K3 surfaces with maximal finite automorphism groups

(joint with C. Bonnafé)

§ Introduction

* In the 80's Nikulin wrote foundational works on autom. of K3 surfaces

Def A K3 surface is a compact cplx mfd X of dim 2 st.

(1) X is simply connected

(2) $H^0(\Omega^2_X) = \mathbb{C}\cdot\omega_X$ ← nowhere zero

(Comp. $K_X \sim 0$)

Def $\sigma \in \mathrm{Aut}(X) = \{ f : X \to X \text{ biholos} \}$

we say σ is <u>symplectic</u> if

$$\sigma^* \omega_X = \omega_X$$

σ is <u>non-symplectic</u> otherwise (i.e. $\sigma^* \omega_X = \alpha \omega_X$)

$\alpha \in \mathbb{C}^*$

1976: Nikulin classified <u>all</u> finite abelian group acting symplectically on K3 surfaces.

14 cases ($\neq \mathrm{id}$): $\frac{\mathbb{Z}}{n\mathbb{Z}}$, $n = 2, .., 8$; $\left(\frac{\mathbb{Z}}{n\mathbb{Z}}\right)^2$ $n = 3, 4$

$\frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{4\mathbb{Z}}$; $\frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{6\mathbb{Z}}$; $\left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^t$ $t = 2, 3, 4$

Several results with chilean colleagues (non-symplectic case).

with: M. Artebani (Concepcion) 2008-2011 : classified prime order autom. acting n.s. on K3 surfaces.
S. Taki

with P. Comparin (Temuco)
N. Priddis

work on progress 2018-

- Study unicity of some order 16 n.s. auto
- Classify n.s. auto of order 16 on K3 surfaces.

In this talk we mostly study finite automorphism acting symplectically.

Mukai (1988), Gray (1996): class of all finite groups acting symplectically on a K3 surface: 11 maximal groups $\subset M_{23}$

Mathieu group.

The biggest group is $M_{20} = A_5 \ltimes \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^4$

$|M_{20}| = 960$

+ several examples

Aim of the talk

Study finite groups $G \curvearrowright X$ K3 s.t.

$G \underset{\neq}{\supseteq} M_{20}$ acts sympl on X.

Q * How big can be G?

* Can we describe the K3 surfaces?

Theorem (Kondo 1999; Bonnafé–S. 2020; Brandhorst–Hashimoto 2020).

X K3, G finite $G \curvearrowright X$, $G \supsetneq M_{20}$ or sympl.
then:

(1) $|G| \leq 3840$

(2) $|G| = 3840 \Rightarrow X = \mathrm{Km}(E_i \times E_i)$, $E_i : y^2 = x^3 + x$
and $G = G_{KO}$ is unique

(Kondo 1999)

(3) If $|G| < 3840$ then there are exactly 2 possibilities (X_i, G_i) $i = 1, 2$ and X_i are Kummer surfaces.

BS + BH indep. $|G_i| = 1920$

Remarks

(1) Take G **any** finite group $\Rightarrow |G| \leq 3840 = 4 \times 960$

If $|G| = 3840 \Rightarrow$ result of Kondo

(2) Take G **any** finite group $\Rightarrow$ the biggest possible groups are G_{3840}, G_1, G_2 $(|G_1| = |G_2| = 1920)$ 2×960

More results on G

2009 K. Frantzen classified all the groups $G_0 \times \mu_2$

$G_0 =$ one of the 11 groups of Mukai

2020: Brandhorst + Hashimoto classify **all** the extension of $G_0 =$ one of the 11 groups of Mukai

