ON THE NÉRON-SEVERI GROUP OF SURFACES WITH MANY LINES

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Abstract. For a binary quartic form ϕ without multiple factors, we classify the quartic K3 surfaces $\phi(x, y) = \phi(z, t)$ whose Néron-Severi group is (rationally) generated by lines. For generic binary forms ϕ, ψ of prime degree without multiple factors, we prove that the Néron-Severi group of the surface $\phi(x, y) = \psi(z, t)$ is rationally generated by lines.

1. Introduction

The study of the Néron-Severi group $\mathrm{NS}(S)$ of a given surface S is interesting for understanding its geometry, but it is not an easy task in general. A first step is to compute its Picard number $\rho(S) := \mathrm{rk}\,\mathrm{NS}(S)$. A second one is to give a family of generators of $\mathrm{NS}(S)$ over $\mathbb{Z}$. To this purpose, it is very useful to find first a nice family of generators of $\mathrm{NS}(S) \otimes_{\mathbb{Z}} \mathbb{Q}$. If one already knows the value of the determinant of $\mathrm{NS}(S)$, this can help deducing a family of generators. If not, the study of the *rational* generators gives non trivial information for the value of the discriminant.

Let ϕ be a binary quartic form without multiple factors. After a suitable linear change of coordinates, we may assume that ϕ is of the form:

$$\phi(x, y) = yx(y - x)(y - \lambda x)$$

for $\lambda \in \mathbb{C}\setminus\{0, 1\}$. Naturally associated to ϕ are the K3 surface $S_\phi : \phi(x, y) = \phi(z, t)$ and the elliptic curve $E_\phi : t^2 = \phi(1, y)$.

Remark 1.1. *Observe that if ϕ, ϕ' are the forms corresponding to λ, λ' and λ' is one of the values $\lambda, \frac{1}{\lambda}, 1 - \lambda, \frac{1}{1-\lambda}, \frac{\lambda}{\lambda-1}, \frac{\lambda-1}{\lambda}$ then there is a linear isomorphism $S_\phi \cong S_{\phi'}$.*

The interplay between the geometry of the K3 surface S_ϕ and the arithmetic of the elliptic curve E_ϕ has been studied by many authors. Of particular interest is the link between the value of the Picard number $\rho(S_\phi)$ and the existence of a complex multiplication on E_ϕ. The following result is classical (see [Kuw95] and references therein):

$$\rho(S_\phi) = \begin{cases} 20 & \text{if } E_\phi \text{ has a complex multiplication,} \\ 19 & \text{otherwise.} \end{cases}$$

We pursue the study by giving numerical conditions for the Néron-Severi group of S_ϕ to be *rationally generated by lines*:

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Notation – Definition. Let $S \subset \mathbb{P}^3_{\mathbb{C}}$ be a smooth surface of degree $d \geq 3$. If L is a line contained in S, by the genus formula the self-intersection of L in S is $L^2 = -d+2$, so the class of L in $\mathrm{NS}(S)$ is not a torsion class. We denote by $\mathrm{LC}(S)$ the sublattice of the torsion-free part of $\mathrm{NS}(S)$ generated by the classes of the lines contained in S. For a generic surface S, it is well-known that $\mathrm{LC}(S) = 0$. If not, these classes are natural candidates as generators of $\mathrm{NS}(S)$ and we say that $\mathrm{NS}(S)$ is *rationally generated by lines* if $\mathrm{rk}\,\mathrm{LC}(S) = \rho(S)$, that is $\mathrm{LC}(S) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathrm{NS}(S) \otimes_{\mathbb{Z}} \mathbb{Q}$.

The most famous examples of surfaces whose Néron-Severi group is rationally generated by lines are certain Fermat surfaces (see [Shi81]). The surfaces we study here are a natural generalization of them. We prove (§2):

Theorem 1.2. *The Néron-Severi group of S_ϕ is rationally generated by lines exactly in the following cases:*

(1) $\lambda \notin \overline{\mathbb{Q}}$;
(2) $\lambda \in \{-1, 2, \frac{1}{2}, \frac{1+i\sqrt{3}}{2}, \frac{1-i\sqrt{3}}{2}\}$;
(3) $\lambda \in \overline{\mathbb{Q}} \setminus \{-1, 2, \frac{1}{2}, \frac{1+i\sqrt{3}}{2}, \frac{1-i\sqrt{3}}{2}\}$ *and* $\rho(S_\phi) = 19$.

