observe that $\phi(q) = 0$. Now let

$$\begin{array}{rcl} t & = & z_0 z_1 (z_0^4 - z_1^4), \\ W & = & z_0^8 + 14 z_0^4 z_1^4 + z_1^8, \\ \chi & = & z_0^{12} - 33(z_0^8 z_1^4 + z_0^4 z_1^8) + z_1^{12} \end{array}$$

denote the $\tilde{T}$-invariant polynomials of degree $6, 8$ and 12 and let

$$\begin{array}{rcl} f & = & z_0 z_1 (z_0^{10} + 11 z_0^5 z_1^5 - z_1^{10}), \\ H & = & -(z_0^{20} + z_1^{20}) + 228(z_0^{15} z_1^5 - z_0^5 z_1^{15}) - 494 z_0^{10} z_1^{10}, \\ \mathcal{T} & = & (z_0^{30} + z_1^{30}) + 522(z_0^{25} z_1^5 - z_0^5 z_1^{25}) - 10005(z_0^{20} z_1^{10} + z_0^{10} z_1^{20}) \end{array}$$

be the $\tilde{I}$-invariant polynomials of degree $12, 20, 30$ given by Klein in [K] p. 51-58. Put $t_1 = t(z_0, z_1)$, $t_2 = t(z_2, z_3)$, $W_1 = W(z_0, z_1)$, $W_2 = W(z_2, z_3)$ and analogously for the other invariants.

Proposition 3.1. *If $p \in H^0(\mathcal{O}_{\mathbb{P}^3}(n))$ is G_6-invariant, then:*

$$\phi(p) = \sum_I \alpha_I t_1^{\alpha_1} t_2^{\alpha_1'} W_1^{\alpha_2} W_2^{\alpha_2'} \chi_1^{\alpha_3} \chi_2^{\alpha_3'}$$

If p is G_{12}-invariant, then:

$$\phi(p) = \sum_J \beta_J f_1^{\beta_1} f_2^{\beta_1'} H_1^{\beta_2} H_2^{\beta_2'} T_1^{\beta_3} T_2^{\beta_3'}$$

where

$$I = \{(\alpha_1, \alpha_1', \alpha_2, \alpha_2', \alpha_3, \alpha_3') | \alpha_i, \alpha_i' \in \mathbb{N}, 6\alpha_1 + 8\alpha_2 + 12\alpha_3 = n,\ 6\alpha_1' + 8\alpha_2' + 12\alpha_3' = n\},$$
$$J = \{(\beta_1, \beta_1', \beta_2, \beta_2', \beta_3, \beta_3') | \beta_i, \beta_i' \in \mathbb{N}, 12\beta_1 + 20\beta_2 + 30\beta_3 = n,\ 12\beta_1' + 20\beta_2' + 30\beta_3' = n\}.$$

Proof. Put

$$\phi(p) = p'(z_0, z_1, z_2, z_3).$$

An element $g = (g_1, g_2)$ in G_6 or G_{12} operates on $(x_0 : x_1 : x_2 : x_3) \in \mathbb{P}_3$ by the matrix multiplication

$$g_1 \begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix} g_2^{-1}$$

and on the matrix $\mathcal{Z}$ of (2) by

$$g_1 \begin{pmatrix} z_0 z_2 & z_0 z_3 \\ z_1 z_2 & z_1 z_3 \end{pmatrix} g_2^{-1} \quad = \quad g_1 \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \cdot \begin{pmatrix} z_2 & z_3 \end{pmatrix} g_2^{-1}.$$

Clearly if p is G_6- or G_{12}-invariant then also the projection $\phi(p)$ with the previous operation is. In particular for $g = (g_1, 1)$ in $\tilde{T} \times 1$, resp. in $\tilde{I} \times 1$ the polynomial p' is $\tilde{T} \times 1$-, respectively $\tilde{I} \times 1$-invariant as polynomial in the coordinates $(z_0 : z_1) \in \mathbb{P}_1$ and for any $(z_2 : z_3) \in \mathbb{P}_1$. On the other hand for $g = (1, g_2)$ in $1 \times \tilde{T}$, resp. in $1 \times \tilde{I}$ the polynomial p' is $1 \times \tilde{T}$-, respectively $1 \times \tilde{I}$-invariant as polynomial in the coordinate $(z_2 : z_3) \in \mathbb{P}_1$ and for any $(z_0 : z_1) \in \mathbb{P}_1$. Hence p' must be in the form of the statement. □

By a direct computation in Section 4 we prove the following

Proposition 3.2. *The quadric q does not divide the polynomials F_i, Γ_j. Moreover, F_6 does not divide F_{12}.*

Corollary 3.1. *We have $\phi(q) = 0$, $\phi(F_6) = t_1 \cdot t_2$, $\phi(F_8) = W_1 \cdot W_2$, $\phi(F_{12}) = \chi_1 \cdot \chi_2$, $\phi(\Gamma_{12}) = f_1 \cdot f_2$, $\phi(\Gamma_{20}) = H_1 \cdot H_2$, $\phi(\Gamma_{30}) = T_1 \cdot T_2$ (up to some scalar factor).*

Proof. This follows from Proposition 3.1 and 3.2

Proposition 3.3. *The polynomials q, F_6, F_8, F_{12}, resp. $q, \Gamma_{12}, \Gamma_{20}, \Gamma_{30}$ are algebraically independent.*

Proof. Let $\sum_I \alpha_I q^{i_1} F_6^{i_2} F_8^{i_3} F_{12}^{i_4} = 0$ and $\sum_J \beta_J q^{j_1} \Gamma_{12}^{j_2} \Gamma_{20}^{j_3} \Gamma_{30}^{j_4} = 0$ be algebraic relations, $I = (i_1, i_2, i_3, i_4) \in \mathbb{N}^4$, $J = (j_1, j_2, j_3, j_4) \in \mathbb{N}^4$, $\alpha_I, \beta_J \in \mathbb{R}$, then

$$\begin{aligned} 0 &= \phi(\textstyle\sum_I \alpha_I q^{i_1} F_6^{i_2} F_8^{i_3} F_{12}^{i_4}) \\ &= \textstyle\sum_{I'} \alpha_{I'} \phi(F_6)^{i_2'} \phi(F_8)^{i_3'} \phi(F_{12})^{i_4'} \\ &= \textstyle\sum_{I'} \alpha_{I'} t_1^{i_2'} t_2^{i_2'} W_1^{i_3'} W_2^{i_3'} \chi_1^{i_4'} \chi_2^{i_4'} \end{aligned} \tag{4}$$

similarly

$$
(5)\qquad
\begin{aligned}
0 &= \phi(\textstyle\sum_J \beta_J q^{j_1}\Gamma_{12}^{j_2}\Gamma_{20}^{j_3}\Gamma_{30}^{j_4})\\
&= \textstyle\sum_{J'} \beta_{J'}\phi(\Gamma_{12})^{j_2'}\phi(\Gamma_{20})^{j_3'}\phi(\Gamma_{30})^{j_4'}\\
&= \textstyle\sum_{J'} \beta_{J'} f_1^{j_2'} f_2^{j_2'} H_1^{j_3'} H_2^{j_3'} \mathcal{T}_1^{j_4'} \mathcal{T}_2^{j_4'}.
\end{aligned}
$$

If the polynomials t_1, W_1, χ_1 are fixed, we obtain a relation between t_2, W_2 and χ_2, which is the same relation as for t_1, W_1 and χ_1 if we fix t_2, W_2 and χ_2. The same holds for the polynomials $f_1, H_1, \mathcal{T}_1$ and $f_2, H_2, \mathcal{T}_2$. From [K] p. 55 and p. 57 there are only the relations

$$108\, t_1^4 - W_1^3 + \chi_1^2 = 0,\ 108\, t_2^4 - W_2^3 + \chi_2^2 = 0$$

and

$$\mathcal{T}_1^2 + H_1^3 - 1728 f_1^5 = 0,\ \mathcal{T}_2^2 + H_2^3 - 1728 f_2^5 = 0$$

between these polynomials. By multiplying these relations, however, it is not possible to obtain expressions like (4) and (5). □

Corollary 3.2. *The polynomials q, F_6, F_8, F_{12}, resp. $q, \Gamma_{12}, \Gamma_{20}, \Gamma_{30}$ generate rationally the ring of invariant polynomials of* $[3,4,3]$*, resp.* $[3,3,5]$.

Proof. (cf. [C1] p. 775) By Proposition 3.3 and Proposition 3.2 these are algebraically independent, moreover the products of their degrees are

$$2\cdot 6\cdot 8\cdot 12 = 1152 \text{ and } 2\cdot 12\cdot 20\cdot 30 = 14400,$$

which are equal to the order of the groups $[3,4,3]$ and $[3,3,5]$. By [C1] this implies the assertion. □

4. Explicit computations

We recall the following matrices of $SO(4)$ (cf. [Sa]):

$$(q_2,1) = \begin{pmatrix} 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 \end{pmatrix},\quad (1,q_2) = \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$(p_3,1) = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ -1 & 1 & 1 & -1\\ 1 & 1 & 1 & 1 \end{pmatrix},\quad (1,p_3) = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 & 1\\ -1 & 1 & -1 & -1\\ 1 & 1 & 1 & -1\\ -1 & 1 & 1 & 1 \end{pmatrix},$$

$$(p_4,1) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 & 0 & 0\\ 1 & 1 & 0 & 0\\ 0 & 0 & 1 & -1\\ 0 & 0 & 1 & 1 \end{pmatrix},\quad (1,p_4) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0\\ -1 & 1 & 0 & 0\\ 0 & 0 & 1 & -1\\ 0 & 0 & 1 & 1 \end{pmatrix},$$

$$(p_5, 1) = \frac{1}{2}\begin{pmatrix} \tau & 0 & 1-\tau & -1 \\ 0 & \tau & -1 & \tau-1 \\ \tau-1 & 1 & \tau & 0 \\ 1 & 1-\tau & 0 & \tau \end{pmatrix},$$

$$(1, p_5) = \frac{1}{2}\begin{pmatrix} \tau & 0 & \tau-1 & 1 \\ 0 & \tau & -1 & \tau-1 \\ 1-\tau & 1 & \tau & 0 \\ -1 & 1-\tau & 0 & \tau \end{pmatrix},$$

where $\tau = \frac{1}{2}(1+\sqrt{5})$. Then we have

Group	Generators
G_6	$(q_2, 1), (1, q_2), (p_3, 1), (1, p_3)$
G_8	$(q_2, 1), (1, q_2), (p_3, 1), (1, p_3), (p_4, 1), (1, p_4)$
G_{12}	$(q_2, 1), (1, q_2), (p_3, 1), (1, p_3), (p_5, 1), (1, p_5)$

Now we can write down the equations of the fix lines on $\mathbb{P}_1 \times \mathbb{P}_1$ and those of the planes which are generated by a couple of lines. The products of planes of Section 2.1 in the case of the group G_6 are

$$\begin{aligned}
f_{11}^{(6)} \cdot f_{22}^{(6)} \cdot \ldots \cdot f_{66}^{(6)} &= (x_2 - ix_3)(x_1 + ix_3)(x_2 + ix_3)(x_1 - ix_2)(x_1 - ix_3)(x_1 + ix_2), \\
f_{11}^{(8)} \cdot f_{22}^{(8)} \cdot \ldots \cdot f_{88}^{(8)} &= (x_1 + ax_2 - bx_3)(x_1 + bx_2 - ax_3)(x_1 - ax_2 - bx_3)(x_1 - ax_3 - bx_2) \\
&\quad (x_2 + bx_1 - ax_3)(x_2 + ax_1 - bx_3)(x_2 - bx_1 + ax_3)(x_2 + bx_3 - ax_1), \\
f_{11}^{(12)} \cdot f_{22}^{(12)} \cdot \ldots \cdot f_{1212}^{(12)} &= (x_3 - x_1 + cx_2)(x_3 - x_1 - cx_2)(x_2 + x_3 - cx_1)(x_2 + x_3 + cx_1) \\
&\quad (x_3 - x_2 + cx_1)(x_3 - x_2 - cx_1)(x_1 + x_2 + cx_3)(x_1 + x_2 - cx_3) \\
&\quad (x_1 + x_3 - cx_2)(x_1 + x_3 + cx_2)(x_1 - x_2 + cx_3)(x_1 - x_2 - cx_3),
\end{aligned}$$

with $a = (1/2)(1 + i\sqrt{3}), b = (1/2)(1 - i\sqrt{3}), c = i\sqrt{2}$.

Then the G_6-invariant polynomials F_6, F_8 and F_{12} have the following expressions

$$\begin{aligned}
F_6 &= x_0^6 + x_1^6 + x_2^6 + x_3^6 + 5x_0^2x_1^2(x_0^2 + x_1^2) + 5x_1^2x_3^2(x_1^2 + x_3^2) + 5x_1^2x_2^2(x_1^2 + x_2^2) \\
&\quad + 6x_0^2x_2^2(x_0^2 + x_2^2) + 6x_0^2x_3^2(x_0^2 + x_3^2) + 6x_3^2x_2^2(x_2^2 + x_3^2) + 2x_0^2x_2^2x_3^2,
\end{aligned}$$

$$F_8 = 3\sum x_i^8 + 12\sum x_i^6x_j^2 + 30\sum x_i^4x_j^4 + 24\sum x_i^4x_j^2x_k^2 + 144x_0^2x_1^2x_2^2x_3^2,$$

$$\begin{aligned}
F_{12} &= \frac{123}{8}\sum x_i^{12} + \frac{231}{4}\sum x_i^{10}x_j^2 + \frac{21}{8}\sum x_i^8x_j^4 - \sum\frac{255}{2}\sum x_i^6x_j^6 + \frac{949}{2}\sum x_i^8x_j^2x_k^2 \\
&\quad + \frac{1839}{2}\sum x_i^6x_j^4x_k^2 + \frac{6111}{4}\sum x_i^4x_j^4x_k^4 + 1809\sum x_i^6x_j^2x_k^2x_h^2 + \frac{7281}{2}\sum x_i^4x_j^4x_k^2x_h^2.
\end{aligned}$$

Here the sums run over all the indices $i, j, k, h = 0, 1, 2, 3$, always being different when appearing together. By applying the map ϕ, a computer computation with MAPLE shows that

$$\begin{aligned}\phi(F_6) &= -\frac{13}{16}\,t_1\cdot t_2,\\ \phi(F_8) &= \frac{3}{64}\,W_1\cdot W_2,\\ \phi(F_{12}) &= \frac{3}{256}\,\chi_1\cdot\chi_2\end{aligned}$$

as claimed in Corollary 3.1.
Proof of Proposition 2.1, 2. The polynomials $F_6,\ F_8,\ F_{12}$ remain invariant by interchanging x_2 with x_3, which is what the matrix C' does. $\square$

Proof of Proposition 3.2. We write the computations just in the case of the $[3,4,3]$-invariant polynomials. Consider the points $p_1 = (i\sqrt{2} : 1 : 1 : 0)$ and $p_2 = (1 : i : 0 : 0)$, then $q(p_1) = q(p_2) = 0$ and by a computer computation with MAPLE we get $F_6(p_1) = 26$, $F_8(p_2) = 12$ and $F_{12}(p_2) = 32$. This shows that q does not divide the polynomials. Since $F_6(p_2) = 0$, F_6 does not divide F_{12}. $\square$

Remark 4.1. Observe that an equation for a $[3,4,3]$-invariant polynomial of degree six and for a $[3,3,5]$-invariant polynomial of degree twelve was given by the author in [Sa] by a direct computer computation with MAPLE.

5. Final Remarks

1. The zero sets of the polynomials which are described in this paper define algebraic surfaces in $\mathbb{P}_3(\mathbb{C})$ with many symmetries. Such surfaces are expected to have many interesting geometrical properties: many lines, many singularities, etc. In [Sa] it is shown that the projective one-dimensional families of surfaces with equations $F_6 + \lambda q^3 = 0$ and $\Gamma_{12} + \lambda q^6 = 0$, $\lambda \in \mathbb{P}_1$ contain each four surfaces with many nodes. The article also describes a one-dimensional $[3,4,3]$-invariant family of surfaces of degree 8. The family contains four surfaces with A_1-singularities and it is also G_8-symmetric. In Figure 1 we show the picture of a surface with 144 nodes. But in fact the whole $[3,4,3]$-invariant family of surfaces of degree 8 is projectively two-dimensional with equation $F_8 + \lambda F_6\cdot q + \mu q^4 = 0$, $(\lambda,\mu) \in \mathbb{P}_2$. It would be interesting to describe more surfaces in this family and in the families of $[3,4,3]$-symmetric surfaces of degree 12 and of $[3,3,5]$-symmetric surfaces of degree 20 and 30.

2. Another interesting problem is to study the quotients of the previous surfaces by the groups. In [BS] it is shown that the G_6-quotient, resp. the G_{12}-quotient of a surface in the family defined by $F_6 + \lambda q^3 = 0$, resp. defined by $\Gamma_{12} + \lambda q^6 = 0$ is a K3-surface. It would be interesting to identify the quotients by the groups $[3,4,3]$, resp. $[3,3,5]$ which contain the groups G_6, resp. G_{12}. And in general, to describe more quotients.

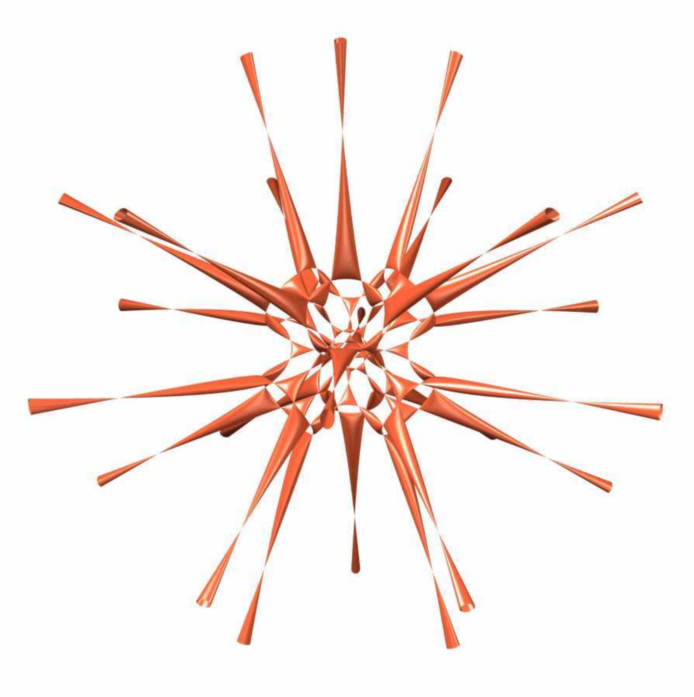

Fig. 1. $[3,4,3]$-symmetric octic with 144 nodes

$$x_0^8+x_1^8+x_2^8+x_3^8+14\left(x_0^4x_1^4+x_0^4x_2^4+x_0^4x_3^4+x_1^4x_2^4+x_1^4x_3^4+x_2^4x_3^4\right)+$$

$$+168x_0^2x_1^2x_2^2x_3^2-\tfrac{9}{16}(x_0^2+x_1^2+x_2^2+x_3^2)^4=0$$

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Counting lines on surfaces

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COUNTING LINES ON SURFACES

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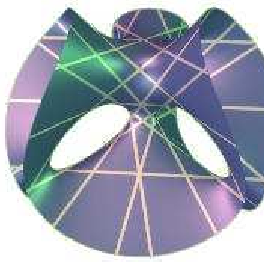
Abstract. This paper deals with surfaces with many lines. It is well-known that a cubic contains 27 of them and that the maximal number for a quartic is 64. In higher degree the question remains open. Here we study classical and new constructions of surfaces with high number of lines. We obtain in particular a symmetric octic with 352 lines.



Cubic surface with 27 lines[1]

1. Introduction

Motivation for this paper is the article in 1943 of Segre [12] which studies the following classical problem: What is the maximum number of lines a surface of degree d in $\mathbb{P}_3$ can have? Segre answers this question for $d = 4$ by using some nice geometry, showing that it is exactly 64. For the degree three it is a classical result that each smooth cubic in $\mathbb{P}_3$ contains 27 lines, but for $d \geq 5$ this number is still not known. In this case, Segre shows in *loc.cit.* that the maximal number is less or equal to $(d-2)(11d-6)$ but this bound is far from beeing sharp. Indeed, already in degree four it gives 76 lines which is not optimal. So from one hand one can try to improve the upper bound for the number of lines $\ell(d)$ a surface of degree d in $\mathbb{P}_3$ can have, on the other hand it is interesting to construct surfaces with as many lines as possible to give a lower bound for $\ell(d)$.

It is notoriously difficult to construct examples of surfaces with many lines. Good examples so far are the surfaces of the kind $F(x, y, z, t) = \phi(x, y) - \psi(z, t) = 0$ where ϕ and ψ are homogeneous polynomials of degree d. Segre in [13] studies the case of $\deg F = 4$ showing that in this case the possible numbers of lines are $16, 32, 48, 64$. He finds these numbers by studying the automorphisms of $\mathbb{P}_1$ between the two sets of four points $\phi = 0$ and $\psi = 0$. Caporaso-Harris-Mazur in [3], by using similar methods as Segre, then study the maximal number of lines N_d on such surfaces in any degree d showing that $N_d \geq 3d^2$ for each d and $N_4 \geq 64$, $N_6 \geq 180$, $N_8 \geq 256$, $N_{12} \geq 864$, $N_{20} \geq 1600$. In this paper we show the exactness of these results. First we note that it is enough to consider surfaces of the kind $\phi(x, y) - \phi(z, t) = 0$ and by a careful analysis of the automorphisms of the set of points $\phi = 0$ on $\mathbb{P}_1$ we can list all the possible numbers of lines on surfaces of this kind for all d and then we prove:

Proposition 3.3 *The maximal numbers of lines on $F = \phi(x, y) - \psi(z, t) = 0$ are:*

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[1] `http://enriques.mathematik.uni-mainz.de/surf/logo.jpg`

- $N_d = 3d^2$ *for* $d \geq 3$, $d \neq 4, 6, 8, 12, 20$*;*
- $N_4 = 64$, $N_6 = 180$, $N_8 = 256$, $N_{12} = 864$, $N_{20} = 1600$.

It is well-known that the Fermat surfaces $(x^d - y^d) - (z^d - t^d) = 0$ have $3d^2$ lines. Our proof provides a method to write equations of surfaces $\phi(x, y) - \psi(z, t) = 0$ with each possible number of lines. In particular, our proposition shows that it is not possible, with these surfaces, to obtain better examples and a better lower bound for $\ell(d)$. So, in order to find better examples, one has to use new methods. In this paper we explore the following kinds of surfaces:

- d-covering of the plane $\mathbb{P}_2$ branched over a curve of degree d.
- Symmetric surfaces in $\mathbb{P}_3$.

We show that the first method cannot give more than $3d^2$ lines (Proposition 4.2).
The second method is based on the following idea: if a surface has many automorphisms (many symmetries) then possibly it contains many orbits of lines. This idea was used successfully in the study of surfaces with many nodes. In this paper we find a G_8-invariant octic with 352 lines, where $G_8 \subset \mathrm{PGL}(3, \mathbb{C})$ has order 576 (Proposition 5.2). This shows $\ell(8) \geq 352$, improving the previous bound of 256.
As stated before, one can also try to improve the upper bound for $\ell(d)$. Following the idea of Segre [12] and imposing some extra conditions on the lines on a surface, we can find the bound $d(7d-12)$ which surprisingly agrees with the maximal examples in degrees $4, 6, 8, 12$ (Section 6).
Finally a related problem to this is to determine the maximal number $m(d)$ of skew-lines a surface of degree d in $\mathbb{P}_3$ can have. It is well-known that $m(3) = 6$ and $m(4) = 16$. For $d \geq 5$, this value is not known. An upper bound $m(d) \leq 2d(d-2)$ is given by Miyaoka in [7], which is sharp for $d = 3, 4$. There are results of Rams [9, 10] giving examples of surfaces with $d(d-2) + 2$ skew-lines ($d \geq 5$) and with 19 skew-lines for $d = 5$. In Proposition 8.2 we improve his examples for $d \geq 7$ and $\gcd(d, d-2) = 1$ to $d(d-2) + 4$.
The paper is organized as follows. In Section 2 we give an overview of known results. In Sections 3 and 4 we describe completely the surfaces of the kind $\phi(x, y) - \psi(z, t) = 0$ and the d-coverings of the plane $t^d = f(x, y, z)$. Section 5 is devoted to the investigation of symmetric surfaces, and in particular of an octic with 352 lines. In Section 6 we present the uniform bound $d(7d-12)$ and Section 7 is an application to the problem of the number of rational points on curves. Finally, Section 8 deals with the skew-lines: we give an overview of known results and some new examples.

Acknowledgements. We thank Duco van Straten for suggesting us this nice problem and for interesting discussions.

2. General results

Our objective is to investigate the number of lines contained in a smooth surface in $\mathbb{P}_3$. We first recall classical results: the generic situation and the bound of Segre.

2.1. **Generic situation.**

It is a well-known fact that each smooth quadric surface in $\mathbb{P}_3$ contains an infinite number of lines and each smooth cubic surface in $\mathbb{P}_3$ contains exactly 27 lines. What happens for surfaces of higher degree? Generically:

Proposition 2.1. *A generic smooth surface of degree* $d \geq 4$ *in* $\mathbb{P}_3$ *contains no line.*

We briefly recall the proof, following [1, 2].

Proof. Let V be the vector space of degree d homogeneous polynomials in the coordinates x, y, z, t and G be the Grassmannian of 2-planes in $\mathbb{C}^4$. Consider the incidence variety $F := \{(L, f) \subset G \times V \mid f_{|L} \equiv 0\}$ with its projections $p : F \to G$ and $q : F \to V$.
• Let $L \in G$ and assume that L is generated by the vectors $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$ in $\mathbb{C}^4$. Consider the affine neighbourhood of L in G:

$$\mathcal{U} := \mathrm{Span}\{(1, 0, a, b), (0, 1, c, d)\}$$

where a, b, c, d are local coordinates. If $f \in p^{-1}(\mathcal{U})$, then

$$f\left(\lambda(1, 0, a, b) + \mu(0, 1, c, d)\right) = 0 \qquad \forall \lambda, \mu \in \mathbb{C},$$

and denoting $f = \sum_{i+j+k+l=d} a_{i,j,k,l} x^i y^j z^k t^l$, the equation

$$\sum_{i+j+k+l=d} a_{i,j,k,l} \lambda^i \mu^j (\lambda a + \mu c)^k (\lambda b + \mu d)^l = 0 \qquad \forall \lambda, \mu \in \mathbb{C}$$

gives $d+1$ linear equations in the coordinates $(a_{i,j,k,l})$ of $f \in V$ whose rank at $a = b = c = d = 0$ is $d+1$: hence locally in a neighbouhood of L, the system has rank $d+1$ so p is a locally trivial bundle of rank: $\dim V - (d+1)$.
• Let X be a surface of degree d in $\mathbb{P}_3$, given by a polynomial $f \in V$. Then the *Fano scheme* parametrizing the lines contained in X is $F(X) := p\left(q^{-1}(f)\right)$.
• Consider the map $q : F \to V$. Since:

$$\dim F = \dim V - (d+1) + \dim G = \dim V - (d-3),$$

for $d \geq 4$ one has $\dim F < \dim V$ hence the map q is not dominant. This means that the generic fibre of q is empty. Otherwise stated, $F(X)$ is empty for X generic. □

We shall see in the next section that the number of lines a smooth surface of degree $d \geq 4$ can have is always finite, and bounded. This leads to the problem of finding surfaces with an optimal number of lines.

2.2. **Upper bound for lines.**

The best upper bound known so far for the number of lines on a smooth surface of degree $d \geq 4$ in $\mathbb{P}_3$ is given by Segre:

Theorem 2.2 (Segre [12]).

- *The number of lines lying on a smooth surface of degree $d \geq 4$ does not exceed $(d-2)(11d-6)$.*
- *The maximum number of lines lying on a quartic surface is exactly* 64.

This bound is effective for $d = 4$ (see for instance maximal examples in Section 3.1) but for $d \geq 5$ it is believed that it could be improved. For instance, already for $d = 4$ the uniform bound $(d-2)(11d-6)$ is too big. The next sections are devoted to the study of some families of surfaces with particular properties, containing many lines.

3. Surfaces of the kind $\phi(x, y) = \psi(z, t)$

We consider a surface $\mathcal{S}$ given by an equation of the kind:

$$F(x, y, z, t) := \phi(x, y) - \psi(z, t)$$

for two homogeneous polynomials ϕ, ψ of degree d. Segre gave a complete description of the possible and maximal numbers of lines in the case $d = 4$ ([13, §VIII]). We generalize the method to all degrees: we treat in details the configuration of lines, give a description of all possible numbers, and conclude with the maximal numbers of lines for such surfaces.

3.1. **Configuration of the lines.**
Let $Z(\phi)$, resp. $Z(\psi)$ denote the set of zeros of $\phi(x,y)$, resp. $\psi(z,t)$ in $\mathbb{P}_1$.

Theorem 3.1. *Let $F(x,y,z,t)=\phi(x,y)-\psi(z,t)$ be the equation of a smooth surface $\mathcal{S}$ of degree d in $\mathbb{P}_3$. The number N_d of lines on $\mathcal{S}$ is exactly:*

$$N_d = d(d+\alpha_d)$$

where α_d is the order of the group of isomorphisms of $\mathbb{P}_1$ mapping $Z(\phi)$ to $Z(\psi)$.

Proof.
• Let L be the line $z=t=0$ and L' be the line $x=y=0$. Then $\mathcal{S}\cap L = Z(\phi)$ and $\mathcal{S}\cap L' = Z(\psi)$. Since the surface $\mathcal{S}$ is smooth, the homogeneous polynomials ϕ and ψ have simple zeros. Indeed, for example in the case of the polynomial ϕ, if $[a:b]\in\mathbb{P}_1$ is such that ϕ can be factorized by $(bx-ay)^2$, then $\partial_x\phi(a,b)=\partial_y\phi(a,b)=0$ and the point $[a:b:0:0]$ is a singular point of $\mathcal{S}$ (the inverse also holds: if both ϕ and ψ have only simple zeros, then $\mathcal{S}$ is smooth). Set $Z(\phi) := \{P_1,\ldots,P_d\}$ and $Z(\psi) := \{P'_1,\ldots,P'_d\}$.
• Each line $L_{i,j}$ joining a P_i to a P'_j is contained in $\mathcal{S}$: if $P_i=[x_i:y_i:0:0]$ and $P'_j=[0:0:z'_j:t'_j]$ the line joining them consists in points $[\lambda x_i : \lambda y_i : \mu z'_j : \mu t'_j]$, $\lambda,\mu\in\mathbb{C}$, which are all contained in the surface, by homogeneity of the polynomials ϕ and ψ. This gives d^2 lines.
• Each line contained in $\mathcal{S}$ and intersecting L and L' is one of the previous lines. Indeed, if D is such a line, set $D\cap L=\{[a:b:0:0]\}$ and $D\cap L'=\{[0:0:c:d]\}$. Then $F(a,b,0,0)=\phi(a,b)=0$ so $[a:b:0:0]$ is one of the points P_i and similarly $[0:0:c:d]$ is one P'_j.
• Let D be a line contained in $\mathcal{S}$ and not intersecting L. Then D does not intersect L' (and *vice-versa*). Indeed, an equation of such a line D is given by two independent equations:

$$\begin{cases} ax+by+cz+dt=0 \\ a'x+b'y+c'z+d't=0 \end{cases}$$

Since D does not intersect L, the system

$$\begin{cases} ax+by=0 \\ a'x+b'y=0 \end{cases}$$

has rank two, so we can rewrite the equations of D as the following independent equations:

$$\begin{cases} x=\alpha z+\beta t \\ y=\gamma z+\delta t \end{cases}$$

Then D does not intersect L' otherwise the matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ would have rank one.
• Therefore, the equations of the line D define a linear isomorphism between the lines L' and L inducing a bijection between $Z(\psi)$ and $Z(\phi)$. Indeed, seting $P'_j=[0:0:c:d]$, then $a:=\alpha c+\beta d$ and $b:=\gamma c+\delta d$ have the property that $[a:b:c:d]\in D\subset\mathcal{S}$ so $\phi(a,b)=F(a,b,c,d)+\psi(c,d)=0$ hence $[a:b:0:0]$ is a zero of ϕ.
• Conversely, let $\sigma: L'\to L$ be an isomorphism mapping the points P'_j to the points P_i, and $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ a matrix defining σ. Consider the smooth quadric $Q_\sigma : x(\gamma z+\delta t)-y(\alpha z+\beta t)=0$. Its first ruling is the family of lines $(p,\sigma(p))$ for $p\in L'$. For $p=[c:d]$, these lines are

given by the equations

$$\mathrm{I}_{[c:d]} : \begin{cases} (\gamma c + \delta d)x - (\alpha c + \beta d)y = 0 \\ dz - ct = 0 \end{cases}$$

Its second ruling consists in the family of lines of equations

$$\mathbb{I}_{[a:b]} : \begin{cases} ax - b(\alpha z + \beta t) = 0 \\ ay - b(\gamma z + \delta t) = 0 \end{cases}$$

for $[a : b] \in \mathbb{P}_1$. To this ruling belong the lines L ($[a : b] = [0 : 1]$), L' ($[a : b] = [1 : 0]$) and D ($[a : b] = [1 : 1]$). It is not true *a priori* that *this* D is contained in $\mathcal{S}$, since the matrix σ is defined up to a scalar factor.
In each ruling, the lines are disjoint to each other, and each line of one ruling intersects each line of the other ruling. Since the intersection $\mathcal{S} \cap Q$ contains exactly the d different lines $(P'_j, \sigma(P'_j))$ of the first ruling, it contains also d lines of the second ruling: Consider a line in the first ruling not contained in $\mathcal{S}$, then it intersects $\mathcal{S}$ in d points, and through each of this points is attached a line of the second ruling, which also intersects the d lines of the first ruling contained in $\mathcal{S}$, so these lines of the second ruling intersect $\mathcal{S}$ at $d+1$ points, so are contained in $\mathcal{S}$. But it is not clear *a priori* with our argument that these lines in the second ruling are different. Denote by $\mathcal{U}_d$ the group of d-th roots of the unit. The group $\mathcal{U}_d \times \mathcal{U}_d$ acts on $\mathbb{P}_3$ by $(\xi, \eta) \cdot [x : y : z : t] = [\xi x : \xi y : \eta z : \eta t]$, leaving the surface $\mathcal{S}$ globally invariant since the polynomials ϕ and ψ are homogeneous of degree d. Observe that the lines of the first ruling are invariant for the action, but for the second ruling, $(\xi, \eta) \cdot \mathbb{I}_{[a:b]} = \mathbb{I}_{[\xi^{-1}a:\eta^{-1}b]}$ so each line of the second ruling produces a length d orbit through the action. Since the surface $\mathcal{S}$ contains at least one line of the second ruling, it contains the whole orbit, this gives us d different lines.
Therefore, each isomorphism $\sigma : L' \to L$ mapping $Z(\psi)$ to $Z(\phi)$ gives d lines, and there are no other lines. Furthermore, for two different isomorphisms, the corresponding lines are different since the matrix defining the isomorphims are not proportional.
• Denote by α_d the number of isomorphims $\sigma : L' \to L$ mapping $Z(\psi)$ to $Z(\phi)$. The preceding discussion shows that the exact number of lines contained in the surface $\mathcal{S}$ is:

$$N_d = d^2 + \alpha_d d.$$

□

Remark 3.2. *In the proof of* [3, Lemma 5.1]*, Caporaso-Harris-Mazur proved with a similar argument that the number of lines is at least $d(d+\alpha_d)$ and described some special values. Our argument includes the exactness. In the next subsections we give a full description of the possible values of α_d, in particular its maximal values for each d.*

3.2. **The possible numbers of lines.**

Now we want to find the possible and maximal values of N_d, or equivalently α_d. If there is at least one isomorphism σ (see the proof above), then by composing by σ^{-1} we are lead to the problem of determining the possible numbers of automorphisms of $\mathbb{P}_1$ (or *projectivities*) acting on a given set of d points on $\mathbb{P}_1$. Since a projectivity is defined by its value on three points, we have always $\alpha_3 = 6$, and for $d \geq 4$ there is only a finite number of such isomorphisms, depending on the relative position of the points, encoded in their cross-ratios. The case $d = 4$ was studied by Segre [13] with this point of view. We give a different argument for the general case. The set Γ_d of isomorphims of $\mathbb{P}_1$ acting on d points defines a finite group of automorphisms of $\mathbb{P}_1$. First recall the classical classification:

Polyhedral groups. There are five types of finite subgroups of $\mathrm{SO}(3,\mathbb{R})$, or equivalently of $\mathrm{PGL}(2,\mathbb{C})$, called *polyhedral groups*:

- the cyclic groups $C_k \cong \mathbb{Z}/k\mathbb{Z}$ of order $k \geq 2$, isomorphic to the group of isometries of a regular polygon with k vertices in the plane;
- the dihedral groups $D_k \cong \mathbb{Z}/k\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ of order $2k$, $k \geq 2$, isomorphic to the group of isometries of regular polygon with k vertices in the space;
- the group $\mathcal{T}$ of positive isometries of a regular tetrahedra, isomorphic to the alternate group $\mathfrak{A}_4$ of order twelve;
- the group $\mathcal{O}$ of positive isometries of a regular octahedra or a cube, isomorphic to the symmetric group $\mathfrak{S}_4$ of order 24;
- the group $\mathcal{I}$ of positive isometries of a regular icosahedra or a regular dodecahedra, isomorphic to the alternate group $\mathfrak{A}_5$ of order 60.

In the sequel, we shall describe generators of these groups and their orbits on $\mathbb{P}_1$, in order to get explicit constructions of surfaces.

We now proceed to the description of all possible groups of isomorphisms ($d \geq 4$):

(1) $\Gamma_d = \{\mathrm{id}\}$. This is not possible for $d = 4$ since there are always at least four automorphisms of a set of four points in $\mathbb{P}_1$ (their cross-ratio takes generically six different values under permutation).

(2) Γ_d *is a cyclic group:* $\Gamma_d \cong \mathbb{Z}/k\mathbb{Z}$ ($k \geq 2$) with generator $\sigma(t) = \xi t$ where ξ is a primitive k-th root of the unit. The action of σ on $\mathbb{P}_1$ has two fix points $\{0,\infty\}$ and all other points generate a length k orbit. So, depending whether the fix points are in the given set of d points or not we have the decomposition:
$$d = \alpha + \beta k$$
with $\alpha \in \{0,1,2\}$ and $\beta \geq 1$:

- $\alpha = 0$. The points are[2]:
$$\{\mu_1, \mu_1\xi, \ldots, \mu_1\xi^{k-1}\}, \ldots, \{\mu_\beta, \mu_\beta\xi, \ldots, \mu_\beta\xi^{k-1}\}.$$
This forces $\beta \geq 3$ since: if $\beta = 1$ or $\beta = 2$ then $t \mapsto 1/t$ or $t \mapsto \mu_2/(\mu_1 t)$ generate a dihedral group. For $\beta \geq 3$ there are no other isomorphisms.
- $\alpha = 1$. The points are:
$$\{0\}, \{\mu_1, \mu_1\xi, \ldots, \mu_1\xi^{k-1}\}, \ldots, \{\mu_\beta, \mu_\beta\xi, \ldots, \mu_\beta\xi^{k-1}\}.$$
There is no other isomorphism whenever $d = 1+\beta k \geq 5$. For $k = 3$ and $\beta = 1$ there are other isomorphism (a tetrahedral group).
- $\alpha = 2$. The points are:
$$\{0,\infty\}, \{\mu_1, \mu_1\xi, \ldots, \mu_1\xi^{k-1}\}, \ldots, \{\mu_\beta, \mu_\beta\xi, \ldots, \mu_\beta\xi^{k-1}\}.$$
As before, this forces $\beta \geq 3$.

To summarize, for the group Γ_d be a cyclic group $\mathbb{Z}/k\mathbb{Z}$ ($d \geq 4$, $k \geq 2$):

- $d = \beta k$, $\beta \geq 3$, *e.g.* $\phi(x,y) = \prod_{i=1}^{\beta}(x^k - \lambda_i y^k)$;
- $d = 1 + \beta k \geq 5$, $\beta \geq 1$ if $k = 3$, *e.g.* $\phi(x,y) = x\prod_{i=1}^{\beta}(x^k - \lambda_i y^k)$;

[2] Here and in the sequel, the μ_i's are assumed to be generic: they are distinct and in particular they are not the β-th roots of the unit and their k-powers $\lambda_i := \mu_i^k$ are distincts.

- $d = 2 + \beta k$, $\beta \geq 3$ *e.g.* $\phi(x,y) = xy \prod_{i=1}^{\beta}(x^k - \lambda_i y^k)$.

(3) Γ_d *is a dihedral group:* $\Gamma_d \cong \mathbb{Z}/k\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ $(k \geq 2)$ with generators $\sigma(t) = \xi t$ and $s(t) = 1/t$ where ξ is a primitive k-th root of the unit. The action of the dihedral group on $\mathbb{P}_1$ has one length 2 orbit $\{0, \infty\}$ and one length k orbit generated by 1. So we have the decomposition:

$$d = 2\alpha + \beta k + \gamma 2k$$

with $\alpha, \beta \in \{0,1\}$, $\gamma \geq 0$:

- $\gamma = 0$, $\alpha = 0$ and $\beta = 1$. The points are:

$$\{1, \xi, \ldots, \xi^{k-1}\}$$

Then $d = k$ and $\phi(x,y) = x^k - y^k$. This gives the Fermat surface.

- $\gamma = 0$, $\alpha = 1$ and $\beta = 1$. The points are:

$$\{0, \infty\}, \{1, \xi, \ldots, \xi^{k-1}\}$$

This forces $k \neq 2, 4$: if $k = 2$, the configuration is isomorphic to the preceding case (with $2k$) and contains more isomorphims, and if $k = 4$ there are other isomorphisms generating an octahedral group. Then $d = 2 + k$ and $\phi(x,y) = xy(x^k - y^k)$.

- $\gamma \neq 0$. Then $d \in \{2k\gamma, 2{+}2k\gamma, k{+}2k\gamma, 2{+}k{+}2k\gamma\}$ and ϕ contains, besides the factors given in the preceding cases, γ factors of the kind $(x^k - \lambda y^k)(x^k - \frac{1}{\lambda}y^k)$.

(4) Γ_d *is a tetrahedral group* $\mathcal{T}$. The group $\mathcal{T}$ is generated by:

$$\sigma(t) = \omega t, \quad s(t) = \frac{1-t}{1+2t}$$

acting on the set $\{0, 1, \omega, \omega^2\}$ where ω is a primitive third root of the unit. The action of $\mathcal{T}$ on $\mathbb{P}_1$ has two length four orbits:

$$\{0, 1, \omega, \omega^2\}, \{\infty, \frac{-1}{2}, \frac{-1}{2}\omega, \frac{-1}{2}\omega^2\}$$

and one length six orbit generated by the fix point $w = \frac{-1-\sqrt{3}}{2}$ of[3] s. These are all the orbits of lengths four or six since the conjugacy classes in $\mathcal{T}$ are generated by $\mathrm{id}, s, \sigma, \sigma^2$. So we have the decomposition:

$$d = 4\alpha + 6\beta + 12\gamma$$

with $\alpha \in \{0,1,2\}, \beta \in \{0,1\}, \gamma \geq 0$:

- $\gamma = 0$, $\beta = 0$ and $\alpha = 1$: the group of isomorphisms is $\mathcal{T}$.
- $\gamma = 0$, $\beta = 0$ and $\alpha = 2$: the group of isomorphisms would be $\mathcal{O}$ since $t \mapsto -1/(2t)$ interchanges the two length four orbits.
- $\gamma = 0$, $\beta = 1$ and $\alpha = 0$: the group of isomorphisms would be $\mathcal{O}$ since the length six orbit is stabilized by $t \mapsto -1/(2t)$.
- $\gamma = 0$, $\beta = 1$ and $\alpha = 1$: the group of isomorphisms is $\mathcal{T}$, because it is not contained in any dihedral group and the groups $\mathcal{O}$ or $\mathcal{I}$ have no length four or ten orbit.
- $\gamma = 0$, $\beta = 1$ and $\alpha = 2$: as before the group of isomorphisms is $\mathcal{O}$.
- For $\gamma \neq 0$, in general the group of isomorphisms is $\mathcal{T}$ but for special points this could be $\mathcal{O}$ or $\mathcal{I}$.

[3] The second fix point $w' = \frac{-1+\sqrt{3}}{2}$ belongs to the same orbit since $w = \sigma^2 s \sigma(w')$.

For example, for the tetrahedral group consider $\phi(x,y)=x(x^3-y^3)$.

(5) Γ_d *is an octahedral group* $\mathcal{O}$. The group $\mathcal{O}$ is generated by:

$$\sigma(t)=\mathrm{i}t,\quad s(t)=\frac{1}{t},\quad a(t)=\frac{t+\mathrm{i}}{t-\mathrm{i}}$$

acting on the set $\{0,\infty,1,\mathrm{i},-1,-\mathrm{i}\}$. The action of $\mathcal{O}$ on $\mathbb{P}_1$ has one length six orbit, one length eight orbit generated by the fix point $w=\frac{1+\mathrm{i}-\sqrt{3}-\mathrm{i}\sqrt{3}}{2}$ of[4] a, and one length twelve orbit generated by the fix point $z=-1+\sqrt{2}$ of the isomorphism[5] $r(t)=\frac{1-t}{1+t}$. These are all orbits of lengths six, eight or twelve since the conjugacy classes in $\mathcal{O}$ are generated by $\mathrm{id}, s, \sigma, a, r$. So we have the decomposition:

$$d=6\alpha+8\beta+12\gamma+24\delta$$

with $\alpha,\beta,\gamma\in\{0,1\}$, $\delta\geq 0$. Since the group $\mathcal{O}$ is not contained in $\mathcal{I}$ nor in any dihedral group, all choices of $\alpha,\beta,\gamma,\delta$ are possible to get $\Gamma_d\cong\mathcal{O}$.

(6) Γ_d *is a icosahedral group* $\mathcal{I}$. The group $\mathcal{I}$ is generated by:

$$p_5(t):=\frac{\tau t+\tau-1+\mathrm{i}}{(-\tau+1+\mathrm{i})t+\tau},\quad q_1(t):=-t,\quad q_2(t):=-\frac{1}{t}$$

where $\tau:=\frac{1+\sqrt{5}}{2}$. The only length twelve orbit is generated by a fix point of p_5, the length 20 orbit is generated by a fix point of $p_5^2q_2$ (which has order three) and the length 30 orbit is generated by a fix point of q_1. Since the conjugacy classes in $\mathcal{I}$ are generated by $\mathrm{id}, p_5, p_5^2, p_5^2q_2, q_1$ there are no other orbits. So we have the decomposition:

$$d=12\alpha+20\beta+30\gamma+60\delta$$

with $\alpha,\beta,\gamma\in\{0,1\}$, $\delta\geq 0$. All choices give $\Gamma_d\cong\mathcal{I}$.

3.3. **Maximal number of lines.**

As a corollary of Theorem 3.1 and the preceding discussion of cases, we get the following maximality result:

Proposition 3.3. *The maximal numbers of lines on $\mathcal{S}$ are:*

- $N_d=3d^2$ *for* $d\geq 3$, $d\neq 4,6,8,12,20$*;*
- $N_4=64$, $N_6=180$, $N_8=256$, $N_{12}=864$, $N_{20}=1600$.

Proof. Looking up at the discussion above, it appears that $\alpha_d=2d$ is maximal when the group of automorphisms can not be a group $\mathcal{T}$, $\mathcal{O}$ or $\mathcal{I}$ and that $\alpha_4=12$, $\alpha_6=\alpha_8=24$ and $\alpha_{12}=\alpha_{20}=60$ are maximal. For other values of d, if the automorphism group is $\mathcal{T}$, resp. $\mathcal{O}$, resp. $\mathcal{I}$ then the number of lines is:

$$d^2+12d,\quad \text{resp. } d^2+24d,\quad \text{resp. } d^2+60d$$

and these numbers are bigger than $3d^2$ only if

$$d<6,\quad \text{resp. } d<12,\quad \text{resp. } d<30.$$

So it just remains to check that the degree $d=10$ is not possible for $\mathcal{O}$ and $\mathcal{I}$ and that the degrees $d=14,16,18,22,24,26,28$ are not possible for $\mathcal{I}$, that is we cannot decompose such a d as a sum of lengths of orbits for the groups $\mathcal{O}$ or $\mathcal{I}$. This is clear with the restrictions on the numbers of orbits of each type. □

[4] The second fix point $w'=\frac{1+\mathrm{i}+\sqrt{3}+\mathrm{i}\sqrt{3}}{2}$ belongs to the same orbit since $w'=sa\sigma s(w)$.

[5] The second fix point $z'=-1-\sqrt{2}$ belongs to the same orbit since $z'=\sigma r\sigma a(z)$.

Remark 3.4. *Although this result was expected, one has to pass through the study of §3.2 to prove it.*

3.4. **Examples.**

(1) For d generic, the Fermat surface $F(x,y,z,t) = (x^d - y^d) - (z^d - t^d)$ gives the best example for surfaces of the kind $\phi(x,y) - \psi(z,t)$.
(2) For $d = 4$, $\Gamma_4 \in \{\emptyset, D_2, D_4, \mathcal{T}\}$ so the possible numbers of lines for such surfaces are: $16, 32, 48, 64$. This agrees with Segre's result and 64 is the maximal possible number of lines on a quartic surface.
(3) For $d = 5$, $\Gamma_5 \in \{\emptyset, \{\mathrm{id}\}, C_4, D_3, D_5\}$ so the possible numbers of lines for such surfaces are: $25, 30, 45, 55, 75$. The general bound of Segre gives 147.
(4) For $d = 6$, $\Gamma_6 \in \{\emptyset, \{\mathrm{id}\}, C_2, D_2, D_3, D_6, \mathcal{O}\}$ so the possible numbers of lines for such surfaces are: $36, 42, 48, 60, 72, 108, 180$. The general bound of Segre gives 240.
(5) The discussion of §3.2 gives explicit constructions of surfaces of each group Γ_d. For the groups $\mathcal{O}$ and $\mathcal{I}$, see also Section 5.

3.5. **Real lines.**
It is an interesting problem to find surfaces of any degree d with as many real lines as possible. For surfaces of the kind $\phi(x,y) - \phi(z,t) = 0$, if the zeros of ϕ are all real, one gets already d^2 real lines (see proof of Theorem 3.1). Then, for each isomophism in the group Γ_d represented by a real matrix, one gets one more real line if d is odd and two more real lines if d is even.

4. Surfaces of the kind $t^d = f(x,y,z)$

We consider smooth surfaces of degree $d \geq 3$ given as covering of $\mathbb{P}_2$ ramified along a plane curve. Let $\mathcal{C} : f(x,y,z) = 0$ be a plane curve defined by a homogeneous polynomial f of degree d and consider the surface $\mathcal{S}$ in $\mathbb{P}_3$ given by the equation:

$$F(x,y,z,t) := t^d - f(x,y,z).$$

Note that the surface $\mathcal{S}$ is smooth if and only if the curve $\mathcal{C}$ is.
Set $p = [0:0:0:1] \in \mathbb{P}_3$. The projection:

$$(\mathbb{P}_3 - \{p\}) \to \mathbb{P}_2,\ [x:y:z:t] \mapsto [x:y:z]$$

induces a d-covering $\pi : \mathcal{S} \to \mathbb{P}_2$ ramified along the curve $\mathcal{C}$.
Recall that a point $x \in \mathcal{C}$ is a *d-point* (or *total inflection point*) if the intersection multiplicy of $\mathcal{C}$ and its tangent line at x is equal to d.

Proposition 4.1.

(1) *Suppose L is a line contained in $\mathcal{S}$. Then $\pi(L)$ is a line.*
(2) *Let $x \in \mathcal{C}$ and L the tangent at $\mathcal{C}$ in x, then the preimage $\pi^{-1}(L)$ consists in d different lines contained in $\mathcal{S}$ if and only if x is a d-point.*
(3) *Let L be a line in $\mathbb{P}_2$. Then $\pi^{-1}(L)$ contains a line if and only if L is tangent to $\mathcal{C}$ at a d-point.*

Proof.

(1) It is clear from the definition of the projection π.
(2) Assume x is a d-point. Let Δ be a line of equation δ intersecting L at x. Then $d \cdot (\Delta \cdot L) = (\mathcal{C} \cdot L)$ so after restriction to L one has up to a scalar factor $f_{|_L} = \delta^d_{|_L}$ showing that the covering restricted to L is trivial and $\pi^{-1}(L)$ consists in the d lines $t - \xi^i \delta_L = 0$, $i = 1, \ldots, d$ where ξ is a primitive d-th root of the unit.

Conversely, if the covering splits, there exists a section $\gamma \in H^0(L, \mathcal{O}_L(1))$ such that $\gamma^d = f_{|_L} \in H^0(L, \mathcal{O}_L(d))$ so L intersects $\mathcal{C}$ at x with multiplicity d.

(3) If L is the tangent to $\mathcal{C}$ at a d-point the assertion follows from (2). Assume now that $\pi^{-1}(L)$ contains a line. Let L be given by a linear function $z = l(x,y)$. Then the equation of $\pi^{-1}(L)$ is $t^d - f(x,y,l(x,y)) = 0$. Since it contains a line the equation splits as

$$t^d - f(x,y,l(x,y)) = (t - w(x,y))F_{d-1}(t,x,y)$$

where $w(x,y)$ is a linear form. By comparing the coefficients in t one obtains $f(x,y,l(x,y)) = w(x,y)^d$ hence the preimage consists in the d lines:

$$t^d - f(x,y,l(x,y)) = \prod_{i=0}^{d-1}(t - \xi^i w(x,y))$$

where ξ is a primitive d-th root of the unit. This means that the covering is trivial over L so by (2) x is a d-point.

□

We deduce the number of lines contained in such surfaces:

Proposition 4.2. *Let $\mathcal{C} : f(x,y,z) = 0$ be a smooth plane curve of degree d with β total inflection points. Let $\mathcal{S}$ the surface in $\mathbb{P}_3$ given by the equation:*

$$F(x,y,z,t) := t^d - f(x,y,z).$$

Then $\mathcal{S}$ contains exactly $\beta \cdot d$ lines. In particular, it contains no more than $3d^2$ lines.

Proof. The first assertion follows directly from the lemma. For the second one, the inflection points are the intersections of $\mathcal{C}$ with its Hessian curve $\mathcal{H}$ of degree $3(d-2)$ and at a total inflection point the intersection multiplicity of $\mathcal{C}$ and $\mathcal{H}$ is $d-2$, so by Bezout one gets $\beta \leq 3d$. □

Remark 4.3.

- *For $d = 3$, it is well-known that each cubic has nine inflection points, then the induced surface has $3 \cdot 9 = 27$ lines.*
- *The Fermat curves $x^d + y^d + z^d = 0$ have $3d$ total inflection points hence the Fermat surfaces are examples of surfaces with $3d^2$ lines.*

5. Symmetric surfaces

We consider surfaces with many symmetries, since one can expect that such surfaces contain many lines. Indeed, if the surface contains a line then it contains the whole orbit, and if the symmetry group is big, hopefully this orbit has big length. To this purpose, we first take $G \subset \mathrm{PGL}(4,\mathbb{C})$ be a finite group of linear transformations acting on $\mathbb{P}_3$ and construct smooth G-invariant surfaces.

5.1. **Surfaces with cyclic symmetries.**

Denote by $\mathcal{U}_d$ the group of d-th roots of the unit. The group $\mathcal{U}_d \times \mathcal{U}_d$ acts on $\mathbb{C}[x,y,z,t]$ by $\mathrm{diag}(\xi,\xi,\mu,\mu)$ for $(\xi,\mu) \in \mathcal{U}_d \times \mathcal{U}_d$. The graded space of invariant polynomials decomposes as:

$$\mathbb{C}[x,y,z,t]^{\mathcal{U}_d \times \mathcal{U}_d} \cong \mathbb{C}[x,y]^{\mathcal{U}_d} \otimes \mathbb{C}[z,t]^{\mathcal{U}_d}.$$

Since $\mathbb{C}[x,y]_k^{\mathcal{U}_d} = 0$ for $d \nmid k$ and $\mathbb{C}[x,y]_k^{\mathcal{U}_d} = \mathbb{C}[x,y]_k$ otherwise, all invariant polynomials of degree d for the action of $\mathcal{U}_d \times \mathcal{U}_d$ are of the kind $\phi(x,y) - \psi(z,t)$ for ϕ and ψ homogeneous polynomials of degree d. These surfaces were studied in Section 3.

5.2. **Surfaces with polyhedral symmetries.**
We consider again surfaces of the kind $\phi(x,y) = \phi(z,t)$: we studied such surfaces and their configuration of lines in Section 3. We adopt here a different point of view. Let Γ be the group of isomorphisms of $\mathbb{P}_1$ permuting the zeros of ϕ in $\mathbb{P}_1$. Then ϕ is a projective invariant for the action of Γ on $\mathbb{C}^2$, *i.e.* $\phi(g(x,y)) = \lambda_g \phi(x,y)$ for $g \in \Gamma$ and $\lambda_g \in \mathbb{C}^*$. This implies that the surface $F(x,y,z,t) = \phi(x,y) - \phi(z,t)$ is invariant for the diagonal action of Γ given by $g(x,y,z,t) = (g(x,y), g(z,t))$. Its number of lines is given by Theorem 3.1. By using this observation, we can find easily equations for surfaces of this kind with the symmetries of the groups $\mathcal{T}, \mathcal{O}, \mathcal{I}$. The projective invariants are computed for example in Klein [6, I.2,§11-12-13]:

(1) A surface of degree six with octahedral symmetries and 180 lines:
$$\phi(x,y) = xy(x^4 - y^4).$$
(2) A surface of degree eight with octahedral symmetries and 256 lines:
$$\phi(x,y) = x^8 + 14x^4y^4 + y^8.$$
(3) A surface of degree twelve with octahedral symmetries and 432 lines:
$$\phi(x,y) = x^{12} - 33x^8y^4 - 33x^4y^8 + y^{12}.$$
(4) A surface of degree twelve with icosahedral symmetries and 864 lines:
$$\phi(x,y) = xy(x^{10} + 11x^5y^5 - y^{10}).$$
(5) A surface of degree 20 with icosahedral symmetries and 1600 lines:
$$\phi(x,y) = -(x^{20} + y^{20}) + 228(x^{15}y^5 - x^5y^{15}) - 494x^{10}y^{10}.$$
(6) A surface of degree 30 with icosahedral symmetries and 2700 lines:
$$\phi(x,y) = (x^{30} + y^{30}) + 522(x^{25}y^5 - x^5y^{25}) - 10005(x^{20}y^{10} + x^{10}y^{20}).$$

5.3. **Surfaces with bipolyhedral symmetries.**
First recall the construction of the bipolyhedral groups. Start from the exact sequence:
$$0 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{SU}(2) \xrightarrow{\phi} \mathrm{SO}(3,\mathbb{R}) \longrightarrow 0.$$
For any polyhedral group $G \subset \mathrm{SO}(3,\mathbb{R})$, the inverse image $\widetilde{G} := \phi^{-1}G$ is called a *binary polyhedral group*. Now consider the exact sequence:
$$0 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{SU}(2) \times \mathrm{SU}(2) \xrightarrow{\sigma} \mathrm{SO}(4,\mathbb{R}) \longrightarrow 0.$$
For $\widetilde{G}$ a binary polyhedral group, the direct image $\sigma(\widetilde{G} \times \widetilde{G}) \subset \mathrm{SO}(4,\mathbb{R})$ is called a *bipolyhedral group*. We shall make use of the following particular groups:

- $G_6 = \sigma(\widetilde{\mathcal{T}} \times \widetilde{\mathcal{T}})$ of order 288;
- $G_8 = \sigma(\widetilde{\mathcal{O}} \times \widetilde{\mathcal{O}})$ of order 1152;
- $G_{12} = \sigma(\widetilde{\mathcal{I}} \times \widetilde{\mathcal{I}})$ of order 7200.

The polynomial invariants of these groups were studied by Sarti in [11]. First note that the quadratic form: $Q := x^2 + y^2 + z^2 + t^2$ is an invariant of the action of these groups.

Theorem 5.1 (Sarti [11, §4])**.** *For $d = 6, 8, 12$, there is a one-dimensional family of G_d-invariant surfaces of degree d. The equation of the family is $S_d + \lambda Q^{d/2} = 0$. The base locus of the family consists in $2d$ lines, d in each ruling of Q. The general member of each family is smooth and there are exactly five singular surfaces in each family.*

From this theorem immediately follows that each member of the family contains at least $2d$ lines.

• **The group G_8.** Denote by $\mathcal{S}_8$ the surface $S_8 = 0$ where:

$$\begin{aligned} S_8 =& x^8 + y^8 + z^8 + t^8 + 168x^2y^2z^2t^2 \\ &+ 14(x^4y^4 + x^4z^4 + x^4t^4 + y^4z^4 + y^4t^4 + z^4t^4). \end{aligned}$$

Proposition 5.2. *The surface $\mathcal{S}_8$ contains exactly 352 lines.*

Proof. The proof goes as follows: first we introduce Plücker coordinates for the lines in $\mathbb{P}_3$, then we compute explicitly all the lines contained in the surface.

•*Plücker coordinates.* Let $\mathbb{G}(1,3)$ be the Grassmannian of lines in $\mathbb{P}_3$, or equivalently of 2-planes in $\mathbb{C}^4$. Such a line L is given by a rank-two matrix:

$$\begin{pmatrix} a & e \\ b & f \\ c & g \\ d & h \end{pmatrix}.$$

The 2-minors (*Plücker coordinates*):

$$\begin{aligned} &p_{12} := af - be && p_{13} := ag - ce && p_{14} := ah - de \\ &p_{23} := bg - cf && p_{24} := bh - df && p_{34} := ch - dg \end{aligned}$$

are not simultaneously zero, and induce a regular map $\mathbb{G}(1,3) \longrightarrow \mathbb{P}_5$. This map is injective, and its image is the hypersurface $p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0$. In order to list once all lines with these coordinates, we inverse the Plücker embedding in the *Plücker stratification*:

(1)	(2)	(3)
$p_{12} = 1$	$p_{12} = 0, p_{13} = 1$	$p_{12} = 0, p_{13} = 0, p_{14} = 1$
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -p_{23} & p_{13} \\ -p_{24} & p_{14} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ p_{23} & 0 \\ 0 & 1 \\ -p_{34} & p_{14} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ p_{24} & 0 \\ p_{34} & 0 \\ 0 & 1 \end{pmatrix}$
(4)	(5)	(6)
$p_{12} = 0, p_{13} = 0$ $p_{14} = 0, p_{23} = 1$	$p_{12} = 0, p_{13} = 0, p_{14} = 0$ $p_{23} = 0, p_{24} = 1$	$p_{12} = 0, p_{13} = 0, p_{14} = 0$ $p_{23} = 0, p_{24} = 0, p_{34} = 1$
$\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ -p_{34} & p_{24} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ p_{34} & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$

• *Counting the lines.* The line L is contained in the surface $\mathcal{S}_8$ if and only if the function $(u, v) \mapsto S_8(ua + ve, ub + vf, uc + vg, ud + vh)$ is identically zero, or equivalently if all coefficients of this polynomial in u, v are zero. The conditions for the line to be contained in the surface is then given by a set of polynomial equations in a, b, c, d, e, f, g, h. In order to count the lines, we restrict the equations to each Plücker stratum and compute the solutions (this computation is not difficult if left to SINGULAR [4]).

(1) *The stratum* $p_{12} = 1$. Set $p_{23} = c$, $p_{24} = d$, $p_{13} = g$, $p_{14} = h$. The equations for such a line to be contained in the surface are:

$$c^7 g + d^7 h + 7c^3 g + 7d^3 h + 7c^4 d^3 h + 7c^3 g d^4 = 0$$

$$c^6 g^2 + d^6 h^2 + 3c^4 d^2 h^2 + 8c^3 g d^3 h + 3c^2 g^2 d^4 + 6c^2 d^2 + 3c^2 g^2 + 3d^2 h^2 = 0$$

$$c^5 g^3 + d^5 h^3 + c^4 d h^3 + c g^3 d^4 + 6c^3 g d^2 h^2 + 6c^2 g^2 d^3 h + c g^3 + d h^3 + 6c^2 d h + 6c g d^2 = 0$$

$$1 + g^4 + 5c^4 g^4 + 5d^4 h^4 + c^4 + d^4 + c^4 h^4 + g^4 d^4 + 16c^3 g d h^3 + 36c^2 g^2 d^2 h^2 + 16c g^3 d^3 h + h^4 + 12c^2 h^2 + 12g^2 d^2 + 48c g d h = 0$$

$$c^3 g + d^3 h + c^3 g h^4 + c^3 g^5 + d^3 h^5 + 6c^2 g^2 d h^3 + g^4 d^3 h + 6c g^3 d^2 h^2 + 6c g h^2 + 6g^2 d h = 0$$

$$3c^2 g^2 + 3d^2 h^2 + 3c^2 g^2 h^4 + 3g^4 d^2 h^2 + c^2 g^6 + d^2 h^6 + 8c g^3 d h^3 + 6g^2 h^2 = 0$$

$$c g^7 + d h^7 + 7c g^3 + 7d h^3 + 7c g^3 h^4 + 7g^4 d h^3 = 0$$

$$1 + g^8 + h^8 + 14g^4 + 14h^4 + 14g^4 h^4 = 0$$

After simplification of the ideal with SINGULAR (that we do not reproduce here), the solutions give 320 lines of the kind $z = cx + gy, t = dx + hy$.

(2) *The stratum* $p_{12} = 0$, $p_{13} = 1$. Set $p_{23} = b$, $-p_{34} = d$, $p_{14} = h$. The equations for such a line to be contained in the surface are (after simplification):

$$d = 0$$

$$b^4 h^2 - b^2 h^4 - b^2 + h^2 = 0$$

$$b^6 - h^6 + 13b^2 - 13h^2 = 0$$

$$h^8 + 14h^4 + 1 = 0$$

$$b^2 h^6 + b^4 + 13b^2 h^2 + 1 = 0$$

The solutions give 32 lines of the kind $y = bz, t = hx$, since there are eight possible values for h, and for each of them there are four values of b.

An easy computation shows that the other strata contain no line, so there are exactly 352 lines on the surface. $\square$

Remark 5.3. *To our knowledge, this is the best example so far of an octic surface with many lines. This improves widely the bound* 256 *of Caporaso-Harris-Mazur* [3].

• **The group** G_6**.** We take:

$$S_6 = x^6 + y^6 + z^6 + t^6 + 15(x^2 y^2 z^2 + x^2 y^2 t^2 + x^2 z^2 t^2 + y^2 z^2 t^2).$$

Proposition 5.4. *The surface* $8S_6 - 5Q^3 = 0$ *contains exactly* 132 *lines.*

There are surfaces with more lines (see §3.4), but this shows the existence of a surface with 132 lines. This result can be shown in a similar way as in the G_8 case.

6. A uniform bound

As we mentioned before, the uniform bound $(d-2)(11d-6)$ of Segre is too big already in degree four. We propose here another lower uniform bound, which interpolates all maximal numbers of lines known so far, including the octic of Section 5. Although there is no reason for this bound to be maximal, it seems reasonable to expect that an effective construction of a surface with this number of lines is possible in all degrees.
Let $\mathcal{S}$ be a smooth surface of degree $d \geq 3$ and C a line contained in $\mathcal{S}$. Let $|H|$ be the linear system of planes H passing through C. Then $H \cap \mathcal{S} = C \cup \Gamma$ where Γ is a curve of degree $d-1$. The system $|\Gamma|$ is described by Segre in [12]: it is base-point free and any curve Γ does not contain C as a component. Then:

Proposition 6.1 (Segre [12])**.** *Either each curve Γ intersects C in $d-1$ points which are inflections for Γ, or the points of C each of which is an inflection for a curve Γ are $8d-14$ in number. In particular, in this case C is met by no more than $8d-14$ lines lying on $\mathcal{S}$.*

Following Segre, C is called a line of the *second kind* if it intersects each Γ in $d-1$ inflections. A generalization of Segre's argument in [12, §9] gives the following result:

Proposition 6.2. *Assume that $\mathcal{S}$ contains d coplanar lines, none of them of the second kind. Then $\mathcal{S}$ contains at most $d(7d-12)$ lines.*

Proof. Let P be the plane containing these d distinct lines. Then they are the complete intersection of P with $\mathcal{S}$. Hence each other line on $\mathcal{S}$ must intersect P in some of the lines. By Proposition 6.1, each of the d lines in the plane meets at most $8d-14$ lines, so $8d-14-(d-1)$ lines not on the plane. The total number of lines is at most:

$$d + d(7d-13) = d(7d-12).$$

□

This bound takes the following values:

d	4	5	6	7	8	9	10	11	12	20
$7d^2-12d$	64	115	180	259	352	459	580	715	864	2560

Note that this bound matches perfectly with the maximal known examples in degrees $4, 6, 8, 12$.

7. Number of rational points on a plane curve

We give an application of our results to the *universal bound conjecture*, following Caporaso-Harris-Mazur [3]:
Universal bound conjecture. *Let $g \geq 2$ be an integer. There exists a number $N(g)$ such that for any number field K there are only finitely many smooth curves of genus g defined over K with more than $N(g)$ K-rational points.*
As mentioned in *loc.cit.* an interesting way to find a lower bound of $N(g)$, or of the limit:

$$\overline{N} := \limsup_{g\to\infty} \frac{N(g)}{g}$$

is to consider plane sections of surfaces with many lines. Indeed, over the common field K of definition of the surface and its lines, a generic plane section is a curve containing at least as many K-rational points as the number of lines. In particular, they show that $N(21) \geq 256$. Since we obtain an octic surface with 352 lines and a generic plane section of this surface is a smooth curve of genus 21, we get:

Corollary 7.1. $N(21) \geq 352$.

As we remarked in Section 6, it seems to be possible to construct surfaces with $d(7d-12)$ lines. This would improve the lower bound of $N(g)$ for many g's. In particular, this would improve the known estimate $\overline{N} \geq 8$ to $\overline{N} \geq 14$.

8. Sequences of skew-lines

A natural question related to the number of lines on a surface is the study of maximal sequences of pairwise disjoint lines on a smooth surface in $\mathbb{P}_3$. We recall the bound of Miyaoka and give some examples.

8.1. Upper bound for skew-lines.

The best upper bound known so far for the maximal length of a sequence of disjoint lines on a smooth surface of degree $d \geq 4$ in $\mathbb{P}_3$ is given by Miyaoka:

Theorem 8.1 (Miyaoka [7, §2.2]). *The maximal length of a sequence of skew-lines is $2d(d-2)$ for $d \geq 4$.*

For $d = 3$, each cubic surface contains a maximal sequence of 6 skew lines. This comes from the study of the configuration of the 27 lines (see for example [5, Theorem V.4.9] and references therein). For $d = 4$, Kummer surfaces contain a maximal sequence of 16 skew lines (see for example [8] and references therein) so the bound is optimal.
But for $d \geq 5$, it is not known if it is sharp.

8.2. On Miyaoka's bound.

We give a quick sketch of the argument of Miyaoka for the bound on the number of skew lines, following [7, §2 Examples 2.1,2.2].
Let X be a smooth surface of degree $d \geq 4$ in $\mathbb{P}_3$. Assume X contains r disjoint lines $D_1, \ldots, D_r$. By adjunction formula, they have self-intersection $-n = -(d-2)$. By contracting these lines one gets a surface Y with r isolated singular points which locally look like the quotient of $\mathbb{C}^2$ by a finite group of order n.
Write $K_X + \sum_{i=1}^{r} D_i = P + N'$ with:

$$P := K_X + \sum_{i=1}^{r} \frac{n-2}{n} D_i \quad \text{and} \quad N' := \sum_{i=1}^{r} \frac{n-2}{n} D_i.$$

This provides a Zariski decomposition in $\mathrm{Pic}(X) \otimes \mathbb{Q}$ of $K_X + \sum_{i=1}^{r} D_i$.
Set $\nu := 2 - 1/n$, by [7, Theorem 1.1], one has the inequality:

$$r\nu \leq c_2(X) - \frac{1}{3}P^2.$$

Using that $c_2(X) = d(d^2 - 4d + 6)$ and $K_X^2 = d(d-4)^2$ one gets $r \leq 2d(d-2)$.

8.3. **Examples.**
In [10], Rams considers the surfaces $x^{d-1}y + y^{d-1}z + z^{d-1}t + t^{d-1}x = 0$ and proves that they contain a family of $d(d-2)+2$ skew-lines for any d. In [9, Example 2.3], he also gives an example of a surface of degree five containing a sequence of 19 skew-lines. We generalize his result, improving the number of skew-lines to $d(d-2)+4$ in the case $d \geq 7$ and $\gcd(d, d-2) = 1$.
Consider the surface $\mathcal{R}_d : x^{d-1}y + xy^{d-1} + z^{d-1}t + zt^{d-1} = 0$. By our study in Section 3.1, this surface contains exactly $3d^2 - 4d$ lines if $d \neq 6$ and 180 lines for $d = 6$. We prove:

Proposition 8.2. *The surface* $\mathcal{R}_d$ *with* $\gcd(d, d-2) = 1$ *contains a sequence of* $d(d-2)+4$ *disjoint lines.*

Proof. Denote by ϵ, γ the primitive roots of the unit of degrees $d-2$ and d, and let $\eta := \epsilon^l \gamma^s$, with $0 \leq l \leq d-3$, $0 \leq s \leq d-1$. Since $\gcd(d, d-2) = 1$ we have $d(d-2)$ such η. Now consider the points

$$(0:1:0:-\eta^{d-1}), (-\eta:0:1:0)$$

then the line through the two points is

$$C_{l,s} : (-\eta\lambda : \mu : \lambda : -\eta^{d-1}\mu)$$

An easy computation shows that these lines are contained in $\mathcal{R}_d$ and are $d(d-2)$. This form a set of $d(d-2)+4$ skew lines together with the lines

$$\{x=0, z+\epsilon t=0\}, \{y=0, z+t=0\},$$
$$\{z=0, x+\epsilon y=0\}, \{t=0, x+y=0\}.$$

□

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On varieties that are uniruled by lines

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ON VARIETIES THAT ARE UNIRULED BY LINES

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Abstract. Using the $\sharp$-minimal model program of uniruled varieties we show that for any pair $(X, \mathcal{H})$ consisting of a reduced and irreducible variety X of dimension $k \geq 3$ and a globally generated big line bundle $\mathcal{H}$ on X with $d := \mathcal{H}^k$ and $n := h^0(X, \mathcal{H}) - 1$ such that $d < 2(n-k) - 4$, then X is uniruled of $\mathcal{H}$-degree one, except if $(k, d, n) = (3, 27, 19)$ and a $\sharp$-minimal model of $(X, \mathcal{H})$ is $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$. We also show that the bound is optimal for threefolds.

0. Introduction

It is well-known that an irreducible nondegenerate complex variety $X \subseteq \mathbb{P}^n$ of degree d satisfies $d \geq n - \dim X + 1$. Varieties for which equality is obtained are the well-known *varieties of minimal degree*, which are completely classified.

Varieties for which d is "small" compared to n have been the objects of intensive study throughout the years, see e.g. [Ha, Ba, F1, F2, F3, Is, Io, Ho, Re, M2]. One of the common features is that such varieties are covered by rational curves.

More generally one can study pairs $(X, \mathcal{H})$ where X is an irreducible k-dimensional variety (possibly with some additional assumptions on its singularities) and $\mathcal{H}$ a line bundle on X which is sufficiently "positive" (e.g. ample or (birationally) very ample or big and nef). Naturally we set $d := \mathcal{H}^k$ and $n := \dim |\mathcal{H}|$. The difference between d and n is measured by the Δ-genus: $\Delta(X, \mathcal{H}) := d + k - n - 1$, introduced by Fujita (cf. [F1] and [F2]), who in fact shows that $\Delta(X, \mathcal{H}) \geq 0$ for X smooth and $\mathcal{H}$ ample and that $\mathcal{H}$ is very ample if equality holds, so that the cases with $\Delta(X, \mathcal{H}) = 0$ are the varieties of minimal degree. The cases with $\Delta(X, \mathcal{H}) = 1$ have been classified by Fujita [F3, F4, F5] and Iskovskih [Is].

If $\mathcal{H}$ is globally generated we can consider the morphism $\varphi_{\mathcal{H}} : X \longrightarrow X' \subseteq \mathbb{P}^n$ defined by $|\mathcal{H}|$. One has $d = (\deg \varphi_{\mathcal{H}})(\deg X')$ and $\deg X' \geq n - k + 1$. If $d < 2(n-k) + 2$ the morphism $\varphi_{\mathcal{H}}$ is forced to be birational and $\deg X' = d$. Hence in the range $d < 2(n-k)+2$ studying nondegenerate degree d varieties in $\mathbb{P}^n$, or pairs $(X, \mathcal{H})$ with $\mathcal{H}$ globally generated and big, is equivalent. Moreover, as the property of being globally generated and big is preserved from $\mathcal{H}$ to $f^*\mathcal{H}$ under a resolution of singularities f, this approach is suitable also to study singular varieties.

The notion of being covered by rational curves is incorporated in the concept of a variety being *uniruled*: A variety is uniruled if through any point there passes a rational curve. With the notation above, $d < k(n-k) + 2$ is an optimal bound for uniruledness by [M2, Thm. A.3 and Exmpl. A.4].

In many ways uniruled varieties are the natural generalizations to higher dimensions of ruled surfaces. In the Mori program they play an important role, because - like in the case

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of ruled surfaces - these are the varieties for which the program does not yield a minimal model, but a Mori fiber space. Uniruled varieties can also be considered to be the natural generalizations to higher dimensions of surfaces of negative Kodaira dimension: in fact it is conjectured that a (smooth) variety is uniruled if and only if its Kodaira dimension is negative. The conjecture has been established for threefolds by Miyaoka [Mi].

With the evolution of a structure theory for higher dimensional varieties in the past decades, namely the Mori program, the geometry of rational curves on varieties has gained new importance. The main idea is to obtain information about varieties by studying the rational curves on them (cf. e.g. [Ko]).

To measure the "degree" of the rational curves which cover X we say in addition that X is uniruled of $\mathcal{H}$-degree at most m if the covering curves all satisfy $\Gamma \cdot \mathcal{H} \leq m$. Returning to the case where $\mathcal{H}$ is globally generated and the morphism $\varphi_{\mathcal{H}} : X \longrightarrow X' \subseteq \mathbb{P}^n$ is birational as above, we see that X is uniruled of $\mathcal{H}$-degree at most m if and only if X' is covered by rational curves of degrees $\leq m$.

For surfaces Xiao [Xi] and Reid [Re] independently found bounds on the uniruledness degree of $(X, \mathcal{H})$ depending on d and n. For instance they showed that an irreducible, nondegenerate surface $X \subseteq \mathbb{P}^n$ is uniruled by lines if $d < \frac{4}{3}(n-2)$, except when $n = 9$ and $(X, \mathcal{O}_X(1)) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$. The same result was obtained by Horowitz [Ho] using a different approach. In particular, it immediately follows (by taking surface sections and using that $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$ cannot be a hyperplane section of any threefold other than a cone) that an irreducible, nondegenerate k-dimensional variety $X \subseteq \mathbb{P}^n$ is uniruled by lines for $k \geq 3$ if $d < \frac{4}{3}(n-k)$. (Note that if one assumes X smooth, one gets the better bound $d < \frac{3}{2}(n-k-1)$, since X is ruled by planes or quadrics in this range by [Ho, Cor. p. 668].) However it is to be expected that this "naive" inductive procedure does not yield an optimal bound.

The purpose of this article is to obtain a bound for uniruledness degree one which is optimal for threefolds and independent of singularities. In fact we show:

Theorem 0.1. *Let $(X, \mathcal{H})$ be a pair consisting of a reduced and irreducible three-dimensional variety X and a globally generated big line bundle $\mathcal{H}$ on X. Set $d := \mathcal{H}^3$ and $n := h^0(X, \mathcal{H}) - 1$.*
If $d < 2n - 10$ then X is uniruled of $\mathcal{H}$-degree one, except when $(d, n) = (27, 19)$ and a $\sharp$-minimal model of $(X, \mathcal{H})$ is $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$.

(For the definition of a $\sharp$-minimal model we refer to Definition 1.4 below.)

The bound in Theorem 0.1 is sharp since there are pairs satisfying $d = 2n-10$ for infinitely many d and n, namely $(\mathbb{P}^2 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^2}(2) \boxtimes \mathcal{O}_{\mathbb{P}^1}(a))$ for $a \geq 2$ (cf. Example 2.5 below), which are not uniruled of $\mathcal{H}$-degree one.

Observe that in Remark 2.3 below we obtain a better bound than in Theorem 0.1 for $8 \leq n \leq 12$.

As a consequence of Theorem 0.1 we get the following result for higher dimensional varieties, which is probably far from being sharp:

Corollary 0.2. *Let $(X, \mathcal{H})$ be a pair consisting of a reduced and irreducible k-dimensional variety X, $k \geq 4$, and a globally generated big line bundle $\mathcal{H}$ on X. Set $d := \mathcal{H}^k$ and $n := h^0(X, \mathcal{H}) - 1$. If $d < 2(n-k) - 4$ then X is uniruled of $\mathcal{H}$-degree one.*

For those preferring the notion of Δ-genus, the condition $d < 2(n-k) - 4$ is equivalent to $\Delta(X, \mathcal{H}) < n - k - 5 = h^0(X, \mathcal{H}) - \dim X - 6$.

The above results have the following corollary for embedded varieties:

Corollary 0.3. *Let $X \subset \mathbb{P}^n$ be a nondegenerate reduced and irreducible variety of dimension $k \geq 3$ and degree d. If $d < 2(n-k) - 4$ then X is uniruled by lines, except when $(k, d, n) = (3, 27, 19)$ and a $\sharp$-minimal model of $(X, \mathcal{O}_X(1))$ is $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$.*

Note that the condition $d < 2(n-k) - 4$ implicitly requires $n \geq k + 6$ in the three results above.
To prove these results we use the $\sharp$-minimal model program of uniruled varieties introduced for surfaces by Reid in [Re] and developed for threefolds by Mella in [M2]. The main advantage of the $\sharp$-minimal model program is that one does not only work with birational modifications along the minimal model program but also uses a polarizing divisor. Under certain assumptions one manages to follow every step of the program on an effective divisor, i.e. a (smooth) surface in the case of threefolds.
Our method of proof uses the classification results in [M2] and borrows ideas from [Re]. The crucial point is a careful investigation of pairs $(X, \mathcal{H})$ such that the output of the $\sharp$-minimal model program is a particular type of Mori fiber space which we call a *terminal Veronese fibration* (see Definition 3.1 below): this is roughly speaking a terminal threefold marked by a line bundle with at most base points fibered over a smooth curve with general fibers being smooth Veronese surfaces (with respect to the marking line bundle) and having at most finitely many fibers being cones over a smooth quartic curve. We find a lower bound on the degree of such a threefold (in fact on every marked terminal threefold having a terminal Veronese fibration as a $\sharp$-minimal model) and on the number of degenerate fibers of the members of the marking linear system.
The precise statement, which we hope might be of independent interest, is the following:

Proposition 0.4. *Let $(X, \mathcal{H})$ be a three-dimensional terminal Veronese fibration (see Definition 3.1) over a smooth curve B and set $n := h^0(\mathcal{H}) - 1$ and $d := \mathcal{H}^3$. Then $d \geq 2n - 10$ and the general member of $|\mathcal{H}|$ is a smooth surface fibered over B with $\geq \frac{n-5}{2}$ fibers which are unions of two conics (with respect to $\mathcal{H}$) intersecting in one point (the other fibers are smooth quartics).*

Observe that both equalities are obtained by $(\mathbb{P}^2 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^2}(2) \boxtimes \mathcal{O}_{\mathbb{P}^1}(a))$, cf. Examples 2.5 and 3.4.
In Section 1 we set notation and give all central definitions. Moreover we introduce, after [M2], the $\sharp$-minimal models of pairs $(X, \mathcal{H})$ where X is a terminal, $\mathbb{Q}$-factorial threefold and $\mathcal{H} \in \operatorname{Pic} X$ such that the general element in $|\mathcal{H}|$ is a smooth surface of negative Kodaira dimension (Theorem 1.2) and obtain results that are essential for the rest of the paper in Lemmas 1.1 and 1.5.
In Section 2 we first obtain an "easy bound" on d such that a threefold is uniruled in degree one (Proposition 2.1) and then we show how to reduce the proofs of our main results Theorem 0.1 and its two corollaries to a result about uniruled threefolds having a terminal Veronese fibration as a $\sharp$-minimal model, namely Proposition 2.4.
The proofs of Proposition 2.4 and of Proposition 0.4 are then settled in Section 3.
Finally, in Section 4 we give some final remarks, including a slight improvement of a result in [M2] and of Theorem 0.1 and Corollary 0.2.

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1. $\sharp$-minimal models of uniruled threefolds

We work over the field of complex numbers.
A reduced and irreducible three-dimensional variety will be called a *threefold*, for short.
A k-dimensional projective variety X is called *uniruled* if there is a variety Y of dimension $k-1$ and a generically finite dominant rational map $p : Y \times \mathbb{P}^1 - - \to X$. In particular, such a variety is covered by rational curves (cf. [Ko, IV 1.4.4]).
If $\mathcal{H}$ is a nef line bundle on X and $m \in \mathbb{Q}$ we say that X is *uniruled of* $\mathcal{H}$*-degree at most* m if $\deg (p^*\mathcal{H})_{|\mathbb{P}^1\times\{y\}} \leq m$ for every $y \in Y$, or equivalently if there is a dense open subset $U \subseteq X$ such that every point in U is contained in a rational curve C with $C \cdot \mathcal{H} \leq m$ (cf. [Ko, IV 1.4.]). A consequence is that in fact *every* point in X is contained in a rational curve C with $C \cdot \mathcal{H} \leq m$ (cf. [Ko, IV 1.4.4]). In particular, if $X \subseteq \mathbb{P}^n$ we say that X is uniruled by lines if $m = 1$ with respect to $\mathcal{H} := \mathcal{O}_X(1)$.
For a pair $(X, \mathcal{H})$ where X is terminal $\mathbb{Q}$-factorial and $\mathcal{H}$ is a line bundle on X with $|\mathcal{H}| \neq \emptyset$, the *threshold* of the pair is defined as

$$\rho(X,\mathcal{H}) := \sup \{m \in \mathbb{Q} \, : \, rmK_X \text{ is Cartier and } |r(\mathcal{H}+mK_X)| \neq \emptyset \text{ for some } r \in \mathbb{Z}_{>0}\} \geq 0$$

(cf. [Re, (2.1)] and [M2, Def. 3.1]).
Moreover we set

$$d(X,\mathcal{H}) := \mathcal{H}^{\dim X} \text{ and } n(X,\mathcal{H}) := h^0(X,\mathcal{H}) - 1 = \dim |\mathcal{H}|. \tag{1}$$

In these terms the Δ-genus, introduced by Fujita (cf. [F1] and [F2]), is

$$\Delta(X,\mathcal{H}) := d(X,\mathcal{H}) + \dim X - n(X,\mathcal{H}) - 1 \tag{2}$$

and all the results in the paper can be equivalently formulated with the Δ-genus.
Recall that a surjective morphism $f : X \longrightarrow Y$ with connected fibers between normal varieties is called a *Mori fiber space* if $-K_X$ is f-ample, rk Pic $(X/Y) = 1$ and $\dim X > \dim Y$.
The following easy consequence of Clifford's theorem will be useful for our purposes:

Lemma 1.1. *Let $(X, \mathcal{H})$ be a pair with X a terminal $\mathbb{Q}$-factorial threefold and $\mathcal{H}$ a globally generated and big line bundle on X. Set $d := d(X, \mathcal{H})$ and $n := n(X, \mathcal{H})$.*
If $d < 2n - 4$, then:

- (i) *the general surface $S \in |\mathcal{H}|$ is smooth with negative Kodaira dimension. In particular X is uniruled and $\rho(X, \mathcal{H}) < 1$.*
- (ii) *for any smooth irreducible $S \in |\mathcal{H}|$ and for any irreducible curve $D \in |\mathcal{H}_{|S}|$ we have*

$$D \cdot K_S \leq d - 2n + 2. \tag{3}$$

Proof. The general element $S \in |\mathcal{H}|$ is a smooth irreducible surface by Bertini's theorem, as X has isolated singularities (cf. [M2, 2.3]).
Pick any irreducible curve $D \in |\mathcal{O}_S(\mathcal{H})|$. Then $\deg \mathcal{O}_D(\mathcal{H}) = \mathcal{H}^3 = d$ and from

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{H} \longrightarrow \mathcal{O}_S(\mathcal{H}) \longrightarrow 0 \tag{4}$$

and

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S(\mathcal{H}) \longrightarrow \mathcal{O}_D(\mathcal{H}) \longrightarrow 0 \tag{5}$$

we get

$$h^0(\mathcal{O}_D(\mathcal{H})) \geq h^0(\mathcal{O}_S(\mathcal{H})) - 1 \geq h^0(\mathcal{H}) - 2 = n - 1. \tag{6}$$

Hence

$$\deg \mathcal{O}_D(\mathcal{H}) - 2(h^0(\mathcal{O}_D(\mathcal{H})) - 1) \leq d - 2(n-2) < 0, \tag{7}$$

whence by Clifford's theorem on irreducible singular curves (see the appendix of [EKS]) we must have $h^1(\mathcal{O}_D(\mathcal{H})) = 0$, so that $\chi(\mathcal{O}_D(\mathcal{H})) = h^0(\mathcal{O}_D(\mathcal{H})) \geq n-1$. From (5) we get

$$\chi(\mathcal{O}_S(D)) - \chi(\mathcal{O}_S) = \chi(\mathcal{O}_D(\mathcal{H})) = h^0(\mathcal{O}_D(\mathcal{H})) \geq n - 1.$$

Combining with Riemann-Roch we get

$$\begin{aligned} D \cdot K_S &= D^2 - 2(\chi(\mathcal{O}_S(D)) - \chi(\mathcal{O}_S)) \\ &\leq d - 2n + 2 < -2, \end{aligned}$$

proving (ii) and showing that $\kappa(S) < 0$. (The latter fact also follows from [M2, Theorem A.3].) Now the fact that X is uniruled with $\rho(X, \mathcal{H}) < 1$ follows from [M2, Def. 5.1 and Lemma 5.2]. $\square$

In the following theorem we collect all the results of Mella [M2] that will be useful to us.