§ Facts on finite automorphisms groups

Recall we have an exact sequence:

$$1 \longrightarrow G_0 \longrightarrow G \xrightarrow{\alpha} \mathbb{C}^*$$

$$g \longmapsto \alpha(g)$$

def by: $g^*\omega_X = \alpha(g)\,\omega_X$

$G_0 = \{g \in G \mid g^*\omega_X = \omega_X\}$ Symplectic auto in G

$Im(\alpha) < \mathbb{C}^* \Longrightarrow Im(\alpha) = \mu_m$ = some group of m-roots of unity

$$1 \longrightarrow \underline{G_0} \longrightarrow G \longrightarrow\!\!\!\!\rightarrow \underline{\mu_m} \longrightarrow 1$$

Important Fact

$$T_X \subset H^2(X, \mathbb{Z})^{G_0}$$

$$\| $$

$$\{ c \in H^2(X, \mathbb{Z}) \mid g^* c = c \}$$

invariant lattice

$$T_X = \operatorname{Pic}(X)^{\perp} \cap H^2(X, \mathbb{Z})$$

= Transc. lattice.

($H^2(X, \mathbb{Z})$ is a lattice with cup-product)

So that:

$$\operatorname{Pic}(X) \supset \left(H^2(X, \mathbb{Z})^{G_0}\right)^{\perp}$$

proof Take $v \in T_X$ consider

$$\langle v, \omega_X \rangle = \langle g^* v, g^* \omega_X \rangle \quad \forall g \in G_0$$

$$= \langle g^* v, \omega_X \rangle$$

$$\Rightarrow \langle v - g^* v, \omega_X \rangle = 0$$

$$v - g^* v \in \mathrm{Pic}(X) \cap T_X = \{0\}$$

$$\mathrm{Pic}(X) = \omega_X^{\perp} \cap H^2(X, \mathbb{Z})$$

$$\Rightarrow v - g^* v = 0 \qquad T_X \subset H^2(X, \mathbb{Z})^{G_0}$$

$\blacksquare$

Apply to $G_0 = \Gamma_{20}$, $G_0 \curvearrowright X$ K3

$$T_X \subset H^2(X,\mathbb{Z})^{\Gamma_{20}} \underset{\text{Kondo}}{\overset{\perp}{=}} \begin{pmatrix} 4 & 0 & -2 \\ 0 & 4 & -2 \\ -2 & -2 & 12 \end{pmatrix} =: \mathbb{L}_{20}$$

$\mathrm{rk}\left(H^2(X,\mathbb{Z})^{\Gamma_{20}}\right)^{\perp} = 19 \Rightarrow \mathrm{rk}\,\mathrm{Pic}(X) = 19 + 1 = 20.$

because X is projective $\quad (X \circlearrowleft G \supsetneq \Gamma_{20}.)$

So that $\mathrm{rk}\, T_X = 2$ sgn $(2,0)$

We can write $T_X = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$ + condition on $a, b, c \in \mathbb{Z}$

but $T_X \subset \mathbb{L}_{20} = L'(2)$ for some lattice L'

In fact $T_X = \begin{pmatrix} 4a' & 2b' \\ 2b' & 4c' \end{pmatrix}$ + cond. $a', b', c' \in \mathbb{Z}$

$\overset{\text{Shioda-Mitani 1974}}{\Longrightarrow}$ X Kummer surface.

Prop G finite, $G \curvearrowright X$ K3, $|G| \leq 3840$

$$1 \to G_0 \to G \to \mu_m \to 1$$

Idea proof ① $G_0 = M_{20}$

Result of Nikulin $\varphi(m) \mid rk\, T_X = 2$

↑ Euler function

$m \in \{1, 2, ~~3~~, 4, ~~6~~\}$

* One can test $\frac{G}{M_{20}}$ acts on $H^2(X, \mathbb{Z})^{G_0} = \mathbb{L}_{20}$.

isom. of $\mathbb{L}_{20}$ →

\+ $|O(\mathbb{L}_{20})| = 16$

(2) If $G_0 \neq M_{20} \Rightarrow$ one uses Xiao's list to see that one gets $|G| < 3860$. ☒