Looking now for a set of generators of the Néron-Severi group, we prove (§3):

Theorem 1.3. *The Néron-Severi group of S_ϕ is generated by lines only in case* (2).

Generalizing the construction, one can consider two binary forms ϕ, ψ of degree d without multiple factors and the associated surface $S^d_{\phi,\psi} : \phi(x, y) = \psi(z, t)$. One can prove that $\rho(S^d_{\phi,\psi}) \geq (d-1)^2 + 1$ with equality for d prime and ϕ, ψ generic (see [Sas68]). We prove (§4):

Theorem 1.4. *For d prime and ϕ, ψ generic, the Néron-Severi group of $S^d_{\phi,\psi}$ is rationally generated by lines.*

In Theorem 1.2 we do not consider the quartics $S^4_{\phi,\psi}$ for $\phi \neq \psi$ since, although $\rho(S^4_{\phi,\psi}) = 18$ (see again [Kuw95]), Proposition 4.1 below says that their 16 lines generate an intersection matrix of rank 10, so such surfaces do not enter in our context.

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2. Proof of Theorem 1.2

The result follows from the following proposition:

Proposition 2.1. *If* $\lambda \in \{-1, 2, \frac{1}{2}, \frac{1+i\sqrt{3}}{2}, \frac{1-i\sqrt{3}}{2}\}$, *then* $\mathrm{rk}\,\mathrm{LC}(S_\phi) = 20$, *otherwise* $\mathrm{rk}\,\mathrm{LC}(S_\phi) = 19$.

Proof of Theorem 1.2. Assuming Proposition 2.1, we prove Theorem 1.2. The key argument is that if E_ϕ has a complex multiplication, then its j-invariant is algebraic over $\overline{\mathbb{Q}}$ (see [Sil94]). Since $j(E_\phi) = \frac{256(1-\lambda+\lambda^2)^3}{\lambda^2(\lambda-1)^2}$, $j(E_\phi) \in \overline{\mathbb{Q}}$ if and only if $\lambda \in \overline{\mathbb{Q}}$. Then:

- If $\lambda \notin \overline{\mathbb{Q}}$, E_ϕ has no complex multiplication so $\rho(S_\phi) = 19$ and by Proposition 2.1, $\mathrm{rk}\,\mathrm{LC}(S_\phi) = 19$. This proves (1).
- If $\lambda \in \{-1, 2, \frac{1}{2}, \frac{1+i\sqrt{3}}{2}, \frac{1-i\sqrt{3}}{2}\}$, by Proposition 2.1 we have $\mathrm{rk}\,\mathrm{LC}(S_\phi) = 20$ so $\rho(S_\phi) = 20$. This proves (2).
- If $\lambda \in \overline{\mathbb{Q}} \setminus \{-1, 2, \frac{1}{2}, \frac{1+i\sqrt{3}}{2}, \frac{1-i\sqrt{3}}{2}\}$, then $\rho(S_\phi) \in \{19, 20\}$ and $\mathrm{rk}\,\mathrm{LC}(S_\phi) = 19$. This gives (3). □

Remark 2.2. *In case (3) of Theorem 1.2, one can not be more precise since:*

- *When $j(E_\phi) \in \overline{\mathbb{Q}}$ (so $\lambda \in \overline{\mathbb{Q}}$), it is not clear whether E_ϕ admits a complex multiplication or not.*
- *There is a dense and numerable set of $\lambda \in \overline{\mathbb{Q}}$ such that $\rho(S_\phi) = 20$ (see* [Ogu]*).*

Proof of Proposition 2.1. The description of the lines on S_ϕ comes from Segre [Seg47]. We follow the presentation given in [BS07].

