CONSTRUCTIONS OF KUMMER STRUCTURES ON GENERALIZED KUMMER SURFACES

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Abstract. We study generalized Kummer surfaces $\mathrm{Km}_3(A)$, by which we mean the K3 surfaces obtained by desingularization of the quotient of an abelian surface A by an order 3 symplectic automorphism group. Such a surface carries 9 disjoint configurations of two smooth rational curves C, C' with $CC' = 1$. This $9\mathbf{A}_2$-configuration plays a role similar to the Nikulin configuration of 16 disjoint smooth rational curves on (classical) Kummer surfaces. We study the (generalized) question of T. Shioda: suppose that $\mathrm{Km}_3(A)$ is isomorphic to $\mathrm{Km}_3(B)$, does that imply that A and B are isomorphic? We answer by the negative in general, by two methods: by a link between that problem and Fourier-Mukai partners of A, and by construction of $9\mathbf{A}_2$-configurations on $\mathrm{Km}_3(A)$ which cannot be exchanged under the automorphism group.

1. Introduction

A Kummer surface $\mathrm{Km}(A)$ is the minimal desingularization of the quotient of an abelian surface A by the standard involution $[-1]$. It is a K3 surface containing 16 disjoint (-2)-curves, which lie over the 16 singularities of $A/\langle[-1]\rangle$. Such set of curves is called a Kummer (or $16\mathbf{A}_1$) configuration. A well-known result of Nikulin [22] gives the converse: if a K3 surface contains a $16\mathbf{A}_1$-configuration, then it is the Kummer surface of an abelian surface A, such that the 16 (-2)-curves lie over the singularities of $A/\langle[-1]\rangle$.

In 1977 Shioda [30] asked the following question: *if two abelian surfaces A and B satisfy $\mathrm{Km}(A) \simeq \mathrm{Km}(B)$, is it true that $A \simeq B$?*

Gritsenko and Hulek [12] gave a negative answer to that question in general. In [26, 27], we studied and constructed examples of two $16\mathbf{A}_1$-configurations on the same Kummer surface such that their associated abelian surfaces are not isomorphic.

Kummer surfaces have natural generalizations to quotients of an abelian surface A by other symplectic groups $G \subseteq \mathrm{Aut}(A)$. If $G \cong \mathbb{Z}/3\mathbb{Z}$, then the quotient surface A/G for the action of G on A has 9 cusp singularities, in bijection with the fixed points of G. Its minimal desingularization, denoted by $\mathrm{Km}_3(A)$, is a K3 surface which contains what we call a *generalized Kummer configuration* (or $9\mathbf{A}_2$-*configuration*), which means that the surface contains 9 disjoint $\mathbf{A}_2$-configurations, i.e. pairs (C, C') of (-2)-curves

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such that $CC'=1$. Barth [2] proved that if a K3 surface contains a $9\mathbf{A}_2$-configuration, then there exists an abelian surface A and a symplectic order 3 automorphism group such that $X=\mathrm{Km}_3(X)$. It is then natural to ask the *generalized Shioda's question*: does an isomorphism $\mathrm{Km}_3(A)\simeq\mathrm{Km}_3(B)$ between two generalized Kummer surfaces implies that the abelian surfaces A and B are isomorphic?

A generalized Kummer structure on a K3 surface X is an isomorphism class of pairs (A,G) of abelian surfaces equipped with an order 3 symplectic automorphism subgroup $G\subset\mathrm{Aut}(A)$, such that $X\simeq\mathrm{Km}_3(A)$, where $\mathrm{Km}_3(A)$ is the minimal desingularization of A/G. Thus Shioda's question is if there is only one generalized Kummer structure on X. In [25], we study the number of such Kummer structures. In [14], we proved that there is a one-to-one correspondence between Kummer structures on X and $\mathrm{Aut}(X)$-orbits of $9\mathbf{A}_2$-configurations. In the present paper, we obtain the first explicit examples of generalized Kummer surfaces which possess two distinct generalized Kummer structures. For that aim, we construct two $9\mathbf{A}_2$-configurations $\mathcal{C}$, $\mathcal{C}'$ on the Kummer surface, and prove that there is no automorphism sending one configuration to the other. A generalized Kummer surface $X=\mathrm{Km}_3(A)$ has a natural $9\mathbf{A}_2$-configuration

$$\mathcal{C}=\{A_1,B_1,\dots,A_9,B_9\}.$$

We suppose that X is generic projective, so that its Picard number is 19. Let L be the big and nef generator of the orthogonal complement of the curves in the Néron-Severi group. By a result of Barth [2], one has either $L^2=6k+2$ or $L^2=6k$, for k an integer. We suppose that $6L^2$ is not a square, so that the two Pell-Fermat equations

$$x^2-12(3k+1)y^2=1 \text{ and } x^2-4ky^2=1$$

have non-trivial solutions. Let us denote by (x_0,y_0) the fundamental solution according to these cases and let us define accordingly:

$$\begin{array}{l} B_1'=3y_0L-(\frac{1}{2}(x_0+1)A_1+x_0B_1) \text{ if } L^2=6k+2,\\ B_1'=y_0L-(\frac{1}{2}(x_0+1)A_1+x_0B_1) \text{ if } L^2=6k.\end{array}$$

The class B_1' is a (-2)-class in the Néron-Severi group of X (i.e. $B_1'^2=-2$) such that $B_1'A_1=B_1A_1=1$. Our main result is

Theorem 1. *Suppose $L^2=2\bmod 6$ or $L^2\neq 0\bmod 18$ or $3|y_0$. Then B_1' is the class of a (-2)-curve such that $B_1'A_1=1$ and the 18 (-2)-curves*

$$\mathcal{C}'=\{A_1,B_1',A_2,B_2,\dots,A_9,B_9\}$$

form a $9\mathbf{A}_2$-configuration.
Suppose moreover that $L^2=2\bmod 6$ and $x_0\neq\pm1\bmod 2t$, or $L^2=6$ or $12\bmod 18$ and $x_0\neq\pm1\bmod 2k$. There are no automorphisms sending $\mathcal{C}$ to $\mathcal{C}'$. For these cases, there are (at least) two generalized Kummer structures on the generalized Kummer surface X.

As pointed out in Example 4.1 the first values for which our theorem produces new generalized Kummer structures are:

$$20, 44, 68, 84, 92, 104, 110, 116, 120, 126, 132, 140, 164, 168, 176, 188.$$

The paper is structured as follows: in Section 2, we give a precise description of the Néron-Severi group of a generalized Kummer surface and how a divisor can be written in the $\mathbb{Q}$-basis $L, A_1, B_1, \ldots, A_9, B_9$. We also obtain that $\mathrm{Km}_3(A) \simeq \mathrm{Km}_3(B)$ if and only if A and B are Fourier-Mukai partners. In Section 3, according to the cases $L^2 = 0$ or $2 \bmod 6$, we construct two generalized Kummer configurations. In Section 4, we give a sufficient condition on which these Kummer configurations give rise to two Kummer structures. In the last section we describe the projective model of the K3 surface determined by L and we recall some known constructions in the literature. We describe more in details the case $L^2 = 20$ which is the first case for which our Theorem gives two non–equivalent generalized Kummer structures.

We aim to study more projective models in a forthcomig paper.