Theorem 1.2. *Let $(X, \mathcal{H})$ be a pair with X a terminal $\mathbb{Q}$-factorial threefold and $\mathcal{H}$ a globally generated and big line bundle on X, such that the general element in $|\mathcal{H}|$ is a smooth surface of negative Kodaira dimension.*
Then there exist a pair $(X^\sharp, \mathcal{H}^\sharp)$ and a birational map $\phi : X - - \to X^\sharp$ such that:

- (i) *$X^\sharp$ is terminal and $\mathbb{Q}$-factorial, $\mathcal{H}^\sharp \in$ Pic $X^\sharp$, $|\mathcal{H}^\sharp|$ has at most base points, and $\rho(X^\sharp, \mathcal{H}^\sharp) = \rho(X, \mathcal{H}) =: \rho$.*
- (ii) *ϕ is a finite composition of Mori extremal contractions and flips, and $\rho K_{X^\sharp} + \mathcal{H}^\sharp$ is $\mathbb{Q}$-nef;*
- (iii) *for any smooth irreducible $S \in |\mathcal{H}|$, $f := \phi_{|S}$ is a birational morphism, and $S^\sharp := f(S)$ is a smooth surface in $|\mathcal{H}^\sharp|$;*
- (iv) *if $X^\sharp$ is uniruled of $\mathcal{H}^\sharp$-degree at most m, then X is uniruled of $\mathcal{H}$-degree at most m;*
- (v) *$(X^\sharp, \mathcal{H}^\sharp)$ belongs to the following list:*
  - (I) *a $\mathbb{Q}$-Fano threefold with $K_{T^\sharp} \sim -(1/\rho)\mathcal{H}^\sharp$, belonging to Table 1 below.*
  - (II) *a bundle over a smooth curve with generic fiber $(F, \mathcal{H}^\sharp_{|F}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ and with at most finitely many fibers $(G, \mathcal{H}^\sharp_{|G}) \cong (\mathbf{S_4}, \mathcal{O}_{\mathbf{S_4}}(\mathbf{1}))$, where $\mathbf{S_4} \subset \mathbb{P}^5$ is the cone over the normal quartic curve. ($\rho = 2/3$)*
  - (III) *a quadric bundle with at most cA_1 singularities and $\mathcal{H}^\sharp_{|F} \sim \mathcal{O}_F(1)$ for every fiber F. ($\rho = 1/2$)*
  - (IV) *$(\mathbb{P}(\mathcal{E}), \mathcal{O}(1))$ where $\mathcal{E}$ is a rank 3 vector bundle over a smooth curve. ($\rho = 1/3$)*
  - (V) *$(\mathbb{P}(\mathcal{E}), \mathcal{O}(1))$ where $\mathcal{E}$ is a rank 2 vector bundle over a surface of negative Kodaira dimension. ($\rho = 1/2$)*

Proof. By [M2, Def. 5.1 and Lemma 5.2] we have $\rho(X, \mathcal{H}) < 1$. Now the existence of a pair $(X^\sharp, \mathcal{H}^\sharp)$ and a map satisfying conditions (i)-(iii) follows combining [M2, Thm. 3.2, Prop. 3.6 and Cor. 3.10] observing that it is implicitly shown in the proof of [M2, Thm 3.2] that $\rho(X^\sharp, \mathcal{H}^\sharp) = \rho(X, \mathcal{H})$.
Property (v) follows from [M2, Thm. 5.3 and Def. 5.1], noting that the values of ρ are explicitly given in each of the cases in the course of the proof of [M2, Thm. 5.3].

TABLE 1. $\mathbb{Q}$-Fano threefolds

Type	$X^\sharp$	general $S^\sharp \in \lvert\mathcal{H}^\sharp\rvert$	ρ	$-\frac{\rho}{\rho-1}$	$K^2_{S^\sharp}$	$d(X^\sharp,\mathcal{H}^\sharp)$	$n(X^\sharp,\mathcal{H}^\sharp)$
(a)	$\mathbb{P}(1,1,2,3)$	$H_6 \subset \mathbb{P}(1,1,2,3)$	6/7	6	1	36	22
(b)	$T_6 \subset \mathbb{P}(1,1,2,3,3)$	$T_6 \cap \{x_4=0\})$	3/4	3	1	9	7
(c)	$T_6 \subset \mathbb{P}(1,1,2,3,4)$	$T_6 \cap \{x_4=0\}$	4/5	4	1	16	11
(d)	$T_6 \subset \mathbb{P}(1,1,2,3,5)$	$T_6 \cap \{x_4=0\}$	5/6	5	1	25	16
(e)	$T_6 \subset \mathbb{P}(1,1,2,2,3)$	$T_6 \cap \{x_3=0\}$	2/3	2	1	4	4
(f)	$T_6 \subset \mathbb{P}(1,1,1,2,3)$	$T_6 \cap \{x_0=0\}$	1/2	1	1	1	2
(g)	$\mathbb{P}(1,1,1,2)$	$H_4 \subset \mathbb{P}(1,1,1,2)$	4/5	4	2	32	21
(h)	$T_4 \subset \mathbb{P}(1,1,1,2,2)$	$T_4 \cap \{x_4=0\}$	2/3	2	2	8	7
(i)	$T_4 \subset \mathbb{P}(1,1,1,2,3)$	$T_4 \cap \{x_4=0\}$	3/4	3	2	18	13
(j)	$T_4 \subset \mathbb{P}(1,1,1,1,2)$	$T_4 \cap \{x_0=0\}$	1/2	1	2	2	3
(k)	$\mathbb{P}^3$	$H_3 \subset \mathbb{P}^3$	3/4	3	3	27	19
(l)	$T_3 \subset \mathbb{P}(1,1,1,1,2)$	$T_3 \cap \{x_4=0\}$	2/3	2	3	12	10
(m)	$T_3 \subset \mathbb{P}^4$	$T_3 \cap \{x_0=0\}$	1/2	1	3	3	4
(n)	$T_2 \subset \mathbb{P}^4$	$H_{2,2} \subset T_2$	2/3	2	4	16	13
(o)	$T_{2,2} \subset \mathbb{P}^5$	$T_{2,2} \cap \{x_0=0\}$	1/2	1	4	4	5
(p)	$\mathbb{P}^6 \cap \mathbb{G}(1,4)$	$\mathbb{P}^6 \cap \mathbb{G}(1,4) \cap \{x_0=0\}$	1/2	1	5	5	6
(q)	$T_2 \subset \mathbb{P}^4$	$T_2 \cap \{x_0=0\} \simeq \mathbb{P}^1 \times \mathbb{P}^1$	1/3	1/2	8	2	4
(r)	$\mathbb{P}^3$	$\mathbb{P}^1 \times \mathbb{P}^1 \simeq H_2 \subset \mathbb{P}^3$	1/2	1	8	8	9
(s)	$\mathbb{P}^3$	$\{x_0=0\} \simeq \mathbb{P}^2 \subset \mathbb{P}^3$	1/4	1/3	9	1	3
(t)	$\mathbb{P}(1,1,1,2)$	$\{x_3=0\} \simeq \mathbb{P}^2 \subset \mathbb{P}(1,1,1,2)$	2/5	2/3	9	4	6

We have left to prove (iv). By assumption $X^\sharp$ is covered by a family of rational curves $\{\Gamma\}$ such that $\Gamma \cdot \mathcal{H}^\sharp \leq m$. The strict transform $\tilde{\Gamma}$ on X of each such Γ then satisfies $\tilde{\Gamma} \cdot S \leq m$ by [M2, Lemma 3.15]. □

In the cases (I) the general $S^\sharp \in |\mathcal{H}^\sharp|$ is a smooth del Pezzo surface and $\mathcal{O}_{S^\sharp}(\mathcal{H}^\sharp) \simeq \frac{\rho}{\rho-1} K_{S^\sharp}$. A list of such threefolds (with corresponding values for ρ) is given in [CF]. Moreover one can easily calculate $d(T^\sharp,\mathcal{H}^\sharp)$ and $n(T^\sharp,\mathcal{H}^\sharp)$. Indeed

$$d(X^\sharp,\mathcal{H}^\sharp) := (\mathcal{H}^\sharp)^3 = (\mathcal{O}_{S^\sharp}(\mathcal{H}^\sharp))^2 = \frac{\rho^2}{(\rho-1)^2} K^2_{S^\sharp}, \tag{8}$$

and by Riemann-Roch

$$n(X^\sharp,\mathcal{H}^\sharp) := h^0(\mathcal{H}^\sharp) - 1 = h^0(\mathcal{O}_{S^\sharp}(\mathcal{H}^\sharp)) = \frac{\rho}{2(\rho-1)^2} K^2_{S^\sharp} + 1. \tag{9}$$

In Table 1 we list all the cases (see [CF, p. 81]). In the table $\mathbb{P}(w_1,\dots,w_n)$ denotes the weighted projective space with weight w_i at the coordinate x_i. The hyperplane given by x_i is denoted $\{x_i=0\}$. Moreover T_a (resp. $T_{a,b}$) denotes a hypersurface of degree a (resp. a complete intersection of two hypersurfaces of degrees a and b) and similarly for H_a and $H_{a,b}$. The variety $\mathbb{G}(1,4)$ is the Grassmannian parameterizing lines in $\mathbb{P}^4$, embedded in $\mathbb{P}^9$ by the Plücker embedding.

Definition 1.3. Following [M2, Def. 3.3] we will call $(X^\sharp,\mathcal{H}^\sharp)$ a $\sharp$*-minimal model* of the pair $(X,\mathcal{H})$. In particular, by Lemma 1.1, it exists when $d(X,\mathcal{H}) < 2n(X,\mathcal{H}) - 4$.

Note that a $\sharp$-minimal model exists for any $(X,\mathcal{H})$ with X a terminal $\mathbb{Q}$-factorial uniruled threefold and $\mathcal{H}$ nef with $h^0(n\mathcal{H}) > 1$ for some $n > 0$ by [M2, Thm. 3.2], but it will in general not have all the nice properties (i)-(v) in Theorem 1.2 above. We will not need

the $\sharp$-minimal model in complete generality, but only in the version stated in Theorem 1.2 above.
The following explains the terminology used in Theorem 0.1 and Corollary 0.3:

Definition 1.4. For any pair $(X, \mathcal{H})$ consisting of a threefold X and a big and globally generated line bundle $\mathcal{H}$ on X, with $d(X, \mathcal{H}) < 2n(X, \mathcal{H}) - 4$, we will by a *$\sharp$-minimal model of* $(X, \mathcal{H})$ mean a $\sharp$-minimal model of $(\tilde{X}, f^*\mathcal{H})$, where $f : \tilde{X} \longrightarrow X$ is a minimal resolution of singularities. (Observe that $d(\tilde{X}, f^*\mathcal{H}) = d(X, \mathcal{H})$ and $n(\tilde{X}, f^*\mathcal{H}) \geq n(X, \mathcal{H})$, so a $\sharp$-minimal model exists and satisfies the properties (i)-(v) of Theorem 1.2.)

Lemma 1.5. *With the same notation and assumptions as in Theorem 1.2, let S, $S^\sharp$ and f be as in (iii) and set $D := \mathcal{O}_S(\mathcal{H})$.*

(a) *We have $n(X^\sharp, \mathcal{H}^\sharp) \geq n(X, \mathcal{H})$ and $d(X^\sharp, \mathcal{H}^\sharp) \geq d(X, \mathcal{H})$. In particular $\mathcal{H}^\sharp$ is big and nef.*

(b) *Let l be the total number of irreducible curves contracted by f. If $\rho \geq 1/3$, then*

$$(D + \frac{\rho}{1-\rho} K_S)^2 \geq -l(\frac{\rho}{1-\rho})^2. \tag{10}$$

(c) *If $(X^\sharp, \mathcal{H}^\sharp)$ is of type (I) in Theorem 1.2(v), then*

$$d(X, \mathcal{H}) - n(X, \mathcal{H}) + 1 = \frac{\rho(2\rho - 1)}{2(\rho - 1)^2} K^2_{S^\sharp}. \tag{11}$$

Proof. We first observe that $n(X^\sharp, \mathcal{H}^\sharp) \geq n(X, \mathcal{H})$ as $S^\sharp = \phi_* S$.
We have a commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\phi"] & X^\sharp \\ S \arrow[u, hook] \arrow[r, "f"] & S^\sharp \arrow[u, hook] \end{tikzcd}$$

where $f := \phi_{|S}$ is well defined and birational by Theorem 1.2(iii). As observed in [M2, Prop. 3.6] one can describe each step in the $\sharp$-minimal model program in a neighborhood of S. More precisely, set $X_0 := X$, $S_0 := S$, $X_m := X^\sharp$ and $S_m = S^\sharp := \phi_* S$. Denote by $\phi_i : X_{i-1} - - \rightarrow X_i$ for $i = 1, \ldots, m$ each birational modification in the $\sharp$-minimal model program relative to $(X, \mathcal{H})$ and define inductively $S_i := \phi_* S_{i-1}$. Then each S_i is smooth, and setting $f_i := \phi_{i|S_i}$ we can factorize f as:

$$S \xrightarrow{f_1} S_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{m-1}} S_{m-1} \xrightarrow{f_m} S_m = S^\sharp,$$

where each f_i contracts l_i disjoint (-1)-curves $E^i_1, \ldots, E^i_{l_i}$ with $l_i \geq 0$ by [M2, Prop. 3.6]. The total number of contracted curves is $l = \sum_{i=1}^m l_i$. We set $D_i := \mathcal{O}_{S_i}(S_i)$ and $D^\sharp := \mathcal{O}_{S^\sharp}(S^\sharp)$.
If ϕ_i is a flip then S_i is disjoint from the flipping curves by [M2, Claim 3.7], so that f_i is an isomorphism.
If ϕ_i contracts a divisor onto a curve then it is shown in [M2, Case 3.8] that the fiber F_i of ϕ_i satisfies $S_i \cdot F_i = 0$, whence $D_i \cdot F_i = 0$, which means that all E^i_j satisfy $E^i_j \cdot D_i = 0$.
If ϕ_i contracts a divisor onto a point then it is shown in [M2, Case 3.9] that f_i is a contraction of a single (-1)-curve $E_i = E^i_1$ which satisfies $E_i \cdot D_i = 1$.

In other words, for every i we have three possibilities:

$$\begin{array}{lll} l_i = 0; & & \text{or} \\ l_i > 0 & \text{and} & E^i_j \cdot D_i = 0 \text{ for all } j \in \{1, \ldots, l_i\}; \text{ or} \\ l_i = 1 & \text{and} & E^i_1 \cdot D_i = 1. \end{array} \tag{12}$$

Now denote by L^i_j the total transform of E^i_j on S. Then $(L^i_j)^2 = -1$ and $L^i_j \cdot L^{i'}_{j'} = 0$ for $(i,j) \neq (i',j')$. We have

$$K_S = f^*K_{S^\sharp} + \sum L^i_j \tag{13}$$

and by (12),

$$D = f^*D^\sharp - \sum \mu^i_j L^i_j \text{ with } \mu^i_j \in \{0,1\}. \tag{14}$$

In particular

$$d(X,\mathcal{H}) = D^2 = (D^\sharp)^2 - \sum (\mu^i_j)^2 = d(X^\sharp,\mathcal{H}^\sharp) - \sum \mu^i_j \leq d(X^\sharp,\mathcal{H}^\sharp), \tag{15}$$

finishing the proof of (a).
From (13) and (14) we get

$$D + \frac{\rho}{1-\rho}K_S = f^*(D^\sharp + \frac{\rho}{1-\rho}K_{S^\sharp}) + \sum (\frac{\rho}{1-\rho} - \mu^i_j)L^i_j,$$

and since there are l terms in the sum we get

$$(D + \frac{\rho}{1-\rho}K_S)^2 = (D^\sharp + \frac{\rho}{1-\rho}K_{S^\sharp})^2 - l(\frac{\rho}{1-\rho})^2 + \sum \mu^i_j(\frac{2\rho}{1-\rho} - \mu^i_j). \tag{16}$$

By definition and invariance of ρ (cf. Theorem 1.2(i)) we have that $\rho K_{T^\sharp} + \mathcal{H}^\sharp$ is $\mathbb{Q}$-effective. From Theorem 1.2(ii) we have that it is also $\mathbb{Q}$-nef, whence its restriction to $S^\sharp$ is also $\mathbb{Q}$-effective and $\mathbb{Q}$-nef. Since $S^\sharp$ is Cartier we get by adjunction that $(\rho K_{T^\sharp} + \mathcal{H}^\sharp)_{|S^\sharp} \simeq (1-\rho)(\frac{\rho}{1-\rho}K_{S^\sharp} + D^\sharp)$, whence by $\mathbb{Q}$-nefness

$$(D^\sharp + \frac{\rho}{1-\rho}K_{S^\sharp})^2 \geq 0. \tag{17}$$

Moreover the assumption $\rho \geq \frac{1}{3}$ is equivalent to $\frac{\rho}{1-\rho} \geq \frac{1}{2}$, whence

$$\sum \mu^i_j(\frac{2\rho}{1-\rho} - \mu^i_j) \geq \mu^i_j(1 - \mu^i_j) \geq 0. \tag{18}$$

Now (10) in (b) follows combining (16)-(18)
We have left to prove (c). Since $S^\sharp$ is a smooth del Pezzo surface, we have $h^1(\mathcal{O}_S) = h^1(\mathcal{O}_{S^\sharp}) = 0$. It is then easily seen by the proof of Lemma 1.1 that equality holds in (3) (note that we have $h^1(\mathcal{O}_X) \leq h^1(\mathcal{O}_X(-\mathcal{H})) + h^1(\mathcal{O}_S) = 0$ by Kawamata-Viehweg vanishing). Using (13), (14) and (15) we therefore get, for $D \in |\mathcal{O}_S(\mathcal{H})|$:

$$\begin{aligned} d(X,\mathcal{H}) - 2n(X,\mathcal{H}) + 2 &= D \cdot K_S = (f^*D^\sharp - \sum \mu^i_j L^i_j) \cdot (f^*K_{S^\sharp} + \sum L^i_j) \\ &= D^\sharp \cdot K_{S^\sharp} + \sum \mu^i_j = \frac{\rho}{\rho - 1}K^2_{S^\sharp} + \sum \mu^i_j \\ &= \frac{\rho}{\rho-1}K^2_{S^\sharp} + d(X^\sharp,\mathcal{H}^\sharp) - d(X,\mathcal{H}). \end{aligned}$$

Now using (8) we obtain

$$2d(X,\mathcal{H}) - 2n(X,\mathcal{H}) + 2 = \frac{\rho(2\rho-1)}{(\rho-1)^2}K^2_{S^\sharp},$$

proving (c). $\square$

2. Bounds for uniruledness degree one

As a "warming up" before proceeding with the proofs of the main results we give the proof of the following bound.

Proposition 2.1. *Let $(X,\mathcal{H})$ be a pair consisting of a terminal $\mathbb{Q}$-factorial threefold X and a globally generated and big line bundle $\mathcal{H}$ on X. Set $d := d(X,\mathcal{H})$ and $n := n(X,\mathcal{H})$. If $n \geq 4$, $d < \frac{4}{3}n - \frac{4}{3}$ and $d \neq n-1$ for $n \leq 9$, then X is uniruled of $\mathcal{H}$-degree one.*

Proof. Since $n \geq 4$ we have $2n - 4 \geq \frac{4}{3}n - \frac{4}{3}$, whence $d < 2n-4$, so by Lemma 1.1(i) the general $S \in |\mathcal{H}|$ is smooth of negative Kodaira dimension.
Moreover for any irreducible $D \in |\mathcal{H}_{|S}|$ we have, by Lemma 1.1(ii),

$$\begin{aligned} D\cdot\Big(\frac{3}{2}\mathcal{H}+K_X\Big) &= D\cdot(\mathcal{H}+K_X)+\frac{1}{2}D\cdot\mathcal{H} = D\cdot K_S+\frac{1}{2}d \\ &\leq d-2n+2+\frac{1}{2}d = \frac{3}{2}d-2n+2 \\ &< \frac{3}{2}\Big(\frac{4}{3}n-\frac{4}{3}\Big)-2n+2 = 0, \end{aligned}$$

whence $\rho(X,\mathcal{H}) < 2/3$.
It follows that the $\sharp$-minimal model $(X^\sharp,\mathcal{H}^\sharp)$ is in the list of Theorem 1.2(v) and moreover it cannot be as in (II) since $\rho(X,\mathcal{H}) = 2/3$ in this case. In the cases (III)-(V) one immediately sees that $(X^\sharp,\mathcal{H}^\sharp)$ is uniruled of $\mathcal{H}^\sharp$-degree one, whence $(X,\mathcal{H})$ is also uniruled of $\mathcal{H}$-degree one by Theorem 1.2(iv).
We have $n(T^\sharp,\mathcal{H}^\sharp) \geq n \geq 4$ by Lemma 1.5(a), and by using Table 1 we see that the cases in (I) where $n(X^\sharp,\mathcal{H}^\sharp) \geq 4$ and $\rho < 2/3$ are the cases (m), (o), (p), (q), (r) and (t). Among these all but (r) are clearly uniruled of $\mathcal{H}^\sharp$-degree one.
By (11) and Table 1 we have $d-n+1 = 0$ in case (r) and by Lemma 1.5(a) we have $n \leq n(X^\sharp,\mathcal{H}^\sharp) = 9$. $\square$

Corollary 2.2. *Let $(X,\mathcal{H})$ be a pair consisting of a threefold X and a globally generated and big line bundle $\mathcal{H}$ on X. Set $d := d(X,\mathcal{H})$ and $n := n(X,\mathcal{H})$.
If $d < \frac{4}{3}n - \frac{4}{3}$, $d \neq n$ when $5 \leq n \leq 8$ and $d \neq n-1$ for $n \leq 9$, then X is uniruled of $\mathcal{H}$-degree one.*

Proof. Let $\pi : \tilde{X} \longrightarrow X$ be a resolution of the singularities of X. Then $\pi^*\mathcal{H}$ is globally generated and big with $d(\tilde{X},\pi^*\mathcal{H}) = \mathcal{H}^3 = d$ and $n(\tilde{X},\pi^*\mathcal{H}) = \dim|\pi^*\mathcal{H}| \geq n$ and we can apply Proposition 2.1. The additional cases $d = n$ for $5 \leq n \leq 8$ occur since equality does not need to occur in $n(\tilde{X},\pi^*\mathcal{H}) \geq n$. $\square$

Remark 2.3. We note that the last corollary improves Theorem 0.1 for $n \leq 12$. Moreover, the cases $n = 3,4$ are trivial, as are the cases $n = 5,6,7$, since then $\varphi_{\mathcal{H}}(X) \subseteq \mathbb{P}^n$ has minimal degree. Hence the relevant statement, combining Theorem 0.1 and Corollary 2.2 is: X is uniruled of $\mathcal{H}$-degree one in the following cases:

- $n = 8$ and $d = 6$ or 9;
- $n = 9$ and $d = 7$, 9 or 10;

- $n = 10$ and $d \leq 11$;
- $n = 11$ or 12 and $d \leq n + 2$;
- $n \geq 13$ and $d \leq 2n - 11$, $(d, n) \neq (27, 19)$.

Now we give the main ideas and the strategy of the proof of Theorem 0.1. The main result we will need to prove is the following:

Proposition 2.4. *Let $(X, \mathcal{H})$ be a pair consisting of a terminal $\mathbb{Q}$-factorial threefold X and a globally generated, big line bundle $\mathcal{H}$ on X. Set $d := d(X, \mathcal{H})$ and $n := n(X, \mathcal{H})$. If a $\sharp$-minimal model of $(X, \mathcal{H})$ is of type (II) in Theorem 1.2(v), then $d \geq 2n - 10$.*

The proof of this result will be given in Section 3 below, after a careful study of the threefolds of type (II) in Theorem 1.2(v). We will now give the proofs of Theorem 0.1 and Corollaries 0.2 and 0.3 assuming Proposition 2.4.

Proof of Theorem 0.1. Let X be a reduced and irreducible 3-dimensional variety and $\mathcal{H}$ a globally generated big line bundle on X. Set $d := \mathcal{H}^3$ and $n := h^0(X, \mathcal{H}) - 1$ and assume $d < 2n - 10$.
Let $\pi : \tilde{X} \longrightarrow X$ be a resolution of the singularities of X. Then $\pi^*\mathcal{H}$ is globally generated and big with $d(\tilde{X}, \pi^*\mathcal{H}) = \mathcal{H}^3 = d$ and $n(\tilde{X}, \pi^*\mathcal{H}) = \dim |\pi^*\mathcal{H}| \geq n$. Since $(d, n) = (27, 19)$ satisfies $d = 2n - 11$ we can reduce to the case where X is smooth. Therefore we assume X is smooth.
By Lemma 1.1(i), any $\sharp$-minimal model $(X^\sharp, \mathcal{H}^\sharp)$ of $(X, \mathcal{H})$ is in the list of Theorem 1.2(v). Moreover, by Proposition 2.4, it cannot be of type (II).
We easily see that the cases (III)-(V) are uniruled of $\mathcal{H}^\sharp$-degree one. In the cases (I) we have, by Lemma 1.5,

$$\frac{\rho(2\rho - 1)}{2(\rho - 1)^2} K^2_{S^\sharp} = d - n + 1 \leq n - 10 \leq n(X^\sharp, \mathcal{H}^\sharp) - 10. \tag{19}$$

By checking Table 1 one finds that we can only be in case (k), with equalities all the way in (19). Hence $n = n(X^\sharp, \mathcal{H}^\sharp) = 19$ and $d = 2n - 11 = 27$. Now the result follows from Theorem 1.2(iv). □

Proof of Corollary 0.2. Let X be a reduced and irreducible variety of dimension $k \geq 4$ and $\mathcal{H}$ a globally generated big line bundle on X with $d := \mathcal{H}^k$ and $n := h^0(X, \mathcal{H}) - 1$.
As just mentioned in the proof of Theorem 0.1 we can assume X is smooth.
Setting $X_k := X$ and $\mathcal{H}_k := \mathcal{H}$, we recursively choose general smooth "hyperplane sections" $X_{i-1} \in |\mathcal{H}_i|$ and define $\mathcal{H}_{i-1} := \mathcal{H}_i \otimes \mathcal{O}_{X_{i-1}}$, for $2 \leq i \leq k$. (Note that $\dim X_i = i$ and $\mathcal{H}_i$ is a line bundle on X_i.)
Let $n_3 := h^0(\mathcal{H}_3) - 1$. Then from the exact sequence

$$0 \longrightarrow \mathcal{O}_{X_i} \longrightarrow \mathcal{H}_i \longrightarrow \mathcal{H}_{i-1} \longrightarrow 0 \tag{20}$$

we have

$$n_3 \geq n - (k - 3) = n - k + 3. \tag{21}$$

Together with the condition $d < 2(n - k) - 4$ this implies $d < 2n_3 - 10$ and it follows from Theorem 0.1 that either $(X_3, \mathcal{H}_3)$ is uniruled of degree one or $(d, n_3) = (27, 19)$ and $(X_3^\sharp, \mathcal{H}_3^\sharp)$ is $(\mathbb{P}^3, \mathcal{O}(3))$.
In the second case we have equality in (21), i.e.

$$19 = n_3 = h^0(\mathcal{H}_3) - 1 = h^0(\mathcal{H}) - (k - 3) - 1 = n - k + 3. \tag{22}$$

Denote by $\phi : X_3 - - \to X_3^\sharp = \mathbb{P}^3$ the birational map of the $\sharp$-minimal model program. By Theorem 1.2(iii) its restriction f to $S := X_2$ is a birational morphism onto a smooth surface $S^\sharp \in |\mathcal{O}_{\mathbb{P}^3}(3)|$.
We have

$$19 = h^0(\mathcal{O}_{S^\sharp}(3)) \geq h^0(\mathcal{H}_2) \geq h^0(\mathcal{H}_3) - 1 = 19$$

by Lemma 1.5(a) and (22), whence $|\mathcal{O}_{S^\sharp}(3)| = f_*|\mathcal{H}_2|$ and this can only be base point free if every curve E contracted by f satisfies $E \cdot \mathcal{H}_2 = 0$. Denoting by $\varphi_{\mathcal{H}_2}$ and $\varphi_{\mathcal{O}_{S^\sharp}(3)}$ the morphisms defined by $|\mathcal{H}_2|$ and $|\mathcal{O}_{S^\sharp}(3)|$ respectively, this implies that $\varphi_{\mathcal{O}_{S^\sharp}(3)} \circ f = \varphi_{\mathcal{H}_2}$, in other words $S' := \varphi_{\mathcal{H}_2}(S) = \varphi_{\mathcal{O}_{S^\sharp}(3)}(S^\sharp) \simeq S^\sharp \subseteq \mathbb{P}^{18}$. Moreover, by (22), the natural map $H^0(\mathcal{H}) \to H^0(\mathcal{H}_2)$ is surjective, so $S' = \varphi_{\mathcal{H}}(S)$, where $\varphi_{\mathcal{H}} : X \longrightarrow \mathbb{P}^n$ is the morphism defined by $|\mathcal{H}|$. Note that $\varphi_{\mathcal{H}}$ is birational for reasons of degree. Setting $X' := \varphi_{\mathcal{H}}(X) \subseteq \mathbb{P}^n$ we therefore have that $S' \subseteq X'$ is a smooth, linear, transversal surface section (recall that $S \subseteq X$ is a complete intersection of $(k-2)$ *general* elements of $|\mathcal{H}|$).
We now apply the theorem of Zak (unpublished, cf. [Za]) and L'vovski (cf. [L1] and [L2]) which says the following (cf. [L2, Thm. 0.1]): if $V \subsetneq \mathbb{P}^N$ is a smooth, nondegenerate variety which is not a quadric and satisfies $h^0(\mathcal{N}_{V/\mathbb{P}^N}(-1)) < 2N+1$; $Y \subseteq \mathbb{P}^{N+m}$ is a nondegenerate, irreducible $(m+\dim V)$-dimensional variety with $m > h^0(\mathcal{N}_{V/\mathbb{P}^N}(-1)) - N - 1$; and $L = \mathbb{P}^N \subseteq \mathbb{P}^{N+m}$ is a linear subspace such that $V = L \cap Y$ (scheme-theoretically), then Y is a cone.
Since a cone is uniruled by lines, the corollary will follow if we show that $h^0(\mathcal{N}_{S'/\mathbb{P}^{18}}(-1)) \leq 20$, with S' being the 3-uple embedding of a smooth cubic surface S_0 in $\mathbb{P}^3$.
We argue as in [GLM, p. 160-161] to compute $h^0(\mathcal{N}_{S'/\mathbb{P}^{18}}(-1))$. We give the argument for the sake of the reader.
From the Euler sequence and tangent bundle sequence

$$0 \longrightarrow \mathcal{O}_{S'}(-1) \longrightarrow \mathbb{C}^{19} \otimes \mathcal{O}_{S'} \longrightarrow \mathcal{T}_{\mathbb{P}^{18}}(-1) \otimes \mathcal{O}_{S'} \longrightarrow 0$$
$$0 \longrightarrow \mathcal{T}_{S'}(-1) \longrightarrow \mathcal{T}_{\mathbb{P}^{18}}(-1) \otimes \mathcal{O}_{S'} \longrightarrow \mathcal{N}_{S'/\mathbb{P}^{18}}(-1) \longrightarrow 0$$

we find

$$h^0(\mathcal{N}_{S'/\mathbb{P}^{18}}(-1)) \leq h^0(\mathcal{T}_{\mathbb{P}^{18}}(-1) \otimes \mathcal{O}_{S'}) + h^1(\mathcal{T}_{S'}(-1)) = 19 + h^1(\mathcal{T}_{S_0}(-3)). \tag{23}$$

From the tangent bundle sequence of $S_0 \subseteq \mathbb{P}^3$

$$0 \longrightarrow \mathcal{T}_{S_0}(-3) \longrightarrow \mathcal{T}_{\mathbb{P}^3}(-3) \otimes \mathcal{O}_{S_0} \longrightarrow \mathcal{N}_{S_0/\mathbb{P}^3}(-3) \longrightarrow 0 \tag{24}$$

and the fact that $\mathcal{N}_{S_0/\mathbb{P}^3}(-3) \simeq \mathcal{O}_{S_0}$, we find

$$h^1(\mathcal{T}_{S_0}(-3)) \leq 1 + h^1(\mathcal{T}_{\mathbb{P}^3}(-3) \otimes \mathcal{O}_{S_0}).$$

In view of (23) it will suffice to show that $h^1(\mathcal{T}_{\mathbb{P}^3}(-3) \otimes \mathcal{O}_{S_0}) = 0$.
Now observe that $\mathcal{T}_{\mathbb{P}^3}(-3) \simeq (\Omega^1_{\mathbb{P}^3})^\vee \otimes K_{\mathbb{P}^3} \otimes \mathcal{O}_{\mathbb{P}^3}(1) \simeq \Omega^2_{\mathbb{P}^3}(1)$ so using Bott vanishing on $\mathbb{P}^3$ and Serre duality one gets $h^1(\mathcal{T}_{\mathbb{P}^3}(-3)) = 0$ and

$$h^2(\mathcal{T}_{\mathbb{P}^3}(-6)) = h^2((\Omega^1_{\mathbb{P}^3})^\vee \otimes K_{\mathbb{P}^3} \otimes \mathcal{O}_{\mathbb{P}^3}(-2)) = h^1(\Omega^2_{\mathbb{P}^3}(2)) = 0.$$

This yields $h^1(\mathcal{T}_{\mathbb{P}^3}(-3) \otimes \mathcal{O}_{S_0}) = 0$.
This concludes the proof of the corollary. □

It is immediate that Corollary 0.3 follows from Theorem 0.1 and Corollary 0.2.
As we already noted in the introduction, Theorem 0.1 is sharp by the following example:

Example 2.5. The bound of Theorem 0.1 is sharp. In fact consider $X = \mathbb{P}^2 \times \mathbb{P}^1$ with projections p and q respectively and let $\mathcal{H} := p^*\mathcal{O}_{\mathbb{P}^2}(2) \otimes q^*\mathcal{O}_{\mathbb{P}^1}(a)$ for an integer $a > 0$. We have $n := h^0(\mathcal{H}) - 1 = h^0(\mathcal{O}_{\mathbb{P}^2}(2)) \cdot h^0(\mathcal{O}_{\mathbb{P}^1}(a)) - 1 = 6(a+1) - 1$ and $d := \mathcal{H}^3 = (p^*\mathcal{O}(2) \otimes q^*\mathcal{O}(a))^3 = 3(p^*\mathcal{O}(2))^2 \cdot q^*\mathcal{O}(a) = 12a$, whence $d = 2n - 10$.
If $a \geq 2$, then clearly any curve C on X satisfies $C \cdot \mathcal{H} = C \cdot p^*\mathcal{O}_{\mathbb{P}^2}(2) + C \cdot q^*\mathcal{O}_{\mathbb{P}^1}(a) \geq 2$, with equality obtained for the lines in the $\mathbb{P}^2$-fibers, so that X is uniruled of $\mathcal{H}$-degree two and not uniruled of $\mathcal{H}$-degree one.
If $a = 1$, then X is clearly uniruled of $\mathcal{H}$-degree one, and since $d = 12$ and $n = 11$, this also follows from Remark 2.3.

3. Terminal Veronese fibrations

In this section we will prove Propositions 0.4 and 2.4.
Since we will have to study the threefolds as in (II) of Theorem 1.2(v) we find it convenient to make the following definition:

Definition 3.1. Let $(T, \mathcal{L})$ be a pair satisfying the following:

(i) T is a terminal $\mathbb{Q}$-factorial threefold with a Mori fiber space structure $p : T \longrightarrow B$, where B is a smooth curve.
(ii) $\mathcal{L}$ is a line bundle on T such that the system $|\mathcal{L}|$ contains a smooth surface and has at most base points and $\mathcal{L}^3 > 0$.
(iii) The general fiber of p is $(V, \mathcal{L}_{|V}) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ and the rest are at most finitely many fibers $(G, \mathcal{L}_{|G}) \simeq (\mathbf{S}_4, \mathcal{O}_{\mathbf{S}_4}(1))$, where $\mathbf{S}_4 \subset \mathbb{P}^5$ is the cone over a normal quartic curve.

Such a Mori fiber space will be called a *(three-dimensional) terminal Veronese fibration.*

The threefolds of type (II) in Theorem 1.2(v) are terminal Veronese fibrations.
The easiest examples of terminal Veronese fibrations are the smooth ones in Example 2.5. But there are also singular such varieties and these were erroneously left out in both [M1, Prop. 3.7] and [CF, Prop. 3.4], as remarked by Mella in [M2, Rem. 5.4]: Take $\mathbb{P}^2 \times \mathbb{P}^1$ and blow up a conic C in a fiber and contract the strict transform of C, thus producing a Veronese cone singularity.
Although our main aim is to prove Proposition 2.4 we believe that terminal Veronese fibrations are interesting in their own rights. In order to prove Proposition 2.4 we will study "hyperplane sections" of T, i.e. surfaces in $|\mathcal{L}|$, and show that the desired bound on the degree follows since the general such surface has to have a certain number of degenerate fibers, i.e. unions of two conics (with respect to $\mathcal{L}$). What we first prove in this section is the following, which is part of the statement in Proposition 0.4:

Proposition 3.2. *Let $(T, \mathcal{L})$ be a three-dimensional terminal Veronese fibration and set $n := h^0(\mathcal{L}) - 1$ and $d := \mathcal{L}^3$.*
Then any smooth member of $|\mathcal{L}|$ is a surface fibered over B with $k \geq \frac{n-5}{2}$ fibers which are unions of two smooth rational curves intersecting in one point (the other fibers are smooth rational curves).

Proof. Denote by $\mathcal{V}$ the numerical equivalence class of a fiber. Let $S \in |\mathcal{L}|$ be a smooth surface. Then, since T is terminal, we have $S \cap \operatorname{Sing} T = \emptyset$ (cf. [M2, (2.3)]).
By property (iii) any fiber of S over B is either a smooth quartic, a union of two conics intersecting in one point, or a double conic, all with respect to $\mathcal{L}$. Denote by F the numerical equivalence class in S of a fiber over B. Then $F^2 = 0$.

If a fiber were a double conic, we could write $F \equiv 2F_0$ in Num S. However, in this case we would get the contradiction $F_0 \cdot (F_0 + K_S) = -1$, so this case does not occur.
In the case of a fiber which is a union of two conics intersecting in one point, we have $F \equiv F_1 + F_2$ in Num S, whence by adjunction both F_i are (-1)-curves. Since S is smooth its general fiber over B is a smooth quartic (with $F \cdot K_S = -2$ by adjunction), whence S has a finite number k of degenerate fibers which are unions of two conics and since these are all (-1)-curves we can blow down one of these curves in every fiber and reach a minimal model R for S which is a ruled surface over B. Let g be the genus of B. Then

$$k = K_R^2 - K_S^2 = 8(1-g) - K_S^2 = 8(1-g) - (K_T + \mathcal{L})^2 \cdot \mathcal{L},$$

which only depends on the numerical equivalence class of S. Therefore, any smooth surface in $|\mathcal{L}|$ has the same number of degenerate fibers.
Now note that the fibers of S over the finitely many points of B over which T has singular fibers are all smooth quartics, since $S \cap \operatorname{Sing} T = \emptyset$.
We now consider the birational map of T to a smooth projective bundle $\tilde{T}$, as described in [M2, p. 699].
Around one singular fiber $\mathbf{S_4}$ of T over a point $p \in B$ this map is given by a succession of blow ups ν_i and contractions μ_i:

(25)
$$\begin{tikzcd}[column sep=small]
 & Y_1 \arrow[dl, "\nu_1"'] \arrow[dr, "\mu_1"] & & \cdots \arrow[dl, "\nu_2"'] \arrow[dr, "\mu_{s-1}"] & & Y_s \arrow[dl, "\nu_s"'] \arrow[dr, "\mu_s"] & \\
T = T_0 & & T_1 & & T_{s-1} & & T_s = \tilde{T}
\end{tikzcd}$$

where the procedure ends as soon as some T_s has a fiber over $p \in B$ which is a smooth Veronese surface.
For every ν_i the exceptional divisor E_i is either a smooth Veronese surface or a cone over a rational normal quartic curve, and the strict transform of the singular fiber $\mathbf{S_4}$ of T_{i-1} over p is $G_i \simeq \mathbb{F}_4$, the desingularization of $\mathbf{S_4}$. These two intersect along a smooth quartic C_i. Then μ_i contracts G_i onto a smooth quartic curve $C_i' = \mu_i(C_i)$ and T_i is smooth along the exceptional locus of the contraction.
Following S throughout the procedure, we see that S stays out of the exceptional locus of every ν_i and in the contraction it is mapped to a surface having C_i' as fiber over p.
In other words the procedure of desingularizing one singular fiber of T maps every smooth surface in $|\mathcal{L}|$ to a smooth surface passing through a unique smooth quartic over p.
Doing the same procedure for all the other singular fibers of T we therefore reach a smooth projective bundle $\mathbb{P}(\mathcal{E})$ over B and under this process $|\mathcal{L}|$ is "mapped" to a (not necessarily complete) linear system on $\mathbb{P}(\mathcal{E})$ having smooth quartics over the corresponding points of B as base curves. Denote the corresponding line bundle on $\mathbb{P}(\mathcal{E})$ by $\mathcal{L}'$. Since we have not changed the number of degenerate fibers of any smooth surface in $|\mathcal{L}|$ over B, we see that every smooth surface in $|\mathcal{L}'|$ still has k degenerate fibers over B. Since clearly $\dim |\mathcal{L}'| \geq \dim |\mathcal{L}|$ it is now sufficient to show that any smooth surface in $|\mathcal{L}'|$ has $k \geq \frac{h^0(\mathcal{L}')-6}{2}$ fibers which are unions of two conics (with respect to $\mathcal{L}'$) intersecting in one point. This is the content of the following proposition. □

Proposition 3.3. *Let $f : T \simeq \mathbb{P}(\mathcal{E}) \longrightarrow B$ be a three-dimensional projective bundle over a smooth curve of genus g. Assume $\mathcal{L}$ is a line bundle on T satisfying:*

(i) $\mathcal{L}_{|V} \simeq \mathcal{O}_{\mathbb{P}^2}(2)$ *for every fiber* $V \simeq \mathbb{P}^2$,
(ii) $|\mathcal{L}|$ *is nonempty with general element a smooth irreducible surface,*

(iii) *the only curves in the base locus of* $|\mathcal{L}|$*, if any, are smooth quartics (with respect to* $\mathcal{L}$*) in the fibers.*

Then any smooth surface in $|\mathcal{L}|$ *is fibered over* B *with* k *fibers which are unions of two conics (with respect to* $\mathcal{L}$*) intersecting in one point, where*

$$k = \frac{1}{4}\mathcal{L}^3 \geq \frac{h^0(\mathcal{L}) - 6}{2}. \tag{26}$$

Proof. We only have to prove (26).

Denote by $\mathcal{V}$ the numerical equivalence class of a fiber. Since every fiber of T over B is a $\mathbb{P}^2$ we have $(K_T)_{|\mathbb{P}^2} \simeq K_{\mathbb{P}^2} \simeq \mathcal{O}_{\mathbb{P}^2}(-3)$ so we can choose a very ample line bundle $\mathcal{G} \in \operatorname{Pic} T$ such that

$$\operatorname{Num} T \simeq \mathbb{Z}\mathcal{G} \oplus \mathbb{Z}\mathcal{V}, \ \mathcal{G}^2 \cdot \mathcal{V} = 1, \ \mathcal{G} \cdot \mathcal{V}^2 = \mathcal{V}^3 = 0, \tag{27}$$

$$K_T \equiv b\mathcal{V} - 3\mathcal{G}, \ b \in \mathbb{Z} \tag{28}$$

and

$$\mathcal{L} \equiv a\mathcal{V} + 2\mathcal{G}, \ a \in \mathbb{Z}. \tag{29}$$

The general element $G \in |\mathcal{G}|$ is a smooth ruled surface over B; in particular

$$8(1-g) = K_G^2 = (K_T + \mathcal{G})^2 \cdot \mathcal{G} = (b\mathcal{V} - 2\mathcal{G})^2 \cdot \mathcal{G} = 4\mathcal{G}^3 - 4b,$$

that is

$$\mathcal{G}^3 = 2(1-g) + b. \tag{30}$$

Let now $S \in |\mathcal{L}|$ be any smooth surface. Clearly (as discussed in the proof of the previous proposition) $K_S^2 = 8(1-g) - k$. We compute, using (27) and (30),

$$\begin{aligned} K_S^2 &= (K_T + \mathcal{L})^2 \cdot \mathcal{L} = ((a+b)\mathcal{V} - \mathcal{G})^2 \cdot (a\mathcal{V} + 2\mathcal{G}) \\ &= 2\mathcal{G}^3 - 3a - 4b = 2(2(1-g) + b) - 3a - 4b \\ &= 4(1-g) - 3a - 2b, \end{aligned}$$

so that

$$k = 4(1-g) + 3a + 2b. \tag{31}$$

At the same time we have

$$\begin{aligned} \mathcal{L}^3 &= (a\mathcal{V} + 2\mathcal{G})^3 = 12a + 8\mathcal{G}^3 \\ &= 12a + 8b + 16(1-g) = 4k, \end{aligned} \tag{32}$$

proving the equality in (26).

The inequality in (26) we have left to prove is $\mathcal{L}^3 \geq 2h^0(\mathcal{L}) - 12$. We therefore assume, to get a contradiction, that

$$\mathcal{L}^3 \leq 2h^0(\mathcal{L}) - 13. \tag{33}$$

Since the 1-dimensional part of the base locus of $|\mathcal{L}|$ can only consist of smooth quartics (with respect to $\mathcal{L}$) in the fibers of f, we can write, on S,

$$\mathcal{L}_{|S} \sim H_0 + (f_{|S})^*\mathfrak{v} \equiv H_0 + cF,$$

for some nonnegative integer c, where $\mathfrak{v}$ is an effective divisor of degree c on B; F denotes the numerical equivalence class of a fiber of $f_{|S} : S \longrightarrow B$; and $|H_0|$ is the moving part of $|\mathcal{L}_{|S}|$. If the general element $C_0 \in |H_0|$ were not reduced and irreducible, then by Bertini's theorem $|H_0|$ would be composed with a pencil, whence $H_0 \equiv mH_0'$, for some $H_0' \in \operatorname{Pic} S$

with ${H'_0}^2 = 0$ and $m \geq 2$. Now $4 = F \cdot \mathcal{L} = F \cdot H_0 = mF \cdot H'_0$ implies $m \leq 4$. By (33) and the short exact sequence

$$0 \longrightarrow \mathcal{O}_T \longrightarrow \mathcal{O}_T(\mathcal{L}) \longrightarrow \mathcal{O}_S(\mathcal{L}) \longrightarrow 0 \tag{34}$$

(using the fact that $\mathcal{L}^3 \geq 0$ by (32)) we get the contradiction

$$5 \geq m+1 \geq h^0(H_0) = h^0(\mathcal{O}_S(\mathcal{L})) \geq h^0(\mathcal{L}) - 1 \geq 6.$$

Therefore C_0 is a reduced and irreducible curve (possibly singular).
From

$$0 \longrightarrow \mathcal{O}_S(f^*\mathfrak{v}) \longrightarrow \mathcal{O}_S(\mathcal{L}) \longrightarrow \mathcal{O}_{C_0}(\mathcal{L}) \longrightarrow 0. \tag{35}$$

and (34) we get, using (33):

$$\begin{aligned} h^0(\mathcal{O}_{C_0}(\mathcal{L})) &\geq h^0(\mathcal{O}_S(\mathcal{L})) - h^0(\mathcal{O}_S(f^*\mathfrak{v})) \\ &\geq h^0(\mathcal{O}_T(\mathcal{L})) - h^0(B, \mathfrak{v}) - 1 \\ &\geq \frac{1}{2}(\mathcal{L}^3 + 13) - c - 2 = \frac{1}{2}\mathcal{L}^3 - c + \frac{9}{2}. \end{aligned} \tag{36}$$

Moreover deg $\mathcal{O}_{C_0}(\mathcal{L}) = \mathcal{L}^2 \cdot (\mathcal{L} - c\mathcal{V}) = \mathcal{L}^3 - c\mathcal{L}^2 \cdot \mathcal{V} = \mathcal{L}^3 - 4c$, so that

$$\deg \mathcal{O}_{C_0}(\mathcal{L}) - 2(h^0(\mathcal{O}_{C_0}(\mathcal{L})) - 1) \leq \mathcal{L}^3 - 4c - (\mathcal{L}^3 - 2c + 7) = -2c - 7 < 0.$$

By Clifford's theorem on singular curves (see the appendix of [EKS]) we must therefore have

$$h^1(\mathcal{O}_{C_0}(\mathcal{L})) = 0. \tag{37}$$

Also note that since T is a projective bundle over a smooth curve of genus g, we have

$$h^0(\mathcal{O}_T) = 1,\ h^1(\mathcal{O}_T) = g,\ h^2(\mathcal{O}_T) = h^3(\mathcal{O}_T) = 0. \tag{38}$$

From Riemann-Roch on S and the fact that $h^2(\mathcal{O}_S(f^*\mathfrak{v})) = h^0(K_S - f^*_{|S}\mathfrak{v}) = 0$ (since F is nef with $F.(K_S - f^*\mathfrak{v}) = F.(K_S - cF) = -2$) we find

$$\begin{aligned} h^1(\mathcal{O}_S(f^*\mathfrak{v})) &= -\chi(\mathcal{O}_S(f^*\mathfrak{v})) + h^0(\mathcal{O}_S(f^*\mathfrak{v})) + h^2(\mathcal{O}_S(f^*\mathfrak{v})) \\ &= -\frac{1}{2}cF \cdot (cF - K_S) + g - 1 + h^0(\mathcal{O}_S(f^*\mathfrak{v})) \\ &\leq -c + g - 1 + c + 1 = g. \end{aligned} \tag{39}$$

Combining all (34)-(39) we find

$$\begin{aligned} h^1(\mathcal{L}) &\leq h^1(\mathcal{O}_T) + h^1(\mathcal{O}_S(\mathcal{L})) \\ &\leq h^1(\mathcal{O}_T) + h^1(\mathcal{O}_S(f^*\mathfrak{v})) + h^1(\mathcal{O}_{C_0}(\mathcal{L})) \\ &\leq g + g + 0 = 2g, \end{aligned} \tag{40}$$

together with

$$h^2(\mathcal{L}) = h^3(\mathcal{L}) = 0. \tag{41}$$

From (28)-(30), (34), (38) and Riemann-Roch on S, we get

$$\begin{aligned} \chi(\mathcal{L}) &= \chi(\mathcal{O}_S(\mathcal{L})) + \chi(\mathcal{O}_T) = \frac{1}{2}\mathcal{O}_S(\mathcal{L}) \cdot \Big(\mathcal{O}_S(\mathcal{L}) - K_S\Big) + \chi(\mathcal{O}_S) + \chi(\mathcal{O}_T) \\ &= \frac{1}{2}\Big(\mathcal{L}^3 - \mathcal{L}^2 \cdot (K_T + \mathcal{L})\Big) + \chi(\mathcal{O}_S) + \chi(\mathcal{O}_T) = -\frac{1}{2}\mathcal{L}^2 \cdot K_T + 2(1-g) \\ &= -\frac{1}{2}(a\mathcal{V} + 2\mathcal{G})^2 \cdot (b\mathcal{V} - 3\mathcal{G}) + 2(1-g) = 14(1-g) + 6a + 4b. \end{aligned}$$

Comparing with (32) we see that

$$\mathcal{L}^3 = 2\chi(\mathcal{L}) - 12(1-g),$$

whence, using (40) and (41),

$$\begin{aligned} \mathcal{L}^3 &= 2\Big(h^0(\mathcal{L}) - h^1(\mathcal{L}) + h^2(\mathcal{L}) - h^3(\mathcal{L})\Big) - 12(1-g) \\ &\geq 2\Big(h^0(\mathcal{L}) - 2g\Big) - 12(1-g) = 2h^0(\mathcal{L}) - 12 + 8g \geq 2h^0(\mathcal{L}) - 12, \end{aligned}$$

contradicting (33).
This shows that (33) cannot hold, proving (26). □

Example 3.4. As in Example 2.5 take $X = \mathbb{P}^2 \times \mathbb{P}^1$ and $\mathcal{H} := p^*\mathcal{O}_{\mathbb{P}^2}(2) \otimes q^*\mathcal{O}_{\mathbb{P}^1}(a)$. Then we have an embedding given by $|\mathcal{H}|$:

$$\mathbb{P}^2 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^{6(a+1)-1}$$

A hyperplane section of X in $\mathbb{P}^{6(a+1)-1}$ has equation

$$\sum_{i,j=0,1,2,0\leq k\leq a} l_{ijk} x_i x_j y_0^k y_1^{a-k} = 0,$$

where $(x_0 : x_1 : x_2)$ are the coordinates on $\mathbb{P}^2$ and $(y_0 : y_1)$ are the coordinates on $\mathbb{P}^1$ and l_{ijk} are coefficients. The section is degenerate on some Veronese surface $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ if the determinant of the matrix of the coefficients of the $x_i x_j$ is zero. This determinant is a polynomial of degree $3a$ in y_0, y_1, hence in general we find $3a$ distinct zeros. This means that a general hyperplane section has $3a = \frac{6(a+1)-1-5}{2}$ degenerate fibers, which is the smallest possible number of degenerate fibers for a terminal Veronese fibration as stated in Proposition 3.2.

Proofs of Propositions 0.4 and 2.4. We note that by Proposition 3.2 the only statement left to prove in Proposition 0.4 is a special case of Proposition 2.4.
As in Proposition 2.4 let $(X, \mathcal{H})$ be a pair consisting of a terminal $\mathbb{Q}$-factorial threefold X and a globally generated, big line bundle $\mathcal{H}$ on X, with $d := d(X, \mathcal{H})$ and $n := n(X, \mathcal{H})$.
Assume that a $\sharp$-minimal model $(X^\sharp, \mathcal{H}^\sharp)$ is of type (II) in Theorem 1.2(v), i.e. a terminal Veronese fibration over a smooth curve B of genus g.
Let $f : S \longrightarrow S^\sharp$ be as in Theorem 1.2(iii). We have $n^\sharp := \dim|\mathcal{H}^\sharp| \geq \dim|\mathcal{H}| = n$ by Lemma 1.5(a). By Proposition 3.2, $S^\sharp$ is fibered over B with general fiber a smooth quartic and $k \geq \frac{n^\sharp - 5}{2}$ fibers being a union of two rational curves intersecting in one point, which are both (-1)-curves. Therefore

$$K_{S^\sharp}^2 = 8(1-g) - k \leq 8(1-g) - \frac{n^\sharp - 5}{2} \leq 8(1-g) - \frac{n-5}{2}. \tag{42}$$

We want to show that $d \geq 2n - 10$. Assume, to get a contradiction, that

$$d \leq 2n - 11. \tag{43}$$

Note that $\rho := \rho(X, \mathcal{H}) = \frac{2}{3}$, so we can apply Lemma 1.5(b). Let l be the total number of irreducible curves contracted by f. Then $K_S^2 = K_{S^\sharp}^2 - l$. Pick any smooth irreducible curve $D \in |\mathcal{O}_S(\mathcal{H})|$. Then by (3), (10) and (42) we have

$$\begin{aligned} 0 &\leq 4l + (D+2K_S)^2 = 4l + 4K_S^2 + 4K_S \cdot D + D^2 \\ &\leq 4l + 4\Big(8(1-g) - \frac{n-5}{2} - l\Big) + 4\Big(d - 2n + 2\Big) + d \\ &= 32(1-g) - 2(n-5) + 4d - 8n + 8 + d \\ &= 32(1-g) + 5d - 10n + 18 \leq 5d - 10n + 50 = 5(d - 2n + 10), \end{aligned}$$

contradicting (43).
We have therefore proved that $d \geq 2n - 10$ and this finishes the proofs of Propositions 0.4 and 2.4. □

4. Final Remarks

To conclude, we remark that a closer look at the proofs of of Propositions 0.4 and 2.4 shows that if we assume only $d < 2n - 4$ instead of (43), we get $g = 0$ as the only possibility. This shows that:
A three-dimensional terminal Veronese fibration over a smooth curve of genus $g > 0$ must satisfy $d \geq 2n - 4$.
Consequently:
If a pair $(X, \mathcal{H})$ consisting of a terminal $\mathbb{Q}$-factorial threefold X and a globally generated, big line bundle $\mathcal{H}$ on X has a $\sharp$-minimal model being of type (II) in Theorem 1.2(v) over a smooth curve of genus $g > 0$, then $d \geq 2n - 4$.
If now $(X^\sharp, \mathcal{H}^\sharp)$ is a $\sharp$-minimal model of a pair $(X, \mathcal{H})$ consisting of a terminal $\mathbb{Q}$-factorial threefold X and a globally generated big line bundle $\mathcal{H}$, then $\mathcal{H}^\sharp$ is still big and nef by Lemma 1.5(a), so that $h^1(\mathcal{O}_X) = h^1(\mathcal{O}_S) = h^1(\mathcal{O}_{S^\sharp}) = h^1(\mathcal{O}_{X^\sharp})$. We have seen that this is zero if $X^\sharp$ is of type (I) in Theorem 1.2(v) and equal to g, the genus of B, if $X^\sharp$ is of type (II) in Theorem 1.2(v).
We have therefore obtained an improvement of [M2, Thm. 5.8] (cf. Theorem 1.2(v)):

Proposition 4.1. *Let $(X, \mathcal{H})$ be a pair consisting of a terminal $\mathbb{Q}$-factorial threefold X and a globally generated big line bundle $\mathcal{H}$ on X. Set $d := \mathcal{H}^3$ and $n := h^0(X, \mathcal{H}) - 1$.*
If $d < 2n - 10$ (resp. $d < 2n - 4$ and $h^1(\mathcal{O}_X) > 0$), then $(X^\sharp, \mathcal{H}^\sharp)$ is of one of the types (i)-(iv) (resp. (ii)-(iv)) below:

(i) $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$ *(with $(d, n) = (27, 19)$),*
(ii) *a quadric bundle with at most cA_1 singularities of type $f = x^2 + y^2 + z^2 + t^k$, for $k \geq 2$, and $\mathcal{H}^\sharp_{|F} \sim \mathcal{O}_F(1)$ for every fiber F,*
(iii) $(\mathbb{P}(E), \mathcal{O}(1))$ *where E is a rank 3 vector bundle over a smooth curve,*
(iv) $(\mathbb{P}(E), \mathcal{O}(1))$ *where E is a rank 2 vector bundle over a surface of negative Kodaira dimension.*

Consequently we have the following slight improvement of Theorem 0.1 and Corollary 0.2:

Corollary 4.2. *Let $(X, \mathcal{H})$ be a pair consisting of a reduced and irreducible k-dimensional variety X, $k \geq 3$, and a globally generated line bundle $\mathcal{H}$ on X. Set $d := \mathcal{H}^k$ and $n = h^0(X, \mathcal{H}) - 1$.*
If $h^1(\mathcal{O}_{\tilde{X}}) > 0$ for a resolution of singularities $\tilde{X}$ of X and $d < 2(n-k)+2$, then X is uniruled of $\mathcal{H}$-degree one.

Proof. In the proof of Theorem 0.1, use Proposition 4.1 in place of Theorem 1.2(v). Then, in the proof of Corollary 0.2, note that $h^1(\mathcal{O}_{X_i}) = h^1(\mathcal{O}_{X_{i-1}})$ as $\mathcal{H}_i$ is big and nef. □

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Polyhedral groups and pencils of K3-surfaces with maximal Picard Number

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POLYHEDRAL GROUPS AND PENCILS OF K3-SURFACES WITH MAXIMAL PICARD NUMBER

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Abstract. A K3-surface is a (smooth) simply-connected surface with trivial canonical bundle. In this note we investigate three particular pencils of K3-surfaces with maximal Picard number. To be precise: The general member in each pencil has Picard number 19. And each pencil contains precisely five surfaces with singularities. Four of them are also singular in the sense that their Picard number is 20. Our surfaces are minimal resolutions of quotients X/G, where $G \subset SO(4,\mathbb{R})$ is a finite group and X a G-invariant surface. The singularities of X/G come from fix-points of G on X or from double points of X. In any case these singularitites are A-D-E. The rational curves resolving them together with some even, resp. 3-divisible sets of rational curves generate the Neron-Severi group.

1. Introduction

The aim of this note is to present three particular pencils of K3-surfaces with Picard-number ≥ 19. These three pencils are related to the three polyhedral groups T, O, resp. I, (the rotation groups of the platonic solids tetrahedron, octahedron and icosahedron) as follows: It is classical that the group $SO(4,\mathbb{R})$ contains central extensions

	G_6	G_8	G_{12}
of	$T \times T$	$O \times O$	$I \times I$

by ± 1. Each group G_n, $n = 6, 8, 12$, has the obvious invariant $q := x_0^2 + x_1^2 + x_2^2 + x_3^2$. In [S] it is shown that each group G_n admits a second non-trivial invariant s_n of degree n. (The existence of these invariants seems to have been known before [Ra,C], but not their explicit form as computed in [S].) The pencil

$$X_\lambda \subset \mathbb{P}_3(\mathbb{C}): \quad s_n + \lambda q^{n/2} = 0$$

therefore consists of degree-n surfaces admitting the symmetry group G_n. We consider here the pencil of quotient surfaces

$$Y'_\lambda := X_\lambda / G_n \subset \mathbb{P}_3/G_n.$$

It is - for us - quite unexpected that these (singular) surfaces have minimal resolutions Y_λ, which are K3-surfaces with Picard-number ≥ 19.

In [S] it is shown that the general surface X_λ is smooth and that for each $n = 6, 8, 12$ there are precisely four singular surfaces X_λ, $\lambda \in \mathbb{C}$. The singularities of these surfaces are ordinary nodes (double points A_1) forming one orbit under G_n.

For a smooth surface X_λ the singularities on the quotient surface Y'_λ originate from fix-points of subgroups of G_n. Using [S, sect. 7] it is easy to enumerate these fix-points and to determine the corresponding quotient singularities. On the minimal resolution Y_λ of Y'_λ we find enough rational curves to generate a lattice in $NS(Y_\lambda)$ of rank 19. In sect. 5 we

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show that the minimal desingularisation Y_λ is K3 and that the structure of this surface varies with λ. This implies that the general surface Y_λ has Picard number 19. Then in sect. 6.1 we use even sets [N], resp. 3-divisible sets [B, T] of rational curves to determine completely the Picard-lattice of these surfaces Y_λ.
If X_λ is one of the four nodal surfaces in the pencil, there is an additional rational curve on Y_λ. This surface then has Picard-number 20. (Such K3-surfaces usually are called singular [SI].) We compute the Picard-lattice for the surfaces Y_λ in all twelve cases (sect. 6.2).

2. Notations and conventions

The base field always is $\mathbb{C}$. We abbreviate complex roots of unity as follows:

$$\omega = e^{2\pi i/3} = \frac{1}{2}(-1+\sqrt{-3}), \quad \epsilon := e^{2\pi i/5}, \quad \gamma := e^{2\pi i/8} = \frac{1}{\sqrt{2}}(1+i).$$

By $G \subset SO(3)$ we always denote one of the (ternary) polyhedral groups T, O or I, and by $\tilde{G} \subset SU(2)$ the corresponding binary group. By

$$\sigma : SU(2) \times SU(2) \to SO(4)$$

we denote the classical $2:1$ covering. The group $G_n \subset SO(4), n = 6, 8, 12,$ is the image $\sigma(\tilde{G} \times \tilde{G})$ for $\tilde{G} = \tilde{T}, \tilde{O}, \tilde{I}$. Usually we are interested more in the group

$$PG_n = G_n/\{\pm 1\} \subset PGL(4).$$

For $n = 6, 8, 12$ it is isomorphic with $T \times T, O \times O, I \times I$ having the order $12^2 = 144, 24^2 = 576$, resp. $60^2 = 3600$.

Definition 2.1. *a) Let $id \neq g \in PG_n$. A* fix-line *for g is a line $L \subset \mathbb{P}_3$ with $gx = x$ for all $x \in L$. The* fix-group *$F_L \subset PG_n$ is the subgroup consisting of all $h \in PG_n$ with $hx = x$ for all $x \in L$. The* order *$o(L)$ of L is the order of this group F_L.*
b) The stabilizer group *$H_L \subset PG_n$ is the subgroup consisting of all $h \in PG_n$ with $hL = L$. The* length *$\ell(L)$ is the length*

$$|PG_n|/|H_L|$$

of the G_n-orbit of L.
c) We shall encounter fix-lines of orders 2, 3, 4 and 5. We define their types *by*

order	2	3	4	5
type	M	N	R	S

We shall denote by $X_\lambda : s_n + \lambda q^{n/2} = 0$ the symmetric surface with parameter $\lambda \in \mathbb{C}$. All these surfaces are smooth, but for four parameters λ_i. These four singular parameters in the normalization of [S, p.445, p.449] are

$n=6$				$n=8$				$n=12$			
λ_1	λ_2	λ_3	λ_4	λ_1	λ_2	λ_3	λ_4	λ_1	λ_2	λ_3	λ_4
-1	$-\frac{2}{3}$	$-\frac{7}{12}$	$-\frac{1}{4}$	-1	$-\frac{3}{4}$	$-\frac{9}{16}$	$-\frac{5}{9}$	$-\frac{3}{32}$	$-\frac{22}{243}$	$-\frac{2}{25}$	0

Sometimes we call the surface X_λ of degree n and parameter λ_i just $X_{n,i}$, or refer to it as the case n,i.

3. Fixpoints

In this section we determine the fix-points for elements $id \neq g \in PG_n$.
Recall that each $\pm 1 \neq p \in \tilde{G}$ has precisely two eigen-spaces in $\mathbb{C}^2$ with the product of its eigen-values $= det(p) = 1$.
In coordinates $x_0, ..., x_3$ on $\mathbb{R}^4$ the morphism $\sigma : \tilde{G} \times \tilde{G} \to SO(4, \mathbb{R})$ is defined by $\sigma(p_1, p_2) : (x_k) \mapsto (y_k)$ with

$$\begin{pmatrix} y_0 + iy_1 & y_2 + iy_3 \\ -y_2 + iy_3 & y_0 - iy_1 \end{pmatrix} = p_1 \cdot \begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix} \cdot p_2^{-1}.$$

The quadratic invariant

$$q = x_0^2 + x_1^2 + x_2^2 + x_3^2 = det \begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix}$$

vanishes on tensor-product matrices

$$\begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix} = \begin{pmatrix} v_0 w_0 & v_0 w_1 \\ v_1 w_0 & v_1 w_1 \end{pmatrix} = v \otimes w.$$

The action of $\tilde{G} \times \tilde{G}$ on the quadric

$$Q := \{q = 0\} = \mathbb{P}_1 \times \mathbb{P}_1$$

is induced by the actions of the group $\tilde{G}$ on the tensor factors v and $w \in \mathbb{C}^2$

$$\sigma(p_1, p_2) : v \otimes w \mapsto (p_1 v) \otimes (\bar{p_2} w).$$

The fix-points for $\pm 1 \neq \sigma(p_1, p_2) \in G_n$ on $\mathbb{P}_3$ come in three kinds:

1) *Fix-points on the quadric:* $\pm 1 \neq p_1 \in \tilde{G}$ has two independent eigenvectors v, v'. The spaces $v \otimes \mathbb{C}^2$ and $v' \otimes \mathbb{C}^2$ determine on the quadric two fix-lines for $\sigma(p_1, \pm 1)$ belonging to the same ruling. In this way $\tilde{G}$-orbits of fix-points for elements $p_1 \in \tilde{G}$ determine G_n-orbits of fix-lines in the same ruling of the following lengths:

order of p	4	6	8	10
G_6	6	4, 4	–	–
G_8	12	8	6	–
G_{12}	30	20	–	12

In the same way fix-points for $p_2 \in \tilde{G}$ determine fix-lines for $\sigma(\pm 1, p_2) \in G_n$ in the other ruling. In [S, p.439] it is shown that the base locus of the pencil X_λ consists of $2n$ such fix-lines, n lines in each ruling, say Λ_k, Λ'_k, $k = 1, ..., n$. The fix-group F_{Λ_k} for the general point on each line Λ_k, Λ'_k then is cyclic of order $s := |G|/n$:

n	6	8	12
s	2	3	5

Where a fix-line for $\sigma(p_1, \pm 1)$ meets a fix-line for $\sigma(\pm 1, p_2)$ we obviously have an isolated fix-point x for the group generated by these two symmetries. We denote by t the order of the (cyclic) subgroup of $P(\sigma(\pm 1, \tilde{G}))$ fixing x. The number of H_{Λ_k}-orbits on each line Λ_k

of such points is

n	s	$t=2$	3	4	5
6	2	1	2	–	–
8	3	1	1	1	–
12	5	1	1	–	1

2) *Fix-lines off the quadric:* Let $L \subset \mathbb{P}_3$ be a fix-line for $\sigma(p_1, p_2) \in G_n$ with $p_1, p_2 \neq \pm 1$. It meets the quadric in at least one fix-point defined by a tensor product $v \otimes w$ with v, w eigenvectors for p_1, p_2 respectively. The group $< \sigma(p_1, \pm 1) > \subset H_L$ centralizes $\sigma(p_1, p_2)$. Therefore there is a second fix-point on L for this group. Necessarily it lies on the quadric, being determined by a tensor-product $v' \otimes w'$ with v', w' eigenvectors for p_1, p_2 respectively. Let α, α' be the eigen-values for p_1 on v, v' and β, β' those for p_2 on w, w' respectively. Then

$$\alpha \cdot \alpha' = \beta \cdot \beta' = 1.$$

Since all points on L have the same eigen-value under $\sigma(p_1, p_2)$ we find

$$\alpha \cdot \beta = \alpha' \cdot \beta' = (\alpha \cdot \beta)^{-1}.$$

So $\alpha \cdot \beta = \pm 1$ and $g := \sigma(p_1, p_2)$ acts on this line by an eigen-value ± 1. In particular p_1 and $\pm p_2 \in \tilde{G}$ have the same order.
We reproduce from [S, p. 443] the table of G_n-orbits of fix-lines off the quadric by specifying a generator $g \in G_n$ of F_L. For this generator we use the notation of [S]. There it is also given the length $\ell(L)$. This length determines the order $|H_L| = |PG_n|/\ell(L)$ of the stabilizer group and the length $|H_L|/|F_L|$ of the general H_L-orbit on L:

n	6			8					12		
g	σ_{24}	$\pi_3\pi_3'$	$\pi_3^2\pi_3'$	$\pi_3\pi_4\pi_3'\pi_4'$	$\pi_3\pi_4\sigma_4$	$\sigma_2\pi_3'\pi_4'$	$\pi_3\pi_3'$	$\pi_4\pi_4'$	σ_{24}	$\pi_3\pi_3'$	$\pi_5\pi_5'$
F_L	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_4$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_5$
type	M	N	N'	M	M'	M''	N	R	M	N	S
$\ell(L)$	18	16	16	72	36	36	32	18	450	200	72
$\|H_L\|/\|F_L\|$	4	3	3	4	8	8	6	8	4	6	10

3) *Intersections of fix-lines off the quadric:* From [S, p.450] one can read off the G_n-orbits of intersections of these lines outside of the quadric and the value of the parameter λ for the surface X_λ passing through this intersection point. An intersection point is a fix-point for the group generated by the transformations leaving fixed the intersecting lines. In the following table we give these (projective) groups (D_n denoting the dihedral group of order $2n$), the orders of the fix-group of intersecting lines, the generators of these groups, as well

as the numbers of lines meeting:

n	λ	group	orders	generators	numbers
6	λ_1	T	$2,3$	$\sigma_{24},\ \pi_3\pi_3'$	$3,4$
	λ_4	T	$2,3$	$\sigma_{24},\ \pi_3^2\pi_3'$	$3,4$
8	λ_1	O	$2,3,4$	$\pi_3\pi_4\pi_3'\pi_4',\ \pi_3\pi_3',\ \pi_4\pi_4'$	$6,4,3$
	λ_2	D_4	$2,2,4$	$\pi_3\pi_4\sigma_4,\ \sigma_2\pi_3'\pi_4',\ \pi_4\pi_4'$	$2,2,1$
	λ_3	$\mathbb{Z}_2\times\mathbb{Z}_2$	$2,2,2$	$\pi_3\pi_4\sigma_4,\ \sigma_2\pi_3'\pi_4',\ \pi_3\pi_4\pi_3'\pi_4'$	$1,1,1$
	λ_4	D_3	$2,3$	$\pi_3\pi_4\pi_3'\pi_4',\ \pi_3\pi_3'$	$3,1$
12	λ_1	T	$2,3$	$\sigma_{24},\ \pi_3\pi_3'$	$3,4$
	λ_2	D_3	$2,3$	$\sigma_{24},\ \pi_3\pi_3'$	$3,1$
	λ_3	D_5	$2,5$	$\sigma_{24},\ \pi_5\pi_5'$	$5,1$
	λ_4	I	$2,3,5$	$\sigma_{24},\ \pi_3\pi_3',\ \pi_5\pi_5'$	$15,10,6$

4. Quotient singularities

Singularities in the quotient surface $Y' = Y'_\lambda$ originate from fix-points of the group action (or from singularities on X, but the latter are included in the fix-points, see [S, (6.4)]). We distinguish four types of fix-points on $X = X_\lambda$ for elements of G_n:

1) Points of the base locus Λ of the pencil, n lines in each of the two rulings of the invariant quadric Q, the (projective) fix-group being $\mathbb{Z}_s$ from section 1;
2) points on a line Λ_k or Λ'_k in the base locus, fixed by the group $\mathbb{Z}_s = <\sigma(p,1)>$ from section 1 and by some non-trivial subgroup $\mathbb{Z}_t \subset P(\sigma(1,\tilde{G}))$;
3) isolated fixed points on the intersection of a fix-line and a smooth surface X_λ;
4) nodes of a surface X_λ.

1) All points of Λ_i are fixed by the cyclic group $\mathbb{Z}_s$ from section 1. The quotient map here is a cyclic covering of order s. The quotient by $\mathbb{Z}_s$ is smooth.

2) Since G_n acts on Λ_i as the ternary polyhedral group G, there are orbits of points on Λ_i, fixed under some none-trivial subgroup of G. We have to disinguish two cases:
Case 1: The n points, where the line Λ_i meets some line $\Lambda'_k \subset \Lambda$. Here the stabilizer group is $\mathbb{Z}_s \times \mathbb{Z}_s$ acting on X by reflections in the two lines Λ_i, Λ'_k. In such points the quotient surface Y' is smooth.
Case 2: The fix-points of other non-trivial subgroups of G. The lengths of these orbits and their stabilizer subgroups $\mathbb{Z}_t \subset G$ are given in section 1:

t	2	3	4
G_6	–	4, 4	–
G_8	12	–	6
G_{12}	30	20	–

The total stabilizer is the direct product $\mathbb{Z}_s \times \mathbb{Z}_t$. Let v, v' be eigen-vectors for $\mathbb{Z}_s$ and w, w' eigen-vectors for $\mathbb{Z}_t$. Let $v \otimes w$ determine the fix-point in question. The surface X is smooth there, containing the line $\mathbb{P}(v \otimes \mathbb{C}^2)$, and intersecting the quadric Q transversally. This implies that the tangent space of X is the plane

$$y_0 \cdot v \otimes w + y_1 \cdot v \otimes w' + y_2 \cdot v' \otimes w', \quad y_0, y_1, y_2 \in \mathbb{C}.$$

Let $\sigma(p_1, \pm 1) \in \mathbb{Z}_s$ and $\sigma(\pm 1, p_2) \in \mathbb{Z}_t$ be generators. Let them act by

$$\sigma(p_1,1)v = \alpha v,\ \sigma(p_1,1)v' = \alpha^{-1}v',\ \sigma(1,p_2)w = \beta w,\ \sigma(1,p_2)w' = \beta^{-1}w'.$$

These transformations act on the coordinates y_ν of the tangent plane as

	y_0	y_1	y_2	$z_1 := y_1/y_0$	$z_2 := y_2/y_0$
$\sigma(p_1,1)$	α	α	α^{-1}	1	α^{-2}
$\sigma(1,p_2)$	β	β^{-1}	β^{-1}	β^{-2}	β^{-2}

We introduce local coordinates on X in which the group acts as on z_1, z_2, and in fact use again z_1, z_2 to denote these local coordinates on X. We locally form the quotient $X/(\mathbb{Z}_s \times \mathbb{Z}_t)$ dividing first by the action of $\mathbb{Z}_s$

$$(z_1, z_2) \mapsto (z_1^s, z_2).$$

Then we trace the action of $\mathbb{Z}_t$ on z_1^s and z_2. A generator $\sigma(1,p_2)$ of $\mathbb{Z}_t$ acts by

y_0	y_1	y_2	z_1	z_2
ω	ω^2	ω^2	ω	ω
i	$-i$	$-i$	-1	-1
γ	γ^7	γ^7	$-i$	$-i$

The resulting singularities on Y' are

n	s	z_1	z_2	z_1^s	z_2	quotient singularity
6	2	ω	ω	ω^2	ω	A_2
8	3	-1	-1	-1	-1	A_1
		$-i$	$-i$	i	$-i$	A_3
12	5	-1	-1	-1	-1	A_1
	5	ω	ω	ω^2	ω	A_2

3) Let $L \subset \mathbb{P}_3$ be a fix-line for $\sigma(p_1,p_2) \in G_n$, not lying on the quadric. Assume that $\sigma(p_1,p_2)$ is chosen as a generator for the group F_L. By sect. 1 there are eigen-vectors v, v' for p_1 with eigen-values α, α^{-1} and w, w' for p_2 with eigen-values β, β^{-1} satisfying

$$\alpha\beta = \pm 1, \quad \alpha\beta = \alpha^{-1}\beta^{-1} = \pm 1,$$

such that L is spanned by $v \otimes w$ and $v' \otimes w'$. The general surface X meets this line in n distinct points. If the line has order s, two of these points lie on the base locus Λ. So the number of points not in the quadric Q cut out on L by X is

n	6		8			12		
$o(L)$	2	3	2	3	4	2	3	5
number	4	6	8	6	8	12	12	10

These points fall into orbits under the stabilizer group H_L. The lengths of these orbits are given in sect. 1.
To identify the quotient singularity we have to trace the action of $\sigma(p_1,p_2)$ on the tangent plane $T_x(X)$. For general X this plane will be transversal to L. So it must be the plane spanned by $x, v \otimes w', v' \otimes w$. By continuity this then is the case also for all smooth X. In particular, all smooth X meet L in n distinct points, i.e., the intersections always are transversal. And by continuity again, the numbers and lengths of H_L-orbits in $X \cap L$ are the same for all smooth X. Since $\sigma(p_1,p_2)$ acts

on	$v \otimes w'$	$v' \otimes w$
by	$\alpha\beta^{-1}$	$\alpha^{-1}\beta,$

the eigen-values for $\sigma(p_1,p_2)$ on $T_x(X)$ are

$$\frac{\alpha^{-1}\beta}{\alpha\beta} = \alpha^{-2} \quad \text{and} \quad \frac{\alpha\beta^{-1}}{\alpha\beta} = \beta^{-2} = (\pm\frac{1}{\alpha})^2 = \alpha^2.$$

The resulting quotient singularity on Y' therefore is of type A_r, where r is the order of L. We collect the results in the following table. It shows in each case length and number of H_L-orbits, the number and type(s) of the quotient singularity(ies).

n	6			8					12		
$o(L)$	2	3	3	2	2	2	3	4	2	3	5
type	M	N'	N''	M'	M''	M	N	R	M	N	S
length	4	3	3	8	8	4	6	8	4	6	10
number	1	2	2	1	1	2	1	1	3	2	1
singularities	A_1	$2A_2$	$2A_2$	A_1	A_1	$2A_1$	A_2	A_3	$3A_1$	$2A_2$	A_4

4) Finally we consider the nodal surfaces X. All the intersections of fix-lines considered in sect. 2 are nodes on the surfaces X. There are just two invariant surfaces with nodes not given there, because through their nodes passes just one fix-line. They are G_6-invariants. Their parameters are as follows:

λ	group	generator
λ_2	$\mathbb{Z}_3$	$\pi_3\pi_3'$
λ_3	$\mathbb{Z}_3$	$\pi_3\pi_3'^2$

We use this to collect the data for the twelve singular surfaces X in the next table. We include the number ns of nodes on the surface and specify the group $F \subset PSL(4)$ fixing the node. For each type we give the number of lines meeting in the node. So e.g. $3M$ means that there are three lines of type M meeting at the node.

n	6				8				12			
λ	λ_1	λ_2	λ_3	λ_4	λ_1	λ_2	λ_3	λ_4	λ_1	λ_2	λ_3	λ_4
ns	12	48	48	12	24	72	144	96	300	600	360	60
F	T	$\mathbb{Z}_3$	$\mathbb{Z}_3$	T	O	D_4	$\mathbb{Z}_2 \times \mathbb{Z}_2$	D_3	T	D_3	D_5	I
	$3M$	$1N'$	$1N''$	$3M$	$6M$	$2M'$	$1M'$	$3M$	$3M$	$3M$	$5M$	$15M$
	$4N'$			$4N''$	$4N$	$2M''$	$1M''$	$1N$	$4N$	$1N$	$1S$	$10N$
					$3R$	$1R$	$1M$					$6S$

Lemma 4.1. *Let $G \subset SO(3)$ be a finite subgroup of order ≥ 3.*
a) Up to G-equivariant linear coordinate change, there is a unique G-invariant quadratic polynomial defining a non-degenerate cone with top at the origin.
b) If X is a G-invariant surface, having a node at the origin, then there is a G-equivariant change of local (analytic) coordinates, such that X is given in the new coordinates by $x^2 + y^2 + z^2 = 0$.

Proof. a) We distinguish two cases:
i) $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ generated by the symmetries

$$(x, y, z) \mapsto (x, -y, -z) \text{ and } (x, y, z) \mapsto (-x, y, -z).$$

The quadratic G-invariants are generated by the squares x^2, y^2 and z^2. The invariant polynomial then is of the form $ax^2 + by^2 + cz^2$ with $a, b, c \neq 0$. The coordinate change

$$x' := \sqrt{a}x,\ y' := \sqrt{b}y,\ z' = \sqrt{c}z$$

is G-equivariant and transforms the polynomial into $x'^2 + y'^2 + z'^2$.
ii) G contains an element g of order ≥ 3. Let it act by

$$(x, y, z) \mapsto (cx - sy,\ sx + cy, z)$$

with $c = cos(\alpha)$, $s = sin(\alpha)$ and $\alpha \neq 0, \pi$. The quadratic invariants of g are generated by $x^2 + y^2$ and z^2. The invariant polynomial must be of the form $a(x^2 + y^2) + bz^2$ with $a, b \neq 0$. The G-equivariant transformation $x' := \sqrt{a}x$, $y' := \sqrt{a}y$, $z' := \sqrt{b}z$ transforms it into the same normal form as in i). This proves the assertion if $G =< g >$ is cyclic or if G is dihedral.
In the three other cases $G = T, O$ or I, it is well-known that $x^2+y^2+z^2$, up to a constant factor, is the unique quadratic G-invariant.
b) Let X be given locally at the origin by an equation $f(x, y, z) = 0$ with f some power series. Since X is G-invariant, so is the tangent cone of X at the origin. By a) we therefore may assume

$$f = x^2 + y^2 + z^2 + f_3(x, y, z)$$

with a power series f_3 containing monomials of degrees ≥ 3 only. It is well-known that there is a local biholomorphic map $\varphi : (x, y, z) \mapsto (x', y', z')$ mapping X to its tangent cone, i.e., with the property $\varphi^*(x'^2 + y'^2 + z'^2) = f$. For the derivative $\varphi'(0)$ this implies $\varphi'(0)^*(x'^2+y'^2+z'^2) = x^2+y^2+z^2$. After replacing φ by $\varphi'(0)^{-1}\circ\varphi$ we even may assume $\varphi'(0) = id$.
Now consider the local G-equivariant holomorphic map

$$\Phi : (x, y, z) \mapsto \frac{1}{|G|} \sum_{h\in G} h \circ \varphi \circ h^{-1}.$$

Using the G-invariance of f and $x'^2 + y'^2 + z'^2$ one easily checks $\Phi^*(x'^2 + y'^2 + z'^2) = f$. It remains to show, that Φ locally at the origin is biholomorphic. But this follows from

$$\Phi'(0) = \frac{1}{|G|} \sum_{h\in G} h \circ \varphi'(0) \circ h^{-1} = id.$$

□

Now consider the automorphism

$$\mathbb{C}^2 \to \mathbb{C}^2, \quad v = (v_0, v_1) \mapsto v^\perp := (v_1, -v_0).$$

For $q \in SU(2)$ it is easy to check that $(qv)^\perp = \bar{q}v^\perp$. Map $\mathbb{C}^2 \to \mathbb{C}^3$ via $v \mapsto v \otimes v^\perp$. Consider $\mathbb{C}^3$ as the space of traceless complex matrices

$$\begin{pmatrix} ix & y + iz \\ -y + iz & -ix \end{pmatrix}.$$

Then

$$v \otimes v^\perp = \begin{pmatrix} v_0v_1 & -v_0^2 \\ v_1^2 & -v_0v_1 \end{pmatrix}$$

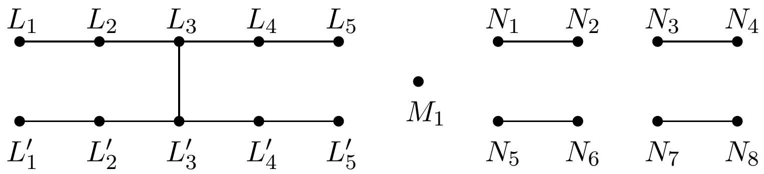
is a matrix of determinant $x^2 + y^2 + z^2 = 0$. One easily checks that the map $v \mapsto v \otimes v^\perp$ is $2 : 1$ onto the cone of equation $x^2 + y^2 + z^2 = 0$, identifying this cone with the quotient $\mathbb{C}^2/ < -id >$. And this map is $SU(2)$-equivariant with respect to the $2 : 1$ cover $SU(2) \to SO(3)$. If $\tilde{G} \subset SU(2)$ is some finite group, then the quotient $\mathbb{C}^2/\tilde{G}$ via this map is identified with the quotient of the cone by the corresponding ternary group $G \subset SO(3)$. Together with lemma 3.1 this shows:

Proposition 4.1. *Let $X = X_\lambda$ be a nodal surface with G the fix-group G of the node. Then the image of this node on X/G_n is a quotient singularity locally isomorphic with $\mathbb{C}^2/\tilde{G}$.*

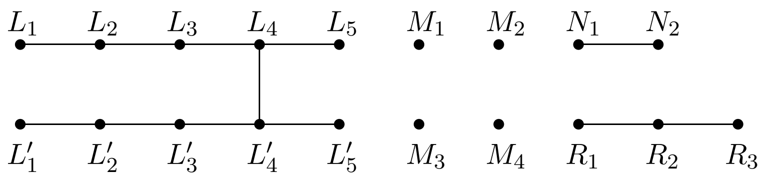
5. Rational curves

We denote by $X = X_\lambda \to Y' = Y'_\lambda$ the quotient map for G_n acting on X and by $Y = Y_\lambda \to Y'$ the minimal resolution of the quotient singularities on Y coming from the orbits of isolated fixed points in sect. 2. The n lines $\Lambda_i, \Lambda'_i \subset Q$ in each ruling map to one smooth rational curve in Y'. We denote those by L, L'. Both these curves meet transversally in a smooth point of Y'. All quotient singularities are rational double points. Resolving them introduces more rational curves in Y. For each singularity A_k we get a chain of k smooth rational (-2)-curves. Since the group $\mathbb{Z}_t$ from sect. 3 acts on X with Λ_i, resp. Λ'_i defining an eigen-space in the tangent space of X, the curves L, L' meet the A_{t-1}-string in an end curve of this string, avoiding the other curves of the string.
All lines L of the types $M, M', M'', N, N', N'', R, S$ form one orbit under G_n. We denote by M_i etc. the rational curves resolving the A_r-singularity on the image of $L \cap X$. If $L \cap X$ consists of more than one H_L-orbit we get in this way more than one A_r-configuration of rational curves coming from $L \cap X$.

