Automorphisms of irreducible holomorphic symplectic manifolds, Enriques manifolds and the Morrison-Kawamata cone conjecture

§ IHS manifolds

1. Elliptic curves : $E = \dfrac{\mathbb{C}}{\Lambda_2}$

Λ_2 = rk 2 lattice.

$\Omega^1_E = \mathcal{O}_E$ is trivial.

$$\boxed{H^0(\Omega^1_E) = \mathbb{C} \cdot dz}$$

(not simply connected)

In dim 2 2 ways to generalize elliptic curves

Tori

$T = \dfrac{\mathbb{C}^2}{\Lambda_4} \leftarrow$ rk 4 lattice

S: K3 surfaces

Most easy ex: $x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$ in $\mathbb{P}^3$

Both Cos Trivial
canonical div.

$K_T \sim 0$ non simply conn.

$K_S \sim 0$ simply conn.

<u>Def</u>: A K3 surface S is a compact cplx smooth surface s.t.

* $\pi_1(S) = \{id\}$
* $H^0(S, \Omega^2_S) = \mathbb{C}\,\omega_S$

Kähler forms

<u>CONSEQUENCES</u>

* $K_S \sim 0$

<u>Rmk</u> X cpct. cplx mf.

$$c_1(X) \simeq c_1^{\mathbb{R}}(X) \in H^2(X, \mathbb{R})$$

$$c_1^{\mathbb{R}}(X) = 0 \iff \left(\Omega_X^{\dim X}\right)^{\otimes m} = \mathcal{O}_X$$

for some m.

Prop X surface cpt. cplex Kähler, $c_1^{\mathbb{R}}(X)=0$

$\Rightarrow X$ is an étale quotient of a Torus or a K3 surface.

Ex Enriques surfaces $\mathcal{E}$ have $2K_{\mathcal{E}} \sim 0$ ($K_{\mathcal{E}} \not\sim 0$) and are quotients of K3 S by a fixed point free curvol.

$$\mathcal{E} = \frac{S}{\langle i \rangle}$$

Prop 1 Is Cmplex true? Yes!

Theorem (Beauville-Bogomolov)

X cpct. cplx Kähler mnf., $c_1^{\mathbb{R}}(X)=0$ $\Rightarrow \exists$ an étale finite covering $\tilde{X}$ of X s.t. $\tilde{X} = T \times \prod V_i \times \prod X_j$

where:

- T = Torus ; V_i = CY mnf.
- X_j = IHS mnf.

Rmk In dim 2 : CY = IHS

dim > 2 : CY $\neq$ IH

Def (IHS manifold). X cpt, cplx Kähler is IHS if

* $\pi_1(X) = \{id\}$
* $\exists!$ global holo 2-form ω_X everywhere non-deg.

$$H^0(X, \Omega^2_X) = \mathbb{C} \cdot \omega_X$$

Consequences:

* $\dim X$ is even
* $K_X \sim 0$

Examples Let $V = A$ or S abelian or K3

Consider the Hilbert scheme of n pts on V:

$$V^n := \underbrace{V \times \dots \times V}_{n\text{-times}} \circlearrowleft \Sigma_n$$

$$\begin{array}{ccc} & & \downarrow \\ V^{[n]} & \xrightarrow{E} & V^n/\Sigma_n \end{array}$$

Hilbert chow

res. of sing.

1 case $V = S$ K3 $\Rightarrow V^{[n]}$ is an IHS (Beauville – Fujiki) | dim $2n$
~'80 $\quad n=2$

2 case $V = A$ abelian $\Rightarrow A^{[n]}$ is NOT IHS
(not simply connected)

$$\begin{tikzcd} A^{[n]} \arrow[r, "\varepsilon"] \arrow[rr, bend left, "\psi"] & A^n/\Sigma^n \arrow[r, "\text{summation}"] & A \ni 0 \end{tikzcd}$$

$$(x_1, \dots, x_n) \longmapsto \sum x_i$$

$K_n(A) := \psi^{-1}(0)$ is IHS

$\dim K_n(A) = 2n-2$

Generalized Kummer manifolds

($\dim K_2(A) = 2$: $K_2(A)$ is a Kummer surface)

§ Automorphisms

X cpt cpx Kähler

$\mathrm{Aut}(X) = \{\sigma : X \to X \,/\, \sigma \text{ is bihol}\}$

If $X = SK3$: a lot of work
(started by Nikulin '70 - '80)

If $X = IHS$: more recent work
Beauville generalised the works
of Nikulin '80

<u>Easy ex. on K3</u>

- $S = \{x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0\} \subset \mathbb{P}^3_{\mathbb{C}}$.