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2. The Néron–Severi lattice and its properties, Fourier-Mukai partners

2.1. **Construction of the Néron-Severi lattice of X.** Let A be an abelian surface with an action of a group $G := \mathbb{Z}/3\mathbb{Z}$ that leaves invariant the space $H^0(A, \Omega_A^2)$ (we call this action *symplectic*). It is well known that the quotient A/G has $9\mathbf{A}_2$ singularities. The minimal resolution denoted by $X := \mathrm{Km}(A, G)$ is a K3 surface, called a *generalized Kummer surface*, which carries a configuration of rational curves with Dynkin diagram $9\mathbf{A}_2$. Observe that the abelian surface A has Picard number at least 3, see [2, Proposition on p. 10] and the K3 surface X has generically Picard number 19. Let $\mathcal{K}_3$ denotes the minimal primitive sub-lattice of the K3 lattice, Λ_{K3}, that contains the 9 configurations $\mathbf{A}_2$. This is a rank 18 negative definite even lattice of discriminant 3^3, which is described as follows. Denote by A_j, B_j, $j = 1, \ldots, 9$ the nine couples of (-2)-curves generating the nine $\mathbf{A}_2$. Then by [3, Proof of Proposition 1.3] the lattice $\mathcal{K}_3$ is generated by the classes $A_1, B_1, \ldots, A_9, B_9$ and the three classes

$$\begin{aligned}
v_1 &= \tfrac{1}{3}(\textstyle\sum_{i=1}^{9}(A_i - B_i)),\\
v_2 &= \tfrac{1}{3}((A_2 - B_2) + 2(A_3 - B_3) + A_6 - B_6 + 2(A_7 - B_7) + A_8 - B_8 + 2(A_9 - B_9)),\\
v_3 &= \tfrac{1}{3}((A_4 - B_4) + 2(A_5 - B_5) + A_6 - B_6 + 2(A_7 - B_7) + 2(A_8 - B_8) + A_9 - B_9),
\end{aligned}$$

with intersection matrix:

$$\begin{pmatrix} -6 & -6 & -6 \\ -6 & -10 & -6 \\ -6 & -6 & -10 \end{pmatrix}.$$

The discriminant group $\mathcal{K}_3^\vee/\mathcal{K}_3$ is generated by the classes

$$\begin{aligned} w_1 &= \tfrac{1}{3}(A_5 - B_5 + A_7 - B_7 + A_8 - B_8), \\ w_2 &= \tfrac{1}{3}(2(A_4 - B_4) + A_6 - B_6 + 2(A_7 - B_7) + A_8 - B_8), \\ w_3 &= \tfrac{1}{3}(A_3 - B_3 + A_5 - B_5 + A_6 - B_6), \end{aligned}$$

with intersection matrix:

$$\begin{pmatrix} -2 & -2 & -\frac{2}{3} \\ -2 & -\frac{20}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & -2 \end{pmatrix}.$$

Theorem 2. *Assume $\rho(\mathrm{Km}(A,G)) = 19$ and let L be a generator of $\mathcal{K}_3^\perp \subset \mathrm{NS}(\mathrm{Km}(A,G))$, with $L^2 > 0$. Then $L^2 \equiv 0 \bmod 6$ or $L^2 \equiv 2 \bmod 6$. We denote by $\mathcal{K}_{6k}$ the lattice $\mathbb{Z}L \oplus \mathcal{K}_3$ in the first case, and by $\mathcal{K}_{6k+2}$ the same lattice in the second case. Then*

(1) *If $L^2 \equiv 0 \bmod 6$ then*

$$\mathrm{NS}(\mathrm{Km}(A,G)) = \mathcal{K}'_{6k}$$

where $\mathcal{K}'_{6k}$ is generated by $\mathcal{K}_{6k}$ and by a class $\frac{L+v_{6k}}{3}$ where $\frac{v_{6k}}{3} \in \mathcal{K}_3^\vee/\mathcal{K}_3$ with $L^2 = -v_{6k}^2 \bmod 18$, moreover $\mathcal{K}_{6k}$ is the unique even lattice, up to isometry, such that $[\mathcal{K}'_{6k} : \mathcal{K}_{6k}] = 3$ and $\mathcal{K}_3$ is a primitive sublattice of $\mathcal{K}'_{6k}$, so that we can assume that

(a) *If $L^2 \equiv 0 \bmod 18$ then $v_{6k}^2 \equiv 0 \bmod 18$.*

(b) *If $L^2 \equiv 12 \bmod 18$ then $v_{6k}^2 \equiv -12 \bmod 18$.*

(c) *If $L^2 \equiv 6 \bmod 18$ then $v_{6k}^2 \equiv -6 \bmod 18$.*

(2) *If $L^2 \equiv 2 \bmod 6$ then*

$$\mathrm{NS}(\mathrm{Km}(A,G)) = \mathcal{K}_{6k+2}.$$

Proof. The fact that $L^2 \equiv 0 \bmod 6$ or $L^2 \equiv 2 \bmod 6$ follows from [2, Section 2.2]. In the first case if $L^2 \equiv 0 \bmod 6$ then the discriminant group of the lattice $\mathcal{K}_{6k}$ contains the generators w_1, w_2, w_3 and $\frac{L^2}{6k}$. Recall that for a K3 surface the discriminant group of the Néron-Severi group is the same as the discriminant group of the transcendental lattice, with the quadratic form which changes the sign. Since here the rank of the transcendental lattice is three, the number of independent generators of the discriminant group does not exceed three, hence a class of the discriminant group of the lattice $\mathcal{K}_{6k}$ is contained in the Néron-Severi group. This class is of the form $\frac{L+v_{6k}}{n}$, where $\frac{v_{6k}}{n}$ belongs to the discriminant group of $\mathcal{K}_3$. By the previous description we necessarily have that $n = 3$. Moreover since the class $\frac{L+v_{6k}}{3}$ is now in the Néron-Severi group, then $(\frac{L+v_{6k}}{3})^2 \in 2\mathbb{Z}$ so that $L^2 + v_{6k}^2 \in 18\mathbb{Z}$ since $L^2 = 6k$ for some integer k, thus we get the cases for L and v_{6k} listed in the statement. Moreover if $\frac{L+v'_{6k}}{3}$ is another class contained in the Néron-Severi group as before then $\frac{v_{6k}-v'_{6k}}{3}$ belongs to the Néron-Severi group and so to $\mathcal{K}_3$ but $\frac{v_{6k}-v'_{6k}}{3} \in \mathcal{K}_3^\vee/\mathcal{K}_3$ so it must be a zero class.

For the unicity statement, we use a similar argument as in [17, Proposition 2.2]. First, one computes some generators of the isometry group $O(\mathcal{K}_3)$ of the negative definite lattice $\mathcal{K}_3$. These elements act on the discriminant group $\mathcal{K}_3^\vee/\mathcal{K}_3 \simeq (\mathbb{Z}/3\mathbb{Z})^3$ and one obtains that the image of $O(\mathcal{K}_3)$ in $O(\mathcal{K}_3^\vee/\mathcal{K}_3)$ is a group isomorphic to $\mathbb{Z}/2\mathbb{Z} \times S_4$, which has exactly 4 orbits: $O_z = \{0\}, O_0, O_1, O_2$, where the elements in O_i ($i \in \{0,1,2\}$) have square $\frac{2i}{3} \bmod 2$. Therefore the isometry class of the gluing $\mathcal{K}'_{6k}$ does not depend on the choice of the element v_{6k} and is unique for each of the respective cases (a), (b) or (c).