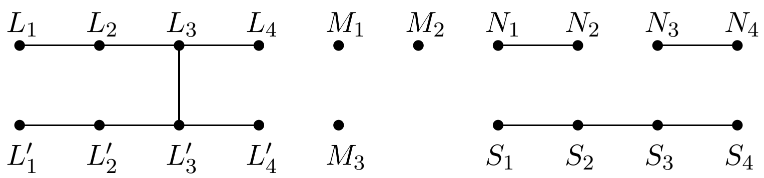
5.1. **The general case.** First we consider the quotients of the smooth surfaces X: The striking fact is that the number of the additional rational curves is 17. We give the dual graphs of the collections of 19 rational curves on Y, changing the notation L, L' to L_3, L'_3 for $n = 6, 12$ and to L_4, L'_4 for $n = 8$:



n=6: M_1 coming from M, $N_1, ..., N_4$ from N', $N_5, ..., N_8$ from N''



n=8: M_1 from M', M_2 from M'', M_3, M_4 from M, N_i from N, R_i from R



n=12: M_i from M, N_i from N, S_i from S

Proposition 5.1. *In each case the 19 rational curves specified generate a sub-lattice of $NS(Y)$ of rank 19.*

Proof. We compute the discriminant d of the lattice. The connected components of the dual graph define sub-lattices, the direct sum of which is the lattice in question. We compute the discriminant block-wise using the sub-lattices

$$L :=< L_i, L_i' >, \quad M :=< M_i >, \quad N :=< N_i >, \quad R :=< R_i >, \quad S :=< S_i >$$

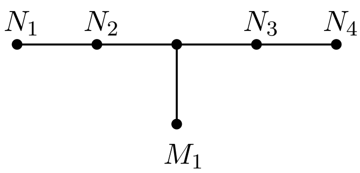
and find

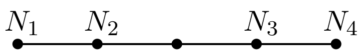
n	$d(L)$	$d(M)$	$d(N)$	$d(R)$	$d(S)$	d
6	-45	-2	3^4			$2 \cdot 3^6 \cdot 5$
8	-28	2^4	3	-4		$2^8 \cdot 3 \cdot 7$
12	-11	-2^3	3^2		5	$2^3 \cdot 3^2 \cdot 5 \cdot 11$

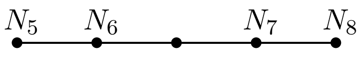
□

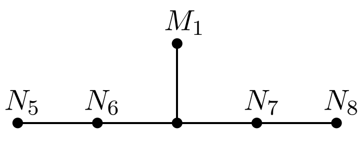
5.2. **The special cases.** Here we consider the desingularized quotients Y for the twelve singular surfaces X. The image of the nodes on X will be on Y a quotient singularity for the binary group corresponding to the ternary group F from sect. 3. There we also gave the lines passing through this node on X. The nodes of X on such a line fall into orbits under the group H fixing the line. If there is just one H-orbit of intersection points of the general surface X with this line, it is clear that this orbit converges to the orbit of nodes. We say: The quotient singularity *swallows* the orbit. If however there are more than one H-orbits, we have to analyze the situation more carefully. We use the map onto $\mathbb{P}_1$ of this line induced by the parameter λ. Nodes of X on the given line will be branch points of this map.

Degree 6: On lines of type M there ist just one orbit of four points. On lines of type N', N'' there are two orbits of length 3. The parameter λ induces on each N'- or N''-line some $6 : 1$ cover over $\mathbb{P}_1$. Each fibre of six points decomposes into two orbits of three points. The total ramification degree is $-2 - 6 \cdot (-2) = 10$. The intersection with Q consists of two points of ramification order 2. So outside of the quadric Q we will have total ramification order six, hence it will happen twice, that two orbits of three points are swallowed by a quotient singularity. This must happen on the surfaces $X_{6,1}$ and $X_{6,2}$ for N', and for N'' on $X_{6,3}$ and $X_{6,4}$. We give the rational curves from 4.1 disappearing in Y, being replaced by rational curves in the minimal resolution of the quotient surface. Here we do not mean that e.g. the curve N_1 indeed converges to the curve denoted by N_1 in the dual graph of the resolution. We just mean that all the curves denoted by letters in the dual graph disappear:

$6,1:$


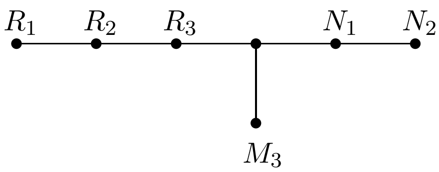
$6,2:$


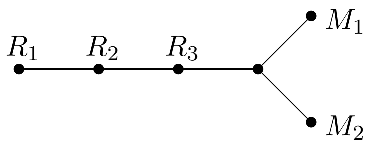
$6,3:$


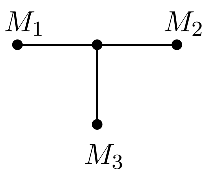
$6,4:$


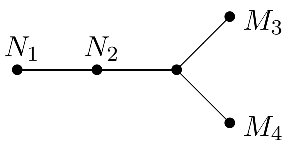
Degree 8: The only lines with two H-orbits are those of type M. The map to $\mathbb{P}_1$ there has degree eight and total ramification order 14. The intersection with Q counts for two points with ramification order three each. So there will be total ramification of order eight off the quadric. The surface $X_{8,1}$ has $24 \cdot 6/72 = 2$ nodes on such a line, it swallows at least one orbit. The surface $X_{8,3}$ has $144/72 = 2$ nodes too and swallows at least one orbit too. The surface $X_{8,4}$ has $96 \cdot 3/72 = 4$ nodes and swallows at least two orbits. Since the total branching order adds up to at least $2+2+4 = 8$, the bounds for the numbers of orders in

fact are exact numbers. The dual graphs for the resolution of quotient singularities and the curves swallowed are as follows:

$8,1:$ 

$8,2:$ 

$8,3:$ 

$8,4:$ 

Notice, that it is not necessary here to distinguish between M_3 and M_4. In fact it is even impossible, since the two corresponding orbits of intersections of the line M with the surface X_λ are interchanged by monodromy.

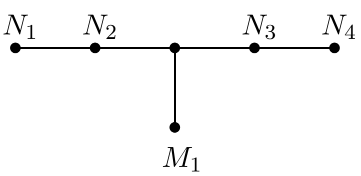
Degree 12: Now a line of type M contains three H-orbits of length four. The total branching order for the λ-map is 22 on such a line. The intersection with Q consists of two six-fold points and decreases the branching order by 10. So the total branching order off the quadric is 12. On such a line there are

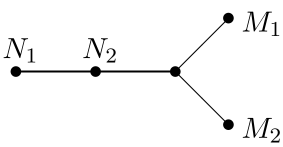
on the surface	nodes	orbits swallowed
$X_{12,1}$	$300 \cdot 3/450 = 2$	≥ 1
$X_{12,2}$	$600 \cdot 3/450 = 4$	≥ 2
$X_{12,3}$	$360 \cdot 5/450 = 4$	≥ 2
$X_{12,4}$	$60 \cdot 15/450 = 2$	≥ 1

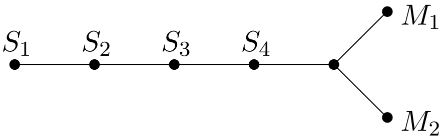
Since the total branching order must add up to twelve, the number given is indeed the number of swallowed orbits.
A line of type N contains two H-orbits of length six. Just as in the preceding case one computes the following numbers

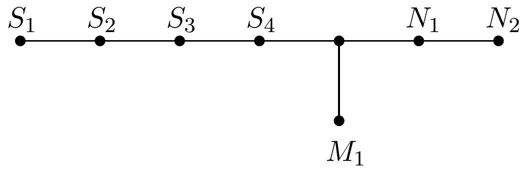
on the surface	nodes	orbits swallowed
$X_{12,1}$	$300 \cdot 4/200 = 6$	≥ 2
$X_{12,2}$	$600 \cdot 1/200 = 3$	≥ 1
$X_{12,4}$	$60 \cdot 10/200 = 3$	≥ 1

Again the total branching order adds up to twelve. Therefore the estimates give the precise number of orbits swallowed.

$12,1:$ 

$12,2:$ 

12, 3 : 

12, 4 : 

Again, by monodromy it is impossible to distinguish between the curves M_1, M_2 and M_3, and likewise between the pairs $\{N_1, N_2\}$ and $\{N_3, N_4\}$.

6. K3-surfaces

In this section we show that the desingularized quotient surfaces Y_λ are K3 and that their structure is not constant in λ. We start with a crude but effective blow-up of $\mathbb{P}_3$. Let

$$\Xi := \{(x, \lambda) \in \mathbb{P}_3 \times \mathbb{C} : s_n(x) + \lambda q^{n/2}(x) = 0\}.$$

In addition we put:

- $\bar{\Xi} \subset \mathbb{P}_3 \times \mathbb{P}_1$ the closure of Ξ. It is a divisor of bidegree (n,1).
- $\tau : \Xi \to \mathbb{P}_3$ the natural projection onto the first factor;
- $f : \Xi \to \mathbb{C}$ the projection onto the second factor. It is given by the function λ.
- $\tilde{\Lambda} := \tau^{-1}\Lambda$. This pull-back of the base-locus is the zero-set of $\tau^* q$ on Ξ;
- $\Xi^0 \subset \Xi$ the complement of the finitely many points in Ξ lying over the nodes of the four nodal surfaces X_λ.
- $\Upsilon' := \Xi / G_n$ the quotient threefold. Notice that the action of G_n on $\mathbb{P}_3$ lifts naturally to an action on Ξ.
- $h : \Upsilon' \to \mathbb{C}$ the map induced by f;
- Υ^0 the image of Ξ^0.

Lemma 6.1. *a) The threefold $\Xi \subset \mathbb{P}_3 \times \mathbb{C}$ is smooth.*
b) If $M \subset \mathbb{P}_3$, $M \not\subset Q$, is a fix-line for an element $\pm 1 \neq g \in G_n$ and $\tilde{M} \subset \Xi$ is its proper transform, then $\tilde{M}$ does not meet $\tilde{\Lambda}$ in Ξ.

Proof. a) By $\partial_\lambda(s_n + \lambda q^{n/2}) = q^{n/2}$ singularities of Ξ can lie only on $\tau^{-1}\Lambda$. But there

$$\partial_{x_i}(s_n + \lambda q^{n/2}) = \partial_{x_i} s_n.$$

Since $s_n = 0$ is smooth along Λ, this proves that Ξ is smooth.
b) The assertion is obvious, if M does not meet the base locus Λ. If however $M \cap \Lambda = \{x_1, x_2\}$ is nonempty, we use the fact, observed in sect. 3, that the polynomial $s_n + tq^{n/2}|M$ vanishes in x_i to the first order for all smooth surfaces $X :\ s_n + tq^{n/2} = 0$. On $\tilde{M}$ however we have $s_n = -\lambda q^{n/2}$ with $n/2 > 1$. So $\tilde{M}$ will not meet $\tau^{-1}\{x_1, x_2\}$ in Ξ. □

The G_n-action on Ξ has the following kinds of fix-points:

1) Fix-points on $\tilde{\Lambda}$ for the group $\mathbb{Z}_s$;
2) Fix-points for the group $\mathbb{Z}_s \times \mathbb{Z}_s$ on the fibre $\tau^{-1}(x)$ over some intersection of lines Λ_i, Λ'_j in the base locus Λ;
3) Fix-points for a group $\mathbb{Z}_s \times \mathbb{Z}_t$ on the fibre $\tau^{-1}(x)$ over a point x, where a line in the base locus meets some line M of fix-points not in the base locus. By lemma 5.1 b) $\tau^{-1}(x)$ and $\tilde{M}$ do not intersect in Ξ.
4) Fix-curves $\tilde{L}$ away from $\tilde{\Lambda}$ lying over fix-lines L not contained in the base-locus. All these curves are disjoint, when considered in Ξ^0.

The quotient three-fold $\Upsilon' = \Xi/G_n$ is smooth in the image points of fix-points of types 1) or 2). It has quotient singularities A_t in the image curves of the curves $\tau^{-1}(x)$ of type 3). To be precise: The singularities there locally are products of an A_t surface singularity with a copy of the complex unit disc. Additional such cyclic quotient singularities A_k occur on the image curves of curves $\tilde{L}$ of type 4). Where two such curves meet we have higher singularities. But such points are removed in Υ^0. So Υ^0 is singular along finitely many smooth irreducible rational curves. The singularities along each curve are products with some cyclic surface quotient A_k.
Let $\Upsilon \to \Upsilon^0$ be the minimal desingularisation of Υ^0 along these singular curves. Locally this is the product of the unit disc with a minimal resolution of the surface singularity A_k. Since the surfaces Y'_λ intersect the singular curves transversally, the proper transforms $Y_\lambda \subset \Upsilon$ are smooth, minimally desingularized. They are the fibres of the map induced by h. For $\lambda_i, i = 1, ..., 4$, we denote by Y_{λ_i} the minimal resolutions of the quotient surfaces X_{λ_i}/G_n. We do not (and cannot) consider them as surfaces in Υ.

Proposition 6.1. *The surfaces Y_λ are (minimal) K3-surfaces.*

Proof. All cyclic quotient singularities on Υ^0 are gorenstein. So there is a dualizing sheaf ω_{Υ^0} pulling back to the canonical bundle K_Υ on Υ. Under the quotient map $\Xi^0 \to \Upsilon^0$ it pulls back to the canonical bundle K_Ξ, except for points on the divisor $\tilde{\Lambda}$. There we form the quotient in two steps, as in sect. 3, first dividing by $\mathbb{Z}_s$ and then by $\mathbb{Z}_t$. The pull-back via the quotient by $\mathbb{Z}_t$ is the canonical bundle of $\Xi/\mathbb{Z}_s$. The quotient map for $\mathbb{Z}_s$ is branched along $\tilde{\Lambda}$ to the order s. So the adjunction formula shows: The dualizing sheaf ω_{Υ^0} pulls back to

$$K_{\Xi^0} \otimes \mathcal{O}_{\Xi^0}((1-s)\tilde{\Lambda}) = K_{\Xi^0} \otimes \tau^*(\mathcal{O}_{\mathbb{P}_3}(2-2s)).$$

The divisor $\bar{\Xi} \subset \mathbb{P}_3 \times \mathbb{P}_1$ is a divisor of bi-degree $(n, 1)$. Hence $\bar{\Xi}$ has a dualizing sheaf

$$\omega_{\bar{\Xi}} = \mathcal{O}_{\mathbb{P}_3 \times \mathbb{P}_1}(n-4, -2).$$

Now the miracle happens:

$$n - 4 = 2s - 2.$$

This implies: The pull-back of ω_{Υ^0} to Ξ^0 equals the restriction of $\mathcal{O}_{\mathbb{P}_3 \times \mathbb{P}_1}(0, -2)$, i.e. it is trivial on Ξ^0.
We distinguish two cases:
a) $\lambda \neq \lambda_i$, $i = 1, ..., 4$: The adjunction formula for $Y' = Y'_\lambda = X_\lambda/G_n$ shows

$$\omega_{Y'} = \omega_{\Upsilon^0}|Y'.$$

So the pull-back of $\omega_{Y'}$ to X is trivial. This implies: $deg(\omega_{Y'})|C = 0$ for all irreducible curves $C \subset Y'$ and then $deg(K_Y|C) = 0$ for all irreducible curves $C \subset Y$. The surfaces Y have canonical bundles, which are numerically trivial. In particular those surfaces are all minimal. By the classification of algebraic surfaces [BPV p. 168] they are abelian, K3, hyper-elliptic or Enriques. Since we specified in sect. 6.1 rational curves on Y spanning a lattice of rank 19 in $NS(Y)$ the only possibility is K3.
b) $\lambda = \lambda_i$, $i = 1, ..., 4$: The proof of a) shows $deg(K_Y|C) = 0$ for all irreducible curves $C \subset Y$ not passing through the exceptional locus of the minimal desingularization $Y \to Y'$. In particular this holds for all curves C which are proper transforms of ample curves $D \subset Y'$. Now an arbitrary curve $C \subset Y$ is linearly equivalent to $E + C_1 - C_2$ with E exceptional and C_i proper transforms of ample curves $D_i \subset Y'$. Since all singularities on Y' are rational double points of type A,D,E, we have $K_Y.E = 0$. The method from a) then applies here too. ◻

Proposition 6.2. *The structure of the K3-surfaces Y_λ varies with λ.*

Proof. We restrict to surfaces near some surface Y, with Y' the quotient of a smooth surface X. Here we may assume that the total space Υ is smooth. If all surfaces near Y were isomorphic, locally near Y the fibration would be trivial [FG]. I.e., there would be an isomorphism $\Phi : Y \times D \to \Upsilon$ respecting the fibre structure. Here D is a copy of the complex unit disc. By the continuity of the induced map

$$Y = Y \times \lambda \to Y_\lambda$$

there is an isomorphism $Y \to Y_\lambda$ mapping the 19 rational curves from sect. 4.1 on Y to the corresponding curves on $Y_\lambda, \lambda \in D$. The covering $X \to Y'$ is defined by a subgroup in the fundamental group of the complement in Y of these rational curves. This implies that the isomorphism $Y \to Y_\lambda$ induces an isomorphism of the coverings $X \to X_\lambda$ equivariant with respect to the G_n-action.
Now this isomorphism must map the canonical bundle $\mathcal{O}_X(n-4)$ to the canonical bundle $\mathcal{O}_{X_\lambda}(n-4)$. Since the surfaces X_λ are simply-connected, the isomorphism maps $\mathcal{O}_X(1)$ to $\mathcal{O}_{X_\lambda}(1)$, i.e., it is given by a projectivity. This is in conflict with the following. $\square$

Lemma 6.2. *For general $\lambda \neq \mu$ there is no projectivity $\varphi : \mathbb{P}_3 \to \mathbb{P}_3$ inducing some G_n-equivariant isomorphism $X_\lambda \to X_\mu$.*

Proof. Assume that such an isomorphism φ exists. Equivariance means for each $g \in G_n$ and $x \in X_\lambda$ that $\varphi g(x) = g\varphi(x)$ or $\varphi^{-1}g^{-1}\varphi g(x) = x$. Since X_λ spans $\mathbb{P}_3$ this implies the same property for all $x \in \mathbb{P}_3$, i.e., the map φ is G_n-equivariant on all of $\mathbb{P}_3$. In particular, if $L \subset \mathbb{P}_3$ is a fixline for $g \in G_n$, then so is $\varphi(L)$. Then we may as well assume $\varphi(L) = L$. We obtain a contradiction by showing that the point sets $X_\lambda \cap L$ and $X_\mu \cap L$ in general are not projectively equivalent.
The cases n=6 and 12: We use the fix-line $L := \{x_0 = x_1 = 0\}$ of type M, fixed under $\sigma_{1,3} = \sigma(q_1, q_1)$ (notation of [S, p. 432]). The group H_L has order 8, containing in addition the symmetries $\sigma(q_1, 1)$ and $\sigma(q_1q_2, q_1q_2)$ sending a point $x = (0 : 0 : x_2 : x_3) \in L$ to

$$\sigma(q_1, 1)(x) = (0 : 0 : x_2 : -x_3), \quad \sigma(q_1q_2, q_1q_2)(x) = (0 : 0 : -x_3 : x_2).$$

Omitting the first two coordinates and putting $x_2 = 1, x_3 = u$, we find that a general H_L-orbit on L consists of points

$$(1 : u),\ (1 : 1/u),\ (1 : -u),\ (1 : -1/u).$$

The cross-ratio of these four points

$$CR = \frac{2u}{u + 1/u} : \frac{1/u + u}{2/u} = \frac{4u^2}{(1+u^2)^2}$$

varies with u. The intersection of $X_{6,\lambda}$ with L consists of one such orbit, the intersection of $X_{12,\lambda}$ of three orbits. This implies the assertion for $n = 6$ and 12.
The case n=8: Here we use the fix-line $L := \{x_1 = x_3, x_2 = 0\}$ of type M for $\pi_3\pi_4\pi_3'\pi_4'$. Again H_L has order 8 containing in addition $\pi_3\pi_4$ and $\sigma(q_1q_2, q_1q_2)$. They send a point $x = (u : 1 : 0 : 1) \in L$ to

$$\pi_3\pi_4(x) = (-2 : u : 0 : u), \quad \sigma(q_1q_2, q_1q_2)(x) = (u : -1 : 0 : -1).$$

Omitting the coordinates x_3 and x_4 we find that a general H_L-orbit consists of

$$(u : 1),\ (-u : 1),\ (2/u : 1),\ (-2/u : 1).$$

Their cross-ratio

$$CR = \frac{u-2/u}{u+2/u} : \frac{-u-2/u}{-u+2/u} = \frac{(u^2-2)^2}{(u^2+2)^2}$$

varies with u. The intersection of $X_{8,\lambda}$ consists of two such orbits. □

Corollary 6.1. *The general K3-surface Y_λ has Picard-number 19.*

7. Picard-Lattices

Here we compute the Picard lattices of our quotient K3-surfaces Y.

7.1. **The general case.** Denote by $V \subset H^2(Y,\mathbb{Z})$ the rank-19 lattice spanned (over $\mathbb{Z}$) by the rational curves from sect. 4.1. For $n = 6$ and 8 this lattice V is not the total Picard lattice:

Proposition 7.1. *a) (n=6) After perhaps interchanging curves N_{2i-1} and N_{2i} the two divisor-classes*

$$\begin{aligned} L &:= L_1 - L_2 + L_4 - L_5 + N_1 - N_2 + N_3 - N_4 + N_5 - N_6 + N_7 - N_8, \\ L' &:= L_1' - L_2' + L_4' - L_5' + N_1 - N_2 + N_3 - N_4 - N_5 + N_6 - N_7 + N_8 \end{aligned}$$

are divisible by 3 in NS(Y). Together with V the classes $L/3$ and $L'/3$ span a rank-19 lattice with discriminant $2 \cdot 3^2 \cdot 5$.
b) (n=8) After perhaps interchanging M_1 and M_2 the two classes

$$\begin{aligned} L &:= L_1 + L_3 + L_5 + M_1 + M_3 + M_4 + R_1 + R_3, \\ L' &:= L_1' + L_3' + L_5' + M_2 + M_3 + M_4 + R_1 + R_3 \end{aligned}$$

are divisible by 2 in NS(Y). Together with V the classes $L/2$ and $L'/2$ span a rank-19 lattice with discriminant $2^4 \cdot 3 \cdot 7$.

Proof. a) Consider reduction modulo 3

$$\varphi_3 : \mathbb{Z}^{22} = H^2(Y,\mathbb{Z}) \to H^2(Y,\mathbb{F}_3) = \mathbb{F}_3^{22}.$$

Because of

$$M_1^2 = -2, \quad M_1.L_i' = 0, \quad det(L_i', L_j')_{i,j=1,..,4} = 5,$$

the images of $M_1, L_1', L_2', L_3', L_4'$ span a subspace of $H^2(Y,\mathbb{F}_3)$ on which the intersection form has rank 5. The orthogonal complement C of this lattice in $H^2(Y,\mathbb{F}_3)$ has dimension 17 and the form is non-degenerate there. This C contains the classes $mod\, 3$ of the twelve curves

$$L_1, L_2, L_4, L_5, N_1, ..., N_8$$

Assume that

$$D_1 := \varphi_3 < L_1, L_2, L_4, L_5, N_1, ..., N_8 >$$

has $\mathbb{F}_3$-dimension 12. Then

$$D_2 := \varphi_3 < L_1 - L_2, L_4 - L_5, N_1 - N_2, N_3 - N_4, N_5 - N_6, N_7 - N_8 >$$

has dimension six. Since $D_1 \perp D_2$, this is a contradiction. We have shown: A non-trivial linear combination of the twelve classes $L_1, L_2, L_4, L_5, N_1, ..., N_8$ lies in the kernel of φ_3. By [T] such a 3-divisible class contains at least 12 curves. Hence we may assume the class is

$$L := \lambda_1(L_1 - L_2) + \lambda_4(L_4 - L_5) + \sum \nu_i(N_{2i-1} - N_{2i})$$

with $\lambda_j, \nu_i = \pm 1$ modulo 3. W.l.o.g. we put $\lambda_1 = 1$. Intersecting with L_3 we find $\lambda_4 = 1$ too. And after perhaps interchanging curves N_{2i-1} with N_{2i} we may assume $\nu_1 = ... = \nu_4 = 1$.
Exactly in the same way we find a class

$$L' := L'_1 - L'_2 + L'_4 - L'_5 + \sum \nu'_i (N_{2i-1} - N_{2i}), \quad \nu'_i = \pm 1 \, mod \, 3,$$

which is 3-divisible in $NS(Y)$. Then $L + L'$ is 3-divisible too, and by [T] contains precisely 12 curves. This implies that precisely two coefficients ν'_i cancel against the corresponding coefficients of L. If these are the coefficients ν'_3 and ν'_4, we are done. If this should not be the case, after perhaps interchanging $\{N_1, N_2\}$ with $\{N_3, N_4\}$, $\{N_5, N_6\}$ with $\{N_7, N_8\}$ we may assume $\nu'_1 = \nu'_3 = 1$ and $\nu'_2 = \nu'_4 = -1$. Denote by $T_2 : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})$, resp. $H^2(Y, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ the monodromy about $X_{6,2}$ (circling the parameter λ_2 in the parameter space) and by T_3 the monodromy about $X_{6,3}$. So T_2 interchanges $\{N_1, N_2\}$ with $\{N_3, N_4\}$, leaving fixed $\{N_5, N_6\}, \{N_7, N_8\}$ with T_3 doing just the opposite. $NS(Y)$ contains the classes (coefficients modulo 3)

	$\frac{L_1-L_2+L_4-L_5}{3}$	$\frac{L'_1-L'_2+L'_4-L'_5}{3}$	$\frac{N_1-N_2}{3}$	$\frac{N_3-N_4}{3}$	$\frac{N_5-N_6}{3}$	$\frac{N_7-N_8}{3}$
L	1	0	1	1	1	1
L'	0	1	1	-1	1	-1
$L + L'$	1	1	-1	0	-1	0
$T_2(L + L')$	1	1	0	± 1	-1	0
$T_3(L + L')$	1	1	-1	0	0	± 1

These classes would span in $NS(Y)/V$ a subgroup of order 3^4, in conflict with $d(V) = 2 \cdot 3^6 \cdot 5$, contradiction.
b) Here we consider reduction modulo 2

$$\varphi_2 : \mathbb{Z}^{22} = H^2(Y, \mathbb{Z}) \to H^2(Y, \mathbb{F}_2) = \mathbb{F}_2^{22}.$$

The subspace

$$C := \varphi_2 < L_1, L_3, L_5, M_1, M_2, M_3, M_4, R_1, R_3 > \subset H^2(Y, \mathbb{F}_2)$$

is totally isotropic. It is orthogonal to $D := \varphi_2 < L'_1, L'_2, L'_3, L'_4, N_1, N_2 >$. Because of

$$det(L'_i.L'_j)_{i,j=1,...,4} = 5, \quad det(N_i.N_j)_{i,j=1,2} = 3,$$

the intersection form on D is non-degenerate, and $D^\perp$ is non-degenerate of rank 16. This implies $dim \, C \leq 8$. So there is a class

$$L := \sum \lambda_i L_i + \mu_i M_i + \rho_i R_i$$

in the kernel of φ_2. By [N] it has precisely eight coefficients $= 1$. Intersecting

with	we find
L_2, L_4	$\lambda_1 = \lambda_3 = \lambda_5 =: \lambda$
R_2	$\rho_1 = \rho_3 =: \rho$

This implies that precisely one coefficient μ_i will vanish and $\lambda = \rho = 1$. In the same way one finds a class

$$L' := L'_1 + L'_3 + L'_5 + \sum \mu'_i M_i + R_1 + R_3$$

in the kernel of φ_2 with precisely one μ'_1 vanishing. The class

$$L + L' = L_1 + L_3 + L_5 + L'_1 + L'_3 + L'_5 + \sum (\mu_i + \mu'_i) M_i$$

also is divisible by 2 and has precisely eight non-zero coefficients. It follows that precisely two of the non-zero coefficients from μ_i and μ'_i coincide. If $\mu_3 = \mu_4 = \mu'_3 = \mu'_4 = 1$ we are done (perhaps after interchanging M_1 and M_2). If this is not the case, assume e.g. $\mu_1 = \mu_2 = \mu_4 = 1$, $\mu_3 = 0$. Denote by T the monodromy about the surface $X_{8,4}$ (circling the parameter λ_4 in the parameter space). It interchanges M_3 and M_4. So the two classes

$$\frac{L}{2} = \frac{1}{2}(L_1 + L_3 + L_5 + M_1 + M_2 + M_4 + R_1 + R_3)$$

$$\frac{T(L)}{2} = \frac{1}{2}(L_1 + L_3 + L_5 + M_1 + M_2 + M_3 + R_1 + R_3)$$

would belong to $NS(Y)$. However this contradicts

$$\frac{L}{2} \cdot \frac{T(L)}{2} = -\frac{7}{2} \notin \mathbb{Z}. \qquad \square$$

Theorem 7.1. *If the Neron-Severi group of Y has rank 19, it is generated by V and*

n	
6	$L/3, L'/3,$
8	$L/2, L'/2,$
12	*no other classes.*

Proof. Denote by $W \subset NS(Y)$ the lattice spanned by the 19 rational curves from sect. 4.1 and by $L/3$, $L'/3$ from prop. 6.1 a) (if $n = 6$) resp. $L/2$, $L'/2$ from prop. 6.1 b) (if $n = 8$). If $NS(Y) \neq W$ there would be an integral lattice W' with $W \subset W' \subset NS(Y)$ and $p := [W' : W]$ a prime such that p^2 divides $d(W)$. The only possibilities are $p = 2$ or $= 3$. The following table gives in each case generators for the p-subgroup $(W^\vee/W)^p$ of $W^\vee/W$:

n	p	generators for $(W^\vee/W)^p$
6	3	$(N_1 - N_2 - N_3 + N_4)/3$, $(N_5 - N_6 - N_7 + N_8)/3$
8	2	$(M_1 + M_2 + M_3)/2$, $(M_1 + M_2 + M_4)/2$, $(M_1 + M_2)/2 + (R_1 + 2R_2 + 3R_3)/4$
12	2	$M_1/2$, $M_2/2$, $M_3/2$
12	3	$(N_1 - N_2)/3$, $(N_3 - N_4)/3$

By [N] a divisor consisting of m disjoint rational curves on Y can be divisible by 2 only if $m = 8$ or $= 16$. For $n = 12$, $p = 2$ there are only three such curves, while for $n = 8$, $p = 2$ there are only the six curves $M_1, M_2, M_3, M_4, R_1, R_3$. These cases are excluded. By [T] a divisor consisting of m disjoint pairs of rational curves, each pair meeting in one point, is divisible by 3 only if $m = 6$ or $= 9$. This excludes the cases $p = 3$ and $n = 6$ or $= 12$. $\square$

7.2. **The special cases.** Just as before we denote by $V \subset NS(Y)$ the sub-lattice spanned by the rational curves from sect. 4.1. Now it has rank 20. In the same way, as in sect. 6.1 we check, that for $n = 6$ the classes $L/3, L'/3$ and for $n = 8$ the classes $L/2, L'/2$ in $NS(Y)$ exist. Intersecting with the twentieth rational curve we find, that the curves can be labelled as in the diagrams of sect. 4.2.

Theorem 7.2. *In all cases $NS(Y)$ is spanned by the classes from sect. 6.1 and the twentieth rational curve. The discriminants of the lattices are*

case	6,1	6,2	6,3	6,4	8,1	8,2	8,3	8,4	12,1	12,2	12,3	12,4
d	−15	−60	−60	−15	−28	−84	−168	−112	−660	−1320	−792	−132

Proof. Denote by C the twentieth rational curve and by W the lattice spanned by V and C. The discriminants in the table above are those of the lattice W. We have to show $W = NS(Y)$. If this would be not the case, there would be a lattice W' with $W \subset W' \subset NS(Y)$ such that $p := [W' : W]$ is a prime with p^2 dividing $d(W)$. In the following table we collect the possibilities and give in each case generators for the p-subgroup $(W^\vee/W)^p$ of $W^\vee/W$. (The cases 6,2 and 6,3 are essentially the same.)

case	p	generators
$6,2$	2	$M_1/2$, $(N_1 + C + N_4)/2$
$8,1$	2	$(M_1 + M_2 + M_4)/2$, $(M_3 + M_4 + R_1 + R_3)/2$
$8,2$	2	$(M_1 + M_2 + M_3)/2$, $(M_1 + M_2 + M_4)/2$
$8,3$	2	$(M_1 + M_2 + M_4)/2$, $(M_1 + M_2)/2 + (R_1 + 2R_2 + 3R_4)/4$
$8,4$	2	$(M_1 + M_2)/2 + (2N_1 + 2C + M_3 + 3M_4)/4$, $(M_1 + M_2)/2 + (R_1 + 2R_2 + 3R_3)/4$
$12,1$	2	$M_2/2$, $M_3/2$
$12,2$	2	$M_3/2$, $(2N_1 + 2C + M_1 + 3M_2)/4$
$12,3$	2	$M_3/2$, $(2S_1 + 2S_3 + 2C + M_1 + 3M_2)/4$
	3	$(N_1 - N_2)/3$, $(N_3 - N_4)/3$
$12,4$	2	$M_2/2$, $M_3/2$

In each single case there are not enough rational curves to meet the conditions [N] for a divisor divisible by 2 or [T] for a divisor divisible by 3. □

8. Comments

1) Denote by M_k the moduli-space of abelian surfaces with level-(1,k) structure In [Mu] the quotients $\mathbb{P}_3/G_6$, resp. $\mathbb{P}_3/G_8$ are identified with the Satake-compactification of M_3, resp M_4, and $\mathbb{P}_3/G_{12}$ is shown to be birationally equivalent with the Satake-compactification of M_5. However the proof there is not very explicit. It is desirable to have an explicit identification of the quotient $\mathbb{P}_3/G_n$ with the corresponding moduli space. The pencil Y'_λ on $\mathbb{P}_3/G_n$ might be useful.
2) We did not consider the quotient threefold $\mathbb{P}_3/G_n$. We just identified the minimal non-singular model Y_λ for each quotient Y'_λ. Of course it would be desirable to have a global resolution of $\mathbb{P}_3/G_n$ and to view our K3-surfaces as a pencil on this smooth threefold. One would need a particular crepant resolution of the singularities of Υ. Such resolutions are given e.g. in [I, IR, Ro]. We would need a resolution, where the behaviour of the K3-surfaces can be controlled, to identify the partial resolutions of the four special surfaces.
3) Our quotient surfaces admit a natural involution induced by the symmetry C from [S, p. 433] normalizing G_n, but not belonging to $SL(4,\mathbb{C})$. It would be interesting to identify the quotients.
4) By [Mo] each K3-surface with Picard number 19 admits a Nikulin-involution, an involution with eight isolated fix-points. We do not know how to identify it in our cases. It cannot be the involution from 3), because this has a curve of fix-points. It is also not clear to us, whether this Nikulin-involution exists globally, i.e. on the total space Υ of our fibration. This Nikulin-involution is related to the existence of a sub-lattice $E_8 \perp E_8 \subset NS(Y)$. We did not manage to identify such a sub-lattice.
5) It seems remarkable that the Picard group of the general surface in a pencil of K3-surfaces can be identified so explicitly, as it is done in sect. 6. It is also remarkable that the quotient K3-surfaces have Picard number ≥ 19. Such pencils have been studied in [Mo] and [STZ]. We expect our surfaces to have some arithmetical meaning. In particular

the prime factor $n-1=5,7,11$ in the discriminant of the Picard lattices draws attention. In fact, the same prime factor appears in each polynomial s_n, $n=6,8,12$ from [S]. It can be found too in the cross-ratio $CR(\lambda_1,...,\lambda_4)$ of the four special parameters in each pencil X_λ and together with strange prime factors in the absolute invariant j:

n	6	8	12
CR	$\frac{5^2}{3^2}$	$\frac{7^2}{2^4\cdot 3}$	$\frac{11^2}{2^5\cdot 3}$
j	$\frac{13^3\cdot 37^3}{2^8\cdot 3^4\cdot 5^4}$	$\frac{13^3\cdot 181^3}{2^8\cdot 3^2\cdot 7^4}$	$\frac{12\,241^3}{2^{10}\cdot 3^2\cdot 5^4\cdot 11^4}$

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GROUP ACTIONS, CYCLIC COVERINGS AND FAMILIES OF K3-SURFACES

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Abstract. In this paper we describe six pencils of $K3$-surfaces which have large Picard-number ($\rho = 19, 20$) and contain each precisely five special fibers: four have A-D-E singularities and one is non-reduced. In particular we describe these surfaces as cyclic coverings of the $K3$-surfaces of [BS]. In many cases using 3-divisible sets, resp. 2-divisible sets of rational curves and lattice theory we describe explicitly the Picard-lattices.

0. Introduction

In the last years using various methods (toric geometry, mirror symmetry, etc.), many families of $K3$-surfaces with large Picard-Number and small number of special fibers have been constructed and studied (see e.g. [D], [VY] and [Be]). In these notes using group actions and cyclic coverings we describe six new families where the generic surface has Picard-number 19 and we identifies four surfaces with Picard-number 20. These six pencils are related to three families of K3-surfaces studied by Barth and the author in [BS], the generic surface has Picard-number 19 and the pencils contain four surfaces with singularities of $A-D-E$ type and $\rho = 20$ and one non-reduced fiber. The families arise as minimal resolutions of quotients X^n_λ/G_n were G_n is a special finite subgroup of $SO(4,\mathbb{R})$ containing the Heisenberg group and $\{X^n_\lambda\}_{\lambda\in\mathbb{P}_1}$ is a G_n-invariant pencil of surfaces in $\mathbb{P}_3$, the latter are described in [S1] (we recall some facts in section 1). In section 1 and section 2 we describe six normal subgroups H of G_n which contain the Heisenberg group, we describe the fix points of H on X^n_λ and we show that the minimal resolutions are pencils of K3-surfaces which contain five special surfaces. Then in section 3 we show that the new families are certain cyclic coverings of the surfaces of [BS]. Then, by a classical result of Inose, [I, Cor. 1.2], they have the same Picard-number, hence the general surface in each of the six pencils has Picard number 19 and we have four surfaces with Picard-number 20. In section 4 by using the rational curves on the minimal resolutions and 2-divisible and 3-divisible sets of rational curves, we describe completely the Picard-lattice of many of the surfaces.

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1. Notations and preliminaries

There are two classical $2:1$ coverings:

$$SU(2) \to SO(3,\mathbb{R}) \text{ and } \sigma : SU(2)\times SU(2) \to SO(4,\mathbb{R}).$$

Denote by $T, O \subset SO(3,\mathbb{R})$, the rotation group of tetrahedron and octahedron, by $\widetilde{T}, \widetilde{O}$ the corresponding binary subgroups of $SU(2)$ and let $G_6 := \sigma(\tilde{T}\times\tilde{T})$, $G_8 := \sigma(\tilde{O}\times\tilde{O})$. We denote an element of $SU(2)\times SU(2)$ and its image in $SO(4,\mathbb{R})$ by (p_1, p_2). Let

$X_\lambda^6 = s_6 + \lambda q^3$ and $X_\lambda^8 = s_8 + \lambda q^4$ denote the pencils of G_6- and of G_8-invariant surfaces in $\mathbb{P}_3$, which are described in [S1], s_6 denotes a G_6-invariant homogeneous polynomial of degree six and s_8 denotes a G_8-invariant homogeneous polynomial of degree eight, $q := x_0^2 + x_1^2 + x_2^2 + x_3^2$ is the equation of the quadric $\mathbb{P}_1 \times \mathbb{P}_1$ in $\mathbb{P}_3$. The base locus of the pencils X_λ^n are $2n$ lines on the quadric, n in each ruling and each pencil contains exactly four nodal surfaces (cf. [S1]). Now recall the matrix:

$$C := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \in O(4, \mathbb{R}),$$

which operates on an element $(p_1, p_2) \in G_1 \times G_2$ by:

$$C^{-1}(p_1, p_2)C = (p_2, p_1).$$

Moreover we specify the following matrices of $SO(4, \mathbb{R})$:

$$(q_1, 1) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad (q_2, 1) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$(p_3, 1) = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad (p_4, 1) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Using these matrices the groups have the following generators:

Group	Generators
G_6	$(q_2, 1), (1, q_2), (p_3, 1), (1, p_3)$
G_8	$(q_2, 1), (1, q_2), (p_3, 1), (1, p_3), (p_4, 1), (1, p_4)$

Denote by PG the image of a subgroup $G \subset SO(4, \mathbb{R})$ in $\mathbb{P}GL(4, \mathbb{R})$. We define the *types* of lines in $\mathbb{P}_3$ which are fixed by elements $(p_1, p_2) \in PG$ of order $2, 3$ or 4 in the following way:

order	2	3	4
type	M	N	R

1.1. **Normal subgroups.** In [S2] the author classifies all the subgroups of $SO(4, \mathbb{R})$ which contain the Heisenberg group $V \times V$. Here we consider all the normal subgroups of G_6 and of G_8 which contain the subgroup $V \times V$, resp. G_6. We denote by H such a normal subgroup, by $o(H)$ its order and by $i(H) = [G_n : H]$ the index of H in G_n. We list below all the groups H and their generators, following the notation of [S2]. Moreover we do not consider separately the groups H and $C^{-1}HC$ or, in general, groups which are conjugate in $O(4, \mathbb{R})$. The group $T \times T$ is in fact the same as G_6, but to avoid confusion we use this notation when we consider it as subgroup of G_8.

$H \subset G_6$	generators	$o(H)$	$i(H)$	$H \subset G_8$	generators	$o(H)$	$i(H)$
$T \times V$	$(q_1,1),(1,q_1)$ $(p_3,1)$	96	3	$O \times T$	$(q_1,1),(1,q_1)$ $(p_3,1),(1,p_3)$ $(p_4,1)$	576	2
$(TT)'$	$(q_1,1),(1,q_1)$ $(q_2,1),(1,q_2)$ (p_3,p_3)	96	3	$(OO)''$	$(q_1,1),(1,q_1)$ $(p_3,1),(1,p_3)$ (p_4q_2,p_4q_2)	576	2
$V \times V$	$(q_1,1),(1,q_1)$ $(q_2,1),(1,q_2)$	32	9	$T \times T$	$(q_1,1),(1,q_1)$ $(p_3,1),(1,p_3)$	288	4

1.2. **Fix-points.** We analyze the different kind of fix-points for elements of the subgroups $PH \subset PG$ in the same way as in [BS]. Recall that the elements of the form $(p,1)$ or $(1,p)$ have each two disjoint lines of fix points contained in one ruling, respectively in the other ruling of the quadric (cf. [S1, 5.4 p. 439)]).

1) *Fix-points on the quadric.* The subgroups $G_1 \times 1$ and $1 \times G_2$ of PH operate on the two rulings of the quadric and determine orbits of lines. We give the lengths of the orbits in the following tables. In the first row we write the order of the element which fixes two lines of the orbit:

order of $(p,1)$	2	3	4
$T \times V$	6	4, 4	–
$O \times T$	12	8	6
$(TT)'$	6	–	–
$(OO)''$	6	8	–
$V \times V$	2, 2, 2	–	–
$T \times T$	6	4, 4	–

order of $(1,p)$	2	3	4
$T \times V$	2, 2, 2	–	–
$O \times T$	6	4, 4	–
$(TT)'$	6	–	–
$(OO)''$	6	8	–
$V \times V$	2, 2, 2	–	–
$T \times T$	6	4, 4	–

In particular observe that in the case of the groups $(TT)'$ and $(OO)''$ the meeting points of the fix-lines of the two rulings of $\mathbb{P}_1 \times \mathbb{P}_1$ split into three orbits of length 12 and two orbits of length 32, in the other cases these meeting-points form just one orbit.

2) *Fix-points off the quadric.* We denote by F_L the fix-group of a line L of $\mathbb{P}_3$ in PH and by H_L the stabilizer group of L in PH, i.e.

$$\begin{aligned} F_L &:= \{h \in PH \ s.t. \ hx = x \ for \ all \ x \in L\} \\ H_L &:= \{h \in PH \ s.t. \ hL = L\}. \end{aligned}$$

Moreover denote by $\ell(L)$ the length of the H-orbit of the line L and by g a representative of a conjugacy class in H:

group	$T \times V$			$(TT)'$				$V \times V$
g	(q_1,q_1)	(q_1,q_2)	(q_1,q_3)	(q_1,q_1)	(q_1,q_2)	(q_1,q_3)	(p_3,p_3)	(q_i,q_j)
F_L	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_2$
type	M_1	M_2	M_3	M_1	M_2	M_3	N	M_{ij}
$\ell(L)$	6	6	6	6	6	6	16	2
$\lvert H_L\rvert/\lvert F_L\rvert$	4	4	4	4	4	4	1	4

Here we denote by $q_3 \in SU(2)$ the product of q_1 and q_2. In the last column of the table the sum runs over $i,j = 1,2,3$. In this case we have nine distinct conjugacy classes with just one element.

group	$O\times T$			$(OO)''$				$T\times T$		
g	(q_1,q_1)	(p_3,p_3)	(p_4q_2,q_2)	(p_4,p_4)	(p_3,p_3)	(p_3^2,p_3)	(p_4q_2,p_4q_2)	(q_2,q_2)	(p_3,p_3)	(p_3^2,p_3)
F_L	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_3$	$\mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_3$
type	M	N	M'	R	N	N'	M	M	N	N'
$\ell(L)$	18	32	36	18	16	16	72	18	16	16
$\lvert H_L\rvert/\lvert F_L\rvert$	8	3	3	4	8	8	2	4	3	3

Remark 1.1. *By taking the generator (p_3^2,p_3) for $(TT)'$ instead of (p_3,p_3) we find a group $(TT)''$ which is conjugate in $O(4,\mathbb{R})$ to $(TT)'$. The description of the fix points is similar as in the case of $(TT)'$.*

2. Quotient surfaces

2.1. **Quotient singularities.** We consider now the projections:

$$\pi_H : X_\lambda^6 \longrightarrow X_\lambda^6/H, \quad \pi_{H'} : X_\lambda^8 \longrightarrow X_\lambda^8/H'$$

with $H = T\times V$, $(TT)'$ or $V\times V$; $H' = O\times T$, $(OO)''$ or $T\times T$. In this section we run the same program as in [BS], section 3 and describe the singularities of the quotients (for the details cf. [BS]) .

1) *Fix-lines on q.* The image in the quotient of the lines of the base locus of the pencils X_λ^6 and X_λ^8 and of the intersection points of the lines of the base locus are smooth. Observe that the points of intersection of the lines of the base locus of the pencils form one orbit under the action of $T\times V$, $V\times V$, $O\times T$ and $T\times T$. In the case of the groups $(TT)'$ we have three orbits and in the case of the group $(OO)''$ we have two orbits, as described in 1.2, this means that the lines in the quotient will meet three times and two times. Now we consider the points of intersection of the lines of the base locus with the other fix-lines on q. In the table below we do not write the groups $(TT)'$ and $V\times V$ because they do not have other fix-points on q other than the lines of the base locus. We denote by $\mathrm{Fix}(P)$ the fix-group in PG of a point P. In the next table we write the length and the number of orbits of fix-points, and we describe which kind of singularities do we have in the quotient:

group	$T\times V$	$O\times T$			$(OO)''$		$T\times T$	
$\mathrm{Fix}(P)$	$\mathbb{Z}_3\times\mathbb{Z}_2$	$\mathbb{Z}_3\times\mathbb{Z}_2$	$\mathbb{Z}_4\times\mathbb{Z}_3$	$\mathbb{Z}_2\times\mathbb{Z}_3$	$\mathbb{Z}_2\times\mathbb{Z}_3$	$\mathbb{Z}_3\times\mathbb{Z}_2$	$\mathbb{Z}_3\times\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_3$
length	8	48	24	48	48	48	24	24
number	6	1	2	2	1	1	2	2
sing.	$6A_2$	$1A_1$	$2A_3$	$2A_1$	$1A_1$	$1A_1$	$2A_1$	$2A_1$

2)*Fix-lines off q.* Denote by $o(L)$ the order of the fix-group F_L of L. The number of points not on q cut out on X_λ^n by L is:

group	$T\times V$	$(TT)'$		$V\times V$	$O\times T$		$(OO)''$		$T\times T$	
$o(L)$	2	2	3	2	2	3	4	3	2	3
number	4	4	6	4	8	6	8	6	8	6

In the next table we show in each case length and number of H_L-orbits, the number and type(s) of the quotient singularity(ies):

group	$T\times V$			$(TT)'$				$V\times V$
$o(L)$	2	2	2	2	2	2	3	2
type	M_1	M_2	M_3	M_1	M_2	M_3	N	M_{ij}
length	4	4	4	4	4	4	1	4
number	1	1	1	1	1	1	6	1
singularities	A_1	A_1	A_1	A_1	A_1	A_1	$6A_2$	A_1

group	$O\times T$			$(OO)''$				$T\times T$		
$o(L)$	2	3	2	4	3	3	2	2	3	3
type	M	N	M'	R	N	N'	M	M	N	N'
length	8	3	3	4	6	6	2	4	3	3
number	1	2	2	2	1	1	4	2	2	2
singularities	A_1	$2A_2$	$2A_1$	$2A_3$	A_2	A_2	$4A_1$	$2A_1$	$2A_2$	$2A_2$

3) *The singular surfaces.* We denote by ns the number of nodes on the surfaces and by F the fix-group of a node in H. In the table below, we give the number of orbits of nodes and their fix-groups in PH, PH' and we describe the singularities in the quotient. We recall [BS, proposition 3.1]:

Proposition 2.1. *Let X be a nodal surface with $F\subset SO(3)$ the fix-group of the node. Then the image of this node on X/H is a quotient singularity locally isomorphic with $\mathbb{C}^2/\tilde F$, where $\tilde F\subset SU(2)$ is the binary group which corresponds to F.*

group	$T\times V$				$(TT)'$				$V\times V$			
λ	λ_1	λ_2	λ_3	λ_4	λ_1	λ_2	λ_3	λ_4	λ_1	λ_2	λ_3	λ_4
ns	12	48	48	12	12	48	48	12	12	48	48	12
orbit	1	1	1	1	3	3	1	1	3	3	3	3
F	$\mathbb{Z}_2\times\mathbb{Z}_2$	id	id	$\mathbb{Z}_2\times\mathbb{Z}_2$	T	$\mathbb{Z}_3$	id	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2$	id	id	$\mathbb{Z}_2\times\mathbb{Z}_2$
lines meeting	$1M_1$ $1M_2$ $1M_3$	–	–	$1M_1$ $1M_2$ $1M_3$	$3M_i$ $4N$	$1N$	–	$1M_1$ $1M_2$ $1M_3$	$3M_{ij}$	–	–	$3M_{ij}$
sing.	D_4	A_1	A_1	D_4	$3E_6$	$3A_5$	A_1	D_4	$3D_4$	$3A_1$	$3A_1$	$3D_4$

group	$O\times T$				$(OO)''$				$T\times T$			
λ	λ_1	λ_2	λ_3	λ_4	λ_1	λ_2	λ_3	λ_4	λ_1	λ_2	λ_3	λ_4
ns	24	72	144	96	24	72	144	96	24	72	144	96
orbit	1	1	1	1	2	1	1	2	2	1	1	2
F	T	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	O	$\mathbb{Z}_4$	$\mathbb{Z}_2$	D_3	T	$\mathbb{Z}_2$	id	$\mathbb{Z}_3$
lines meeting	$3M$ $4N$	$1M$ $2M'$	$1M'$	$1N$	$3R$ $4N(N')$ $6M$	$1R$	$1M$	$1N(N')$ $3M$	$3M$ $4N(N')$	$1M$	–	$1N(N')$
sing.	E_6	D_4	A_3	A_5	$2E_7$	A_7	A_3	$2D_5$	$2E_6$	A_3	A_1	$2A_5$

2.2. **Rational curves.** Let

$$\mu : Y_{\lambda,H} \longrightarrow X_\lambda^n/H$$

be the minimal resolution of the singularities of X_λ^n/H. In the following table we give the number of rational curves coming from the curves of the base locus of X_λ^n (denote it by ν_1) and from the resolution of the singularities. The latter are of three kinds: those coming from the intersection points of the lines of the base locus with other fix-lines on q, those coming from fix-points which are off q and do not come from nodes of X^n_λ, and those coming from the nodes. We denote their numbers by ν_2, ν_3 and ν_4, then the total number of rational curves is $\nu := \nu_1 + \nu_2 + \nu_3 + \nu_4$. The configurations of some of the curves are then given in the figures in section 6. In the table we write the discriminant, d, of the intersection matrix too, this is easy to compute since we know the configurations of the rational curves. Since in each case $d \neq 0$ the classes of the curves are independent in $NS(Y_{\lambda,H})$.

1. The smooth X^n_λ.

group	$T\times V$	$(TT)'$	$V\times V$	$O\times T$	$(OO)''$	$T\times T$
ν_1	4	2	6	3	2	4
ν_2	12	–	–	9	2	4
ν_3	3	15	9	7	14	10
ν	19	17	15	19	18	18
d	$2^5\cdot3^3\cdot5$	$2^3\cdot3^6\cdot5$	$2^{13}\cdot5$	$2^5\cdot3^3\cdot7$	$-2^8\cdot3^2\cdot7$	$-2^2\cdot3^6\cdot7$

2. The singular X^n_λ. In this case the surfaces X^n_λ do not have extra singularities on q, hence the number ν_1 and ν_2 remain the same as above and we do not write them again.

group	$T\times V$				$(TT)'$			
λ	λ_1	λ_2	λ_3	λ_4	λ_1	λ_2	λ_3	λ_4
ν_3	–	3	3	–	–	3	15	12
ν_4	4	1	1	4	18	15	1	4
ν	20	20	20	20	20	20	18	18
d	$-2^4\cdot3^3\cdot5$	$-2^6\cdot3^3\cdot5$	$-2^6\cdot3^3\cdot5$	$-2^4\cdot3^3\cdot5$	$-3^3\cdot5$	$-2^6\cdot3^3\cdot5$	$-2^4\cdot3^6\cdot5$	$-2^2\cdot3^6\cdot5$

group	$V\times V$				$O\times T$			
λ	λ_1	λ_2	λ_3	λ_4	λ_1	λ_2	λ_3	λ_4
ν_3	–	9	9	–	2	4	5	3
ν_4	12	3	3	12	6	4	3	5
ν	18	18	18	18	20	20	20	20
d	$-2^{10}\cdot5$	$-2^{16}\cdot5$	$-2^{16}\cdot5$	$-2^{10}\cdot5$	$-2^4\cdot3^2\cdot7$	$-2^4\cdot3^3\cdot7$	$-2^5\cdot3^3\cdot7$	$-2^6\cdot3^2\cdot7$

group	$(OO)''$				$T\times T$			
λ	λ_1	λ_2	λ_3	λ_4	λ_1	λ_2	λ_3	λ_4
ν_3	–	8	12	6	–	8	10	2
ν_4	16	7	3	10	12	3	1	10
ν	20	19	19	20	20	19	19	20
d	$-2^4\cdot7$	$2^7\cdot3^2\cdot7$	$2^8\cdot3^2\cdot7$	$-2^8\cdot7$	$-3^4\cdot7$	$2^2\cdot3^6\cdot7$	$2^3\cdot3^6\cdot7$	$-2^4\cdot3^4\cdot7$

2.3. **K3-surfaces.** Since the groups H and H' contain the subgroups $V\times V$ of G_6 resp. $T\times T$ of G_8 the projections π_H and $\pi_{H'}$ are ramified on the lines of the base locus of the families X^n_λ with ramification index two and three. By using Hurwitz-formula and the fact that in each case the previous rational curves are independent in the Neron-Severi group, the same computation as in [BS, section 5] shows that the minimal resolutions of the quotients are $K3$-surfaces, a direct proof of this fact is given in the next section.

3. Cyclic Coverings

We give another description of the pencils of K3-surfaces by using cyclic coverings.
We consider the pairs G_n and H so that G_n/H is cyclic, in our cases either $|G_n/H| = 3$ or $|G_n/H| = 2$, and we consider the map:

$$\pi : X_\lambda^n/H \longrightarrow X_\lambda^n/G_n$$

3.1. **The general case.** For the moment assume that X_λ^n is smooth. The group G_n/H acts on the points of the fiber $\pi^{-1}(P)$. If the point P is not fixed by G_n/H then the map is $3:1$ or $2:1$ there. If P is fixed by G_n/H then we have a singularity on X_λ^n/H, more precisely an A_2 or an A_1, now the fiber $\pi^{-1}(P)$ is one point and the map has multiplicity 2 or 3 there (cf. [M2, Lemma 3.6 p. 80]). We have a rational map between the minimal resolutions of X_λ^n/H and X_λ^n/G_n:

$$\gamma : Y_{\lambda,H} - - - \rightarrow Y_{\lambda,G_n}$$

which is $3:1$ or $2:1$. Observe that this map is not defined over the (-2)-rational curves in the blow up of the singular points of X_λ^n/G_n which comes from fix-points of G_n/H on X_λ^n/H. The surfaces $Y_{\lambda,H}$ are K3-surfaces as well and by [I, Cor. 1.2] these have the same Picard-number $\rho(Y_{\lambda,H}) = \rho(Y_{\lambda,G_n})= 19$.
In this section we describe the map γ by using cyclic coverings. For the general theory about 2-cyclic coverings and 3-cyclic coverings we send back to the article [N] of Nikulin and to the articles [M1] of Miranda and [T] of Tan. For the convenience of the reader in Figure 1, section 6, we recall the configurations of (-2)-rational curves on the smooth surfaces Y_{λ,G_6} and on Y_{λ,G_8} given in [BS].
By [BS, proposition 6.1] the following classes are 3-divisible in $NS(Y_{\lambda,G_6})$:

$$\begin{aligned}\mathcal{L} &:= L_1 - L_2 + L_4 - L_5 + N_1 - N_2 + N_3 - N_4 + N_5 - N_6 + N_7 - N_8,\\ \mathcal{L}' &:= L_1' - L_2' + L_4' - L_5' + N_1 - N_2 + N_3 - N_4 - N_5 + N_6 - N_7 + N_8,\end{aligned}$$

and also:

$$\begin{aligned}\mathcal{L} - \mathcal{L}' &= L_1 - L_2 + L_4 - L_5 - L_1' + L_2' - L_4' + L_5' + 2(N_5 - N_6 + N_7 - N_8),\\ \mathcal{L} + \mathcal{L}' &= L_1 - L_2 + L_4 - L_5 + L_1' - L_2' + L_4' - L_5' + 2(N_1 - N_2 + N_3 - N_4).\end{aligned}$$

Making reduction modulo three we find the classes:

$$\begin{aligned}\mathcal{M} &:= L_1 - L_2 + L_4 - L_5 - L_1' + L_2' - L_4' + L_5' - (N_5 - N_6 + N_7 - N_8),\\ \mathcal{M}' &:= L_1 - L_2 + L_4 - L_5 + L_1' - L_2' + L_4' - L_5' - (N_1 - N_2 + N_3 - N_4).\end{aligned}$$

In $NS(Y_{\lambda,G_8})$ the following classes are 2-divisible:

$$\begin{aligned}\mathcal{L} &:= L_1 + L_3 + L_5 + M_1 + M_3 + M_4 + R_1 + R_3,\\ \mathcal{L}' &:= L_1' + L_3' + L_5' + M_2 + M_3 + M_4 + R_1 + R_3.\end{aligned}$$

Consider also the classes $\mathcal{L} + \mathcal{L}'$ and $\mathcal{L} - \mathcal{L}'$, which after reduction modulo two are the same as:

$$\mathcal{M} := L_1 + L_3 + L_5 + L_1' + L_3' + L_5' + M_1 + M_2.$$

These classes consist of six disjoint A_2-configurations of curves and of eight disjoint A_1-configurations of curves (according to [T] and [N]). These are the resolutions of A_2 and A_1 singularities of X_λ^n/G_n which arise by doing the quotient of X_λ^n/H by G_n/H. We construct the 3-cyclic coverings and the 2-cyclic coverings by using the divisors $\mathcal{L}, \mathcal{L}', \mathcal{M}, \mathcal{M}'$. We recall briefly the construction in the case of 3-cyclic coverings, then in the case of 2-cyclic coverings it is similar. First to avoid to produce singularities, we have to blow up the meeting points of the A_2-configurations. Call Y^0_{λ,G_6} the surface which we obtain after these blow-ups. The meeting points are replaced by (-1)-curves and the two (-2)-curves become now (-3)-curves. Denote by $\phi : Y^1_{\lambda,G_6} \longrightarrow Y^0_{\lambda,G_6}$ the 3-cyclic covering with branching divisor $\mathcal{L}$, $\mathcal{L}'$ or $\mathcal{M}$, $\mathcal{M}'$ then

Proposition 3.1. *A configuration of curves on* Y^0_{λ,G_6}*:*

$$\overset{-3}{\bullet}\text{———}\overset{-1}{\bullet}\text{———}\overset{-3}{\bullet}$$

becomes a configuration:

$$\underset{-1}{\bullet}\text{———}\underset{-3}{\bullet}\text{———}\underset{-1}{\bullet}$$

on Y^1_{λ,G_6}.

Proof. We do the computation for one configuration of curves $L_1 - L_2$, this is the same in the other cases. Denote again by L_1 and L_2 the curves on Y^0_{λ,G_6} which now are (-3)-curves and denote by E the exceptional (-1)-curve. By the properties of cyclic coverings we have $\phi^* L_i = 3\tilde{L_i}$, where $\tilde{L_i}$ is the strict transform of L_i. Then:

$$9(\tilde{L_i})^2 = (\phi^* L_i)^2 = (\deg\phi) L_i^2 = -9.$$

Hence $(\tilde{L_i})^2 = -1$. Since $E\cdot(L_1 - L_2) = 0$ the map ϕ is not ramified on E and the restriction $\phi_{|\tilde{E}}$ is $3:1$ onto E. Hence we have $\phi^* E = \tilde{E}$ and $\tilde{E}^2 = (\phi^* E)^2 = (\deg\phi) E^2 = 3E^2 = -3$. $\square$

Our surface Y^1_{λ,G_6} is now no more minimal. By blowing down the (-1)-curves, the curve $\tilde{E}$ becomes also a (-1)-curve so we blow it down too. By construction the surfaces which we obtain are minimal K3-surfaces and are exactly the surfaces $Y_{\lambda,H}$ which are obtained as the minimal resolutions of X^6_λ/H, in fact we have a commutative diagram:

$$\begin{array}{ccccc} & Y^1_{\lambda,G_6} & \xrightarrow{\phi} & Y^0_{\lambda,G_6} & \\ \swarrow & & & & \searrow \\ Y_{\lambda,H} & & \overset{\gamma}{- - \rightarrow} & & Y_{\lambda,G_6} \\ \searrow & & & & \swarrow \\ & X^6_\lambda/H & \xrightarrow{\pi} & X^6_\lambda/G_6 & \end{array}$$

The construction is similar in the case of 2-cyclic coverings of the surfaces Y_{λ,G_8}. This gives another description of the families of K3-surfaces $Y_{\lambda,T\times V}$, $Y_{\lambda,(TT)'}$ and $Y_{\lambda,O\times T}$, $Y_{\lambda,(OO)''}$ as finite coverings of the families Y_{λ,G_6} and Y_{λ,G_8}.

Remark 3.1. *By using the divisors* $\mathcal{L}'$ *and* $\mathcal{M}'$ *on* Y_{λ,G_6} *for the coverings we obtain the surfaces* $Y_{\lambda,V\times T}$ *and* $Y_{\lambda,(TT)''}$ *and by taking the divisor* $\mathcal{L}'$ *on* Y_{λ,G_8} *we obtain the surface* $Y_{\lambda,T\times O}$. *We do not discuss these surfaces separately in the sequel.*

3.2. **The special cases.** In these cases, the situation is a little more complicated. Now in the counterimage $\pi^{-1}(P)$ of some singular point P of X^n_λ/G_n coming from the A_1-singularities of X^n_λ we have singularities on X^n_λ/H too. In the following table we give the singularities in the quotient X^n_λ/G_n, $n = 6, 8$ and the type and the number of singularities in the counterimage on X^n_λ/H. As in [BS] we donte by $6,1,\ldots,8,4$ the special surfaces in the families.

	$6,1$	$6,2$	$6,3$	$6,4$		$8,1$	$8,2$	$8,3$	$8,4$
G_6	E_6	A_5	A_5	E_6	G_8	E_7	D_6	D_4	D_5
$T \times V$	D_4	A_1	A_1	D_4	$O \times T$	E_6	D_4	A_3	A_5
$(TT)'$	$3E_6$	$3A_5$	A_1	D_4	$(OO)''$	$2E_7$	A_7	A_3	$2D_5$
$V \times V$	$3D_4$	$3A_1$	$3A_1$	$3D_4$	$T \times T$	$2E_6$	A_3	A_1	$2A_5$

By resolving the quotients we get a map like γ as before and so again by the result of Inose the minimal resolutions of the special K3-surfaces are K3-surfaces too with Picard-number 20. We can describe this map as before by using cyclic coverings. In the case of the special surfaces in the family Y_{λ,G_6} we construct 3-cyclic covering as in the general case by using the divisors $\mathcal{L}$, $\mathcal{L}'$, $\mathcal{M}$, $\mathcal{M}'$, which are in the case of the special surfaces 3-divisible too, cf. [BS, 6.2]. By taking $\mathcal{L}$ and $\mathcal{L}'$ we obtain the special $K3$-surfaces in the families $Y_{\lambda,T\times V}$, resp. $Y_{\lambda,V\times T}$, by taking $\mathcal{M}$ and $\mathcal{M}'$ we obtain the special $K3$-surfaces in the covering $Y_{\lambda,(TT)'}$ and $Y_{\lambda,(TT)''}$. In the case of the special surfaces in the family Y_{λ,G_8} we take the divisors $\mathcal{L}$, $\mathcal{L}'$, $\mathcal{M}$ and we do 2-cyclic coverings. By taking $\mathcal{L}$ or $\mathcal{L}'$ we find the singular surfaces in the family $Y_{\lambda,O\times T}$ resp. $Y_{\lambda,T\times O}$ and by taking $\mathcal{M}$ we find the singular surfaces in the family $Y_{\lambda,(OO)''}$.

4. Picard-lattices

We compute the Picard-lattices of the general K3-surface in the families $Y_{\lambda,T\times V}$, $Y_{\lambda,O\times V}$ and of the special surfaces with $\rho = 20$ in each pencil. First we recall some facts. Denote by W the lattice spanned by the curves of section 2, 2.2. If W is not the total Picard-lattice, which we call NS there is an integral lattice W' s.t. $W \subset W' \subset NS$ with $p := [W' : W]$ a prime number. Denote by $d(W)$, $d(W')$ the discriminant of the lattices W, W'. Since $[W' : W]^2 = d(W) \cdot d(W')^{-1}$ (cf. [BPV, Lemma 2.1, p. 12]) we find that p^2 divides the discriminant of W. Denote by $(W^\vee/W)^p$ the p-subgroup of $(W^\vee/W) \subset NS^\vee/NS$ and denote by T the transcendental lattice orthogonal to the Picard-lattice. Since the discriminant groups $T^\vee/T$ and $NS^\vee/NS$ are isomorphic (cf. e.g. [BPV, p. 13 Lemma 2.5]), they have the same rank which is $\leq \mathrm{rk}(T)$. It follows that also $\mathrm{rk}(W^\vee/W)^p \leq \mathrm{rk}(T)$.

Proposition 4.1. *The Picard-lattices of the generic surface $Y_{\lambda,T\times V}$ and $Y_{\lambda,O\times T}$ are generated by the 19 rational curves of section 2, 2.2 and the classes:*

$$\frac{\bar{L}'}{3} := \frac{L_1 - L_2 + L_4 - L_5 + L_1' - L_2' + L_4' - L_5' + L_1'' - L_2'' + L_4'' - L_5''}{3}$$
$$\frac{h_1}{2} := \frac{L_1 + L_3 + L_5 + L_1' + L_3' + L_5' + M_1 + M_2}{2},$$
$$\frac{h_2}{2} := \frac{L_1 + L_3 + L_5 + L_1'' + L_3'' + L_5'' + M_1 + M_3}{2}$$

of $NS(Y_{\lambda,T\times V})$*, then the lattice has discriminant* $2\cdot 3\cdot 5$*; resp. the classes*

$$\frac{\bar{L}''}{2} := \frac{L_1+L_3+L_5+L_1'+L_3'+L_5'+M_1+M_2}{2}$$
$$\frac{k_1}{3} := \frac{L_1-L_2+L_4-L_5-L_1'+L_2'-L_4'+L_5'+N_1-N_2+N_3-N_4}{3}$$

of $NS(Y_{\lambda,O\times T})$*, then the lattice has discriminant* $2^3\cdot 3\cdot 7$*.*

Proof. 1. The discriminant of the lattice generated by the 19 curves is $2^5\cdot 3^3\cdot 5$ hence we can have 2-divisible classes or 3-divisible classes. The divisor $\bar{L}'$ is 3-divisible since it is the pull back of the divisor $\mathcal{L}'$ on Y_{λ,G_6} which is 3-divisible too. And we cannot have more 3-divisible classes. If there are no 2-divisible classes then the group $(W^\vee/W)^2$ would contain the classes $M_1/2$, $M_2/2$, $M_3/2$, $(L_1+L_3+L_5+L_1'+L_3'+L_5')/2$, $(L_1+L_3+L_5+L_1''+L_3''+L_5'')/2$, $(L_1'+L_3'+L_5'+L_1''+L_3''+L_5'')/2$ which are independent classes with respect to the intersection form. Since the rank of $(W^\vee/W)^2$ is less or equal as the rank of $T^\vee/T$ which is at most three, it can not happen that we find five classes as before. Hence some combination of them must be contained in the Neron-Severi group. So we have

$$\frac{1}{2}(\lambda(L_1+L_3+L_5)+\lambda'(L_1'+L_3'+L_5')+\lambda''(L_1''+L_2''+L_3'')+\mu_1M_1+\mu_2M_2+\mu_3M_3)\in NS$$

for some parameters $\lambda,\lambda',\lambda'',\mu_1,\mu_2,\mu_3\in\mathbb{Z}_2$.
By Nikulin [N] such a 2-divisible set contains 8 curves. So putting $\lambda''=0$ and $\mu_3=0$ we get the divisor $h_1/2$, putting $\lambda'=0$ and $\mu_2=0$ we get the divisor $h_2/2$. The discriminant of the lattice W together with these three classes now change into $2\cdot 3\cdot 5$, hence we cannot have more torsion classes.
2. Again the class $\bar{L}''$ is the pull back of the class $\mathcal{M}'$ on Y_{λ,G_8} hence it is 2-divisible. If there are no 3-divisible classes then the group $(W^\vee/W)^3$ would contain the classes $N_1-N_2/3$, $N_3-N_4/3$ and $(L_1-L_2+L_4-L_5+L_1'-L_2'+L_4'-L_5')/3$ which are independent. By specializing to the surfaces $Y^{(8,2)}_{\lambda,O\times O}$ and $Y^{(8,3)}_{\lambda,O\times O}$ we find also here these three independent classes and so $\mathrm{rk}(W^\vee/W)^3\geq 3$. This is not possible in fact on these surfaces we have rk(W)=20 which implies $\mathrm{rk}(W^\vee/W)^3\leq 2$. This means that the three classes fit together giving a 3-divisible class in $NS(Y^{(8,2)}_{\lambda,O\times O})$ and $\mathrm{NS}(Y^{(8,3)}_{\lambda,O\times T})$ and so in $NS(Y_{\lambda,O\times T})$ (cf. [vGT, Lemma 2.3]). $\square$

In the same way as before we can compute the Picard-lattices of the special surfaces in the families. We give the results leaving the proofs to the reader.

Proposition 4.2. *1. The Picard-lattice of the special surfaces in* $Y_{\lambda,T\times V}$ *and* $Y_{\lambda,O\times T}$ *is generated in all the cases but* $Y^{(8,4)}_{\lambda,O\times T}$ *by the curves of section 2, 2.2 and by the classes* $\bar{L}'/3$*,* $h_1/2$*,* $h_2/2$*, resp.* $\bar{L}''/2$*,* $k_1/3$ *of proposition 4.1. In the case of* $Y^{(8,4)}_{\lambda,O\times T}$ *the class:*

$$\frac{L_1+L_3+L_5+N_1+C+N_4+R_2+M_1}{2}$$

is a generator too, and they span the 20-dimensional Picard-lattice.
2. In the case of $Y^{(6,1)}_{\lambda,(TT)'}$ *and of* $Y^{(6,2)}_{\lambda,(TT)'}$ *the class:*

$$\frac{\bar{L}}{3} := \frac{N_1-N_2+N_3-N_4+N_5-N_6+N_7-N_8+N_9-N_{10}+N_{11}-N_{12}}{3}$$

is in the Neron-Severi group and in the case of $Y^{(6,2)}_{\lambda,(TT)'}$ *the classes:*

$$\frac{N_1+C_1+N_4+N_5+C_2+N_8+M_1+M_2}{2},$$
$$\frac{N_1+C_1+N_4+N_9+C_3+N_{12}+M_1+M_3}{2}$$

are in the Neron-Severi group too. These together with the 20 *curves of section 2, 2.2 span the 20-dimensional Picard-lattice.*
3. In the case of $Y^{(8,1)}_{\lambda,(OO)''}$ *and* $Y^{(8,4)}_{\lambda,(OO)''}$, *the class:*

$$\frac{\bar{L}}{2} := \frac{M_1+M_2+M_3+M_4+R_1+R_3+R_1'+R_3'}{2}$$

is in the Neron-Severi group and in the case of $Y^{(8,4)}_{\lambda,(OO)''}$ *the class:*

$$\frac{W}{4} := \frac{R_1+2R_2+3R_3+R_1'+2R_2'+3R_3'+2N_1+2C_1+3M_1+M_2+2N_3+2C_2+3M_3+M_4}{4}$$

is in the Neron-Severi group too.
Again these classes together with the 20 curves of section 2, 2.2 span the 20-dimensional Picard-lattice.
The discriminants of the Picard-lattices then are:

	$Y_{\lambda,T\times V}$				$Y_{\lambda,(TT)'}$	
	$6,1$	$6,2$	$6,3$	$6,4$	$6,1$	$6,2$
d	$-3\cdot 5$	$-2^2\cdot 3\cdot 5$	$-2^2\cdot 3\cdot 5$	$-3\cdot 5$	$-3\cdot 5$	$-2^2\cdot 3\cdot 5$

	$Y_{\lambda,O\times T}$				$Y_{\lambda,(OO)''}$	
	$8,1$	$8,2$	$8,3$	$8,4$	$8,1$	$8,4$
d	$-2^2\cdot 7$	$-2^2\cdot 3\cdot 7$	$-2^3\cdot 3\cdot 7$	$-2^2\cdot 7$	$-2^2\cdot 7$	$-2^4\cdot 7$

4.1. **More cyclic coverings.** Now we can construct the 3-cyclic covering of $Y_{\lambda,T\times V}$, by using the 3-divisible classes $\bar{L}'$ and the 2-cyclic coverings of $Y_{\lambda,O\times T}$, by using the 2-divisible class $\bar{L}''$. We can do this for the general surface in the pencil and for the special surfaces too, in this case we obtain another description of the families $Y_{\lambda,V\times V}$ and $Y_{\lambda,T\times T}$. In particular also in these cases the general surface in the family has Picard-number 19 and we have four surfaces with Picard-number 20. The description of the Picard-lattices of the surfaces with $\rho = 20$ is given in the following proposition (again, we leave the proof to the reader):

Proposition 4.3. *The classes*

$$\frac{L_2-L_4+L_3'-L_1'+N_1-N_2+N_3-N_4+N_5-N_6+N_7-N_8}{3},$$
$$\frac{L_2'-L_4'+L_3-L_1+N_1-N_2-N_3+N_4+N_5-N_6-N_7+N_8}{3}$$

are in $NS(Y^{(8,1)}_{\lambda,T\times T})$ *and in* $NS(Y^{(8,4)}_{\lambda,T\times T})$. *Moreover the class*

$$\frac{N_1+C_1+N_4+N_5+C_2+N_8+M_1+M_2}{2}$$

is in $NS(Y^{(8,4)}_{\lambda,T\times T})$ *too. These classes together with the rational curves of section 2, 2.2 span the 20-dimensional Picard-lattices of the surfaces* $Y^{(8,1)}_{\lambda,T\times T}$ *and* $Y^{(8,4)}_{\lambda,T\times T}$*. Then the lattices have discriminant* -7*, resp.* $-2^2\cdot 7$*.*

5. Final remarks

1. In the section 4 we identify explicitly the Picard-lattice of some K3-surfaces. It is our next aim to compute the transcendental lattices orthogonal to the Picard-lattices to classify the K3-surfaces. In particular by a result of Shioda and Inose, cf. [SI], K3-surfaces with $\rho = 20$ are classified by means of their transcendental lattice.
2. By a result of Morrison, cf. [Mo], each K3-surface with $\rho = 19$ or 20 admits a so called Shioda-Inose structure. This means that there is a Nikulin-involution, an involution with eight isolated fix-points and the quotient is birational to a Kummer-surface. It would be desirable to have an explicit description of this structure for our surfaces.
3. We do not describe the quotients 3-folds $\mathbb{P}_3/G_n$, $\mathbb{P}_3/H$, H a normal subgroup of G_n. It would be interesting to have a global resolution of these spaces and to see our $K3$-surfaces as smooth pencils on the smooth 3-folds.

6. Figures: Configurations of rational curves

In this section we give the configurations of rational curves on the surfaces with Picard-number 19 and 20. In the case of the singular surfaces of the families $Y_{\lambda,T\times V}$ and $Y_{\lambda,O\times T}$ also the curves L_i, L'_i and L''_i are contained on the surfaces, but we do not draw again the picture. Moreover the configurations of curves on the surfaces $Y^{(6,4)}_{\lambda,T\times V}$ and $Y^{(6,3)}_{\lambda,T\times V}$ are the same as on the surfaces $Y^{(6,1)}_{\lambda,T\times V}$ resp $Y^{(6,2)}_{\lambda,T\times V}$ so again we draw only one picture.

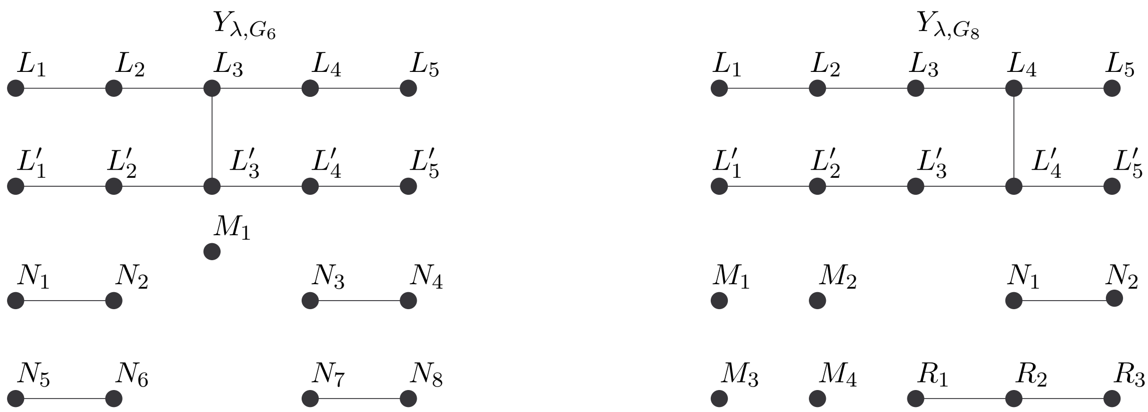


Fig. 1

$Y_{\lambda,T\times V}$ $Y_{\lambda,O\times T}$

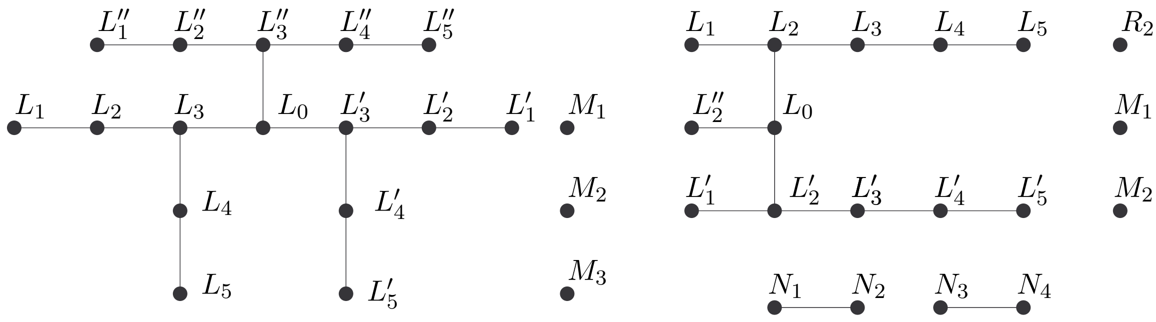
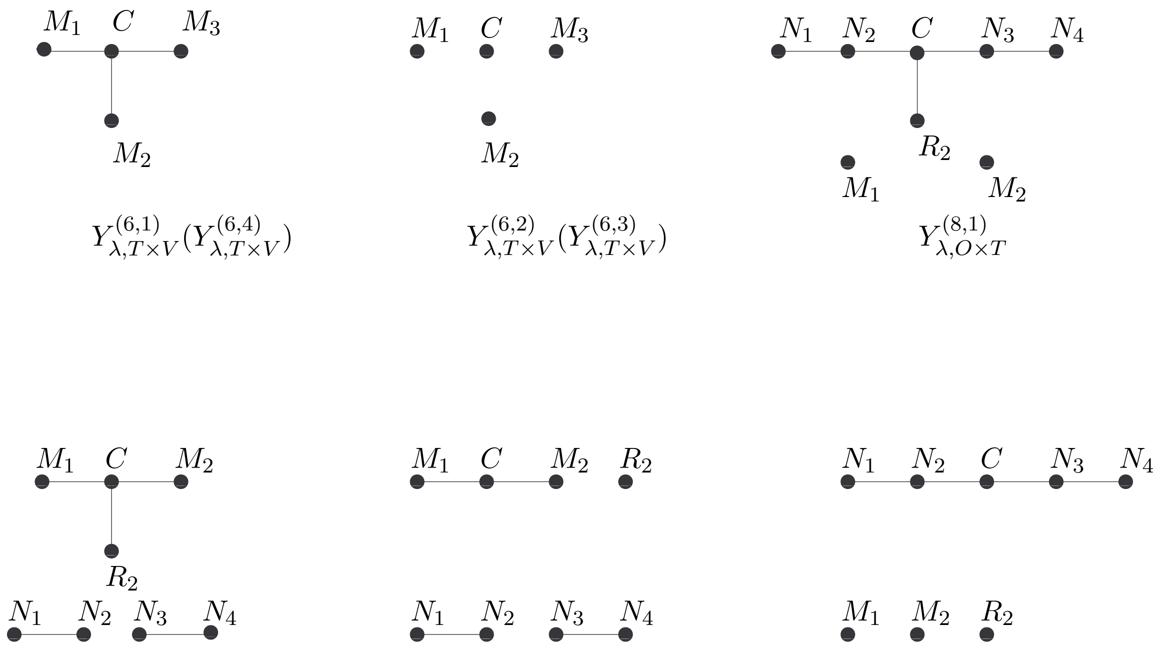


Fig. 2



$Y^{(6,1)}_{\lambda,T\times V}(Y^{(6,4)}_{\lambda,T\times V})$ $Y^{(6,2)}_{\lambda,T\times V}(Y^{(6,3)}_{\lambda,T\times V})$ $Y^{(8,1)}_{\lambda,O\times T}$

$Y^{(8,2)}_{\lambda,O\times T}$ $Y^{(8,3)}_{\lambda,O\times T}$ $Y^{(8,4)}_{\lambda,O\times T}$

Fig. 3

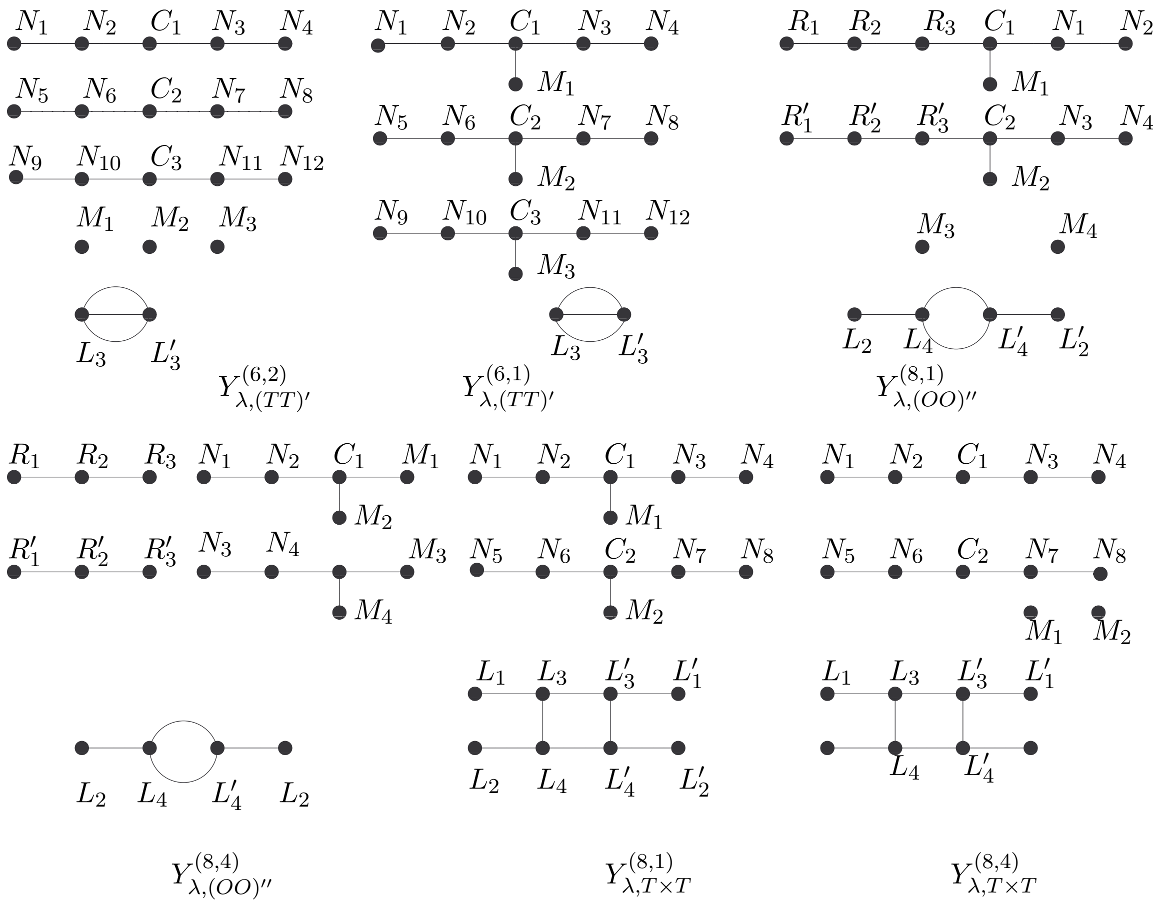


Fig. 4

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Transcendental lattices of some K3-surfaces

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TRANSCENDENTAL LATTICES OF SOME K3-SURFACES

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Abstract. In a previous paper, [S2], we described six families of $K3$-surfaces with Picard-number 19, and we identified surfaces with Picard-number 20. In these notes we classify some of the surfaces by computing their transcendental lattices. Moreover we show that the surfaces with Picard-number 19 are birational to a Kummer surface which is the quotient of a non-product type abelian surface by an involution.