Case 1. If $\lambda \notin \{-1, 2, \frac{1}{2}, \frac{1+i\sqrt{3}}{2}, \frac{1-i\sqrt{3}}{2}\}$, the group of automorphisms of $\mathbb{P}^1_{\mathbb{C}}$ permuting the set $\{\infty, 0, 1, \lambda\}$ is the dihedral group $D_2 = \{id, s_1, s_2, s_1s_2\}$ and the surface S_ϕ contains exactly the following 32 lines:

$$\ell_z(u,v)\colon \begin{cases} vx = uy \\ vt = uz \end{cases} \quad u,v \in \{\infty,0,1,\lambda\} \qquad \ell_{id}(p)\colon \begin{cases} x = pz \\ y = pt \end{cases} \quad p \in \{1,-1,i,-i\} \qquad \ell_{s_1}(p)\colon \begin{cases} x = pz - pt \\ y = \lambda pz - pt \end{cases} \quad p \in \{\tfrac{1}{\sqrt{\lambda-1}}, \tfrac{-1}{\sqrt{\lambda-1}}, \tfrac{i}{\sqrt{\lambda-1}}, \tfrac{-i}{\sqrt{\lambda-1}}\}$$

$$\ell_{s_2}(p)\colon \begin{cases} x = pt \\ y = \lambda pz \end{cases} \quad p \in \{\tfrac{1}{\sqrt{\lambda}}, \tfrac{-1}{\sqrt{\lambda}}, \tfrac{i}{\sqrt{\lambda}}, \tfrac{-i}{\sqrt{\lambda}}\} \qquad \ell_{s_1s_2}(p)\colon \begin{cases} x = -\lambda pz + pt \\ y = -\lambda pz + \lambda pt \end{cases} \quad p \in \{\tfrac{1}{\sqrt{\lambda^2-\lambda}}, \tfrac{-1}{\sqrt{\lambda^2-\lambda}}, \tfrac{i}{\sqrt{\lambda^2-\lambda}}, \tfrac{-i}{\sqrt{\lambda^2-\lambda}}\}$$

The intersection matrix of these 32 lines is easy to compute (we do not reproduce it here), and is independent of λ. One finds that its rank is 19, so $\operatorname{rk} \mathrm{LC}(S_\phi) = 19$.

Case 2. If $\lambda \in \{-1, 2, \frac{1}{2}\}$, the surfaces are isomorphic to each other by Remark 1.1. The group of automorphisms is the dihedral group $D_4 = \langle D_2, r\rangle$. The surface S_ϕ contains exactly 48 lines: the 32 preceding ones and 16 other lines. For $\lambda = -1$ for example, these lines are:

$$\ell_r(p)\colon \begin{cases} x = pz + pt \\ y = -pz + pt \end{cases} \qquad \ell_{r^{-1}}(p)\colon \begin{cases} x = -pz + pt \\ y = -pz - pt \end{cases} \qquad p \in \{\tfrac{1+i}{2}, \tfrac{1-i}{2}, \tfrac{-1+i}{2}, \tfrac{-1-i}{2}\}$$

$$\ell_{rs_1}(p)\colon \begin{cases} x = pt \\ y = pz \end{cases} \qquad \ell_{s_1r}(p)\colon \begin{cases} x = -pz \\ y = pt \end{cases} \qquad p \in \{\tfrac{1+i}{\sqrt{2}}, \tfrac{1-i}{\sqrt{2}}, \tfrac{-1+i}{\sqrt{2}}, \tfrac{-1-i}{\sqrt{2}}\}$$

The rank of the intersection matrix of the 48 lines is $\operatorname{rk} \mathrm{LC}(S_\phi) = 20$.

Case 3. If $\lambda \in \{\frac{1+i\sqrt{3}}{2}, \frac{1-i\sqrt{3}}{2}\}$, the surfaces are isomorphic to each other by Remark 1.1. The group of automorphisms is the tetrahedral group $T = \langle r, s\rangle$. The surface S_ϕ contains exactly the following 64 lines:

$$\ell_z(u,v)\colon \begin{cases} vx = uy \\ vt = uz \end{cases} \qquad \ell_{id}(p)\colon \begin{cases} x = pz \\ y = pt \end{cases} \qquad u,v \in \{\infty,0,1,\lambda\} \quad p \in \{1,-1,i,-i\}$$

$$\ell_r(p)\colon \begin{cases} x = pz \\ y = pz + \lambda^2 pt \end{cases} \qquad \ell_{r^2}(p) : \begin{cases} x = pz \\ y = \lambda pz - \lambda pt \end{cases}$$