In case that $L^2 = 6k+2$ with k an integer, observe that a class of the form $\frac{L+v_{6k+2}}{3}$ must satisfy $(L \cdot \frac{L+v_{6k+2}}{3}) = \frac{6k+2}{3} \in \mathbb{Z}$ which is impossible, so this class does not exist and we get that

$$\mathrm{NS}(\mathrm{Km}(A,G)) = \mathcal{K}_{6k+2},$$

which finishes the proof. □

Remark 3. The classes v_{6k} can be chosen equal to be $3w_1$ or $3w_3$ if $L^2 \equiv 0 \bmod 18$; equal to $3w_2$ if $L^2 \equiv 6 \bmod 18$ and equal to $3(w_3 - w_2)$ if $L^2 \equiv 12 \bmod 18$.

The following result is due to Barth:

Theorem 4. *(*[2, Section 2.2]*). There exist a K3 surface X such that* $\mathrm{NS}(X) = \mathcal{K}'_{6k}$ *(respectively* $\mathrm{NS}(X) = \mathcal{K}_{6k+2}$*) for an integer $k > 0$ (respectively $k \geq 0$) and such a surface X is a generalized Kummer surface.*

Remark 5. i) When the Picard number of X is 19, the integer L^2 determines uniquely the lattice $\mathrm{NS}(X)$. The knowledge of the discriminant group of $\mathrm{NS}(X)$ also determines L^2 and $\mathrm{NS}(X)$.
ii) Theorems 3 and 4 and their proofs (with $L \neq 0$) are also valid when A is a complex non-algebraic torus, equivalently, when $L^2 \leq 0$. If $L = 0$, then $\mathrm{NS}(X) = \mathcal{K}_3$ is negative definite of rank 18, and the K3 surface X is also non-algebraic.

2.2. **Classes and polarizations in the Néron-Severi lattice of X.**

2.2.1. *Notations and divisibility of the classes.* Let us denote the curves forming the generalized Kummer configuration by $A_1, B_1, \dots, A_9, B_9$, where $A_jB_j = 1$ and denote by L the orthogonal complement of these 18 curves in the rank 19 lattice $\mathrm{NS}(X)$, we recall that $L^2 = 2t$, for $t \in \mathbb{N}^*$. Let $a, \alpha_j, \beta_j \in \frac{1}{3}\mathbb{Z}$ be such that the class

$$\Gamma = aL - \sum_{j=1}^{9} (\alpha_j A_j + \beta_j B_j),$$

is in the Néron-Severi lattice $\mathrm{NS}(X)$. The following Corollary is a direct consequence of Theorem 2 and Remark 3:

Corollary 6. *For $j\in\{1,\dots,9\}$, one has $\alpha_j\in\frac{1}{3}\mathbb{Z}\setminus\mathbb{Z}\Leftrightarrow\beta_j\in\frac{1}{3}\mathbb{Z}\setminus\mathbb{Z}$.*
Suppose $L^2=2\bmod 6$. Then $a\in\mathbb{Z}$ and if one coefficient α_j or β_j is in $\frac{1}{3}\mathbb{Z}\setminus\mathbb{Z}$, then there are 12 or 18 coefficients that are in $\frac{1}{3}\mathbb{Z}\setminus\mathbb{Z}$,
Suppose $L^2=0\bmod 6$, and $a\in\mathbb{Z}$. If one coefficient α_j or β_j is in $\frac{1}{3}\mathbb{Z}\setminus\mathbb{Z}$, then there are 12 or 18 coefficients that are in $\frac{1}{3}\mathbb{Z}\setminus\mathbb{Z}$,
Suppose $L^2=0\bmod 18$ (respectively $L^2=6\bmod 18$ and $L^2=12\bmod 18$), and $a\in\frac{1}{3}\mathbb{Z}\setminus\mathbb{Z}$. Then there are at least 6 (respectively 8 and 10) coefficients α_j or β_j that are in $\frac{1}{3}\mathbb{Z}\setminus\mathbb{Z}$.
The group $L^{\perp}/\langle A_1,\dots,B_9\rangle$ (where the orthogonal of L is taken in $\mathrm{NS}(X)$) is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^3$. The 27 elements of that group are: 24 elements which are supported on 6 $\mathbf{A}_2$ blocs, an element S supported on the 9 blocs $\mathbf{A}_2$, the element $2S$, and the zero element.

Let $\Gamma\in\mathrm{NS}(X)$ be the class of a divisor and let us write

$$\Gamma=aL-\frac{1}{3}\sum_{j=1}^{9}\left(a_jA_j+b_jB_j\right).$$

with $a\in\frac{1}{3}\mathbb{Z},\ a_j,b_j\in\mathbb{Z}$. The intersection numbers $\Gamma A_j=\frac{1}{3}(2a_j-b_j)$ and $\Gamma B_j=\frac{1}{3}(2b_j-a_j)$ are integers. Since $2a_j-b_j$ and $2b_j-a_j$ are divisible by 3, there exist integers u_j,v_j such that

$$\begin{cases} a_j=u_j+2v_j\\ b_j=2u_j+v_j\end{cases},$$

so that we can write

$$\Gamma=aL-\frac{1}{3}\sum_{j=1}^{9}\left((u_j+2v_j)A_j+(2u_j+v_j)B_j\right),$$

with $u_j,v_j\in\mathbb{Z}$, which is also

$$\Gamma=aL-\frac{1}{3}\sum_{j=1}^{9}\left(u_jF_j+v_jG_j\right)$$

for $F_j=A_j+2B_j$, $G_j=2A_j+B_j$. We have $F_j^2=G_j^2=-6$, $F_jG_j=-3$, so that

$$\Gamma^2=2ta^2-\frac{2}{3}\sum_{j=1}^{9}\left(u_j^2+u_jv_j+v_j^2\right).$$

Let us suppose moreover that Γ is the class of an irreducible curve which is not among the 18 curves $A_1,\dots,B_9$. Then $a\in\frac{1}{3}\mathbb{N}^*$ and the intersection numbers $\Gamma A_j,\Gamma B_j$ are positive or zero:

$$\begin{cases} 2a_j\geq b_j\\ 2b_j\geq a_j\end{cases},$$

these inequalities are equivalent to $u_j \geq 0$, and $v_j \geq 0$.

2.2.2. *Polarizations.* Let be $u \in \mathbb{Z}$ and define

$$D = uL - \sum_{j=1}^{9}(A_j + B_j).$$

The 18 curves $A_1, B_1, \dots, A_9, B_9$ have degree 1 for D: $DA_k = DB_k = 1$. With the same notations, we show:

Proposition 7. *The minimal integer u_0 such that for $u \geq u_0$ the divisor D is ample is given in the following table (according to the cases of L^2):*

$L^2 = 2 \bmod 6$	$L^2 = 2$	$L^2 = 8$ *or* 14	$L^2 \geq 20$
u_0	4	2	1
$L^2 = 0 \bmod 18$	$L^2 = 18$	$L^2 \geq 36$	
u_0	2	1	
$L^2 = 6 \bmod 18$	$L^2 = 6$	$L^2 \geq 24$	
u_0	3	1	
$L^2 = 12 \bmod 18$	$L^2 = 12$	$L^2 \geq 30$	
u_0	2	1	

Proof. We start by defining $D = u_0 L - \sum_{j=1}^{9}(A_j + B_j)$ with u_0 as defined in the Table. We check that $D^2 > 0$ and $D^\perp$ contains no vector with square -2. Using that the ample cone is a fundamental domain for the Weyl group (the reflection group generated by reflection by vectors of square -2), we can choose D as an ample class. We have $DA_j = 1$; if $A_j = C + C'$ with effective divisors C, C', then DC or DC' is 0 thus, since D is ample, C or C' is 0, which prove that A_j is a (-2)-curve. Using that fact, one easily check that L is nef. Then for $u \geq u_0$, the divisor $(u - u_0)L + D$ is also ample, which proves the result. $\square$

2.3. **Fourier-Mukai partners and generalized Kummer structures.** Recall that for a K3 surface (resp. an abelian surface) X, a Fourier-Mukai partner of X is a K3 surface (resp. an abelian surface) Y such that there is an isomorphism of Hodge structures

$$(T(Y), \mathbb{C}\omega_Y) \simeq (T(X), \mathbb{C}\omega_X),$$

where ω_X is a generator of $H^0(X, \Omega_X^2)$, and $T(X)$ is the transcendental lattice. The set of isomorphism classes of Fourier-Mukai partners of X is denoted by $\mathrm{FM}(X)$.