0. Introduction

Given a K3-surface an important step toward its classification in view of the Torelli theorem is to compute the Picard lattice and the transcendental lattice. When the rank of the Picard lattice (i.e. the *Picard-number*, which we denote by ρ) of the K3-surface is 20, the maximal possible, the transcendental lattice has rank two. These $K3$-surfaces are called by Shioda and Inose *singular*. In [SI], Shioda and Inose classified such surfaces in terms of their transcendental lattice, more precisely they show the following:

Theorem 0.1. [SI, Theorem 4, §4] *There is a natural one-to-one correspondence from the set of singular $K3$-surfaces to the set of equivalence classes of positive-definite even integral binary quadratic forms with respect to $SL_2(\mathbb{Z})$.*

When the Picard-number is 19 the transcendental lattice has rank three and by results of Morrison, [M], and Nikulin, [N], the embedding in the *K3-lattice* $\Lambda := -E_8 \oplus -E_8 \oplus U \oplus U \oplus U$ is unique, hence it identifies the moduli curve classifying the K3-surfaces. In general however it seems to be difficult to compute explicitly the transcendental lattice. In [S2] we describe six families of K3-surfaces with Picard-number 19 and we identify in each family four surfaces with Picard-number 20. The aim of these notes is to compute their transcendental lattice and to classify them. In [S2] we describe completely the Picard lattice of the general surface in two of the families and of the special surfaces and we describe the Picard lattice of six surfaces with Picard-number 20 in the other families. Here by using lattice-theory and results on quadratic forms we compute the transcendental lattices of these surfaces. The methods are similar as the methods used by Barth in [B] for describing the K3-surfaces of [BS].
By a result of Morrison, [M, Cor. 6.4], K3-surfaces with $\rho = 19$ and 20 have a Shioda-Inose structure, in particular this means that there is a birational map from the K3-surface to a Kummer surface. It is well known (cf. [SI]) that if $\rho = 20$, then the Kummer surface is the quotient by an involution of a product-type abelian variety. When $\rho = 19$ this is not always the case. In fact we use the transcendental lattices to show that in our cases the abelian variety is not a product of two elliptic curves. In this case we call the Shioda-Inose structure *simple*.
The paper is organized as follows: in section 1 we recall some basic facts about lattices and quadratic forms and the construction of the families of K3-surfaces. Then section 2

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is entirely devoted to the computations of the transcendental lattices of the K3-surfaces of [S2]. In section 3 we show that the Shioda-Inose structure of the surfaces with $\rho = 19$ is simple. In section 4 we compare our singular K3-surfaces with already known surfaces, more precisely with the Hessians surfaces which are described in [DvG]: we see that all our singular K3-surfaces are Hessians of some cubic surface and we see that some of them are extremal elliptic K3-surfaces in the meaning of [SZ]. Finally in section 5 we recall the rational curves generating the Neron-Severi group of the K3-surfaces over $\mathbb{Q}$.

I would like to thank Wolf Barth for letting me know about his paper [B] *and for many discussions and Slawomir Rams and Bert van Geemen for many useful comments.*

1. Notations and preliminaries

1.1. **Lattices and quadratic forms.** A *lattice* L is a free $\mathbb{Z}$-module of finite rank with a $\mathbb{Z}$-valued symmetric bilinear form:

$$b : L \times L \longrightarrow \mathbb{Z}.$$

An isomorphism of lattices preserving the bilinear form is called an *isometry*, L is said to be *even* if the associate quadratic form to b takes only even values, otherwise it is called *odd*. The *discriminant* $d(L)$ of L is the determinant of the matrix of b, L is said to be *unimodular* if $d(L) = \pm 1$. If L is non-degenerate, i.e. $d(L) \neq 0$, then the *signature* of L is a pair (s_+, s_-) where $s_\pm$ denotes the multiplicity of the eigenvalue ± 1 for the quadratic form on $L \otimes \mathbb{R}$, L is called *positive-definite (negative-definite)* if the quadratic form associate to b takes just positive (negative) values. We will denote by U the *hyperbolic plane* i.e. a free $\mathbb{Z}$-module of rank 2 with bilinear form with matrix:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Moreover we denote by E_8 the unique even unimodular positive definite lattice of rank 8, with bilinear form with matrix:

$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

Let $L^\vee = Hom_{\mathbb{Z}}(L, \mathbb{Z}) = \{v \in L \otimes_{\mathbb{Z}} \mathbb{Q} \mid b(v, x) \in \mathbb{Z} \text{ for all } x \in L\}$ denotes the dual of the lattice L, then there is a natural embedding of L in $L^\vee$ via $c \mapsto b(c, -)$, and we have:

Lemma 1.1. *(cf.* [BPV, Lemma 2.1, p. 12]*) If L is a non-degenerate lattice with bilinear form b. Then*
1. $[L^\vee : L] = |d(L)|$.
2. If M is a submodule of L with rank M=rank L, then

$$[L : M]^2 = d(M)d(L)^{-1}.$$

Let A be a finite abelian group. A quadratic form on A is a map:

$$q : A \longrightarrow \mathbb{Q}/2\mathbb{Z}$$

together with a symmetric bilinear form:

$$b : A \times A \longrightarrow \mathbb{Q}/\mathbb{Z}$$

such that:
1. $q(na) = n^2 q(a)$ for all $n \in \mathbb{Z}$ and $a \in A$
2. $q(a + a') - q(a) - q(a') \equiv 2b(a, a')$ $(mod\ 2\mathbb{Z})$

Let L be a non-degenerate even lattice then the $\mathbb{Q}$-valued quadratic form on $L^\vee$ induces a quadratic form

$$q_L : L^\vee/L \longrightarrow \mathbb{Q}/2\mathbb{Z}$$

called *discriminant-form* of L. By a result of Nikulin [N, Cor. 1.9.4], the signature and the discriminant form of an even lattice determines its *genus* (we do not need the exact definition here, cf. e.g. [CS]).
An embedding of lattices $M \hookrightarrow L$ is *primitive* if L/M is free.

Lemma 1.2. *(cf.* [N, Prop. 1.6.1]*) Let $M \hookrightarrow L$ be a primitive embedding of non-degenerate even lattices and suppose L unimodular then:*
1. There is an isomorphism $M^\vee/M \cong (M^\perp)^\vee/M^\perp$.
2. $q_{M^\perp} = -q_M$.

Let now X be an algebraic $K3$-surface, the group $H^2(X, \mathbb{Z})$ with the intersection pairing has the structure of a lattice and by Poincaré duality it is unimodular. This is isometric to the *K3-lattice*:

$$\Lambda := -E_8 \oplus -E_8 \oplus U \oplus U \oplus U$$

(cf. [BPV, Prop.3.2, p. 241]). The *Neron-Severi group* $NS(X) = H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$ and its orthogonal complement T_X in $H^2(X, \mathbb{Z})$ (the *transcendental lattice*) are primitive sublattice of $H^2(X, \mathbb{Z})$ and have signature $(1, \rho - 1)$ and $(2, 20 - \rho)$, $\rho =$rank$(NS(X))$. By the Lemma 1.2 we have

$$NS(X)^\vee/NS(X) \cong (T_X)^\vee/T_X$$

and the discriminat-forms differ just by their sign. Moreover by the Lemma 1.1 we have $|NS(X)^\vee/NS(X)| = |(T_X)^\vee/T_X| = d(NS(X))$.

We recall some more facts about $K3$-surfaces X with $\rho = 20$ (*singular* K3-surfaces, cf. [SI, p. 128]). Denote by $\mathcal{Q}$ the set of 2×2 positive-definite even integral matrices:

$$Q := \begin{pmatrix} 2a & c \\ c & 2b \end{pmatrix},\ a, b, c \in \mathbb{Z} \tag{1}$$

with $d := 4ab - c^2 > 0$ and $a, b > 0$. We define $Q_1 \sim Q_2$ if and only if $Q_1 = {}^t\gamma Q_2 \gamma$ for some $\gamma \in SL_2(\mathbb{Z})$. Let $[Q]$ be the equivalence class of Q and by $\mathcal{Q}/SL_2(\mathbb{Z})$ the set of these equivalence classes. Then:

Theorem 1.1. *(cf.* [SI, Thm. 4]*). The map $X \mapsto [T_X]$ estabilishes a bijective correspondence from the set of singular K3-surfaces onto $\mathcal{Q}/SL_2(\mathbb{Z})$.*

In particular $K3$-surfaces with $\rho = 20$ are classified in terms of their transcendental lattice. By [Bu, Thm. 2.3, p. 14], we can assume that Q is *reduced*, i.e. $-a \le c \le a \le b$, and so $c^2 \le ab \le d/3$. Recall the following:

Theorem 1.2. *(*[Bu, Theorem 2.4, p. 15]*) With the exception of*

$$1. \begin{pmatrix} 2a & a \\ a & 2b \end{pmatrix} \sim \begin{pmatrix} 2a & -a \\ -a & 2b \end{pmatrix}; \ 2. \begin{pmatrix} 2a & b \\ b & 2a \end{pmatrix} \sim \begin{pmatrix} 2a & -b \\ -b & 2a \end{pmatrix}$$

no distinct reduced quadratic forms are equivalent.

Here the relation "$\sim$" is conjugation with a matrix of $SL_2(\mathbb{Z})$.
It is well known that the number of equivalence classes of forms of a given discriminat d, i.e. the *class number* of d, is finite. If there is only one class we say that d has class number one. In some other cases we have one class per genus. In [Bu, pp.81–82] with the assumption $g.c.d(a,c,b) = 1$ all the discriminants of class number one and of one class per genus are listed. If $g.c.d(a,c,b) \neq 1$ then the form is a multiple of a primitive form.

1.2. **Families of K3-surfaces.** Let $G \subset SO(3)$ denotes the polyhedral group T, O or I, and let $\widetilde{G} \subset SU(2)$ be the corresponding binary groups. Let

$$\sigma : SU(2) \times SU(2) \to SO(4, \mathbb{R})$$

denotes the classical $2:1$ covering. The images $\sigma(\widetilde{T} \times \widetilde{T}) := G_6$, $\sigma(\widetilde{O} \times \widetilde{O}) := G_8$ and $\sigma(\widetilde{I} \times \widetilde{I}) := G_{12}$ in $SO(4,\mathbb{R})$ are studied in [S1], where we show that there are 1-dimensional families in $\mathbb{P}_3(\mathbb{C})$ of G_n-invariant surfaces of degree n, which we denote by X_λ^n, λ a parameter in $\mathbb{P}_1$. In [BS] it is shown that the quotients Y_{λ,G_n}, $n = 6, 8, 12$ are families of $K3$-surfaces where the general surface has Picard-number 19 and there are four surfaces with Picard-number 20. Then in [S2] by taking special normal subgroups of G_n ($n = 6, 8$) and making the quotient of X_λ^6 resp. X_λ^8 by these subgroups we find six more pencils of K3-surfaces, using the notations there the subgroups are

$$\mathcal{G} : \ T \times V \quad (TT)' \quad V \times V \quad O \times T \quad (OO)'' \quad T \times T$$

and the families of K3-surfaces are denoted by $Y_{\lambda,\mathcal{G}}$. Here V denotes the Klein four group in $SO(3,\mathbb{R})$ and the groups $(TT)', (OO)''$ are described in [S2], the others are the images in $SO(4,\mathbb{R})$ of the direct product of binary subgroups of $SU(2)$. Moreover $T \times V$, $(TT)'$ are subgroups of index 3 of G_6 and $V \times V$ has index 3 in $T \times V$, $(TT)'$; $O \times T$, $(OO)''$ are subgroups of index 2 of G_8 and $T \times T$ has index 2 in $O \times T$, $(OO)''$. In the families $Y_{\lambda,T\times V}$ and $Y_{\lambda,O\times T}$ the general surface has Picard-number 19 and we could identify four surfaces with Picard-number 20. We denote them by $Y_{\lambda,\mathcal{G}}^{(n,j)}$, where $n = 6$, $\mathcal{G} = T \times V$ and $j = 1, 2, 3, 4$ or $n = 8$, $\mathcal{G} = O \times T$ and $j = 1, 2, 3, 4$. In the other families we identify the Picard lattice of the following surfaces with $\rho = 20$:

$$Y_{\lambda,(TT)'}^{(6,1)},\ T_{\lambda,(TT)'}^{(6,2)},\ Y_{\lambda,(OO)''}^{(8,1)},\ Y_{\lambda,(OO)''}^{(8,4)}, Y_{\lambda,T\times T}^{(8,1)}, Y_{\lambda,T\times T}^{(8,4)}.$$

We denote by NS the Picard-lattice, by T the transcendental lattice. We denote by $\mathbb{Z}_m(\alpha)$ the cyclic group $\mathbb{Z}_m$ with the quadratic form taking the value $\alpha \in \mathbb{Q}/2\mathbb{Z}$ on the generator of the group.

2. Transcendental Lattices

In this section we identify first the transcendental lattice of the singular $K3$-surfaces then of the surfaces with $\rho = 19$. In each case we proceed as follows:
1. We determine generators for $NS^\vee/NS$ with the help of the intersection pairing $(-,-)$, which is defined on NS (recall that $NS^\vee = \{v \in NS \otimes_{\mathbb{Z}} \mathbb{Q} \mid (v,x) \in \mathbb{Z}$ for all $x \in NS\}$).
2. We determine the discriminant-form of NS.
3. We use Lemma 1.2 to determine the discriminant-form of T.

4. We list all the reduced quadratic forms which have the discriminant $d(T) = d(NS)$ (we will see that in each case the matrices have form 1 or 2 as in the Theorem 1.2).
5. We use the discriminant form to determine T, in fact we see that when the rank is two the discriminants have class number one or one class per genus. When the rank is three in our cases the discriminants are *small*, Def. 2.1, and these have one class per genus.

2.1. **The singular cases.** *The family* $Y_{\lambda,T\times V}$. We recall the following 3-divisible class of NS

$$\bar{L}' = L_1 - L_2 + L_4 - L_5 + L_1' - L_2' + L_4' - L_5' + L_1'' - L_2'' + L_4'' - L_5''$$

and the following 2-divisible classes of NS

$$\begin{aligned} h_1 &= L_1 + L_3 + L_5 + L_1' + L_3' + L_5' + M_1 + M_2, \\ h_2 &= L_1 + L_3 + L_5 + L_1'' + L_3'' + L_5'' + M_1 + M_3. \end{aligned}$$

The general $K3$-surface in the family has $\rho = 19$ and the family contains four singular $K3$-surfaces. The discriminant of the general $K3$-surface in the pencil is $2\cdot 3\cdot 5$ which is the order of $NS^\vee/NS$ by the Lemma 1.1. We specify the following generators:

$$\begin{aligned} &M := M_1 + M_2 + M_3/2, \\ &N := L_1 - L_2 + L_4 - L_5 - L_1' + L_2' - L_4' + L_5'/3, \\ &L := (3L_0 - L_1 - L_1' - L_1'' - 2L_2 - 2L_2' - 2L_2'' - 3L_3 - 3L_3' - 3L_3'' \\ &-2L_4 - 2L_4' - 2L_4'' - L_5 - L_5' - L_5'')/5 \end{aligned}$$

where

$$\begin{aligned} M^2 &= -3/2 = 1/2 \mod 2\mathbb{Z}, \\ N^2 &= -8/3 = 4/3 \mod 2\mathbb{Z}, \\ L^2 &= -18/5 = 2/5 \mod 2\mathbb{Z}. \end{aligned}$$

Hence the dicriminant form of the Picard lattice is

$$\mathbb{Z}_2(1/2) \oplus \mathbb{Z}_3(4/3) \oplus \mathbb{Z}_5(2/5) \cong \mathbb{Z}_{30}(7/30)$$

The singular case $6,1(6,4)$. Here the discriminant is $-3\cdot 5 = -15$ and the generators of $NS^\vee/NS$ are N and L. The dicriminant form is

$$\mathbb{Z}_3(4/3) \oplus \mathbb{Z}_5(2/5) = \mathbb{Z}_{15}(26/15)$$

The singular case $6,2(6,3)$. Here the discriminant is $-2^2\cdot 3\cdot 5 = -60$, and the generators are M, N, L and another class $M' = M_4/2$ with $M'^2 = -1/2 = 3/2 \mod 2\mathbb{Z}$. The discriminant form is

$$\mathbb{Z}_2(1/2) \oplus \mathbb{Z}_2(3/2) \oplus \mathbb{Z}_3(4/3) \oplus \mathbb{Z}_5(2/5) \cong \mathbb{Z}_2(1/2) \oplus \mathbb{Z}_{30}(97/30).$$

The discriminant form of the transcendental lattice differs by the previous form just by the sign, hence in the general case is

$$\mathbb{Z}_{30}(53/30)$$

and in the special cases is

$$\begin{aligned} &6,1\ (6,4): \quad \mathbb{Z}_{15}(4/15), \\ &6,2\ (6,3): \quad \mathbb{Z}_2(3/2) \oplus \mathbb{Z}_{30}(23/30). \end{aligned}$$

Here we identify the transcendental lattices of these four singular $K3$-surfaces, and in the next section of the general $K3$-surface.

The singular case $6,1$ $(6,4)$. We classify all the reduced matrices with discriminant 15 (one representant per class, cf. [Bu, pp.19–20]). We have just the following possibilities

$$\begin{pmatrix} 2 & 1 \\ 1 & 8 \end{pmatrix}, \ A := \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}.$$

By taking the generator $(4/15, -1/15)$ and the bilinear form defined by A, we find a lattice $\mathbb{Z}_{15}(4/15)$ which is exactly the lattice $T^\vee/T$ hence $T = A$.
The singular case $6,2$ $(6,3)$. We classify all the reduced matrices with discriminant 60(cf. [Bu, pp.19–20]). We have just the following possibilities

$$\begin{pmatrix} 2 & 0 \\ 0 & 30 \end{pmatrix}, \ B := \begin{pmatrix} 6 & 0 \\ 0 & 10 \end{pmatrix}, \begin{pmatrix} 4 & 2 \\ 2 & 16 \end{pmatrix}, \begin{pmatrix} 8 & 2 \\ 2 & 8 \end{pmatrix}.$$

By taking the generators $(1/2, 0)$ and $(1/3, 1/10)$ and the quadratic form B we find a lattice $\mathbb{Z}_2(3/2) \oplus \mathbb{Z}_{30}(23/30)$ which is exactly the lattice $T^\vee/T$, hence $T = B$.

The family $Y_{\lambda,(TT)'}$. We recall the following 3-divisible class in NS:

$$\bar{L} = N_1 - N_2 + N_3 - N_4 + N_5 - N_6 + N_7 - N_8 + N_9 - N_{10} + N_{11} - N_{12}.$$

Now we identify the transcendental lattice of $Y^{(6,1)}_{\lambda,(TT)'}$ and of $Y^{(6,2)}_{\lambda,(TT)'}$.
The singular case $6,1$. In this case the discriminant is $-3 \cdot 5 = -15$ and we have the following generators of $NS^\vee/NS$:

$$\begin{aligned} N &:= (N_1 - N_2 + N_3 - N_4 - N_5 + N_6 - N_7 + N_8)/3, \\ L &:= (3L_3 - 3L_3')/5, \end{aligned}$$

where

$$\begin{aligned} N^2 &= -8/3 = 4/3 \mod 2\mathbb{Z}, \\ L^2 &= -18/5 = 2/5 \mod 2\mathbb{Z}. \end{aligned}$$

Hence the transcendental lattice is the same as in the case of $Y^{(6,1)}_{\lambda,T\times V}$.
The singular case $6,2$. Recall the following 2-divisible classes in NS:

$$\begin{aligned} &N_1 + C_1 + N_4 + N_5 + C_2 + N_8 + M_1 + M_2, \\ &N_1 + C_1 + N_4 + N_9 + C_3 + N_{12} + M_1 + M_3. \end{aligned}$$

The discriminant is $-2^2 \cdot 3 \cdot 5 = -60$ and the classes

$$N, \ M = M_1 + M_2 + M_3/2, \ M' = N_5 + C_2 + N_8 + M_1 + M_3/2, \ L$$

are generators for $NS^\vee/NS$. Where

$$\begin{aligned} N^2 &= -8/3 = 4/3 \mod 2\mathbb{Z}, \\ M^2 &= 1/2 \mod 2\mathbb{Z}, \\ M'^2 &= 3/2 \mod 2\mathbb{Z}, \\ L^2 &= -18/5 = 2/5 \mod 2\mathbb{Z}. \end{aligned}$$

Hence the transcendental lattice is the same as in the case of $Y^{(6,2)}_{\lambda,T\times V}$.
The family $Y_{\lambda,O\times T}$. Recall the following 2-divisible class of NS:

$$\bar{L}' = L_1 + L_3 + L_5 + L_1' + L_3' + L_5' + M_1 + M_2,$$

and the following 3-divisible class of NS:

$$k_1 = L_1 - L_2 + L_4 - L_5 - L_1' + L_2' - L_4' + L_5' + N_1 - N_2 + N_3 - N_4.$$

The general surface in the pencil has $\rho = 19$ and we have four surfaces with $\rho = 20$. The discriminant of the general $K3$-surface in the pencil is $2^3 \cdot 3 \cdot 7 = 168$. We specify the following generators of $NS^\vee/NS$:

$$\begin{aligned}
&M := L_1 + L_3 + L_5 + M_2/2,\\
&M' := L_1 + L_3 + L_5 + M_3/2,\\
&R := R_2/2,\\
&N := N_1 - N_2 - N_3 + N_4/3,\\
&L := (2L_2'' + 4L_0 - 2L_1 - 2L_1' + 3L_2 + 3L_2' - 3L_3 - 3L_3' - 2L_4 - 2L_4' - L_5 - L_5')/7
\end{aligned}$$

where

$$\begin{aligned}
&M^2 = -2 = 0 \mod 2\mathbb{Z},\\
&M'^2 = -2 = 0 \mod 2\mathbb{Z},\\
&R^2 = -1/2 = 3/2 \mod 2\mathbb{Z},\\
&N^2 = -4/3 = 2/3 \mod 2\mathbb{Z},\\
&L^2 = -16/7 = 12/7 \mod 2\mathbb{Z}.
\end{aligned}$$

Observe that the classes M, M' and L are not orthogonal to eachother in fact $M \cdot M' = 1/2 \mod 2\mathbb{Z}$ and $M \cdot L = M' \cdot L = 1 \mod 2\mathbb{Z}$. Hence the discriminant form of the Picard lattice is:

$$\mathbb{Z}_2(0) \oplus \mathbb{Z}_2(0) \oplus \mathbb{Z}_2(3/2) \oplus \mathbb{Z}_3(2/3) \oplus \mathbb{Z}_7(12/7)) \cong \mathbb{Z}_2(0) \oplus \mathbb{Z}_2(0) \oplus \mathbb{Z}_{42}(79/42).$$

The singular case $8,1$. Here the discriminant is $-2^2 \cdot 7 = -28$ and the generators for $NS^\vee/NS$ are M, M' and L. The discriminant form is

$$\mathbb{Z}_2(0) \oplus \mathbb{Z}_2(0) \oplus \mathbb{Z}_7(12/7)) \cong \mathbb{Z}_2(0) \oplus \mathbb{Z}_{14}(12/7).$$

The singular case $8,2$. The discriminant is $-2^2 \cdot 3 \cdot 7 = -84$ and the generators for $NS^\vee/NS$ are $M + R$, $M' + R$, N and L. The discriminant form is

$$\mathbb{Z}_2(3/2) \oplus \mathbb{Z}_2(3/2) \oplus \mathbb{Z}_3(2/3)) \oplus \mathbb{Z}_7(12/7) \cong \mathbb{Z}_2(3/2) \oplus \mathbb{Z}_{42}(163/42) = \mathbb{Z}_2(3/2) \oplus \mathbb{Z}_{42}(79/42).$$

The singular case $8,3$. Here the discriminant is $-2^3 \cdot 3 \cdot 7 = -168$ and the generators for $NS^\vee/NS$ are R,

$$R' = M_1 + 2C + 3M_2/4,$$

N and L, where $R'^2 = 1/4 \mod 2\mathbb{Z}$. The discriminant form is

$$\mathbb{Z}_2(3/2) \oplus \mathbb{Z}_4(1/4) \oplus \mathbb{Z}_3(2/3) \oplus \mathbb{Z}_7(12/7)) \cong \mathbb{Z}_2(3/2) \oplus \mathbb{Z}_{84}(221/84) = \mathbb{Z}_2(3/2) \oplus \mathbb{Z}_{84}(53/84).$$

The singular case $8,4$. Recall the 2-divisible class in NS

$$L_1 + L_3 + L_5 + N_1 + C + N_4 + R_2 + M_1$$

The discriminant is $-2^2 \cdot 7 = -28$ and the generators for $NS^\vee/NS$ are $L' + R$,

$$M'' = M_1 + M_2 + R_2/2,$$

and L, where $M''^2 = 1/2 \mod 2\mathbb{Z}$.
The discriminant form is

$$\mathbb{Z}_2(3/2) \oplus \mathbb{Z}_2(1/2) \oplus \mathbb{Z}_7(12/7)) \cong \mathbb{Z}_2(3/2) \oplus \mathbb{Z}_{14}(31/14) \cong \mathbb{Z}_2(3/2) \oplus \mathbb{Z}_{14}(3/14)(mod\ 2\mathbb{Z}).$$

The discriminant of the transcendental lattice differs by the previous form just by the sign, hence in the general case is

$$\mathbb{Z}_2(0) \oplus \mathbb{Z}_2(0) \oplus \mathbb{Z}_{42}(5/42)$$

and in the special cases is

$$\begin{array}{ll} 8,1: & \mathbb{Z}_2(0) \oplus \mathbb{Z}_{14}(2/7), \\ 8,2: & \mathbb{Z}_2(1/2) \oplus \mathbb{Z}_{42}(5/42), \\ 8,3: & \mathbb{Z}_2(1/2) \oplus \mathbb{Z}_{84}(115/84), \\ 8,4: & \mathbb{Z}_2(1/2) \oplus \mathbb{Z}_{14}(25/14). \end{array}$$

Here we identify the transcendental lattice for this four singular cases, and in the next section for the general $K3$-surface.

The singular case $8,1$. We classify all the reduced matrices with discriminant 28 ([Bu, pp.19–20]). We have just the following possibilities:

$$A := \begin{pmatrix} 2 & 0 \\ 0 & 14 \end{pmatrix}, \; B := \begin{pmatrix} 4 & 2 \\ 2 & 8 \end{pmatrix}.$$

Now take the form B and the generators $(0,1/2)$ and $(3/14,1/14)$. These span exactly the lattice we were looking for.

The singular case $8,2$.We classify all the reduced matrices with discriminant 84 ([Bu, pp.19–20]). We have the following four cases:

$$\begin{pmatrix} 2 & 0 \\ 0 & 42 \end{pmatrix}, \begin{pmatrix} 6 & 0 \\ 0 & 14 \end{pmatrix}, \begin{pmatrix} 4 & 2 \\ 2 & 22 \end{pmatrix}, \; C := \begin{pmatrix} 10 & 4 \\ 4 & 10 \end{pmatrix}.$$

Now we take the form C and the generators $(1/2,0)$ and $(8/21,-19/42)$ and we are done.

The singular case $8,3$.We classify all the reduced matrices with discriminant 168 ([Bu, pp.19–20]). We have the following four cases

$$\begin{pmatrix} 2 & 0 \\ 0 & 84 \end{pmatrix}, \begin{pmatrix} 6 & 0 \\ 0 & 28 \end{pmatrix}, \; E := \begin{pmatrix} 12 & 0 \\ 0 & 14 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 42 \end{pmatrix}.$$

Now we take the form E and the generators $(1/2,1/2)$ and $(1/12,1/7)$. These span exactly the lattice we were looking for.

The singular case $8,4$. The discriminant is 28 like in the case of $8,1$. Now by taking the form A and the generators $(1/2,0)$ and $(0,5/14)$ we are done.

The family $Y_{\lambda,(OO)''}$. Recall the following 2-divisible class of NS

$$\bar{L} = M_1 + M_2 + M_3 + M_4 + R_1 + R_3 + R_1' + R_3'$$

We identify the transcendental lattices of the surfaces $Y^{(8,1)}_{\lambda,(OO)''}$ and $Y^{(8,4)}_{\lambda,(OO)''}$.

The singular case $8,1$. In this case the the discriminant is $-2^2 \cdot 7 = -28$ and we have the following generators in $NS^\vee/NS$

$$\begin{array}{l} L := 2L_2 + 4L_4 - 2L_2' - 4L_4'/7, \\ M := R_1 + R_3 + M_1 + M_3/2, \\ M' := R_1 + R_3 + M_1 + M_4/2, \end{array}$$

where

$$\begin{array}{l} L^2 = 12/7 \mod 2\mathbb{Z}, \\ M^2 = M'^2 = 0 \mod 2\mathbb{Z}. \end{array}$$

Hence the transcendental lattice is the same as in the case $Y^{(8,1)}_{\lambda,O\times T}$.

The singular case $8,4$. Recall the following 4-divisible class in NS

$$W := R_1 + 2R_2 + 3R_3 + R_1' + 2R_2' + 3R_3' + 2N_1 + 2C_1 + 3M_1 + M_2 + 2N_3 + 2C_2 + 3M_3 + M_4.$$

Moreover specify the classes:

$$\begin{aligned} v_1 &:= R_1 + 2R_2 + 3R_3/4, \\ v_2 &:= R_1' + 2R_2' + 3R_3'/4, \\ v_3 &:= 2N_1 + 2C_1 + 3M_1 + M_2/4, \\ v_4 &:= 2N_3 + 2C_2 + 3M_3 + M_4/4, . \end{aligned}$$

The discriminant is $-2^4 \cdot 7 = -112$ and the generators of $NS^\vee/NS$ are

$$v_1 + v_3/4,\ v_2 + v_4/4,\ L$$

with

$$(v_1 + v_3/4)^2 = (v_2 + v_4/4)^2 = 0 \mod 2\mathbb{Z}.$$

The discriminant form of the Picard lattice is

$$\mathbb{Z}_4(0) \oplus \mathbb{Z}_4(0) \oplus \mathbb{Z}_7(12/7) = \mathbb{Z}_4(0) \oplus \mathbb{Z}_{28}(12/7).$$

Hence the discriminant form of the transcendental lattice is

$$\mathbb{Z}_4(0) \oplus \mathbb{Z}_{28}(2/7)$$

We classify all the reduced matrices with discriminant 112, these are

$$\begin{pmatrix} 2 & 0 \\ 0 & 56 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 28 \end{pmatrix}, F := \begin{pmatrix} 8 & 0 \\ 0 & 14 \end{pmatrix}, \begin{pmatrix} 8 & 4 \\ 4 & 16 \end{pmatrix}.$$

We take the matrix F and the generators $(1/4, 1/2)$ and $(1/4, 9/14)$, so we are done.
The family $Y_{\lambda,T\times T}$. A similar computation as before shows that in the singular case $8,1$, resp. $8,4$ the transcendental lattice has bilinear form with intersection matrix:

$$\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}, \text{ resp } \begin{pmatrix} 4 & 2 \\ 2 & 8 \end{pmatrix}.$$

Remark 2.1. *Observe that if the reduced matrices had not been as in case 1 or 2 of Theorem 1.2 we would find two different isomorphism classes of K3-surfaces with the same discriminant and the same discriminant form (cf.* [SZ] *p. 3).*

2.2. **The general cases.** Here we identify the transcendental lattice of the general surfaces, $\rho = 19$ in the families $Y_{\lambda,T\times V}$ and $Y_{\lambda,O\times T}$. In the last section we have identified the discriminant form of the transcendental lattice, we use it to determine T. We need the following:

Definition 2.1. *(cf.*[B, Def. 1.1]*) The discriminant $d = d_{NS} = -d_T$ is* small *if $4 \cdot d$ is not divisible by k^3 for any non square natural number k congruent to* 0 *or* 1 *modulo 4.*

Then if d_T is small , the lattice T is uniquely determined by its genus (cf. [CS, Thm. 21, p. 395]), hence by signature and discriminant form.
The family $Y_{\lambda,T\times V}$. The candidate lattice is

$$T_0 := \begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

this has discriminant -30, and taking the generator

$$f_1 := \begin{pmatrix} 4/15 \\ -1/15 \\ 1/2 \end{pmatrix}$$

Table 1. Transcendental Lattices

Family	general surface	singular surfaces			
Y_{λ,G_6}	$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 8 & 0 \\ 0 & 0 & -6 \end{pmatrix}$	$\begin{pmatrix} 2 & 1 \\ 1 & 8 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 \\ 0 & 30 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 \\ 0 & 30 \end{pmatrix}$	$\begin{pmatrix} 2 & 1 \\ 1 & 8 \end{pmatrix}$
d	-90	15	60	60	15
Y_{λ,G_8}	$\begin{pmatrix} 6 & 0 & 0 \\ 0 & 28 & 0 \\ 0 & 0 & -2 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 \\ 0 & 14 \end{pmatrix}$	$\begin{pmatrix} 6 & 0 \\ 0 & 14 \end{pmatrix}$	$\begin{pmatrix} 6 & 0 \\ 0 & 28 \end{pmatrix}$	$\begin{pmatrix} 4 & 0 \\ 0 & 28 \end{pmatrix}$
d	-336	28	84	168	112
$Y_{\lambda,G_{12}}$	$\begin{pmatrix} 4 & 2 & 0 \\ 2 & 34 & 0 \\ 0 & 0 & -30 \end{pmatrix}$	$\begin{pmatrix} 12 & 6 \\ 6 & 58 \end{pmatrix}$	$\begin{pmatrix} 6 & 0 \\ 0 & 220 \end{pmatrix}$	$\begin{pmatrix} 6 & 0 \\ 0 & 132 \end{pmatrix}$	$\begin{pmatrix} 4 & 2 \\ 2 & 34 \end{pmatrix}$
d	-3960	660	1320	792	132
$Y_{\lambda,T\times V}$	$\begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & -2 \end{pmatrix}$	$\begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$	$\begin{pmatrix} 6 & 0 \\ 0 & 10 \end{pmatrix}$	$\begin{pmatrix} 6 & 0 \\ 0 & 10 \end{pmatrix}$	$\begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$
d	-30	15	60	60	15
$Y_{\lambda,O\times T}$	$\begin{pmatrix} 10 & 4 & 0 \\ 4 & 10 & 0 \\ 0 & 0 & -2 \end{pmatrix}$	$\begin{pmatrix} 4 & 2 \\ 2 & 8 \end{pmatrix}$	$\begin{pmatrix} 10 & 4 \\ 4 & 10 \end{pmatrix}$	$\begin{pmatrix} 12 & 0 \\ 0 & 14 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 \\ 0 & 14 \end{pmatrix}$
d	-168	28	84	168	28
$Y_{\lambda,(TT)'}$	-	$\begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$	$\begin{pmatrix} 6 & 0 \\ 0 & 10 \end{pmatrix}$	-	-
d	-	15	60	-	-
$Y_{\lambda,(OO)''}$	-	$\begin{pmatrix} 2 & 0 \\ 0 & 14 \end{pmatrix}$	-	-	$\begin{pmatrix} 8 & 0 \\ 0 & 14 \end{pmatrix}$
d	-	28	-	-	112
$Y_{\lambda,T\times T}$	-	$\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$	-	-	$\begin{pmatrix} 4 & 2 \\ 2 & 8 \end{pmatrix}$
d	-	7	-	-	28

one computes $q_{T_0}(f_1) = -7/30 = 53/30 \mod 2\mathbb{Z}$, hence the discriminant form is $\mathbb{Z}_{30}(53/30)$. Since $d_T = -30$ is small the transcendental lattice of the general K3-surface is T_0.

The family $Y_{\lambda,O\times T}$. The candidate lattice is

$$T_1 := \begin{pmatrix} 10 & 4 & 0 \\ 4 & 10 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

this has discriminant -168, and taking the generators

$$f_1 := \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix}, \ f_2 := \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \end{pmatrix}, \ f_3 := \begin{pmatrix} 8/21 \\ -19/42 \\ 0 \end{pmatrix}$$

we find $q_{T_1}(f_i) = 0 \mod 2\mathbb{Z}$, $i = 1, 2$ and $q_{T_1}(f_3) = 5/42 \mod 2\mathbb{Z}$, hence the discriminant form is $\mathbb{Z}_2(0) \oplus \mathbb{Z}_2(0) \oplus \mathbb{Z}_{42}(5/42)$. Since $d_T = -168$ is small we have $T = T_1$.

We collect the results in the table 1. We recall also the results of [B] about the general surfaces of the families $Y_6(\lambda)$, $Y_8(\lambda)$, $Y_{12}(\lambda)$ and also about the singular surfaces in these pencils, Barth computed the transcendental lattices of the singular surfaces too, but he did not published his result. In the table we write also the discriminants of the lattices.

2.3. **Moduli curve.** Let

$$\Omega = \{[\omega] \in \mathbb{P}(\Lambda \otimes \mathbb{C}) | (\omega, \omega) = 0; (\omega, \bar{\omega}) > 0\},$$

this is an open subset in a quadric in $\mathbb{P}^{21}$. If X is a K3-surface and $\omega_X \in H^{2,0}(X)$, then it is well known that $\omega_X \in \Omega$ and it is called a *period point.* Moreover also the converse is true: each point of Ω occurs as period point of some K3-surface, this is the so called *surjectivity of the period map* (cf. [BPV, Thm. 14.2]). Now let $M \subset \Lambda$ be a sublattice of signature $(1, \rho - 1)$ and define:

$$\Omega_M = \{[\omega] \in \Omega | (\omega, \mu) = 0 \text{ for all } \mu \in M\}.$$

This has dimension $20 - \rho = 20-$rank M. If rank M=19 then this space is a curve. Let X be a K3-surface with $\rho = 19$ since in this case the embedding of T_X in Λ is unique up to isometry of Λ (cf. [M, Cor. 2.10]), T_X determines Ω_M, with $M = T_X^{\perp} = NS(X)$ and so the moduli curve, which classify the K3-surfaces. Hence in our cases the transcendental lattices given in the table 1 identify the moduli curve of the K3-surfaces in the families $Y_{\lambda, T\times V}$ and $Y_{\lambda, O\times T}$ (in the case $\rho = 19$).

3. Shioda-Inose structure

By a result of Morrison $K3$-surfaces with $\rho = 19$ or $\rho = 20$ admit a Shioda-Inose structure. Before discussing our cases we recall some facts.

Definition 3.1. *(cf.* [M, Def. 6.1]*) A K3-surface X admits a Shioda-Inose structure if there is a Nikulin Involution ι on X with rational quotient map $\pi : X - - \rightarrow Y$ such that Y is a Kummer surface, and π_* induces an Hodge isometry $T_X(2) \cong T_Y$.*

Hence we have the following diagram:

(2)
$$\begin{tikzcd}
 & A \arrow[dl] \arrow[dr, dashed] & & X \arrow[dl, dashed] \arrow[dr] & \\
A/i & & Y \arrow[ll] \arrow[rr] & & X/\iota
\end{tikzcd}$$

where A is the complex torus whose Kummer-surface is Y, ι is a Nikulin involution, i.e. an involution with 8 fix-points on X, i is an involution on A with 16 fix-points and the rational maps from A to Y and from X to Y are 2:1. By definition we have $T_X(2) \cong T_Y$ and by [M, Prop. 4.3], we have $T_A(2) \cong T_Y$ hence the diagram induces an Hodge isometry $T_X \cong T_A$.
In our cases the $K3$-surface which we consider are algebraic hence A is an abelian variety (cf. [M, Thm. 6.3, (ii)]). Moreover whenever X is an algebraic $K3$-surface and $\rho(X) = 19$ or 20 then X admits always a Shioda-Inose structure (cf. [M, Cor. 6.4]). Whenever $\rho = 20$ Shioda and Inose show that $A = C_1 \times C_2$ where C_1 and C_2 are elliptic curves

$$C_i = \mathbb{C}/\mathbb{Z} + \mathbb{Z} \cdot \tau_i, \ i = 1, 2$$

whith

$$\tau_1 = (-c + \sqrt{-d})/2a, \ \tau_2 = (c + \sqrt{-d})/2, \ (d = 4ab - c^2)$$

We show that in the case of the general $K3$-surfaces of the families $Y_{\lambda, T\times V}$ and of $Y_{\lambda, O\times T}$ the abelian surface $A(\lambda)$ is *simple*, i.e. it is not a product of elliptic curves, in this case we say that the Shioda-Inose structure is *simple.*
The transcendental lattice $T_{A(\lambda)}$ has rank 3 hence its orthogonal complement $NS_{A(\lambda)}$ in U^3 has rank 3 too and we have $NS(A(\lambda)) \cong T(Y(\lambda))(-1)$ because by [CS, Thm. 21, p. 395],

the lattices are uniquely determined. We use this fact to show:

Theorem 3.1. *For general λ, $A = A(\lambda)$ is not a product of elliptic curves.*

Proof. (cf.[B, Thm. 5.1]) We show that A does not contain any elliptic curve C, i.e. a curve with $C^2 = 0$.
The general surface in $Y_{\lambda,T\times V}$: We have intersection form on the transcendental lattice with matrix

$$T_0 := \begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

hence the form on NS_A is

$$\begin{pmatrix} -4 & -1 & 0 \\ -1 & -4 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

The associated quadratic form is

$$-4x^2 - 2xy - 4y^2 + 2z^2,\ \ x, y, z \in \mathbb{Z}.$$

If A contains an elliptic curve, then there are $x, y, z \in \mathbb{Z}$ with

$$2z^2 = 4x^2 + 2xy + 4y^2$$

hence

$$8z^2 = 16x^2 + 8xy + 16y^2$$

Put $u = 4x + y$, then

$$8z^2 = u^2 + 15y^2. \tag{3}$$

Hence we have $u^2 = 3z^2 \mod 5\mathbb{Z}$, since 3 is not a square modulo 5 we have $u = z = 0 \mod 5\mathbb{Z}$, hence $u = 5u_1$, $z = 5z_1$, so

$$3y^2 = 5(8z^2 - u^2) \tag{4}$$

hence $y = 5y_1$ and substituting in (4) and dividing by 5 we find

$$15y_1^2 = 8z^2 - u^2$$

which is the same as (3).
The general surface in $Y_{\lambda,O\times T}$: We have intersection form on the transcendental lattice with matrix:

$$T_1 := \begin{pmatrix} 10 & 4 & 0 \\ 4 & 10 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Hence the form on NS_A is

$$\begin{pmatrix} -10 & -4 & 0 \\ -4 & -10 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

The quadratic form is

$$-10x^2 - 8xy - 10y^2 = 2z^2,\ \ x, y, z \in \mathbb{Z}$$

If A contains an elliptic curve, then there are $x, y, z \in \mathbb{Z}$ with

$$2z^2 = 10x^2 + 8xy + 10y^2$$

hence dividing by 2 and multiplying by 5 we find

$$5z^2 = 25x^2 + 20xy + 25y^2 = (5x + 2y)^2 + 21y^2.$$

Put $u = 5x + 2y$, then

$$5z^2 = u^2 + 21y^2. \tag{5}$$

Hence we have $u^2 = 5z^2 \mod 7\mathbb{Z}$. Since 5 is not a square modulo 7 we have $u = 7u_1$, $z = 7z_1$, so we obtain

$$3y^2 = 7(5z_1^2 - u_1^2) \tag{6}$$

hence $3y^2 = 0 \mod 7\mathbb{Z}$. Since 3 is not a square modulo 7 we have $y = 7y_1$ and substituting in (6) and dividing by 7 we find

$$21y_1^2 = 5z_1^2 - u_1^2$$

which is again (5). □

4. Hessians and extremal elliptic K3-surfaces

Many of the singular K3-surfaces of this article appear already in other realizations. In [DvG] Dardanelli and van Geemen give a criteria to estabilish if a singular K3-surface is the desingularization of the Hessian of a cubic surface:

Proposition 4.1. *(cf.* [DvG, Prop. 2.4.1]*) Let T be an even lattice of rank 2,*

$$T = \begin{pmatrix} 2n & a \\ a & 2m \end{pmatrix}.$$

There is a primitive embedding $T \hookrightarrow T_{Hess}$ if and only if at least one among a, n and m is even. In this case T embeds in $U \oplus U(2)$.

Here $T_{Hess} = U \oplus U(2) \oplus A_2(-2)$. If we look in table 1 we see that all our singular $K3$-surfaces are desingularizations of Hessians of cubic surfaces. In particular Dardanelli and van Geemen study explicitely the singular K3-surfaces with

$$T = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}.$$

They call the surface X_{10} and show that it is the desingularization of the Hessian of the cubic surface with 10 Eckardt points. The latter has e.g. the following equation in $\mathbb{P}^4$

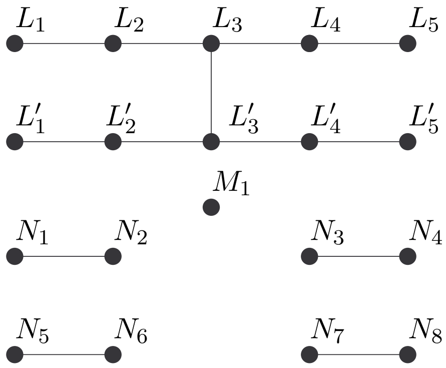
$$\sum_{i=0}^{4} x_i^3 = 0, \ \sum_{i=0}^{4} x_i = 0.$$

Finally observe that the singular surfaces of the families Y_{λ,G_6}, $Y_{\lambda,T\times V}$ and $Y_{\lambda,(TT)'}$ are *extremal elliptic K3-surfaces*, in the sense of Shimada and Zhang (cf. [SZ]), in fact these are the numbers: 322, 173, 102, 148, 276 in their list in [SZ, Table 2, pp. 15-24].

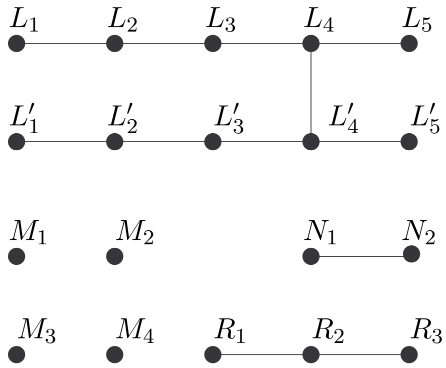
5. Figures: Configurations of rational curves

In this section we recall the configurations of (-2)-rational curves generating the Neron-Severi group over $\mathbb{Q}$. In the case of the families $Y_{\lambda,T\times V}$ and $Y_{\lambda,O\times T}$ the curves L_i, L_i' and L_i'' on the general K3-surface are also contained in the Neron-Severi group of the singular K3-surfaces, but we do not draw their configuration again. Moreover since the singular surfaces $Y^{(6,1)}_{\lambda,T\times V}$ and $Y^{(6,4)}_{\lambda,T\times V}$, as the surfaces $Y^{(6,2)}_{\lambda,T\times V}$ and $Y^{(6,3)}_{\lambda,T\times V}$ have the same graph, we draw just one picture.

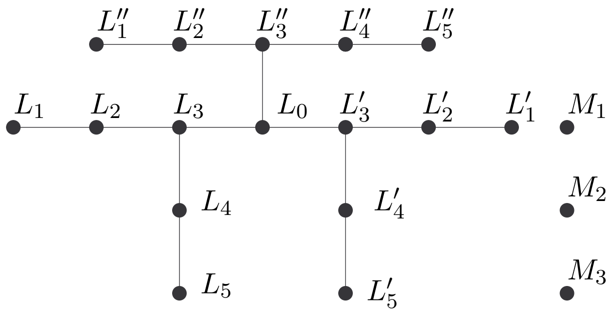
Y_{λ,G_6}



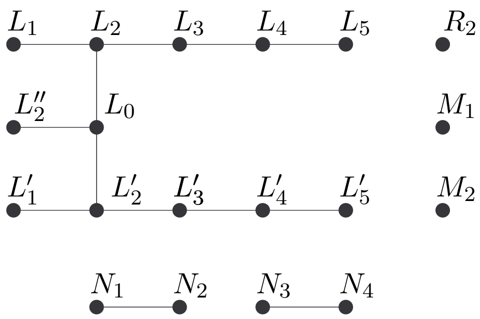
Y_{λ,G_8}

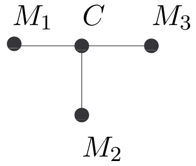


$Y_{\lambda,T\times V}$

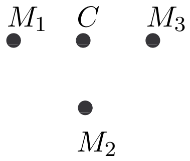


$Y_{\lambda,O\times T}$

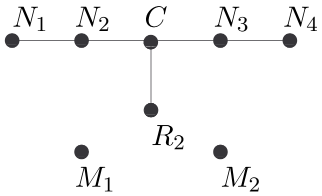




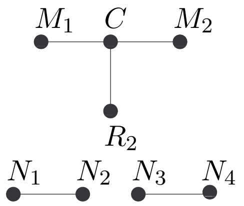
$Y^{(6,1)}_{\lambda,T\times V}(Y^{(6,4)}_{\lambda,T\times V})$



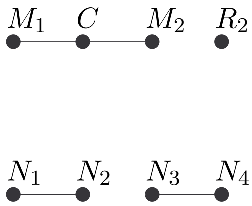
$Y^{(6,2)}_{\lambda,T\times V}(Y^{(6,3)}_{\lambda,T\times V})$



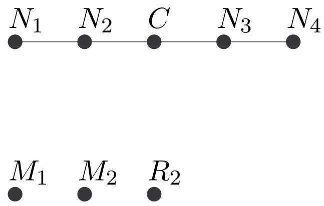
$Y^{(8,1)}_{\lambda,O\times T}$



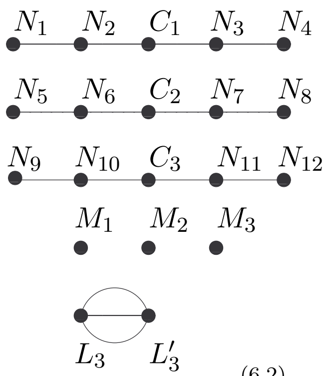
$Y^{(8,2)}_{\lambda,O\times T}$



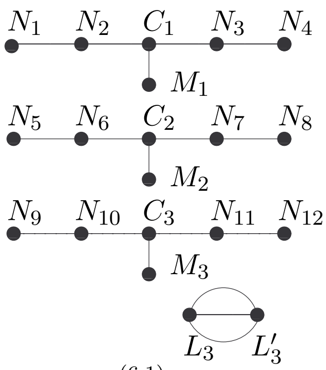
$Y^{(8,3)}_{\lambda,O\times T}$



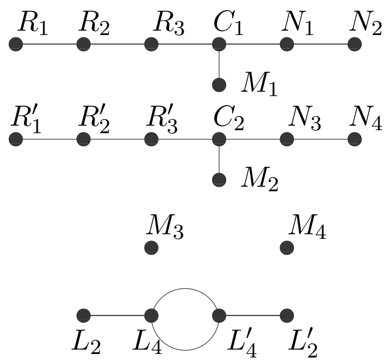
$Y^{(8,4)}_{\lambda,O\times T}$



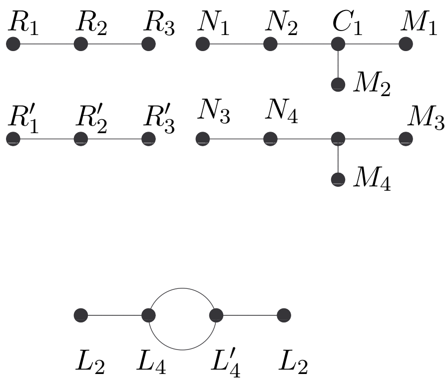
$Y^{(6,2)}_{\lambda,(TT)'}$



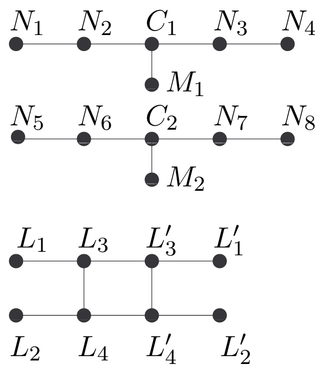
$Y^{(6,1)}_{\lambda,(TT)'}$



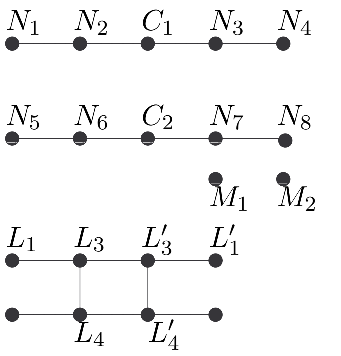
$Y^{(8,1)}_{\lambda,(OO)''}$



$Y^{(8,4)}_{\lambda,(OO)''}$



$Y^{(8,1)}_{\lambda,T\times T}$



$Y^{(8,4)}_{\lambda,T\times T}$

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Nikulin involutions on K3 surfaces

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NIKULIN INVOLUTIONS ON K3 SURFACES

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Abstract. We study the maps induced on cohomology by a Nikulin (i.e. a symplectic) involution on a K3 surface. We parametrize the eleven dimensional irreducible components of the moduli space of algebraic K3 surfaces with a Nikulin involution and we give examples of the general K3 surface in various components. We conclude with some remarks on Morrison-Nikulin involutions, these are Nikulin involutions which interchange two copies of $E_8(-1)$ in the Néron Severi group.

In his paper [Ni1] Nikulin started the study of finite groups of automorphisms on K3 surfaces, in particular those leaving the holomorphic two form invariant, these are called *symplectic*. He proves that when the group G is cyclic and acts symplectically, then $G \cong \mathbf{Z}/n\mathbf{Z}$, $1 \leq n \leq 8$. Symplectic automorphisms of K3 surfaces of orders three, five and seven are investigated in the paper [GS]. Here we consider the case of $G \cong \mathbf{Z}/2\mathbf{Z}$, generated by a symplectic involution ι. Such involutions are called *Nikulin involutions* (cf.[Mo, Definition 5.1]). A Nikulin involution on the K3 surface X has eight fixed points, hence the quotient $\bar{Y} = X/\iota$ has eight nodes, by blowing them up one obtains a K3 surface Y. In the paper [Mo] Morrison studies such involutions on algebraic K3 surfaces with Picard number $\rho \geq 17$ and in particular on those surfaces whose Néron Severi group contains two copies of $E_8(-1)$. These K3 surfaces always admit a Nikulin involution which interchanges the two copies of $E_8(-1)$. We call such involutions *Morrison-Nikulin involutions*.
The paper of Morrison motivated us to investigate Nikulin involutions in general. After a study of the maps on the cohomology induced by the quotient map, in the second section we show that an algebraic K3 surface with a Nikulin involution has $\rho \geq 9$ and that the Néron Severi group contains a primitive sublattice isomorphic with $E_8(-2)$. Moreover if $\rho = 9$ (the minimal possible) then the following two propositions are the central results in the paper:

Proposition 2.2. Let X be a K3 surface with a Nikulin involution ι and assume that the Néron Severi group $NS(X)$ of X has rank nine. Let L be a generator of $E_8(-2)^\perp \subset NS(X)$ with $L^2 = 2d > 0$ and let

$$\Lambda_{2d} := \mathbf{Z}L \oplus E_8(-2) \quad (\subset NS(X)).$$

Then we may assume that L is ample and:
(1) in case $L^2 \equiv 2 \bmod 4$ we have $\Lambda_{2d} = NS(X)$;
(2) in case $L^2 \equiv 0 \bmod 4$ we have that either $NS(X) \cong \Lambda_{2d}$ or $NS(X) \cong \Lambda_{\widetilde{2d}}$ where $\Lambda_{\widetilde{2d}}$ is the unique even lattice containing Λ_{2d} with $\Lambda_{\widetilde{2d}}/\Lambda_{2d} \cong \mathbf{Z}/2\mathbf{Z}$ and such that $E_8(-2)$ is a primitive sublattice of $\Lambda_{\widetilde{2d}}$.

Proposition 2.3. Let $\Gamma = \Lambda_{2d}$, $d \in \mathbf{Z}_{>0}$ or $\Gamma = \Lambda_{\widetilde{2d}}$, $d \in 2\mathbf{Z}_{>0}$. Then there exists a K3 surface X with a Nikulin involution ι such that $NS(X) \cong \Gamma$ and $(H^2(X, \mathbf{Z})^\iota)^\perp \cong E_8(-2)$.

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The coarse moduli space of Γ-polarized K3 surfaces has dimension 11 and will be denoted by $\mathcal{M}_{2d}$ if $\Gamma = \Lambda_{2d}$ and by $\mathcal{M}_{\widetilde{2d}}$ if $\Gamma = \Lambda_{\widetilde{2d}}$.

Thus we classified all the algebraic K3 surfaces with Picard number nine with a Nikulin involution. For the proofs we use lattice theory and the surjectivity of the period map for K3 surfaces. We also study the ι^*-invariant line bundle L on the general member of each family, for example in Proposition 2.7 we decompose the space $\mathbf{P}H^0(X,L)^*$ into ι^*-eigenspaces. This result is fundamental for the description of the ι-equivariant map $X \longrightarrow \mathbf{P}H^0(X,L)^*$. In section three we discuss various examples of the general K3 surface in these moduli spaces, recovering well-known classical geometry in a few cases. We also describe the quotient surface $\bar{Y}$.

In the last section we give examples of K3 surfaces with an elliptic fibration and a Nikulin involution which is induced by translation by a section of order two in the Mordell-Weil group of the fibration. Such a family has only ten moduli, and the minimal resolution of the quotient K3 surface Y is again a member of the same family. By using elliptic fibrations we also give an example of K3 surfaces with a Morrison-Nikulin involution. These surfaces with involution are parametrized by three dimensional moduli spaces. The Morrison-Nikulin involutions have interesting applications towards the Hodge conjecture for products of K3 surfaces (cf. [Mo], [GL]). In section 2.4 we briefly discuss possible applications of the more general Nikulin involutions.

1. General results on Nikulin Involutions

1.1. **Nikulin's uniqueness result.** A Nikulin involution ι of a K3 surface X is an automorphism of order two such that $\iota^*\omega = \omega$ for all $\omega \in H^{2,0}(X)$. That is, ι preserves the holomorphic two form and thus it is a symplectic involution. Nikulin, [Ni1, Theorem 4.7], proved that any abelian group G which acts symplectically on a K3 surface, has a unique, up to isometry, action on $H^2(X,\mathbf{Z})$.

1.2. **Action on cohomology.** D. Morrison ([Mo, proof of Theorem 5.7],) observed that there exist K3 surfaces with a Nikulin involution which acts in the following way on the second cohomology group:

$$\iota^* : H^2(X,\mathbf{Z}) \cong U^3 \oplus E_8(-1) \oplus E_8(-1) \longrightarrow H^2(X,\mathbf{Z}), \qquad (u,x,y) \longmapsto (u,y,x).$$

Thus for *any* K3 surface X with a Nikulin involution ι there is an isomorphism $H^2(X,\mathbf{Z}) \cong U^3 \oplus E_8(-1) \oplus E_8(-1)$ such that ι^* acts as above.

Given a free $\mathbf{Z}$-module M with an involution g, there is an isomorphism

$$(M,g) \cong M_1^s \oplus M_{-1}^t \oplus M_p^r,$$

for unique integers r,s,t (cf. [R]), where:

$$M_1 := (\mathbf{Z}, \iota_1 = 1), \qquad M_{-1} := (\mathbf{Z}, \iota_{-1} = -1), \qquad M_p := \left(\mathbf{Z}^2, \iota_p = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right).$$

Thus for a Nikulin involution acting on $H^2(X,\mathbf{Z})$ the invariants are $(s,t,r) = (6,0,8)$.

1.3. **The invariant lattice.** The invariant sublattice is:

$$H^2(X,\mathbf{Z})^\iota \cong \{(u,x,x) \in U^3 \oplus E_8(-1) \oplus E_8(-1)\} \cong U^3 \oplus E_8(-2).$$

The anti-invariant lattice is the lattice perpendicular to the invariant sublattice:

$$(H^2(X,\mathbf{Z})^\iota)^\perp \cong \{(0,x,-x) \in U^3 \oplus E_8(-1) \oplus E_8(-1)\} \cong E_8(-2).$$

The sublattices $H^2(X,\mathbf{Z})^\iota$ and $(H^2(X,\mathbf{Z})^\iota)^\perp$ are obviously primitive sublattices of $H^2(X,\mathbf{Z})$.

1.4. **The standard diagram.** The fixed point set of a Nikulin involution consists of exactly eight points ([Ni1, section 5]). Let $\beta : \tilde{X} \to X$ be the blow-up of X in the eight fixed points of ι. We denote by $\tilde{\iota}$ the involution on $\tilde{X}$ induced by ι. Moreover, let $\bar{Y} = X/\iota$ be the eight-nodal quotient of X, and let $Y = \tilde{X}/\tilde{\iota}$ be the minimal model of $\bar{Y}$, so Y is a K3 surface. This gives the 'standard diagram':

$$\begin{array}{ccc} X & \stackrel{\beta}{\longleftarrow} & \tilde{X} \\ \downarrow & & \downarrow \pi \\ \bar{Y} & \longleftarrow & Y. \end{array}$$

We denote by E_i, $i = 1, \ldots, 8$ the exceptional divisors in $\tilde{X}$ over the fixed points of ι in X, and by $N_i = \pi(E_i)$ their images in Y, these are (-2)-curves.

1.5. **The Nikulin lattice.** The minimal primitive sublattice of $H^2(Y, \mathbf{Z})$ containing the N_i is called the Nikulin lattice N (cf. [Mo, section 5]). As $N_i^2 = -2$, $N_iN_j = 0$ for $i \neq j$, the Nikulin lattice contains the lattice $< -2 >^8$. The lattice N has rank eight and is spanned by the N_i and a class $\hat{N}$:

$$N = \langle N_1, \ldots, N_8, \hat{N} \rangle, \qquad \hat{N} := (N_1 + \ldots + N_8)/2.$$

A set of 8 rational curves on a K3 surface whose sum is divisible by 2 in the Néron Severi group is called an even set, see [B] and section 3 for examples.

1.6. **The cohomology of $\tilde{X}$.** It is well-known that

$$H^2(\tilde{X}, \mathbf{Z}) \cong H^2(X, \mathbf{Z}) \oplus (\oplus_{i=1}^8 \mathbf{Z}E_i) \cong U^3 \oplus E_8(-1)^2 \oplus < -1 >^8 .$$

For a smooth surface S with torsion free $H^2(S, \mathbf{Z})$, the intersection pairing, given by the cup product to $H^4(S, \mathbf{Z}) = \mathbf{Z}$, gives an isomorphism $H^2(S, \mathbf{Z}) \to Hom_{\mathbf{Z}}(H^2(S, \mathbf{Z}), \mathbf{Z})$.

The map β^* is:

$$\beta^* : H^2(X, \mathbf{Z}) \longrightarrow H^2(\tilde{X}, \mathbf{Z}) = H^2(Y, \mathbf{Z}) \oplus (\oplus_{i=1}^8 \mathbf{Z}E_i), \qquad x \longmapsto (x, 0),$$

and its dual $\beta_* : H^2(\tilde{X}, \mathbf{Z}) \to H^2(X, \mathbf{Z})$ is $(x, e) \mapsto x$.

Let $\pi : \tilde{X} \to Y$ be the quotient map, let $\pi^* : H^2(Y, \mathbf{Z}) \to H^2(\tilde{X}, \mathbf{Z})$ be the induced map on the cohomology and let $\pi_* : H^2(\tilde{X}, \mathbf{Z}) \to H^2(Y, \mathbf{Z})$ be its dual, so:

$$\pi_* a \cdot b = a \cdot \pi^* b \qquad (a \in H^2(\tilde{X}, \mathbf{Z}),\ b \in H^2(Y, \mathbf{Z})).$$

Moreover, as π^* is compatible with cup product we have:

$$\pi^* b \cdot \pi^* c = 2(b \cdot c) \qquad (b, c \in H^2(Y, \mathbf{Z})).$$

1.7. **Lattices.** For a lattice $M := (M, b)$, where b is a $\mathbf{Z}$-valued bilinear form on a free $\mathbf{Z}$-module M, and an integer n we let $M(n) := (M, nb)$. In particular, M and $M(n)$ have the same underlying $\mathbf{Z}$-module, but the identity map $M \to M(n)$ is *not* an isometry unless $n = 1$ or $M = 0$.

1.8. **Proposition.** Using the notations and conventions as above, the map $\pi_* : H^2(\tilde{X}, \mathbf{Z}) \longrightarrow H^2(Y, \mathbf{Z})$ is given by

$$\pi_* : U^3 \oplus E_8(-1) \oplus E_8(-1) \oplus < -1 >^8 \longrightarrow U(2)^3 \oplus N \oplus E_8(-1) \hookrightarrow H^2(Y, \mathbf{Z}),$$

$$\pi_* :\ (u, x, y, z) \longmapsto (u, z, x + y).$$

The map π^*, on the sublattice $U(2)^3 \oplus N \oplus E_8(-1)$ of $H^2(Y, \mathbf{Z})$ is given by:

$$\pi^* : U(2)^3 \oplus N \oplus E_8(-1) \hookrightarrow H^2(\tilde{X}, \mathbf{Z}) \cong U^3 \oplus E_8(-1) \oplus E_8(-1) \oplus < -1 >^8,$$

$$\pi^* : (u, n, x) \longmapsto (2u, x, x, 2\tilde{n}),$$

here if $n = \sum n_i N_i$, $\tilde{n} = \sum n_i E_i$.

Proof. This follows easily from the results of Morrison. In the proof of [Mo, Theorem 5.7], it is shown that the image of each copy of $E_8(-1)$ under π_* is isomorphic to $E_8(-1)$. As $E_8(-1)$ is unimodular, it is a direct summand of the image of π_*. As $\pi_*\iota^* = \pi_*$, we get that $\pi_*(0,x,0,0) = \pi_*(0,0,y,0) \in E_8(-1)$. The $< -1 >^8$ maps into N (the image has index two). As U^3 is a direct summand of $H^2(X,\mathbf{Z})^\iota$, [Mo, Proposition 3.2] gives the first component.

As π_* and π^* are dual maps, $\pi^*a = b$ if for all $c \in H^2(\tilde{X},\mathbf{Z})$ one has $(b \cdot c)_{\tilde{X}} = (a \cdot \pi_* c)_Y$. In particular, if $a \in U(2)^3$ and $c \in U^3$ we get $(\pi^*a \cdot c)_{\tilde{X}} = (a \cdot \pi_* c)_Y = 2(a \cdot c)_{\tilde{X}}$ since we compute in $U(2)^3$, hence $\pi^*a = 2a$. Similarly, $(\pi^*N_i \cdot E_j)_{\tilde{X}} = (N_i \cdot \pi_* E_j)_Y = -2\delta_{ij}$, so $\pi^*N_i = 2E_i$ (this also follows from the fact that the N_i are classes of the branch curves, so π^*N_i is twice the class of $\pi^{-1}(N_i) = E_i$). Finally for $x \in E_8(-1)$ and $(y,0) \in E_8(-1)^2$ we have $(\pi^*x \cdot (y,0))_{\tilde{X}} = (x \cdot \pi_*(y,0))_Y = (x \cdot y)_Y$ and also $(\pi^*x \cdot (0,y))_{\tilde{X}} = (x \cdot y)_Y$, so $\pi^*x = (x,x) \in E_8(-1)^2$. $\square$

1.9. **Extending π^*.** To determine the homomorphism $\pi^* : H^2(Y,\mathbf{Z}) \to H^2(\tilde{X},\mathbf{Z})$ on all of $H^2(Y,\mathbf{Z})$, and not just on the sublattice of finite index $U(2)^3 \oplus N \oplus E_8(-1)$ we need to study the embedding $U(2)^3 \oplus N \hookrightarrow U^3 \oplus E_8(-1)$. This is done below. For any $x \in U^3 \oplus E_8(-1)$, one has $2x \in U(2)^3 \oplus N$ and $\pi^*(2x)$ determined as in Proposition 1.8. As π^* is a homomorphism and lattices are torsion free, one finds π^*x as $\pi^*x = (\pi^*(2x))/2$.

1.10. **Lemma.** The sublattice of $(U(2)^3 \oplus N) \otimes \mathbf{Q}$ generated by $U(2)^3 \oplus N$ and the following six elements, each divided by two, is isomorphic to $U^3 \oplus E_8(-1)$:

$$\begin{array}{lll} e_1 + (N_1 + N_2 + N_3 + N_8), & e_2 + (N_1 + N_5 + N_6 + N_8), & e_3 + (N_2 + N_6 + N_7 + N_8), \\ f_1 + (N_1 + N_2 + N_4 + N_8), & f_2 + (N_1 + N_5 + N_7 + N_8), & f_3 + (N_3 + N_4 + N_5 + N_8), \end{array}$$

here e_i, f_i are the standard basis of the i-th copy of $U(2)$ in $U(2)^3$. Any embedding of $U(2)^3 \oplus N$ into $U^3 \oplus E_8(-1)$ such that the image of N is primitive in $U^3 \oplus E_8(-1)$ is isometric to this embedding.

Proof. The theory of embeddings of lattices can be found in [Ni2, section 1]. The dual lattice M^* of a lattice $M = (M,b)$ is

$$M^* = Hom(M,\mathbf{Z}) = \{x \in M \otimes \mathbf{Q} :\ b(x,m) \in \mathbf{Z}\ \ \forall m \in M\}.$$

Note that $M \hookrightarrow M^*$, intrinsically by $m \mapsto b(m,-)$ and concretely by $m \mapsto m \otimes 1$. If (M,b_M) and (L,b_L) are lattices such that $M \hookrightarrow L$, that is $b_M(m,m') = b_L(m,m')$ for $m,m' \in M$, then we have a map $L \to M^*$ by $l \mapsto b_L(l,-)$. In case M has finite index in L, so $M \otimes \mathbf{Q} \cong L \otimes \mathbf{Q}$, we get inclusions:

$$M \hookrightarrow L \hookrightarrow L^* \hookrightarrow M^*.$$

Therefore L is determined by the image of L/M in the finite group $A_M := M^*/M$, the discriminant group of M.

Since $b = b_M$ extends to a $\mathbf{Z}$-valued bilinear form on $L \subset M^*$ we get $q(l) := b_L(l,l) \in \mathbf{Z}$ for $l \in L$. If L is an even lattice, the discriminant form

$$q_M : A_M \longrightarrow \mathbf{Q}/2\mathbf{Z}, \qquad m^* \longmapsto b_L(m^*,m^*)$$

is identically zero on the subgroup $L/M \subset A_M$. In this way one gets a bijection between even overlattices of M and isotropic subgroups of A_M. In our case $M = K \oplus N$, with $K = U(2)^3$, so $A_M = A_K \oplus A_N$ and an isotropic subgroup of A_M is the direct sum of an isotropic subgroup of A_K and one isotropic subgroup of A_N. We will see that $(A_K, q_K) \cong (A_N, -q_N)$, hence the even unimodular overlattices L of M, with N primitive in L, correspond to isomorphisms $\gamma : A_N \to A_K$ with $q_N = -q_K \circ \gamma$. Then one has that

$$L/M = \{(\gamma(\bar{n}), \bar{n}) \in A_M = A_K \oplus A_N : \ \bar{n} \in A_N\}.$$

The overlattice L_γ corresponding to γ is:

$$L_\gamma := \{(u, n) \in K^* \oplus N^* : \ \gamma(\bar{n}) = \bar{u} \ \}.$$

We will show that the isomorphism γ is unique up to isometries of K and N.
Let e, f be the standard basis of U, so $e^2 = f^2 = 0, ef = 1$, then $U(2)$ has the same basis with $e^2 = f^2 = 0, ef = 2$. Thus $U(2)^*$ has basis $e/2, f/2$ with $(e/2)^2 = (f/2)^2 = 0, (e/2)(f/2) = 2/4 = 1/2$. Thus $A_K = (U(2)^*/U(2))^3 \cong (\mathbf{Z}/2\mathbf{Z})^6$, and the discriminant form q_K on A_K is given by

$$q_K : A_K = (\mathbf{Z}/2\mathbf{Z})^6 \longrightarrow \mathbf{Z}/2\mathbf{Z}, \qquad q_K(x) = x_1x_2 + x_3x_4 + x_5x_6.$$

The Nikulin lattice N contains $\oplus \mathbf{Z}N_i$ with $N_i^2 = -2$, hence $N^* \subset \mathbf{Z}(N_i/2)$. As $N =< N_i, (\sum N_i)/2 >$ we find that $n^* \in \mathbf{Z}(N_i/2)$ is in N^* iff $n^* \cdot (\sum N_i)/2 \in \mathbf{Z}$, that is, $n^* = \sum x_i(N_i/2)$ with $\sum x_i \equiv 0 \bmod 2$. Thus we obtain an identification:

$$A_N = N^*/N = \{(x_1, \dots, x_8) \in (\mathbf{Z}/2\mathbf{Z})^8 : \ \sum x_i = 0\}/ < (1, \dots, 1) >\cong (\mathbf{Z}/2\mathbf{Z})^6,$$

where $(1, \dots, 1)$ is the image of $(\sum N_i)/2$. Any element in A_N has a unique representative which is either 0, $(N_i + N_j)/2$, with $i \neq j$ and $((N_i + N_j)/2)^2 = 1 \bmod 2\mathbf{Z}$, or $(N_1 + N_i + N_j + N_k)/2$ $(= (N_l + N_m + N_n + N_r)/2)$, with distinct indices and with $\{i, \dots, r\} = \{2, \dots, 8\}$ and $((N_1 + N_i + N_j + N_k)/2)^2 = 0 \bmod 2$. The quadratic spaces, over the field $\mathbf{Z}/2\mathbf{Z}$, $((\mathbf{Z}/2\mathbf{Z})^6, q_K)$ and $((\mathbf{Z}/2\mathbf{Z})^6, q_N)$ are isomorphic, an explicit isomorphism is defined by

$$\gamma : A_N \longrightarrow A_K, \qquad \gamma((N_1 + N_2 + N_3 + N_8)/2) = e_1/2,$$

etc. where we use the six elements listed in the lemma.

The orthogonal group of the quadratic space $((\mathbf{Z}/2\mathbf{Z})^6, q_N)$ obviously contains S_8, induced by permutations of the basis vectors in $(\mathbf{Z}/2\mathbf{Z})^8$, and these groups are actually equal cf. [Co]. Thus any two isomorphisms $A_N \to A_K$ preserving the quadratic forms differ by an isometry of A_N which is induced by a permutation of the nodal classes $N_1, \dots, N_8$. A permutation of the 8 nodal curves N_i in N obviously extends to an isometry of N.

This shows that such an even unimodular overlattice of $U(2)^3 \oplus N$ is essentially unique. As these are classified by their rank and signature, the only possible one is $U^3 \oplus E_8(-1)$. Using the isomorphism γ, one obtains the lattice L_γ, which is described in the lemma. □

1.11. **The lattices $N \oplus N$ and Γ_{16}.** Using the methods of the proof of Lemma 1.10 we show that any even unimodular overlattice L of $N \oplus N$ such that $N \oplus \{0\}$ is primitive in L, is isomorphic to the Barnes-Wall lattice $\Gamma_{16}(-1)$ (cf. [Se, Chapter V, 1.4.3]). The lattice $\Gamma_{16}(-1)$ is the unique even unimodular negative definite lattice which is not generated by its roots, i.e. by vectors v with $v^2 = -2$.

The discriminant form q_N of the lattice N has values in $\mathbf{Z}/2\mathbf{Z}$, hence $q_N = -q_N$. Therefore isomorphisms $\gamma : N \to N$ correspond to the even unimodular overlattices L_γ of $N \oplus N$ with $N \oplus \{0\}$ primitive in L_γ. Since $N \oplus N$ is negative definite, so is L_γ. The uniqueness of the overlattice follows, as before, from the fact $O(q_N) \cong S_8$. To see that this overlattice is $\Gamma_{16}(-1)$, recall that

$$\Gamma_{16} = \{x = (x_1, \dots, x_{16}) \in \mathbf{Q}^{16} : \ 2x_i \in \mathbf{Z}, \ x_i - x_j \in \mathbf{Z}, \ \sum x_i \in 2\mathbf{Z} \ \},$$

and the bilinear form on Γ_{16} is given by $\sum x_iy_i$. Let e_i be the standard basis vectors of $\mathbf{Q}^{16}$. As

$$N \oplus N \hookrightarrow \Gamma_{16}(-1), \qquad (N_i, 0) \longmapsto e_i + e_{i+8}, \qquad (0, N_i) \longmapsto e_i - e_{i+8},$$

is a primitive embedding $N \oplus N$ into $\Gamma_{16}(-1)$ (note $(\hat{N},0) \mapsto (\sum e_i)/2 \in \Gamma_{16}$, $(0,\hat{N}) \mapsto ((\sum_{i=1}^{8} e_i) - (\sum_{i=9}^{16} e_i))/2 \in \Gamma_{16}$) the claim follows.

2. Eleven dimensional families of K3 surfaces with a Nikulin involution

2.1. **Néron Severi groups.** As X is a K3 surface it has $H^{1,0}(X) = 0$ and

$$Pic(X) = NS(X) = H^{1,1}(X) \cap H^2(X,\mathbf{Z}) = \{x \in H^2(X,\mathbf{Z}) :\ x \cdot \omega = 0\ \forall \omega \in H^{2,0}(X)\,\}.$$

For $x \in (H^2(X,\mathbf{Z})^\iota)^\perp$ we have $\iota^* x = -x$. As $\iota^*\omega = \omega$ for $\omega \in H^{2,0}(X)$ we get:

$$\omega \cdot x = \iota^*\omega \cdot \iota^* x = -\omega \cdot x \qquad \text{hence} \quad (H^2(X,\mathbf{Z})^\iota)^\perp \subset NS(X).$$

As we assume X to be algebraic, there is a very ample line bundle M on X, so $M \in NS(X)$ and $M^2 > 0$. Therefore the Néron Severi group of X contains $E_8(-2) \cong (H^2(X,\mathbf{Z})^\iota)^\perp$ as a primitive sublattice and has rank at least 9.

The following proposition gives all even, rank 9, lattices of signature $(1+, 8-)$ which contain $E_8(-2)$ as a primitive sublattice. We will show in Proposition 2.3 that any of these lattices is the Néron Severi group of a K3 surface with a Nikulin involution. Moreover, the moduli space of K3 surfaces, which contain such a lattice in the Néron Severi group, is an 11-dimensional complex variety.

2.2. **Proposition.** Let X be a K3 surface with a Nikulin involution ι and assume that the Néron Severi group of X has rank 9. Let L be a generator of $E_8(-2)^\perp \subset NS(X)$ with $L^2 = 2d > 0$ and let

$$\Lambda = \Lambda_{2d} := \mathbf{Z}L \oplus E_8(-2) \quad (\subset NS(X)).$$

Then we may assume that L is ample and:

1. in case $L^2 \equiv 2 \bmod 4$ we have $\Lambda = NS(X)$;
2. in case $L^2 \equiv 0 \bmod 4$ we have that either $NS(X) = \Lambda$ or $NS(X) \cong \tilde{\Lambda}$ where $\tilde{\Lambda} = \Lambda_{\widetilde{2d}}$ is the unique even lattice containing Λ with $\tilde{\Lambda}/\Lambda \cong \mathbf{Z}/2\mathbf{Z}$ and such that $E_8(-2)$ is a primitive sublattice of $\tilde{\Lambda}$.

Proof. As $L^2 > 0$, either L or $-L$ is effective, so may assume that L is effective. As there are no (-2)-curves in $L^\perp = E_8(-2)$, any (-2)-curve N has class $aL + e$ with $a \in \mathbf{Z}_{>0}$ and $e \in E_8(-2)$. Thus $NL = aL^2 > 0$ and therefore L is ample.

From the definition of L and the description of the action of ι on $H^2(X,\mathbf{Z})$ it follows that $\mathbf{Z}L$ and $E_8(-2)$ respectively are primitive sublattices of $NS(X)$. The discriminant group of $< L >$ is $A_L := < L >^* / < L > \cong \mathbf{Z}/2d\mathbf{Z}$ with generator $(1/2d)L$ where $L^2 = 2d$ and thus $q_L((1/2d)L) = 1/2d$. The discriminant group of $E_8(-2)$ is $A_E \cong (1/2)E_8(-2)/E_8(-2) \cong (\mathbf{Z}/2\mathbf{Z})^8$, as the quadratic form on $E_8(-2)$ takes values in $4\mathbf{Z}$, the discriminant form q_E takes values in $\mathbf{Z}/2\mathbf{Z}$.

The even lattices $\tilde{\Lambda}$ which have Λ as sublattice of finite index correspond to isotropic subgroups H of $A_L \oplus A_E$ where $A_L := < L >^* / < L > \cong \mathbf{Z}/2d\mathbf{Z}$. If $E_8(-2)$ is a primitive sublattice of $\tilde{\Lambda}$, H must have trivial intersection with both A_L and A_E. Since A_E is two-torsion, it follows that H is generated by $((1/2)L, v/2)$ for some $v \in E_8(-2)$. As $((1/2)L)^2 = d/2 \bmod 2\mathbf{Z}$ and $(v/2)^2 \in \mathbf{Z}/2\mathbf{Z}$, for H to be isotropic, d must be even. Moreover, if $d = 4m+2$ we must have $v^2 = 8k+4$ for some k and if $d = 4m$ we must have $v^2 = 8k$. Conversely, such a $v \in E_8(-2)$ defines an isotropic subgroup $< (L/2, v/2) > \subset A_L \oplus A_E$ which corresponds to an overlattice $\tilde{\Lambda}$. The group $O(E_8(-2))$ contains $W(E_8)$ (cf. [Co]) which maps onto $O(q_E)$. As $O(q_E)$ has three orbits on A_E, they are $\{0\}$, $\{v/2 : (v/2)^2 \equiv 0\ (2)\}$ and $\{v/2 : (v/2)^2 \equiv 1\ (2)\}$, the overlattice is unique up to isometry. $\square$

2.3. **Proposition.** Let $\Gamma = \Lambda_{2d}$, $d \in \mathbf{Z}_{>0}$ or $\Gamma = \Lambda_{\widetilde{2d}}$, $d \in 2\mathbf{Z}_{>0}$. Then there exists a K3 surface X with a Nikulin involution ι such that $NS(X) \cong \Gamma$ and $(H^2(X,\mathbf{Z})^\iota)^\perp \cong E_8(-2)$.

The coarse moduli space of Γ-polarized K3 surfaces has dimension 11 and will be denoted by $\mathcal{M}_{2d}$ if $\Gamma = \Lambda_{2d}$ and by $\mathcal{M}_{\widetilde{2d}}$ if $\Gamma = \Lambda_{\widetilde{2d}}$.

Proof. We show that there exists a K3 surface X with a Nikulin involution ι such that $NS(X) \cong \Lambda_{\widetilde{2d}}$ and under this isomorphism $(H^2(X,\mathbf{Z})^\iota)^\perp \cong E_8(-2)$. The case $NS(X) \cong \Lambda_{2d}$ is similar but easier and is left to the reader.

The primitive embedding of $\Lambda_{\widetilde{2d}}$ in the unimodular lattice $U^3 \oplus E_8(-1)^2$ is unique up to isometry by [Ni2, Theorem 1.14.1], and we will identify $\Lambda_{\widetilde{2d}}$ with a primitive sublattice of $U^3 \oplus E_8(-1)^2$ from now on. We choose an $\omega \in \Lambda_{\widetilde{2d}}^\perp \otimes_{\mathbf{Z}} \mathbf{C}$ with $\omega^2 = 0$, $\omega\bar{\omega} > 0$ and general with these properties, hence $\omega^\perp \cap (U^3 \oplus E_8(-1)^2) = \Lambda_{\widetilde{2d}}$. By the 'surjectivity of the period map', there exists a K3 surface X with an isomorphism $H^2(X,\mathbf{Z}) \cong U^3 \oplus E_8(-1)^2$ such that $NS(X) \cong \Lambda_{\widetilde{2d}}$.

The involution of $\Lambda = \mathbf{Z}L \oplus E_8(-2)$ which is trivial on L and -1 on $E_8(-2)$, extends to an involution of $\Lambda_{\widetilde{2d}} = \Lambda + \mathbf{Z}(L/2, v/2)$. The involution is trivial on the discriminant group of $\Lambda_{\widetilde{2d}}$ which is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^6$. Therefore it extends to an involution ι_0 of $U^3 \oplus E_8(-1)^2$ which is trivial on $\Lambda_{\widetilde{2d}}^\perp$. As $((U^3 \oplus E_8(-1)^2)^{\iota_0})^\perp = E_8(-2)$ is negative definite, contains no (-2)-classes and is contained in $NS(X)$, results of Nikulin ([Ni1, Theorems 4.3, 4.7, 4.15]) show that X has a Nikulin involution ι such that $\iota^* = \iota_0$ up to conjugation by an element of the Weyl group of X. Since we assume L to be ample and the ample cone is a fundamental domain for the Weyl group action, we do get $\iota^* = \iota_0$, hence $(H^2(X,\mathbf{Z})^\iota)^\perp \cong E_8(-2)$.

For the precise definition of Γ-polarized K3 surfaces we refer to [Do]. We just observe that each point of the moduli space corresponds to a K3 surface X with a primitive embedding $\Gamma \hookrightarrow NS(X)$. The moduli space is a quotient of the 11-dimensional domain

$$\mathcal{D}_\Gamma = \{\omega \in \mathbf{P}(\Gamma^\perp \otimes_{\mathbf{Z}} \mathbf{C}) :\ \omega^2 = 0,\ \omega\bar{\omega} > 0\}$$

by an arithmetic subgroup of $O(\Gamma)$. □

2.4. **Note on the Hodge conjecture.** For a smooth projective surface S with torsion free $H^2(S,\mathbf{Z})$, let $T_S := NS(S)^\perp \subset H^2(S,\mathbf{Z})$ and let $T_{S,\mathbf{Q}} = T_S \otimes_{\mathbf{Z}} \mathbf{Q}$. Then T_S, the transcendental lattice of S, is an (integral, polarized) weight two Hodge structure.