$$\ell_s(p)\colon \begin{cases} x = pt \\ y = \lambda pz \end{cases} \qquad \ell_{rs}(p)\colon \begin{cases} x = pt \\ y = -pz + pt \end{cases} \qquad \ell_{rsr}(p) : \begin{cases} x = pz + \lambda^2 pt \\ y = \lambda^2 pt \end{cases}$$

$$\ell_{r^2s}(p)\colon \begin{cases} x = pt \\ y = -\lambda^2 pz + \lambda pt \end{cases} \qquad \ell_{sr}(p)\colon \begin{cases} x = pz + \lambda^2 pt \\ y = \lambda pz \end{cases} \qquad {\scriptstyle p\in\{\lambda,-\lambda,\mathrm{i}\lambda,-\mathrm{i}\lambda\}}$$

$$\ell_{rsr^2s}(p)\colon \begin{cases} x = -\lambda^2 pz + \lambda pt \\ y = -\lambda^2 pz + \lambda^2 pt \end{cases} \qquad \ell_{r^2srs}(p)\colon \begin{cases} x = -pz + pt \\ y = -\lambda pz + pt \end{cases} \qquad {\scriptstyle p\in\{\lambda^2,-\lambda^2,\mathrm{i}\lambda^2,-\mathrm{i}\lambda^2\}}$$

$$\ell_{srs}(p)\colon \begin{cases} x = -pz + pt \\ y = \lambda pt \end{cases} \qquad \ell_{rsrs}(p)\colon \begin{cases} x = -pz + pt \\ y = -pz \end{cases}$$

The rank of the intersection matrix of the 64 lines is $\operatorname{rk}\mathrm{LC}(S_\phi) = 20$.

□

3. Proof of Theorem 1.3

As we explained in the Introduction, once one has found a nice family of *rational* generators of the Néron-Severi group, the next task is to get information on divisible classes. We call a divisor $\Lambda = \sum_{i=1}^n \alpha_i L_i \in \mathrm{NS}(S)$ 2^m*-divisible* if the class of Λ in $\mathrm{NS}(S)$ is divisible by 2^m; for $m = 1$ we say also that the lines in Λ form an *even set*.

Proof of Theorem 1.3.
Cases (1) *and* (3)*.* For $\lambda \notin \{-1, 2, \frac{1}{2}, \frac{1+\mathrm{i}\sqrt{3}}{2}, \frac{1-\mathrm{i}\sqrt{3}}{2}\}$, with the help of a computer program we obtain that the best choice of a family of 19 lines among the 32 generating rationally the Néron-Severi group gives a determinant of value 2^9. Denoting this lattice by M and its dual by $M^\vee$, the discriminant group is:

$$M^\vee/M = (\mathbb{Z}_2)^{\oplus 2} \oplus (\mathbb{Z}_4)^{\oplus 2} \oplus \mathbb{Z}_8$$

hence we can have only 2^m-divisible classes for $m = 1, 2, 3$. Denote by $(M^\vee/M)_2$ the part of the discriminant group generated by the 2-torsion classes. We have $(M^\vee/M)_2 = (\mathbb{Z}_2)^{\oplus 5}$ hence $\mathrm{rank}(M^\vee/M)_2 = 5$. However, denoting by T the transcendental lattice of S_ϕ, $(\mathrm{NS}(S_\phi)^\vee/\mathrm{NS}(S_\phi))_2 \cong (T^\vee/T)_2$ has rank at most the rank of T, which is three: This shows that $M \subsetneq \mathrm{NS}(S_\phi)$, and that there are at least two even sets of lines in the Néron Severi group. In particular there is no set of 19 lines generating $\mathrm{NS}(S_\phi)$.

Case (2) *for* $\lambda \in \{-1, 2, \frac{1}{2}\}$. By Remark 1.1, the surfaces S_ϕ are isomorphic to each other. The best choice of a family of 20 lines among 48 gives a determinant of value -2^6. Observe that a suitable permutation of the zeros of $x^4 - y^4$ in $\mathbb{P}^1_{\mathbb{C}}$ gives a cross-ratio equal to -1, so our surfaces are isomorphic to the Fermat quartic. It is then well-known that $\det \mathrm{NS}(S_\phi) = -64$, so the lines generate the Néron-Severi group.