We are left with 2 cases:

$$1 \longrightarrow M_{20} \longrightarrow G \longrightarrow \mu_4 \longrightarrow 1$$ KONDO

$$1 \longrightarrow M_{20} \longrightarrow G \longrightarrow \mu_2 \longrightarrow 1$$ BS + BH

§ proof of thm part 3

thm $\Gamma_{20} \curvearrowright X$ sympl. and X admits an involution i, $i^2 \in \Gamma_{20}$. Then we have the following 3 possibilities, $G \supset \langle i, \Gamma_{20} \rangle$

$\mathrm{Pic}(X)^G$	$\langle 40 \rangle$ (not Kummer)	$\langle 4 \rangle$	$\langle 8 \rangle$
T_X	$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$ (Kondo)	$\begin{pmatrix} 4 & 0 \\ 0 & 40 \end{pmatrix}$	$\begin{pmatrix} 8 & 4 \\ 4 & 12 \end{pmatrix}$

Idea of proof One has an action of $i \curvearrowright H_{20} = H^2(X,\mathbb{Z})^{\Gamma_{20}}$

One can show:

$$T_X \oplus H^2(X,\mathbb{Z})^G \xhookrightarrow{\text{index } 2} \mathbb{L}_{20}.$$

$$H^2(X,\mathbb{Z})^G = \langle 4m \rangle$$

$$4 = [\mathbb{L}_{20} : T_X \oplus \langle 4m \rangle]^2 = \frac{\det(T_X \oplus \langle 4m \rangle)}{\det \mathbb{L}_{20}} = \frac{16m\,(4a'c' - b'^2)}{160}$$

$$T_X = \begin{pmatrix} 4a' & 2b' \\ 2b' & 4c' \end{pmatrix}$$

One can show that b' even:

$$b' = 2b''$$

All together this gives

$$m(a'c' - b''^2) = 10$$

$$\Rightarrow m \in \{1, 2, 5, 10\}$$

$$4m \in \{4, 8, \cancel{20}, 40\}$$

↑ not possible

Existence of K3 surfaces

(2) the case of $\mathrm{Pic}(X)^G = \mathbb{Z}L$, $L^2 = 40$: Kondo.

$X = \mathrm{Km}(E_i \times E_i)$

(2) $\mathrm{Pic}(X)^G = \mathbb{Z}L$, $L^2 = 4$

$X \hookrightarrow \mathbb{P}^3$, $T_X = \begin{pmatrix} 4 & 0 \\ 0 & 40 \end{pmatrix}$

$$X = X_{Mu} = \{x_0^4 + \cdots + x_3^4 - 6(x_0^2x_1^2 + \cdots + x_2^2x_3^2) = 0\}$$

studied by Mukai.

Here $G_{Mu} = \mu_2 \times M_{20}$, $G_{Mu} = PG_{29}$, $|G_{Mu}| = 1920$

G_{29} = complex reflection group.

(Shephard-Todd class. 1954)

(3) $\mathrm{Pic}(X)^G \cong \mathbb{Z}L$, $L^2 = 8$, $T_X = \begin{pmatrix} 8 & 4 \\ 4 & 12 \end{pmatrix}$.

$$X \hookrightarrow \mathbb{P}^5$$

$$X = X_{BH} : \begin{cases} x_0^2 + x_3^2 - \phi x_4^2 + \phi x_5^2 = 0 \\ x_1^2 - \phi x_3^2 + x_4^2 - \phi x_5^2 = 0 \\ x_2^2 + \phi x_3^2 - \phi x_4^2 + x_5^2 = 0 \end{cases}$$

$$G_{BH} = \mu_2 \times \mathfrak{A}_{20} \qquad \phi = \frac{1+\sqrt{5}}{2}$$

<u>Remark</u> ① : G_{BH}, G_{mu} are not iso and not subgroups of G_{Ko}

:

② XK3

with
action by G_{KO}, G_{PU}, G_{BH} $\Rightarrow$

$X \cong X_{KO}$, X_{PU} resp. X_{BH}.