Let A be an abelian surface and G_A be an order 3 automorphism group of A acting symplectically. We denote by $X = \mathrm{Km}_3(A)$ (or sometimes $\mathrm{Km}_3(A, G_A)$ or $\mathrm{Km}_3(A, J_A)$, where $\langle J_A \rangle = G_A$) the minimal resolution of the quotient surface A/G_A; since G_A is symplectic, X is a generalized Kummer surface.

An isomorphism class of pairs (B, G_B) where B is an abelian surface and G_B is an order 3 symplectic automorphism group such that $\mathrm{Km}_3(B) \simeq X$ is called a *generalized Kummer structure* on X. Let us denote by $\mathcal{K}(X)$ these isomorphism classes.

Theorem 8. *Let (A, G_A) and (B, G_B) be two abelian surfaces with an order 3 symplectic automorphism group. We have $\mathrm{Km}_3(B) \simeq \mathrm{Km}_3(A) = X$ (i.e. $\{(B, G_B)\} \in \mathcal{K}(X)$) if and only if B is a Fourier-Mukai partner of A.*

The proof is similar to [10, Theorem 0.1] for the classical Kummer surfaces. Let us start with the following

Lemma 9. *The canonical rational map*

$$\pi_A : A \dashrightarrow \mathrm{Km}_3(A)$$

gives an Hodge isometry

$$\pi_{A,*} : (T(A)(3), \mathbb{C}\omega_A) \stackrel{\simeq}{\to} (T(\mathrm{Km}_3(A)), \mathbb{C}\omega_{\mathrm{Km}_3(A)}).$$

Here, for a lattice $L := (L, (\,,\,))$ and a non-zero integer m, we define the lattice $L(m)$ by $L(m) := (L, m(\,,\,))$.

Proof. (Of Lemma 9). The algebraic surfaces A and $\mathrm{Km}_3(A)$ have equal geometric genus. We can therefore apply [13, Proposition 1.1 & Remark after the Proposition] of Inose to the degree 3 rational map π_A, which gives the result. □

Let us prove Theorem 8.

Proof. Let B be an abelian surface with an order 3 symplectic group G_B acting on it. From Lemma 9, there exists an isomorphism of Hodge structures

$$(T(B), \mathbb{C}\omega_B) \simeq (T(A), \mathbb{C}\omega_A)$$

if and only if there is an isomorphism of Hodge structures

$$(T(\mathrm{Km}_3(B)), \mathbb{C}\omega_{\mathrm{Km}_3(B)}) \simeq (T(\mathrm{Km}_3(A)), \mathbb{C}\omega_{\mathrm{Km}_3(A)}).$$

Therefore B is a Fourier-Mukai partner of A if and only if $\mathrm{Km}_3(B)$ is a Fourier-Mukai partner of $\mathrm{Km}_3(A)$. By [11, Corollary 2.6], since $X = \mathrm{Km}_3(A)$ has Picard number $19 > 2 + \ell$ (here ℓ is the length of the discriminant group of $\mathrm{NS}(X)$, which is ≤ 3), one has $\{\mathrm{Km}_3(A)\} = \mathrm{FM}(\mathrm{Km}_3(A))$ (which means that the isomorphism class of $\mathrm{Km}_3(A)$ is the unique Fourier-Mukai partner of $\mathrm{Km}_3(A)$), therefore B is a Fourier-Mukai partner of A if and only if $\mathrm{Km}_3(B) \simeq \mathrm{Km}_3(A)$. □

By [6, Proposition 5.3], the number $|\mathrm{FM}(A)|$ of Fourier-Mukai partners of A is finite. So to find the generalized Kummer structures on X reduces to find the number of conjugacy classes of order 3 symplectic groups G' on a finite number of abelian surfaces B such that $\mathrm{Km}_3(B, G') \simeq \mathrm{Km}_3(A)$.

3. Constructions of Kummer structures

3.1. **(-2)-classes with intersection one with A_1.** Let $A_1, B_1, \dots, A_9, B_9$ be a generalized Nikulin configuration on a generalized Kummer surface X. Let L be the big and nef generator of the orthogonal complement of these 18 curves and let $t \in \mathbb{N}$ such that $L^2 = 2t$. According to the two possible cases, $L^2 = 2 \bmod 6$ and $L^2 = 0 \bmod 6$, we denote by $k \in \mathbb{N}$ the integer such that $t = 1 + 3k$ in the first case, and such that $t = 3k$ in the second case.

Our aim is to construct a (-2)-curve $B_1' \neq B_1$ in the lattice $\mathbb{L}$ generated by L, A_1, B_1 such that $A_1 B_1' = 1$, so that (A_1, B_1') is another $\mathbf{A}_2$-configuration. As we will see in the next Section, under some conditions on t, such a curve exists and is unique. In order to prove this result, we will study the properties of L', the orthogonal complement of A_1, B_1' in $\mathbb{L}$.

By Theorem 2, an element in the lattice generated by L, A_1, B_1 has the form

$$B_1' = aL - (a_1 A_1 + b_1 B_1),$$

for integers a, a_1, b_1. The class B_1' satisfies $B_1' A_1 = 1$ if and only if $2a_1 - b_1 = 1$, which gives

$$B_1' = aL - (a_1 A_1 + (2a_1 - 1)B_1).$$

Moreover since we search for a (-2)-curve, we have

$$B_1'^2 = -2 = 2ta^2 - 6a_1^2 + 6a_1 - 2,$$

which is equivalent to

$$3a_1(a_1 - 1) - ta^2 = 0.$$

We can write that condition as

$$3\left((2a_1 - 1)^2 - 1\right) - 4ta^2 = 0,$$

which is equivalent to

$$3(2a_1 - 1)^2 - 4ta^2 = 3. \tag{3.1}$$

3.1.1. *Case $t = 1 \bmod 3$.* Since we suppose that $t = 1 \bmod 3$, the integer a in Equation (3.1) must be divisible by 3. Let us define the integers x_0, y_0 by

$$x_0 = 2a_1 - 1, \;\; a = 3y_0.$$

Equation (3.1) is then equivalent to the Pell-Fermat equation

$$x_0^2 - 12ty_0^2 = 1. \tag{3.2}$$

Let us suppose that $12t$ is not a square, so that there exist non-trivial solutions, and let us fix such a solution (x_0, y_0) (we observe that x_0 is necessarily odd). Then

$$B_1' = 3y_0 L - (a_1 A_1 + (2a_1 - 1)B_1)$$

with $2a_1 - 1 = x_0$ is such that $B_1'^2 = -2$ and $B_1' A_1 = 1$, and conversely, all (-2)-classes with these two properties are obtained in that way.