Case (2) *for* $\lambda \in \{\frac{1+\mathrm{i}\sqrt{3}}{2}, \frac{1-\mathrm{i}\sqrt{3}}{2}\}$. A computer program shows that the best choice of a family of 20 lines among the 64 contained in the surface, generating rationally the Néron-Severi group, gives a determinant of value $-2^4 \cdot 3$. We show in Appendix B that $\det \mathrm{NS}(S_\phi) = -48$ so the lines generate the Néron-Severi group. □

4. Proof of Theorem 1.4

Since $\rho(S^d_{\phi,\psi}) = (d-1)^2 + 1$ for d prime and ϕ, ψ generic, Theorem 1.4 follows from the following result:

Proposition 4.1. *It is* $\operatorname{rk} \mathrm{LC}(S^d_{\phi,\psi}) = (d-1)^2 + 1$.

Proof of Proposition 4.1. We set $S := S^d_{\phi,\psi}$. Let L be the line $z = t = 0$ and L' be the line $x = y = 0$. The intersection $S \cap L$ is the set of zeros of ϕ, whereas $S \cap L'$ is the set of zeros of ψ. If $p \in L$ is a zero of ϕ and $q \in L'$ a zero of ψ, the line $L_{p,q}$ joining p and q is contained in S: this gives a family of d^2 lines contained in S. The intersection matrix of this family is given by $L^2 = -d + 2$ and $L \cdot L' = 1$ if L and L' intersect, 0 otherwise. Note that:

$$(L_{p,q} \cap L_{p',q'} \neq \emptyset) \iff (p = p' \text{ or } q = q').$$

This implies that after ordering correctly the lines, the intersection matrix is the matrix $M_d := K^d_{-d+2,1,1,0}$ (see the notation in Appendix A). Remark A.5 gives $\operatorname{rk} \mathrm{LC}(S) = \operatorname{rk} M_d = (d-1)^2 + 1$. □

Appendix A. Some linear algebra

Let $a, b, c, d, \ldots$ denote indeterminates. For $d \geq 2$, let $J^d_{a,b}$ be the (d,d)-matrix defined by:

$$J^d_{a,b} := \begin{pmatrix} a & & \text{\Large b} \\ & \ddots & \\ \text{\Large b} & & a \end{pmatrix} = b \cdot \begin{pmatrix} \text{\Large 1} \end{pmatrix} + (a-b) \cdot I_d$$

where I_d denotes the identity (d,d)-matrix. The following lemma is clear:

Lemma A.1. *The following identities hold:*

$$\begin{aligned} J^d_{a,b} + J^d_{a',b'} &= J^d_{a+a',b+b'}; \\ J^d_{a,b} \cdot J^d_{a',b'} &= J^d_{aa'+(d-1)bb',ab'+a'b+(d-2)bb'}. \end{aligned}$$

Let now $K^d_{a,b,c,d}$ be the (d^2,d^2)-matrix defined as the following (d,d)-blocks of (d,d)-matrices:

$$K^d_{a,b,c,d} := \begin{pmatrix} J^d_{a,b} & & \text{\Large $J^d_{c,d}$} \\ & \ddots & \\ \text{\Large $J^d_{c,d}$} & & J^d_{a,b} \end{pmatrix}$$

The following lemma follows easily from Lemma A.1:

Lemma A.2. *The following identity holds:*

$$K^d_{a,b,c,d} \cdot K^d_{a',b',c',d'} = K^d_{\alpha,\beta,\gamma,\delta}$$

where:

$$\begin{aligned}
\alpha &= aa' + (d-1)(bb' + cc') + (d-1)^2 dd';\\
\beta &= ab' + a'b + (d-1)(cd' + c'd) + (d-2)bb' + (d-1)(d-2)dd';\\
\gamma &= ac' + a'c + (d-1)(bd' + b'd) + (d-2)cc' + (d-1)(d-2)dd';\\
\delta &= ad' + a'd + bc' + b'c + (d-2)(cd' + c'd + bd' + b'd) + (d-2)^2 dd'.
\end{aligned}$$

Set $K_d := K^d_{1,1,1,0}$. Its minimal polynomial $\mu_{K_d}(t)$ is given by:

Lemma A.3. $\mu_{K_d}(t) = (t-(d-1)) \cdot (t-(2d-1)) \cdot (t+1)$.