$\sigma : (x_0 : x_1 : x_2 : x_3) \longmapsto (x_0 : x_1 : x_2 : -x_3)$

involution

$Fix_\sigma = \{x_3 = 0\}$ = plane genus 3 curve

- S K3 $\circlearrowleft$ i fixed pt free.

$\Rightarrow \dfrac{S}{\langle i \rangle}$ Enriques Surface.

<u>Ex. on $K_n(A)$</u> Take $A = E \times F$

product of ell. curves.

$E : y^2 = x(x-1)(x-\lambda)$, $\lambda \in \mathbb{C}$

invol on E : $y \longrightarrow -y$

$h = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \curvearrowright E \times F$

Consider a Transl. $t_e(x, y) = (x + e_1, y + e_2)$

$\uparrow$ $E \times F$

where $(e_1, e_2) \in E \times F$ st.

* e_2 is 2 Torsion
* Take n, m integers st. $2m = (n+1)$

e_1 is $(n+1)$-Torsion, $\boxed{m e_1 \notin \mathbb{Z} + \tau\mathbb{Z}}$

$\left(E = \dfrac{\mathbb{C}}{\mathbb{Z} + \tau\mathbb{Z}} \right)$

Take $\varphi = t_a \circ h$, is an invol.

It induces an act o

$\varphi^{[n]} \curvearrowright K_n(A)$ invol.

fixed point free

If we consider the quotient.

$$\frac{K_n(A)}{\langle \varphi^{[n]} \rangle}$$ $\boxed{\text{this is an Enriques manifold}}$

§ Enriques manifolds

Def (OS 2011 – B-NW-S. 2011)

X cpct cplx mfd. is an Enriques mfd. if:

- X is <u>not</u> simply connected
- The univ cover is

even dim.

$$\pi : \tilde{X} \xrightarrow{\text{étale}} X, \quad \tilde{X} \text{ is IHS}$$

(in dim 2 : Enriques mfd = Enriques surfaces)

Consequences

- X smooth, projective, $\dim X = 2n$ even
- $\pi_1(X)$ is finite, can be identif. with $G < \mathrm{Aut}(\tilde{X})$ s.t. $X = \dfrac{\tilde{X}}{G}$

$$G \circlearrowright \tilde{X} \longrightarrow X^{\mathrm{Enr}}$$

S K3 $\circlearrowleft$ invol. i fixed pt free $\Rightarrow \Sigma := \dfrac{S}{\langle i \rangle}$ Enr. surface. $2K_\Sigma \sim 0$ ($K_\Sigma \not\sim 0$)

One can show that G acts on $\tilde{X}$ purely non-symplectically!

$\forall f \in G$, $f^* \omega_{\tilde{X}} = \xi_t \omega_{\tilde{X}}$

$(\mathbb{C} \cdot \omega_{\tilde{X}} = H^0(\tilde{X}, \Omega^2_{\tilde{X}}))$ $t = |f|$

ξ_t = primitive t-root of unity.

$\Longrightarrow$ G is <u>cyclic</u>

Remark

$\Rightarrow X = \dfrac{\tilde{X}}{\langle f \rangle}$ $G = \langle f \rangle$

. $d = |G| = |\langle f \rangle|$ is called the index of the Enriques mfd.

<u>Ex</u>. Enriques Spaces are Enriques mfd of index 2

$S \longrightarrow \left.\dfrac{S}{\langle i \rangle}\right) = \Sigma$ $|i| = 2$

. Also $\dfrac{K_n(E \times F)}{\langle \varphi^{[n]} \rangle}$ Enriques mfd of index 2

. $S^{[n]}$, i = fixed pt. free invol.