Let us search for the class of $L' = \alpha L - (\beta_1 A_1 + \beta_1' B_1)$, $\alpha, \beta_1, \beta_1' \in \mathbb{Z}$, such that L' generates the orthogonal complement of $A_1, B_1', A_2, B_2, \dots, A_9, B_9$. From $L'A_1 = 0$, we obtain

$$\beta_1' = 2\beta_1.$$

Since also $L'B_1' = 0$, we get

$$2\alpha at = 3\beta_1(2a_1 - 1),$$

in other words:

$$2\alpha ty_0 = \beta_1 x_0.$$

Since by equation 3.2 the integers x_0, y_0 are co-primes and x_0 is co-prime to $2t$, we get:

$$\beta_1 = 2ty_0,\ \ \alpha = x_0,$$

and the class $L' = x_0 L - 2ty_0(A_1 + 2B_1)$ is primitive with $L'L > 0$, and such that

$$L'^2 = 2tx_0^2 - 24t^2 y_0^2 = 2t.$$

For any solution (x_0, y_0) of the Pell-Fermat equation (3.2), the classes

$$\begin{array}{l} B_1' = 3y_0 L + (\frac{1}{2}(x_0+1)A_1 + x_0 B_1) \\ L' = x_0 L - 2ty_0(A_1 + 2B_1) \end{array}$$

in $\mathbb{L}$ have the properties $B_1'^2 = -2$, $B_1'A_1 = 1$, $L'A_1 = L'B_1' = 0$, $L'^2 = 2t$, $L'L > 0$ and all the classes (B, L_0) with these properties are obtained in that way.

3.1.2. *Case* $t = 0 \bmod 3$. Let us suppose that $t = 0 \bmod 3$, and let $k \in \mathbb{N}^*$ such that $t = 3k$. Then the equation

$$3(2a_1 - 1)^2 - 4ta^2 = 3$$

is equivalent to

$$(2a_1 - 1)^2 - 4ka^2 = 1.$$

By defining $x_0 = 2a_1 - 1,\ y_0 = a$, we are reduced to the Pell-Fermat equation

$$x_0^2 - 4ky_0^2 = 1, \tag{3.3}$$

(x_0 is necessarily odd). Let us fix such a solution (x_0, y_0). Then

$$B_1' = y_0 L - (a_1 A_1 + (2a_1 - 1)B_1),$$

with $2a_1 - 1 = x_0$ satisfies $B_1'^2 = -2$, $B_1'A_1 = 1$.

Let us search for the class of $L' = \alpha L - (\beta_1 A_1 + \beta_1' B_1)$, $\alpha, \beta_1, \beta_1' \in \mathbb{Z}$, such that L' generates the orthogonal complement of $A_1, B_1', A_2, B_2, \dots, A_9, B_9$. From $L'A_1 = 0$, we obtain

$$\beta_1' = 2\beta_1,$$

and since $L'B_1' = 0$, we get

$$2\alpha at = 3\beta_1(2a_1 - 1),$$

in other words:

$$2\alpha ky_0 = \beta_1 x_0.$$

Since x_0, y_0 are co-primes, and L' is primitive, we get:

$$\beta_1 = 2ky_0,\ \alpha = x_0$$

and $L' = x_0L - 2ky_0(A_1 + 2B_1)$, thus

$$L'^2 = 2tx_0^2 - 24k^2y_0^2 = 2t(x_0^2 - 4ky_0^2) = 2t.$$

For any solution (x_0, y_0) of the Pell-Fermat equation (3.3), the classes

$$\begin{array}{l} B_1' = y_0L + (\frac{1}{2}(x_0+1)A_1 + x_0B_1) \\ L' = x_0L - 2ky_0(A_1 + 2B_1) \end{array}$$

have the properties we required: $B_1'^2 = -2$, $B_1'A_1 = 1$, $L'A_1 = L'B_1' = 0$ and all classes with these properties are obtained in that way.

3.2. **Existence and unicity of B_1'.**

3.2.1. *Case $t = 1 \bmod 3$.* Let (x_0, y_0) be a non-trivial solution of the Pell-Fermat equation (3.2). In Section 3.1.1, we defined the classes

$$\begin{array}{l} L' = x_0L - 2ty_0(A_1 + 2B_1), \\ B_1' = 3y_0L - (\frac{1}{2}(x_0+1)A_1 + x_0B_1). \end{array}$$

Let us prove the following result:

Proposition 10. *Suppose that (x_0, y_0) is the fundamental solution of the Pell-Fermat equation (3.2). The class L' is nef, moreover $L'\Gamma = 0$ for a (-2)-class Γ if and only if $\pm\Gamma \in \{A_1, B_1', A_2, \dots, A_9, B_9\}$.*

Proof. Let Γ be a (-2)-curve, we write it as (see Section 2.2.1)

$$\Gamma = aL - \frac{1}{3}\sum_{j=1}^{9}\left((u_j + 2v_j)A_j + (2u_j + v_j)B_j\right),$$

with $u_j \geq 0$, $v_j \geq 0$ and $u_j, v_j \in \mathbb{Z}$, $a \in \mathbb{Z}$, so that

$$\Gamma^2 = 2ta^2 - \frac{2}{3}\sum_{j=1}^{9}\left(u_j^2 + u_jv_j + v_j^2\right) = -2. \tag{3.4}$$

Suppose that $\Gamma L' \leq 0$. This is equivalent to:

$$\left(aL - \frac{1}{3}\left((u_1 + 2v_1)A_1 + (2u_1 + v_1)B_1\right)\right)(x_0L - 2ty_0(A_1 + 2B_1)) \leq 0$$

which is equivalent to

$$ax_0 \leq (2u_1 + v_1)y_0.$$

By taking the square (recall that $x_0 > 0, y_0 > 0$ and $a \geq 0$), we get

$$x_0^2a^2 \leq \left(4(u_1^2 + u_1v_1 + v_1^2) - 3v_1^2\right)y_0^2,$$

which is equivalent to

$$-4(u_1^2 + u_1v_1 + v_1^2)y_0^2 \leq -x_0^2a^2 - 3v_1^2y_0^2,$$

thus
$$-\frac{2}{3}(u_1^2+u_1v_1+v_1^2)\leq-\frac{1}{6y_0^2}(x_0^2a^2+3v_1^2y_0^2)$$
and using equality $\Gamma^2=-2$ in (3.4), we get
$$-2\leq 2ta^2-\frac{1}{6y_0^2}(x_0^2a^2+3v_1^2y_0^2)-\frac{2}{3}S$$
where here we denote
$$S=\sum_{j=2}^{9}\left(u_j^2+u_jv_j+v_j^2\right)\geq 0.$$
Thus we obtain
$$\tfrac{1}{2}v_1^2+\tfrac{1}{6}a^2(\frac{x_0^2}{y_0^2}-12t)+\tfrac{2}{3}S\leq 2,$$
which is equivalent to
$$\tfrac{1}{2}v_1^2+\tfrac{1}{6}\frac{a^2}{y_0^2}+\tfrac{2}{3}S\leq 2. \tag{3.5}$$
If $a=0$, then $\Gamma=A_1$ or A_j,B_j with $j\geq 2$. We therefore suppose that $a\neq 0$, and then $a>0$ since Γ is effective. Let us suppose that one of the coefficients $\alpha_j=\frac{1}{3}(u_j+2v_j)$, $\beta_j=\frac{1}{3}(2u_j+v_j)$ of Γ is in $\frac{1}{3}\mathbb{Z}\setminus\mathbb{Z}$. Then by Corollary 6, at least 12 of the coefficients α_j,β_j are non-zero, and therefore at least 5 of the u_j or v_j (which are ≥ 0) with the condition $j\geq 2$ are non-zero. That implies that $S\geq 5$, thus $\frac{2}{3}S\geq\frac{10}{3}>2$, which is a contradiction. So all the coefficients of Γ are integers. Suppose that $S>0$. Since $S<3$, there exist one or two indices $j\geq 2$ such that u_j or v_j is equal to 1 (and the other coefficients with index $k\geq 2$ are 0). But then the coefficients of Γ are not integral: the only possibility is $S=0$. Since $a\neq 0$, we have $v_1\in\{0,1\}$. Since $\Gamma^2=-2$, the integers a,u_1,v_1,t satisfies
$$2ta^2-\frac{2}{3}\left(u_1^2+u_1v_1+v_1^2\right)=-2,$$
which is equivalent to
$$u_1^2+u_1v_1+v_1^2=3(ta^2+1).$$
If $v_1=0$, since $t=1\bmod 3$, $ta^2+1=1$ or $2\bmod 3$. But then $3(ta^2+1)$ is not a square, and there is no such a solution. Therefore $v_1=1$, and from inequality (3.5), the integer a is in the range
$$1\leq a\leq 3y_0.$$
The equality
$$u_1^2+u_1+1=3(ta^2+1),$$
is equivalent to
$$(2u_1+1)^2+3=12(ta^2+1),$$
thus to
$$(2u_1+1)^2=3(4ta^2+3).$$