Proof. Note that:

$$\begin{aligned}
K_d - (d-1)I_d &= K^d_{-d+2,1,1,0};\\
K_d - (2d-1)I_d &= K^d_{-2d+2,1,1,0};\\
K_d + I_d &= K^d_{2,1,1,0}.
\end{aligned}$$

Applying Lemma A.2 one gets:

$$\begin{aligned}
K^d_{-d+2,1,1,0} \cdot K^d_{-2d+2,1,1,0} &= K^d_{2(d-1)^2,-2d+2,-2d+2,2};\\
K^d_{-d+2,1,1,0} \cdot K^d_{2,1,1,0} &= K^d_{2,2,2,2};\\
K^d_{-2d+2,1,1,0} \cdot K^d_{2,1,1,0} &= K^d_{-2d+2,-d+2,-d+2,2};\\
K^d_{-d+2,1,1,0} \cdot K^d_{-2d+2,1,1,0} \cdot K^d_{2,1,1,0} &= K^d_{0,0,0,0} = 0.
\end{aligned}$$

□

For $\lambda \in \{d-1, 2d-1, -1\}$, we denote by $V(\lambda)$ the eigenspace of K_d associated to the eigenvalue λ. One computes:

Lemma A.4.

$$\dim V(2d-1) = 1; \quad \dim V(-1) = (d-1)^2; \quad \dim V(d-1) = 2(d-1).$$

Proof. The first two results are a (quite long) direct computation. One deduces the third one using that K_d is diagonalizable (Lemma A.3). □

Remark A.5. *Since $K^d_{\lambda,1,1,0} = K_d - (1-\lambda)I_d$, the matrix $K^d_{\lambda,1,1,0}$ is invertible when $1-\lambda$ is not an eigenvalue of K_d. By Lemma A.3 this is $\lambda \notin \{-d+2, -2d+2, 2\}$. For $\lambda = -d+2$, one has:*

$$\operatorname{rk} K^d_{-d+2,1,1,0} = d^2 - \dim V(d-1) = (d-1)^2 + 1.$$

Appendix B. Results on Kummer surfaces

We recall some classical facts from [Ino76, PŠŠ71, SI77, SM74]. If S is a K3 surface with Picard number 20, we denote by T_S the transcendental lattice and Q_S the intersection matrix of T_S with respect to an oriented basis. Let $\mathcal{Q}$ be the set of positive definite, even integral 2×2 matrices. The class $[Q_S] \in \mathcal{Q}/\operatorname{SL}_2(\mathbb{Z})$ is uniquely determined by S and $\det \operatorname{NS}(S) = -\det Q_S$.

For S_ϕ, let σ be the involution $(x:y:z:t) \mapsto (x:y:-z:-t)$. Then the minimal resolution of S_ϕ/σ is isomorphic to the Kummer surface $Y := \operatorname{Km}(E_\phi \times E_\phi)$ and:

$$Q_{S_\phi} = 2Q_Y = 4Q_A$$

where $A := E_\phi \times E_\phi$ and Q_A is the binary quadratic form associated to A as in [SM74].

For $\lambda = \frac{1+\mathrm{i}\sqrt{3}}{2}$, the group of automorphisms of the elliptic curve E_ϕ fixing a point has order 6 (since $j(\lambda) = 0$) so $E_\phi \cong C_\tau := \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ with $\tau = \frac{-1+\mathrm{i}\sqrt{3}}{2}$. By the construction of [SM74], for $A = C_\tau \times C_\tau$, one has $Q_A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ so $Q_{S_\phi} = \begin{pmatrix} 8 & 4 \\ 4 & 8 \end{pmatrix}$ and $\det \mathrm{NS}(S_\phi) = -\det Q_{S_\phi} = -48$. Moreover, observe that for $A' = C_\tau \times C_{\tau'}$ with $\tau' = \mathrm{i}\sqrt{3}$, one has $Q_{A'} = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$ so $S_\phi \cong \mathrm{Km}(A')$.

Remark B.1. *The same method has been used to compute the determinant of the Néron-Severi group of the Fermat quartic.*

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