The results in section 1 show that $\pi_* \circ \beta^*$ induces an isomorphism of rational Hodge structures:

$$\phi_\iota : T_{X,\mathbf{Q}} \xrightarrow{\cong} T_{Y,\mathbf{Q}},$$

in fact, both are isomorphic to $T_{\tilde{X},\mathbf{Q}}$. Any homomorphism of rational Hodge structures $\phi : T_{X,\mathbf{Q}} \to T_{Y,\mathbf{Q}}$ defines, using projection and inclusion, a map of Hodge structures $H^2(X,\mathbf{Q}) \to T_{X,\mathbf{Q}} \to T_{Y,\mathbf{Q}} \hookrightarrow H^2(Y,\mathbf{Q})$ and thus it gives a Hodge (2,2)-class

$$\phi \in H^2(X,\mathbf{Q})^* \otimes H^2(Y,\mathbf{Q}) \cong H^2(X,\mathbf{Q}) \otimes H^2(Y,\mathbf{Q}) \hookrightarrow H^4(X \times Y,\mathbf{Q}),$$

where we use Poincaré duality and the Künneth formula. Obviously, the isomorphism $\phi_\iota : T_{X,\mathbf{Q}} \to T_{Y,\mathbf{Q}}$ corresponds to the class of the codimension two cycle which is the image of $\tilde{X}$ in $X \times Y$ under (β,π).

Mukai showed that any homomorphism between $T_{S,\mathbf{Q}}$ and $T_{Z,\mathbf{Q}}$ where S and Z are K3 surfaces which is moreover an isometry (w.r.t. the quadratic forms induced by the intersection forms) is induced by an algebraic cycle if $\dim T_{S,\mathbf{Q}} \leq 11$ ([Mu, Corollary 1.10]). Nikulin, [Ni3, Theorem 3], strengthened this result and showed that it suffices that $NS(X)$ contains a class e with $e^2 = 0$. In particular, this implies that any Hodge isometry $T_{S,\mathbf{Q}} \to T_{Z,\mathbf{Q}}$ is induced by an algebraic cycle if $\dim T_{S,\mathbf{Q}} \leq 18$ (cf. [Ni3, proof of Theorem 3]).

The Hodge conjecture predicts that any homomorphism of Hodge structures between $T_{S,\mathbf{Q}}$ and $T_{Z,\mathbf{Q}}$ is induced by an algebraic cycle, *without* requiring that it is an isometry. There are few results in this direction, it is therefore maybe worth noticing that ϕ_ι is not an isometry if T_X has odd rank, see the proposition below. In [GL] a similar result of D. Morrison in a more special case is used to obtain new results on the Hodge conjecture. In Proposition 4.2 we show that there exists a K3 surface X with Nikulin involution where $T_{X,\mathbf{Q}}$ has even rank and $T_{X,\mathbf{Q}}$ *is* isometric to $T_{Y,\mathbf{Q}}$.

2.5. **Proposition.** Let $\phi_\iota : T_{X,\mathbf{Q}} \xrightarrow{\cong} T_{Y,\mathbf{Q}}$ be the isomorphism of Hodge structures induced by the Nikulin involution ι on X and assume that $\dim T_{X,\mathbf{Q}}$ is an odd integer. Then ϕ_ι is not an isometry.

Proof. Let $Q : \mathbf{Q}^n \to \mathbf{Q}$ be a quadratic form, then Q is defined by an $n \times n$ symmetric matrix, which we also denote by Q: $Q(x) := {}^t xQx$. An isomorphism $A : \mathbf{Q}^n \to \mathbf{Q}^n$ gives an isometry between $(\mathbf{Q}^n, Q)$ and $(\mathbf{Q}^n, Q')$ iff $Q' = {}^tA^{-1}QA^{-1})$. In particular, if $(\mathbf{Q}^n, Q) \cong (\mathbf{Q}^n, Q')$ the quotient $\det(Q)/\det(Q')$ must be a square in $\mathbf{Q}^*$.

For a $\mathbf{Z}$-module M we let $M_\mathbf{Q} := M \otimes_\mathbf{Z} \mathbf{Q}$. Let V_X be the orthogonal complement of $E_8(-2)_\mathbf{Q} \subset NS(X)_\mathbf{Q}$, then $\det(NS(X)_\mathbf{Q}) = 2^8 \det(V_X)$ up to squares. Let V_Y be the orthogonal complement of $N_\mathbf{Q} \subset NS(Y)_\mathbf{Q}$ then $\det(NS(Y)_\mathbf{Q}) = 2^6 \det(V_Y)$ up to squares. Now $\beta_*\pi^* : H^2(Y, \mathbf{Q}) \to H^2(X, \mathbf{Q})$ induces an isomorphism $V_X \to V_Y$ which satisfies $(\beta_*\pi^*x)(\beta_*\pi^*y) = 2xy$ for $x, y \in V_Y$, hence $\det(V_X) = 2^d \det(V_Y)$ where $d = \dim V_X = 22 - 8 - \dim T_{X,\mathbf{Q}}$, so d is odd by assumption.

For a K3 surface S, $\det(T_{S,\mathbf{Q}}) = -\det(NS(S)_\mathbf{Q})$ and thus $\det(T_{X,\mathbf{Q}})/\det(T_{Y,\mathbf{Q}}) = 2^{d+2}$ up to squares. As d is odd and 2 is not a square in the multiplicative group of $\mathbf{Q}$, it follows that there exists no isometry between $T_{X,\mathbf{Q}}$ and $T_{Y,\mathbf{Q}}$. □

2.6. **The bundle L.** In case $NS(X)$ has rank 9, the ample generator L of $E_8(-2)^\perp \subset NS(X)$ defines a natural map

$$\phi_L : X \longrightarrow \mathbf{P}^g, \qquad g = h^0(L) - 1 = L^2/2 + 1$$

which we will use to study X and Y. As $\iota^*L \cong L$, the involution ι acts as an involution on $\mathbf{P}^g = |L|^*$ and thus it has two fixed spaces $\mathbf{P}^a, \mathbf{P}^b$ with $(a+1)+(b+1) = g+1$. The fixed points of ι map to these fixed spaces. Even though L is ι-invariant, it is not the case in general that on $\tilde{X}$ we have $\beta^*L = \pi^*M$ for some line bundle $M \in NS(Y)$. In fact, $\beta^*L = \pi^*M$ implies $L^2 = (\beta^*L)^2 = (\pi^*M)^2 = 2M^2$ and as M^2 is even we get $L^2 \in 4\mathbf{Z}$. Thus if $L^2 \notin 4\mathbf{Z}$, the ι-invariant line bundle L cannot be obtained by pull-back from Y. On the other hand, if for example $|L|$ contains a reduced ι-invariant divisor D which does not pass through the fixed points, then $\beta^*D = \beta^{-1}D$ is invariant under $\tilde{\iota}$ on $\tilde{X}$ and does not contain any of the E_i as a component. Then $\beta^*D = \pi^*D'$ where $D' \subset Y$ is the reduced divisor with support $\pi(\beta^{-1}D)$.

The following lemma collects the basic facts on L and the splitting of $\mathbf{P}^g = \mathbf{P}H^0(X, L)^*$.

2.7. **Proposition.**

(1) Assume that $NS(X) = \mathbf{Z}L \oplus E_8(-2)$. Let $E_1, \ldots, E_8$ be the exceptional divisors on $\tilde{X}$.

In case $L^2 = 4n + 2$, there exist line bundles $M_1, M_2 \in NS(Y)$ such that for a suitable numbering of these E_i we have:

$$\beta^*L - E_1 - E_2 = \pi^*M_1, \qquad \beta^*L - E_3 - \ldots - E_8 = \pi^*M_2.$$

The decomposition of $H^0(X,L)$ into ι^*-eigenspaces is:

$$H^0(X,L) \cong \pi^* H^0(Y,M_1) \oplus \pi^* H^0(Y,M_2), \qquad (h^0(M_1) = n+2,\ h^0(M_2) = n+1).$$

and the eigenspaces $\mathbf{P}^{n+1}, \mathbf{P}^n$ contain six, respectively two, fixed points.

In case $L^2 = 4n$, for a suitable numbering of the E_i we have:

$$\beta^*L - E_1 - E_2 - E_3 - E_4 = \pi^* M_1, \qquad \beta^*L - E_5 - E_6 - E_7 - E_8 = \pi^* M_2$$

with $M_1, M_2 \in NS(Y)$. The decomposition of $H^0(X,L)$ into ι^*-eigenspaces is:

$$H^0(X,L) \cong \pi^* H^0(Y,M_1) \oplus \pi^* H^0(Y,M_2), \qquad (h^0(M_1) = h^0(M_2) = n+1).$$

and each of the eigenspaces $\mathbf{P}^n$ contains four fixed points.

(2) Assume that $\mathbf{Z}L \oplus E_8(-2)$ has index two in $NS(X)$. Then there is a line bundle $M \in NS(Y)$ such that:

$$\beta^*L \cong \pi^*M, \qquad H^0(X,L) \cong H^0(Y,M) \oplus H^0(Y, M - \hat{N}),$$

where $\hat{N} = (\sum_{i=1}^8 N_i)/2 \in NS(Y)$ and this is the decomposition of $H^0(X,L)$ into ι^*-eigenspaces. One has $h^0(M) = n+2$, $h^0(M - \hat{N}) = n$, and all fixed points map to the eigenspace $\mathbf{P}^{n+1} \subset \mathbf{P}^{2n+1} = \mathbf{P}^g$.

Proof. The primitive embedding of $\mathbf{Z}L \oplus E_8(-2)$ in the unimodular lattice $U^3 \oplus E_8(-1)^2$ is unique up to isometry by [Ni2, Theorem 1.14.1]. Therefore if $L^2 = 2r$ we may assume that $L = e_1 + rf_1 \in U \subset U^3 \oplus E_8(-1)^3$ where e_1, f_1 are the standard basis of the first copy of U.

In case $r = 2n+1$, it follows from Lemma 1.10 that $(e_1 + (2n+1)f_1 + N_3 + N_4)/2 \in NS(Y)$. By Proposition 1.8, $M_1 := (e_1 + (2n+1)f_1 + N_3 + N_4)/2 - N_3 - N_4$ satisfies $\pi^*M_1 = \beta^*L - E_3 - E_4$. Similarly, let $M_2 = (e_1 + (2n+1)f_1 + N_3 + N_4)/2 - \hat{N} \in NS(Y)$, then $\pi^*M_2 = \beta^*L - (E_1 + E_2 + E_5 + \ldots + E_8)$.

Any two sections $s,t \in H^0(X,L)$ lie in the same ι^*-eigenspace iff the rational function $f = s/t$ is ι-invariant. Thus $s,t \in \pi^*H^0(Y,M_i)$ are ι^*-invariant, hence each of these two spaces is contained in an eigenspace of ι^* in $H^0(X,L)$. If both are in the same eigenspace, then this eigenspace would have a section with no zeroes in the 8 fixed points of ι. But a ι-invariant divisor on X which doesn't pass through any fixed point is the pull back of divisor on Y, which contradicts that L^2 is not a multiple of 4. Thus the $\pi^*H^0(Y,M_i)$ are in distinct eigenspaces. A dimension count shows that $h^0(L) = h^0(M_1) + h^0(M_2)$, hence the $\pi^*H^0(Y,M_i)$ are the eigenspaces.

In case $r = 2n$, again by Lemma 1.10 we have $(e_1 + N_1 + N_2 + N_3 + N_8)/2 \in NS(Y)$. Let $M_1 := nf_1 + (e_1 + N_1 + N_2 + N_3 + N_8)/2 - (N_1 + N_2 + N_3 + N_8)$ then $\pi^*M_1 = \beta^*L - (E_1 + E_2 + E_3 + E_8)$. Put $M_2 = M_1 + \hat{N} - (N_4 + N_5 + N_6 + N_7)$, then $\pi^*M_2 = \beta^*L - (E_4 + E_5 + E_6 + E_7)$. As above, the $\pi^*H^0(Y,M_i)$, $i = 1,2$, are contained in distinct eigenspaces and a dimension count again shows that $h^0(L) = h^0(M_1) + h^0(M_2)$.

If $\mathbf{Z}L \oplus E_8(-2)$ has index two in $NS(X)$, the (primitive) embedding of $NS(X)$ into $U^3 \oplus E_8(-1)$ is still unique up to isometry. Let $L^2 = 4n$. Choose an $\alpha \in E_8(-1)$ with $\alpha^2 = -2$ if n is odd, and $\alpha^2 = -4$ if n is even. Let $v = (0, \alpha, -\alpha) \in E_8(-2) \subset U^3 \oplus E_8(-1)^2$ and let $L = (2u, \alpha, \alpha) \in U^3 \oplus E_8(-1)^2$ where $u = e_1 + (n+1)/2 f_1$ if n is odd and $u = e_1 + (n/2+1)f_1$ if n is even. Note that $L^2 = 4u^2 + 2\alpha^2 = 4n$ and that $(L+v)/2 = (u, \alpha, 0) \in U^3 \oplus E_8(-1)^2$. Thus we get a primitive embedding of $NS(X) \hookrightarrow U^3 \oplus E_8(-1)^2$ which extends the standard one of $E_8(-2) \subset NS(X)$. Proposition 1.8 shows that $\beta^*L = \pi^*M$ with $M = (u, 0, \alpha) \in U^3(2) \oplus N \oplus E_8(-1) \subset H^2(Y, \mathbf{Z})$. For the double cover $\pi : \tilde{X} \to Y$ branched along $2\hat{N} = \sum N_i$ we have as usual: $\pi_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_Y \oplus \mathcal{O}_Y(-\hat{N})$ hence, using the projection formula:

$$H^0(\tilde{X}, \pi^* M) \cong H^0(Y, \pi_*(\pi^* M \otimes \mathcal{O}_{\tilde{X}}) \cong H^0(Y, M) \oplus H^0(Y, M - \hat{N}).$$

Note that the sections in $\pi^* H^0(Y, M - \hat{N})$ vanish on all the exceptional divisors, hence the fixed points of ι map to a $\mathbf{P}^{n+1}$. □

3. Examples

3.1. In Proposition 2.3 we showed that K3 surfaces with a Nikulin involution are parametrized by eleven dimensional moduli spaces $\mathcal{M}_{2d}$ and $\mathcal{M}_{\widetilde{4e}}$ with $d, e \in \mathbf{Z}_{>0}$. For some values of d, e we will now work out the geometry of the corresponding K3 surfaces. We will also indicate how to verify that the moduli spaces are indeed eleven dimensional.

3.2. The case $\mathcal{M}_2$. Let X be a K3 surface with Nikulin involution ι and $NS(X) \cong \mathbf{Z}L \oplus E_8(-2)$ with $L^2 = 2$ and $\iota^* L \cong L$ (cf. Proposition 2.3). The map $\phi_L : X \to \mathbf{P}^2$ is a double cover of $\mathbf{P}^2$ branched over a sextic curve C, which is smooth since there are no (-2)-curves in $L^\perp$. The covering involution will be denoted by $i : X \to X$. The fixed point locus of i is isomorphic to C.

As i^* is $+1$ on $\mathbf{Z}L$, -1 on $E_8(-2)$ and -1 on T_X, whereas ι^* is $+1$ on $\mathbf{Z}L$, -1 on $E_8(-2)$ and $+1$ on T_X, these two involutions commute. Thus ι induces an involution $\bar{\iota}_{\mathbf{P}^2}$ on $\mathbf{P}^2$ (which is ι^* acting on $\mathbf{P}H^0(X, L)^*$) and in suitable coordinates:

$$\bar{\iota}_{\mathbf{P}^2} : \ (x_0 : x_1 : x_2) \longmapsto (-x_0 : x_1 : x_2).$$

We have a commutative diagram

$$\begin{array}{ccccc} C & \hookrightarrow & X & \xrightarrow{\iota} & X \\ \cong\downarrow & & \downarrow \phi & & \downarrow \phi \\ C & \hookrightarrow & \mathbf{P}^2 & \xrightarrow{\bar{\iota}_{\mathbf{P}^2}} & \mathbf{P}^2 = X/i. \end{array}$$

The fixed points of $\bar{\iota}_{\mathbf{P}^2}$ are:

$$(\mathbf{P}^2)^{\bar{\iota}_{\mathbf{P}^2}} = l_0 \cup \{p\}, \qquad l_0 : \ x_0 = 0, \quad p = (1 : 0 : 0).$$

The line l_0 intersects the curve C in six points, which are the images of six fixed points $x_3, \ldots, x_8$ of ι on X. Thus the involution ι induces an involution on $C \subset X$ with six fixed points. The other two fixed points x_1, x_2 of ι map to the point p, so i permutes these two fixed points of ι. In particular, these two points are not contained in C so $p \notin C$ $(\subset \mathbf{P}^2)$, which will be important in the moduli count below. The inverse image $C_2 = \phi^{-1}(l_0)$ is a genus two curve in the system $|L|$. Both ι and i induce the hyperelliptic involution on C_2. By doing then the quotient by ι, since this has six fixed points on C_2 we obtain a rational curve C_0.

To describe the eight nodal surface $\bar{Y} = X/\iota$, we use the involution $\bar{i}_{\bar{Y}}$ of $\bar{Y}$ which is induced by $i \in Aut(X)$. Then we have:

$$Q := \bar{Y}/\bar{i}_{\bar{Y}} \ \cong \ X/ < \iota, i > \cong \ \mathbf{P}^2/\bar{\iota}_{\mathbf{P}^2}.$$

This leads to the following diagrams of double covers and fixed point sets:

$$\begin{tikzcd} & X \arrow[dl] \arrow[d] \arrow[dr] & \\ \mathbf{P}^2 \arrow[dr] & \bar{Y} \arrow[d] & S \arrow[dl] \\ & Q & \end{tikzcd} \qquad \begin{tikzcd} & \{x_1, x_2\} \cup C \cup C_2 \arrow[dl] \arrow[d] \arrow[dr] & \\ \{p\} \cup C \cup l_0 \arrow[dr] & \{y_1, y_2\} \cup C_4 \cup C_0 \arrow[d] & \{p_9\} \cup C_4 \cup D_0 \arrow[dl] \\ & \{q\} \cup C_4 \cup H_0 & \end{tikzcd}$$

The quotient of $\mathbf{P}^2 = X/i$ by $\bar{\iota}_{\mathbf{P}^2}$ is isomorphic to a quadric cone Q in $\mathbf{P}^3$ whose vertex q is the image of the fixed point $(1:0:0)$. In coordinates, the quotient map is:

$$\mathbf{P}^2 \longrightarrow Q = \mathbf{P}^2/\bar{\iota}_{\mathbf{P}^2} \subset \mathbf{P}^3, \qquad (x_0 : x_1 : x_2) \longmapsto (y_0 : \ldots : y_3) = (x_0^2 : x_1^2 : x_1x_2 : x_2^2)$$

and Q is defined by $y_1y_3 - y_2^2 = 0$.

The sextic curve $C \subset \mathbf{P}^2$, which has genus 10, is mapped 2:1 to a curve C_4 on the cone. The double cover $C \to C_4$ ramifies in the six points where C intersects the line $x_0 = 0$. Thus the curve C_4 is smooth, has genus four and degree six (the plane sections of C_4 are the images of certain conic sections of the branch sextic) and does not lie in a plane (so C_4 spans $\mathbf{P}^3$). The only divisor class D of degree $2g-2$ with $h^0(D) \geq g$ on a smooth curve of genus g is the canonical class, hence C_4 is a canonically embedded curve. The image of the line l_0 is the plane section $H_0 \subset Q$ defined by $y_0 = 0$.

The branch locus in Q of the double cover

$$\bar{Y} \longrightarrow Q = \bar{Y}/\bar{i}_{\bar{Y}}$$

is the union of two curves, C_4 and the plane section H_0, these curves intersect in six points, and the vertex q of Q.

To complete the diagram, we consider the involution

$$j := \iota \circ i : X \longrightarrow X, \qquad S := X/j.$$

The fixed point set of j is the (smooth) genus two curve C_2 lying over the line l_0 in $\mathbf{P}^2$ (use $j(p) = p$ iff $\iota(p) = i(p)$ and consider the image of p in $\mathbf{P}^2$). Thus the quotient surface S is a smooth surface. The Riemann-Hurwitz formula implies that the image of C_2 in S is a curve $D_0 \in |-2K_S|$, note that $D_0 \cong C_2$.

The double cover $S \to Q$ branches over the curve $C_4 \subset Q$ and the vertex $q \in Q$. It is well-known that such a double cover is a Del Pezzo surface of degree 1 ([Dem], [DoO]) and the map $S \to Q \subset \mathbf{P}^3$ is given by ϕ_{-2K}, which verifies that the image of D_0 is a plane section.

On the other hand, any Del Pezzo surface of degree 1 is isomorphic to the blow up of $\mathbf{P}^2$ in eight points. The linear system $|-K_S|$ corresponds to the pencil of elliptic curves on the eight points, the ninth base point in $\mathbf{P}^2$ corresponds to the unique base point p_9 of $|-K_S|$ in S. The point p_9 maps to the vertex $q \in Q$ under the 2:1 map ϕ_{-2K} ([DoO, p. 125]). The Néron Severi group of S is thus isomorphic to

$$NS(S) \cong \mathbf{Z}e_0 \oplus \mathbf{Z}e_1 \oplus \ldots \oplus \mathbf{Z}e_8, \qquad e_0^2 = 1, \quad e_i^2 = -1 \quad (1 \leq i \leq 8)$$

and $e_ie_j = 0$ if $i \neq j$. The canonical class is $K_S = -3e_0 + e_1 + \ldots + e_8$. Since $K_S^2 = 1$, we get a direct sum decomposition:

$$NS(S) \cong \mathbf{Z}K_S \oplus K_S^{\perp} \cong \mathbf{Z}K_S \oplus E_8(-1)$$

(cf. [DoO, VII.5]). The surface S has 240 exceptional curves (smooth rational curves E with $E^2 = -1$), cf. [DoO, p.125]. The adjunction formula shows that $EK_S = -1$ and the map $E \mapsto E + K$ gives a bijection between these exceptional curves and the roots of $E_8(-1)$, i.e. the $x \in E_8(-1)$ with $x^2 = -2$. An exceptional divisor $E \subset S$ meets the branch curve D_0 ($\in |-2K_S|$) of $X \to S$ in two points, hence the inverse image of E in X is a (-2)-curve. Thus we get 240 such (-2)-curves. Actually,

$$j^* : NS(S) = \mathbf{Z}K_S \oplus E_8(-1) \longrightarrow NS(X) = \mathbf{Z}L \oplus E_8(-2)$$

is the identity on the $\mathbf{Z}$-modules and $NS(X) \cong NS(S)(2)$. The class of such a (-2)-curve is $L + x$, with $x \in L^\perp = E_8(-2)$, $x^2 = -2$. As $i^*(L + x) = L - x \neq L + x$, these (-2)-curves map pairwise to conics in $\mathbf{P}^2$, which must thus be tangent to the sextic C. As also $\iota(L + x) = L - x$, these conics are invariant under $\bar{\iota}_{\mathbf{P}^2}$ and thus they correspond to plane sections of $Q \subset \mathbf{P}^3$, tangent to C_4, that is tritangent planes. This last incarnation of exceptional curves in S as tritangent planes (or equivalently, odd theta characteristics of C_4) is of course very classical.

Finally we compute the moduli. A $\bar{\iota}_{\mathbf{P}^2}$-invariant plane sextic which does not pass through $p = (1 : 0 : 0)$ has equation

$$\sum a_{ijk} x_0^{2i} x_1^j x_2^k \qquad (2i + j + k = 6,\ a_{000} \neq 0).$$

The vector space spanned by such polynomials is 16-dimensional. The subgroup of $GL(3)$ of elements commuting with $\bar{\iota}_{\mathbf{P}^2}$ (which thus preserve the eigenspaces) is isomorphic to $\mathbf{C}^* \times GL(2)$, hence the number of moduli is $16 - (1 + 4) = 11$ as expected.

Alternatively, the genus four curves whose canonical image lies on a cone have $9 - 1 = 8$ moduli (they have one vanishing even theta characteristic), next one has to specify a plane in $\mathbf{P}^3$, this gives again $8 + 3 = 11$ moduli.

3.3. The case $\mathcal{M}_6$. The map ϕ_L identifies X with a complete intersection of a cubic and a quadric in $\mathbf{P}^4$. According to Proposition 2.7, in suitable coordinates the Nikulin involution is induced by

$$\iota_{\mathbf{P}^4} : \mathbf{P}^4 \longrightarrow \mathbf{P}^4, \qquad (x_0 : x_1 : x_2 : x_3 : x_4) \longmapsto (-x_0 : -x_1 : x_2 : x_3 : x_4).$$

The fixed locus in $\mathbf{P}^4$ is:

$$(\mathbf{P}^4)^{\iota_{\mathbf{P}^4}} = l \cup H, \qquad l:\ x_2 = x_3 = x_4 = 0, \quad H:\ x_0 = x_1 = 0.$$

The points $X \cap l$ and $X \cap H$ are fixed points of ι on X and Proposition 2.7 shows that $\sharp(X \cap l) = 2$, $\sharp(X \cap H) = 6$. In particular, the plane H meets the quadric and cubic defining X in a conic and a cubic curve which intersect transversely. Moreover, the quadric is unique, so must be invariant under $\iota_{\mathbf{P}^4}$, and, by considering the action of $\iota_{\mathbf{P}^4}$ on the cubics in the ideal of X, we may assume that the cubic is invariant as well.

$$\begin{array}{lcl} l_{00}(x_2, x_3, x_4)x_0^2 + l_{11}(x_2, x_3, x_4)x_1^2 + l_{01}(x_2, x_3, x_4)x_0x_1 + f_3(x_2, x_3, x_4) & = & 0 \\ \alpha_{00}x_0^2 + \alpha_{11}x_1^2 + \alpha_{01}x_0x_1 + f_2(x_2, x_3, x_4) & = & 0 \end{array}$$

where the α_{ij} are constants, the l_{ij} are linear forms, and f_2, f_3 are homogeneous polynomials of degree two and three respectively. Note that the cubic contains the line $l : x_2 = x_3 = x_4 = 0$.

The projection from $\mathbf{P}^4$ to the product of the eigenspaces $\mathbf{P}^1 \times \mathbf{P}^2$ maps X to a surface defined by an equation of bidegree $(2, 3)$. In fact, the equations imply that $(\sum l_{ij}x_ix_j)/f_3 = (\sum \alpha_{ij}x_ix_j)/f_2$ hence the image of X is defined by the polynomial: $(\sum l_{ij}x_ix_j)f_2 - (\sum \alpha_{ij}x_ix_j)f_3$. Adjunction shows that a smooth surface of bidegree (2,3) is a K3 surface, so the equation defines $\bar{Y}$. The space of invariant quadrics is $3 + 6 = 9$ dimensional and the space of cubics is $3 \cdot 3 + 10 = 19$ dimensional. Multiplying the quadric by a linear form $a_2x_2 + a_3x_3 + a_4x_4$ gives an invariant cubic. The automorphisms of $\mathbf{P}^4$

commuting with ι form a subgroup which is isomorphic with $GL(2) \times GL(3)$ which has dimension $4 + 9 = 13$. So the moduli space of such K3 surfaces has dimension:

$$(9-1) + (19-1) - 3 - (13-1) = 11$$

as expected.

3.4. The case $\mathcal{M}_4$. The map $\phi_L : X \to \mathbf{P}^3$ is an embedding whose image is a smooth quartic surface. From Proposition 2.7 the Nikulin involution ι on $X \subset \mathbf{P}^3 \cong \mathbf{P}(\mathbf{C}^4)$ is induced by

$$\tilde{\iota} : \mathbf{C}^4 \longrightarrow \mathbf{C}^4, \qquad (x_0, x_1, x_2, x_3) \longmapsto (-x_0, -x_1, x_2, x_3)$$

for suitable coordinates. The eight fixed points of the involution are the points of intersection of these lines $x_0 = x_1 = 0$ and $x_2 = x_3 = 0$ with the quartic surface X.

A quartic surface which is invariant under $\tilde{\iota}$ and which does not contain the lines has an equation which is a sum of monomials $x_0^a x_1^b x_2^c x_3^d$ with $a + b = 0, 2, 4$ and $c + d = 4 - a - b$.

The quadratic polynomials invariant under $\tilde{\iota}$ define a map:

$$\mathbf{P}^3 \longrightarrow \mathbf{P}^5, \qquad (x_0 : \ldots : x_3) \longmapsto (z_0 : z_1 : \ldots : z_5) = (x_0^2 : x_1^2 : x_2^2 : x_3^2 : x_0x_1 : x_2x_3)$$

which factors over $\mathbf{P}^3/\tilde{\iota}$. Note that any quartic invariant monomial is a monomial of degree two in the z_i. Thus if $f = 0$ is the equation of X, then $f(x_0, \ldots, x_3) = q(z_0, \ldots, z_5)$ for a quadratic form q. This implies that

$$\bar{Y} : \qquad q(z_0, \ldots, z_5) = 0, \quad z_0z_1 - z_4^2 = 0, \quad z_2z_3 - z_5^2 = 0$$

is the intersection of three quadrics.

The invariant quartics span a $5 + 9 + 5 = 19$-dimensional vector space. On this space the subgroup H of $GL(4)$ of elements which commute with $\iota_{\mathbf{P}^3}$ acts, it is easy to see that $H \cong GL(2) \times GL(2)$ (in block form). Thus $\dim H = 8$ and we get an $19 - 8 = 11$ dimensional family of quartic surfaces in $\mathbf{P}^3$, as desired. See [I] for some interesting sub-families.

3.5. The case $\mathcal{M}_{\widetilde{4}}$. In this case $\mathbf{Z}L \oplus E_8(-2)$ has index two in $NS(X)$. Choose a $v \in E_8(-2)$ with $v^2 = -4$. Then we may assume that $NS(X)$ is generated by $L, E_8(-2)$ and $E_1 := (L + v)/2$, cf. (the proof of) Proposition 2.2. Let $E_2 := (L - v)/2$, then $E_i^2 = L^2/4 + v^2/4 = 1 - 1 = 0$. By Riemann-Roch we have:

$$\chi(\pm E_i) = E_i^2/2 + 2 = 2$$

and so $h^0(\pm E_i) \geq 2$ so E_i or $-E_i$ is effective. Now $L \cdot E_i = L^2/2 + v/2 \cdot L = 2$, hence E_i is effective. As $p_a(E_i) = 1$ and $E_iN \geq 0$ for all (-2)-curves N, each E_i is the class of an elliptic fibration. As $L = E_1 + E_2$, by [SD, Theorem 5.2] the map ϕ_L is a 2:1 map to a quadric Q in $\mathbf{P}^3$ and it is ramified on a curve B of bi-degree $(4, 4)$. The quadric is smooth, hence isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$, because there are no (-2)-curves in $NS(X)$ perpendicular to L.

Let $i : X \to X$ be the covering involution of $X \to Q$. Then i and the Nikulin-involution ι commute. The elliptic pencils E_1 and E_2 are permuted by ι because $\iota^*L = L, \iota^*v = -v$. This means that the involution $\bar{\iota}_Q$ on $Q = \mathbf{P}^1 \times \mathbf{P}^1$ induced by ι acts as $((s : t), (u : v)) \mapsto ((u : v), (s : t))$. The quotient of $Q/\bar{\iota}_Q$ is well known to be isomorphic to $\mathbf{P}^2$.

The fixed point set of $\bar{\iota}_Q$ in $\mathbf{P}^1 \times \mathbf{P}^1$ is the diagonal Δ. Thus Δ intersects the branch curve B in eight points. The inverse image of these points in X are the eight fixed points of the Nikulin involution.

The diagonal maps to a conic C_0 in $\mathbf{P}^2 = Q/\bar{\iota}_Q$, which gives the representation of a smooth quadric as double cover of $\mathbf{P}^2$ branched along a conic (in equations: $t^2 = q(x, y, z)$). The curve B maps to a plane curve isomorphic to $\bar{B} = B/\iota$. As ι has 8 fixed points on the genus 9 curve B, the genus of $\bar{B}$ is 3 and $\bar{B} \subset \mathbf{P}^2$ is a quartic curve.

Let $j = i\iota = \iota i \in Aut(X)$. The fixed point set of j is easily seen to be the inverse image C_3 of the diagonal $\Delta \subset Q$. As $C_3 \to \Delta$ branches over the 8 points in $B \cap \Delta$, C_3 is a smooth (hyperelliptic) genus three curve. Thus the surface $S := X/j$ is smooth and the image of C_3 in S lies in the linear system $|-2K_S|$. The double cover $S \to \mathbf{P}^2$ is branched over the plane quartic $\bar{B} \subset \mathbf{P}^2$. This implies that S is a Del Pezzo surface of degree 2, cf. [Dem], [DoO].

This leads to the following diagrams of double covers and fixed point sets:

$$\begin{tikzcd}
 & X \arrow[dl] \arrow[d] \arrow[dr] & \\
Q \cong \mathbf{P}^1 \times \mathbf{P}^1 \arrow[dr] & \bar{Y} \arrow[d] & S \arrow[dl] \\
 & \mathbf{P}^2 &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & B \cup C_3 \arrow[dl] \arrow[d] \arrow[dr] & \\
B \cup \Delta \arrow[dr] & \bar{B} \cup C_0 \arrow[d] & \bar{B} \cup C_3 \arrow[dl] \\
 & \bar{B} \cup C_0 &
\end{tikzcd}$$

In particular the eight nodal surface $\bar{Y}$ is the double cover of $\mathbf{P}^2$ branched over the reducible sextic with components the conic C_0 and the quartic $\bar{B}$. The nodes of $\bar{Y}$ map to the intersection points of C_0 and $\bar{B}$.

To count the moduli we note that the homogeneous polynomials of degree two and four in three variables span vector spaces of dimension 6 and 15, as $\dim GL(3) = 9$ we get: $(6-1)+(15-1)-(9-1) = 11$ moduli.

3.6. The case $\mathcal{M}_8$. We have $H^0(X, L) \cong \pi^* H^0(Y, M_1) \oplus \pi^* H^0(Y, M_2)$ and $L^2 = 8$, $M_i^2 = 2$ so $h^0(L) = 6$, $h^0(M_i) = 3$ for $i = 1, 2$. The image of X under ϕ_L is the intersection of three quadrics in $\mathbf{P}^5$ and ι is induced by

$$\tilde{\iota} : \mathbf{C}^6 \longrightarrow \mathbf{C}^6, \qquad (x_0, x_1, x_2, y_0, y_1, y_2) \longmapsto (x_0, x_1, x_2, -y_0, -y_1, -y_2).$$

The multiplication map maps the 21-dimensional space $S^2H^0(X, L)$ onto the 18-dimensional space $H^0(X, 2L)$. Using ι we can get some more information on the kernel of this map, which are the quadrics defining $X \subset \mathbf{P}^5$. We have:

$$S^2H^0(X, L) \cong \left(S^2H^0(Y, M_1) \oplus S^2H^0(Y, M_2)\right) \oplus \left(H^0(Y, M_1) \otimes H^0(Y, M_2)\right),$$

Moreover, as

$$\beta^*(2L) = \pi^* M, \qquad \text{with} \quad M = 2M_1 + N_1 + \ldots + N_4 = 2M_2 + N_5 + \ldots + N_8,$$

(cf. Proposition 2.7) we have the decomposition

$$H^0(X, 2L) \cong \pi^* H^0(Y, M) \oplus \pi^* H^0(Y, M - \hat{N}), \qquad (h^0(M) = (M^2)/2 + 2 = 10,\ h^0(M - \hat{N}) = 8).$$

In particular, the multiplication maps splits as:

$$H^0(M_1) \otimes H^0(M_2) \longrightarrow H^0(Y, M - \hat{N})$$

(vector spaces with dimensions with $3 \cdot 3 = 9$ and 8 resp.) and

$$S^2H^0(Y, M_1) \ \oplus \ S^2H^0(Y, M_2) \longrightarrow H^0(Y, M)$$

(with dimensions 6+6 = 12 and 10 resp.). Each of these two maps is surjective, and as $S^2H^0(Y,M_1) \to H^0(Y,M)$ is injective (ϕ_{M_1} maps Y onto $\mathbf{P}^2$), the quadrics in the ideal of X can be written as:

$$Q_1(x) - Q_2(y) = 0, \qquad Q_3(x) - Q_4(y) = 0, \qquad B(x,y) = 0$$

with Q_i homogeneous of degree two in three variables, and B of bidegree $(1,1)$. Note that each eigenspace intersects X in $2 \cdot 2 = 4$ points.

The surface $\bar{Y}$ maps to $\mathbf{P}^2 \times \mathbf{P}^2$ with the map $\phi_{M_1} \times \phi_{M_2}$, its image is the image of X under the projections to the eigenspaces $\mathbf{P}^5 \to \mathbf{P}^2 \times \mathbf{P}^2$. As $(x_0 : \ldots : y_2) \mapsto Q_1(x)/Q_2(y)$ is a constant rational function on X and similarly for $Q_3(x)/Q_4(y)$, there is a $c \in \mathbf{C}$ such that the image of X is contained in the complete intersection of type $(2,2)$, $(1,1)$ in $\mathbf{P}^2 \times \mathbf{P}^2$ defined by

$$Q_1(x)Q_4(y) - cQ_3(x)Q_2(y) = 0, \qquad B(x,y) = 0.$$

By adjunction, smooth complete intersections of this type are K3 surfaces.

To count the moduli, note that the first two equations come from a $6+6 = 12$-dimensional vector space and the third comes from a $3 \cdot 3 = 9$-dimensional space. The Grassmanian of 2-dimensional subspaces of a 12 dimensional space has dimension $2(12-2) = 20$. The subgroup of $GL(6)$ which commutes with $\iota_{\mathbf{P}^5}$ is isomorphic to $GL(3) \times GL(3)$ and has dimension $9+9 = 18$. Thus we get $20 + (9-1) - (18-1) = 11$ moduli, as expected.

3.7. The case $\mathcal{M}_{\widetilde{8}}$. We have $H^0(X,L) \cong \pi^*H^0(Y,M) \oplus \pi^*H^0(Y,M-\hat{N})$ and $L^2 = 8$, $M^2 = 4$ so $h^0(M) = 4, h^0(M-N) = 2$. The image of X under ϕ_L is is the intersection of three quadrics in $\mathbf{P}^5$ and ι is induced by

$$\tilde{\iota} : \mathbf{C}^6 \longrightarrow \mathbf{C}^6, \qquad (x_0,x_1,x_2,x_3,y_0,y_1) \longmapsto (x_0,x_1,x_2,x_3,-y_0,-y_1).$$

To study the quadrics defining X, that is the kernel of the multiplication map $S^2H^0(X,L) \to H^0(X,2L)$ we again split these spaces into ι^*-eigenspaces:

$$S^2H^0(X,L) \cong \left(S^2H^0(Y,M) \oplus S^2H^0(Y,M-\hat{N})\right) \oplus \left(H^0(Y,M) \otimes H^0(Y,M-\hat{N})\right),$$

(with dimensions $21 = (10+3)+8$) and

$$H^0(X,2L) \cong \pi^*H^0(Y,2M) \oplus \pi^*H^0(Y,2M-\hat{N})$$

(with dimensions $h^0(2M) = 10$, $h^0(2M-\hat{N}) = 8$).

This implies that there are no quadratic relations in the 8 dimensional space $H^0(Y,M) \otimes H^0(Y,M-\hat{N})$. As ϕ_M maps Y onto a quartic surface in $\mathbf{P}^3$ and $M-\hat{N}$ is a map of Y onto $\mathbf{P}^1$, the quadrics in the ideal of X are of the form:

$$y_0^2 = Q_1(x), \qquad y_0y_1 = Q_2(x), \qquad y_1^2 = Q_3(x).$$

The fixed points of the involution are the eight points in the intersection of X with the $\mathbf{P}^3$ defined by $y_0 = y_1 = 0$.

The image of Y by ϕ_M is the image of the projection of X from the invariant line to the invariant $\mathbf{P}^3$, which is defined by $y_0 = y_1 = 0$. The image is the quartic surface defined by $Q_1Q_3 - Q_2^2 = 0$ which can be identified with $\bar{Y}$. The equation is the determinant of a symmetric 2×2 matrix, which also implies that this surface has 8 nodes, (cf. [Ca, Theorem 2.2], [B, section 3]), the nodes form an even set (cf. [Ca, Proposition 2.6]).

We compute the number of moduli. Quadrics of this type span a space U of dimension $3+10 = 13$. The dimension of the Grassmanian of three dimensional subspaces of U is $3(13-3) = 30$. The group of automorphisms of $\mathbf{C}^6$ which commute with $\iota_{\mathbf{P}^5}$ is $GL(2)\times GL(4)$. So we have a $30-(4+16-1)=11$ dimensional space of such K3-surfaces in $\mathbf{P}^5$, as expected.

3.8. The case $\mathcal{M}_{12}$. We have $H^0(X,L)\cong\pi^*H^0(Y,M_1)\oplus\pi^*H^0(Y,M_2)$ and $L^2=12$, $M_i^2=4$ so $h^0(L)=8$, $h^0(M_i)=4$ for $i=1,2$. The image of X under ϕ_L is the intersection of ten quadrics in $\mathbf{P}^7$.

Following Example 3.6, we use ι^* to split the multiplication map from the $36=(10+10)+16$-dimensional space $S^2H^0(X,L)$ onto the $26=14+12$-dimensional space $H^0(X,2L)$, again $\beta^*(2L)=\pi^*M$ for an $M\in NS(Y)$ with $M^2=24$. Thus we find $20-14=6$ quadrics of the type $Q_1(x)-Q_2(y)$ with Q_i quadratic forms in 4 variables, and $16-12=4$ quadratic forms $B_i(x,y)$, $i=1,\dots,4$ where x,y are coordinates on the two eigenspaces in $H^0(X,L)$.

In particular, the projection from $\mathbf{P}^7$ to the product of the eigenspaces $\mathbf{P}^3\times\mathbf{P}^3$ maps X onto a surface defined by 4 equations of bidegree $(1,1)$. Adjunction shows that a complete intersection of this type is a K3 surface, so the four B_i's define $\bar{Y}\subset\mathbf{P}^3\times\mathbf{P}^3$.

Each B_i can be written as: $B_i(x,y)=\sum_j l_{ij}(x)y_j$ with linear forms l_{ij} in $x=(x_0,\dots,x_3)$. The image of $\bar{Y}\subset\mathbf{P}^3\times\mathbf{P}^3$ under the projection to the first factor is then defined by $\det(l_{ij}(x))=0$, which is a quartic surface in $\mathbf{P}^3$ as expected. In fact, a point $x\in\mathbf{P}^3$ has a non-trivial counter image $(x,y)\in X\subset\mathbf{P}^3\times\mathbf{P}^3$ iff the matrix equation $(l_{ij})y=0$ has a non-trivial solution.

As X is not a complete intersection, we omit the moduli count.

3.9. The case $\mathcal{M}_{\widetilde{12}}$. In this case $\beta^*L\cong\pi^*M$, $h^0(L)=8=5+3=h^0(M)+h^0(M-\hat{N})$. We consider again the quadrics in the ideal of X in Example 3.7. The space $S^2H^0(X,L)$ of quadrics on $\mathbf{P}^7$ decomposes as:

$$S^2H^0(X,L)\cong\left(S^2H^0(Y,M)+S^2H^0(Y,M-\hat{N})\right)\oplus\left(H^0(Y,M)\otimes H^0(Y,M-\hat{N})\right),$$

with dimensions $36=(15+6)+15$, whereas the sections of $2L$ decompose as:

$$h^0(2L)=(4L^2)/2+2=26=14+12=h^0(2M)\oplus h^0(2M-\hat{N}).$$

Thus there are $(15+6)-14=7$ independent quadrics in the ideal of $X\subset\mathbf{P}^7$ which are invariant and there are $15-12=3$ quadrics which are anti-invariant under the map

$$\tilde{\iota}:\mathbf{C}^8\longrightarrow\mathbf{C}^8,\qquad(x_0,\dots,x_4,y_0,y_1,y_2)\longmapsto(x_0,\dots,x_4,-y_0,-y_1,-y_2).$$

An invariant quadratic polynomial looks like $q_0(x_0,\dots,x_4)+q_1(y_0,y_1,y_2)$, and since the space of quadrics in three variables is only 6 dimensional, there is one non-zero quadric q in the ideal of the form $q=q(x_0,\dots,x_4)$. An anti-invariant quadratic polynomial is of bidegree $(1,1)$ in x and y. In particular, the image of the projection of X to the product of the eigenspaces $\mathbf{P}^4\times\mathbf{P}^2$ is contained in one hypersurface of bidegree $(2,0)$ and in three hypersurfaces of bidegree $(1,1)$. The complete intersection of four general such hypersurfaces is a K3 surface (use adjunction and $(2+3\cdot1,3\cdot1)=(5,3)$).

The three anti-invariant quadratic forms can be written as $\sum_j l_{ij}(x)y_j$, $i=1,2,3$. The determinant of the 3×3 matrix of linear forms $(l_{ij}(x))$, defines a cubic form which is an equation for the image of X in $\mathbf{P}^4$ (cf. Example 3.8). Thus the projection $\bar{Y}$ of X to $\mathbf{P}^4$ is the intersection of the quadric defined by $q(x)=0$ and a cubic.

The projection to $\mathbf{P}^2$ is 2:1, as it should be, since for general $y\in\mathbf{P}^2$ the three linear forms in x given by $\sum_j l_{ij}(x)y_j$ define a line in $\mathbf{P}^4$ which cuts the quadric $q(x)=0$ in two points.

4. Elliptic fibrations with a section of order two

4.1. Elliptic fibrations and Nikulin involutions. Let X be a K3 surface which has an elliptic fibration $f : X \to \mathbf{P}^1$ with a section σ. The set of sections of f is a group, the Mordell-Weil group MW_f, with identity element σ. This group acts on X by translations and these translations preserve the holomorphic two form on X. In particular, if there is an element $\tau \in MW_f$ of order two, then translation by τ defines a Nikulin involution ι.

In that case the Weierstrass equation of X can be put in the form:

$$X : \qquad y^2 = x(x^2 + a(t)x + b(t))$$

the sections σ, τ are given by the section at infinity and $\tau(t) = (x(t), y(t)) = (0, 0)$. For the general fibration on a K3 surface X, the degrees of a and b are 4 and 8 respectively.

4.2. Proposition. Let $X \to \mathbf{P}^1$ be a general elliptic fibration with sections σ, τ as above in section 4.1. and let ι be the corresponding Nikulin involution on X. These fibrations form a 10-dimensional family.

The quotient K3 surface Y also has an elliptic fibration:

$$Y : \qquad y^2 = x(x^2 - 2a(t)x + (a(t)^2 - 4b(t)),$$

We have:

$$NS(X) \cong NS(Y) \cong U \oplus N, \qquad T_X \cong T_Y \cong U^2 \oplus N.$$

The bad fibers of $X \to \mathbf{P}^1$ are eight fibers of type I_1 (which are rational curves wit a node) over the zeroes of $a^2 - 4b$ and eight fibers of type I_2 (these fibers are the union of two $\mathbf{P}^1$'s meeting in two points) over the zeroes of b. The bad fibers of $Y \to \mathbf{P}^1$ are eight fibers of type I_2 over the zeroes of $a^2 - 4b$ and eight fibers of type I_1 over the zeroes of b.

Proof. Since X has an elliptic fibration with a section, $NS(X)$ contains a copy of the hyperbolic plane U (with standard basis the class of a fiber f and $f + \sigma$). The discriminant of the Weierstrass model of X is $\Delta_X = b^2(a^2 - 4b)$ and the fibers of the Weierstrass model over the zeroes of Δ_X are nodal curves. Thus $f : X \to \mathbf{P}^1$ has eight fibers of type I_1 (which are rational curves with a node) over the zeroes of $a^2 - 4b$ and 8 fibers of type I_2 (these fibers are the union of two $\mathbf{P}^1$'s meeting in two points) over the zeroes of b.

The components of the singular fibers which do not meet the zero section σ, give a sublattice $< -2 >^8$ perpendicular to U. If there are no sections of infinite order, the lattice $U \oplus < -2 >^8$ has finite index in the Néron Severi group of X. Hence X has $22 - 2 - 10 = 10$ moduli. One can also appeal to [Shim] where the Néron Severi group of the general elliptic K3 fibration with a section of order two is determined. To find the moduli from the Weierstrass model, note that a and b depend on $5 + 9 = 14$ parameters. Using transformations of the type $(x, y) \mapsto (\lambda^2 x, \lambda^3 y)$ (and dividing the equation by λ^6) and the automorphism group $\mathbf{P}GL(2)$ of $\mathbf{P}^1$ we get $14 - 1 - 3 = 10$ moduli.

The Shioda-Tate formula (cf. e.g. [Shio, Corollary 1.7]) shows that the discriminant of the Néron Severi group is $2^8/n^2$ where n is the order of the torsion subgroup of MW_f. The curve defined by $x^2 + a(t)x + b(t) = 0$ cuts out the remaining pair of points of order two on each smooth fiber. As it is irreducible in general, MW_f must be cyclic. If there were a section σ of order four, it would have to satisfy $2\sigma = \tau$. But in a fiber of type I_2 the complement of the singular points is the group $G = \mathbf{C}^* \times (\mathbf{Z}/2\mathbf{Z})$ and the specialization $MW_f \to G$ is an injective homomorphism. Now τ specializes to $(\pm 1, \bar{1})$ (the sign doesn't matter) since τ specializes to the node in the Weierstrass model. But there is no $g \in G$ with $2g = (\pm 1, \bar{1})$. We conclude that for general X we have $MW_f = \{\sigma, \tau\} \cong \mathbf{Z}/2\mathbf{Z}$ and that the discriminant of the Néron Severi group of X is 2^6.

The Néron Severi group has $\mathbf{Q}$ basis $\sigma, f, N_1, \ldots, N_8$ where the N_i are the components of the I_2 fibers not meeting σ. As $\tau \cdot \sigma = 0$, $\tau \cdot f = 1$ and $\tau \cdot N_i = 1$, we get:

$$\tau = \sigma + 2f - \hat{N}, \qquad \hat{N} = (N_1 + \ldots + N_8)/2.$$

Thus the smallest primitive sublattice containing the N_i is the Nikulin lattice. Comparing discriminants we conclude that:

$$NS(X) = \langle s, f\rangle \oplus \langle N_1, \ldots, N_8, \hat{N}\rangle \;\cong\; U \oplus N.$$

The transcendental lattice T_X of X can be determined as follows. It is a lattice of signature $(2+, 10-)$ and its discriminant form is the opposite of the one of N, but note that $q_N = -q_N$ since q_N takes values in $\mathbf{Z}/2\mathbf{Z}$. Moreover, $T_X^*/T_X \cong N^*/N \cong (\mathbf{Z}/2\mathbf{Z})^6$. Using [Ni2, Corollary 1.13.3], we find that T_X is uniquely determined by the signature and the discriminant form. The lattice $U^2 \oplus N$ has these invariants, so

$$T_X \cong U^2 \oplus N.$$

As the Nikulin involution preserves the fibers of the elliptic fibration on X, the desingularisation Y of the quotient X/ι has an elliptic fibration $g : Y \to \mathbf{P}^1$, with a section $\bar{\sigma}$, (the image of σ). Observe that given any elliptic curve in Weierstrass form, as explained in [ST, Section 4, p.76 and Proposition p.79] there is a straightforward way to write down the Weierstrass form of its quotient by a translation by a point of order two, and so one can immediately write down the Weierstrass equation of Y.

The discriminant of the Weierstrass model of Y is $\Delta_Y = 4b(a^2 - 4b)^2$ and, reasoning as before, we find the bad fibers of $g : Y \to \mathbf{P}^1$. In particular, the I_1 and I_2 fibers of X and Y are indeed 'interchanged'.

Geometrically, the reason for this is as follows. The fixed points of translation by τ are the eight nodes in the I_1-fibers, blowing them up gives I_2-type fibers which map to I_2-type fibers in Y. The exceptional curves lie in the ramification locus of the quotient map, the other components, which meet σ, map 2:1 to components of the I_2-fibers which meet $\bar{\sigma}$. The two components of an I_2-fiber in X are interchanged and also the two singular points of the fiber are permuted, so in the quotient this gives an I_1-type fiber. □

4.3. Remark. Note that $NS(X) \oplus T_X \cong U^3 \oplus N^2$, however, there is *no* embedding of N^2 into $E_8(-1)^2$, such that $N \oplus \{0\}$ ($\subset NS(X)$) is primitive in $E_8(-1)^2$. However, $N^2 \subset \Gamma_{16}(-1)$ (cf. section 1.11), an even, negative definite, unimodular lattice of rank 16 and $U^3 \oplus \Gamma_{16}(-1) \cong U^3 \oplus E_8(-1)^2$ by the classification of even indefinite unimodular quadratic forms.

4.4. Morrison-Nikulin involutions. D. Morrison observed that a K3 surface X having two perpendicular copies of $E_8(-1)$ in the Néron Severi group has a Nikulin involution which exchanges the two copies of $E_8(-1)$, cf. [Mo, Theorem 5.7]. We will call such an involution a Morrison-Nikulin involution. This involution then has the further property that $T_Y \cong T_X(2)$ where Y is the quotient K3 surface and we have a Shioda-Inose structure on Y (cf. [Mo, Theorem 6.3])

4.5. Moduli. As $E_8(-1)$ has rank eight and is negative definite, a projective K3 surface with a Morrison-Nikulin involution has a Néron Severi group of rank at least 17 and hence has at most three moduli. In case the Néron Severi group has rank exactly 17, we get

$$NS(X) \cong \langle 2n\rangle \oplus E_8(-1) \oplus E_8(-1)$$

since the sublattice $E_8(-1)^2$ is unimodular. Results of Kneser and Nikulin, [Ni2, Corollary 1.13.3], guarantee that the transcendental lattice $T_X := NS(X)^\perp$ is uniquely determined by its signature and discriminant form. As the discriminant form of T_X is the opposite of the one on $NS(X)$ we get

$$T_X \cong \langle -2n \rangle \oplus U^2.$$

In case $n = 1$ such a three dimensional family can be obtained from the double covers of $\mathbf{P}^2$ branched along a sextic curve with two singularities which are locally isomorphic to $y^3 = x^5$. The double cover then has two singular points of type E_8, that is, each of these can be resolved by eight rational curves with incidence graph E_8. As the explicit computations are somewhat lengthy and involved, we omit the details. See [GL, Appendix], [P] and [Deg] for more on double covers of $\mathbf{P}^2$ along singular sextics.

4.6. Morrison-Nikulin involutions on elliptic fibrations. We consider a family of K3 surfaces with an elliptic fibration with a Morrison-Nikulin involution induced by translation by a section of order two. It corresponds to the family with $n = 2$ from section 4.5.

Note that in the proposition below we describe a $K3$ surface Y with a Nikulin involution and quotient K3 surface X such that $T_Y = T_X(2)$, which is the 'opposite' of what would happen if the involution of Y was a Morrison-Nikulin involution. It is not hard to see that there is no primitive embedding $T_Y \hookrightarrow U^3$, so Y does not have a Morrison-Nikulin involution at all (cf. [Mo, Theorem 6.3]).

4.7. Proposition. Let $X \to \mathbf{P}^1$ be a general elliptic fibration defined by the Weierstrass equation

$$X: \qquad y^2 = x(x^2 + a(t)x + 1), \qquad a(t) = a_0 + a_1 t + a_2 t^2 + t^4 \in \mathbf{C}[t].$$

The K3 surface X has a Morrison-Nikulin involution defined by translation by the section, of order two, $t \mapsto (x(t), y(t)) = (0,0)$. Then:

$$NS(X) = \langle 4 \rangle \oplus E_8(-1) \oplus E_8(-1), \qquad T_X = \langle -4 \rangle \oplus U^2.$$

The bad fibers of the fibration are nodal cubics (type I_1) over the eight zeroes of $a^2(t) - 4$ and one fiber of type I_{16} over $t = \infty$.

The quotient K3 surface Y has an elliptic fibration defined by the Weierstrass model:

$$Y: \qquad y^2 = x(x^2 - 2a(t)x + (a(t)^2 - 4)), \qquad T_Y \cong \langle -8 \rangle \oplus U(2)^2.$$

This K3 surface has a Nikulin involution defined by translation by the section $t \mapsto (x(t), y(t)) = (0,0)$ and the quotient surface is X. For general X, the bad fibers of Y are 8 fibers of type I_2 over the same points in $\mathbf{P}^1$ where X has fibers of type I_1 and at infinity Y has a fiber of type I_8.

Proof. As we observed in section 4.1, translation by the section of order two defines a Nikulin involution.

Let $\hat{a}(s) := s^4 a(s^{-1})$, it is a polynomial of degree at most four and $\hat{a}(0) \neq 0$. Then on $\mathbf{P}^1 - \{0\}$, with coordinate $s = t^{-1}$, the Weierstrass model is

$$v^2 = u(u^2 + \hat{a}(s)u + s^8), \qquad \Delta = s^{16}(\hat{a}(s)^2 - 4s^8), \qquad u = s^4 x,\ v = s^6 y,$$

where Δ is the discriminant. The fiber over $s = 0$ is a stable (nodal) curve, so the corresponding fiber X_∞ is of type I_m where m is the order of vanishing of the discriminant in $s = 0$ (equivalently, it is the order of the pole of the j-invariant in $s = 0$). Thus X_∞ is an I_{16} fiber. As the section of order two specializes to the singular point $(u, v, s) = (0,0,0)$, after blow up it will not meet the component of the fiber which meets the zero section.

The group structure of the elliptic fibration induces a Lie group structure on the smooth part of the I_{16} fiber. Taking out the 16 singular points in this fiber, we get the group $\mathbf{C}^* \times \mathbf{Z}/16\mathbf{Z}$. The zero section meets the component C_0, where

$$C_n := \mathbf{P}^1 \times \{\bar{n}\} \hookrightarrow X_\infty,$$

and the section of order two must meet C_8. Translation by the section of order two induces the permutation $C_n \mapsto C_{n+8}$ of the 16 components of the fiber. The classes of the components C_n, with $n = -2, \ldots, 4$, generate a lattice of type $A_7(-1)$ which together with the zero section gives an $E_8(-1)$. The Nikulin involution maps this $E_8(-1)$ to the one whose components are the C_n, $n = 6, \ldots, 12$, and the section of order two. Thus the Nikulin involution permutes two perpendicular copies of $E_8(-1)$ and hence it is a Morrison-Nikulin involution.

The bad fibers over $\mathbf{P}^1 - \{\infty\}$ correspond to the zeroes of $\Delta = a^2(t) - 4$. For general a, Δ has eight simple zeroes and the fibers are nodal, so we have eight fibers of type I_1 in $\mathbf{P}^1 - \{\infty\}$.

By considering the points on $\mathbf{P}^1$ where there are bad fibers it is not hard to see that we do get a three dimensional family of elliptic K3 surfaces with a Morrison-Nikulin involution. Thus the general member of this three dimensional family has a Néron Severi group S of rank 17.

As we constructed a unimodular sublattice $E_8(-1)^2 \subset S$, we get $S \cong < -d > \oplus E_8(-1)^2$ and d (> 0) is the discriminant of S. The Shioda-Tate formula (cf. e.g. [Shio, Corollary 1.7]) gives that $d = 16/n^2$ where n is the order of the group of torsion sections. As n is a multiple of 2 and d must be even it follows that $d = 4$. As the embedding of $NS(X)$ into $U^3 \oplus E_8(-1)^2$ is unique up to isometry it is easy to determine $T_X = NS(X)^\perp$. Finally $T_Y \cong T_X(2)$ by the results of [Mo].

The Weierstrass model of the quotient elliptic fibration Y can be computed with the standard formula cf. [ST, p.79], the bad fibers can be found from the discriminant $\Delta = -4(a^2 - 4)^2$ (and j-invariant). Alternatively, fixed points of the involution on X are the nodes in the I_1-fibers. Since these are blown up, we get 8 fibers of type I_2 over the same points in $\mathbf{P}^1$ where X has fibers of type I_1. At infinity Y has a fiber of type I_8 because the involution on X permutes of the 16 components of the I_{16}-fiber ($C_n \leftrightarrow C_{n+8}$). Now the minimal model of the quotient surface of Y by the translation of order two is again X, cf. [ST, Section 4, p.76]. Indeed consider the generic fiber, this is an elliptic curve C in Weierstrass form, its quotient by a point of order two is an elliptic curve $\overline{C}$. The kernel of the multiplication by two:

$$[2] : C \longrightarrow C$$

contains all points of order two. Hence this map factorizes as

$$C \xrightarrow{2:1} \overline{C} \xrightarrow{2:1} C.$$

□

4.8. Remark. The Weierstrass model we used to define X, $y^2 = x(x^2 + a(t)x + 1)$, exhibits X as the minimal model of the double cover of $\mathbf{P}^1 \times \mathbf{P}^1$, with affine coordinates x and t. The branch curve consists of the the lines $x = 0$, $x = \infty$ and the curve of bidegree $(2, 4)$ defined by $x^2 + a(t)x + 1 = 0$. Special examples of such double covers are studied in section V.23 of [BPV]. In particular, on p.185 the 16-gon appears with the two sections attached and the E_8's are pointed out in the text. Note however that our involution is *not* among those studied there.

4.9. Remark. Note that if $X \longrightarrow \mathbf{P}^1$ is an elliptic fibration with a section, multiplication by n on each smooth fiber gives a fiber preserving rational map $X - - \rightarrow X$ of degree n^2. In the Proposition 4.7 we give an example with $n = 2$ (such self-maps are rather rare for non-rational surfaces).

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Symplectic automorphisms of prime order on K3 surfaces

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SYMPLECTIC AUTOMORPHISMS OF PRIME ORDER ON K3 SURFACES

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Abstract. We study algebraic K3 surfaces (defined over the complex number field) with a symplectic automorphism of prime order. In particular we consider the action of the automorphism on the second cohomology with integer coefficients (by a result of Nikulin this action is independent on the choice of the K3 surface). With the help of elliptic fibrations we determine the invariant sublattice and its perpendicular complement, and show that the latter coincides with the Coxeter-Todd lattice in the case of automorphism of order three.

0. Introduction

In the paper [Ni1] Nikulin studies finite abelian groups G acting symplectically (i.e. $G_{|H^{2,0}(X,\mathbb{C})} = id_{|H^{2,0}(X,\mathbb{C})}$) on K3 surfaces (defined over $\mathbb{C}$). One of his main result is that the action induced by G on the cohomology group $H^2(X,\mathbb{Z})$ is unique up to isometry. In [Ni1] all abelian finite groups of automorphisms of a K3 surface acting symplectically are classified. Later Mukai in [Mu] extends the study to the non abelian case. Here we consider only abelian groups of prime order p which, by Nikulin, are isomorphic to $\mathbb{Z}/p\mathbb{Z}$ for $p = 2, 3, 5, 7$.

In the case of $p = 2$ the group is generated by an involution, which is called by Morrison in [Mo, Def. 5.1] Nikulin involution. This was very much studied in the last years, in particular because of its relation with the Shioda-Inose structure (cf. e.g. [CD], [GL], [vGT], [L], [Mo]). In [Mo] Morrison proves that the isometry induced by a Nikulin involution ι on the lattice $\Lambda_{K3} \simeq U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1)$, which is isometric to $H^2(X,\mathbb{Z})$, switches the two copies of $E_8(-1)$ and acts as the identity on the sublattice $U \oplus U \oplus U$. As a consequence one sees that $(H^2(X,\mathbb{Z})^{\iota^*})^{\perp}$ is the lattice $E_8(-2)$. This implies that the Picard number ρ of an algebraic K3 surface admitting a Nikulin involution is at least nine. In [vGS] van Geemen and Sarti show that if $\rho \geq 9$ and $E_8(-2) \subset NS(X)$ then the algebraic $K3$ surface X admits a Nikulin involution and they classify completely these $K3$ surfaces. Moreover they discuss many examples and in particular those surfaces admitting an elliptic fibration with a section of order two. This section operates by translation on the fibers and defines a Nikulin involution on the K3 surface.

The aim of this paper is to identify the action of a symplectic automorphism σ_p of the remaining possible prime orders $p = 3, 5, 7$ on the K3 lattice Λ_{K3} and to describe such algebraic K3 surfaces with minimal possible Picard number. Thanks to Nikulin's result ([Ni1, Theorem 4.7]), to find the action on Λ_{K3}, it suffices to identify the action in one special case. For this purpose it seemed to be convenient to study algebraic K3 surfaces with an elliptic fibration with a section of order three, five, resp. seven. Then the translation by this section is a symplectic automorphism of the surface of the same order. A

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concrete analysis leads us to the main result of the paper which is the description of the lattices $H^2(X,\mathbb{Z})^{\sigma_p^*}$ and $\Omega_p = (H^2(X,\mathbb{Z})^{\sigma_p^*})^\perp$ given in the Theorem 4.1. The proof of the main theorem consists in the Propositions 4.2, 4.4, 4.6. We describe the lattice Ω_p also as $\mathbb{Z}[\omega_p]$-lattices, where ω_p is a primitive p root of the unity. This kind of lattices are studied e.g. in [Ba], [BS] and [E]. In particular in the case $p = 3$ the lattice Ω_3 is the Coxeter-Todd lattice with the form multiplied by -2, $K_{12}(-2)$, which is described in [CT] and in [CS]. The elliptic surfaces we used to find the lattices Ω_p do not have the minimal possible Picard number. We prove in Proposition 5.1 that for K3 surfaces, X, with minimal Picard number and symplectic automorphism, if L is a class in $NS(X)$ which is invariant for the automorphisms, with $L^2 = 2d > 0$, then either $NS(X) = \mathbb{Z}L \oplus \Omega_p$ or the latter is a sublattice of index p in $NS(X)$. Using this result and the one of the Proposition 5.2 we describe the coarse moduli space of the algebraic K3 surfaces admitting a symplectic automorphism of prime order.
The structure of the paper is the following: in section 1 we compute the number of moduli of algebraic K3 surfaces admitting a symplectic automorphism of order p and their minimal Picard number. In section 2 we give the definition of $\mathbb{Z}[\omega_p]$-lattice and we associate to it a module with a bilinear form, which in some cases is a $\mathbb{Z}$-lattice (we use this construction in section 4 to describe the lattices Ω_p as $\mathbb{Z}[\omega_p]$-lattices). In section 3 we recall some results about elliptic fibrations and elliptic K3 surfaces (see e.g. [Mi1], [Mi2], [Shim], [Shio] for more on elliptic K3 surfaces). In particular we introduce the three elliptic fibrations which we use in section 4 and give also their Weierstrass form. In section 4 we state and proof the main result, Theorem 4.1: we identify the lattices Ω_p and we describe them as $\mathbb{Z}[\omega_p]$-lattices. In section 5 we describe the Néron-Severi group of K3 surfaces admitting a symplectic automorphism and having minimal Picard number (Proposition 5.1). In section 5 we describe the coarse moduli space of the algebraic K3 surfaces admitting a symplectic automorphism and the Néron-Severi group of those having minimal Picard number.
We would like to express our deep thanks to Bert van Geemen for suggesting us the problem and for his invaluable help during the preparation of this paper.

1. Preliminary results

Definition 1.1. *A symplectic automorphism σ_p of order p on a K3 surface X is an automorphism such that:*

1. the group G generated by σ_p is isomorphic to $\mathbb{Z}/p\mathbb{Z}$,
2. $\sigma_p^*(\delta) = \delta$, for all δ in $H^{2,0}(X)$.

We recall that by [Ni1] an automorphism on a K3 surface is symplectic if and only if it acts as the identity on the transcendental lattice T_X. In local coordinates at a fixed point σ_p has the form $\mathrm{diag}(\omega_p, \omega_p^{p-1})$ where ω_p is a primitive p-root of unity. By a result of Nikulin the only possible values for p are 2,3,5,7 see [Ni1, Theorem 4.5] and [Ni1, §5]. The automorphism σ_3 has six fixed points on X, σ_5 has four fixed points and σ_7 has three fixed points. The automorphism σ_p induces a σ_p^* isometry on $H^2(X,\mathbb{Z}) \cong \Lambda_{K3}$. Nikulin proved [Ni1, Theorem 4.7] that if σ_p is symplectic, then the action of σ_p^* is unique up to isometry of Λ_{K3}.
Let ω_p be a primitive p-root of the unity. The vector space $H^2(X,\mathbb{C})$ can be decomposed

in eigenspaces of the eigenvalues 1 and ω_p^i:

$$H^2(X,\mathbb{C}) = H^2(X,\mathbb{C})^{\sigma_p^*} \oplus (\bigoplus_{i=1,\dots,p-1} H^2(X,\mathbb{C})_{\omega_p^i}).$$

We observe that the non rational eigenvalues ω_p^i have all the same multiplicity. So we put: $a_p :=$ multiplicity of the eigenvalue 1, $b_p :=$multiplicity of the eigenvalues ω_p^i.

In the following we find a_p and b_p by using the Lefschetz fixed point formula:

$$\mu_p = \sum_r (-1)^r \mathrm{trace}(\sigma_p^*|H^r(X,\mathbb{C})) \tag{1}$$

where μ_p denotes the number of fixed points. For $K3$ surfaces we obtain

$$\mu_p = 1 + 0 + \mathrm{trace}(\sigma_p^*|H^2(X,\mathbb{C})) + 0 + 1.$$

Proposition 1.1. *Let X, σ_p, a_p, b_p be as above, $p = 3,5,7$. Let ρ_p be the Picard number of X, and let m_p be the dimension of the moduli space of the algebraic K3 surfaces admitting a symplectic automorphism of order p. Then*

$$\begin{array}{llll} a_3 = 10 & b_3 = 6 & \rho_3 \geq 13 & m_3 \leq 7 \\ a_5 = 6 & b_5 = 4 & \rho_5 \geq 17 & m_5 \leq 3 \\ a_7 = 4 & b_7 = 3 & \rho_7 \geq 19 & m_7 \leq 1. \end{array}$$

Proof. The proof is similar in all the cases, here we give the details only in the case $p = 5$. A symplectic automorphism of order five on a K3 surface has exactly four fixed points. Applying the Lefschetz fixed points formula (1), we have $a_5 + b_5(\omega_5 + \omega_5^2 + \omega_5^3 + \omega_5^4) = 2$. Since $\omega_p^{p-1} = -(\sum_{i=0}^{p-2} \omega_p^i)$, the equation becomes $a_5 - b_5 = 2$.

Since $\dim H^2(X,\mathbb{C}) = 22$, a_5 and b_5 have to satisfy:

$$\left\{ \begin{array}{rcr} a_5 - b_5 & = & 2 \\ a_5 + 4b_5 & = & 22. \end{array} \right. \tag{2}$$

We have

$\dim H^2(X,\mathbb{C})^{\sigma_5^*} = 6 = a_5$ and
$\dim H^2(X,\mathbb{C})_{\omega_5} = \dim H^2(X,\mathbb{C})_{\omega_5^2} = \dim H^2(X,\mathbb{C})_{\omega_5^3} = \dim H^2(X,\mathbb{C})_{\omega_5^4} = 4 = b_5$.

Since $T_X \otimes \mathbb{C} \subset H^2(X,\mathbb{C})^{\sigma_5^*}$, $(H^2(X,\mathbb{C})^{\sigma_5^*})^\perp = H^2(X,\mathbb{C})_{\omega_5} \oplus H^2(X,\mathbb{C})_{\omega_5^2} \oplus H^2(X,\mathbb{C})_{\omega_5^3} \oplus H^2(X,\mathbb{C})_{\omega_5^4} \subset NS(X) \otimes \mathbb{C}$. We consider only algebraic $K3$ surfaces and so we have an ample class h on X, by taking $h + \sigma_5^* h + \sigma_5^{*2} h + \sigma_5^{*3} h + \sigma_5^{*4} h$ we get a σ_5-invariant class, hence in $H^2(X,\mathbb{C})^{\sigma_5^*}$. From here it follows that $\rho_p =$rank $NS(X) \geq 16 + 1 = 17$, whence rank $T_X \leq 22 - 17 = 5$. The number of moduli is at most $20 - 17 = 3$. □

Remark. In [Ni1, §10] Nikulin computes $\mathrm{rank}(H^2(X,\mathbb{Z})^{\sigma_p^*})^\perp = (p-1)b_p$ and $\mathrm{rank}(H^2(X,\mathbb{Z})^{\sigma_p^*}) = a_p$. In [Ni1, Lemma 4.2] he also proves that there are no classes with self intersection -2 in the lattices $(H^2(X,\mathbb{Z})^{\sigma_p^*})^\perp$; we describe these lattices in the sections 4.1, 4.4, 4.6 and we find again the result of Nikulin.

2. The $\mathbb{Z}[\omega]$-lattices

In the sections 4.2, 4.5, 4.7 our purpose is to describe $(H^2(X,\mathbb{Z})^{\sigma_p^*})^{\perp}$ as $\mathbb{Z}[\omega_p]$-lattice. We recall now some useful results on these lattices.

Definition 2.1. *Let p be an odd prime and $\omega := \omega_p$ be a primitive p-root of the unity. A $\mathbb{Z}[\omega]$-lattice is a free $\mathbb{Z}[\omega]$-module with an hermitian form (with values in $\mathbb{Z}[\omega]$). Its rank is its rank as $\mathbb{Z}[\omega]$-module.*

Let $\{L, h_L\}$ be a $\mathbb{Z}[\omega]$-lattice of rank n. The $\mathbb{Z}[\omega]$-module L is also a $\mathbb{Z}$-module of rank $(p-1)n$. In fact if e_i, $i=1,\ldots,n$ is a basis of L as $\mathbb{Z}[\omega]$-module, $\omega^j e_i$, $i=1,\ldots,n$, $j=0,\ldots,p-2$ is a basis for L as $\mathbb{Z}$-module (recall that $\omega^{p-1} = -(\omega^{p-2}+\omega^{p-3}+\ldots+1)$). The $\mathbb{Z}$-module L will be called $L_{\mathbb{Z}}$.
Let $\Gamma_p := Gal(\mathbb{Q}(\omega)/\mathbb{Q})$ be the group of the automorphisms of $\mathbb{Q}(\omega)$ which fix $\mathbb{Q}$. We recall that the group Γ_p has order $p-1$ and its elements are automorphisms ρ_i such that $\rho_i(1)=1$, $\rho_i(\omega)=\omega^i$ where $i=1,\ldots,p-1$. We define a bilinear form on $L_{\mathbb{Z}}$

$$b_L(\alpha,\beta) = -\frac{1}{p}\sum_{\rho\in\Gamma_p}\rho(h_L(\alpha,\beta)). \tag{3}$$

Note that b_L takes values in $\frac{1}{p}\mathbb{Z}[\omega]$, so in general $\{L_{\mathbb{Z}}, b_L\}$ is not a $\mathbb{Z}$-lattice. We call it the *associated* module (resp. lattice) of the $\mathbb{Z}[\omega]$-lattice L.

Remark. Remark. By the definition of the bilinear form is clear that

$$b_L(\alpha,\beta) = -\frac{1}{p}Tr_{\mathbb{Q}(\omega)/\mathbb{Q}}(h_L(\alpha,\beta)).$$

For a precise definition of the Trace see [E, page 128]

2.1. **The $\mathbb{Z}$-lattice F_p.** We consider a K3 surface admitting an elliptic fibration. Let p be an odd prime number. Let I_p be a semistable fiber of a minimal elliptic fibration, i.e. (cf. section 3) I_p is a fiber which is a reducible curve, whose irreducible components are the edges of a p-polygon, as described in [Mi1, Table I.4.1], we denote the p-irreducible components by C_i, $i=0,\ldots,p-1$, then

$$C_i\cdot C_j = \begin{cases} -2 & if\ i\equiv j \mod p\\ 1 & if\ |i-j|\equiv 1 \mod p\\ 0 & \text{otherwise.}\end{cases}$$

We consider now the free $\mathbb{Z}$-module F_p with basis the elements of the form $C_i - C_{i+1}$, $i=1,\ldots,p-1$ and with bilinear form b_{F_p} which is the restriction of the intersection form to the basis $C_i - C_{i+1}$, then $\{F_p, b_{F_p}\}$ is a $\mathbb{Z}$-lattice.

2.2. **The $\mathbb{Z}[\omega_p]$-lattice G_p.** Let G_p be the $\mathbb{Z}[\omega]$-lattice $G_p := (1-\omega)^2\mathbb{Z}[\omega]$, with the standard hermitian form: $h(\alpha,\beta)=\alpha\bar{\beta}$. A basis for the $\mathbb{Z}$-module $G_{p,\mathbb{Z}}$ is $(1-\omega)^2\omega^i$, $i=0,\ldots,p-2$.

On $\mathbb{Z}[\omega]$ we consider the bilinear form b_L defined in (3), with values in $\frac{1}{p}\mathbb{Z}$,

$$b(\alpha,\beta) = -\frac{1}{p}\sum_{\rho\in\Gamma_p}\rho(\alpha\bar{\beta}),\qquad \alpha,\beta\in\mathbb{Z}[\omega],$$

then we have

Lemma 2.1. *The bilinear form b restricted to G_p (denoted by b_G) has values in $\mathbb{Z}$ and coincides with the intersection form on F_p by using the map $F_p \to G_p$ defined by $C_i - C_{i+1} \mapsto \omega^i(1-\omega)^2$, $i = 1, \dots, p-1$, $C_p = C_0$.*

Proof. An easy computation shows that we have for $p > 3$:

$$b_G(\omega^k(1-\omega)^2, \omega^h(1-\omega)^2) = \begin{cases} -6 & \text{if } k \equiv h \mod p, \\ 4 & \text{if } |k-h| \equiv 1 \mod p, \\ -1 & \text{if } |k-h| \equiv 2 \mod p, \\ 0 & \text{otherwise,} \end{cases}$$

and for $p = 3$:

$$b_G(\omega^k(1-\omega)^2, \omega^h(1-\omega)^2) = \begin{cases} -6 & \text{if } k \equiv h \mod p, \\ 3 & \text{if } |k-h| \equiv 1 \mod p. \end{cases}$$

The intersection form on F_p is easy to compute (cf. section 2.1) and this computation proves that the map $F_p \to G_p$ defined in the lemma is an isometry. □

In section 4 we apply the results of this section and we find a $\mathbb{Z}[\omega]$-lattice $\{L_p, h_{L_p}\}$ such that

- $\{L_p, h_{L_p}\}$ contains G_p as sublattice;
- $\{L_{p,\mathbb{Z}}, b_{L_p}\}$ is a $\mathbb{Z}$-lattice;
- the $\mathbb{Z}$-lattice $\{L_{p,\mathbb{Z}}, b_{L_p}\}$ is isometric to the $\mathbb{Z}$-lattice $(H^2(X,\mathbb{Z})^{\sigma_p^*})^{\perp}$ for $p = 3, 5, 7$.

3. Some general facts on elliptic fibrations

In the next section we give explicit examples of K3 surfaces admitting a symplectic automorphism σ_p by using elliptic fibrations. Here we recall some general results about these fibrations.
Let X be an elliptic $K3$ surface, this means that we have a morphism

$$f : X \longrightarrow \mathbb{P}^1$$

such that the generic fiber is a (smooth) elliptic curve. We assume moreover that we have a section $s : \mathbb{P}^1 \longrightarrow X$. The sections of X generate the Mordell-Weil group MW_f of X and we take s as zero section. This group acts on X by translation (on each fiber), hence it leaves the two form invariant. We assume that the singular fibers of the fibration are all of type I_m, $m \in \mathbb{N}$. Let F_j be a fiber of type I_{m_j}, we denote by $C_0^{(j)}$ the irreducible components of the fibers meeting the zero section. After choosing an orientation, we denote the other irreducible components of the fibers by $C_1^{(j)}, \dots, C_{m_j-1}^{(j)}$. In the sequel we always consider m_j a prime number, and the notation $C_i^{(j)}$ means $i \in \mathbb{Z}/m_j\mathbb{Z}$. For each section r we define the number $k := k_j(r)$ by

$$r \cdot C_j^{(k)} = 1 \text{ and } r \cdot C_j^{(i)} = 0 \text{ if } i = 0, \dots, m_j - 1 \;\; i \neq k.$$

If the section r is a torsion section and h is the number of reducible fibers of type I_{m_j}, then by [Mi2, Proposition 3.1] we have

$$\sum_{j=1}^{h} k_j(r)\left(1 - \frac{k_j(r)}{m_j}\right) = 4. \tag{4}$$

Moreover we recall the Shioda-Tate formula (cf.[Shio, Corollary 5.3] or [Mi1, p.70])

$$\operatorname{rank}(NS(X)) = 2 + \sum_{j=1}^{h}(m_j - 1) + \operatorname{rank}(MW_f). \tag{5}$$

The $\operatorname{rank}(MW_f)$ is the number of generators of the free part. If there are no sections of infinite order then $\operatorname{rank}(MW_f) = 0$. Assume that X has h fibers of type I_m, $m \in \mathbb{N}$, $m > 1$, and the remaining singular fibers are of type I_1, which are rational curves with one node. Let $U \oplus (A_{m-1})^h$ denote the lattice generated by the zero section, the generic fiber and by the components of the reducible fibers not meeting s. If there are no sections of infinite order then it has finite index in $NS(X)$ equal to n, the order of the torsion part of the group MW_f. Using this remark we find that

$$|\det(NS(X))| = \frac{\det(A_{m-1})^h}{n^2} = \frac{m^h}{n^2}. \tag{6}$$

3.1. **Elliptic fibrations with a symplectic automorphism.** Now we describe three particular elliptic fibrations which admit a symplectic automorphism σ_3, σ_5 or σ_7. Assume that we have a section of prime order $p = 3, 5, 7$. By [Shim, No. 560, 2346, 3256] there exist elliptic fibrations with one of the following configurations of components of singular fibers I_p not meeting s such that all the singular fibers of the fibrations are semistable (i.e. they are all of type I_n for a certain $n \in \mathbb{N}$) and the order of the torsion subgroup of the Mordell-Weil group $o(MW_f) = p$:

$$\begin{array}{lll} p = 3: & 6A_2 & o(MW_f) = 3, \\ p = 5: & 4A_4 & o(MW_f) = 5, \\ p = 7: & 3A_6 & o(MW_f) = 7. \end{array} \tag{7}$$

We can assume that the remaining singular fibers are of type I_1. Since the sum of the Euler characteristic of the fibers must add up to 24, these are six, four, resp. three fibers. Observe that each section of finite order induces a symplectic automorphism of the same order which corresponds to a translation by the section on each fiber, we denote it by σ_p. The nodes of the I_1 fibers are then the fixed points of these automorphisms, whence σ_p permutes the p components of the I_p fibers. For these fibrations we have rank $NS(X) = 14,\ 18,\ 20$ and dimensions of the moduli spaces six, two and zero, which is one less then the maximal possible dimension of the moduli space we have given in the Proposition 1.1.

3.1.1. *Weierstrass forms.* We compute the Weierstrass form for the elliptic fibration described in (7). When X is a K3 surface then this form is

$$y^2 = x^3 + A(t)x + B(t),\ \ t \in \mathbb{P}^1 \tag{8}$$

or in homogeneous coordinates

$$x_3x_2^2 = x_1^3 + A(t)x_1x_3^2 + B(t)x_3^3 \tag{9}$$

where $A(t)$ and $B(t)$ are polynomials of degrees eight and twelve respectively, $x_3 = 0$ is the line at infinity and also the tangent to the inflectional point $(0 : 1 : 0)$.
Fibration with a section of order 3. In this case the point of order three must be an inflectional point (cf. [C, Ex. 5, p.38]), we want to determine $A(t)$ and $B(t)$ in the

equation (8). We start by imposing to a general line $y = l(t)x + m(t)$ to be an inflectional tangent so the equation of the elliptic fibration is

$$y^2 = x^3 + A(t)x + B(t),\ t \in \mathbb{P}^1,\ \text{with}\quad A(t) = \frac{2l(t)m(t) + l(t)^4}{3},\ B(t) = \frac{m(t)^2 - l(t)^6}{3^3}.$$

Since $A(t)$ and $B(t)$ are of degrees eight and twelve, we have $\deg l(t) = 2$ and $\deg m(t) = 6$. The section of order three is

$$t \mapsto \left(\frac{l(t)^2}{3}, \frac{l(t)^3}{3} + m(t)\right).$$

The discriminant $\Delta = 4A^3 + 27B^2$ of the fibration is

$$\Delta = \frac{(5l(t)^3 + 27m(t))(l(t)^3 + 3m(t))^3}{27}$$

hence in general it vanishes to the order three on six values of t and to the order one on other six values. Since A and B in general do not vanish on these values, this equation parametrizes an elliptic fibration with six fibers I_3 (so we have six curves A_2 not meeting the zero section) and six fibers I_1 (cf. [Mi1, Table IV.3.1 pag.41]).

Fibration with a section of order 5. In the same way we can compute the Weierstrass form of the elliptic fibration described in (7) with a section of order five.

In [BM] a geometrical condition for the existence of a point of order five on an elliptic curve is given. For fixed t let the cubic curve be in the form (9) then take two arbitrary distinct lines through O which meet the cubic in two other distinct points each. Call 1, 4 the points on the first line and 2, 3 the points on the second line, then 1 (or any of the other point) has order five if:

-the tangent through 1 meets the cubic in 3,
-the tangent through 4 meets the cubic in 2,
-the tangent through 3 meets the cubic in 4,
-the tangent through 2 meets the cubic in 1.

These conditions give the Weierstrass form:

$$y^2 = x^3 + A(t)x + B(t),\ t \in \mathbb{P}^1,\ \text{with}$$

$$A(t) = \frac{(-b(t)^4 + b(t)^2a(t)^2 - a(t)^4 - 3a(t)b(t)^3 + 3a(t)^3b(t))}{3},$$

$$B(t) = \frac{(b(t)^2 + a(t)^2)(19b(t)^4 - 34b(t)^2a(t)^2 + 19a(t)^4 + 18a(t)b(t)^3 - 18a(t)^3b(t))}{108}$$

where $\deg a(t) = 2$, $\deg b(t) = 2$. The section of order five is

$$t \mapsto ((2b(t)^2 - a(t)^2)2 : 3(a(t) + b(t))(a(t) - b(t))^2 : 6)$$

and the discriminant is

$$\Delta = \frac{1}{16}(b(t)^2 - a(t)^2)^5(11(b(t)^2 - a(t)^2) + 4a(t)b(t)).$$

By a careful analysis of the zeros of the discriminant we can see that the fibration has four fibers I_5 and four fibers I_1 (cf. [Mi1, Table IV.3.1 pag.41]).

Fibration with a section of order 7. To find the Weierstrass form we use also in this case the results of [BM]. We explain briefly the idea to find a set of points of order seven on an elliptic curve. One takes points $0, 3, 4$ and $1, 2, 4$ on two lines in the plane. Then the intersections of a line through 3 different from the lines $\{3, 2\}$, $\{3, 4\}$, $\{3, 1\}$ with the lines $\{1, 0\}$ and $\{2, 0\}$ give two new points -1 and -2. By using the conditions that the tangent through 1 goes through -2 and the tangent through 2 goes through 3 one can determine a cubic having a point of order seven which is e.g. 1. By using these conditions one can find the equation, but since the computations are quite involved, we recall the Weierstrass form given in [T, p.195]

$$y^2 + (1 + t - t^2)xy + (t^2 - t^3)y = x^3 + (t^2 - t^3)x^2.$$

By a direct check one sees that the point of order seven is $(0(t), 0(t))$. This elliptic fibration has three fibers I_7 and three fibers I_1.

4. Elliptic K3 surfaces with an automorphism of prime order

In this section we prove the main theorem:

Theorem 4.1. *For any K3 surface X with a symplectic automorphism σ_p of order $p = 2, 3, 5, 7$ the action on $H^2(X, \mathbb{Z})$ decomposes in the following way:*

$\mathbf{p = 2}:$ $H^2(X,\mathbb{Z})^{\sigma_2^*} = E_8(-2) \oplus U \oplus U \oplus U$, $(H^2(X,\mathbb{Z})^{\sigma_2^*})^\perp = E_8(-2)$.

$\mathbf{p = 3}:$ $H^2(X,\mathbb{Z})^{\sigma_3^*} = U \oplus U(3) \oplus U(3) \oplus A_2 \oplus A_2$

$$(H^2(X,\mathbb{Z})^{\sigma_3^*})^\perp = \left\{ (x_1,\ldots,x_6) \in (\mathbb{Z}[\omega_3])^{\oplus 6} : \begin{array}{l} x_i \equiv x_j \mod (1-\omega_3), \\ \sum_{i=1}^{6} x_i \equiv 0 \mod (1-\omega_3)^2 \end{array} \right\} = K_{12}(-2)$$

with hermitian form $h(\alpha,\beta) = \sum_{i=1}^{6}(\alpha_i\overline{\beta_i})$.