Then $2u_1+1$ must be divisible by 3: let w be such that $3w=2u_1+1$. The above equation is equivalent to

$$3w^2=4ta^2+3,$$

which in turn, since $t=1 \bmod 3$, implies that there exists an integer A such that $a=3A$, then the equation is equivalent to the Pell-Fermat equation

$$w^2-12tA^2=1.$$

Let w_0, A_0 be a solution of that Pell-Fermat equation, then $a=3A_0, u_1=\frac{1}{2}(3w_0-1)$, $v_1=1$ are such that $\Gamma^2=-2$. Since $a\leq 3y_0$, and we now use that (x_0,y_0) is the fundamental solution of the Pell-Fermat equation to conclude that $a=3y_0$ and $u_1=\frac{1}{2}(3x_0-1)$, for which integers one has $\Gamma=B_1'$, and $L'B_1'=0$. That concludes the proof. □

We obtain:

Corollary 11. *The divisor $B_1'=3y_0L-(\frac{1}{2}(x_0+1)A_1+x_0B_1)$ is the class of a (-2)-curve.*
The curves B_1 and B_1' are the unique (-2)-curves in the lattice generated by L, A_1, B_1 which have intersection 1 with A_1.

Proof. A suitable multiple of the big and nef divisor L' defines a map which is generically one to one onto a projective model. It contracts the irreducible curves subjacent to B_1' and the curves $A_1, A_2, B_2, \dots, A_9, B_9$ to ADE singularities. By the genericity assumption on the K3 surface X, the surface has Picard number 19, which forces the image of A_1+B_1' to be a $\mathbf{A}_2$-singularity, therefore the curve B_1' is irreducible.

Let us prove the unicity assumption. By Section 3.1, if $\tilde{B}_1$ is the class of a (-2)-curve such that $\tilde{B}_1A_1=1$, there exists a solution (x,y) of the Pell-Fermat equation (3.2) such that

$$\tilde{B}_1=3yL-(\frac{1}{2}(x+1)A_1+xB_1).$$

Suppose $\tilde{B}_1\neq B_1$ ie $(x,y)\neq(-1,0)$. Then since $\tilde{B}_1$ is effective, one has $x>0, y>0$ and there exists $k\in\mathbb{N}^*$ such that $x=x_k,\ y=y_k$ for

$$x_k+\sqrt{12t}y_k=(x_0+\sqrt{12t}y_0)^k=(x_0+\sqrt{12t}y_0)(x_{k-1}+\sqrt{12t}y_{k-1}),$$

where (x_0,y_0) is the fundamental solution of the Pell-Fermat equation (3.2) (see e.g. [1]). One has

$$\begin{array}{rl}\tilde{B}_1B_1' & =(3yL-(\frac{1}{2}(x+1)A_1+xB_1))(3y_0L-(\frac{1}{2}(x_0+1)A_1+x_0B_1)) \\ & =18yy_0t-\frac{3}{2}xx_0-\frac{1}{2}\end{array},$$

therefore $\tilde{B}_1B_1'<0$ if and only if $xx_0+\frac{1}{3}>12yy_0$. Using an induction on k, one can check that this is the case for all $k\geq 1$. Thus if $k>1$, the (-2)-class $\tilde{B}_1$ cannot be the class of an irreducible curve. We observe moreover that if $k=1$, then $\tilde{B}_1=B_1'$, and that concludes the proof. □

3.2.2. *Case* $t = 0 \bmod 6$. Suppose that $L^2 = 2t = 6k$. Let (x_0, y_0) be the fundamental solution of the Pell-Fermat equation (3.3). Let us search when the (-2)-class

$$B_1' = y_0 L - (\tfrac{1}{2}(x_0+1)A_1 + x_0 B_1),$$

is the class of a (-2)-curve. Recall that

$$L' = x_0 L - 2ky_0(A_1 + 2B_1),$$

generates in $\mathrm{NS}(X)$ the orthogonal complement of $A_1, B_1', A_2, B_2, \dots, A_9, B_9$.

Proposition 12. *Suppose that $L^2 = 6$ or $12 \bmod 18$ or $3|y_0$. The class L' is nef. One has $L'\Gamma = 0$ for an effective (-2)-class Γ if and only if $\Gamma \in \{A_1, B_1', A_2, \dots, A_9, B_9\}$.*
Suppose that $L^2 = 0 \bmod 18$ and $3|y_0$. Up to exchanging the curves A_1 and B_1, the same result holds true.

Let us prove Proposition 12. Let Γ be an effective (-2)-class; we write it as

$$\Gamma = aL - \frac{1}{3}\sum_{j=1}^{9}\left((u_j + 2v_j)A_j + (2u_j + v_j)B_j\right),$$

with $u_j \geq 0$, $v_j \geq 0$, $u_j, v_j \in \mathbb{Z}$, but that time, since $t = 0 \bmod 6$, we allow a to be in $\frac{1}{3}\mathbb{Z}$, (moreover $a > 0$, since Γ is effective). As in Section 3.2.1, let us study if L' is nef. After computations similar to those of Section 3.2.1, we obtain that $\Gamma L' \leq 0$ if and only if

$$\frac{1}{2}v_1^2 + \frac{3a^2}{2y_0^2} + \frac{2}{3}S \leq 2, \tag{3.6}$$

where $S = \sum_{j=2}^{9}\left(u_j^2 + u_j v_j + v_j^2\right) \in \mathbb{N}$.
Suppose that one of the coefficients of Γ is in $\frac{1}{3}\mathbb{Z} \setminus \mathbb{Z}$. By Corollary 6, there are at least $6, 8$ or 10 coefficients $\alpha_j = \frac{1}{3}(u_j + 2v_j), \beta_j = \frac{1}{3}(2u_j + v_j)$ that are in $\frac{1}{3}\mathbb{Z} \setminus \mathbb{Z}$ according if $L^2 = 0$, 6 or $12 \bmod 18$. Thus respectively, at least $3, 4$ or 5 integers u_j, v_j are non-zero, and therefore the sum S (which is over the indices $j \geq 2$) is $\geq 2, 3$ or 4 respectively. Since $S < 3$, by Corollary 6, necessarily $L^2 = 0 \bmod 18$, $S = 2$, $a \in \frac{1}{3}\mathbb{Z} \setminus \mathbb{Z}$ and