$\Rightarrow \dfrac{S^{[n]}}{\langle i^{[n]} \rangle}$ $\boxed{n \text{ odd}}$ Enriques of index 2.

Theorem (OS–BNWS)

$\exists$ Enriques mfd of index 2, 3, 4.

use gen. Kummer

Q Are there more ex?

Restriction: d = index of an Enriques mfd. X

$\dim X = 2n$

$\Longrightarrow$ $d \mid n+1$

- $d K_X \sim 0$, $d' K_X \not\sim 0$ $0 < d' < d$

(project w. G. Pacienza)

By the Morrison-Kawamata cone conj.

X proj $\quad N^1(X) := \mathrm{Div}(X) / \equiv$ ← Cartier, ← num. equiv.

- $\mathrm{Nef}(X) = \overline{\mathrm{Amp}(X)} \subset N^1_{\mathbb{R}}(X) = N^1(X) \otimes \mathbb{R}$

- $\mathrm{Nef}^+(X) :=$ smallest cone cont. $\mathbb{Q}$-rat pts of $\mathrm{Nef}(X)$

Conj: (Morrison–Kawamata '93–'97)

X proj. $/\mathbb{C}$, $K_X \equiv 0$

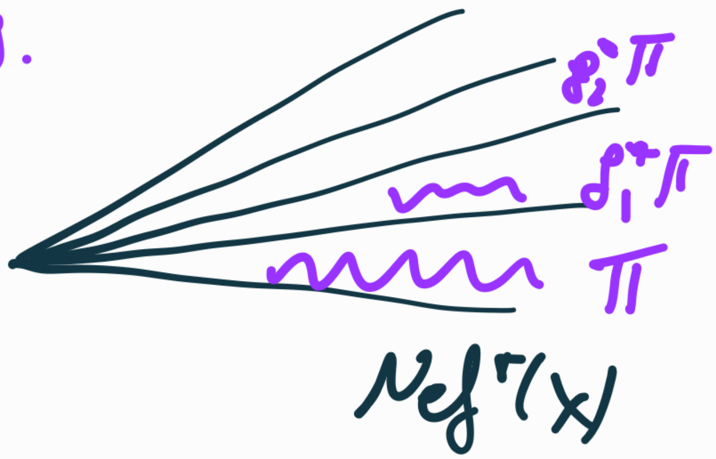
There exists rational polyhedral cone Π which is a fundamental domain for the action of $\mathrm{Aut}(X)$ on $\mathrm{Nef}^+(X)$

this means:

$$\mathrm{Nef}^+(X) = \overline{\bigcup_{g \in \mathrm{Aut}(X)} g^*\Pi}$$

$$\mathrm{int}(\Pi) \cap \mathrm{int}(g^*\Pi) = \emptyset, \quad g^* \neq \mathrm{id}.$$

(if $|\mathrm{Aut}(X)| < +\infty \Rightarrow \mathrm{Nef}^+(X)$ is itself poly.)



Fact MK cone conj holds for IHS (Amerik–Verbitsky 2017)

Q X proj. $K_X \equiv 0 \Rightarrow \exists$

$\tilde{X} \longrightarrow X$ étale cover $\overset{BB}{\,}$ $\tilde{X} = T \times \prod V_i \times \boxed{\prod X_j}$

torus — CY — IHS

If we know HK cone conj. on $\tilde{X}$ what can we say on X?

Theorem (Pacienza–S. 2024)
$X = \tilde{X}/G$ Enriques manifold $\tilde{X} \longrightarrow X$ univ. cover, $\tilde{X}$ IHS

$G \curvearrowright \tilde{X}$, $(G = \langle g \rangle)$

Assume that G acts as the identity on $\mathrm{Pic}(\tilde{X}) \Rightarrow$ the HK cone conj. holds for X

Corollary X very general Enriques mnf. of prime index $p \Rightarrow$ HK cone conj. holds for X.

$$\mathrm{Nef}^e = \underline{\mathrm{Nef} \cap \mathrm{Eff}(X)} \subset \underline{\mathrm{Nef}^+(X)}$$

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