$\mathbf{p = 5}:$ $H^2(X,\mathbb{Z})^{\sigma_5^*} = U \oplus U(5) \oplus U(5)$

$$(H^2(X,\mathbb{Z})^{\sigma_5^*})^\perp = \left\{ (x_1,\ldots,x_4) \in (\mathbb{Z}[\omega_5])^{\oplus 4} : \begin{array}{l} x_1 \equiv x_2 \equiv 2x_3 \equiv 2x_4 \mod (1-\omega_5), \\ (3-\omega_5)(x_1+x_2) + x_3 + x_4 \equiv 0 \mod (1-\omega_5)^2 \end{array} \right\}$$

with hermitian form $h(\alpha,\beta) = \sum_{i=1}^{2}\alpha_i\overline{\beta_i} + \sum_{j=3}^{4} f\alpha_j\overline{f\beta_j}$ *where* $f = 1-(\omega_5^2+\omega_5^3)$.

$\mathbf{p = 7}:$ $H^2(X,\mathbb{Z})^{\sigma_7^*} = U(7) \oplus \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}$

$$(H^2(X,\mathbb{Z})^{\sigma_7^*})^\perp = \left\{ (x_1,x_2,x_3) \in (\mathbb{Z}[\omega_7])^{\oplus 3} : \begin{array}{l} x_1 \equiv x_2 \equiv 6x_3 \mod (1-\omega_7), \\ (1+5\omega_7)x_1 + 3x_2 + 2x_3 \equiv 0 \mod (1-\omega_7)^2 \end{array} \right\}$$

with hermitian form $h(\alpha,\beta) = \alpha_1\overline{\beta_1} + f_1\alpha_2\overline{f_1\beta_2} + f_2\alpha_3\overline{f_2\beta_3}$
where $f_1 = 3 + 2(\omega_7+\omega_7^6) + (\omega_7^2+\omega_7^5)$ *and* $f_2 = 2 + (\omega_7+\omega_7^6)$.

In the case $p = 3$, $K_{12}(-2)$ denotes the Coxeter-Todd lattice with the bilinear form multiplied by -2.
This theorem gives a complete description of the invariant sublattice $H^2(X,\mathbb{Z})^{\sigma_p^*}$ and its orthogonal complement in $H^2(X,\mathbb{Z})$ for the symplectic automorphisms σ_p of all possible prime order $p = 2, 3, 5, 7$ acting on a K3 surface. The results about the order two automorphism is proven by Morrison in [Mo, Theorem 5.7].
We describe the lattices of the theorem and their hermitian forms in the sections from 4.1 to 4.7. The proof is the following: we identify the action of σ_p^* on $H^2(X,\mathbb{Z})$ in the case of X an elliptic K3 surface, this is done in several propositions in these sections, then we apply [Ni1, Theorem 4.7] which assure the uniqueness of this action.

4.1. **A section of order three.** Let X be a K3 surface with an elliptic fibration which admits a section of order three described in (7) of section 3. We recall that X has six reducible fibres of type I_3 and six singular irreducible fibres of type I_1. In the preceding section we have seen that the rank of the Néron-Severi group is 14. We determine now $NS(X)$ and T_X.
Let t_1 denote the section of order three and $t_2 = t_1 + t_1$. Let σ_3 be the automorphism of X which corresponds to the translation by t_1. It leaves each fiber invariant and $\sigma_3^*(s) = t_1$, $\sigma_3^*(t_1) = t_2$, $\sigma_3^*(t_2) = s$. Denoted by $C_0^{(i)}, C_1^{(i)}, C_2^{(i)}$ the components of the $i-th$ reducible fiber ($i = 1, \dots, 6$), we can assume that $C_1^{(i)} \cdot t_1 = C_2^{(i)} \cdot t_2 = C_0^{(i)} \cdot s = 1$.

Proposition 4.1. *A $\mathbb{Z}$-basis for the lattice $NS(X)$ is given by*

$$s, t_1, t_2, F, C_1^{(1)}, C_2^{(1)}, C_1^{(2)}, C_2^{(2)}, C_1^{(3)}, C_2^{(3)}, C_1^{(4)}, C_2^{(4)}, C_1^{(5)}, C_2^{(5)}.$$

Let $U \oplus A_2^6$ be the lattice generated by the section, the fiber and the irreducible components of the six fibers I_3 which do not intersect the zero section s. It has index three in the Néron-Severi group of X, $NS(X)$. The lattice $NS(X)$ has discriminant -3^4 and its discriminant form is

$$\mathbb{Z}_3(\frac{2}{3}) \oplus \mathbb{Z}_3(\frac{2}{3}) \oplus \mathbb{Z}_3(\frac{2}{3}) \oplus \mathbb{Z}_3(-\frac{2}{3}).$$

The transcendental lattice T_X is

$$T_X = U \oplus U(3) \oplus A_2 \oplus A_2$$

and has a unique primitive embedding in the lattice Λ_{K3}.

Proof. It is clear that a $\mathbb{Q}$-basis for $NS(X)$ is given by $s, F, C_1^{(i)}, C_2^{(i)}$, $i = 1, \dots, 6$. This basis generates the lattice $U \oplus A_2^6$. It has discriminant $d(U \oplus A_2^6) = -3^6$. We denote by

$$\begin{array}{ll} c_i = 2C_1^{(i)} + C_2^{(i)}, & C = \sum c_i, \\ d_i = C_1^{(i)} + 2C_2^{(i)}, & D = \sum d_i. \end{array}$$

Since we know that $t_1 \in NS(X)$ we can write

$$t_1 = \alpha s + \beta F + \sum \gamma_i C_1^{(i)} + \sum \delta_i C_2^{(i)}, \quad \alpha, \beta, \gamma_i, \delta_i \in \mathbb{Q}.$$

Then by using the fact that $t_1 \cdot s = t_1 \cdot C_2^{(i)} = 0$ and $t_1 \cdot C_1^{(i)} = t_1 \cdot F = 1$ one obtains that $\alpha = 1$, $\beta = 2$ and $\gamma_1 = -2/3$, $\delta_1 = -1/3$ hence $\frac{1}{3}C \in NS(X)$. A similar computation with t_2 shows that $\frac{1}{3}D \in NS(X)$. So one obtains that

$$\begin{array}{l} t_1 = s + 2F - \frac{1}{3}C \in NS(X), \\ t_2 = s + 2F - \frac{1}{3}D \in NS(X). \end{array} \tag{10}$$

and so

$$3(t_2 - t_1) = \sum_{i=1}^{6} (C_1^{(i)} - C_2^{(i)}) = C - D.$$

We consider now the $\mathbb{Q}$-basis for the Néron-Severi group

$$s, t_1, t_2, F, C_1^{(1)}, C_2^{(1)}, C_1^{(2)}, C_2^{(2)}, C_1^{(3)}, C_2^{(3)}, C_1^{(4)}, C_2^{(4)}, C_1^{(5)}, C_2^{(5)}.$$

By computing the matrix of the intersection form respect to this basis one finds that the determinant is -3^4. By the Shioda-Tate formula we have $|\det(NS(X))| = 3^4$. Hence this is a $\mathbb{Z}$-basis for the Néron-Severi group. We add to the classes which generate $U \oplus A_2^6$ the

classes t_1 and $\sigma_3^*(t_1)=t_2$ given in the formula (10). Since $d(U\oplus A_2^6)=3^6$ and $d(NS(X))=-3^4$ the index of $U\oplus A_2^6$ in $NS(X)$ is 3. Observe that this is also a consequence of a general result given at the end of section 3.
The classes

$$v_i=\frac{C_1^{(i)}-C_2^{(i)}-(C_1^{(5)}-C_2^{(5)})}{3},\qquad i=1,\ldots,4$$

generate the discriminant group, which is $NS(X)^\vee/NS(X)\cong(\mathbb{Z}/3\mathbb{Z})^{\oplus 4}$.
These classes are not orthogonal to each other with respect to the bilinear form, so we take

$$w_1=v_1-v_2,\ w_2=v_3-v_4,\ w_3=v_1+v_2+v_3+v_4,\ w_4=v_1+v_2-(v_3+v_4)$$

which form an orthogonal basis with respect to the bilinear form with values in $\mathbb{Q}/\mathbb{Z}$. And it is easy to compute that $w_1^2=w_2^2=w_3^2=2/3,\ w_4^2=-2/3$. The discriminant form of the lattice $NS(X)$ is then

$$\mathbb{Z}_3(\frac{2}{3})\oplus\mathbb{Z}_3(\frac{2}{3})\oplus\mathbb{Z}_3(\frac{2}{3})\oplus\mathbb{Z}_3(-\frac{2}{3}).\tag{11}$$

The transcendental lattice T_X orthogonal to $NS(X)$ has rank eight. Since $NS(X)$ has signature $(1,13)$, the transcendental lattice has signature $(2,6)$. The discriminant form of the transcendental lattice is the opposite of the discriminant form of the Néron-Severi lattice. So the transcendental lattice has signature $(2,6)$, discriminant 3^4, discriminant group $T_X^\vee/T_X\cong(\mathbb{Z}/3\mathbb{Z})^{\oplus 4}$ and discriminant form $\mathbb{Z}_3(-\frac{2}{3})\oplus\mathbb{Z}_3(-\frac{2}{3})\oplus\mathbb{Z}(-\frac{2}{3})\oplus\mathbb{Z}_3(\frac{2}{3})$. By [Ni2, Cor. 1.13.5] we have $T=U\oplus T'$ where T' has rank six, signature $(1,5)$ and T' has discriminant form as before. These data identify T' uniquely ([Ni2, Corollary 1.13.3]). Hence it is isomorphic to $U(3)\oplus A_2\oplus A_2$ with generators for the discriminant form

$$(e-f)/3,\ (e+f)/3,\ (A-B)/3,\ (A'-B')/3,$$

where e,f,A,B,A',B' are the usual bases of the lattices.
The transcendental lattice

$$T_X=U\oplus U(3)\oplus A_2\oplus A_2$$

has a unique embedding in the lattice Λ_{K3} by [Ni2, Theorem 1.14.4] or [Mo, Corollary 2.10]. $\square$

4.1.1. *The invariant lattice and its orthogonal complement.*

Proposition 4.2. *The invariant sublattice of the Néron-Severi group is isometric to $U(3)$ and it is generated by the classes F and $s+t_1+t_2$.*
The invariant sublattice $H^2(X,\mathbb{Z})^{\sigma_3^*}$ is isometric to $U\oplus U(3)\oplus U(3)\oplus A_2\oplus A_2$.
Its orthogonal complement $\Omega_3:=(H^2(X,\mathbb{Z})^{\sigma_3^*})^\perp$ is the negative definite twelve dimensional

lattice $\{\mathbb{Z}^{12}, M\}$ *where* M *is the bilinear form*

$$\begin{pmatrix} -4 & 2 & -3 & -2 & 0 & -2 & 0 & -2 & 0 & -2 & 0 & -2 \\ 2 & -4 & 3 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ -3 & 3 & -18 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -9 \\ -2 & 1 & 0 & -6 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & -4 & 3 & 2 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 3 & -6 & -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -3 & -4 & 3 & 2 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 & 3 & -6 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -3 & -4 & 3 & 2 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -6 & -3 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 2 & -3 & -4 & 3 \\ -2 & 1 & -9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -6 \end{pmatrix}$$

and it is equal to the lattice $(NS(X)^{\sigma_3^*})^{\perp}$.
The lattice Ω_3 *admits a unique primitive embedding in the lattice* Λ_{K3}.
The discriminant of Ω_3 *is* 3^6 *and its discriminant form is* $(\mathbb{Z}_3(\frac{2}{3}))^{\oplus 6}$.
The isometry σ_3^* *acts on the discriminant group* $\Omega_3^{\vee}/\Omega_3$ *as the identity.*

Proof. It is clear that the isometry σ_3^* fixes the classes F and $s+t_1+t_2$. These generate a lattice $U(3)$ (with basis F and $F+s+t_1+t_2$).
The invariant sublattice $H^2(X,\mathbb{Z})^{\sigma_3^*}$ contains T_X and the invariant sublattice of the Néron-Severi group. So $(H^2(X,\mathbb{Z})^{\sigma_3^*})^{\perp} = (NS(X)^{\sigma_3^*})^{\perp}$, this lattice has signature $(0,12)$ and by [Ni1, p. 133] the discriminant group is $(\mathbb{Z}/3\mathbb{Z})^{\oplus 6}$. Hence by [Ni2, Theorem 1.14.4] there is a unique primitive embedding of $(H^2(X,\mathbb{Z})^{\sigma_3^*})^{\perp}$ in the K3-lattice. By using the orthogonality conditions one finds the following basis of $\Omega_3 = (NS(X)^{\sigma_3^*})^{\perp}$:

$$b_1 = t_2 - t_1,\ \ b_2 = s - t_2,\ \ b_3 = F - 3C_2^{(5)},\ \ b_{2(i+1)} = C_1^{(i)} - C_2^{(i)},\ i = 1,\dots,5$$
$$b_{2j+3} = C_1^{(j)} - C_1^{(j+1)},\ j = 1,\dots,4.$$

An easy computation shows that the Gram matrix of this basis is exactly the matrix M which indeed has determinant 3^6.
Since $H^2(X,\mathbb{Z})^{\sigma_3^*} \supseteq T_X \oplus NS(X)^{\sigma_3^*} = U \oplus U(3) \oplus U(3) \oplus A_2 \oplus A_2$ and these lattices have the same rank, to prove that the inclusion is an equality we compare their discriminants. The lattice $(H^2(X,\mathbb{Z})^{\sigma_3^*})^{\perp}$ has determinant 3^6. So the lattice $(H^2(X,\mathbb{Z})^{\sigma_3^*})$ has determinant -3^6 (because these are primitive sublattices of $H^2(X,\mathbb{Z})$). The lattice $U \oplus U(3) \oplus U(3) \oplus A_2 \oplus A_2$ has determinant exactly -3^6, so

$$H^2(X,\mathbb{Z})^{\sigma_3^*} = U \oplus U(3) \oplus U(3) \oplus A_2 \oplus A_2.$$

Since $NS(X)^{\vee}/NS(X) \subset \Omega_3^{\vee}/\Omega_3$ the generators of the discriminant form of the lattice Ω_3 are classes $w_1,\dots,w_6$ with $w_1,\dots,w_4$ the classes which generate the discriminant form of $NS(X)$ (cf. the proof of the Proposition 4.1) and

$$w_5 = \frac{1}{3}(b_1+2b_2) = \frac{1}{3}(2s-t_1-t_2) \quad w_6 = \frac{1}{3}(b_1+2b_2-2b_3) = \frac{1}{3}(2s-t_1-t_2-2F+6C_2^{(5)}).$$

These six classes are orthogonal, with respect to the bilinear form taking values in $\mathbb{Q}/\mathbb{Z}$, and generate the discriminant form. Their squares are $w_1^2 = w_2^2 = w_3^2 = w_5^2 \equiv \frac{2}{3}$ mod $2\mathbb{Z}$, $w_4^2 = w_6^2 \equiv -\frac{2}{3}$ mod $2\mathbb{Z}$. By replacing w_4, w_6 by $w_4 - w_6$, $w_4 + w_6$ we obtain the discriminant form $(\mathbb{Z}_3(\frac{2}{3}))^{\oplus 6}$.
By computing the image of w_i, $i = 1,\dots,6$ under σ_3^* one finds that $\sigma_3^*(w_i) - w_i \in \Omega_3$. For

example: $\sigma_3^*(w_5) - w_5 = \frac{1}{3}(2t_1 - t_2 - s) - \frac{1}{3}(2s - t_1 - t_2) = t_1 - s$ which is an element of Ω_3 (in fact it is orthogonal to F and to $s + t_1 + t_2$). Hence the action of σ_3^* is trivial on $\Omega_3^\vee/\Omega_3$ as claimed. $\square$

In the next two subsections we apply the results of section 2 about the $\mathbb{Z}[\omega]$-lattices to describe the lattice $\{\Omega_3, M\}$ and to prove that Ω_3 is isomorphic to the lattice $K_{12}(-2)$, where K_{12} is the Coxeter-Todd lattice (cf. e.g. [CT], [CS] for a description of this lattice).

4.2. **The lattice Ω_3.** Let ω_3 be a primitive third root of the unity. In this section we prove the following result (we use the same notations of section 2):

Theorem 4.2. *The lattice Ω_3 is isometric to the $\mathbb{Z}$-lattice associated to the $\mathbb{Z}[\omega_3]$-lattice $\{L_3,\ h_{L_3}\}$ where*

$$L_3 = \left\{ (x_1, \ldots, x_6) \in (\mathbb{Z}[\omega_3])^{\oplus 6} \ : \begin{array}{ll} x_i \equiv x_j & \mod (1-\omega_3), \\ \sum_{i=1}^6 x_i \equiv 0 & \mod (1-\omega_3)^2 \end{array} \right\}$$

and h_{L_3} is the restriction of the standard hermitian form on $\mathbb{Z}[\omega_3]^{\oplus 6}$.

Proof. Let $F = F_3^6$ be the $\mathbb{Z}$-sublattice of $NS(X)$ generated by

$$C_i^{(j)} - C_{i+1}^{(j)}, \quad i = 0,1,2, \quad j = 1, \ldots, 6$$

with bilinear form induced by the intersection form on $NS(X)$.
Let $G = G_3^6$ denote the $\mathbb{Z}[\omega_3]$-lattice $(1-\omega_3)^2\mathbb{Z}[\omega_3]^{\oplus 6}$ with the standard hermitian form. This is a sublattice of $\mathbb{Z}[\omega_3]^{\oplus 6}$. Applying to each component of G the Lemma 2.1 we know that $\{G_{\mathbb{Z}}, b_G\}$ is a $\mathbb{Z}$-lattice isometric to the lattice F. The explicit isometry is given by

$$\begin{array}{l} C_i^{(1)} - C_{i+1}^{(1)} \mapsto (1-\omega_3)^2(\omega_3^{i-1}, 0,0,0,0,0) \\ C_i^{(2)} - C_{i+1}^{(2)} \mapsto (1-\omega_3)^2(0, \omega_3^{i-1}, 0,0,0,0) \\ \vdots \\ C_i^{(6)} - C_{i+1}^{(6)} \mapsto (1-\omega_3)^2(0,0,0,0,0,\omega_3^{i-1}). \end{array}$$

The multiplication by ω_3 of an element $(1-\omega_3)^2 e_j$ (where e_j is the canonical basis) corresponds to a translation by t_1 on a singular fiber, which sends the curve $C_i^{(j)}$ to the curve $C_{i+1}^{(j)}$. Hence we have a commutative diagram:

$$\begin{array}{ccccc} & F & \longrightarrow & G & \\ \sigma_3^* & \downarrow & & \downarrow & \cdot\omega_3 \\ & F & \longrightarrow & G. & \end{array}$$

The elements $C_i^{(j)} - C_k^{(j)}$, $i,k = 0,1,2$, $j = 1,\ldots,6$ are all contained in the lattice $\Omega_3 = (NS(X)^{\sigma_3^*})^\perp$, but they do not generate this lattice. A set of generators for Ω_3 is

$$s - t_1, \quad t_1 - t_2, \quad C_i^{(j)} - C_h^{(k)} \quad i,h = 0,1,2,\ j,k = 1,\ldots,6.$$

From the formula (10) we obtain that

$$s - t_1 = \sum_{j=1}^{6} [\frac{1}{3}(C_1^{(j)} - C_2^{(j)}) + \frac{1}{3}\sigma_3^*(C_1^{(j)} - C_2^{(j)})].$$

After the identification of F with $G_{\mathbb{Z}}$ we have

$$s-t_1=(1-\omega_3)^2(\frac{1}{3}(1+\omega_3))(1,1,1,1,1,1)=(1,1,1,1,1,1).$$

The divisor t_1-t_2, which is the image of $s-t_1$ under the action of σ_3^*, corresponds to the vector $(\omega_3,\omega_3,\omega_3,\omega_3,\omega_3,\omega_3)$. Similarly one can see that the element $C_1^{(1)}-C_1^{(2)}$ corresponds to the vector $(1-\omega_3)(1,-1,0,0,0,0,0)$ and more in general $C_i^{(j)}-C_i^{(k)}$ with $j\neq k$ corresponds to the vector $(1-\omega_3)(\omega_3^{i-1}e_j-\omega_3^{i-1}e_k)$ where e_i is the standard basis. The lattice L_3 generated by the vectors of $G_{\mathbb{Z}}$ and by

$$\omega_3^i(1,1,1,1,1,1)\quad (1-\omega_3)\omega_3^{i-1}(-e_j+\omega_3 e_k)\quad i=0,1,2,\quad j,k=1,\dots 6,$$

is thus isometric to Ω_3.
In conclusion a basis for L_3 is

$$\begin{array}{ll}
l_1=-\omega_3(1,1,1,1,1,1) & l_2=-\omega_3^2(1,1,1,1,1,1)=(1+\omega_3)(1,1,1,1,1,1)\\
l_3=(1-\omega_3)^2(0,0,0,0,1-\omega_3,0) & l_4=(1-\omega_3)^2(1,0,0,0,0,0)\\
l_5=(1-\omega_3)(1,-1,0,0,0,0) & l_6=(1-\omega_3)^2(0,1,0,0,0,0)\\
l_7=(1-\omega_3)(0,1,-1,0,0,0) & l_8=(1-\omega_3)^2(0,0,1,0,0,0)\\
l_9=(1-\omega_3)(0,0,1,-1,0,0) & l_{10}=(1-\omega_3)^2(0,0,0,1,0,0)\\
l_{11}=(1-\omega_3)(0,0,0,1,-1,0) & l_{12}=(1-\omega_3)^2(0,0,0,0,1,0).
\end{array}$$

The identification between Ω_3 and L_3 is given by the map $b_i\mapsto l_i$.
After this identification the intersection form on Ω_3 is exactly the form $b_{|L_3}$ on L_3.
The basis l_i of L_3 satisfies the condition given in the theorem, and so

$$\begin{array}{lll} L_3\subseteq & \{(x_1,\dots,x_6)\in(\mathbb{Z}[\omega_3])^{\oplus 6}\ : & x_i\equiv x_j \mod (1-\omega_3),\\ & & \sum_{i=1}^6 x_i\equiv 0 \mod (1-\omega_3)^2\}.\end{array}$$

Since the vectors $(1-\omega_3)^2e_j$, $(1-\omega_3)(e_i-e_j)$ and $(1,1,1,1,1,1)$ generate the $\mathbb{Z}[\omega_3]$-lattice $\{(x_1,\dots,x_6)\in(\mathbb{Z}[\omega_3])^{\oplus 6}:\ x_i\equiv x_j \mod (1-\omega_3),\ \sum_{i=1}^6 x_i\equiv 0 \mod (1-\omega_3)^2\}$ and since they are all vectors contained in L_3, the equality holds. □

4.3. **The Coxeter-Todd lattice K_{12}.**

Theorem 4.3. *The lattice Ω_3 is isometric to the lattice $K_{12}(-2)$.*

Proof. The lattice K_{12} is described by Coxeter and Todd in [CT] and by Conway and Sloane in [CS]. The lattice K_{12} is the twelve dimensional $\mathbb{Z}$-module associated to a six dimensional $\mathbb{Z}[\omega_3]$-lattice $\Lambda_6^{\omega_3}$.
The $\mathbb{Z}[\omega_3]$-lattice $\Lambda_6^{\omega_3}$ is described in [CS] in four different ways. We recall one of them denoted by $\Lambda^{(3)}$ in [CS, Definition 2.3], which is convenient for us. Let $\theta=\omega_3-\bar{\omega_3}$, then $\Lambda_6^{\omega_3}$ is the $\mathbb{Z}[\omega_3]$-lattice

$$\Lambda_6^{\omega_3}=\{(x_1,\dots,x_6):\ x_i\in\mathbb{Z}[\omega_3], x_i\equiv x_j \mod \theta,\ \sum_{i=1}^6 x_i\equiv 0 \mod 3\}$$

with hermitian form $\frac{1}{3}{}^t x\bar{y}$. We observe that $\theta=\omega_3(1-\omega_3)$. The element ω_3 is a unit in $\mathbb{Z}[\omega_3]$ so the congruence modulo θ is the same as the congruence modulo $(1-\omega_3)$. Observing that $-3=\theta^2$ it is then clear that the $\mathbb{Z}[\omega_3]$-module $\Lambda_6^{\omega_3}$ is the $\mathbb{Z}[\omega_3]$-module L_3. The $\mathbb{Z}$-modules K_{12} and $L_{3,\mathbb{Z}}$ are isomorphic since they are the twelve dimensional

$\mathbb{Z}$-lattices associated to the same $\mathbb{Z}[\omega_3]$-lattice. The bilinear form on the $\mathbb{Z}$-module K_{12} is given by

$$b_{K_{12}}(x,y)=\frac{1}{3}x\bar{y}=\frac{1}{6}Tr(x\bar{y})$$

and the bilinear form on $L_{3,\mathbb{Z}}$ is given by

$$b_{|L_3}(x,y)=-\frac{1}{3}Tr(x\bar{y}).$$

So the $\mathbb{Z}$-lattice $\{L_{3,\mathbb{Z}},b_{|L_3}\}$ is isometric to $K_{12}(-2)$. □

Remark. 1) In [CT] Coxeter and Todd give an explicit basis of the $\mathbb{Z}$-lattice K_{12}. By a direct computation one can find the change of basis between the basis described in [CT] and the basis $\{b_i\}$ given in the proof of Proposition 4.2.
2) The lattice Ω_3 does not contain vectors of norm -2 (cf. [Ni1, Lemma 4.2]), but has 756 vectors of norm -4, 4032 of norm -6 and 20412 of norm -8. Since these properties define the lattice $K_{12}(-2)$, (cf. [CS, Theorem 1]), this is another way to prove the equality between Ω_3 and $K_{12}(-2)$.
3) The lattice $K_{12}(-2)$ is generated by vectors of norm -4, [PP, Section 3].

4.4. **Section of order five.** Let X be a K3 surface with an elliptic fibration which admits a section of order five as described in section 3. We recall that X has four reducible fibres of type I_5 and four singular irreducible fibres of type I_1. We have seen that the rank of the Néron-Severi group is 18. We determine now $NS(X)$ and T_X.
We label the fibers and their components as described in the section 3. Let t_1 denote the section of order five which meets the first singular fiber in $C_1^{(1)}$. By the formula (4) of section 3 up to permutation of the fibers only the following situations are possible:

$$t_1\cdot C_1^{(1)}=t_1\cdot C_1^{(2)}=t_1\cdot C_2^{(3)}=t_1\cdot C_2^{(4)}=1 \text{ and } t_1\cdot C_i^{(j)}=0 \text{ otherwise;}$$
$$\text{or } t_1\cdot C_1^{(1)}=t_1\cdot C_4^{(2)}=t_1\cdot C_2^{(3)}=t_1\cdot C_3^{(4)}=1 \text{ and } t_1\cdot C_i^{(j)}=0 \text{ otherwise.}$$

Observe that these two cases describe the same situation if we change the "orientation" on the last two fibers, so we assume to be in the first case. Let σ_5 be the automorphism of order five which leaves each fiber invariant and is the translation by t_1, so $\sigma_5^*(s)=t_1$, $\sigma_5^*(t_1)=t_2$, $\sigma_5^*(t_2)=t_3$, $\sigma_5^*(t_3)=t_4$, $\sigma_5^*(t_4)=s$.

Proposition 4.3. *A $\mathbb{Z}$-basis for the lattice $NS(X)$ is given by*

$$s,t_1,t_2,t_3,t_4,F,C_1^{(1)},C_2^{(1)},C_3^{(1)},C_4^{(1)},C_1^{(2)},C_2^{(2)},C_3^{(2)},C_4^{(2)},C_1^{(3)},C_2^{(3)},C_3^{(3)},C_4^{(3)}.$$

Let $U\oplus A_4^4$ be the lattice generated by the section, the fiber and the irreducible components of the four fibers I_5 which do not intersect the zero section s. It has index five in the Néron-Severi group of X, $NS(X)$.
The lattice $NS(X)$ has discriminant -5^2 and its discriminant form is

$$\mathbb{Z}_5(\frac{2}{5})\oplus\mathbb{Z}_5(-\frac{2}{5}).$$

The transcendental lattice is

$$T_X=U\oplus U(5)$$

and has a unique primitive embedding in the lattice Λ_{K3}.

Proof. The proof is similar to the proof of Proposition 4.1. So we sketch it briefly. The classes $s, F, C_i^{(j)}$, $i = 1, \dots, 4$, $j = 1, \dots, 4$, generate $U \oplus A_4^4$. By using the intersection form, or by the result of [Mi2, p. 299], we find

$$\begin{array}{rl} t_1 = & s + 2F - \frac{1}{5}\left[\sum_{i=1}^{2}(4C_1^{(i)} + 3C_2^{(i)} + 2C_3^{(i)} + C_4^{(i)}) + \right. \\ & \left. + \sum_{j=3}^{4}(3C_1^{(j)} + 6C_2^{(j)} + 4C_3^{(j)} + 2C_4^{(j)})\right]. \end{array} \tag{12}$$

A $\mathbb{Z}$-basis is $s, t_1, t_2, t_3, t_4, F, C_1^{(1)}, C_2^{(1)}, C_3^{(1)}, C_4^{(1)}, C_1^{(2)}, C_2^{(2)}, C_3^{(2)}, C_4^{(2)}, C_1^{(3)}, C_2^{(3)}, C_3^{(3)}, C_4^{(3)}$. Since $d(NS(X)) = -5^2$ and $d(U \oplus A_4^4) = -5^4$, the index of $U \oplus A_4^4$ in $NS(X)$ is five. Let w_1 and w_2 be

$$\begin{array}{l} w_1 = \frac{1}{5}(2C_1^{(1)} + 4C_2^{(1)} + C_3^{(1)} + 3C_4^{(1)} + 4C_1^{(3)} + 3C_2^{(3)} + 2C_3^{(3)} + C_4^{(3)}); \\ w_2 = \frac{1}{5}(3C_1^{(2)} + C_2^{(2)} + 4C_3^{(2)} + 2C_4^{(2)} + C_1^{(3)} + 2C_2^{(3)} + 3C_3^{(3)} + 4C_4^{(3)}). \end{array}$$

The classes $v_1 = w_1 - w_2$, $v_2 = w_1 + w_2$ are orthogonal classes and generate the discriminant group of $NS(X)$, the discriminant form is

$$\mathbb{Z}_5(\frac{2}{5}) \oplus \mathbb{Z}_5(-\frac{2}{5}).$$

The transcendental lattice T_X has rank four, signature $(2,2)$ and discriminant form $\mathbb{Z}_5(-\frac{2}{5}) \oplus \mathbb{Z}_5(\frac{2}{5})$. Since in this case T_X is uniquely determined by signature and discriminant form (cf. [Ni2, Corollary 1.13.3]) this is the lattice

$$T_X = U \oplus U(5).$$

The transcendental lattice has a unique embedding in the lattice Λ_{K3} by [Ni2, Theorem 1.14.4] or [Mo, Corollary 2.10]. □

4.4.1. *The invariant lattice and its orthogonal complement.*

Proposition 4.4. *The invariant sublattice of the Néron-Severi lattice is isometric to the lattice $U(5)$ and it is generated by the classes F and $s + t_1 + t_2 + t_3 + t_4$.*
The invariant lattice $H^2(X, \mathbb{Z})^{\sigma_5^*}$ is isometric to $U \oplus U(5) \oplus U(5)$ and its orthogonal complement $\Omega_5 = (H^2(X, \mathbb{Z})^{\sigma_5^*})^{\perp}$ is the negative definite sixteen dimensional lattice $\{\mathbb{Z}^{16}, M\}$ where M is the bilinear form

$$\left(\begin{array}{rrrrrrrrrrrrrrrr}
-4 & 2 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 1 & -1 & 0 \\
2 & -4 & 2 & 0 & 5 & 2 & -1 & 0 & 0 & 2 & -1 & 0 & 1 & -1 & 1 & 1 \\
0 & 2 & -4 & 2 & -5 & -1 & 2 & -1 & 0 & -1 & 2 & -1 & 1 & -1 & 0 & -1 \\
0 & 0 & 2 & -4 & 0 & 0 & -1 & 2 & 0 & 0 & -1 & 2 & -1 & 1 & 1 & -1 \\
0 & 5 & -5 & 0 & -50 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -15 \\
-1 & 2 & -1 & 0 & 0 & -6 & 4 & -1 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 4 & -6 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & 0 & -1 & 4 & -6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -3 & 1 & 0 & -4 & 3 & -1 & 0 & 2 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 3 & -6 & 4 & -1 & -3 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & -1 & 4 & -6 & 4 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -6 & 0 & 0 & 0 & 0 \\
-1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 2 & -3 & 1 & 0 & -4 & 3 & -1 & 0 \\
1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -6 & 4 & -1 \\
-1 & 1 & 0 & 1 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -6 & 4 \\
0 & 1 & -1 & -1 & -15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -6
\end{array}\right)$$

and it is equal to the lattice $(NS(X)^{\sigma_5^*})^{\perp}$.
The lattice Ω_5 *admits a unique primitive embedding in the lattice* Λ_{K3}.
The discriminant of Ω_5 *is* 5^4 *and its discriminant form is* $(\mathbb{Z}_5(\frac{2}{5}))^{\oplus 4}$.
The isometry σ_5^* *acts on the discriminant group* $\Omega_5^{\vee}/\Omega_5$ *as the identity.*

Proof. As in the case of an elliptic fibration with a section of order three, it is clear that σ_5^* fixes the classes F and $s+t_1+t_2+t_3+t_4$. These classes generate the lattice $U(5)$, and so $H^2(X,\mathbb{Z})^{\sigma_5^*} \supseteq U(5)\oplus T_X = U(5)\oplus U(5)\oplus U$. Using Nikulin's result in [Ni1, p. 133] we find that the lattice $H^2(X,\mathbb{Z})^{\sigma_5^*}$ has determinant -5^4, which is exactly the determinant of $U(5)\oplus U(5)\oplus U$. Since these have the same rank, we conclude that $H^2(X,\mathbb{Z})^{\sigma_5^*} = U(5)\oplus U(5)\oplus U$.
The orthogonal complement $(H^2(X,\mathbb{Z})^{\sigma_5^*})^{\perp}$ is equal to $(NS(X)^{\sigma_5^*})^{\perp}$ as in Proposition 4.2. It has signature $(0,16)$ and by [Ni1, p. 133] the discriminant group is $(\mathbb{Z}/5\mathbb{Z})^{\oplus 4}$. Hence by [Ni2, Theorem 1.14.4] there is a unique primitive embedding of $(H^2(X,\mathbb{Z})^{\sigma_5^*})^{\perp}$ in the K3-lattice. By using the orthogonality conditions one finds the following basis of $\Omega_5 = (NS(X)^{\sigma_5^*})^{\perp}$:

$$\begin{array}{l}
b_1 = s - t_1,\ b_2 = t_1 - t_2,\ b_3 = t_2 - t_3,\ b_4 = t_3 - t_4,\ b_5 = F - 5C_4^{(3)},\\
b_i = C_{i-5}^{(1)} - C_{i-4}^{(1)},\ i = 6,7,8, \qquad b_9 = C_1^{(1)} - C_1^{(2)},\\
b_i = C_{i-9}^{(2)} - C_{i-8}^{(2)},\ i = 10,11,12, \qquad b_{13} = C_1^{(2)} - C_1^{(3)},\\
b_i = C_{i-13}^{(3)} - C_{i-12}^{(3)},\ i = 14,15,16.
\end{array}$$

The Gram matrix of this basis is exactly the matrix M.
The generators of the discriminant group of Ω_5 are the classes v_1, v_2 of the discriminant form of $NS(X)$ and the classes

$$\begin{array}{l}
v_3 = \frac{1}{5}(b_3 + 2b_1 + 3b_4 + 4b_2),\\
v_4 = \frac{1}{5}(b_3 + 2b_1 + 3b_4 + 4b_2 - b_5).
\end{array}$$

These have $v_3^2 = -2/5 \mod 2\mathbb{Z}$, $v_4^2 = 2/5 \mod 2\mathbb{Z}$. The generators v_1, $2v_2 - 4v_3 - v_4$, $2v_3$, v_4 are orthogonal to each other and have self-intersection $2/5$. □

4.5. **The lattice Ω_5.** Let ω_5 be a primitive fifth root of the unity. In this section we prove the following result

Theorem 4.4. *The lattice* Ω_5 *is isometric to the* $\mathbb{Z}$*-lattice associated to the* $\mathbb{Z}[\omega_5]$*-lattice* $\{L_5,\ h_{L_5}\}$ *where*

$$L_5 = \left\{ (x_1,\ldots,x_4) \in (\mathbb{Z}[\omega_5])^{\oplus 4} : \begin{array}{ll} x_1 \equiv x_2 \equiv 2x_3 \equiv 2x_4 & \mod (1-\omega_5) \\ (3-\omega_5)x_1 + (3-\omega_5)x_2 + x_3 + x_4 \equiv 0 & \mod (1-\omega_5)^2 \end{array} \right\}$$

with the hermitian form

$$h_{L_5}(\alpha,\beta) = \sum_{i=1}^{2} \alpha_i\bar{\beta}_i + \sum_{j=3}^{4} f\alpha_j\overline{f\beta_j} = \sum_{i=1}^{2}\alpha_i\bar{\beta}_i + \tau\sum_{j=3}^{4}\alpha_j\overline{\beta_j}\ , \tag{13}$$

where $\alpha,\beta \in L_5 \subset \mathbb{Z}[\omega_5]^{\oplus 4}$, $f = 1-(\omega_5^2+\omega_5^3)$ *and* $\tau = f\overline{f} = 2-3(\omega_5^2+\omega_5^3)$.

Proof. The strategy of the proof is the same as in the case with an automorphism of order three, but the situation is more complicated because the section t_1 does not meet all the fibers I_5 in the same component. For this reason the hermitian form of the $\mathbb{Z}[\omega_5]$-lattice L_5 is not the standard hermitian form on all the components. It is possible to repeat the

construction used in the case of order three, but with the hermitian form (13). We explain now how we find this hermitian form.
Let $F := F_5^4$ be the lattice generated by the elements $C_i^{(j)} - C_{i+1}^{(j)}$, $i = 0, \ldots, 4$, $j = 1, \ldots, 4$. This is a sublattice of $(NS(X)^{\sigma_5 *})^{\perp}$. A basis is

$$\begin{array}{l} d_{1+i} = (\sigma_5^*)^i (C_1^{(1)} - C_2^{(1)}), \quad d_{5+i} = (\sigma_5^*)^i (C_1^{(2)} - C_2^{(2)}), \quad d_{9+i} = (\sigma_5^*)^i (C_1^{(3)} - C_2^{(3)}), \\ d_{13+i} = (\sigma_5^*)^i (C_1^{(4)} - C_2^{(4)}), \quad i = 0, \ldots, 3 \end{array}$$

and the bilinear form is thus the diagonal block matrix $Q = \mathrm{diag}(A, A, B, B)$

$$A = \begin{pmatrix} -6 & 4 & -1 & -1 \\ 4 & -6 & 4 & -1 \\ -1 & 4 & -6 & 4 \\ -1 & -1 & 4 & -6 \end{pmatrix}, \quad B = \begin{pmatrix} -6 & -1 & 4 & 4 \\ -1 & -6 & -1 & 4 \\ 4 & -1 & -6 & -1 \\ 4 & 4 & -1 & -6 \end{pmatrix}.$$

We want to identify the multiplication by ω_5 in the lattice G with the action of the isometry σ_5^* on the lattice F. We consider the $\mathbb{Z}[\omega_5]$-module $G = (1-\omega)^2 \mathbb{Z}[\omega]^{\oplus 4}$. Now we consider the $\mathbb{Z}$-module $G_{\mathbb{Z}}$. The map

$$\begin{array}{rrcl} \phi: & (\sigma_5^*)^i (C_1^{(1)} - C_2^{(1)}) & \mapsto & (1-\omega_5)^2 \omega_5^i (1,0,0,0) \\ & (\sigma_5^*)^i (C_1^{(2)} - C_2^{(2)}) & \mapsto & (1-\omega_5)^2 \omega_5^i (0,1,0,0) \\ & (\sigma_5^*)^i (C_1^{(3)} - C_2^{(3)}) & \mapsto & (1-\omega_5)^2 \omega_5^i (0,0,1,0) \\ & (\sigma_5^*)^i (C_1^{(4)} - C_2^{(4)}) & \mapsto & (1-\omega_5)^2 \omega_5^i (0,0,0,1) \end{array}$$

is an isomorphism between the $\mathbb{Z}$-modules $G_{\mathbb{Z}}$ and F.
Now we have to find a bilinear form b_G on G such that $\{G_{\mathbb{Z}}, b_G\}$ is isometric to $\{F, Q\}$. On the first and second fiber the action of σ_5^* is $\sigma_5^*(C_i^{(j)}) = C_{i+1}^{(j)}$, $j = 1, 2$, $i = 0, \ldots 4$, so $(\sigma_5^*)^i (C_1^{(j)} - C_2^{(j)}) = C_{i+1}^{(j)} - C_{i+2}^{(j)}$. Hence the map ϕ operates on the first two fibers in the following way:

$$\begin{array}{rrcl} \phi: & C_{i+1}^{(1)} - C_{i+2}^{(1)} & \mapsto & (1-\omega_5)^2 \omega_5^i (1,0,0,0) \\ & C_{i+1}^{(2)} - C_{i+2}^{(2)} & \mapsto & (1-\omega_5)^2 \omega_5^i (0,1,0,0). \end{array}$$

This identification is exactly the identification described in Lemma 2.1, so on these generators of the lattices F and G we can choose exactly the form described in the lemma.
On the third and fourth fiber the action of σ_5^* is different (because σ_5^* is the translation by t_1 and it meets the first and second fiber in the component C_1 and the third and fourth fiber in the component C_2). In fact $(\sigma_5^*)^i (C_1^{(j)} - C_2^{(j)}) = C_{2i+1}^{(j)} - C_{2i+2}^{(j)}$, $j = 3, 4$, $i = 0, \ldots, 4$ and so

$$\begin{array}{rrcl} \phi: & C_{2i+1}^{(3)} - C_{2i+2}^{(3)} & \mapsto & (1-\omega_5)^2 \omega_5^i (0,0,1,0) \\ & C_{2i+1}^{(4)} - C_{2i+2}^{(4)} & \mapsto & (1-\omega_5)^2 \omega_5^i (0,0,0,1). \end{array} \tag{14}$$

A direct verification shows that the map ϕ defines an isometry between the module generated by $(\sigma_p^*)^i (C_1^{(j)} - C_2^{(j)})$, $i = 0, \ldots, 4$ and $(1-\omega_5)^2 \mathbb{Z}[\omega_5]$, $(j = 3, 4)$ if one considers on $(1-\omega_5)^2 \mathbb{Z}[\omega_5]$ the bilinear form associated to the hermitian form $h(\alpha, \beta) = \tau \alpha \overline{\beta}$ where $\tau = (2 - 3(\omega_5^2 + \omega_5^3))$. The real number τ is the square of $f = 1 - (\omega_5^2 + \omega_5^3)$, so the hermitian form above is also $h(\alpha, \beta) = \tau \alpha \overline{\beta} = f \alpha \overline{f \beta}$. So now we consider the $\mathbb{Z}[\omega_5]$-lattice $\mathbb{Z}[\omega_5]^{\oplus 4}$ with the hermitian form h given in (13) and G as a sublattice of $\{\mathbb{Z}[\omega_5]^{\oplus 4}, h\}$. We show that $L_5 = \Omega_5$. We have to add to the lattice F some classes to obtain the lattice Ω_5, and so we have to add some vectors to the lattice G to obtain the lattice L_5. It is sufficient to add to F the classes $s-t_1$, $C_1^{(1)}-C_1^{(2)}$, $C_1^{(2)}-C_1^{(3)}$, $C_1^{(3)}-C_1^{(4)}$ and their images under σ_5^*. These classes correspond to the following vectors in $\mathbb{Z}[\omega_5]^{\oplus 4}$:

$$\begin{array}{l} s-t_1=(1,1,c,c),\\ C_1^{(1)}-C_1^{(2)}=(1-\omega_5)(1,-1,0,0),\\ C_1^{(2)}-C_1^{(3)}=(1-\omega_5)(0,1,-(1+\omega_5^3),0),\\ C_1^{(3)}-C_1^{(4)}=(1-\omega_5)(0,0,(1+\omega_5^3),-(1+\omega_5^3)) \end{array}$$

where $c=\omega_5(2\omega_5^2-\omega_5+2)$. A basis for the lattice L_5 is then

$$\begin{array}{ll} l_1=(1,1,c,c) & l_2=\omega_5 l_1\\ l_3=\omega_5^2 l_1 & l_4=\omega_5^3 l_1\\ l_5=(1-\omega_5)^2(0,0,2+4\omega_5+\omega_5^2+3\omega_5^3,0) & l_6=(1-\omega_5)^2(1,0,0,0)\\ l_7=\omega_5 l_6 & l_8=\omega_5^2 l_6\\ l_9=(1-\omega_5)(1,-1,0,0) & l_{10}=(1-\omega_5)^2(0,1,0,0)\\ l_{11}=\omega_5 l_{10} & l_{12}=\omega_5^2 l_{10}\\ l_{13}=(1-\omega_5)(0,1,-(1+\omega_5^3),0) & l_{14}=(1-\omega_5)^2(0,0,1,0)\\ l_{15}=\omega_5 l_{14} & l_{16}=\omega_5^2 l_{14}. \end{array}$$

The identification between Ω_5 and L_5 is given by the map $b_i\mapsto l_i$. After this identification the intersection form on Ω_5 is exactly the form $b_{|L_5}$ on L_5. □

Remark. 1) We recall that the density of a lattice L of rank n is $\Delta=V_n/\sqrt{\det L}$ where V_n is the volume of the n dimensional sphere of radius r (called *packing radius* of the lattice), $V_n=r^n\pi^{n/2}/(n/2)!$, $r=\sqrt{\mu}/2$ and μ is the minimal norm of a vector of the lattice.
The density of Ω_5 is $\Delta=\frac{\pi^8}{8!}\frac{1}{5^2}\approx 0.0094$.
2) The lattice Ω_5 does not admit vectors of norm -2 and can be generated by vectors of norm -4, and a basis is b_1, b_2, b_3, b_4, $b_5-b_{13}-2b_{14}-3b_{15}-4b_{16}$, b_6, b_7, b_8, b_9, $b_{10}+b_{11}$, $b_{11}+b_{12}$, $b_{10}+b_{11}+b_{12}$, b_{13}, $b_{14}+b_{15}$, $b_{15}+b_{16}$, $b_{14}+b_{15}+b_{16}$.

4.6. **Section of order seven.** Let X be a K3 surface with an elliptic fibration which admits a section of order seven as described in section 3. We recall that X has three reducible fibres of type I_7 and three singular irreducible fibres of type I_1. We have seen that the rank of the Néron-Severi group is 20. We determine now $NS(X)$ and T_X.
We label the fibers and their components as described in the section 3. Let t_1 denote the section of order seven which meets the first fiber in $C_1^{(1)}$. Again by the formula (4) of section 3 we have

$$t_1\cdot C_1^{(1)}=1,\ t_1\cdot C_2^{(2)}=1,\ t_1\cdot C_3^{(3)}=1,\ \text{and } t_1\cdot C_i^{(j)}=0 \text{ otherwise.}$$

Let σ_7 denote the automorphism of order seven which leaves each fiber invariant and is the translation by t_1, so $\sigma_7^*(s)=t_1$, $\sigma_7^*(t_1)=t_2$, $\sigma_7^*(t_2)=t_3$, $\sigma_7^*(t_3)=t_4$, $\sigma_7^*(t_4)=t_5$, $\sigma_7^*(t_5)=t_6$, $\sigma_7^*(t_6)=s$. The proofs of the next two propositions are very similar to those of the similar propositions in the case of the automorphisms of order three and five, so we omit them.

Proposition 4.5. *A $\mathbb{Z}$-basis for the lattice $NS(X)$ is given by*

$$s,t_1,t_2,t_3,t_4,t_5,t_6,F,C_1^{(1)},C_2^{(1)},C_3^{(1)},C_4^{(1)},C_5^{(1)},C_6^{(1)},C_1^{(2)},C_2^{(2)},C_3^{(2)},C_4^{(2)},C_5^{(2)},C_6^{(2)}.$$

Let $U\oplus A_6^3$ be the lattice generated by the section, the fiber and the irreducible components of the three fibers I_7 which do not intersect the zero section s. It has index seven in the

Néron-Severi group of X, $NS(X)$.
The lattice $NS(X)$ has discriminant -7 and its discriminant form is $\mathbb{Z}_7(-\frac{4}{7})$.
The transcendental lattice T_X is the lattice $\{\mathbb{Z}^{\oplus 2}, \Upsilon\}$ where

$$\Upsilon := \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}$$

and it has a unique primitive embedding in the lattice Λ_{K3}.

4.6.1. *The invariant lattice and its orthogonal complement.*

Proposition 4.6. *The invariant sublattice of the Néron-Severi lattice is isometric to the lattice $U(7)$ and it is generated by the classes F and $s+t_1+t_2+t_3+t_4+t_5+t_6$.*
The invariant lattice $H^2(X,\mathbb{Z})^{\sigma_7^*}$ is isometric to $U(7)\oplus T_X$. Its orthogonal complement $\Omega_7 := (H^2(X,\mathbb{Z})^{\sigma_7^*})^{\perp}$ is the negative definite eighteen dimensional lattice $\{\mathbb{Z}^{18}, M\}$ where M is the bilinear form

$$\begin{pmatrix}
-4 & 2 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
2 & -4 & 2 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & 0 \\
0 & 2 & -4 & 2 & 0 & 0 & 7 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 \\
0 & 0 & 2 & -4 & 2 & 0 & -7 & 0 & -1 & 2 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 2 & -4 & 2 & 0 & 0 & 0 & -1 & 2 & -1 & -1 & 1 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & -4 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 & 1 & 1 & -1 \\
0 & 0 & 7 & -7 & 0 & 0 & -98 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & -21 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & -6 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 & 4 & -6 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & 0 & -1 & 4 & -6 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & -1 & 4 & -6 & 4 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & -1 & 4 & -6 & 3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 3 & -4 & 3 & -1 & 0 & 0 & 0 \\
1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -6 & 4 & -1 & 0 & 0 \\
-1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -6 & 4 & -1 & 0 \\
0 & 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -6 & 4 & -1 \\
0 & -1 & 1 & 0 & 0 & 1 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -6 & 4 \\
0 & 0 & 1 & -1 & 0 & -1 & -21 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -6
\end{pmatrix}$$

and it is equal to the lattice $(NS(X)^{\sigma_7^*})^{\perp}$.
The lattice Ω_7 admits a unique primitive embedding in the lattice Λ_{K3}.
The discriminant of Ω_7 is 7^3 and its discriminant form is $(\mathbb{Z}_7(\frac{4}{7}))^{\oplus 3}$.
The isometry σ_7^* acts on the discriminant group $\Omega_7^{\vee}/\Omega_7$ as the identity.

The basis of $(NS(X)^{\sigma_7^*})^{\perp}$ associated to the matrix M is $b_1 = s-t_1$, $b_2 = t_1-t_2$, $b_3 = t_2-t_3$, $b_4 = t_3-t_4$, $b_5 = t_4-t_5$, $b_6 = t_5-t_6$ $b_7 = F - 7C_6^{(2)}$, $b_i = C_{i-7}^{(1)} - C_{i-6}^{(1)}$, $i = 8, \dots, 12$, $b_{13} = C_1^{(1)} - C_1^{(2)}$, $b_i = C_{i-13}^{(2)} - C_{i-8}^{(2)}$, $i = 14, \dots, 18$.

4.7. **The lattice Ω_7.** Let ω_7 be a primitive seventh root of the unity. In this section we prove the following result

Theorem 4.5. *The lattice Ω_7 is isometric to the $\mathbb{Z}$-lattice associated to the $\mathbb{Z}[\omega_7]$-lattice $\{L_7,\ h_{L_7}\}$ where*

$$L_7 = \left\{ (x_1,x_2,x_3) \in (\mathbb{Z}[\omega_7])^{\oplus 3} \ : \ \begin{array}{l} x_1 \equiv x_2 \equiv 6x_3 \qquad \mod (1-\omega_7), \\ \\ (1+5\omega_7)x_1 + 3x_2 + 2x_3 \equiv 0 \qquad \mod (1-\omega_7)^2 \end{array} \right\}$$

with the hermitian form

$$h_{L_7}(\alpha,\beta)=\alpha_1\bar{\beta}_1+f_1\alpha_2\overline{f_1\beta_2}+f_2\alpha_3\overline{f_2\beta_3}, \tag{15}$$

where $f_1=3+2(\omega_7+\omega_7^6)+(\omega_7^2+\omega_7^5)$, $f_2=2+(\omega_7+\omega_7^6)$.

Proof. As in the previous cases we define the lattice $F:=F_7^3$. We consider the hermitian form

$$h(\alpha,\beta)=\alpha_1\bar{\beta}_1+f_1\alpha_2\overline{f_1\beta_2}+f_2\alpha_3\overline{f_2\beta_3}$$

on the lattice $\mathbb{Z}[\omega_7]^{\oplus 3}$, and define G to be the sublattice $G=(1-\omega_7)^2\mathbb{Z}[\omega_7]^{\oplus 3}$ of $\{\mathbb{Z}[\omega_7]^{\oplus 3},h\}$.
The map $\phi: F\to G$

$$\begin{array}{rrcl}\phi: & (\sigma_7^*)^i(C_1^{(1)}-C_2^{(1)}) & \mapsto & (1-\omega_7)^2\omega_7^i(1,0,0)\\ & (\sigma_7^*)^i(C_1^{(2)}-C_2^{(2)}) & \mapsto & (1-\omega_7)^2\omega_7^i(0,1,0)\\ & (\sigma_7^*)^i(C_1^{(3)}-C_2^{(3)}) & \mapsto & (1-\omega_7)^2\omega_7^i(0,0,1)\end{array}$$

is an isomorphism between the $\mathbb{Z}$-lattice $G_{\mathbb{Z}}$, with the bilinear form induced by the hermitian form, and F with the intersection form. We have to add to G some vectors to find a lattice L_7 isomorphic to Ω_7. These vectors are

$$\begin{array}{l}s-t_1=(1,c,k),\\ C_1^{(1)}-C_1^{(2)}=(1-\omega_7)(1,-(1+\omega_7^4),0),\\ C_1^{(2)}-C_1^{(3)}=(1-\omega_7)(0,(1+\omega_7^4),-(1+\omega_7^3+\omega_7^5)),\end{array}$$

where $c=1+3\omega_7+3\omega_7^4+\omega_7^5$ and $k=-5+\omega_7-5\omega_7^2-3\omega_7^4-3\omega_7^5$. A basis for the lattice L_7 is

$$\begin{array}{ll}l_1=(1,c,k) & l_2=\omega_7 l_1\\ l_3=\omega_7^2 l_1 & l_4=\omega_7^3 l_1\\ l_5=\omega_7^4 l_1 & l_6=\omega_7^5 l_1\\ l_7=(1-\omega_7)^2(0,2+4\omega_7+6\omega_7^2+\omega_7^3+3\omega_7^4+5\omega_7^5,0) & l_8=(1-\omega_7)^2(1,0,0)\\ l_9=\omega_7 l_8 & l_{10}=\omega_7^2 l_8\\ l_{11}=\omega_7^3 l_8 & l_{12}=\omega_7^4 l_8\\ l_{13}=(1-\omega_7)(1,-(1+\omega_7^4),0) & l_{14}=(1-\omega_7)^2(0,1,0)\\ l_{15}=\omega_7 l_{14} & l_{16}=\omega_7^2 l_{14}\\ l_{17}=\omega_7^3 l_{14} & l_{18}=\omega_7^4 l_{14}\end{array}.$$

The identification between Ω_7 and L_7 is given by the map $b_i\mapsto l_i$. After this identification the intersection form on Ω_7 is exactly the form $b_{|L_7}$ on L_7 induced by the hermitian form (15). □

Remark. 1) The density of Ω_7 is $\Delta=\frac{\pi^9}{9!}\frac{1}{\sqrt{7^3}}\approx 0.0044$.
2) As in the previous cases the lattice Ω_7 does not admit vectors of norm -2 and can be generated by vectors of norm -4, and a basis is b_1, b_2, b_3, b_4, b_5, b_6, $b_7-b_{13}-2b_{14}-3b_{15}-4b_{16}-5b_{17}-6b_{18}$, b_8+b_9, b_9+b_{10}, $b_{10}+b_{11}$, $b_{11}+b_{12}$, $b_{10}+b_{11}+b_{12}$, b_{13}, $b_{14}+b_{15}$, $b_{15}+b_{16}$, $b_{16}+b_{17}$, $b_{17}+b_{18}$, $b_{16}+b_{17}+b_{18}$.

5. Families of K3 surfaces with a symplectic automorphism of order p

In the previous sections we used elliptic K3 surfaces to describe some properties of the automorphism σ_p. All these K3 surfaces have Picard number $\rho_p + 1$, where ρ_p is the minimal Picard number found in the Proposition 1.1. In this section we want to describe algebraic K3 surfaces with symplectic automorphism of order p and with the minimal possible Picard number. Recall that the values of ρ_p are

p	3	5	7
ρ_p	13	17	19,

and Ω_p denote the lattices described in the sections 4.1, 4.4, 4.6.

Proposition 5.1. *Let X be a K3 surface with symplectic automorphism of order $p = 3, 5, 7$ and Picard number ρ_p as above. Let L be a generator of $\Omega_p^{\perp} \subset NS(X)$, with $L^2 = 2d > 0$ and let*

$$\mathcal{L}_{2d}^p := \mathbb{Z}L \oplus \Omega_p.$$

Then we may assume that L is ample and
(1) if $L^2 \equiv 2, 4, \dots, 2(p-1) \mod 2p$, then $\mathcal{L}_{2d}^p = NS(X)$,
(2) if $L^2 \equiv 0 \mod 2p$, then either $\mathcal{L}_{2d}^p = NS(X)$ or $NS(X) = \widetilde{\mathcal{L}_{2d}^p}$ with $\widetilde{\mathcal{L}_{2d}^p}/\mathcal{L}_{2d}^p \simeq \mathbb{Z}/p\mathbb{Z}$ and in particular $\widetilde{\mathcal{L}_{2d}^p}$ is generated by an element $(L/p, v/p)$ with $v^2 \equiv 0$ mod $2p$ and $L^2 + v^2 \equiv 0$ mod $2p^2$.

Proof. Since $L^2 > 0$ by Riemann Roch theorem we can assume L or $-L$ effective. Hence we assume L effective. Let N be an effective (-2) curve then $N = \alpha L + v'$, with $v' \in \Omega_p$ and $\alpha > 0$ since Ω_p do not contains (-2)-curves. We have $L \cdot N = \alpha L^2 > 0$, and so L is ample. Moreover recall that L and Ω_p are primitive sublattices of $NS(X)$. Since the discriminant group of $\mathcal{L}_{2d}^p := \mathbb{Z}L \oplus \Omega_p$ is $(\mathbb{Z}/2d\mathbb{Z}) \oplus (\mathbb{Z}/p\mathbb{Z})^{\oplus n_p}$, with $n_3 = 6$, $n_5 = 4$, $n_7 = 3$ an element in $NS(X)$ not in $\mathcal{L}_{2d}^p$ is of the form $(\alpha L/2d, v/p)$, $v \in \Omega_p$ and satisfy the following conditions:
(a) $p \cdot (\alpha L/2d, v/p) \in NS(X)$,
(b) $(\alpha L/2d, v/p) \cdot L \in \mathbb{Z}$,
(c) $(\alpha L/2d, v/p)^2 \in \mathbb{Z}$.
By using the condition (a) we obtain $p \cdot (\alpha L/2d, v/p) - (0, v) \in NS(X)$ and so

$$\frac{p\alpha L}{2d} \in NS(X).$$

Hence by the primitivity of L in $NS(X)$ follows that $d \equiv 0$ mod p, $d = pd'$, $d' \in \mathbb{Z}_{>0}$ and so

$$\frac{\alpha L}{2d'} \in NS(X)$$

which gives $\alpha = 2d'$ and the class (if there is) is $(L/p, v/p)$. Now condition (b) gives

$$(L/p, v/p) \cdot L = L^2/p \in \mathbb{Z}$$

and so $L^2 = 2p \cdot r$, $r \in \mathbb{Z}_{>0}$, since the lattice is even. And so if $NS(X) = \widetilde{\mathcal{L}_{2d}^p}$, then $L^2 \equiv 0$ mod $2p$. Finally condition (c) gives

$$(L/p, v/p)^2 = \frac{L^2 + v^2}{p^2}$$

and so since a square is even $L^2 + v^2 \equiv 0$ mod $2p^2$. □

In the sections 4.1, 4.4, 4.6 we defined a symplectic automorphism σ_p, $p = 3,5,7$ of order p on some special $K3$ surfaces and we found the lattices $\Omega_p = (\Lambda_{K3}^{\sigma_p^*})^{\perp}$. Now we consider more in general an isometry on Λ_{K3} defined as σ_p^* (we call it again σ_p^*). In the next theorem we prove that if X is a K3 surface such that $NS(X) = \mathcal{L}_{2d}^p$ or $NS(X) = \widetilde{\mathcal{L}_{2d}^p}$, then this isometry is induced by a symplectic automorphism of the surface X.

Proposition 5.2. *Let $\mathcal{L}_p = \mathcal{L}_{2d}^p$ or $\mathcal{L}_p = \widetilde{\mathcal{L}_{2d}^p}$ if $p = 3,5$ and let $\mathcal{L}_7 = \widetilde{\mathcal{L}_{2d}^7}$. Then there exists a K3 surface X with symplectic automorphism σ_p of order p such that $NS(X) = \mathcal{L}_p$ ($p = 3,5,7$) and $(H^2(X,\mathbb{Z})^{\sigma_p^*})^{\perp} = \Omega_p$.*
Moreover there are no K3 surfaces with Néron-Severi group isometric to $\mathcal{L}_{14d}^7$.

Proof. Let σ_p^*, $p = 3,5,7$, be an isometry as in the sections 4.1, 4.4, 4.6. We make the proof in several steps.
Step 1: there exists a marked K3 surface X such that $NS(X)$ is isometric to $\mathcal{L}_p$, and there are no K3 surfaces with Néron-Severi group isometric to $\mathcal{L}_{14d}^7$. By [Ni2, Theorem 1.14.4] the lattices $\mathcal{L}_{2d}^p$ $\widetilde{\mathcal{L}_{2d}^3}$, $\widetilde{\mathcal{L}_{2d}^5}$ have a unique primitive embedding in the K3 lattice. The lattice $T_5 = U(5) \oplus U(5) \oplus \langle -2d \rangle$ has a unique primitive embedding in Λ_{K3}, again by [Ni2, Theorem 1.14.4]. Its signature is $(2,3)$ and its discriminant form is the opposite of the discriminant form of $\mathcal{L}_{2d}^5$. Since, by [Ni2, Corollary 1.13.3], $\mathcal{L}_{2d}^5$ is uniquely determined by its signature and discriminant form, it is the orthogonal of T_5 in Λ_{K3} and then $\mathcal{L}_{2d}^5$ admits a primitive embedding in Λ_{K3}. The lattice $\widetilde{\mathcal{L}_{2d}^7}$ is a primitive sublattice of the Néron-Severi group of the K3 surface described in the section 4.6, so it is a primitive sublattice of Λ_{K3} (the same argument can be applied to the lattices $\widetilde{\mathcal{L}_{2d}^p}$, $p = 3,5$). Let now $\omega \in \mathcal{L}_p^{\perp} \otimes \mathbb{C} \subseteq \Lambda_{K3} \otimes \mathbb{C}$, with $\omega\omega = 0$, $\omega\bar{\omega} > 0$. We choose ω generic with these properties. By the surjectivity of the period map of K3 surfaces, ω is the period of a K3 surface X with $NS(X) = \omega^{\perp} \cap \Lambda_{K3} = \mathcal{L}_p$.
The rank of the lattice $\mathcal{L}_{14d}^7$ is 19 and its discriminant group has four generators. If $\mathcal{L}_{14d}^7$ was the Néron-Severi group of a K3 surface, the transcendental lattice of this surface should be a rank three lattice with a discriminant group generated by four elements. This is clearly impossible.
Step 2: the isometry σ_p^* fixes the sublattice $\mathcal{L}_p$. Since $\sigma_p^*(\Omega_p) = \Omega_p$ and $\sigma_p^*(L) = L$ (because $L \in \Omega_p^{\perp}$ which is the invariant sublattice of Λ_{K3}), if $\mathcal{L}_p = \mathcal{L}_{2d}^p = \mathbb{Z}L \oplus \Omega_p$ it is clear that $\sigma_p^*(\mathcal{L}_p) = \mathcal{L}_p$. Now we consider the case $\mathcal{L}_p = \widetilde{\mathcal{L}_{2d}^p}$. The isometry σ_p^* acts trivially on $\Omega_p^{\vee}/\Omega_p$ (cf. Propositions 4.2, 4.4, 4.6) and on $(\mathbb{Z}L)^{\vee}/\mathbb{Z}L$. Let $\frac{1}{p}(L,v') \in \mathcal{L}_p$, with $v' \in \Omega_p$. This is also an element in $(\Omega_p \oplus L\mathbb{Z})^{\vee}/(\Omega_p \oplus L\mathbb{Z})$. So we have $\sigma_p^*(\frac{1}{p}(L,v')) \equiv \frac{1}{p}(L,v')$ mod $(\Omega_p \oplus \mathbb{Z}L)$, which means

$$\sigma_p^*(\frac{1}{p}(L,v')) = \frac{1}{p}(L,v') + (\beta L, v''), \qquad \beta \in \mathbb{Z}, \quad v'' \in \Omega_p.$$

Hence we have $\sigma_p^*(\mathcal{L}_p) = \mathcal{L}_p$.
Step 3: The isometry σ_p^* is induced by an automorphism of the surface X. The isometry σ_p^* fixes the sublattice $\mathcal{L}_p^{\perp}$ of Λ_{K3}, so it is an Hodge isometry. By the Torelli theorem an effective Hodge isometry of the lattice Λ_{K3} is induced by an automorphism of the K3 surface (cf. [BPV, Theorem 11.1]). To apply this theorem we have to prove that σ_p^* is an effective isometry. An effective isometry on a surface X is an isometry which preserves the set of effective divisors. By [BPV, Corollary 3.11] σ_p^* preserves the set of the effective divisors if and only if it preserves the ample cone. So if σ_p^* preserves the ample

cone it is induced by an automorphism of the surface. This automorphism is symplectic by construction (it is the identity on the transcendental lattice $T_X \subset \Omega_p^{\perp}$), and so if σ_p^* preserves the ample cone, the theorem is proven.
Step 4: The isometry σ_p^* preserves the ample cone $\mathcal{A}_X$. Let $\mathcal{C}_X^+$ be one of the two connected components of the set $\{x \in H^{1,1}(X,\mathbb{R}) \mid (x,x) > 0\}$. The ample cone of a K3 surface X can be described as the set $\mathcal{A}_X = \{x \in \mathcal{C}_X^+ \mid (x,d) > 0$ for each d such that $(d,d) = -2,\ d$ effective$\}$. First we prove that σ_p^* fixes the set of the effective (-2)-curves. Since there are no (-2)-curves in Ω_p, if $N \in \mathcal{L}_p$ has $N^2 = -2$ then $N = \frac{1}{p}(aL, v')$, $v' \in \Omega_p$, for an integer $a \neq 0$. Since $\frac{1}{p}aL^2 = L \cdot N > 0$, because L and N are effective divisor, we obtain $a > 0$. The curve $N' = \sigma_p^*(N)$ is a (-2)-curve because σ_p^* is an isometry, hence N' or $-N'$ is effective. Since $N' = \sigma_p^*(N) = (aL, \sigma_p^*(v'))$ we have $-N' \cdot L = -aL^2 < 0$ and so $-N'$ is not effective. Using the fact that σ_p^* has finite order it is clear that σ_p^* fixes the set of the effective (-2)-curves.
Now let $x \in \mathcal{A}_X$ then $\sigma_p^*(x) \in \mathcal{A}_X$, in fact $(\sigma_p^*(x), \sigma_p^*(x)) = (x,x) > 0$ and for each effective (-2)-curve d there exists an effective (-2)-curve d' with $d = \sigma_p^*(d')$, so we have $(\sigma_p^*(x), d) = (\sigma_p^*(x), \sigma_p^*(d')) = (x, d') > 0$. Hence σ_p^* preserves $\mathcal{A}_X$ as claimed. □

Corollary 5.1. *The coarse moduli space of $\mathcal{L}_p$-polarized $K3$ surfaces (cf.* [Do] *for the definition) $p = 3,5,7$ has dimension seven, three, respectively one and is a quotient of*

$$\mathcal{D}_{\mathcal{L}_p} = \{\omega \in \mathbb{P}(\mathcal{L}_p^{\perp} \otimes_{\mathbb{Z}} \mathbb{C}) :\ \omega^2 = 0,\ \omega\bar{\omega} > 0\}$$

by an arithmetic group $O(\mathcal{L}_p)$.

Remark. In particular the moduli space of K3 surfaces admitting a symplectic automorphism of order $p = 3,5,7$ has dimension respectively seven, three and one.

6. Final remarks

1. In Proposition 5.2 it would be interesting to prove the unicity of the lattices $\widetilde{\mathcal{L}^p_{2d}}$, this requires some careful analysis of the automorphism group of the lattices Ω_p, $p = 3,5,7$.
2. It is not difficult to give examples of K3 surfaces (not elliptic) in some projective space with a symplectic automorphism of order three or five. Consider for example the surfaces of $\mathbb{P}^3$:

$$S1:\quad q_4(x_0,x_1) + q_2(x_0,x_1)x_2x_3 + l_1(x_0,x_1)x_2^3 + l_1'(x_0,x_1)x_3^3 + ax_2^2x_3^2 = 0$$
$$S2:\quad a_{01}x_0^2x_1^2 + a_{23}x_2^2x_3^2 + a_{0123}x_0x_1x_2x_3 + a_{02}x_0^3x_2 + a_{13}x_1^3x_3 + a_{12}x_1x_2^3 + a_{03}x_0x_3^3 = 0.$$

where q_i is homogeneous of degree i, l_1, l_1' are linear forms, and $a_{ij} \in \mathbb{C}$. The surfaces S_1 resp. S_2 admit symplectic automorphisms of order three resp. of order five induced by the automorphisms of $\mathbb{P}^3$ given by $\sigma_3 : (x_0 : x_1 : x_2 : x_3) \longrightarrow (x_0 : x_1 : \omega_3 x_2 : \omega_3^2 x_3)$ and $\sigma_5 : (x_0 : x_1 : x_2 : x_3) \longrightarrow (\omega_5 x_0 : \omega_5^4 x_1 : \omega_5^2 x_2 : \omega_5^3 x_3)$. The automorphisms of $\mathbb{P}^3$ commuting with σ_3 resp. σ_5 form a space of dimension six, resp. four, since the equations depend on 13, resp. seven parameters the dimension of the moduli space is seven, resp. three as expected (this is the minimal possible dimension). In a similar way one can costruct many more examples. In the case of order seven automorphisms it is more difficult to give such examples. Already in the case of a polarization $L^2 = 2$, the K3 surface is the minimal resolution of the double covering of $\mathbb{P}_2$ ramified on a sextic with singular points and these are the fixed points of the automorphisms. One should resolve the singularities and analyze the action on the resolution before doing the double cover.

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Projective models of K3 surfaces with an even set

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PROJECTIVE MODELS OF K3 SURFACES WITH AN EVEN SET

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Abstract. The aim of this paper is to describe algebraic K3 surfaces with an even set of rational curves or of nodes. Their minimal possible Picard number is nine. We completely classify these K3 surfaces and after a careful analysis of the divisors contained in the Picard lattice we study their projective models, giving necessary and sufficient conditions to have an even set. Moreover we investigate their relation with K3 surfaces with a Nikulin involution.

0. Introduction

It is a classical problem in algebraic geometry to determine when a set of (-2)-rational curves on a surface is even. This means the following: let $L_1, \ldots, L_N$ be rational curves on a surface X then they form an *even set* if there is $\delta \in Pic(X)$ such that

$$L_1 + \ldots + L_n \sim 2\delta.$$

This is equivalent to the existence of a double cover of X branched on $L_1 + \ldots + L_n$. This problem is related to the study of even sets of nodes, in fact a set of nodes is even if the (-2)-rational curves in the minimal resolution are an even set. In particular the study of even sets on surfaces plays an important role in determining the maximal number of nodes a surface can have (cf. e.g. [Be], [JR]). Here we restrict our attention to K3 surfaces.

In a famous paper of 1975 [N1] Nikulin shows that an even set of disjoint rational curves (resp. of distinct nodes) on a K3 surface contains 0, 8 or 16 rational curves (nodes). If the even set on the K3 surface X is made up by sixteen rational curves, the surface covering X is birational to a complex torus A and X is the Kummer surface of A. This situation is studied by Nikulin in [N1]. If the even set on X is made up by eight rational curves then the surface covering X is also a K3 surface. There are some more general results about even sets of curves not necessarily disjoint. More recently in [B1] Barth studies the case of even sets of rational curves on quartic surfaces (i.e. K3 surfaces in $\mathbb{P}^3$) also in the case that the curves meet each other and he finds sets containing six or ten lines too.

In the paper [B2] he discusses some particular even sets of disjoint lines and nodes on K3 surfaces whose projective models are a double cover of the plane, a quartic in $\mathbb{P}^3$ or a double cover of the quadric $\mathbb{P}^1 \times \mathbb{P}^1$, and he gives necessary and sufficient conditions to have an even set.

Our purpose is to study algebraic K3 surfaces admitting an even set of eight disjoint rational curves. We investigate their Picard lattices, moduli spaces and projective models. The minimal possible Picard number is nine, and we restrict our study to the surfaces with this Picard number. The techniques used by Barth in his article are mostly geometric, here we use lattice theory: we investigate first the Picard lattices of the K3 surfaces and the ampleness of certain divisors, then we study the projective models. We find again the cases studied by Barth and we discuss many new cases, with a special attention to complete intersections. We

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give also an explicit relation between the Picard lattice of an algebraic K3 surface with an even set and the Picard lattice of the K3 surface which is its double cover. More precisely if X admits an even set of eight disjoint rational curves, then by [N1], it is the desingularization of the quotient of a K3 surface by a Nikulin involution (i.e. a symplectic automorphism of order two). The Nikulin involutions are well known and are studied by Morrison in [M] and by van Geemen and Sarti in [vGS]. In [vGS] the authors describe also some geometric properties of the quotient by a Nikulin involution and so of K3 surfaces with an even set of eight nodes. In [N1] Nikulin proves that a sufficient condition on a K3 surface to be a Kummer surface (and so to have an even set made up by sixteen disjoint rational curves) is that a particular lattice (the so called Kummer lattice) is primitively embedded in the Néron Severi group of the surface. Here we prove a similar result: a sufficient condition on a K3 surface X to be the desingularization of the quotient of another K3 surface with a Nikulin involution (and so to have an even set made up by eight disjoint rational curves) is that a particular lattice (the so called Nikulin lattice) is primitively embedded in the Néron Severi group of X. This result is essential to describe the coarse moduli space of a K3 surface with an even set of eight disjoint rational curves.

In the Section 1 we recall some known results on even sets on surfaces, in particular on K3 surfaces. In the Section 2 we study algebraic K3 surfaces X with Picard number nine. If X admits an even set of eight disjoint rational curves, then its Néron Severi group has rank at least nine (it has to contain the eight rational curves of the even set and a polarization, because the K3 surface is algebraic). The main results of this section (and also two of the main results of this paper) are the complete description of the possible Néron Severi groups of rank nine of algebraic K3 surfaces admitting an even set and the complete description of the coarse moduli space of the algebraic K3 surfaces with an even set of eight disjoint rational curves. Moreover using the results of [vGS] we describe the relation between the Néron Severi group of an algebraic K3 surface Y admitting a Nikulin involution ι and the Néron Severi group of a K3 surface admitting an even set, which is the desingularization of Y/ι (Corollary 2.2). In the Section 3 we analyze the ampleness of some divisors (or more in general the nefness). These classes are used in the Section 4 to describe projective models of algebraic K3 surfaces with an even set of eight disjoint rational curves. In particular we describe the following projective models:

• **double covers of** $\mathbb{P}^2$: these branch along a sextic with eight nodes (Paragraph 4.1) or along a smooth sextic (Paragraph 4.4, a)) (these two situations are studied also by Barth in [B2], first and second cases), or along a sextic with four nodes (Paragraph 4.7);

• **quartic surfaces in** $\mathbb{P}^3$: these have an even set of nodes (Paragraph 4.2) or an even set of lines (Paragraph 4.5, a)) (these two situations are studied also by Barth in [B2], third and forth cases), or it has a mixed even set of nodes and conics (Paragraph 4.8, b));

• **double covers of a cone :** these branch along a conic and a sextic on the cone, which intersect in six points (Paragraph 4.3);

• **complete intersections of a hyperquadric and a cubic hypersurface in** $\mathbb{P}^4$: these have an even set of nodes (Paragraph 4.4, b)) or an even set of lines (Paragraph 4.7, a));

• **complete intersections of three hyperquadrics in** $\mathbb{P}^5$: these have an even set of nodes (Paragraph 4.5, b) and Paragraph 4.6, b)) or an even set of lines (Paragraph 4.8, a) and Paragraph 4.9, a));

• **double covers of a smooth quadric**: these branch along a curve of bidegree $(4, 4)$ (Paragraph 4.6) (this case is studied also by Barth in [B2], sixth case);

• We study also the following complete intersections (c.i.) of hypersurfaces of bidegree (a, b) in $\mathbb{P}^n \times \mathbb{P}^m$:

space	c.i.	paragraph
$\mathbb{P}^1 \times \mathbb{P}^2$	$(2,3)$	4.9 *b)*
$\mathbb{P}^4 \times \mathbb{P}^2$	$(2,0),(1,1),(1,1),(1,1)$	4.4 *c)*
$\mathbb{P}^2 \times \mathbb{P}^2$	$(1,2),(2,1)$	4.10
$\mathbb{P}^3 \times \mathbb{P}^3$	$(1,1),(1,1),(1,1),(1,1)$	4.11

In Section 4 we describe moreover geometric properties of these K3 surfaces with an even set. In Section 5 we use these properties to give sufficient conditions for a K3 surface to have an even set.

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1. K3 surfaces with an even set of nodes and of rational curves

Definition 1.1. *Let X be a surface. A set of m disjoint (-2)-rational smooth curves, N_1, …,N_m, on X, is an* even set of rational curves *if there is a divisor $\delta \in Pic(X)$ such that*

$$N_1 + \ldots + N_m \sim 2\delta,$$

where $''\sim''$ denotes linear equivalence.

Definition 1.2. *Let $\bar{X}$ be a surface and let $\mathcal{N} = \{p_1, \ldots, p_m\}$ be a set of nodes on $\bar{X}$. Let $\tilde{\beta} : X \longrightarrow \bar{X}$ be the minimal resolution of the nodes of X and let $N_i = \tilde{\beta}^{-1}(p_i)$, $i = 1, \ldots, m$. These are (-2)-rational curves on X. The set $\mathcal{N}$ is an* even set of nodes *if $N_1, \ldots, N_m$ are an even set of rational curves.*

In the case of $K3$ surfaces linear equivalence is the same as algebraic equivalence (which we denote by $\equiv$) and $Pic(X) = NS(X)$.

The existence of an even set $N_1, \ldots, N_m$ on a surface X is equivalent to the existence of a double cover $\pi : \widetilde{Y} \to X$ from a surface $\widetilde{Y}$ to X branched on $N_1 + \ldots + N_m$ [BPV, Lemma 17.1].

Let Y be a surface and ι be an involution on Y with exactly m distinct fixed points $q_1, \ldots, q_m$ and let $\widetilde{Y}$ be the blow up of Y at the points $q_1, \ldots, q_m$. The involution ι induces an involution $\widetilde{\iota}$ on $\widetilde{Y}$. Let $\bar{X}$ be the quotient surface Y/ι and $\pi' : Y \to \bar{X}$ be the projection. The surface $\bar{X}$ has m nodes in $\pi'(q_i)$, $i = 1, \ldots, m$. Let $\widetilde{\beta} : X \to \bar{X}$ be the minimal resolution of $\bar{X}$. Then the following diagram commutes

$$\begin{array}{ccc} \widetilde{Y} & \stackrel{\beta}{\longrightarrow} & Y \\ \pi \downarrow & & \downarrow \pi' \\ X & \stackrel{\tilde{\beta}}{\longrightarrow} & \bar{X}. \end{array} \tag{1}$$

The double cover $\pi : \widetilde{Y} \to X$ is branched on $N_1 + \ldots + N_m$ where N_i are the (-2)-curves such that $\widetilde{\beta}(N_i) = \pi'(q_i)$, $i = 1, \ldots, m$ and these form an even set.

Conversely if $\pi : \widetilde{Y} \to X$ is a double cover of X branched on the divisor $N_1 + \ldots + N_m$ where N_i are (-2)-rational curves, then there is a diagram as (1).

We recall some facts about even sets on K3 surfaces:

- If $N_1, \dots, N_m$ is an even set of disjoint curves on a K3 surface, by a result of Nikulin [N1, Lemma 3] we have $m = 0, 8$ or 16.
- If $m = 16$ the surface Y in the diagram (1) is a torus of dimension two ([N1, Theorem 1]), the involution ι is defined on Y as $y \mapsto -y$, $y \in Y$ and has sixteen fixed points. So X is the Kummer surface associated to the surface Y (a Kummer surface is by definition the K3 surface obtained as the desingularization of the quotient of a torus Y by the involution $y \mapsto -y$, $y \in Y$). If Y is an algebraic torus (so an Abelian surface), then X is an algebraic K3 surface and its Picard number is $\rho \geq 17$.
- If $m = 8$ then the surface Y is a K3 surface and the cover involution has eight isolated fixed points (it is a *Nikulin involution*, cf. Definition 1.3 below). If Y is an algebraic K3 surface, then X is algebraic and its Picard number is $\rho \geq 9$.

Definition 1.3. *Let Y be a K3 surface. Let ι be an involution of Y. The involution ι is called* Nikulin involution *if $\iota_{|H^{2,0}(X,\mathbb{C})} = id_{|H^{2,0}(X,\mathbb{C})}$.*

We recall some facts about the Nikulin involutions:

- An involution ι on a K3 surface is a Nikulin involution if and only if it has eight isolated fixed points [N2, Section 5].
- The Nikulin involutions are the unique involutions on a K3 surface Y such that the desingularization of $Y/\iota = \bar{X}$ is a K3 surface. In fact let $\widetilde{Y}$ be the blow up of Y on the fixed points of the involution ι. In this way we obtain more algebraic classes on $\widetilde{Y}$, but the transcendental classes are the same and so $H^{2,0}(Y)^{\iota^*} = H^{2,0}(\widetilde{Y})^{\tilde{\iota}^*}$. Since an automorphism of a K3 surface induces a Hodge isometry on the second cohomology group we have $\iota^*(H^{2,0}(Y)) = H^{2,0}(Y) \simeq \mathbb{C}$ and since $H^{2,0}(\widetilde{Y})^{\tilde{\iota}^*} = H^{2,0}(X) \simeq \mathbb{C}$ it follows that ι^* is the identity on $H^{2,0}(Y)$, so ι is a symplectic automorphism.

2. Even sets and Nikulin involutions

Let $N_1, \dots, N_8$ be an even set of eight disjoint smooth rational curves on a K3 surface X, then by adjunction $N_i^2 = -2$ and Morrison shows in [M, Lemma 5.4] that the minimal primitive sublattice of $H^2(X, \mathbb{Z})$ containing these (-2)-curves is isomorphic to the Nikulin lattice:

Definition 2.1. [M, Definition 5.3] *The* Nikulin lattice *is an even lattice N of rank eight generated by $\{N_i\}_{i=1}^8$ and $\hat{N} = \frac{1}{2}\sum N_i$, with bilinear form induced by*

$$N_i \cdot N_j = -2\delta_{ij}.$$

Observe that $\hat{N}^2 = -4$ and $\hat{N} \cdot N_i = -1$. This lattice is a negative definite lattice of discriminant 2^6 and discriminant group $(\mathbb{Z}/2\mathbb{Z})^{\oplus 6}$.
From now on X is an algebraic K3 surface. A K3 surface has an even set of eight disjoint rational curves if there are eight disjoint rational curves spanning a copy of N in $NS(X)$ (then rank $NS(X) \geq 8$). Since X is algebraic the signature of the Néron Severi group $NS(X)$ is $(1, \rho - 1)$, where ρ is the Picard number of X (i.e. the rank of $NS(X)$). So the Néron Severi group of X has signature $(1, \rho - 1)$ and has to contain the negative lattice N of rank eight, so $NS(X)$ contains also a class with positive self intersection. Clearly $\rho \geq 9$ and we will see that the generic algebraic K3 surface with an even set has $\rho = 9$ and that the number of moduli is $20 - 9 = 11$ (Corollary 2.3). Here we study the case of algebraic K3 surfaces with Picard number nine.

Proposition 2.1. *Let X be an algebraic K3 surface with an even set of eight disjoint rational curves and with Picard number nine, let L be a divisor generating $N^\perp \subset NS(X)$, $L^2 > 0$.*

Let d be a positive integer such that $L^2 = 2d$ and let

$$\mathcal{L}_{2d} = \mathbb{Z}L \oplus N.$$

Then
(1) if $L^2 \equiv 2 \mod 4$ then $NS(X) = \mathcal{L}_{2d}$,
(2) if $L^2 \equiv 0 \mod 4$ then either $NS(X) = \mathcal{L}_{2d}$ or $NS(X) = \mathcal{L}'_{2d}$, where $\mathcal{L}'_{2d}$ is generated by $\mathcal{L}_{2d}$ and by a class $(L/2, v/2)$, with

- $v^2 \in 4\mathbb{Z}$,
- $v \cdot N_i \in 2\mathbb{Z}$ *($v \in N$ but $v/2 \notin N$),*
- $L^2 \equiv -v^2 \mod 8$.

Proof. The discriminant group of $\mathcal{L}_{2d} = \mathbb{Z}L \oplus N$ is $(\mathbb{Z}/2d\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})^{\oplus 6}$, hence an element in the Néron Severi group of X but not in $\mathcal{L}_{2d}$ is of the form $(\alpha L/2d, v/2)$ with $\alpha \in \mathbb{Z}$, $v \in N$. Since $2 \cdot (\alpha L/2d, v/2) - v \in NS(X)$ we can assume that $\alpha = d$ and so the element is $(L/2, v/2)$. We can write $v = \sum \alpha_i N_i + \beta \hat{N}$, $\beta \in \{0, 1\}$, we have

$$(\frac{L}{2}, \frac{v}{2}) \cdot N_i \in \mathbb{Z}.$$

Hence by doing the computations it follows

$$\frac{1}{2}(-2\alpha_i - \beta) \in \mathbb{Z},$$

hence $\beta \in 2\mathbb{Z}$, and so we may assume $\beta = 0$. We have also

$$(\frac{L}{2}, \frac{v}{2}) \cdot \hat{N} \in \mathbb{Z}$$

so

$$-\frac{1}{2}(\sum \alpha_i) \in \mathbb{Z}$$

hence $\alpha_1 + \ldots + \alpha_8 \in 2\mathbb{Z}$ and so $\alpha_1^2 + \ldots + \alpha_8^2 \in 2\mathbb{Z}$ too. We have

$$\begin{array}{rcl} v^2 & = & -2\sum \alpha_i^2 - 4\beta^2 - 2\beta \sum \alpha_i \\ & = & -2(\sum \alpha_i^2). \end{array}$$

It follows that $v^2 \in 4\mathbb{Z}$ and $v \cdot N_i \in 2\mathbb{Z}$.
Since the Néron Severi lattice of a K3 surface is even we have

$$(\frac{L}{2}, \frac{v}{2})^2 = \frac{L^2 + v^2}{4} \in 2\mathbb{Z}$$

which gives $L^2 \in 4\mathbb{Z}$, so d must be even and $L^2 + v^2 \equiv 0 \mod 8$.
Assume now that there is another class $(L/2, v'/2) \in NS(X)$, then the class $(L/2, v/2) - (L/2, v'/2) = (v - v')/2 \in NS(X)$ too. Since N is primitive $(v - v')/2 \in N$. So there is a $\delta \in N$ s.t. $v - v' = 2\delta$. So $(L/2, v'/2) \in NS(X)$ if and only if $(L/2, v'/2) = (L/2, v/2) + \delta$ for certain $\delta \in N$. This concludes the proof of the proposition. □

Proposition 2.2. *Under the assumptions of the Proposition 2.1, $\mathcal{L}'_{2d}$ is the unique even lattice (up to isometry) such that $[\mathcal{L}'_{2d} : \mathcal{L}_{2d}] = 2$ and N is a primitive sublattice of $\mathcal{L}'_{2d}$.*

Proof. We describe briefly the group $O(N)$ of isometries of N. These must preserve the intersection form, so the image of each (-2)-vector under an isometry is a (-2)-vector. The only (-2)-vectors in the Nikulin lattice N up to the sign are the eight vectors N_i and so if $\sigma \in O(N)$ then $\sigma(N_i) = \pm N_j$, $i, j = 1, \ldots, 8$. In particular the group of permutation of eight elements Σ_8 is contained in $O(N)$. This group fixes the class $\hat{N}$.