$$\frac{1}{2}v_1^2 + \frac{3a^2}{2y_0^2} \leq \frac{2}{3}.$$

Let us suppose that $v_1 = 1$ and define $a' \in \mathbb{Z}$ such that $a = \frac{a'}{3}$ (since $a \in \frac{1}{3}\mathbb{Z} \setminus \mathbb{Z}$, a' is coprime to 3). Inequality (3.6) implies $a' \leq y_0$. Since $S = 2$ and $v_1 = 1$, the (-2)-class Γ has the form:

$$\Gamma = \tfrac{1}{3}a'L - \tfrac{1}{3}(u_1 F_1 + G_1 + H + H')$$

with H, H' in $\{F_j, G_j \,|\, j \geq 2\}$ such that $HH' = 0$. Equality $\Gamma^2 = -2$ is equivalent to

$$\frac{2}{3}ka'^2 - \frac{2}{3}(u_1^2 + u_1 + 1) - \frac{2}{3} - \frac{2}{3} = -2,$$

which is equivalent to

$$(u_1^2+u_1+1)-ka'^2=1,$$

(observe here that since $L^2=0 \bmod 18$, one has $k=0 \bmod 3$, thus $u_1^2+u_1=0 \bmod 3$) and finally, to

$$(2u_1+1)^2-4ka'^2=1.$$

Since $a'\leq y_0$ and (x_0,y_0) is the fundamental solution of the Pell-Fermat equation (3.3), we have $a'=y_0$, (thus $a=\frac{1}{3}y_0$) and $u_1=\frac{1}{2}(x_0-1)$, so that the class Γ is

$$\Gamma_0=\tfrac{1}{3}y_0L-\tfrac{1}{3}\left(\tfrac{1}{2}(x_0-1)F_1+G_1+H+H'\right).$$

By Corollary 6, this class can be in the Néron-Severi group only if y_0 is coprime to 3. Conversely, suppose y_0 is coprime to 3, let us search when is Γ_0 in $\mathrm{NS}(X)$.

If such a pair $\{H,H'\}$ exists, it is unique, otherwise the difference between the two obtained (-2)-classes Γ_0 would have at least 2 and at most 8 coefficients in $\frac{1}{3}\mathbb{Z}\setminus\mathbb{Z}$, with the coefficient of L in $\mathbb{Z}$, but this is impossible by Corollary 6.

Suppose that the (-2)-class

$$\Gamma_0'=\tfrac{1}{3}y_0L-\tfrac{1}{3}\left(\tfrac{1}{2}(x_0-1)G_1+F_1+H_1+H_1'\right)$$

is also in the Néron-Severi group, for some classes $H_1,H_1'\in\{F_j,G_j\,|\,j\geq 2\}$. Then

$$\Gamma_0-\Gamma_0'=\tfrac{1}{3}((u_1-1)(G_1-F_1)+H+H'-H_1-H_1')$$

has at least 2 (because $u_1\neq 1 \bmod 3$) and at most 10 coefficients in $\frac{1}{3}\mathbb{Z}\setminus\mathbb{Z}$. By Corollary 6, this is impossible, we thus obtain that either Γ_0 or Γ_0' is not in the Néron-Severi group. Thus up to exchanging A_1 and B_1, one can suppose that Γ_0' is not in the Néron-Severi group.

Remark 13. If $\Gamma_0\in\mathrm{NS}(X)$, then

$$B_1'=3\Gamma_0+B_1+H+H',$$

and the curve B_1' is not irreducible.

If 3 divides y_0, there is no such class Γ_0. One can check moreover that the coefficient v_1 of Γ cannot be 0.
The case $a\in\frac{1}{3}\mathbb{Z}\setminus\mathbb{Z}$ and $v_1=0$ leads to $L^2=0$ mod 18 and Γ has the form

$$\Gamma=\frac{2}{3}y_0L-\frac{1}{3}\left(x_0(A_1+2B_1)+H+H'\right),$$

with H,H' in $\{F_j,G_j\,|\,j\geq 2\}$ such that $HH'=0$.

Let us therefore assume that either $L^2=6$ or $12 \bmod 18$, or 3 divides y_0 if $L^2=0 \bmod 18$, or that Γ_0 is not the class of a (-2)-curve. Then by the previous discussion, the coefficient a of Γ is an integer, and therefore all the coefficients of Γ are integers by Corollary 6, moreover $S=0$, $v_1\in\{0,1\}$, and

$$\Gamma = aL - \frac{1}{3}(u_1F_1+v_1G_1)$$

with

$$6ka^2-\frac{2}{3}(u_1^2+u_1v_1+v_1^2)=-2, \tag{3.7}$$

where we put $L^2=6k$. Suppose $v_1=0$, then Equation (3.7) is equivalent to $u_1^2=3(1+3ka^2)$, which has no solutions. Therefore $v_1=1$, and by defining $U=\frac{1}{3}(2u_1+1)$, Equation (3.7) is equivalent to

$$U^2-4ka^2=1.$$

Since by (3.6), $a\leq y_0$ and (x_0,y_0) is the fundamental solution of the above Pell-Fermat equation, we obtain that $a=y_0$, $u_1=\frac{1}{2}(3x_0-1)$ and

$$\Gamma = y_0L-(\tfrac{1}{2}(x_0+1)A_1+x_0B_1)=B_1'.$$

That finishes the proof of Proposition 12.

As for the case $L^2=2\bmod 6$, we get :

Corollary 14. *Suppose that (x_0,y_0) is the fundamental solution of the Pell-Fermat equation 3.3 and that $L^2=6$ or $12\bmod 18$ or $3|y_0$. The divisor $B_1'=y_0L-(\frac{1}{2}(x_0+1)A_1+x_0B_1)$ is the class of a (-2)-curve.*
The curves B_1 and B_1' are the unique (-2)-curves in the lattice generated by L,A_1,B_1 which have intersection one with A_1.
Up to exchanging the role of the curves A_1,B_1, the same result holds true for $L^2=0\bmod 18$.

Proof. For proving that B_1' is a (-2)-curve we use the same argument as in the proof of Corollary 11. Let $\tilde{B}_1$ be a (-2)-curve such that $\tilde{B}_1A_1=1$. Suppose $\tilde{B}_1\neq B_1$, there exists (x,y) with $x>0,y>0$ solution of the Pell-Fermat equation 3.3 such that

$$\tilde{B}_1 = yL-\left(\frac{1}{2}(x+1)A_1+xB_1\right).$$

One has $\tilde{B}_1B_1'<0$ if and only if

$$4kyy_0<xx_0+\frac{1}{3}.$$

We proceed as in the proof of Corollary 11 and obtain that this inequality holds for any such (x,y), therefore $(x,y)=(x_0,y_0)$. □

4. Existence of two Kummer structures

4.1. **A theoretical approach.** Let X be a generalized Kummer surface, we keep the notations as before, in particular the polarization L generates the orthogonal complement to the 18 curves $A_1,\dots,B_9$. Through this section we suppose that $6L^2$ is not a square. For $L^2=2\bmod 6$ (respectively for