Each class v in N is $v=\sum_{i=1}^{8}\alpha_iN_i+a\hat{N}$, $\alpha_i,\ a\in\mathbb{Z}$. We consider two different elements v and v' such that $\mathcal{L}_{2d}$ together with the class $(L/2,v/2)$ or with the class $(L/2,v'/2)$ generate an overlattice of $\mathcal{L}_{2d}$. We want to prove that there exists an isometry σ of N such that $\sigma(v)=v'$. From the conditions given on v, or v', (in particular from the fact that $v\cdot N_i\in 2\mathbb{Z}$), $v=\sum_{i=1}^{8}\alpha_iN_i$, $\alpha_i\in\mathbb{Z}$ and $v'=\sum_{i=1}^{8}\beta_iN_i$, $\beta_i\in\mathbb{Z}$, we may assume that $\alpha_i,\beta_i\in\{0,1\}$. The only possibilities for v^2 (or v'^2) are -4,-8,-12, (by the condition on v^2 given in the previous proof) and this depends only on the number of $\alpha_i's$ (resp. $\beta_i's$) equal to one.
We distinguish two different cases: $v^2=v'^2$ and $v^2\neq v'^2$ but $v^2\equiv v'^2 \mod 8$ (since $-v'^2\equiv L^2\equiv -v^2 \mod 8$).
The case $v^2=v'^2$. This condition implies that there are the same number of α_i and β_i equal to one. Hence there is a permutation $\sigma\in\Sigma_8\subset O(N)$ of the N_i, s.t. $\sigma(v)=v'$.
Observe that if $L^2\equiv 0 \mod 8$, then it is clear from the description above that $v^2=v'^2=-8$ and so we are in this case.
The case $v^2\neq v'^2$, $v^2\equiv v'^2 \mod 8$ *and* $L^2\equiv 4 \mod 8$. If $L^2\equiv 4 \mod 8$ then v^2 and v'^2 are -4 or -12. So we can assume $v^2=-4$ and $v'^2=-12$ and $v=N_1+N_2$ $v'=N_3+N_4+N_5+N_6+N_7+N_8$ (up to isometry of the lattice). Observe that $v'/2=\hat{N}-v/2$ hence the lattice generated by $\mathcal{L}_{2d}$ and by $(L/2,v/2)$ or by $\mathcal{L}_{2d}$ and by $(L/2,v'/2)$ are the same. □

Corollary 2.1. *Let* $L^2\equiv 0 \mod 4$ *and* $NS(X)=\mathcal{L}'_{2d}$. *Then there are two possibilities:*

- $L^2\equiv 4 \mod 8$. *In this case one can assume that* $v=-N_1-N_2$ *and* $(L-N_3-\ldots-N_8)/2=(L+v)/2+\hat{N}-(N_3+\cdots+N_8)$ *is in* $NS(X)$ *too.*
- $L^2\equiv 0 \mod 8$. *In this case one can assume that* $v=-(N_1+N_2+N_3+N_4)$ *and* $(L-N_5-N_6-N_7-N_8)/2$ *is in* $NS(X)$ *too.*

Proposition 2.3. *Let* $\Gamma=\mathcal{L}_{2d}$ *or* $\mathcal{L}'_{2d}$ *then there exists a K3 surface* X *with an even set of eight disjoint rational* (-2)*-smooth curves, such that* $NS(X)=\Gamma$.

In the proof of this proposition we will use the relations between the Néron Severi group of a K3 surface Y with a Nikulin involution and the Néron Severi group of a K3 surface X which is the desingularization of the quotient of Y by the Nikulin involution. Here we recall the two following Propositions of [vGS] in which the properties of the Néron Severi group of a K3 surface with a Nikulin involution are described (we use the notation of the Diagram 1).
Proposition [vGS, Proposition 2.2] *Let* Y *be an algebraic K3 surface admitting a Nikulin involution and with Picard number nine. Let* M *be a divisor generating* $E_8(-2)^{\perp}\subset NS(Y)$, $M^2=2d'>0$ *and let*

$$\mathcal{M}_{2d'}=\mathbb{Z}M\oplus E_8(-2).$$

Then M *is ample, and*
(1) if $M^2\equiv 2 \mod 4$ *then* $NS(Y)=\mathcal{M}_{2d'}$,
(2) if $M^2\equiv 0 \mod 4$ *then either* $NS(Y)=\mathcal{M}_{2d}$ *or* $NS(Y)=\mathcal{M}'_{2d'}$, *where* $\mathcal{M}'_{2d'}$ *is generated by* $\mathcal{M}_{2d'}$ *and by a class* $(M/2,v/2)$, *with* $v\in E_8(-2)$.
Proposition [vGS, Proposition 2.7] *(1) Assume that* $NS(Y)=\mathbb{Z}M\oplus E_8(-2)=\mathcal{M}_{2d'}$. *Let* $E_1,\ldots,E_8$ *be the exceptional divisors on* $\tilde{Y}$. *Then: (i) In case* $M^2=4n+2$, *there exist line bundles* $L_1,L_2\in NS(X)$ *such that for a suitable numbering of these* E_i *we have:*
$\beta^*M-E_1-E_2=\pi^*L_1,\qquad \beta^*M-E_3-\ldots-E_8=\pi^*L_2.$
The decomposition of $H^0(Y,M)$ *into* ι^**-eigenspaces is:*
$H^0(Y,M)\cong\pi^*H^0(X,L_1)\oplus\pi^*H^0(X,L_2),\qquad (h^0(L_1)=n+2,\ h^0(L_2)=n+1)$
and the eigenspaces $\mathbb{P}^{n+1},\mathbb{P}^n$ *contain six, respectively two, fixed points.*

(ii) In case $M^2 = 4n$, for a suitable numbering of the E_i we have:
$\beta^*M - E_1 - E_2 - E_3 - E_4 = \pi^*L_1, \qquad \beta^*M - E_5 - E_6 - E_7 - E_8 = \pi^*L_2$ *with* $L_1, L_2 \in NS(X)$.
The decomposition of $H^0(Y, M)$ into ι^*-eigenspaces is:
$H^0(Y, M) \cong \pi^*H^0(X, L_1) \oplus \pi^*H^0(X, L_2), \qquad (h^0(L_1) = h^0(L_2) = n + 1).$
and each of the eigenspaces $\mathbb{P}^n$ contains four fixed points.
(2) Assume $NS(Y) = \mathcal{M}'_{2d'}$. Then there is a line bundle $L \in NS(X)$ such that: $\beta^*M \cong \pi^*L$. The decomposition of $H^0(Y, M)$ into ι^*-eigenspaces is:
$H^0(Y, M) \cong H^0(X, L) \oplus H^0(X, L - \hat{N}), \qquad (h^0(L) = n + 2,\ h^0(L - \hat{N}) = n)$
and all fixed points map to the eigenspace $\mathbb{P}^{n+1} \subset \mathbb{P}^{2n+1}$.

Proof of Proposition 2.3. First observe that the lattices $\mathcal{L}_{2d}$ and $\mathcal{L}'_{2d}$ are primitively embedded in the K3 lattice by [N3, Theorem 1.14.1], so we can identify them with sublattices of $U^3 \oplus E_8(-1)^2$.
1) We consider first the case of $\Gamma = \mathcal{L}_{2d} = \mathbb{Z}L \oplus N$ in this case $L^2 \equiv 2 \mod 4$ or $L^2 \equiv 0 \mod 4$. We show that there exists a K3 surface with an even set of (-2)-smooth curves s.t. $NS(X) = \mathcal{L}_{2d}$. Let Y be a K3 surface with $\rho(Y) = 9$, with Nikulin involution and Néron Severi group containing the lattice $\mathbb{Z}M \oplus E_8(-2)$ with index two and with $M^2 \equiv 0 \mod 4$; such a K3 surface exists by [vGS, Proposition 2.2, 2.3] (in this case its Néron-Severi group is $\mathcal{M}'_{2d'}$, for some non-negative integer d'). We have a diagram like Diagram 1, and so a K3 surface X, which is the minimal resolution of the quotient of Y by the Nikulin involution. Since $\rho(Y) = 9$ then $\rho(X) = 9$ too. By [vGS, Proposition 2.7] there is a line bundle L, $L \in NS(X)$ with $\pi^*L = \beta^*M$. By the properties of the map π^*, $2L^2 = (\pi^*L)^2 = (\beta^*M)^2 = M^2 \equiv 0 \mod 4$ and so $L^2 \equiv 2 \mod 4$ or $L^2 \equiv 0 \mod 4$. Moreover X has an even set made up by the eight curves in the resolutions of the nodes of the quotient $\bar{X}$.
If $L^2 \equiv 2 \mod 4$ then by the Proposition 2.1 $NS(X) = \mathcal{L}_{2d}$, where $L^2 = 2d$, as required.
If $L^2 \equiv 0 \mod 4$ we must exclude that $NS(X) = \mathcal{L}'_{2d}$. Assume that we have an element $L_1 = (L - N_1 - N_2)/2 \in NS(X)$. If $NS(Y) = \mathcal{M}'_{2d'}$ the primitive embedding of $NS(Y)$ in $U^3 \oplus E_8(-1)^2$ is unique up to isometry. Assume that $M^2 = 4n$ and choose an $\alpha \in E_8(-1)$ with $\alpha^2 = -2$ if n is odd and $\alpha^2 = -4$ if n is even. Let $v \in E_8(-2) \subset U^3 \oplus E_8(-1)^2$ be $v = (0, \alpha, -\alpha)$ and let M be $M = (2u, \alpha, \alpha) \in U^3 \oplus E_8(-1)^2$ where $u = e_1 + \frac{(n+1)}{2} f_1$ if n is odd, and $u = e_1 + (\frac{n}{2} + 1) f_1$ if n is even (here e_1, f_1 denotes the standard basis of the first copy of U). Then $M^2 = 4n$ and $(M + v)/2 = (u, \alpha, 0) \in U^3 \oplus E_8(-1)^2$. This gives a primitive embedding of $NS(Y)$ in $U^3 \oplus E_8(-1)^2$, which extends the standard one of $E_8(-2) \subset U^3 \oplus E_8(-1)^2$. Now we can assume that $L = (u, 0, \alpha) \in U(2) \oplus N \oplus E_8(-1) \subset H^2(X, \mathbb{Z})$, so by [vGS, Proposition 1.8] we have $\beta^*M = \pi^*L$. Now $(L - N_1 - N_2)/2 = (u, -N_1 - N_2, \alpha)/2 \in NS(X)$. By using [vGS, Proposition 1.8] again we obtain $\pi^*((L - N_1 - N_2)/2) = (u, \frac{\alpha}{2}, \frac{\alpha}{2}, -E_1 - E_2) \in NS(\tilde{Y})$ and so $(u, \frac{\alpha}{2}, \frac{\alpha}{2}) \in NS(Y)$, this means that $M/2 \in NS(Y)$ which is not the case. Hence $(L - N_1 - N_2)/2 \notin NS(X)$, in a similar way one shows that $(L - N_1 - N_2 - N_3 - N_4)/2 \notin NS(X)$ and so we conclude that $NS(X) = \mathcal{L}_{2d}$.
2) Assume now that $\Gamma = \mathcal{L}'_{2d}$. In this case we have either
a) $L^2 \equiv 4 \mod 8$ and so $(L - N_1 - N_2)/2$ and $(L - N_3 - \ldots - N_8)/2$ are in Γ or
b) $L^2 \equiv 0 \mod 8$ and so $(L - N_1 - N_2 - N_3 - N_4)/2$ and $(L - N_5 - N_6 - N_7 - N_8)/2$ are in Γ. We do the proof assuming that we are in case a), for the case b) the proof is very similar. Let Y be a K3 surface with $\rho(Y) = 9$, Nikulin involution, Néron Severi group $NS(Y) = \mathbb{Z}M \oplus E_8(-2)$(which is $\mathcal{M}_{2d'}$ for a non-negative integer d') and $M^2 = 4n+2$, such a K3 surface exists by [vGS, Proposition 2.2, 2.3]. Moreover by [vGS, Proposition 2.7] there are line bundles L_1 and L_2 in $NS(X)$ with $\beta^*M - E_1 - E_2 = \pi^*L_1$, $\beta^*M - E_3 - \ldots - E_8 = \pi^*L_2$. Since the

embedding of $\mathbb{Z}L \oplus E_8(-2)$ in the K3 lattice is unique we may assume that $M = e_1+(2n+1)f_1$ and $L_1 = (e_1 + (2n + 1)f_1 + N_1 + N_2)/2 - N_1 - N_2 \in NS(X)$, by [vGS, Proposition 1.8] we have $\beta^*M - E_1 - E_2 = \pi^*L_1$. The class $U(2) \ni (e_1 + (2n + 1)f_1) = 2L_1 + N_1 + N_2$ is in $NS(X)$, is orthogonal to the N_i and has self intersection $8n'+4$, we call it L. By Proposition 2.1 we have $NS(X) = \mathcal{L}'_{2d}$ with $d = 4n + 2$, so we are done. □

Remark. By using the *surjectivity of the period map* one can show the existence of a K3 surface X with $NS(X) = \Gamma$, it is however difficult to show that there is an embedding of the classes N_i as irreducible (-2)-smooth curves in $NS(X)$. This is assured by the previous proposition.

From the Proposition 2.3 follows a relation between the Néron Severi group of the K3 surface Y admitting a Nikulin involution and the Néron Severi group of a K3 surface X which is the desingularization of the quotient.

Corollary 2.2. *Let Y be an algebraic K3 surface with $\rho(Y) = 9$ admitting a Nikulin involution, and let X be the desingularization of its quotient.*
(1) $NS(Y) = \mathcal{M}_{2d}$ if and only if $NS(X) = \mathcal{L}'_{4d}$;
(2) $NS(Y) = \mathcal{M}'_{4d}$ if and only if $NS(X) = \mathcal{L}_{2d}$.

Proof. The proof follows from [vGS, Proposition 2.7] and Proposition 2.3. We sketch it briefly. The proof of the direction $\Leftarrow$ of the statement follows immediately from the proof of Proposition 2.3. For the other direction we distinguish three cases (we use the notation of *loc. cit.*):
(a) Case (1), (i). Clearly $(\beta^*M - E_1 - E_2)^2 = (\pi^*L_1)^2$ and in the proofs of Proposition 2.3, case (2), and of [vGS, Proposition 2.7] it is proved that

$$L_1 = (L - N_1 - N_2)/2, \quad L_2 = (L - N_3 - \ldots - N_8)/2. \tag{2}$$

Since π is a $2:1$ map to X the previous equality becomes $4n+2-1-1 = \frac{1}{2}(L-N_1-N_2)^2 = \frac{1}{2}(L^2 - 4)$ and so $L^2 = 2(4n + 2)$. By the Proposition 2.1, where we describe the possible Néron-Severi groups of K3 surfaces with an even set, we obtain that $NS(X) = \mathcal{L}'_{2d}$, $d \equiv 2 \mod 4$.
(b) Case (1), (ii). As before, in the proof of [vGS, Proposition 2.7] it is proved that:

$$L_1 = (L - N_1 - \ldots - N_4)/2, \; L_2 = (L - N_5 - \ldots - N_8)/2. \tag{3}$$

So we obtain $4n-4 = (\beta^*M - E_1 - E_2 - E_3 - E_4)^2 = 2((L - N_1 - \ldots - N_4)/2)^2 = \frac{1}{2}(L^2 - 8)$ and so $L^2 = 2(4n)$. By the Proposition 2.1 we obtain that $NS(X) = \mathcal{L}'_{2d}$, $d \equiv 0 \mod 4$.
(c) Case (2), $M^2 = 2L^2$, and so by an argumentation as in the proof of the Proposition 2.3, case (1), we have $NS(X) = \mathcal{L}_{2d}$. □

Some explicit correspondence between the K3 surfaces Y and X are shown in the Table 1.

Remark. Let X be a K3 surface such that the lattice $\Gamma = \mathcal{L}_{2d}$ or $\mathcal{L}'_{2d}$ is primitively embedded in $NS(X)$ and $\rho(X) \geq 9$. There exists a deformation of the K3 surface $\{X_t\}$ such that $X_{\bar{t}} = X$ and X_0 is such that $NS(X_0) = \Gamma$. Let Y_0 be the K3 surface such that the desingularization of its quotient by a Nikulin involution is X_0. The Néron Severi group of Y_0 is either $\mathcal{M}'_{4d}$ or $\mathcal{M}_d$. The deformation on X induces a deformation $\{Y_t\}$ of Y_0 such that the surface $Y_{\bar{t}}$ admits a Nikulin involution and the desingularization of its quotient by the Nikulin involution is $X_{\bar{t}}$. This means that $X_{\bar{t}}$ admits an even set of eight disjoint rational curves.

In particular if X is an algebraic K3 surface such that $\mathcal{L}_{2d}$ (resp. $\mathcal{L}'_{2d}$) is primitively embedded in $NS(X)$, then X is the minimal resolution of the quotient of a K3 surface Y such that $\mathcal{M}'_{4d}$ (resp. $\mathcal{M}_d$) is primitively embedded in $NS(Y)$.

Corollary 2.3. *The coarse moduli space of* Γ*-polarized* $K3$ *surfaces (cf.* [D, p.5] *for the definition) is the quotient of*

$$\mathcal{D}_\Gamma = \{\omega \in \mathbb{P}(\Gamma^\perp \otimes_{\mathbb{Z}} \mathbb{C}) : \ \omega^2 = 0, \ \omega\bar{\omega} > 0\}$$

by an arithmetic group $O(\Gamma)$ *and has dimension eleven. The generic K3 surface with an even set of eight disjoint rational curves has Picard number nine.*

Proof. By Proposition 2.1, each K3 surface with an even set is contained in this space, on the other hand, by Proposition 2.3 each point of this space corresponds to a K3 surface with an even set of irreducible (-2)-curves. □

3. Ampleness and nefness of some divisors on X

Our next aim (cf. Section 4) is to describe projective models of K3 surfaces with an even set of eight disjoint rational curves. Here we give some results on ampleness and on nefness of divisors on such K3 surfaces. We prove moreover that the associated linear systems have no base points. These properties guarantee that the maps induced by the linear systems are regular (in fact birational) maps.

Definition 3.1. *A divisor* L *on a surface* S *is:*

- ***nef*** *if* $L^2 \geq 0$ *and* $L \cdot C \geq 0$ *for each irreducible curve* C *on* S,
- ***pseudo ample*** *(or **big and nef**)* *if* $L^2 > 0$ *and* $L \cdot C \geq 0$ *for each irreducible curve* C *on* S,
- ***ample*** *if* $L^2 > 0$ *and* $L \cdot C > 0$ *for each irreducible curve* C *on* S.

If X is a K3 surface with a line bundle L such that $L^2 \geq 0$, the condition $L \cdot C \geq 0$ for each irreducible curve C on X is equivalent to the condition $L \cdot \delta \geq 0$ for each irreducible (-2)-curve δ on X (cf. [BPV, Proposition 3.7]).

Let H be an effective divisor on a K3 surface. The intersection of H with each curve C is non-negative except when C is a component of H and C is a (-2)-curve. If the linear system $|H|$ does not have fixed components and if $H^2 > 0$, then the generic element in $|H|$ is smooth and irreducible and H is a pseudo ample divisor (cf. [SD, Proposition 2.6]). The fixed components of a linear system on a K3 surface are always (-2)-curves [SD, Paragraph 2.7.1]. Recall that by [R, Theorem p.79] if H is pseudo ample (or ample) then either $|H|$ has no fixed components or $H = aE + \Gamma$ where $|E|$ is a free pencil and Γ is an irreducible (-2)-curve such that $E\Gamma = 1$. Finally in [SD, Corollary 3.2] Saint-Donat proves that a linear system on a K3 surface has no base points outside its fixed components.

Let now H be a pseudo ample divisor on X. If $|H|$ has a fixed component B, then $H = B + M$, where M is the moving part of the linear system $|H|$. The linear system $|H|$ defines a map ϕ_H and if $H = M + B$ then $\phi_H = \phi_M$. Now we assume that $|H|$ has no fixed components (and hence no base points). The system $|H|$ defines the map:

$$\phi_H : X \longrightarrow \mathbb{P}^{p_a(H)}$$

where $p_a(H) = H^2/2 + 1$ and there are two cases (cf. [SD, Paragraph 4.1]):
(i) either ϕ_H is of degree two and its image has degree $p_a(H) - 1$ (ϕ_H is *hyperelliptic*),
(ii) or ϕ_H is birational and its image has degree $2p_a(H) - 2$.
In particular in the second case if H is ample (i.e. does not contract (-2)-curves) ϕ_H is an embedding, so H is very ample.

Proposition 3.1. *Let X be as in the Proposition 2.1. Then we may assume that L is pseudo ample and it has no fixed components.*

Proof. Since X and Y are algebraic, by using the notations of the Diagram 1, the surface $\bar{X}$ is embedded in some projective space and has eight nodes. The generic hyperplane section of $\bar{X}$ is a smooth and irreducible curve (it does not pass through the nodes). Its pull back on X is then orthogonal to $N_1, \ldots, N_8$, we call it $\mathcal{H}$, observe that $\mathcal{H} = \alpha L$ for some integer α. Since $\mathcal{H}$ is pseudo ample then L is pseudo ample too, in particular observe that $L\Gamma > 0$ for each (-2)-curve which is not one of the N_i's. If L has fixed components then by [R, Theorem p.79] it is $L = aE + \Gamma$ where $|E|$ is a free pencil and Γ an irreducible (-2)-curve such that $E\Gamma = 1$. If $\Gamma \neq N_i$ for each $i = 1, \ldots, 8$, then $0 < L\Gamma = a - 2$, which gives $a > 2$. Now $0 = LN_i = aEN_i + \Gamma N_i$, since $a > 2$, $EN_i \geq 0$, $\Gamma N_i \geq 0$ we obtain $EN_i = 0$ and $\Gamma N_i = 0$ for each i, so Γ is in $(N)^\perp$ which is not possible. If $\Gamma = N_i$ for some i, then $0 = LN_i = a - 2$ so $a = 2$ then $L = 2E + N_i$ and so $(L - N_i)/2$ is in the Néron Severi group too which is not the case. So by [SD, Proposition 2.6] we can assume that L is smooth and irreducible. □

Proposition 3.2. *Let X be as in the Proposition 2.1. If $d \geq 3$, i.e. $L^2 \geq 6$, then the class $L - \hat{N}$ in the Néron Severi group is an ample class.*

Proof. The self intersection of $L - \hat{N}$ is $(L - \hat{N})^2 = 2d - 4$, which is positive for each $d \geq 3$. So to prove that $L - \hat{N}$ is ample we have to prove that for each irreducible (-2)-curve C the intersection number $C \cdot (L - \hat{N})$ is positive.
In the proof we use the inequality:

$$(\sum_{i=1}^{n} x_i)^2 \leq n \sum_{i=1}^{n} x_i^2 \tag{4}$$

which is true for every $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Suppose that there exists an effective irreducible curve C such that $C \cdot (L - \hat{N}) \leq 0$, then we prove that $C \cdot C < -2$.
We observe that each element in the Néron Severi group is a linear combination of L and N_i with coefficients in $\frac{1}{2}\mathbb{Z}$. We consider the curve $C = aL + \sum_{i=1}^{8} b_i N_i$ where $a, b_i \in \frac{1}{2}\mathbb{Z}$. If $a = 0$ the only possible (-2)-curves are the N_i's and $N_i \cdot (L - \hat{N}) = 1$. So we can assume that $a \neq 0$. Since C is an irreducible curve, it has a non-negative intersection with all effective divisors. Hence $C \cdot L = 2da \geq 0$, so $a > 0$, and $C \cdot N_i = -2b_i \geq 0$, so $b_i \leq 0$.
Now we assume that $C \cdot (L - \hat{N}) \leq 0$, then

$$(aL + \sum_{i=1}^{8} b_i N_i) \cdot (L - \hat{N}) = 2da + \sum_{i=1}^{8} b_i \leq 0.$$

Since $b_i \leq 0$, $2da - \sum_{i=1}^{n} |b_i| \leq 0$ and so $2da \leq \sum_{i=1}^{n} |b_i|$, where each member is non negative. So it is possible to pass to the square of the relation, obtaining $4d^2a^2 \leq (\sum_{i=1}^{n} |b_i|)^2$. Using the relation (4) one has

$$4d^2a^2 \leq (\sum_{i=1}^{n} |b_i|)^2 \leq 8 \sum_{i=1}^{8} b_i^2. \tag{5}$$

Now we compute the square of C and we use the inequality (5) to estimate it:

$$C \cdot C = 2da^2 - 2\sum_{i=1}^{8}(b_i^2) \leq 2da^2 - d^2a^2.$$

If $d \geq 5$ then $\sqrt{\frac{2}{d^2-2d}} < \frac{1}{2}$, and so for $d \geq 5$ we have $C \cdot C < -2$ (because $a \geq \frac{1}{2}$). This proves the theorem in the case $d \geq 5$.
More in general for each $d \geq 3$, $\sqrt{\frac{2}{d^2-2d}} < 1$, so for the cases $d = 3$ and $d = 4$ one has to study only the case $a = \frac{1}{2}$.
For $d = 3$, then $L^2 = 6$ and so in $NS(X)$ all the elements are of the form $aL + \sum_{i=1}^{8} b_i N_i$ with $a \in \mathbb{Z}$ (and not in $\frac{1}{2}\mathbb{Z}$). Then the theorem is proved exactly in the same way as before.
Let $d = 4$. By the previous computations follows that the only possible irreducible (-2)-curves with a negative intersection with $(L - \hat{N})$ are of the form $\frac{1}{2}(L + N_1 + N_2 + N_3 + N_4) + \sum_{i=1}^{8} \beta_i N_i$ with $\beta_i \in \mathbb{Z}$, $\beta_1, \ldots, \beta_4 \leq -1$ and $\beta_5, \ldots, \beta_8 \leq 0$. It is easy to see that the only (-2)-curves of this type are $\frac{L+N_1+N_2+N_3+N_4}{2} - N_1 - N_2 - N_3 - N_4 - N_j$, $j = 5, 6, 7, 8$ and these curves have a positive intersection with $L - \hat{N}$. Then the proposition is proved also for $d = 4$. □

Proposition 3.3. *In the situation of Proposition 3.2, $m(L - \hat{N})$ and $mL - \hat{N}$ for $m \in \mathbb{Z}_{>0}$, are ample. If $d = 2$, i.e. $(L - \hat{N})^2 = 0$, then $m(L - \hat{N})$ is nef and $mL - \hat{N}$ is ample for $m \geq 2$.*

Proof. It is a similar computation as in the proof of Proposition 3.2. □

Proposition 3.4. *The divisors $L - \hat{N}$, $mL - \hat{N}$ and $m(L - \hat{N})$, $m \in \mathbb{Z}_{>0}$, do not have fixed components for $d \geq 2$.*

Proof. We proof the proposition for the divisor $L - \hat{N}$. The proof in the other cases is essentially the same.
For $d = 2$ we have $(L - \hat{N})^2 = 0$ and is nef by the Proposition 3.3 so by [R, Theorem p. 79, (b)] $L - \hat{N} = aE$ where $|E|$ is a free pencil, and so the assertion is proved in this case.
Assume $d \geq 3$, then for [R, Theorem p. 79, (d)] we have either $L - \hat{N}$ has no fixed components or $L - \hat{N} = aE + \Gamma$, where $|E|$ is a free pencil and Γ is an irreducible (-2)-curve such that $E\Gamma = 1$. We assume we are in the second case, then since $L - \hat{N}$ is ample we have

$$0 < \Gamma(L - \hat{N}) = a - 2$$

and so $a > 2$. We distinguish two cases:
1. $\Gamma = \alpha L + \sum \beta_j N_j$, $\Gamma \neq N_i$ for each i, so $\alpha \neq 0$. For each i we have:

$$1 = N_i(L - \hat{N}) = aEN_i + \Gamma N_i$$

Since $EN_i \geq 0$ and $a > 2$ then $EN_i = 0$ and so

$$1 = \Gamma N_i = (\alpha L + \sum_j \beta_j N_j)N_i = -2\beta_i.$$

We obtain $\beta_j = -1/2$ for all j, so

$$\Gamma = \alpha L - \frac{N_1 + \ldots + N_8}{2} = \alpha L - \hat{N}.$$

By considering the self-intersection of Γ we obtain

$$-2 = \alpha^2 2d - 4 \geq 6\alpha^2 - 4$$

which is positive since α is a non zero integer. So this case is not possible.
2. $\Gamma = N_i$ for some $i = 1, \ldots, 8$. We have

$$1 = N_i(L - \hat{N}) = N_i(aE + N_i) = aEN_i - 2 = a - 2$$

so $a = 3$, $L - \hat{N} = 3E + N_i$. For $j \neq i$ we have

$$1 = (L - \hat{N})N_j = 3EN_j$$

but this is impossible. Hence $L - \hat{N}$ has no base components. □

Lemma 3.1. *The map $\phi_{L-\hat{N}}$ is*

- *an embedding if $L^2 \geq 10$,*
- *a 2:1 map to $\mathbb{P}^1 \times \mathbb{P}^1$ if $L^2 = 8$,*
- *a 2:1 map to $\mathbb{P}^2$ if $L^2 = 6$.*

Proof. By the Proposition 3.2 $L - \hat{N}$ is ample and by the Proposition 3.4 $|L - \hat{N}|$ has no fixed components; for a K3 surface this implies that $|L - \hat{N}|$ has no base points too (cf. [SD, Corollary 3.2]), and it defines a map $\phi_{L-\hat{N}}$. The assertion for $L^2 = 6$ is clear since $(L - \hat{N})^2 = 2$ and hence the map $\phi_{L-\hat{N}}$ defines a double cover of $\mathbb{P}^2$. We show that in the case $L^2 = 2d \geq 10$, i.e. $d \geq 5$, the map is not hyperelliptic. By [SD, Theorem 5.2] $L - \hat{N}$ is hyperelliptic iff (i) there is an elliptic irreducible curve E with $E \cdot (L - \hat{N}) = 2$ or (ii) there is an irreducible curve B, with $p_a(B) = 2$ and $L - \hat{N} = \mathcal{O}(2B)$. The case (ii) would implies $L - \hat{N} \equiv 2B$ and so $\frac{1}{2}(L - \hat{N}) \in NS(X)$ which is not possible by the description of $NS(X)$ of Proposition 2.1. We have to exclude (i). We argue in a similar way as in Proposition 3.2. Assume that there is $E = aL + \sum b_i N_i$ an irreducible curve with $E \cdot (L - \hat{N}) = 2$. Then we show $E^2 \neq 0$. Since E is the class of an irreducible curve, $a \in \frac{1}{2}\mathbb{Z}_{>0}$ and $b_i \in \frac{1}{2}\mathbb{Z}_{\leq 0}$. We have $2 = E \cdot (L - \hat{N}) = 2da + \sum_{i=1}^{8} b_i$ and so $2da - 2 = -\sum_{i=1}^{8} b_i$ which gives together with the inequality (4):

$$4(da-1)^2 = (\sum_{i=1}^{8} |b_i|)^2 \leq 8 \sum_{i=1}^{8} |b_i|^2$$

and so $(da - 1)^2 \leq 2\sum_{i=1}^{8} b_i^2$. On the other hand we have

$$E^2 = 2da^2 - 2\sum_{i=1}^{8} b_i^2 \leq 2da^2 - (da-1)^2 = 2da^2 - d^2a^2 - 1 + 2da.$$

We have $E^2 < 0$ for $a < \frac{d-\sqrt{2d}}{d(d-2)}$ or $a > \frac{d+\sqrt{2d}}{d(d-2)}$, since $a > 1/2$ and $\frac{d-\sqrt{2d}}{d(d-2)} < \frac{1}{2}$ for each $d \geq 5$ and $\frac{d+\sqrt{2d}}{d(d-2)} < \frac{1}{2}$ for each $d \geq 6$, we obtain $E^2 < 0$ for $d \geq 6$. We analyze the case of $d = 5$. Here $L^2 = 10$ and so $a \in \mathbb{Z}_{>0}$, for $d = 5$ we have $\frac{d+\sqrt{2d}}{d(d-2)} = \frac{5+\sqrt{10}}{15} < 1$. In conclusion for each $d \geq 5$ we obtain $E^2 < 0$. In the case of $d = 4$, then we have $L^2 = 8$ and the classes $E_1 = \frac{L-N_1-N_2-N_3-N_4}{2}$, $E_2 = \frac{L-N_5-N_5-N_7-N_8}{2}$ are in the Néron Severi group. We have $E_1^2 = E_2^2 = 0$, $E_1 \cdot E_2 = 2$ and $L - \hat{N} = E_1 + E_2$, so $\phi_{L-\hat{N}}$ defines a 2:1 map to a quadric in $\mathbb{P}^1 \times \mathbb{P}^1$ (cf. [SD, Proposition 5.7]). □

Proposition 3.5. *1) Let D be the divisor $D = L - (N_1 + \ldots + N_r)$ (up to relabel the indices), $1 \leq r \leq 8$.*

- *If $NS(X) = \mathcal{L}_{2d}$, then D is pseudo ample for $d > r$;*
- *if $NS(X) = \mathcal{L}'_{2d}$, then D is nef for $d = r + 4$ and pseudo ample for $d > r + 4$,*
- *if D is pseudo ample and $NS(X) = \mathcal{L}_{2d}$ then it does not have fixed components.*

2) Let $NS(X) = \mathcal{L}'_{2d}$. Let $\bar{D} = (L - (N_1 + \ldots + N_r))/2$ with $r = 2, 6$ if $2d \equiv 4 \mod 8$ and $r = 4$ if $2d \equiv 0 \mod 8$. Then

- *the divisor $\bar{D}$ is nef and is pseudo ample whenever it has positive self intersection,*
- *if $\bar{D}$ is pseudo ample then it does not have fixed components, if $\bar{D}^2 = 0$ then the generic element in $|\bar{D}|$ is an elliptic curve.*

Proof. The arguments are similar as those used in the the proof of the Proposition 3.2 for the ampleness properties and in the proof of the Proposition 3.4 for the absence of fixed components. □

Corollary 3.1. *Let D and $\bar{D}$ be divisors as in the Proposition 3.5. We suppose that $D^2 > 0$, $\bar{D}^2 > 0$. Let C be a (-2)-curve with $C \cdot D = 0$ or $C \cdot \bar{D} = 0$. Then $C = N_i$ for some $i = 1, \dots, 8$.*

Lemma 3.2. *With the same notation as in Proposition 3.5, we have:*

- *for $NS(X) = \mathcal{L}_{2d}$ and $D^2 \geq 4$ the map ϕ_D is birational,*
- *for $\bar{D}^2 \geq 4$ the map $\phi_{\bar{D}}$ is birational.*

Proof. The proof is very similar to the proof of Lemma 3.1 and is left to the reader. □

We prove the following proposition which is a generalization of [C, Proposition 2.6] to the case of surfaces in $\mathbb{P}^n$.

Proposition 3.6. *Let F be a surface in $\mathbb{P}^n$ and let $\mathcal{N}$ be a subset of the set of nodes of F. Let $G \subset \mathbb{P}^n$ be a hypersurface s.t. $div_F(G) = 2C$ (here $div_F(G)$ denotes the divisor cut out by G on F), with C a divisor on F which is not Cartier at the points of $\mathcal{N}$. Then $\mathcal{N}$ is an even set of nodes iff G has even degree. Conversely if $\mathcal{N}$ is an even set of nodes then there is an hypersurface G as above.*

Proof. The proof is identical to the proof of [C, Proposition 2.6], we recall it briefly.
Let $\tilde{F} \longrightarrow F$ be the minimal resolution of the singularities of F, let H denote the pull-back on $\tilde{F}$ of the hyperplane section on F and let $\deg G = m$ then

$$mH \sim 2\tilde{C} + \sum \alpha_i N_i$$

where $\tilde{C}$ denotes the strict transform of C on $\tilde{F}$ and the N_i's denote the exceptional curves over the nodes. Since C is not Cartier at the singular points, the α_i are odd. Hence

$$\sum N_i \sim \delta H + 2\left([\frac{m}{2}]H - \tilde{C} - \sum [\frac{\alpha_i}{2}] N_i\right)$$

where $\delta = 0, 1$ according to m even or odd. Now if $\sum N_i$ is an even set then $\delta = 0$ and m is even. If m is even then $\delta = 0$ and so $\sum N_i$ is an even set.
On the other hand, if $\mathcal{N}$ is an even set then

$$2B \sim \sum N_i.$$

For $B \in Pic(\tilde{F})$ choose r such that $rH - B$ is linearly equivalent to an effective divisor $\tilde{C}$. Then

$$2rH \sim 2B + 2\tilde{C} = 2\tilde{C} + \sum N_i$$

so there is a hypersurface G ($\sim 2rH$) with the properties of the statement. □

From this follows a geometrical characterization of even set of nodes on K3 surfaces.

Corollary 3.2. *Let $\bar{X} \subset \mathbb{P}^{d+1}$, $d \geq 2$, be a surface of degree $2d$ with a set of m nodes $\mathcal{N}$, s.t. its minimal resolution is a K3 surface. Then*
(i) if $m = 8$, then $\mathcal{N}$ is even iff G is a quadric,
(ii) if $m = 16$ and $d \geq 3$, then $\mathcal{N}$ is even iff G is a quadric.

Proof. (*i*) Let $L := H \cap \bar{X}$ be the generic hyperplane section with $2d = L^2$ and let $\mathcal{N}$ be an even set of nodes. Then the lattice $\mathbb{Z}L \oplus N \subset NS(X)$ and we have $2\hat{N} \equiv \sum N_i$. Since the self-intersection of $L - \hat{N}$ is $2d - 4 \geq 0$ by the theorem of Riemann-Roch $L - \hat{N}$ or $-(L - \hat{N})$ is effective. Since $(L - \hat{N}) \cdot L \geq 0$, $L - \hat{N}$ is effective, so $2L \equiv 2(L - \hat{N}) + \sum N_i$. And so $G \in |2L|$ and $div_{\bar{X}}(G) = 2C = 2(L - \hat{N})$. The converse follows from the Proposition 3.6.
(*ii*) The proof in this case is essentially the same. We use the Kummer lattice K instead of the Nikulin lattice N. The lattice K is generated over $\mathbb{Q}$ by the sixteen disjoint rational (-2)-curves $K_1, \ldots, K_{16}$ and it contains the class $(K_1 + \ldots + K_{16})/2$, which we use in the proof above instead of the class $\hat{N}$ (for a precise definition of the lattice K see [N1]). □

Remark. In particular this means that if $\bar{X}$ is a K3 surface with an even set of nodes, then there exists a quadric cutting a curve on $\bar{X}$ with multiplicity two, passing through the even set of nodes (this is the condition $div_F(G) = 2C$ of the theorem).

4. Projective models

In this section we determine projective models of K3 surfaces with an even set of nodes and Picard number nine. These were already partially studied by Barth in [B2]. Here we recover, with different methods, some of these examples and we discuss many new examples. Observe that some of the cases that Barth describes require Picard number at least ten (these are case five and case four in his list (cf. Paragraph 4.5 below)).
In Section 3 we proved that the divisors L, $L - \hat{N}$, and $(L - N_1 - \ldots - N_m)/2$, $m = 2$ or $m = 4$ or $m = 6$ on X define regular maps. We use these divisors to give projective models of a K3 surface X with Néron Severi group isometric to $\mathcal{L}_{2d}$ or to $\mathcal{L}'_{2d}$ and in general we study the projective models of the same surface by using different polarizations. In particular in each case one can use as polarization L or $L - \hat{N}$, if $L^2 > 4$. The first polarization contracts the curves of the even sets to eight nodes on the surface, the second one sends these curves to lines on the projective model.
In case (1) of the Corollary 2.2 it is also possible to study the projective models given by the maps ϕ_{L_1}, resp. ϕ_{L_2} (for $L_i^2 > 0$) where L_1 and L_2 are the divisors defined in (2) or in (3). They give projective models of X in the projective space $\mathbb{P}(H^0(X, L_1))$, resp. $\mathbb{P}(H^0(X, L_2))$ or give $2:1$ maps to the images of X in these spaces. If the maps are not $2:1$, the image of X contains nodes and lines, which on X form an even set. The image of X under $\phi_{L_1} \times \phi_{L_2} : X \to \mathbb{P}(H^0(X, L_1)) \times \mathbb{P}(H^0(X, L_2))$ is a surface, which is the image of $Y \hookrightarrow \mathbb{P}^{h^0(L_1)+h^0(L_2)-1}$ under the projection to the eigenspaces: $\mathbb{P}^{h^0(L_1)+h^0(L_2)-1} \longrightarrow \mathbb{P}(H^0(X, L_1)) \times \mathbb{P}(H^0(X, L_2))$. Indeed put $h^0(L_1) = m_1 + 1$, $h^0(L_2) = m_2 + 1$, then $h^0(M) = m_1 + m_2 + 2$ and we have a commutative diagram:

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, dashed] & \mathbb{P}^{m_1+m_2+1} \arrow[r, "v"] & \mathbb{P}^r \supset V_{m_1+m_2+1} \arrow[d, "p"] \\
X \arrow[r] & \mathbb{P}^{m_1} \times \mathbb{P}^{m_2} \arrow[r, "s"] & \mathbb{P}^{r'} \supset S_{m_1+m_2}
\end{tikzcd}$$

Here the rational map between Y and X follows from Diagram 1, v is the Veronese embedding and so $r = \frac{(m_1+m_2+2)(m_1+m_2+3)}{2}$, s is the Segre embedding and so $r' = (m_1+1)(m_2+1) - 1$, p is the projection, $V_{m_1+m_2+1}$ is the image of $\mathbb{P}^{m_1+m_2+1}$ in $\mathbb{P}^r$ and $S_{m_1+m_2}$ is the image of $\mathbb{P}^{m_1} \times \mathbb{P}^{m_2}$ in $\mathbb{P}^{r'}$. The Nikulin involution on $\mathbb{P}^{m_1+m_2+1}$ operates as:

$$(x_0 : \ldots : x_{m_1} : y_0 : \ldots : y_{m_2}) \mapsto (x_0 : \ldots : x_{m_1} : -y_0 : \ldots : -y_{m_2})$$

which induces an operation on the coordinates of $\mathbb{P}^r$ as

$$(x_0^2 : x_1^2 : \ldots : y_0^2 : y_1^2 : \ldots : y_{m_2-1}y_{m_2} : x_0y_1 : x_0y_2 : \ldots : x_{m_1}y_{m_2}) \mapsto$$
$$(x_0^2 : x_1^2 : \ldots : y_0^2 : y_1^2 : \ldots : y_{m_2-1}y_{m_2} : -x_0y_1 : -x_0y_2 : \ldots : -x_{m_1}y_{m_2}).$$

The projection p goes to the invariant space $\mathbb{P}^{r'}$ with coordinates $(x_0y_1 : x_0y_2 : \ldots : x_{m_1}y_{m_2})$ and so the equations of the image of Y in $V_{m_1+m_2+1}$ in these coordinates give equations for the image of the surface X in $S_{m_1+m_2}$ (cf. also [vGS, Proposition 2.7]).
Observe that the sum of the divisors L_1 and L_2 is exactly $L-\hat{N}$. From now on if $NS(X) = \mathcal{L}'_{2d}$ and $d/2$ is odd then $L_1 := (L - N_1 - N_2)/2$, $L_2 := (L - N_3 - \ldots - N_8)/2$, if $NS(X) = \mathcal{L}'_{2d}$ and $d/2$ is even then $L_1 := (L - N_1 - \ldots - N_4)/2$, $L_2 := (L - N_5 - \ldots - N_8)/2$.

In the case $NS(X) = \mathcal{L}_{2d}$ the construction above holds if instead of L_1 and L_2 we take L and $L - \hat{N}$.

4.1. **The case of $L^2 = 2$, $NS(X) = \mathcal{L}_2$, the polarization L.** Since L is pseudo ample by the Proposition 3.1 the linear system $|L|$ defines a $2:1$ map $X' \longrightarrow \mathbb{P}^2$ ramified on a sextic curve with eight nodes where X' is the surface X after contraction of the (-2)-curves. More precisely we have a commutative diagram:

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ \widetilde{\mathbb{P}}^2 & \longrightarrow & \mathbb{P}^2 \end{array}$$

where $\widetilde{\mathbb{P}}^2$ is the blow up of $\mathbb{P}^2$ at the eight double points of the sextic. By general results on cyclic coverings the pull back of the branching sextic on X is $3L - (N_1 + \ldots + N_8) = 3L - 2\hat{N} = (L - \hat{N}) + (2L - \hat{N})$. Now $(L - \hat{N})^2 = 2 - 4 = -2$, $L \cdot (L - \hat{N}) = 2$ and so by using Riemann-Roch Theorem the divisor $L - \hat{N}$ is effective, and is a rational curve of degree two on $\mathbb{P}^2$. Observe that its image is an irreducible conic, in fact we are assuming that X' has exactly eight nodes and no other singularities. On the other hand $(2L - \hat{N})^2 = 8 - 4 = 4$ so by Proposition 3.4 and by [SD, Proposition 2.6] the generic member in $|2L - \hat{N}|$ is an irreducible curve of genus three, and its image in $\mathbb{P}^2$ is a curve of degree four (and in fact genus $3 = (4-1)(4-2)/2$). In both cases we have $(L - \hat{N}) \cdot N_i = (2L - \hat{N}) \cdot N_i = 1$ and so the curves intersect at the points which are the images of the curves N_i in $\mathbb{P}^2$. This is the first case in the paper of Barth, [B2].

4.2. **The case of $L^2 = 4$, $NS(X) = \mathcal{L}_4$, the polarization L.** By the Proposition 3.1 the linear system $|L|$ defines a birational map ϕ_L from X to a quartic surface in $\mathbb{P}^3$, the curves N_i are contracted to nodes. In this case $(L - \hat{N})^2 = 0$ and by the Proposition 3.4 $|L - \hat{N}|$ has no base components, so the generic member in the system is an irreducible elliptic curve (observe that by the structure of the Néron Severi group it cannot be the multiple of an elliptic curve). Since $L \cdot (L - \hat{N}) = 4$ the elliptic curve is sent to a quartic curve in $\mathbb{P}^3$ and is a complete intersection of two quadrics (observe that it cannot be a plane quartic since this has genus three). Moreover since $(L - \hat{N}) \cdot N_i = 1$, the quartic contains the nodes. There is a third quadric passing through the nodes, in fact $h^1(L - \hat{N}) = 0$ ([SD, Proposition 2.6]) hence $h^0(L - \hat{N}) = 2$ and again by loc. cit. $h^0(2(L - \hat{N})) = 3$. Now $2(L - \hat{N}) = 2L - (N_1 + \ldots + N_8)$ and the image of these divisors are precisely the quadrics which vanish on the eight singular points (cf. Corollary 3.2). Let s_1, s_2 be a basis of $H^0(L - \hat{N})$ then s_1^2, s_1s_2, s_2^2 is a basis of $H^0(2(L - \hat{N}))$ and these are the three quadrics through the nodes. This is the case three of Barth [B2] and by the Table 1 it corresponds to the case of $\mathcal{M}'_8$ of [vGS].

TABLE 1. Néron Severi lattices and projective models

	X		Y
	$NS(X) = \mathcal{L}_2$		$NS(Y) = \mathcal{M}'_4$
ϕ_L	double plane (singular sextic)	ϕ_M	smooth quartic in $\mathbb{P}^3$
	$NS(X) = \mathcal{L}_4$		$NS(Y) = \mathcal{M}'_8$
ϕ_L	quartic with even set of nodes	ϕ_M	complete intersection in $\mathbb{P}^5$
	$NS(X) = \mathcal{L}'_4$		$NS(Y) = \mathcal{M}_2$
ϕ_L	double cover of a cone	ϕ_M	double plane
ϕ_{L_1}	elliptic fibration		
	$NS(X) = \mathcal{L}_6$		$NS(Y) = \mathcal{M}'_{12}$
ϕ_L	singular complete intersection in $\mathbb{P}^4$	ϕ_M	projective model in $\mathbb{P}^7$
$\phi_{L-\hat{N}}$	double plane (smooth sextic)		
$\phi_L \times \phi_{L-\hat{N}}$	complete intersection in $\mathbb{P}^4 \times \mathbb{P}^2$		
	$NS(X) = \mathcal{L}_8$		$NS(Y) = \mathcal{M}'_{16}$
ϕ_L	singular complete intersection in $\mathbb{P}^5$	ϕ_M	projective model in $\mathbb{P}^9$
$\phi_{L-\hat{N}}$	smooth quartic in $\mathbb{P}^3$		
	$NS(X) = \mathcal{L}'_8$		$NS(Y) = \mathcal{M}_4$
ϕ_L	singular complete intersection in $\mathbb{P}^5$	ϕ_M	smooth quartic in $\mathbb{P}^3$
$\phi_{L-\hat{N}}$	double cover of a quadric		
	$NS(X) = \mathcal{L}_{10}$		$NS(Y) = \mathcal{M}'_{20}$
$\phi_{L-\hat{N}}$	smooth complete intersection in $\mathbb{P}^4$	ϕ_M	projective model in $\mathbb{P}^{11}$
$\phi_{L-\sum_{i=1}^4 N_i}$	double cover of a plane		
	$NS(X) = \mathcal{L}_{12}$		$NS(Y) = \mathcal{M}'_{24}$
$\phi_{L-\hat{N}}$	smooth complete intersection in $\mathbb{P}^5$	ϕ_M	projective model in $\mathbb{P}^{13}$
$\phi_{L-\sum_{i=1}^4 N_i}$	singular quartic in $\mathbb{P}^3$		
	(mixed even set with conics)		
	$NS(X) = \mathcal{L}'_{12}$		$NS(Y) = \mathcal{M}_6$
$\phi_{L-\hat{N}}$	smooth complete intersection in $\mathbb{P}^5$	ϕ_M	complete intersection in $\mathbb{P}^4$
$\phi_{L_2} \times \phi_{L_1}$	surface of bidegree $(2,3)$ in $\mathbb{P}^1 \times \mathbb{P}^2$		
	$NS(X) = \mathcal{L}'_{16}$		$NS(Y) = \mathcal{M}_8$
$\phi_{L_1} \times \phi_{L_2}$	complete intersection in $\mathbb{P}^2 \times \mathbb{P}^2$	ϕ_M	complete intersection in $\mathbb{P}^5$
	$NS(X) = \mathcal{L}'_{24}$		$NS(Y) = \mathcal{M}_{12}$
$\phi_{L_1} \times \phi_{L_2}$	complete intersection in $\mathbb{P}^3 \times \mathbb{P}^3$	ϕ_M	complete intersection in $\mathbb{P}^7$

4.3. The case of $L^2 = 4$, $NS(X) = \mathcal{L}'_4$.

(a) The polarization L. We may assume that the class $(L/2, v/2)$ is equal to $(L/2, (-N_1 - N_2)/2)$. By the Proposition 3.5 and [SD, Proposition 2.6] this defines a pencil of elliptic curves which we denote by E. Observe that $L = 2E + N_1 + N_2$ with $N_i \cdot E = 1$, hence by [SD, Proposition 5.7, (iii), a)] L defines a $2:1$ map to a cone of $\mathbb{P}^3$. The pencil $|E|$ corresponds to the system of lines through the vertex of the cone under this map. The class $C_2 := E - \hat{N} + N_1 + N_2 = L/2 + (-N_3 - \ldots - N_8)/2$ is effective with $C_2 \cdot N_i = 1$, similarly we may assume that the class $C_6 := 3L/2 + (-N_3 - \ldots - N_8)/2$ is an irreducible curve (it follows by Proposition 3.5), with $C_6 \cdot N_i = 1$, moreover $C_2 \cdot C_6 = 0$, $C_2 \cdot L = 2$ and $C_6 \cdot L = 6$. Let $c_2 := \varphi_L(C_2)$ and $c_6 := \varphi_L(C_6)$. These two curves meet on the cone at the images of N_i, $i = 3, \ldots, 8$. Their union is a curve of degree eight, which is the branch divisor of the covering. In fact if C_2 is not a component of the branch divisor then $\varphi_L(C_2)$ has degree one and so is a line. But this means that $C_2 \in |E|$ which is not the case. Hence C_2 is a component of the branch divisor and c_2 is a conic. If now C_6 is not in the branch divisor, we have $\deg c_6 = 3$, and $c_2 \cdot c_6 = 6$, but then c_6 is contained in the plane of c_2 and on the cone too, which is impossible. Hence C_6 is also a component of the branch divisor. Finally observe that $\varphi_L(N_i) = Q$, for $i = 1, 2$ where Q is the vertex of the cone. This surface is also described in [vGS, Paragraph 3.2].

(b) An elliptic fibration. Now we describe the elliptic fibration on X defined by the divisor E. We consider the rational curve $C_2 = L/2 + (-N_3 - \ldots - N_8)/2$, it has intersection one with the class of the fiber E. So C_2 is a section of the fibration $\phi_E : X \to \mathbb{P}^1$ and the classes E and C_2 generate a lattice isometric to U.

Since the six (-2)-curves $N_3,\ N_4, \ldots, N_8$ are orthogonal to E, they are the components of some reducible fibers. All these curves intersect the section C_2 so they are components of six different reducible fibers. The rational curve N_1 is another section of the fibration (because its intersection with E is one). The Néron Severi group is generated over $\mathbb{Q}$ by the classes E of the fiber, by C_2, by the components N_i, $i = 3, \ldots, 8$ of the reducible fibers and by the other section N_1. The Néron Severi group of an elliptic fibration admitting a section is generated by the class of the fiber, by the zero section, by the irreducible components of the reducible fibers (not meeting the zero section) and by other sections. Since the Picard number is nine the six reducible fibers containing N_i, $i = 3, \ldots, 8$ are the only reducible fibers of the fibration, they are all of type I_2 (two rational curves meeting in two distinct points). The Euler characteristic of a K3 surface is 24 and is the sum of the Euler characteristics of the singular fibers. The singular irreducible fibers in the generic case are of type I_1 (singular irreducible curve with a node). Each fiber of type I_1 has Euler characteristic one, and each fiber of type I_2 has Euler characteristic equal to two. By the computation on the Euler characteristic it is clear that there are twelve singular fibers of type I_1 and six of type I_2. There are two independent sections, so the rank of the Mordell-Weil lattice is one. One of these sections (the zero section) is the curve C_2, mapped by ϕ_L to the conic in the branch locus on the cone. Other sections correspond to the curves N_1 and N_2, these are both mapped to the vertex of the cone.

Since X is a double cover of a cone, it admits an involution j. This involution fixes the classes N_i, $i = 3, \ldots, 8$, because they correspond to the intersection points between the conic and the sextic in the branch locus on the cone; it fixes the class C_2, and switches the classes N_1 and N_2. The involution j fixes also the class L which defines the double cover ϕ_L. On the fibration the involution j fixes the class of the fiber E and so it acts on the base, $\mathbb{P}^1$, of the fibration as the indentity, it fixes the zero section, which corresponds to the class C_2, and switches the other two independent sections N_1 and N_2. On the reducible fiber the involution j clearly fixes the component N_i and it fixes the other component $E - N_i$ too, since it fixes the fiber and a reducible fiber has two components. On $E - N_i$ the involution j switches the points P_1 and P_2, which are the points of intersection between the fiber and the sections N_1, respectively N_2.

The surface X admits an even set of eight disjoint rational curves, so it is the minimal resolution of the quotient of a K3 surface Y by a Nikulin involution. The elliptic fibration of X on $\mathbb{P}^1$ induces a fibration of $\widetilde{Y}$ (the blow up of Y) on $\mathbb{P}^1$ and so of Y (cf. Diagram 1). Let E denote the generic fiber of the fibration on X and A the generic fiber of the fibration on

$\widetilde{Y}$. By the Hurwitz formula, we have

$$2g(A)-2=2(2g(E)-2)+\deg R$$

where R is the branch divisor. Since X has an elliptic fibration we have $g(E)=1$ and $\deg R=2$ because the involution ramifies on the points of intersection $E\cap N_1$ and $E\cap N_2$. So we find $2g(A)-2=2(2-2)+2$, hence the generic fiber of the fibration $\widetilde{Y}\to\mathbb{P}^1$ is hyperelliptic of genus two.

4.4. **The case of $L^2=6$, $NS(X)=\mathcal{L}_6$.**

(a) The polarization $L-\hat{N}$. In this case $(L-\hat{N})^2=2$ by Lemma 3.1 it defines a $2:1$ map to $\mathbb{P}^2$. The curves N_i are mapped to lines in the plane. Let $l:=\phi_{L-\hat{N}}(L-\hat{N})$, then for each curve C in the plane we have the formula $\phi^*_{L-\hat{N}}(l)\cdot\phi^*_{L-\hat{N}}(C)=2(C\cdot l)$. Since $(L-\hat{N})\cdot N_i=1$ the curves N_i are contained in the preimage $\phi^{-1}_{L-\hat{N}}(T_i)$ where T_i are lines which are tritangents to the branch divisor, and so $\varphi^*_{L-\hat{N}}(T_i)=N_i+N_i'$. The curves N_i,N_i' meet in three points. Barth in [B2, Paragraph 2] shows that there is a quartic in $\mathbb{P}^2$ meeting the branch sextic at the tangency points.

(b) The polarization L. We consider the projective model of X as complete intersection of a cubic and a quadric hypersurface, it has eight nodes and the map is $\phi_L:X\to\mathbb{P}^4$. The curve $L-\hat{N}$ (cf. Proposition 3.4) has degree $6=L\cdot(L-\hat{N})$ and genus $(L-\hat{N})^2/2+1=2$. Since $(L-\hat{N})\cdot N_i=1$, $i=1,\dots,8$ its image in $\mathbb{P}^4$ passes through the eight singular points. This curve is contained in the intersection of three quadrics in $\mathbb{P}^4$, in fact $h^0(2L-(L-\hat{N}))=(L+\hat{N})^2/2+2=3$. The eight singular points of the surface are contained in three more quadrics, in fact $h^0(2L-(\sum_{i=1}^8 N_i))=6$ (cf. Corollary 3.2).

We consider now the linear system $|L-\hat{N}|$ associated to the hyperplane sections passing through the eight singular points of the image of X in $\mathbb{P}^4$. We have $h^0(L-\hat{N})=3$ and let l_1, l_2, l_3 be its generators. The six elements l_1^2, l_2^2, l_3^2, $l_1\cdot l_2$, $l_1\cdot l_3$, $l_2\cdot l_3$ span $|2L-\sum_{i=1}^8 N_i|\cong\mathbb{P}^5$ (these are the quadrics passing through the nodes).

(c) The map $\phi_L\times\phi_{L-\hat{N}}$. In [vGS, Paragraph 3.9] the K3 surface Y admitting a Nikulin involution with Néron Severi group $\mathcal{M}_{12}'$ is described. Its quotient $\bar{X}$ is birational to a K3 surface which is complete intersection of a hypersurface of bidegree $(2,0)$ and three hypersurfaces of bidegree $(1,1)$ in $\mathbb{P}^4\times\mathbb{P}^2$ (for a more detailed description of this complete intersection see the Section 5). The minimal resolution of the quotient $\bar{X}$ is the K3 surface X with $NS(X)=\mathcal{L}_6$. The projection of X to the first factor is defined by the divisor L and to the second one by $L-\hat{N}$. The first projection contracts eight disjoint rational curves and the same curves are sent to eight lines by the second projection (the $2:1$ map to $\mathbb{P}^2$).

4.5. **The case of $L^2=8$, $NS(X)=\mathcal{L}_8$.**

(a) The polarization $L-\hat{N}$. We have $(L-\hat{N})^2=4$ and the map $\phi_{L-\hat{N}}:X\longrightarrow\mathbb{P}^3$ exhibits X as a quartic surface in $\mathbb{P}^3$ with eight disjoint lines. This case is studied by Barth in [B2]. He describes two conditions to have an even set. The second one is not satisfied in our case, since it requires Picard number at least ten. In fact he shows that in this case there are two skew lines Z_1, Z_2 on the quartic surface with Z_1 meeting four lines and skipping the other four lines, and viceversa for Z_2. An easy computation shows that the intersection matrix of the hyperplane section, of the lines N_i and of Z_1 (or Z_2) has rank ten.

Barth's first condition says that there is an elliptic quartic curve in $\mathbb{P}^3$ which meets in two points four rational curves and skips the other four. In term of classes in the Néron Severi lattice this means that there is a curve $E=\alpha L+\sum b_iN_i$ with $E^2=0$, $EN_i=2$ for $i=1,\dots,4$

and $EN_i = 0$ for $i = 5, \dots, 8$. By using the intersection products we obtain $\alpha = 1$, $b_i = -1$ for $i = 1, \dots, 4$ and $b_i = 0$ for $i = 5, \dots, 8$. So the elliptic curve is $E = L - N_1 - \dots - N_4$ and in fact $E \cdot (L - \hat{N}) = 4$ which is the degree of E in $\mathbb{P}^3$. Similarly the curve $L - N_5 - \dots - N_8$ meets the other four curves and skips the first four. Finally observe that these divisors are not studied in Proposition 3.5, with the notation there this is the case $d = r$ and the proof does not work in this case.

We describe briefly the elliptic fibration defined by E. Since $E \cdot N_i = 0$, for $i = 5, \dots, 8$ these are components of reducible fibers. On the other hand the curves N_i, $i = 1, \dots, 4$ are bisections of the fibration. The curves $L - N_2 - N_3 - N_4 - N_j - N_k$ with $j \neq k$ and $j, k = 5, 6, 7, 8$ are rational (-2)-curves which meet E in two points and N_j, N_k in two points as well. Hence they are also bisections of the fibration, and since a bisection meets also the singular fiber in two points, the curves $N_5, \dots, N_8$ are contained in four different singular fibers, which are of type I_2. The remaining singular fibers are of type I_1, and we have 16 of them. This fibration does not admit sections. In this case the even set consists of four bisections and of four components of the singular fibers I_2 (these are all disjoint).

(b) The polarization L. We consider the projective model of X given by the map $\phi_L : X \to \mathbb{P}^5$, this is a complete intersection of three quadrics and has eight nodes. The generic element in $|L - \hat{N}|$ is a curve of degree $8 = L \cdot (L - \hat{N})$ and genus $(L - \hat{N})^2/2 + 1 = 3$ (cf. Proposition 3.4). Since $(L - \hat{N}) \cdot N_i = 1$, $i = 1, \dots, 8$ the image of the curve $L - \hat{N}$ in $\mathbb{P}^5$ passes through the eight singular points, moreover this divisor is not Cartier at the nodes. By the Corollary 3.2 there exists a quadric G which cuts on the surface the curve $L - \hat{N}$ passing through all the singular points, so this curve is contained in the intersection of four quadrics in $\mathbb{P}^5$, in fact $h^0(2L - (L - \hat{N})) = (L + \hat{N})^2/2 + 2 = 4$. The quadric G must cut the image of $L - \hat{N}$ with multiplicity two since $\deg(X) = 8$ and the intersection has degree 16. The eight singular points of the surface are contained in the intersection of ten quadrics, in fact $h^0(2L - (\sum_{i=1}^{8} N_i)) = 10$. We consider the linear system $|L - \hat{N}|$ associated to the hyperplane section passing through the eight singular points. We have $h^0(L - \hat{N}) = 4$ and we call l_1, l_2, l_3, l_4 its generators. The ten elements l_1^2, l_2^2, l_3^2, l_4^2, l_1l_2, l_1l_3, l_1l_4, l_2l_3, l_2l_4, l_3l_4 span $|2L - \sum_{i=1}^{8} N_i|$.

(c) The map $\phi_L \times \phi_{L-\hat{N}}$. This K3 surface is the minimal resolution of the quotient of a K3 surface Y by a Nikulin involution. The Néron Severi group of Y is $\mathcal{M}'_{16}$ by the Table 1, and M is the ample class on Y with $M^2 = 16$. This gives an immersion of Y in $\mathbb{P}^9$, and the action of the Nikulin involution is induced by $(x_0 : \dots : x_5 : y_0 : \dots : y_3) \mapsto (x_0 : \dots : x_5 : -y_0 : \dots : -y_3)$. By the projection formula we have $H^0(Y, M) \cong H^0(X, L) \oplus H^0(X, L - \hat{N})$, with $h^0(X, L) = 6$, $h^0(X, L - \hat{N}) = 4$. Now

$$S^2H^0(Y, M) = (S^2H^0(X, L) \oplus S^2H^0(X, L - \hat{N})) \oplus (H^0(X, L) \otimes H^0(X, L - \hat{N})).$$

This has dimension $55 = (21 + 10) + 24$. On the other hand

$$H^0(Y, 2M) \cong H^0(X, 2L) \oplus H^0(X, 2L - \hat{N})$$

and the dimensions are $34 = 18 + 16$. This shows that there are $(21 + 10) - 18 = 13$ invariant quadrics and $24 - 16 = 8$ antiinvariant quadrics $Q_i(x, y)$, $i = 1 \dots, 8$ in the ideal of Y. Since the quadrics in four variables are only ten, there are three quadrics $q_1(x_0, \dots, x_5)$, $q_2(x_0, \dots, x_5)$, $q_3(x_0, \dots, x_5)$ in the ideal of Y. The map $\phi_L \times \phi_{L-\hat{N}}$ sends X to the product $\mathbb{P}^5 \times \mathbb{P}^3$ and its image is the image of $Y \subset \mathbb{P}^9$ into the product of the eigenspaces, hence it is contained in three quadrics $q_1(x_0, \dots, x_5)$, $q_2(x_0, \dots, x_5)$, $q_3(x_0, \dots, x_5)$ of bidegree $(2, 0)$ and eight quadrics $Q_i(x, y)$, $i = 1, \dots, 8$, of bidegree $(1, 1)$ in particular it is not a complete intersection of quadrics. The quadrics $q_1(x_0,\ldots,x_5)$, $q_2(x_0,\ldots,x_5)$, $q_3(x_0,\ldots,x_5)$ define the image of Y in $\mathbb{P}^5$, which is X with the polarization L. Since the fixed points of the Nikulin involution are contained in the space $y_0=\ldots=y_3=0$, then the projection of the ten quadrics of the kind $q(x)-q'(y)=0$ to $\mathbb{P}^5$ are ten quadrics cutting out the set of nodes on $X\subset\mathbb{P}^5$. The projection to $\mathbb{P}^3$ is X with the polarization $L-\hat{N}$ and is a quartic. One can obtain an equation for the quartic in the following way: a point $x\in X\subset\mathbb{P}_5$ has a non-trivial counterimage if there is a non-trivial solution of $Q_i(x,y)=\sum_{j=0}^{5}a_{ij}(y)x_j=0$, $i=1,\ldots,8$ which for a fixed x is a linear system of eight equations in six variables. Hence all the 6×6 minors of the matrix $(a_{ij}(y))$ are zero. Each of these is a sextic surface of $\mathbb{P}^3$ vanishing on $X\subset\mathbb{P}^3$. Since this is a surface of degree four, each of them splits into a product $q(x)\cdot p_4(x)$ where $p_4(x)=0$ is an equation of $X\subset\mathbb{P}^3$.

4.6. **The case of $L^2=8$, $NS(X)=\mathcal{L}_8'$.**
(a) The polarization $L-\hat{N}$. In this case we have the divisor $E_1:=(L/2,v/2)$ with $v^2=-8$ and $v=-N_1-N_2-N_3-N_4$, and also the divisor $E_2:=(L/2,v'/2)$ with $v'=-N_5-N_6-N_7-N_8$ so $(L/2,v/2)^2=(L/2,v'/2)^2=0$ and $L-\hat{N}=(L/2,v/2)+(L/2,-v/2)$ is the sum of two elliptic curves (cf. Proposition 3.5). This is a $2:1$ map to $\mathbb{P}^1\times\mathbb{P}^1$ (by Lemma 3.1) and the curves N_i are sent to lines on the quadric. Moreover since $E_1\cdot N_i=1$ and $E_2\cdot N_i=0$ for $i=1,2,3,4$ the images of these lines belong to the same ruling on the quadric and the images of N_i, $i=5,6,7,8$ belong to the other ruling. By a similar computation as in Paragraph 4.4 the curves N_i are one of the two components of the preimage of a curve on the quadric which splits on X, hence $\phi_{L-\hat{N}}(N_i)=T_i$ and these are bitangents to the branch curve of bidegree $(4,4)$. Let $\phi^*_{L-\hat{N}}(T_i)=N_i+N_i'$ then

$$\mathcal{O}_{B'}(N_1+\ldots+N_8+N_1'+\ldots+N_8')=\mathcal{O}_{B'}(4(L-\hat{N}))=\mathcal{O}_{B'}(2(2(L-\hat{N})))$$

By Proposition 3.4 $2(L-\hat{N})$ is a curve, and has bidegree $(2,2)$ on the quadric. Hence the divisor cut out by $N_1+\ldots+N_8'$ is two times the divisor cut out by $2(L-\hat{N})+\mu$ where μ is a 2-torsion element in the Picard group. Barth shows in [B2], case six, that such an element does not exist. This implies that the tangency points of the T_i on the quadric are cut out by a curve of bidegree $(2,2)$.
(b) The polarization L. All the considerations of the Paragraph 4.5, case **(b)** are true. Moreover there are two elliptic curves $L_1=\frac{L-N_1-N_2-N_3-N_4}{2}$, $L_2=\frac{L-N_5-N_6-N_7-N_8}{2}$ passing through four of the eight singular points each and not passing through the other four. Obviously also in this case the image of $2(L-\hat{N})=2(L_1+L_2)$ is cut out by a quadric. By the Table 1 this case corresponds to a K3 surface Y with $NS(Y)=\mathcal{M}_4$. After a change of coordinates the surface X can be written in the form

$$q(z_0,\ldots,z_5)=0,\ z_0z_1-z_4^2=0,\ z_2z_3-z_5^2=0$$

and there are four singularities on $z_0=z_1=z_4=0$ and four on $z_2=z_3=z_5=0$ (the two copies of $\mathbb{P}^2$ which are the vertices of the cones). Now the quadrics of the kind $z_iz_j=0$ with $i=0,1$ and $j=2,3$ meet the K3 surface in two curves C_i, C_j with multiplicity two, hence $2C_i\in|L-(N_1+\ldots+N_4)|$ and $2C_j\in|L-(N_5+\ldots+N_8)|$ and so C_i, C_j are in the linear system of L_1, resp. of L_2.

4.7. **The case of $L^2=10$, $NS(X)=\mathcal{L}_{10}$.**
(a) The polarization $L-\hat{N}$. Since $(L-\hat{N})^2=6$, then the projective model of X is a complete intersection of a quadric and a cubic hypersurfaces in $\mathbb{P}^4$ with an even set of eight lines (cf. Lemma 3.1).

(b) The polarizations $L - N_1 - N_2 - N_3 - N_4$ and $L - N_5 - N_6 - N_7 - N_8$. The divisors $L-N_1-N_2-N_3-N_4$ and $L-N_5-N_6-N_7-N_8$ are pseudo ample classes by the Proposition 3.5. They define two maps $2:1$ to $\mathbb{P}^2$. Each of these maps contracts four curves of the eight rational curves N_i and maps the other four in four conics.

4.8. **The case of $L^2 = 12$, $NS(X) = \mathcal{L}_{12}$.**
(a)The polarization $L - \hat{N}$. Since $(L-\hat{N})^2 = 8$ the projective model of X is a K3 surface in $\mathbb{P}^5$ with an even set of eight disjoint lines.
(b) The polarizations $L - N_1 - N_2 - N_3 - N_4$ and $L - N_5 - N_6 - N_7 - N_8$. The curves $E_1 = L - N_1 - N_2 - N_3 - N_4$ and $E_2 = L - N_5 - N_6 - N_7 - N_8$ have self intersection four, so they define two maps to $\mathbb{P}^3$ (by Lemma 3.2). The map ϕ_{E_1} contracts the four curves N_i, $i = 1, \dots, 4$ and sends the other in four conics. The map ϕ_{E_2} contracts the other four curves and sends N_i, $i = 1, \dots, 4$ in conics.

4.9. **The case of $L^2 = 12$, $NS(X) = \mathcal{L}'_{12}$.**
(a)The polarization $L - \hat{N}$. Observe that the considerations of Paragraph 4.8, **(a)** are true also in this case. Moreover there are two curves $C_1 = \frac{L-N_1-N_2-N_3-N_4-N_5-N_6}{2}$ and $C_2 = \frac{L-N_7-N_8}{2}$ intersecting respectively six and two of the lines N_i in one point. The curve C_1 has degree three and genus one. The curve C_2 has degree five and genus two.
(b) The map $\phi_{L_1} \times \phi_{L_2}$. The intersection properties of L_1 and L_2 are $L_1 \cdot L_1 = 2$, $L_2 \cdot L_2 = 0$ and $L_1 \cdot L_2 = 3$. The K3 surface X is the minimal rsolution of the quotient $\bar{X}$ of a K3 surface Y admitting a Nikulin involution with $NS(Y) = \mathcal{M}_6$ which is described in [vGS, Paragraph 3.3]. The surface $\bar{X}$ has bidegree $(2,3)$ in $\mathbb{P}^1 \times \mathbb{P}^2$. The maps ϕ_{L_1} and ϕ_{L_2} are respectively the projection to the second and to the first projective space.
The map $\phi_{L_1} : X \to \mathbb{P}^2$ is a 2:1 map. It contracts the six rational curves $N_3, \dots, N_8$ to six nodes of the branch sextic and the two curves N_1 and N_2 are mapped to lines in $\mathbb{P}^2$ which are tritangent to the branch locus. The map $\phi_{L_2} : X \to \mathbb{P}^1$ is an elliptic fibration, it contracts the two rational curves N_1, N_2, whence the curves $N_3, \dots, N_8$ are six independent sections of the fibration. This fibration has two reducible fibers of type I_2 (made up by the classes N_1, $E_2 - N_1$ and N_2, $E_2 - N_2$).
The Segre map s sends $\mathbb{P}^1 \times \mathbb{P}^2$ in $\mathbb{P}^5$.

$$\begin{tikzcd} & \mathbb{P}^1 & & \\ X \arrow[ur, "\phi_{L_2}"] \arrow[dr, "\phi_{L_1}"'] \arrow[rr, "\phi_{L_2}\times\phi_{L_1}"] & & \mathbb{P}^1\times\mathbb{P}^2 \arrow[ul] \arrow[dl] \arrow[r, "s"] & \mathbb{P}^5 \\ & \mathbb{P}^2 & & \end{tikzcd}$$

Observe that the map $s \circ (\phi_{L_2} \times \phi_{L_1}) : X \longrightarrow \mathbb{P}^5$ is the map $\phi_{L_1+L_2} = \phi_{L-\hat{N}}$ (since $L_1 + L_2 = L - \hat{N}$). Indeed let s_1, s_2, s_3 be a basis of $H^0(L_1)$, and s_4, s_5 be a basis of $H^0(L_2)$. Then the products s_1s_4, s_1s_5, s_2s_4, s_2s_5, s_3s_4, s_3s_5 are linear independent sections in $H^0(L_1 + L_2)$ and define the Segre embedding of X in $\mathbb{P}^1 \times \mathbb{P}^2$. Since $h^0(L_1 + L_2) = 6$ then the map $\phi_{L_1+L_2}$ is exactly the map $s \circ (\phi_{L_2} \times \phi_{L_1})$.

4.10. **The case of $L^2 = 16$, $NS(X) = \mathcal{L}'_{16}$, the map $\phi_{L_1} \times \phi_{L_2}$.** The intersection properties of L_1 and L_2 are $L_1 \cdot L_1 = 2$, $L_2 \cdot L_2 = 2$ and $L_1 \cdot L_2 = 4$. In [vGS, Paragraph 3.6] is described the K3 surface Y admitting a Nikulin involution with Néron Severi group $\mathcal{M}_8$. Its quotient $\bar{X}$ is the complete intersection of a hypersurface of bidegree $(1,1)$ and of a hypersurface of bidegree $(2,2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$, its minimal resolution is X. A K3 surface which is a complete intersection of a bidegree $(1,1)$ and a bidegree $(2,2)$ hypersurface in $\mathbb{P}^2 \times \mathbb{P}^2$ is a *Wehler surface* (cf. [W]). We describe this surface more in details in the Section 5. The surface X has a projective model in $\mathbb{P}^2\times\mathbb{P}^2$ and the map associated to the divisors L_1 and L_2 are respectively the projection to the first and to the second projective space. The map $\phi_{L_1}: X\rightarrow\mathbb{P}^2$ is a $2:1$ map. It contracts the four rational curves $N_5,\ldots,N_8$ to nodes of the branch sextic of the double cover. The four curves $N_1,\ldots,N_4$ are mapped to lines in $\mathbb{P}^2$. Since their intersection with L_1 is equal to one, each of them is one of the two components of the pullback of a line in $\mathbb{P}^2$. So their image under the map ϕ_{L_1} is a line tritangent to the branch curve. The branch curve has degree six and has four nodes so its genus is $(6-1)(6-2)/2-4=6$. The curve R_1 on X such that $\phi_{L_1}(R_1)$ is the branch curve, has degree six and genus six (because it is the branch curve, so the genus of the curve on X is the genus of its image on $\mathbb{P}^2$), this implies that the curve R_1 has self-intersection ten ($g=R_1^2/2+1$) and its intersection with L_1 is six. The curve R_1 has to intersect the curves N_i, $i=1,\ldots,4$ in three points (because N_i are mapped to tritangent to the sextic) and the curves N_i, $i=5,\ldots,8$ in two points (because the branching curve has nodes in the points which are the images of these curves). So we find

$$R_1=3\left(\frac{L-N_1-N_2-N_3-N_4}{2}\right)-(N_5+N_6+N_7+N_8).$$

Exactly in the same way one sees that the branch curve of the second projection is

$$R_2=3\left(\frac{L-N_5-N_6-N_7-N_8}{2}\right)-(N_1+N_2+N_3+N_4).$$

The equation of a generic K3 surface which is complete intersection of a $(1,1)$ and a $(2,2)$ hypersurface in $\mathbb{P}^2\times\mathbb{P}^2$ is given by the system

$$\left\{\begin{array}{l}\sum_{i,j=0,1,2}q_{ij}(x_0:x_1:x_2)y_iy_j=0\\ \sum_{i=0,1,2}l_i(x_0:x_1:x_2)y_i=0\end{array}\right.$$

where q_{ij} and l_i, $i,j=0,1,2$ are homogeneous polynomial of degree respectively two and one in the variables x_j, which are the coordinates of the first copy of $\mathbb{P}^2$ and y_j denote coordinates of the second copy of $\mathbb{P}^2$.

For a generic point $(\overline{x_0}:\overline{x_1}:\overline{x_2})$ of $\mathbb{P}^2$ the system has two solutions in $(y_0:y_1:y_2)$ and this gives the 2:1 map to $\mathbb{P}^2$. If the point $(\overline{x_0}:\overline{x_1}:\overline{x_2})$ is such that $\sum_{i=0,1,2}l_i(\overline{x_0}:\overline{x_1}:\overline{x_2})y_i=0$ for each $(y_0:y_1:y_2)$, then the fiber on it is the quadric $\sum_{i,j=0,1,2}q(\overline{x_0}:\overline{x_1}:\overline{x_2})y_iy_j=0$. Otherwise if $\sum_{i,j=0,1,2}q(\overline{x_0}:\overline{x_1}:\overline{x_2})y_iy_j=0$ for each $(y_0:y_1:y_2)$ then the fiber on $(\overline{x_0}:\overline{x_1}:\overline{x_2})$ is a line.

Since in our case each map to $\mathbb{P}^2$ contracts four lines, then for each copy of $\mathbb{P}^2$ there are four points in which $\sum_{i,j=0,1,2}q(x_0:x_1:x_2)y_iy_j=0$ are identically satisfied. Up to a projective transformation one can suppose that the four points with dimension one fiber are, on each $\mathbb{P}^2$, $(1:0:0),(0:1:0),(0:0:1),(1:1:1)$. This implies that the equation $\sum_{i,j=0,1,2}q(x_0:x_1:x_2)y_iy_j=0$ is of the form

$$\begin{array}{c}y_0y_1(x_0x_1+ax_0x_2-(a+1)x_1x_2)+y_0y_2(bx_0x_1+cx_0x_2-(b+c)x_1x_2)+\\ +y_1y_2(-(1+b)x_0x_1-(a+c)x_0x_2+(a+b+c+1)x_1x_2)=0.\end{array}$$

and so it depends on three projective parameters. The equation of type $(1,1)$ are

$$x_0y_0+dx_0y_1+ex_0y_2+fx_1y_0+gx_1y_1+hx_1y_2+lx_2y_0+mx_2y_1+nx_2y_2=0$$

and so depends on eight parameters (we can not apply other projective transformations because we have chosen the points on which there are lines as fibers). So a Wehler K3 surface such that the projection ϕ_{L_1} to $\mathbb{P}^2$ contracts four rational curves of the K3 surface and ϕ_{L_2}

contracts four other rational curves (disjoint from the previous curves) depends exactly on eleven parameters.

4.11. **The case of $L^2 = 24$, $NS(X) = \mathcal{L}'_{24}$, the map $\phi_{L_1} \times \phi_{L_2}$.** The intersection properties of L_1 and L_2 are $L_1 \cdot L_1 = 4$, $L_2 \cdot L_2 = 4$ and $L_1 \cdot L_2 = 6$. Each of them defines a map from X to $\mathbb{P}^3$ (by Lemma 3.2). Each map ϕ_{L_i}, $i = 1, 2$ contracts four rational curves and sends the other in four lines. The curve L_1 is sent by ϕ_{L_2} to a curve of degree six, and viceversa.
In [vGS, Paragraph 3.8] it is described the K3 surface Y admitting a Nikulin involution with Néron-Severi group $\mathcal{M}_{12}$. Its quotient is $\bar{X}$ and it is a complete intersection of four varieties of bidegree $(1, 1)$ in $\mathbb{P}^3 \times \mathbb{P}^3$, the minimal resolution is X (cf. Table 1). The projections to the two copies of $\mathbb{P}^3$ are ϕ_{L_1} and ϕ_{L_2}.

5. Geometric conditions to have an even set.

In this section we describe geometrical properties of K3 surfaces which imply the presence of an even set. These even sets can be of eight nodes, of eight rational curves (lines or conics) or of some nodes and some rational curves. The following results are in a certain sense the converse of the results of the previous section, where we supposed that a K3 surface admits an even set and we described its geometry. To prove the existence of an even set on S we will prove that either the lattice $\mathcal{L}_{2d}$ or $\mathcal{L}'_{2d}$ is embedded in $NS(S)$ and that the sublattice N of $\mathcal{L}_{2d}$ (or of $\mathcal{L}'_{2d}$) is generated over $\mathbb{Q}$ by (-2)-irreducible curves. Since rank $\mathcal{L}_{2d}$ =rank $\mathcal{L}'_{2d} = 9$ and since the K3 surfaces with Néron Severi group equal to $\mathcal{L}_{2d}$ or $\mathcal{L}'_{2d}$ have an even set, then the number of moduli of the families of K3 surfaces that we describe here is eleven.

5.1. **Double cover of a cone with an even set.** Let S be a K3 surface which is a double cover of a cone, then by [SD, Proposition 5.7 case iii)] the map from S to the cone is given by a class L' in $NS(S)$ such that either

a) $L' = 2E' + \Gamma_0 + \Gamma_1$ with $\Gamma_0 \cdot E' = \Gamma_1 \cdot E' = 1$ and $\Gamma_0 \cdot \Gamma_1 = 0$ or
b) $L' = 2E' + 2\Gamma_0 + \ldots + 2\Gamma_n + \Gamma_{n+1} + \Gamma_{n+2}$, with $E' \cdot \Gamma_0 = \Gamma_i \cdot \Gamma_{i+1} = 1$ $i = 0, \ldots, n-1$, $\Gamma_n \cdot \Gamma_{n+1} = \Gamma_n \cdot \Gamma_{n+2} = 1$ and the other intersections are equal to zero.

The Γ_i's are irreducible (-2)-curves. If we are in the case a), then we can give a sufficient condition for S to have an even set of eight disjoint rational curves.

Proposition 5.1. *Let S be a K3 surface such that there exist a map $\phi_{L'} : S \stackrel{2:1}{\to} Z$, where Z is a cone and $L' = 2E' + \Gamma_0 + \Gamma_1$ with $\Gamma_0 \cdot E' = \Gamma_1 \cdot E' = 1$ and $\Gamma_0 \cdot \Gamma_1 = 0$. If the branch locus of the double cover is the union of a conic and a sextic meeting in six distinct points and not passing through the vertex of the cone, then S admits an even set of eight disjoint rational curves.*

Proof. We prove that under the hypothesis the lattice $\mathcal{L}'_4$ is embedded in the Néron Severi lattice of S. In particular there exist eight disjoint rational curves in $NS(S)$ generating on $\mathbb{Q}$ a copy of N in the Néron Severi lattice. This implies that S admits an even set made up by these eight disjoint rational curves.
By the hypothesis the classes L', E' and Γ_0 are linearly independent and are in $NS(S)$. The map $\phi_{L'}$ is a $2:1$ map to the cone, which contracts the two rational curves Γ_0 and Γ_1 to the vertex of the cone. The (smooth) K3 surface S is the double cover of the blow up of the cone in the vertex and in the six singular points of the ramification locus. On S there are six rational curves Γ_i, $i = 2, \ldots, 7$ on the six singular points of the ramification locus, and the two rational curves Γ_0 and Γ_1 on the blow up of the vertex of the cone (since this is not in the ramification locus we obtain two curves).

Let C_2' be the curve such that $\phi_{L'}(C_2')=c_2'$ is the conic of the branching locus. Since c_2' is a conic, $C_2'\cdot L=2$ and since it does not pass through the vertex then $C_2'\cdot\Gamma_0=C_2'\cdot\Gamma_1=0$, so $C_2'\cdot E'=1$, moreover C_2' is a rational curve and so $C_2'^2=-2$. Since on the cone c_2' passes through the six singular points of the ramification locus, on the K3 surface we have $C_2'\cdot\Gamma_i=1$, $i=2,\ldots,7$.
The classes L', E', Γ_0, C_2, Γ_i, $i=2,\ldots,7$ spans a lattice R which is isometric to the lattice $\mathcal{L}_4'$. In fact a basis for $\mathcal{L}_4'$ is given by $(L-N_1-N_2)/2$, $\hat{N}$ and N_i, $i=1,\ldots 7$. The map

$$E'\mapsto(L-N_1-N_2)/2,\quad C_2+E'-L'\mapsto\hat{N},\quad \Gamma_i\mapsto N_{i+1}\ \ i=0,\ldots,7.$$

gives the explicit change of basis from R to $\mathcal{L}_4'$. □

5.2. **Complete intersection of one $(2,0)$ and three $(1,1)$ hypersurfaces in $\mathbb{P}^4\times\mathbb{P}^2$.** If S is a complete intersection of a hypersurface of bidegree $(2,0)$ and three hypersurfaces of bidegree $(1,1)$ in $\mathbb{P}^4\times\mathbb{P}^2$, by the adjunction formula S is a K3 surface. The Néron Severi group of a generic K3 surface which is a complete intersection of this type is generated by the two divisors D_1 and D_2 associated to the two projections. The family of the K3 surfaces of this type has Picard number two and so it has 18 moduli. To give the complete description of the Néron Severi group we compute the intersection $D_1\cdot D_2$. We describe here how to find $D_1\cdot D_2$ as explained in [vG, Section 5]. On the K3 surface the divisors D_1 and D_2 correspond to the restriction to S of the pull back of the hyperplane section of $\mathbb{P}^4$, respectively of $\mathbb{P}^2$. We put $h=\mathbb{P}^3\times\mathbb{P}^2$ and $k=\mathbb{P}^4\times\mathbb{P}^1$. It is clear that $h^3=\{point\}\times\mathbb{P}^2$, and so $h^4=0$ because in $\mathbb{P}^4$ it corresponds to the intersection between a point and a space. In the same way one computes that $k^2=\mathbb{P}^4\times\{point\}$ (intersection of two lines in $\mathbb{P}^2$) and $k^3=0$, $h^3k^2=1$ ($\{point\}\times\{point\}$). The hypersurface of bidegree $(2,0)$ corresponds to $2h$ (has degree two with respect to the first factor, so with respect to h, and zero with respect to the second factor, k) and the hypersurfaces of bidegree $(1,1)$ correspond to the divisor $h+k$. Since X is the complete intersection of one hypersurface of bidegree $(2,0)$ and three hypersurfaces of bidegree $(1,1)$, X corresponds in $\mathbb{P}^4\times\mathbb{P}^2$ to the divisor $2h(h+k)^3$. We want to compute $D_1\cdot D_2$ which is $h\cdot k$ restricted to $2h(h+k)^3$. Then $D_1\cdot D_2$ is equal to $hk(2h)(h+k)^3$ in the six dimensional space $\mathbb{P}^4\times\mathbb{P}^2$. The terms h^ik^j with $i+j=6$ correspond to the intersections of codimension six and so are a finite number of points. The sum of the coefficients of these terms is exactly the number of points, so $D_1\cdot D_2=6$.
Hence the general K3 surface which is complete intersection of a $(2,0)$ hypersurface and three $(1,1)$ hypersurfaces in $\mathbb{P}^4\times\mathbb{P}^2$ has Néron-Severi lattice isometric to $\{\mathbb{Z}^2,\begin{bmatrix}6&6\\6&2\end{bmatrix}\}$. This is a sublattice of the Néron Severi lattice of any $K3$ surface which is a complete intersection of a $(2,0)$ hypersurface and three $(1,1)$ hypersurfaces in $\mathbb{P}^4\times\mathbb{P}^2$.

Proposition 5.2. *Let S be a complete intersection of one hypersurface of bidegree $(2,0)$ and three hypersurfaces of bidegree $(1,1)$ in $\mathbb{P}^4\times\mathbb{P}^2$. Let ϕ_{A_1} and ϕ_{A_2} be the projections to the first and to the second factor associated to the pseudo ample class A_1, with $A_1^2=6$, and to the pseudo ample class A_2, with $A_2^2=2$. If there exist eight curves R_i, $i=1,\ldots,8$ such that ϕ_{A_1} contracts all these curves to eight nodes of the image and ϕ_{A_2} sends these curves in lines on $\mathbb{P}^2$, then R_i, $i=1,\ldots,8$ form an even set.*

Proof. The idea of the proof is similar to the proof of Proposition 5.1 and is based on the presence of certain divisors in $NS(S)$. The divisors A_j, $j=1,2$, R_i, $i=1,\ldots,8$ are contained in the Néron Severi group. Nine of these classes are linearly independent. The lattice generated by A_1, A_2, R_1, R_2, R_3, R_4, R_5, R_6, R_7 is embedded in $NS(S)$ and a computation shows that it is isometric to the lattice $\mathcal{L}_6$. Since the lattice $\mathcal{L}_6$ contains an even set, also in the Néron Severi group of S there is an even set made up by $R_1, \ldots, R_7, 2A_2 - 2A_1 + R_1 + \ldots + R_7$.□

Remark. Observe that Proposition 5.2 gives a sufficient condition for a K3 surface complete intersection in $\mathbb{P}^4$ to have an even set of nodes (or of eight rational curves in the minimal resolution).

5.3. **Complete intersection of three quadrics in $\mathbb{P}^5$ with an even set of nodes.** We give two different sufficient conditions for a K3 surface in $\mathbb{P}^5$ to have an even set of nodes. These two possibilities correspond to the fact that the Néron Severi group of such a K3 surface, with Picard number nine, is equal either to the lattice $\mathcal{L}_8$ or $\mathcal{L}'_8$.

Proposition 5.3. *Let S be a K3 surface admitting two maps ϕ_{A_1}, ϕ_{A_2} associated to the pseudo ample class A_1 with $A_1^2 = 8$ and to the ample class A_2 with $A_2^2 = 4$. If there exist eight curves R_i, $i = 1, \ldots, 8$ such that ϕ_{A_1} contracts all these curves to eight nodes and ϕ_{A_2} sends these curves to lines on the quartic in $\mathbb{P}^3$, then R_i, $i = 1, \ldots, 8$ form an even set.*

Proof. One can prove that $\mathcal{L}_8$ is primitively embedded in $NS(S)$ as in the Propositions 5.1 and 5.2. □

Proposition 5.4. *Let X be a K3 surface in $\mathbb{P}^5$ having eight nodes. These nodes form an even set if X is the complete intersection of a smooth quadric and two quadrics, which are singular in two planes $H \cong \mathbb{P}^2$ and $K \cong \mathbb{P}^2$, $H \cap K = \emptyset$ and four of the points are contained in H and the other four in K.*

Proof. Let $h_0 = h_1 = h_2 = 0$ and $k_0 = k_1 = k_2 = 0$ be the equations defining H resp. K in $\mathbb{P}^5$, then we can write the equations of the two cones as $h_0h_1 - h_2^2 = 0$ and $k_0k_1 - k_2^2 = 0$. The quadrics $h_ik_j = 0$, $i = 0, 1$, $j = 0, 1$ meet the K3 surface in two curves C_i, C_j with multiplicity two, which passes through four singular points, resp. to the other four. So $2(C_i + C_j) \in |2L - (N_1 + \ldots + N_8)|$, which shows that $N_1 + \ldots + N_8$ form an even set. □

5.4. **Double covers of $\mathbb{P}^2$.** Here we consider two different K3 surfaces with an even set which admit maps $2:1$ to $\mathbb{P}^2$. The first one is a Wehler surface, the second one is not. In the first case the curves of the even sets are contracted to singular points of the branch locus or are sent to lines of $\mathbb{P}^2$ which are tritangent to the ramification locus, in the second case they are contracted or sent to conics. Other double covers of $\mathbb{P}^2$ with an even set are described in [B2].

5.4.1. *The first case: complete intersections of bidegree $(1,1)$, $(2,2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$.* The complete intersections of bidegree $(1,1)$ and $(2,2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ are the Wehler surfaces. The projections to the two copies of $\mathbb{P}^2$ are 2:1 maps. It is known (but can also be computed as in Section 5.2) that the Néron Severi group of the generic member of this family is the two dimensional lattice $\{\mathbb{Z}^2, \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}\}$. The number of moduli of the family of the Wehler K3 surfaces is 18.

Proposition 5.5. *Let S be a Wehler K3 surface such that the first projection π_1 contracts four rational disjoint curves R_l, $l = 1, \ldots, 4$ on S and the second projection π_2 contracts other four rational disjoint curves R_l, $l = 5, \ldots, 8$. Moreover the map π_1 sends the curves contracted by π_2 to lines on $\mathbb{P}^2$ and viceversa. Then the eight rational curves R_l, $l = 1, \ldots, 8$ form an even set on S.*

Proof. One can prove that $\mathcal{L}'_{16}$ is primitively embedded in $NS(S)$ as in the Propositions 5.1 and 5.2, and that $N \subset \mathcal{L}'_{16}$ is generated over $\mathbb{Q}$ by the curves R_i. □

5.4.2. *The second case.*

Proposition 5.6. *Let S be a K3 surface admitting two maps $2:1$ to $\mathbb{P}^2$. If there exist eight curves R_i, $i=1,\ldots,8$ such that the map on the first copy of $\mathbb{P}^2$ contracts the curves R_i, $i=1,\ldots,4$ and sends the others in four conics and the map on the second copy of $\mathbb{P}^2$ contracts the curves R_i $i=5,\ldots,8$ and sends the others in conics, then R_i, $i=1,\ldots,8$ is an even set of eight disjoint rational curves.*

Proof. One can prove that $\mathcal{L}_{10}$ is primitively embedded in $NS(S)$ as in the Propositions 5.1 and 5.2. □

5.5. **A mixed even set.**

Proposition 5.7. *Let S be a K3 surface admitting two maps to $\mathbb{P}^3$. If there exist eight curves R_i, $i=1,\ldots,8$ such that the map on the first copy of $\mathbb{P}^3$ contracts the curves R_i, $i=1,\ldots,4$ and sends the others in four conics and the map on the second copy of $\mathbb{P}^3$ contracts the curves R_i, $i=5,\ldots,8$ and sends the others in conics, then R_i, $i=1,\ldots,8$ is an even set of eight disjoint rational curves.*

Proof. One can prove that $\mathcal{L}_{12}$ is primitively embedded in $NS(S)$ as in the Propositions 5.1 and 5.2. □

In this case we have on a quartic in $\mathbb{P}^3$ a *mixed even set*, in fact it consists of four nodes and of four conics.

5.6. **Surfaces of bidegree $(2,3)$ in $\mathbb{P}^1\times\mathbb{P}^2$.** A K3 surface in $\mathbb{P}^1\times\mathbb{P}^2$ has bidegree $(2,3)$ by the adjunction formula. These K3 surfaces are studied in [vG, Paragraph 5.8]. The family has 18 moduli, in fact the Néron Severi group of such a K3 surface has to contain two classes D_1, D_2 giving the regular maps ϕ_{D_1} and ϕ_{D_2}, which correspond to the projections to $\mathbb{P}^1$ and to $\mathbb{P}^2$. We can compute the intersection properties of D_1 and D_2 as in the Paragraph 5.2 (these computations can be found also in [vG, Paragraph 5.8]). The general K3 surface which has bidegree $(2,3)$ in $\mathbb{P}^1\times\mathbb{P}^2$ has Néron-Severi lattice isometric to $\{\mathbb{Z}^2, \begin{bmatrix} 0 & 3 \\ 3 & 2 \end{bmatrix}\}$. This is a sublattice of the Néron Severi lattice of all the $K3$ surfaces which have bidegree $(2,3)$ in $\mathbb{P}^1\times\mathbb{P}^2$.

Proposition 5.8. *Let S be a K3 surface of bidegree $(2,3)$ in $\mathbb{P}^1\times\mathbb{P}^2$ and such that the projection to the first space p_1 (which gives an elliptic fibration) contracts two disjoint rational curves and the projection p_2 to the second space contracts other six disjoint rational curves. If the curves contracted by p_1 are sent to lines by p_2 and the curves contracted by p_2 are sent by p_1 to two sections of the elliptic fibration, then the eight rational curves on S form an even set.*

Proof. One can prove that $\mathcal{L}'_{12}$ is primitively embedded in $NS(S)$ as in the Propositions 5.1 and 5.2. □

5.7. **Complete intersections in $\mathbb{P}^3\times\mathbb{P}^3$.** The Néron Severi group of a complete intersection of four hypersurfaces of bidegree $(1,1)$ in $\mathbb{P}^3\times\mathbb{P}^3$ is generated by the two divisors D_1 and D_2 associated to the two projections. The divisors D_1 and D_2 have self intersection equal to four, computing as before the intersection between D_1 and D_2 one finds $D_1\cdot D_2=6$, so the Néron Severi lattice of the generic K3 surface which is a complete intersection of four bidegree $(1,1)$ hypersurfaces in $\mathbb{P}^3\times\mathbb{P}^3$ is $\{\mathbb{Z}^2, \begin{bmatrix} 4 & 6 \\ 6 & 4 \end{bmatrix}\}$.

Proposition 5.9. *Let S be a complete intersection of four bidegree $(1,1)$ hypersurfaces in $\mathbb{P}^3 \times \mathbb{P}^3$. Let A_1 and A_2 be two pseudo ample divisors defining two maps to $\mathbb{P}^3$. If the map ϕ_{A_1}, respectively ϕ_{A_2}, contracts four rational curves R_1, R_2, R_3, R_4, respectively R_5, R_6, R_7, R_8, and sends the others four rational curves in lines, then R_i, $i = 1, \ldots, 8$ is an even set on X.*

Proof. One can prove that $\mathcal{L}'_{24}$ is primitively embedded in $NS(S)$ as in the Propositions 5.1 and 5.2. □

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Contraction of excess fibres between the Mckay correspondence in dimensions two and three

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CONTRACTION OF EXCESS FIBRES BETWEEN THE MCKAY CORRESPONDENCES IN DIMENSIONS TWO AND THREE

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Abstract. The quotient singularities of dimensions two and three obtained from polyhedral groups and the corresponding binary polyhedral groups admit natural resolutions of singularities as Hilbert schemes of regular orbits whose exceptional fibres over the origin reveal similar properties. We construct a morphism between these two resolutions, contracting exactly the excess part of the exceptional fibre. This construction is motivated by the study of some pencils of K3-surfaces arising as minimal resolutions of quotients of nodal surfaces with high symmetries.

1. Introduction

Consider a binary polyhedral group $\widetilde{G} \subset \mathrm{SU}(2)$ corresponding to a polyhedral group $G \subset \mathrm{SO}(3,\mathbb{R})$ through the double-covering $\mathrm{SU}(2) \to \mathrm{SO}(3,\mathbb{R})$. The group $\widetilde{G}$ acts freely on $\mathbb{C}^2 - \{0\}$ and the quotient $\mathbb{C}^2/\widetilde{G}$ is a surface singularity with an isolated singular point at the origin. The exceptional divisor of its minimal resolution of singularities $\mathcal{X} \to \mathbb{C}^2/\widetilde{G}$ is a tree of smooth rational curves of self-intersection -2, intersecting transversely, whose intersection graph is an A-D-E Dynkin diagram. The classical McKay correspondence ([23]) relates this intersection graph to the representations of the group $\widetilde{G}$, associating bijectively each exceptional curve to a non-trivial irreducible representation of the group: the correspondence in fact identifies the intersection graph with the McKay quiver of the action of $\widetilde{G}$ on $\mathbb{C}^2$. Among these irreducible representations we find all irreducible representations of the group G: we call them *pure* and the remaining ones *binary*. Since $\widetilde{G}/G \cong \{\pm 1\}$, one can produce a G-invariant cone $\mathbb{C}^2/\{\pm 1\} \xrightarrow{\sim} K \hookrightarrow \mathbb{C}^3$ whose quotient K/G is isomorphic to $\mathbb{C}^2/\widetilde{G}$. In this note, we prove the following result, conjectured by W. P. Barth:

Theorem 1.1. *There exists a crepant resolution of singularities of $\mathbb{C}^3/G$ containing a partial resolution $\mathcal{Y} \to K/G$ with the property that the intersection graph of its exceptional locus is precisely the McKay quiver of the action of G on $\mathbb{C}^3$, together with a resolution map $\mathcal{X} \to \mathcal{Y}$ mapping isomorphically the exceptional curves corresponding to pure representations and contracting those associated with binary representations to ordinary nodes.*

We make this construction in the framework of the Hilbert schemes of regular orbits of Nakamura ([25]) providing, thanks to the Bridgeland-King-Reid theorem ([5]), the natural candidates for the resolutions of singularities in dimensions two and three. We produce a morphism $\mathscr{S}$ between these two resolutions of singularities, define our partial resolution

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$\mathcal{Y}$ as the image of this map and study the effect of $\mathscr{S}$ on the exceptional fibres:

$$\begin{tikzcd}
\widetilde{G}\text{-Hilb}\left(\mathbb{C}^2\right) \arrow[rr, "\mathscr{S}"] \arrow[dr, "\mathscr{S}"'] \arrow[dd, "\widetilde{\pi}"'] & & G\text{-Hilb}\left(\mathbb{C}^3\right) \arrow[dd, "\pi"] \\
& \mathcal{Y} \arrow[ur, hook] \arrow[d] & \\
\mathbb{C}^2/\widetilde{G} \arrow[r, "\sim"] & K/G \arrow[r, hook] & \mathbb{C}^3/G
\end{tikzcd}$$

Although the exceptional fibres can be described very explicitly in all cases (see [19]), by principle our proof avoids any case-by-case analysis. Therefore, the key point consists in a systematic modular interpretation of the objects at issue.

From the strict point of view of the McKay correspondence, this construction shows some new properties revealing again the fertility of the geometric construction of the McKay correspondence following Gonzales-Sprinberg and Verdier [14], Ito-Nakamura [19], Ito-Najakima [18] and Reid [26]. The beginning of the story was devoted to the study of all situations in dimension two and three, in general by a case-by-case analysis. Then efforts were made to understand how to get all these cases by one general geometric construction ([18, 5]). The development followed then the cohomological direction in great dimensions in a symplectic setup ([21, 11]), leading to an explicit study of a family of examples of increasing dimension for the specific symmetric group problem ([3]). The new point of view in the present paper consists in working between two situations of different dimensions for different - but related - groups and construct a relation between them. This may be considered as a concrete application of some significant results in this area coming again at the beginning of the story, dealing with a now quite classical material approached by natural transformations between moduli spaces.

This study is motivated by previous works of Sarti [27] and Barth-Sarti [2] studying special pencils of surfaces in $\mathbb{P}_3$ with bipolyhedral symmetries. The minimal resolutions of the associated quotient surfaces are K3-surfaces with maximal Picard numbers. For some special fibres of these pencils, the resolution looks locally like the quotient of a cone by a polyhedral group, and our result gives a local interpretation of the exceptional locus in these cases.

The structure of the paper is as follows: in Section 2 we introduce the notations and we recall some basic facts about clusters and in Section 3 we recall the construction of the Hilbert schemes of points and clusters. The Sections 4, 5 and 6 give a brief survey on polyhedral, binary polyhedral and bipolyhedral groups, their representations and the classical Mckay correspondences in dimensions two and three. In Section 7 we start the study of the map $\mathscr{S}$. First we show that it is well defined (lemma 7.1) and then that it is a regular projective map, which induces a map between the exceptional fibres (proposition 7.2). In Section 8 the theorem 8.1 is the fundamental step for proving the main theorem 1.1: we show that the map $\mathscr{S}$ contracts the curves corresponding to the binary representations and maps the curves corresponding to the pure representations isomorphically to the exceptional curves downstairs. In Section 9, as an example we describe in details the case when $\widetilde{G}$ is a cyclic group. Finally the Section 10 is devoted to an application to resolutions of pencils of K3-surfaces.

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2. Clusters

In the sequel, we aim to study a link between the two-dimensional and the three-dimensional McKay correspondences. In order to avoid confusion, we shall use different sets of letters for the corresponding algebraic objects at issue in both situations. In this section, we fix the notations and the terminology.

2.1. **General setup.** Let V be a n-dimensional complex vector space and $\mathfrak{G}$ a finite subgroup of $\mathrm{SL}(V)$. We denote by $\mathcal{O}(V) := \mathrm{S}^*(V^\vee)$ the algebra of polynomial functions on V, with the induced left action $g \cdot f := f \circ g^{-1}$ for $f \in \mathcal{O}(V)$ and $g \in \mathfrak{G}$.
We choose a basis $X_1, \ldots, X_n$ of linear forms on V, denote the ring of polynomials in n indeterminates by $S := \mathbb{C}[X_1, \ldots, X_n]$ and identify $\mathcal{O}(V) \cong S$. The ring S is given a graduation by the total degree of a polynomial, where each indeterminate X_i has degree 1. In particular, the action of the group $\mathfrak{G}$ on S preserves the degree.
Let $\mathfrak{m}_S := \langle X_1, \ldots, X_n \rangle$ be the maximal ideal of S at the origin. We denote by $S^{\mathfrak{G}}$ the subring of $\mathfrak{G}$-invariant polynomials, by $\mathfrak{m}_{S^{\mathfrak{G}}}$ its maximal ideal at the origin and by $\mathfrak{n}_{\mathfrak{G}} := \mathfrak{m}_{S^{\mathfrak{G}}} \cdot S$ the ideal of S generated by the non-constant $\mathfrak{G}$-invariant polynomials vanishing at the origin. The quotient ring of *coinvariants* is by definition $S_{\mathfrak{G}} := S/\,\mathfrak{n}_{\mathfrak{G}}$.
An ideal $\mathfrak{I} \subset S$ is called a $\mathfrak{G}$-*cluster* if it is globally invariant under the action of $\mathfrak{G}$ and the quotient $S/\mathfrak{I}$ is isomorphic, as a $\mathfrak{G}$-module, to the regular representation of $\mathfrak{G}$: $S/\mathfrak{I} \cong \mathbb{C}[\mathfrak{G}]$. A closed subscheme $Z \subset \mathbb{C}^n$ is called a $\mathfrak{G}$-*cluster* if its defining ideal $\mathfrak{I}(Z)$ is a $\mathfrak{G}$-cluster. Such a subscheme is then zero-dimensional and has length $|\mathfrak{G}|$. For instance, a free $\mathfrak{G}$-orbit defines a $\mathfrak{G}$-cluster. In particular, a $\mathfrak{G}$-cluster contains only one orbit: the support of a cluster is a union of orbits, and any function constant on one orbit and vanishing on another one would induce a different copy of the trivial representation in the quotient $S/\mathfrak{I}$.
We are particularly interested in $\mathfrak{G}$-clusters supported at the origin. Then $\mathfrak{I} \subset \mathfrak{m}_S$ and in fact this condition is enough to assert that the cluster is supported at the origin: else, the support of the cluster would consist in more than one orbit. Furthermore, one has automatically $\mathfrak{n}_{\mathfrak{G}} \subset \mathfrak{I}$, since any non-constant function $f \in \mathfrak{n}_{\mathfrak{G}}$ not contained in $\mathfrak{I}$ would induce a new copy of the trivial representation in the quotient $S/\mathfrak{I}$, different from the one already given by the constant functions. Hence we wish to understand the structure of the $\mathfrak{G}$-clusters $\mathfrak{I}$ such that $\mathfrak{n}_{\mathfrak{G}} \subset \mathfrak{I} \subset \mathfrak{m}_S$, equivalent to the study of the quotient ideals $\mathfrak{I}/\mathfrak{n}_{\mathfrak{G}} \subset \mathfrak{m}_S/\mathfrak{n}_{\mathfrak{G}} \subset S/\mathfrak{n}_{\mathfrak{G}} = S_{\mathfrak{G}}$, with the exact sequence:

$$0 \longrightarrow \mathfrak{I}/\mathfrak{n}_{\mathfrak{G}} \longrightarrow S_{\mathfrak{G}} \longrightarrow S/\mathfrak{I} \longrightarrow 0. \tag{1}$$

From now on, we assume that the group $\mathfrak{G}$ is a subgroup of index 2 of a group $\mathfrak{R} \in \mathrm{GL}(V)$ generated by *reflections* (we follow here the terminology of [7]), *i.e.* elements $g \in \mathfrak{R}$ such that $\mathrm{rk}(g - \mathrm{Id}_V) = 1$.
The structure of the action of $\mathfrak{R}$ on S has the following properties (see [7]):

- The algebra of invariants $S^{\mathfrak{R}}$ is a polynomial algebra generated by exactly n algebraically independent homogeneous polynomials $f_1, \ldots, f_n$ of degrees d_i.
- $|\mathfrak{R}| = d_1 \cdot \ldots \cdot d_n$.
- The set of degrees $\{d_1, \ldots, d_n\}$ is independent of the choice of the homogeneous generators.

- The algebra of coinvariants is isomorphic to the regular representation: $S_{\mathfrak{R}} \cong \mathbb{C}[\mathfrak{R}]$.

As a byproduct, we get that the algebra of coinvariants $S_{\mathfrak{R}}$ is a graded finite-dimensional algebra.
From this and the fact that $\mathfrak{G} = \mathfrak{R} \cap \mathrm{SL}(V)$, one deduces the structure of the action of $\mathfrak{G}$ on S (see [4, 12, 13]):

- There exists a homogeneous $\mathfrak{R}$-skew-invariant polynomial $f_{n+1} \in S$, *i.e.* such that $g \cdot f_{n+1} = \det(g).f_{n+1}$ for all $g \in \mathfrak{R}$, unique up to a multiplicative constant, dividing any $\mathfrak{R}$-skew-invariant polynomial: hence the set $f_{n+1} \cdot S^{\mathfrak{R}}$ is precisely the set of $\mathfrak{R}$-skew-invariants. A natural choice for this element is $f_{n+1} = \mathrm{Jac}(f_1, \ldots, f_n)$.
- $S^{\mathfrak{G}} = \mathbb{C}[f_1, \ldots, f_n, f_{n+1}]$.
- $\mathfrak{n}_{\mathfrak{G}} = \mathfrak{n}_{\mathfrak{R}} \oplus \mathbb{C} f_{n+1}$.
- $S_{\mathfrak{R}} = S_{\mathfrak{G}} \oplus \mathbb{C} f_{n+1}$.

Note that, as a $\mathfrak{G}$-module, $\mathbb{C}[\mathfrak{R}]$ is isomorphic to two copies of $\mathbb{C}[\mathfrak{G}]$. It follows that $\mathfrak{m}_S/\mathfrak{n}_{\mathfrak{G}}$ is a graded finite-dimensional algebra which, as a $\mathfrak{G}$-module, consists exactly of each non-trivial representation ρ of $\mathfrak{G}$ repeated $2\dim\rho$ times: one can denote the occurrences of each representation ρ by $V^{(1)}(\rho), \ldots, V^{(2\dim\rho)}(\rho)$ where each $V^{(i)}(\rho)$ is given by homogeneous polynomials modulo $\mathfrak{n}_{\mathfrak{G}}$.
Thanks to the exact sequence (1), giving a $\mathfrak{G}$-cluster supported at the origin consists in choosing, for each non-trivial representation ρ of $\mathfrak{G}$, $\dim\rho$ copies of ρ in $\mathfrak{m}_S/\mathfrak{n}_{\mathfrak{G}}$. But this gives many choices since any linear combination of some $V^{(i)}(\rho)$ and $V^{(j)}(\rho)$ provides such a copy. The ground idea is that one does not have to make all these choices in order to define $\mathfrak{I}$ (see §9 for an explicit example).
For such an ideal $\mathfrak{I}$ with $\mathfrak{n}_{\mathfrak{G}} \subset \mathfrak{I} \subset \mathfrak{m}_S$, we consider the finite-dimensional $\mathfrak{G}$-modules $W \subset S$ generating $\mathfrak{I}$ in the sense that $\mathfrak{I} = W \cdot S + \mathfrak{n}_{\mathfrak{G}}$. Such modules do exist thanks to the preceding construction. Among these choices, we consider the minimal ones, *i.e.* such that no strict $\mathfrak{G}$-submodule of them generate $\mathfrak{I}$ in the preceding sense.
If W is a generator in this sense, then

$$\mathfrak{I} = W \cdot S + \mathfrak{n}_{\mathfrak{G}} = W + \mathfrak{m}_S \cdot W + \mathfrak{n}_{\mathfrak{G}} = W + \mathfrak{m}_S \cdot \mathfrak{I} + \mathfrak{n}_{\mathfrak{G}}.$$

This means that the $\mathbb{C}$-linear map $W \to \mathfrak{I}/(\mathfrak{m}_S \cdot \mathfrak{I} + \mathfrak{n}_{\mathfrak{G}})$ is surjective. Also, since W is a $\mathfrak{G}$-module and since $\mathfrak{m}_S \cdot \mathfrak{I} + \mathfrak{n}_{\mathfrak{G}}$ is $\mathfrak{G}$-stable, this map is $\mathfrak{G}$-linear. If W is a minimal set of generators, it satisfies in particular $W \cap (\mathfrak{m}_S \cdot \mathfrak{I} + \mathfrak{n}_{\mathfrak{G}}) = \{0\}$ since this intersection would provide a $\mathfrak{G}$-submodule whose complementary in W is a smaller $\mathfrak{G}$-submodule generating $\mathfrak{I}$. Hence, for W minimal one gets an isomorphism of $\mathfrak{G}$-modules $W \cong \mathfrak{I}/(\mathfrak{m}_S \cdot \mathfrak{I} + \mathfrak{n}_{\mathfrak{G}})$.
We set then $V(\mathfrak{I}) := \mathfrak{I}/(\mathfrak{m}_S \cdot \mathfrak{I} + \mathfrak{n}_{\mathfrak{G}})$. The set of generators of $V(\mathfrak{I})$ may not be uniquely determined, but its structure as a $\mathfrak{G}$-module is unique. The important issue, that will be the core of the classification, will be to determine whether $V(\mathfrak{I})$ is irreducible or not, although it is a minimal set of generators.

2.2. **Notations for the two- and three-dimensional cases.** When applying the preceding constructions in dimensions two or three, we fix the following notations:

- For $n = 2$, the polynomial ring is denoted by $A := \mathbb{C}[x, y]$, the group by $\widetilde{G}$ and any ideal by I.
- For $n = 3$, the polynomial ring is denoted by $B := \mathbb{C}[a, b, c]$, the group by G and any ideal by J.

3. Moduli space of clusters

We recall here the constructions of the Hilbert schemes of points or clusters.

3.1. **Hilbert scheme of points.** Let $X \subset \mathbb{P}^n_{\mathbb{C}}$ be a quasi-projective scheme and N a positive integer. Consider the contravariant functor $\mathcal{H}ilb^N_X$ from the category of schemes to the category of sets

$$\mathcal{H}ilb^N_X : (Schemes) \to (Sets)$$

which is given by

$$\mathcal{H}ilb^N_X(T) := \left\{ Z \subset T \times X \;\middle|\; \begin{array}{ll} (a) & Z \text{ is a closed subscheme} \\ (b) & \text{the morphism } Z \hookrightarrow T \times X \xrightarrow{p} T \text{ is flat} \\ (c) & \forall t \in T, Z_t \subset X \text{ is a closed subscheme} \\ & \text{of dimension 0 and length } N \end{array} \right\}$$

By a theorem of Grothendieck ([15]), this functor is representable by a quasi-projective scheme $\mathrm{Hilb}^N(X)$ equipped with a *universal family* $\Xi^X_N \subset \mathrm{Hilb}^N(X) \times X$. In the sequel, we shall always denote by p the projection to the moduli space (here $\mathrm{Hilb}^N(X)$) and by q the projection to the base (here X). When X is projective, the scheme $\mathrm{Hilb}^N(X)$ is projective and comes with a very ample line bundle (for $\ell \gg 0$):

$$\det \left(p_* \left(\mathcal{O}_{\Xi^X_N} \otimes q^* \mathcal{O}_X(\ell) \right) \right).$$

When $X = \mathbb{C}^n$, one gets an open immersion $\mathrm{Hilb}^N(\mathbb{C}^n) \hookrightarrow \mathrm{Hilb}^N(\mathbb{P}^n_{\mathbb{C}})$ corresponding to the restriction of the universal family. The induced restriction of the preceding determinant line bundle provides us the very ample line bundle $\det \left(p_* \mathcal{O}_{\Xi^{\mathbb{C}^n}_N} \right)$ on $\mathrm{Hilb}^N(\mathbb{C}^n)$.

There exists a natural projective morphism from $\mathrm{Hilb}^N(X)$ to the symmetric product $\mathrm{S}^N(X)$ sending a closed subscheme to the corresponding 0-cycle describing its support, called the *Hilbert-Chow* morphism:

$$\mathscr{H} : \mathrm{Hilb}^N(X) \longrightarrow \mathrm{S}^N(X).$$

By a theorem of Fogarty ([10]), the scheme $\mathrm{Hilb}^N(X)$ is connected. For $\dim X = 2$, it is reduced, smooth and the morphism $\mathscr{H}$ is a resolution of singularities.

3.2. **Hilbert scheme of regular orbits.** We consider the sub-functor $\mathfrak{G}\text{-}\mathcal{H}ilb_{\mathbb{C}^n}$ of $\mathcal{H}ilb^{|\mathfrak{G}|}_{\mathbb{C}^n}$ given by

$$\mathfrak{G}\text{-}\mathcal{H}ilb_{\mathbb{C}^n}(T) := \left\{ Z \in \mathcal{H}ilb^{|\mathfrak{G}|}_{\mathbb{C}^n}(T) \,|\, \forall t \in T, Z_t \subset \mathbb{C}^n \text{ is a } \mathfrak{G}\text{-cluster} \right\}.$$

This functor is representable by a quasi-projective scheme $\mathfrak{G}\text{-Hilb}(\mathbb{C}^n)$ called the *Hilbert scheme of $\mathfrak{G}$-regular orbits*, which is a union of some connected components of the subscheme of $\mathfrak{G}$-fixed points $\left(\mathrm{Hilb}^{|\mathfrak{G}|}(\mathbb{C}^n)\right)^{\mathfrak{G}}$. Furthermore, the quotient $\mathbb{C}^n/\mathfrak{G}$ can be identified with a closed subscheme of $\mathrm{S}^{|\mathfrak{G}|}(\mathbb{C}^n)$ and since the support of a $\mathfrak{G}$-cluster consists exactly of one orbit through $\mathfrak{G}$, the restriction of the Hilbert-Chow morphism factorizes through a projective morphism (see [5, 18, 28]):

$$\mathscr{H} : \mathfrak{G}\text{-Hilb}(\mathbb{C}^n) \longrightarrow \mathbb{C}^n/\mathfrak{G}.$$

There is a unique irreducible component of $\mathfrak{G}\text{-Hilb}(\mathbb{C}^n)$ containing the free $\mathfrak{G}$-orbits and mapping birationally onto $\mathbb{C}^n/\mathfrak{G}$. This component is taken as the definition of the Hilbert scheme of $\mathfrak{G}$-regular orbits in [25]. By the theorem of Bridgeland-King-Reid [5], if $n \leq 3$, then $\mathfrak{G}\text{-Hilb}(\mathbb{C}^n)$ is already irreducible, reduced, smooth and the map $\mathscr{H}$ a crepant resolution of singularities of the quotient $\mathbb{C}^n/\mathfrak{G}$. Moreover, $\mathscr{H}$ is an isomorphism over the open subset of free $\mathfrak{G}$-orbits. As a byproduct, the two definitions coincide.

As before, the scheme $\mathfrak{G}\text{-Hilb}(\mathbb{C}^n)$ is equipped with a universal family $\mathcal{Z}_{\mathfrak{G}}$ which is the restriction of the universal family $\Xi_{|\mathfrak{G}|}^{\mathbb{C}^n}$ corresponding to the closed immersion $\mathfrak{G}\text{-Hilb}(\mathbb{C}^n) \hookrightarrow \text{Hilb}^{|\mathfrak{G}|}(\mathbb{C}^n)$. The induced restriction of the determinant line bundle provides us, by naturality of the construction of the determinant of a family (see [17, §8.1]), the very ample line bundle $\det(p_* \mathcal{O}_{\mathcal{Z}_{\mathfrak{G}}})$ on $\mathfrak{G}\text{-Hilb}(\mathbb{C}^n)$.

4. Rotation groups

4.1. **Polyhedral groups.** Let $\mathrm{SO}(3,\mathbb{R})$ be the group of rotations in $\mathbb{R}^3$. Up to conjugation, there are five different types of finite subgroups of $\mathrm{SO}(3,\mathbb{R})$, called *polyhedral groups*:

- the cyclic groups $C_n \cong \mathbb{Z}/n\mathbb{Z}$ of order $n \geq 1$;
- the dihedral groups $D_n \cong \mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ of order $2n$, $n \geq 1$;
- the group $\mathcal{T}$ of positive isometries of a regular tetrahedra, isomorphic to the alternate group $\mathfrak{A}_4$ of order 12;
- the group $\mathcal{O}$ of positive isometries of a regular octahedra or a cube, isomorphic to the symmetric group $\mathfrak{S}_4$ of order 24;
- the group $\mathcal{I}$ of positive isometries of a regular icosahedra or a regular dodecahedra, isomorphic to the alternate group $\mathfrak{A}_5$ of order 60.

4.2. **Binary polyhedral groups.** Let $\mathbb{H}$ be the real algebra of quaternions, with basis $(1,i,j,k)$. The *norm* of a quaternion $q = a\cdot 1 + b\cdot i + c\cdot j + d\cdot k$ is $N(q) := a^2 + b^2 + c^2 + d^2$, $a,b,c,d \in \mathbb{R}$. Let $\mathbb{S}$ be the three-dimensional sphere of quaternions of length 1 and H the three-dimensional vector subspace of *pure* quaternions (*i.e.* $a = 0$). For $q \in \mathbb{S}$, the action by conjugation $\phi(q) : H \to H$, $x \mapsto q \cdot x \cdot q^{-1}$ is an isometry. Since the group $\mathbb{S}$ is isomorphic to $\mathrm{SU}(2)$ by the identification

$$q = \begin{pmatrix} a + \mathrm{i}b & c + \mathrm{i}d \\ -c + \mathrm{i}d & a - \mathrm{i}b \end{pmatrix},$$

one gets an exact sequence

$$0 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{SU}(2) \xrightarrow{\phi} \mathrm{SO}(3,\mathbb{R}) \longrightarrow 0.$$

For any finite subgroup $G \subset \mathrm{SO}(3,\mathbb{R})$, the inverse image $\widetilde{G} := \phi^{-1}G$ is called a *binary polyhedral group.* It is a finite subgroup of $\mathrm{SU}(2)$ or equivalently, up to conjugation, of $\mathrm{SL}(2,\mathbb{C})$:

- the binary cyclic groups $\widetilde{C}_n \cong C_{2n}$ have order $2n$;
- the binary dihedral groups $\widetilde{D}_n$ have order $4n$;
- the binary tetrahedral group $\widetilde{\mathcal{T}}$ has order 24;
- the binary octahedral group $\widetilde{\mathcal{O}}$ has order 48;
- the binary icosahedral group $\widetilde{\mathcal{I}}$ has order 120.

4.3. **Representations of polyhedral groups.** Consider a binary polyhedral group $\widetilde{G}$, the associated polyhedral group G and set $\tau := \{\pm 1\}$:

$$0 \longrightarrow \tau \longrightarrow \widetilde{G} \xrightarrow{\phi} G \longrightarrow 0.$$

This exact sequence induces an injection of the set of irreducible representations of G in the set of irreducible representations of $\widetilde{G}$: if $\rho : G \to \mathrm{GL}(V)$ is an irreducible representation of G, it induces by composition a representation of $\widetilde{G}$ which is τ-invariant, *i.e.* such that $\rho(-g) = \rho(g)$ for all $g \in \widetilde{G}$. Thanks to this property, if the representation ρ would admit a non-trivial $\widetilde{G}$-submodule, it would also be a non-trivial G-submodule after going to the quotient $\widetilde{G}/\tau \cong G$. This shows also that the image of the injection (since G is a quotient of $\widetilde{G}$):

$$\mathrm{Irr}(G) \hookrightarrow \mathrm{Irr}(\widetilde{G})$$

consists precisely on those irreducible representations which are τ-invariant. These representations are called *pure* and the remaining representations are called *binary*. More precisely, if $\rho : \widetilde{G} \to \mathrm{GL}(V)$ is an irreducible representation of $\widetilde{G}$, the subspace

$$V^\tau := \{v \in V \mid v = \rho(-1)v\}$$

is a $\widetilde{G}$-submodule of V. Hence either $V^\tau = V$ and the representation ρ is pure, or ρ is binary and $V^\tau = \{0\}$.
For each type of binary polyhedral group, we draw the list of the irreducible representations with their dimension. The binary representations are labelled by a "$\widetilde{\ }$" and the trivial representation is denoted by χ_0 in all cases:

- binary cyclic group $\widetilde{C}_n$, $n \geq 1$:

representation	χ_0	$\{\chi_j\}_{j=1,\ldots,n-1}$	$\{\widetilde{\chi}_j\}_{j=1,\ldots,n}$
dimension	1	1	1

- binary dihedral group $\widetilde{D}_n$ for $n = 2\ell + 1$, $\ell \geq 1$:

representation	χ_0	χ_1	$\{\tau_j\}_{j=1,\ldots,\ell}$	$\widetilde{\chi}_1$	$\widetilde{\chi}_2$	$\{\widetilde{\sigma}_j\}_{j=1,\ldots,\ell}$
dimension	1	1	2	1	1	2

- binary dihedral group $\widetilde{D}_n$ for $n = 2\ell$, $\ell \geq 1$:

representation	χ_0	χ_1	χ_2	χ_3	$\{\tau_j\}_{j=1,\ldots,\ell-1}$	$\{\widetilde{\sigma}_j\}_{j=1,\ldots,\ell}$
dimension	1	1	1	1	2	2

- binary tetrahedral group $\widetilde{\mathcal{T}}$:

representation	χ_0	χ_1	χ_2	χ_3	$\widetilde{\chi}_1$	$\widetilde{\chi}_2$	$\widetilde{\chi}_3$
dimension	1	1	1	3	2	2	2

- binary octahedral group $\widetilde{\mathcal{O}}$:

representation	χ_0	χ_1	χ_2	χ_3	χ_4	$\widetilde{\chi}_1$	$\widetilde{\chi}_2$	$\widetilde{\chi}_3$
dimension	1	1	2	3	3	2	2	4

- binary icosahedral group $\widetilde{\mathcal{I}}$:

representation	χ_0	χ_1	χ_2	χ_3	χ_4	$\widetilde{\chi}_1$	$\widetilde{\chi}_2$	$\widetilde{\chi}_3$	$\widetilde{\chi}_4$
dimension	1	3	3	4	5	2	2	4	6

4.4. **Bipolyhedral groups.** For $p, q \in \mathbb{S}$, the action $\sigma(p,q) : \mathbb{H} \to \mathbb{H}, x \mapsto p \cdot x \cdot q^{-1}$ is an isometry and one gets an exact sequence

$$0 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{SU}(2) \times \mathrm{SU}(2) \xrightarrow{\sigma} \mathrm{SO}(4,\mathbb{R}) \longrightarrow 0.$$

For any binary polyhedral group $\widetilde{G}$, the direct image $\sigma(\widetilde{G} \times \widetilde{G}) \subset \mathrm{SO}(4,\mathbb{R})$ is called a *bipolyhedral group*. In §10, we shall make use of the following particular groups:

- $G_6 = \sigma(\widetilde{\mathcal{T}} \times \widetilde{\mathcal{T}})$ of order 288;
- $G_8 = \sigma(\widetilde{\mathcal{O}} \times \widetilde{\mathcal{O}})$ of order 1152;
- $G_{12} = \sigma(\widetilde{\mathcal{I}} \times \widetilde{\mathcal{I}})$ of order 7200.

5. Graph-theoretic intuition

5.1. **McKay quivers.** If $\mathfrak{G} \subset \mathrm{SL}(n,\mathbb{C})$ is a finite subgroup, it defines a natural faithful representation $\mathcal{Q}$ of $\mathfrak{G}$. Let $\{V_0, \ldots, V_k\}$ be a complete set of irreducible representations of $\mathfrak{G}$, where V_0 denotes the trivial one. For each such representation, one may decompose the tensor products

$$\mathcal{Q} \otimes V_i \cong \bigoplus_{j=0}^{k} V_j^{\oplus a_{i,j}}$$

for some non-negative integers $a_{i,j}$. If the character of the representation $\mathcal{Q}$ is real-valued, then $a_{i,j} = a_{j,i}$ for all i, j. One defines the *McKay quiver* as the graph with vertices $V_0, V_1, \dots, V_k$ and $a_{i,j}$ edges between the vertices V_i and V_j. In particular, this quiver may contain some loops. For our purpose, we only consider the *reduced* McKay quiver with vertices $V_1, \dots, V_k$ and one edge between V_i and V_j if $i \neq j$ and $a_{i,j} \neq 0$: this means that we remove from the McKay quiver the vertex V_0, all edges starting from it, all loops and all multiple edges. When there is an edge joining V_i and V_j, the vertices are called *adjacent*.

One may check that all finite subgroups of $\mathrm{SL}(2)$ or $\mathrm{SO}(3, \mathbb{R})$ enter in this context since their natural representation $\mathcal{Q}$ is real-valued.

5.2. **McKay quivers for the polyhedral groups.** For each binary polyhedral group $\widetilde{G} \subset \mathrm{SU}(2)$ and its corresponding polyhedral group $G \subset \mathrm{SO}(3, \mathbb{R})$, we draw the reduced McKay quiver with our conventions. For the binary polyhedral groups, we denote by a white vertex the pure representations and by a black vertex the binary ones. We get (see for example [14, 12, 13]) the graphs of figure 1.

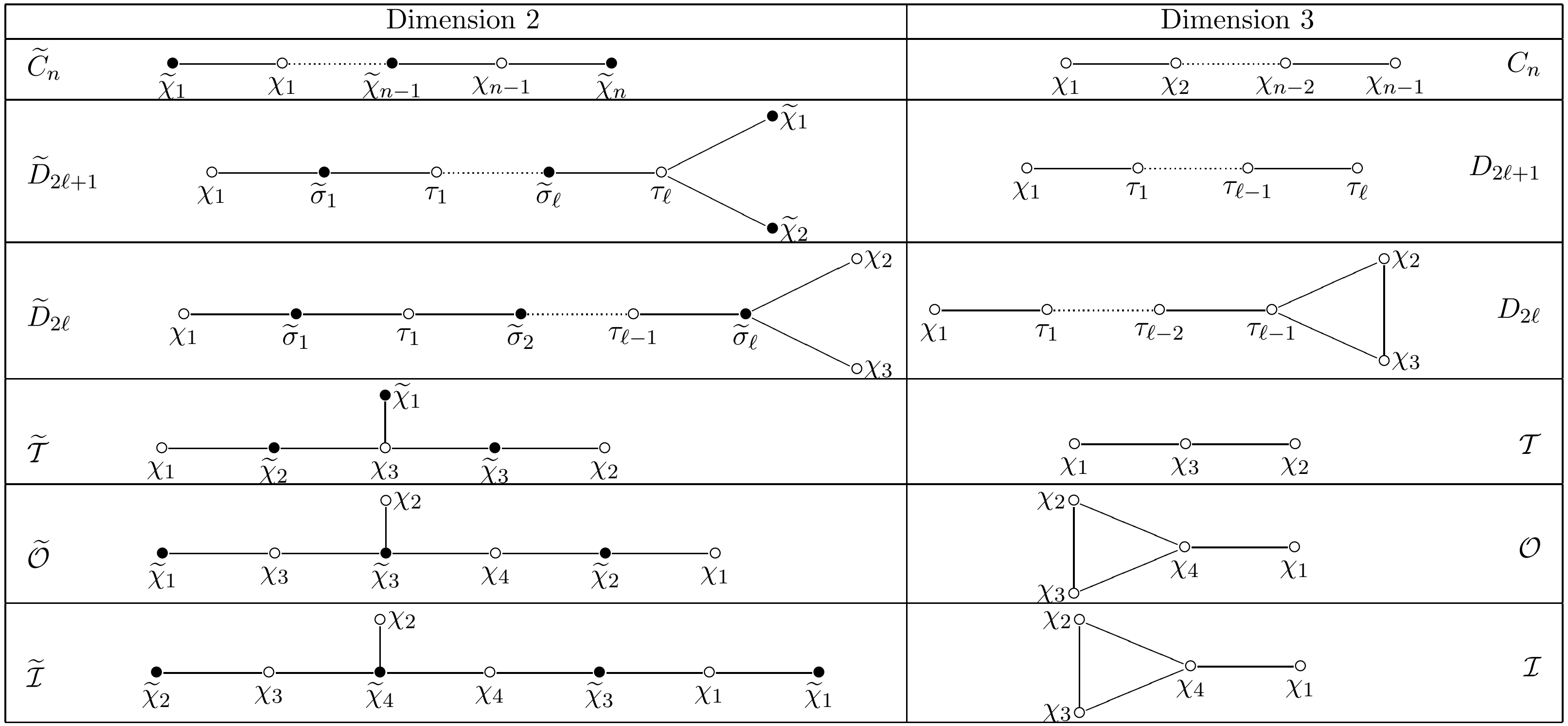

FIGURE 1. Reduced McKay quivers

In the sequel, we shall interpret these graphs as the intersection graphs of a family of smooth rational curves meeting transversally. One may then get the following intuition: looking at the two-dimensional graphs, if one contracts the curves associated to a binary representation (black nodes), then one gets as intersection graph precisely the corresponding graph in dimension three!
Another property of the two-dimensional quivers is that no two pure representations and no two binary representations are adjacent. This means that the preceding idea of contraction contracts only one curve each time.

6. Exceptional fibres in dimensions two and three

Considering the Hilbert-Chow morphism $\mathscr{H} : \mathfrak{G}\text{-Hilb}(\mathbb{C}^n) \longrightarrow \mathbb{C}^n/\mathfrak{G}$, our purpose is to describe the exceptional fibre $\mathscr{H}^{-1}(O)$ over the origin $O \in \mathbb{C}^n/\mathfrak{G}$ in the two- and three-dimensional cases. Note that all finite subgroups of $\mathrm{SL}(2,\mathbb{C})$ or $\mathrm{SO}(3,\mathbb{R})$ enter in the context of §2 since they are subgroups of index 2 of a reflection group (see [13, §2.7]). Hence we may apply the general procedure for the study of the clusters supported at the origin.
The understanding of the exceptional fibre in these cases was achieved by Ito-Nakamura [19, 20] in dimension two and by Gomi-Nakamura-Shinoda [12, 13] in dimension three, by a case-by-case analysis. For the two-dimensional case, there is another proof by Crawley-Boevey [8] avoiding this case-by-case analysis. We recall the results.
For any finite group $\mathfrak{G}$, $\mathrm{Irr}^*(\mathfrak{G})$ denotes the set of irreducible representations but the trivial one.

6.1. **Structure of the exceptional fibre in dimension two.** Let $\widetilde{G} \subset \mathrm{SL}(2,\mathbb{C})$ be a binary polyhedral group and denote the Hilbert-Chow morphism by

$$\widetilde{\pi} : \widetilde{G}\text{-Hilb}\big(\mathbb{C}^2\big) \longrightarrow \mathbb{C}^2/\widetilde{G}.$$

For each non-trivial irreducible representation ρ of $\widetilde{G}$, set

$$E(\rho) := \{I \in \widetilde{\pi}^{-1}(O)_{\mathrm{red}} \,|\, V(I) \supset \rho\}.$$

Theorem 6.1. ([19, Theorem 3.1]

- *Each $E(\rho)$ is a smooth rational curve of self-intersection -2.*
- $\widetilde{\pi}^{-1}(O)_{red} = \bigcup_\rho E(\rho)$ *and* $\widetilde{\pi}^{-1}(O) = \sum_\rho \dim\rho \cdot E(\rho)$ *as a Cartier-divisor,* $\rho \in \mathrm{Irr}^*(\widetilde{G})$.
- *If $I \in E(\rho)$ and $I \notin E(\rho')$ for all $\rho \neq \rho'$, then $V(I) \cong \rho$.*
- *If $I \subset E(\rho) \cap E(\rho')$, then $V(I) \cong \rho \oplus \rho'$ and the curves $E(\rho)$ and $E(\rho')$ intersect transversally at I.*
- *The intersection graph of these curves is the reduced McKay quiver of the group $\widetilde{G}$.*

In particular, a generator $V(I)$ does not contain more than one copy of any irreducible representation, and $E(\rho)\cap E(\rho') \neq \emptyset$ if and only if the representations ρ and ρ' are adjacent.

6.2. **Structure of the exceptional fibre in dimension three.** Let $G \subset \mathrm{SO}(3,\mathbb{R})$ be a polyhedral group and denote the Hilbert-Chow morphism by

$$\pi : G\text{-Hilb}\big(\mathbb{C}^3\big) \longrightarrow \mathbb{C}^3/G.$$

For each non-trivial irreducible representation ρ of G, set

$$C(\rho) := \{J \in \pi^{-1}(O)_{\mathrm{red}} \,|\, V(J) \supset \rho\}.$$

Theorem 6.2. ([13, Theorem 3.1])

- *Each $C(\rho)$ is a smooth rational curve.*
- $\pi^{-1}(O)_{red} = \bigcup_\rho C(\rho)$, $\rho \in \mathrm{Irr}^*(G)$.
- *If $J \in C(\rho)$ and $J \notin C(\rho')$ for all $\rho \neq \rho'$, then $V(J) \cong \rho$.*
- *The intersection graph of these curves is the reduced McKay quiver of the group G.*

6.3. **Explicit parameterizations.** Let us explain briefly the explicit parameterizations of the exceptional curves obtained in *loc.cit.* This description holds both in dimensions two and three so we do it with our general notations. The example of the cyclic group is treated in §9. As we explained in §2,

$$\mathfrak{m}_S/\mathfrak{n}_{\mathfrak{G}} \cong \bigoplus_{\substack{\rho\in\mathrm{Irr}(\mathfrak{G})\\ \rho\neq\rho_0}} \bigoplus_{i=1}^{2\dim\rho} V^{(i)}(\rho)$$

where ρ_0 denotes the trivial representation. Thanks to the exact sequence

$$0 \longrightarrow \mathfrak{I}/\mathfrak{n}_{\mathfrak{G}} \longrightarrow \mathfrak{m}_S/\mathfrak{n}_{\mathfrak{G}} \longrightarrow \mathfrak{m}_S/\mathfrak{I} \longrightarrow 0,$$

if one wants to parameterize a flat family of clusters over $\mathbb{P}_1$, one has to choose, in the trivial sheaf:

$$\mathcal{O}_{\mathbb{P}^1} \otimes \bigoplus_{\substack{\rho\in\mathrm{Irr}(\mathfrak{G})\\ \rho\neq\rho_0}} \bigoplus_{i=1}^{2\dim\rho} V^{(i)}(\rho),$$

a locally free $\mathfrak{G}$-equivariant sheaf affording the regular representation on each fibre whose quotient is also locally free. The parameterizations are then produced as follows: one chooses *one* non trivial subbundle

$$\mathcal{O}_{\mathbb{P}_1}(-1) \otimes \rho \hookrightarrow \mathcal{O}_{\mathbb{P}_1} \otimes (V^{(i)}(\rho) \oplus V^{(j)}(\rho))$$

for some appropriate choice of the indices, and shows that this gives the required family whose points $\mathfrak{I}$ are characterized by their generator

$$V(\mathfrak{I}) \subset \mathbb{P}(V^{(i)}(\rho) \oplus V^{(j)}(\rho)).$$

That is: once one choice has been made, the other choices are automatic, and we shall see that they always correspond to a trivial subbundle (see 8.4).

7. Geometric construction

Let $\widetilde{G}$ be a binary polyhedral group acting on $A = \mathbb{C}[x, y]$. Set $\tau := \langle \pm 1\rangle \subset \widetilde{G}$ and $G := \widetilde{G}/\tau$ the associated polyhedral group as before. It is important for the sequel to begin so, and not to choose the group G with its action on some coordinates first, as we shall see. We aim to define a regular map

$$\mathscr{S} : \widetilde{G}\text{-Hilb}\big(\mathbb{C}^2\big) \longrightarrow G\text{-Hilb}\big(\mathbb{C}^3\big)$$

inducing a map between the exceptional fibres over the origin.
Since $A^\tau = \mathbb{C}[x^2, y^2, xy]$, we consider the following composition of ring morphisms, with $B = \mathbb{C}[a, b, c]$:

$$\sigma : B \twoheadrightarrow B \big/ \langle ab - c^2\rangle \xrightarrow{\sim} A^\tau \longrightarrow A \tag{2}$$

where the identification is defined by $a = x^2, b = y^2, c = xy$. The action of $\widetilde{G}$ on A induces an action of G on A^τ. Using the identification, we can define an action of G on the coordinates a, b, c, inducing an action on B with the property that the cone $K = \langle ab - c^2 \rangle$ is G-invariant. This is the reason why we did not fix the action of G at first: another choice of identification would induce another action of G.

Let I be an ideal of A and $J := \sigma^{-1}(I)$ the corresponding ideal of B. Observe the following property of the map σ:

Lemma 7.1. *If I is a $\widetilde{G}$-cluster in A, then J is a G-cluster in B. Furthermore, if I is supported at the origin, then so is J.*

Proof. If I is a $\widetilde{G}$-cluster, then $A/I \cong \mathbb{C}[\widetilde{G}]$. Since the group τ is finite, we have isomorphisms:

$$B/J \cong A^\tau / I^\tau \cong (A/I)^\tau \cong \mathbb{C}[\widetilde{G}]^\tau \cong \mathbb{C}[G],$$

hence J is a G-cluster in B. Furthermore, note that $\sigma^{-1}\mathfrak{m}_A = \mathfrak{m}_B$ hence if I is a $\widetilde{G}$-cluster supported at the origin, one has $I \subset \mathfrak{m}_A$ and then $J \subset \mathfrak{m}_B$, which implies that J is also supported at the origin (see §2.1). $\square$

Therefore, this construction defines set-theoretically a map between the two moduli spaces of clusters $\mathscr{S} : \widetilde{G}\text{-Hilb}\big(\mathbb{C}^2\big) \longrightarrow G\text{-Hilb}\big(\mathbb{C}^3\big)$ by $\mathscr{S}(I) \overset{\text{Def}}{=} J$. It remains to see that this map is a regular morphism.

Proposition 7.2. *The map $\mathscr{S}$ is regular, projective, and induces a map between the exceptional fibres.*

Proof.

$\diamond$ In order to get that the map $\mathscr{S}$ is regular, we show that it is induced by a natural transformation between the two functors of points

$$\widetilde{G}\text{-}\mathcal{H}ilb_{\mathbb{C}^2}(\cdot) \Longrightarrow G\text{-}\mathcal{H}ilb_{\mathbb{C}^3}(\cdot).$$

Let T be a scheme and $Z \in \widetilde{G}\text{-}\mathcal{H}ilb_{\mathbb{C}^2}(T)$. Then $Z \subset T \times \mathbb{C}^2$ is a flat family of $\widetilde{G}$-clusters over T and the map $Z \hookrightarrow T \times \mathbb{C}^2$ is τ-equivariant (for a trivial action on T). It induces a family

$$Z/\tau \hookrightarrow T \times (\mathbb{C}^2/\tau) \hookrightarrow T \times \mathbb{C}^3$$

where the quotient $\mathbb{C}^2/\tau$ is considered as the cone $\langle ab - c^2 \rangle$ in $\mathbb{C}^3$. If T is a point, this is precisely our set-theoretic construction since then if Z is given by an ideal I, Z/τ is given by the ideal I^τ.

In order to show that $Z/\tau \in G\text{-}\mathcal{H}ilb_{\mathbb{C}^3}(T)$, we have to prove that this family is flat over T. Since this problem is local in T, we may assume that T is an affine scheme, say $T = \operatorname{Spec} R$. Then the family Z is given by a τ-equivariant quotient $R \otimes A \twoheadrightarrow Q$ so that the composition $R \hookrightarrow R \otimes_{\mathbb{C}} A \twoheadrightarrow Q$ makes Q a flat R-module. The family Z/τ is then given by the quotient

$$R \hookrightarrow R \otimes_{\mathbb{C}} B \twoheadrightarrow R \otimes_{\mathbb{C}} A^\tau \twoheadrightarrow Q^\tau,$$

where the quotient $R \otimes_{\mathbb{C}} B \twoheadrightarrow R \otimes_{\mathbb{C}} A^\tau$ is induced by tensorization of the quotient $B \twoheadrightarrow A^\tau$. We have to show that this makes Q^τ a flat R-module. By hypothesis, the functor $Q \otimes_R -$ in the category of R-modules is exact. Since τ is finite, the functor $(-)^\tau$ is also exact in this category, and we note that the functor $Q^\tau \otimes_R -$ is the composition of this two functors since

$$Q^\tau \otimes_R N = (Q \otimes_R N)^\tau$$

for any R-module N. Hence the functor $Q^\tau \otimes_R -$ is exact, which means that the family is flat.

$\diamond$ The composition of ring morphisms (2) gives an equivariant ring morphism

$$\begin{tikzcd} \mathbb{C}[a,b,c] \arrow[r, "\sigma"] \arrow[loop below, "G"] & \mathbb{C}[x,y] \arrow[loop below, "\widetilde{G}"] \end{tikzcd}$$

inducing a surjective map at the level of the invariants: $\mathbb{C}[a,b,c]^G \twoheadrightarrow \mathbb{C}[x,y]^{\widetilde{G}}$, hence a closed immersion

$$\eta : \mathbb{C}^2/\widetilde{G} = \operatorname{Spec}\mathbb{C}[x,y]^{\widetilde{G}} \longrightarrow \operatorname{Spec}\mathbb{C}[a,b,c]^G = \mathbb{C}^3/G.$$

Taking more care of the cone $K = \mathbb{C}^2/\tau$ (in the notations of the introduction), the equivariant map

$$\begin{tikzcd} \mathbb{C}^2 \arrow[r] \arrow[loop below, "\widetilde{G}"] & \mathbb{C}^2/\tau \arrow[r] \arrow[loop below, "G"] & \mathbb{C}^3 \arrow[loop below, "G"] \end{tikzcd}$$

induces the η map between the quotients:

$$\eta : \mathbb{C}^2/\widetilde{G} \xrightarrow{\sim} \left(\mathbb{C}^2/\tau\right)/\,G \longrightarrow \mathbb{C}^3/G$$

sending the origin $O \in \mathbb{C}^2/\widetilde{G}$ to the origin $O \in \mathbb{C}^3/G$ and by definition of $\mathscr{S}$ the following diagram is commutative:

$$\begin{tikzcd} \widetilde{G}\text{-Hilb}\left(\mathbb{C}^2\right) \arrow[r, "\mathscr{S}"] \arrow[d, "\widetilde{\pi}"'] & G\text{-Hilb}\left(\mathbb{C}^3\right) \arrow[d, "\pi"] \\ \mathbb{C}^2/\widetilde{G} \arrow[r, "\eta"] & \mathbb{C}^3/G \end{tikzcd}$$

This implies that $\mathscr{S}$ induces a map between the exceptional fibres

$$\widetilde{\pi}^{-1}(O) \xrightarrow{\mathscr{S}} \pi^{-1}(O).$$

$\diamond$ We prove that the map $\mathscr{S}$ is proper by applying the valuative criterion of properness. Let K be any field over $\mathbb{C}$ and $R \subset K$ any valuation ring with quotient field K. Consider a commutative diagram:

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "\phi"] \arrow[d, "i"] & \widetilde{G}\text{-Hilb}\left(\mathbb{C}^2\right) \arrow[d, "\mathscr{S}"] \\ \operatorname{Spec} R \arrow[r, "\psi"] & G\text{-Hilb}\left(\mathbb{C}^3\right) \end{tikzcd}$$

We have to show that there exists a unique factorization

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "\phi"] \arrow[d, "i"] & \widetilde{G}\text{-Hilb}\left(\mathbb{C}^2\right) \arrow[d, "\mathscr{S}"] \\ \operatorname{Spec} R \arrow[r, "\psi"] \arrow[ur, "\tilde{\phi}"] & G\text{-Hilb}\left(\mathbb{C}^3\right) \end{tikzcd}$$

making the whole diagram commute.

By modular interpretation, the data of the map ϕ consists in an ideal $I \subset K[x,y]$ such that $K[x,y]/I \cong \mathbb{C}[\widetilde{G}] \otimes_{\mathbb{C}} K$ and $K[x,y]/I$ is K-flat (it is here trivial since K is a field). Similarly, the data of the map ψ consists in an ideal $J \subset R[a,b,c]$ such that $R[a,b,c]/J \cong$

$\mathbb{C}[G] \otimes_{\mathbb{C}} R$ and $R[a,b,c]/J$ is R-flat. The commutativity $\mathscr{S} \circ \phi = \psi \circ i : \operatorname{Spec} K \to G\text{-Hilb}\big(\mathbb{C}^3\big)$ means the following. Consider the diagram of ring morphisms induced by natural extension of scalars and base-change from the map σ:

$$\begin{tikzcd} R[a,b,c] \arrow[r, "\sigma_R"] \arrow[d, hook] & R[x,y] \arrow[d, hook] \\ K[a,b,c] \arrow[r, "\sigma_K"] & K[x,y] \end{tikzcd}$$

Then the commutativity condition means that $\sigma_K^{-1}(I) = J \cdot K[a,b,c]$.

We are looking for a map $\tilde{\phi}$ such that $\tilde{\phi} \circ i = \phi$ and $\mathscr{S} \circ \tilde{\phi} = \psi$, *i.e.* for an ideal $\widetilde{I} \subset R[x,y]$ such that $R[x,y]/\widetilde{I} \cong \mathbb{C}[\widetilde{G}] \otimes_{\mathbb{C}} R$ and $R[x,y]/\widetilde{I}$ is R-flat, satisfying the conditions $\widetilde{I} \cdot K[x,y] = I$ and $\sigma_R^{-1}(\widetilde{I}) = J$.

A natural candidate is $\widetilde{I} \stackrel{\text{Def}}{=} I \cap R[x,y]$. We have to prove that it satisfies all the conditions and that it is unique for these properties. Denote by $\nu : K - \{0\} \to H$ the valuation with values in a totally ordered group H, satisfying the properties:

$$\nu(x \cdot y) = \nu(x) + \nu(y) \text{ and } \nu(x+y) \geq \min(\nu(x), \nu(y)) \quad \text{for } x, y \in K - \{0\}$$

and such that $R = \{x \in K \mid \nu(x) \geq 0\} \cup \{0\}$. Recall that R is by definition integral and that a R-module is flat if and only if it is torsion-free (see for instance [1, 16]).

- It is already clear that $\widetilde{I} \cdot K[x,y] \subset I$. Conversely, Let $P = \sum_{i,j} p_{i,j} x^i y^j \in I$ and $p \in \{p_{i,j}\}$ an element of minimal valuation. If $\nu(p) \geq 0$, then $P \in \widetilde{I}$. Else all coefficients of $p^{-1}P$ have positive valuation and so $p^{-1}P \in \widetilde{I}$. So $P = p \cdot (p^{-1}P) \in \widetilde{I} \cdot K[x,y]$, hence the equality.
- By commutativity of the above diagram,

$$\begin{aligned} \sigma_R^{-1}(\widetilde{I}) &= \sigma_R^{-1}(I \cap R[x,y]) \\ &= \sigma_K^{-1}(I) \cap R[a,b,c] \\ &= (J \cdot K[a,b,c]) \cap R[a,b,c]. \end{aligned}$$

  It is already clear that $J \subset (J \cdot K[a,b,c]) \cap R[a,b,c]$. Conversely, let $P \in (J \cdot K[a,b,c]) \cap R[a,b,c]$, decomposed as $P = \sum_\ell U_\ell \cdot V_\ell$ with $U_\ell \in J$ and $V_\ell \in K[a,b,c]$. As before, there exists a coefficient q in all V_ℓ's of minimal valuation, and we assume $\nu(q) < 0$ (else there is no problem). Then $q^{-1}P \in J$. By assumption, the R-module $R[a,b,c]/J$ is torsion-free, so the multiplication by $q^{-1} \in R$ is injective. This means that $P \in J$.
- By definition, we have an R-linear inclusion $R[x,y]/\widetilde{I} \hookrightarrow K[x,y]/I$, which shows that $R[x,y]/\widetilde{I}$ is torsion-free, hence flat. It inherits an action of $\widetilde{G}$ and since $K[x,y]/I \cong \mathbb{C}[\widetilde{G}] \otimes_{\mathbb{C}} K$, there exists a subrepresentation V of $\mathbb{C}[\widetilde{G}]$ such that $R[x,y]/\widetilde{I} \cong V \otimes_{\mathbb{C}} R$ (this uses the flatness, see [20, lemma 9.4]). By the isomorphism of R-modules $R[x,y]/\widetilde{I} \otimes_R K \cong K[x,y]/I$, the representation V is such that $V \otimes_R K = \mathbb{C}[\widetilde{G}] \otimes_{\mathbb{C}} K$, which forces $V \cong \mathbb{C}[\widetilde{G}]$.
- The uniqueness of the candidate follows from the condition $\tilde{I} \cdot K[x,y] = I$ since as we already noted:

$$I \cap R[x,y] = (\tilde{I} \cdot K[x,y]) \cap R[x,y] = \tilde{I}$$

  so our natural candidate is the only possibility.

$\diamond$ To finish with, remark that any proper map between to quasi-projective varieties is automatically a projective map. $\square$

8. Contracted versus non-contracted fibres

Theorem 8.1. *Consider the restriction of the map $\mathscr{S} : \widetilde{G}\text{-Hilb}(\mathbb{C}^2) \longrightarrow G\text{-Hilb}(\mathbb{C}^3)$ to a reduced curve $E(\rho)$. Then:*

1. *If the representation ρ is pure, then $\mathscr{S}$ maps isomorphically the curve $E(\rho)$ onto the curve $C(\rho)$.*
2. *If the representation ρ is binary, then $\mathscr{S}$ contracts the curve $E(\rho)$ to a point.*

Proof. Let $E(\rho)$ be any exceptional curve. Since the map $\mathscr{S}$ sends this curve to the bunch of curves $\pi^{-1}(O)$, the image lies in some irreducible component C and the restricted morphism $\mathscr{S} : E(\rho) \to C$ is a proper map. We prove that:

- if the representation ρ is binary, then the map $\mathscr{S} : E(\rho) \to C$ contracts the curve to a point;
- if the representation ρ is pure, then $C = C(\rho)$ and the restricted map $\mathscr{S} : E(\rho) \to C(\rho)$ is an isomorphism.

The parameterizations of the two curves $E(\rho)$ and C defines a composite proper map f whose properties reflect those of the restriction of $\mathscr{S}$:

$$\begin{tikzcd}
\mathbb{P}_1 \arrow[r, "\sim", "\phi"'] \arrow[d, "f"] & E(\rho) \subset \widetilde{G}\text{-Hilb}(\mathbb{C}^2) \arrow[d, "\mathscr{S}"] \\
\mathbb{P}_1 \arrow[r, "\sim", "\psi"'] & C \subset G\text{-Hilb}(\mathbb{C}^3)
\end{tikzcd}$$

We know (see [16, II.6.8,II.6.9]) that either the map f contracts the curve to a point, or it is a finite surjective map. The basic idea in order to determine which case occurs is to suppose given an ample line bundle $\mathcal{O}_{\mathbb{P}_1}(a)$ on the target (with $a > 0$): if the map f contracts the curve to a point, then $f^*\mathcal{O}_{\mathbb{P}_1}(a)$ is trivial and else $f^*\mathcal{O}_{\mathbb{P}_1}(a) \cong \mathcal{O}_{\mathbb{P}_1}(\deg(f)\cdot a)$ is ample.

The natural candidate for an ample line bundle over the curve C is the determinant $\det(p_*\mathcal{O}_{Z(C)})$ obtained by restriction of the universal family $Z(C) := \mathcal{Z}_G|_C$.

The parameterization $\mathbb{P}_1 \xrightarrow{\phi} \widetilde{G}\text{-Hilb}(\mathbb{C}^2)$ of the curve $E(\rho)$ corresponds to a flat family $Z_{\widetilde{G}}(\rho) \subset \mathbb{P}_1 \times \mathbb{C}^2$ which is the restriction to $E(\rho)$ of the universal family $\mathcal{Z}_{\widetilde{G}}$ over $\widetilde{G}\text{-Hilb}(\mathbb{C}^2)$. The direct image $p_*\mathcal{O}_{Z_{\widetilde{G}}(\rho)}$ is a vector bundle of rank $|\widetilde{G}|$ over $\mathbb{P}_1$ equipped with an action of $\widetilde{G}$ affording the regular representation on each fibre. It admits an isotypical decomposition over the irreducible representation of $\widetilde{G}$ and we recall the well-known explicit decomposition:

Lemma 8.2.

$$p_*\mathcal{O}_{Z_{\widetilde{G}}(\rho)} \cong \left(\mathcal{O}_{\mathbb{P}_1}(1) \oplus \mathcal{O}_{\mathbb{P}_1}^{\oplus \dim\rho - 1}\right) \otimes \rho \oplus \bigoplus_{\substack{\rho' \in \mathrm{Irr}(\widetilde{G}) \\ \rho' \neq \rho}} \mathcal{O}_{\mathbb{P}_1}^{\oplus \dim\rho'} \otimes \rho'$$

Proof of the lemma. This is an equivalent form of [22, §2.1 lemma] or [18, Proposition 6.2(3)]. We recall briefly the argument. Since this bundle is a quotient of $\mathcal{O}_{\mathbb{P}_1} \otimes A$ (see §6.3), it is generated by its global sections, hence it is a sum of line bundles $\mathcal{O}_{\mathbb{P}_1}(a)$ for

$a \geq 0$. By the classical observation $\deg(p_*\mathcal{O}_{Z_{\widetilde{G}}(\rho)}) = 1$ (see [14]), all line bundles are trivial but one, of degree one. □

In particular, note that $\det(p_*\mathcal{O}_{Z_{\widetilde{G}}(\rho)}) \cong \mathcal{O}_{\mathbb{P}_1}(\dim\rho)$ is the ample determinant line bundle in dimension two.
Thanks to the functorial definition of the map $\mathscr{S}$, the composition

$$\mathbb{P}_1 \xrightarrow{\phi} \widetilde{G}\text{-Hilb}\big(\mathbb{C}^2\big) \xrightarrow{\mathscr{S}} G\text{-Hilb}\big(\mathbb{C}^3\big)$$

parameterizes the flat family $Z_{\widetilde{G}}(\rho)/\tau$ whose structural sheaf is $\mathcal{O}_{Z_{\widetilde{G}}(\rho)/\tau} = \Big(\mathcal{O}_{Z_{\widetilde{G}}(\rho)}\Big)^\tau$ and one gets:

$$f^*(\det(p_*\mathcal{O}_{Z(C)})) = \det\Big((p_*\mathcal{O}_{Z_{\widetilde{G}}(\rho)})^\tau\Big).$$

Now, as we noticed in §4.3, taking the invariants under τ keeps invariant the pure representations and kills the binary ones. Hence:

- If the representation ρ is binary, then:

$$\Big(p_*\mathcal{O}_{Z_{\widetilde{G}}(\rho)}\Big)^\tau \cong \bigoplus_{\rho'\in\mathrm{Irr}(G)} \mathcal{O}_{\mathbb{P}_1}^{\oplus\dim\rho'} \otimes \rho'$$

  hence $\det(p_*\mathcal{O}_{Z_{\widetilde{G}}(\rho)})^\tau \cong \mathcal{O}_{\mathbb{P}_1}$ is trivial;
- If the representation ρ is pure, then:

$$\Big(p_*\mathcal{O}_{Z_{\widetilde{G}}(\rho)}\Big)^\tau \cong \Big(\mathcal{O}_{\mathbb{P}_1}(1) \oplus \mathcal{O}_{\mathbb{P}_1}^{\oplus\dim\rho-1}\Big) \otimes \rho \oplus \bigoplus_{\substack{\rho'\in\mathrm{Irr}(G)\\ \rho'\neq\rho}} \mathcal{O}_{\mathbb{P}_1}^{\oplus\dim\rho'} \otimes \rho'$$

  hence $\det(p_*\mathcal{O}_{Z_{\widetilde{G}}(\rho)})^\tau \cong \mathcal{O}_{\mathbb{P}_1}(\dim\rho)$ is ample.

This achieves the first part of the proof. It remains to show that in the case of a pure representation ρ, the target curve is $C = C(\rho)$ and that the finite surjective map f is an isomorphism. We do it by hand. A point $I \in E(\rho)$ is characterized by the choice of $V(I)$ and generically $V(I) \cong \rho$. For a pure representation ρ, the polynomials defining $V(I)$ are even hence:

$$V(I^\tau) = V\left((A \cdot V(I) + \mathfrak{n}_A)^\tau\right) \supset V(I)$$

so generically $V(I^\tau) = V(I)$ (only modified by setting $a = x^2, b = y^2, c = xy$). This means that $C = C(\rho)$ and if $I \neq J \in E(\rho)$, then $V(I) \neq V(J)$ hence the images are also different, so the map is generically injective. This concludes the proof. □

As a byproduct of our argument, we get the following equivalent in dimension three of the lemma 8.2 which, to our knowledge, does not appear explicitly in the literature:

Corollary 8.3. *For any finite subgroup $G \subset \mathrm{SO}(3,\mathbb{R})$ and any non-trivial representation ρ of G, the restriction of the tautological bundle to the exceptional curve $C(\rho)$ decomposes as:*

$$p_*\mathcal{O}_{Z_G(\rho)} \cong \Big(\mathcal{O}_{\mathbb{P}_1}(1) \oplus \mathcal{O}_{\mathbb{P}_1}^{\oplus\dim\rho-1}\Big) \otimes \rho \oplus \bigoplus_{\substack{\rho'\in\mathrm{Irr}(G)\\ \rho'\neq\rho}} \mathcal{O}_{\mathbb{P}_1}^{\oplus\dim\rho'} \otimes \rho'$$

Proof. The same argument as in the proof of lemma 8.2 shows that this bundle in generated by its global sections. The bijectivity of the map f on the curves associated to pure representations (in the notation of the proof of theorem 8.1) implies that $\det(p_*\mathcal{O}_{Z_G(\rho)}) \cong \mathcal{O}_{\mathbb{P}_1}(\dim\rho)$, hence in the isotypical decomposition there is only one non-trivial line bundle, of degree one, and we already know by the explicit parameterizations that the isotypical component corresponding to ρ is not trivial. $\square$

Remark 8.4. *In the decomposition of the lemma 8.2, the unique presence of the $\mathcal{O}_{\mathbb{P}_1}(1)$ corresponds to the choice of the line $V(I)$ in a projective space $\mathbb{P}(\rho \oplus \rho)$ as explicitly described in §6.3. The fact that no other ample bundle occurs reflects the property that once one choice has been made, the other generators of the ideal do not involve the choice any more, as one can easily notice from the explicit computations of* [20, §13,§14] *(see §9 in this paper for an example). In the three-dimensional case, the same situation occurs thanks to the corollary 8.3.*

We get now the theorem 1.1 presented in the introduction as a corollary of the theorem 8.1:

Corollary 8.5. *The image $\mathcal{Y} := \mathscr{S}(\widetilde{G}\text{-Hilb}(\mathbb{C}^2))$ projects onto the quotient K/G, inducing a partial resolution of singularities containing only the exceptional curves corresponding to pure representations. The map $\mathscr{S} : \widetilde{G}\text{-Hilb}(\mathbb{C}^2) \longrightarrow \mathcal{Y}$ is a resolution of singularities contracting the excess exceptional curves to ordinary nodes.*

Proof. The projection $\pi : \mathcal{Y} \longrightarrow \mathbb{C}^3/G$ factors through K/G by construction of $\mathcal{Y}$. The other assertions result from theorem 8.1. The excess curves contract to ordinary nodes since, as one checks with the figure 1, each excess (-2)-curve is contracted to a different point. $\square$

9. Example: the cyclic group case

Let the cyclic group $\widetilde{C}_n \cong \mathbb{Z}/(2n)\mathbb{Z}$ act on $\mathbb{C}^2$ with generator:

$$\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \quad \text{with } \xi = e^{\frac{2\pi \mathrm{i}}{(2n)}}.$$

The choice of coordinates made in §7 implies that the group $C_n \cong \mathbb{Z}/n\mathbb{Z}$ acts on $\mathbb{C}^3$ with generator:

$$\begin{pmatrix} \xi^2 & 0 & 0 \\ 0 & \xi^{-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The irreducible representations of the cyclic group $\widetilde{C}_n$ are given by the matrices (ξ^i), $i = 0, \dots, 2n-1$. For i even, they are also the irreducible representations of C_n. There are then n pure and n binary representations. With the notations of §4.3, we set $\chi_i := \rho_{2i}$ and $\widetilde{\chi}_i = \rho_{2i+1}$ for $i = 0, \dots, n-1$. By Theorem 8.1, the exceptional curves on $\widetilde{C}_n\text{-Hilb}(\mathbb{C}^2)$ corresponding to the binary representations are contracted by $\mathscr{S}$ to a node on $\mathscr{S}(\widetilde{C}_n\text{-Hilb}(\mathbb{C}^2))$ whereas the curves corresponding to the pure representations are in $1:1$ correspondence with the exceptional curves downstairs (see figure 2). In this section, we check this by a direct computation.

The ring of invariants $\mathbb{C}[x,y]^{\widetilde{C}_n}$ is generated by x^{2n}, y^{2n}, xy and $\mathbb{C}[a,b,c]^{C_n}$ is generated by c, a^n, b^n, ab. Recall the description of the exceptional curves of $\widetilde{C}_n\text{-Hilb}(\mathbb{C}^2)$ following [20, Theorem 12.3]. We sort the basis of the algebra of coinvariants with respect to each irreducible representation:

$$\{1\}, \{x, y^{2n-1}\}, \dots, \{x^i, y^{2n-i}\}, \dots, \{x^{2n-1}, y\}.$$

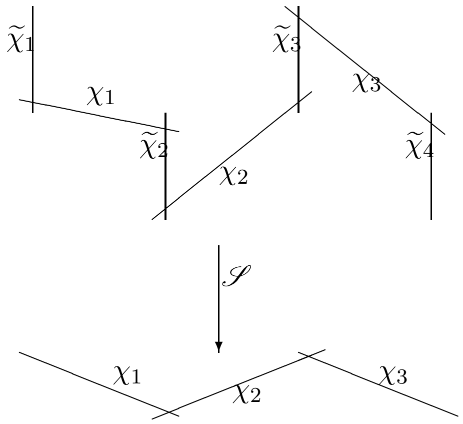

FIGURE 2. Contracted fibres for $\widetilde{C}_4$

To choose a cluster $I/\mathfrak{n}_A$ supported at the origin amounts in choosing one copy of each non-trivial representation, *i.e.* for all $i = 1, \ldots, 2n-1$ a point $(p_i : q_i) \in \mathbb{P}_1$ defining the ideal by the generators:

$$\langle p_1 x - q_1 y^{2n-1}, \ldots, p_i x^i - q_i y^{2n-i}, \ldots, p_{2n-1} x^{2n-1} - q_{2n-1} y \rangle.$$

But the point is that one only needs *one* choice. Suppose there exists an index i such that $p_i q_i \neq 0$, and take the smaller i with this property. Set $p = p_i, q = q_i$ and $v = px^i - qy^{2n-i}$. Then since xy is invariant, $x^{i+1}, \ldots, x^{2n-1} \in I/\mathfrak{n}_A$ and $y^{2n-i+1}, \ldots, y^{2n-1} \in I/\mathfrak{n}_A$ so all our other choices were trivial, and $V(I) = \mathbb{C} \cdot v$. More formally, we parameterized the exceptional curve $E(\rho_i)$ by a subbundle:

$$\mathcal{O}_{\mathbb{P}_1}(-1) \otimes \rho_i \oplus \bigoplus_{j \neq i} \mathcal{O}_{\mathbb{P}_1} \otimes \rho_j \hookrightarrow \bigoplus_j (\mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}) \otimes \rho_j.$$

If there is no such index, suppose x^i is the minimal power of x in the choice: in order to find once each non-trivial representation one has to choose y^{2n-i+1} and the minimal set of generators $V(I) = \mathbb{C} \cdot x^i \oplus \mathbb{C} \cdot y^{2n-i+1}$ contains two adjacent representations.
Otherwise stated, a $\widetilde{C}_n$-cluster at the origin takes the form:

$$I_j(p:q) := \langle px^j - qy^{2n-j}, xy, x^{j+1}, y^{2n-j+1} \rangle, \quad 1 \leq j \leq 2n-1, \ (p:q) \in \mathbb{P}_1$$

(the above expression contains enough generators to include the two possible cases) and

$$E(\rho_j) = \{I_j(p:q)\}.$$

By the same method, one sees easily that a C_n-cluster at the origin takes the form:

$$J_k(s:t) := \langle sa^k - tb^{n-k}, c, a^{k+1}, b^{n-k+1}, ab \rangle, \quad 1 \leq k \leq n-1, \ (s:t) \in \mathbb{P}_1$$

and

$$C(\chi_k) = \{J_k(s:t)\}.$$

Recall that with the construction (2) we have to compute $\sigma^{-1}(I_j(p:q))$. Denoting by $\bar{\sigma}$ the map $B \big/ \langle ab - c^2 \rangle \longrightarrow A$, it is equivalent to compute $\bar{\sigma}^{-1}(I_j(p:q))$. First we compute $I_j(p:q)^\tau \in A^\tau$. We distinguish two cases:

- j *even, i.e.* $j = 2j'$, $j' = 1, \dots, n-1$. In this case we have
$$I_j(p:q)^\tau = I_j(p:q) = \langle p(x^2)^{j'} - q(y^2)^{n-j'}, xy, (x^2)^{j'+1}, (y^2)^{n-j'+1}\rangle$$
expressed in $A^\tau = \mathbb{C}[x^2, y^2, xy]$. Then
$$\begin{aligned}\bar\sigma^{-1}(I_j(p:q)) &= \langle pa^{j'} - qb^{n-j'}, c, a^{j'+1}, b^{n-j'+1}\rangle \\ &= J_{j'}(p:q).\end{aligned}$$

- j *odd, i.e.* $j = 2j'+1$, $j' = 0, \dots, n-1$. Observe that $xy \in I_j(p:q)^\tau$ and $(x^2)^{j'+1}$, $y^{n-j'} \in I_j(p:q)^\tau$, but $px^{2j'+1} - qy^{2n-2j'-1} \notin I_j(p:q)^\tau$. So
$$\bar\sigma^{-1}(I_j(p:q)) = \langle a^{j'+1}, b^{n-j'}, c\rangle.$$
We observe then that
$$\bar\sigma^{-1}(I_j(p:q)) \in C(\rho_{j'}) \cap C(\rho_{j'+1})$$
since
$$\bar\sigma^{-1}(I_j(p:q)) = J_{j'}(0:1) = J_{j'+1}(1:0).$$

The curves $E(\rho_j)$ with j even correspond to the pure representations and are not contracted by $\mathscr{S}$ as the previous computation shows, the curves with j odd correspond to the binary representations: these are contracted by $\mathscr{S}$.

10. Application

10.1. **Pencils of symmetric surfaces.** Let $\mathbb{H}_\mathbb{C} := \mathbb{H} \otimes_\mathbb{R} \mathbb{C}$ be the complexification of the space of quaternions. By the choice of the coordinates $q = a\cdot 1 + b\cdot i + c\cdot j + d\cdot k$, $a, b, c, d \in \mathbb{C}$, one gets an isomorphism $\mathbb{P}_3 \cong \mathbb{P}(\mathbb{H}_\mathbb{C})$ such, that for $n = 6, 8, 12$ the bipolyhedral group G_n acts linearly on $\mathbb{P}_3$, leaving invariant the quadratic polynomial $Q := a^2 + b^2 + c^2 + d^2$. In [27] it is shown that the next non-trivial invariant is a homogeneous polynomial S_n of degree n. Consider then the following pencil of G_n-symmetric surfaces in $\mathbb{P}_3$:
$$X_n(\lambda) = \{S_n + \lambda Q^{n/2} = 0\}, \quad \lambda \in \mathbb{C}.$$
In [27] it is proved that the general surface $X_n(\lambda)$ is smooth and that for each n there are precisely four singular surfaces in the corresponding pencil: the singularities of these surfaces are ordinary nodes forming one orbit through G_n.
Consider now the pencil of quotient surfaces in $\mathbb{P}_3/\,G_n$:
$$\{\,X_n(\lambda)/\,G_n\}, \quad \lambda \in \mathbb{C}.$$
In [2] it is proved that these quotient surfaces have only A-D-E singularities and that the minimal resolutions of singularities $Y_n(\lambda) \to X_n(\lambda)/\,G_n$ are K3-surfaces with Picard number greater than 19. For the four nodal surfaces in each pencil, a careful study of the stabilizers of the nodes shows that, if X denotes one of these nodal surfaces, the image of the node on $X/G_n \subset \mathbb{P}_3/\,G_n$ is a particular quotient singularity locally isomorphic to $\mathbb{C}^2/\widetilde{G} \subset \mathbb{C}^3/G$ for some polyhedral group G explicitly computed (see [2, §3, Proposition 3.1]):

- for $n = 6$: $C_3, \mathcal{T}$;
- for $n = 8$: $D_2, D_3, D_4, \mathcal{O}$;
- for $n = 12$: $D_3, D_5, \mathcal{T}, \mathcal{I}$.

Therefore, our theorem 1.1 gives locally a group-theoretic interpretation of the exceptional curves of the K3-surfaces $Y_n(\lambda)$ over the particular singularities of the nodal surfaces.

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