$L^2 = 0 \bmod 6$), let (x_0, y_0) be the fundamental solution of the Pell-Fermat equation

$$x^2 - 12ty^2 = 1,$$

for t such that $L^2 = 2t$ (respectively

$$x^2 - 4ky^2 = 1,$$

for k such that $L^2 = 6k$). We remark that $x_0^2 = 1 \bmod 12t$ (respectively $x_0^2 = 1 \bmod 4k$). Let B_1' be the (-2)-class

$$B_1' = 3y_0 L - (\tfrac{1}{2}(x_0+1)A_1 + x_0 B_1),$$

(respectively

$$B_1' = y_0 L - (\frac{1}{2}(x_0+1)A_1 + x_0 B_1))\,).$$

Let us suppose that B_1' is the class of a (-2)-curve. This is guarantied by Corollaries 11 and 14 if $L^2 \neq 0 \bmod 18$, or if $3|y_0$, or up to exchanging the role of A_1 and B_1 if $L^2 = 0 \bmod 18$. Then, we know two generalized Nikulin configurations

$$\mathcal{C} = \{A_1, B_1, \dots, A_9, B_9\},\ \mathcal{C}' = \{A_1, B_1', A_2, B_2, \dots, A_9, B_9\}.$$

Let us prove the following:

Theorem 15. *Suppose that $x_0 \neq \pm 1 \bmod 2t$ (respectively $x_0 \neq \pm 1 \bmod 2k$). There is no automorphism sending the configuration $\mathcal{C}$ to the configuration $\mathcal{C}'$. As a consequence, there are (at least) two generalized Kummer structures on the generalized Kummer surface X.*

Proof. Let us suppose that such an automorphism g sending $\mathcal{C}$ to $\mathcal{C}'$ exists. The automorphism g induces an isometry on $\mathrm{NS}(X)$, it therefore sends the orthogonal complement of $\mathcal{C}$ to the orthogonal complement of $\mathcal{C}'$. Since L, L' are the positive generators of these complements, it maps L to L'. Suppose that L is such that $L^2 = 2 \bmod 6$. We recall that $L' = x_0 L - 2ty_0(A_1 + 2B_1)$ and that by Section 2.1, the Néron-Severi lattice is

$$\mathrm{NS}(X) = \mathbb{Z}L \oplus \mathcal{K}_3,$$

therefore $\frac{1}{2t}L$ is in the dual of $\mathrm{NS}(X)$ (we recall that $L^2 = 2t$). Since we know that $g(L) = L'$, the action of g on the class of $\frac{1}{2t}L$ in the discriminant group is

$$g^*(\frac{1}{2t}L) = \frac{x_0}{2t}L - y_0(A_1 + 2B_1) = \frac{x_0}{2t}L \in NS(X)^\vee / NS(X).$$

However, the action of an automorphism on the discriminant group must be $\pm$ identity (see e.g. [29, Section 8.1]). Since the hypothesis is that $x_0 \neq \pm 1 \bmod 2t$, such a g does not exist.

Suppose that $L^2 = 0 \bmod 6$, i.e. $L^2 = 6k$ ($k \in \mathbb{N}^*$), then $L' = x_0 L - 2ky_0(A_1 + 2B_1)$. The Néron-Severi lattice is generated by the lattice $\mathbb{Z}L \oplus \mathcal{K}_3$ (see section 2.1) and by a vector $\frac{1}{3}(L + v_{6k})$, $v_{6k} \in \mathcal{K}_3$. The class $\frac{1}{2k}L$ is thus in the dual lattice of $\mathrm{NS}(X)$. The action of g on the class of $\frac{1}{2k}L$ in the discriminant group is

$$g^*(\frac{1}{2k}L) = \frac{x_0}{2k}L - y_0(A_1 + 2B_1) = \frac{x_0}{2k}L.$$

Again, since we supposed that $x_0 \neq \pm 1 \bmod 2k$, this is impossible. □

Remark 16. Suppose for simplicity that $L^2 = 2 \bmod 6$. One could play again the same game: pick-up an $\mathbf{A}_2$ configuration C_1, D_1 in $\mathcal{C}'$, then Corollary 11 implies that $D_1' = 3y_0L - (\frac{1}{2}(x_0+1)C_1 + x_0D_1)$ is irreducible and we obtain in that way a new $9\mathbf{A_2}$-configuration $\mathcal{C}''$, with orthogonal complement L''. Again there is no automorphism sending $\mathcal{C}'$ to $\mathcal{C}''$. However, the coefficient on L of L'' will be x_0^2, which is congruent to 1 modulo $2t$, therefore there could be an automorphism sending $\mathcal{C}$ to $\mathcal{C}''$, and in fact, in all the tested cases, computations show that this always happens.

Example 17. The first values of L^2 for which Theorem 15 provides two Kummer structures on the generalized Kummer surface are:

$$20, 44, 68, 84, 92, 104, 110, 116, 120, 126, 132, 140, 164, 168, 176, 188. \tag{4.1}$$

The following infinite series of examples was given to us by Olivier Ramaré:

Example 18. Let k be an integer, and let $a = 8 + 12k$. It is easy to check that there exists $t \in \mathbb{N}$ such that $a^2 + a = 12t$. The pair $(2a+1, 2)$ is the fundamental solution of the Pell-Fermat equation $x^2 - 12ty^2 = 1$. Since moreover $2a + 1 \neq \pm 1 \bmod 2t$, we can apply Theorem 15 for such t's.

The next Section suggests that the criteria in Theorem 15 for having two Kummer structures is quite sharp.

4.2. **A computational approach.** Suppose that the polarization L on the generalized Kummer surface X is such that $6L^2$ is not a square and $L^2 \neq 0 \bmod 18$ so that the curve B_1' obtained in Corollaries 11 and 14 is irreducible. Then, we know two generalized Nikulin configurations

$$\mathcal{C} = \{A_1, B_1, \dots, A_9, B_9\},\ \mathcal{C}' = \{A_1, B_1', A_2, B_2, \dots, A_9, B_9\}.$$

Let L and L' be the orthogonal complements of $\mathcal{C}$ and $\mathcal{C}'$ respectively. Suppose that there is an automorphism g sending $\mathcal{C}$ to $\mathcal{C}'$. Then it induces an isometry g^* of the lattice $\mathrm{NS}(X)$, in particular it sends L to L'. Using the Torelli Theorem for K3 surfaces, one can test all linear maps

$$\psi : \mathrm{NS}(X) \otimes \mathbb{Q} \to \mathrm{NS}(X) \otimes \mathbb{Q}$$

which satisfies to the following conditions:
i) it sends the $\mathbb{Q}$-base $\{L\} \cup \mathcal{C}$ to the $\mathbb{Q}$-base $\{L'\} \cup \mathcal{C}'$ and sends $\mathbf{A}_2$-configurations in $\mathcal{C}$ to $\mathbf{A}_2$-configurations in $\mathcal{C}'$,
ii) it preserves the Néron-Severi lattice
iii) it acts on the discriminant group of $\mathrm{NS}(X)$ by $\pm Id$,
iv) it sends an ample class to an ample class.
About that last point, we remark that it is always satisfied by such a ψ.

Indeed, the nef divisor L' generates the orthogonal complement of a $9\mathbf{A}_2$ configuration $\mathcal{C}'$, thus by Section 2.2.2, for $u_0 \geq 4$, the divisor

$$D' = u_0 L' - \sum_{C \in \mathcal{C}'} C$$

is ample, but D' is also the image by ψ of the ample class $D = u_0 L - \sum_{C \in \mathcal{C}} C$.

There are

$$9!2^9 = 185794560$$

maps ψ satisfying i) and it can be quite long to sort among them the maps that satisfy ii) and iii). However, recall that from Corollary 6, the divisors supported only on the $\mathbf{A}_2$-blocs A_j, B_j ($j \in \{1, \dots, 9\}$) have restrictions: they form in $\mathrm{NS}(X) \otimes \mathbb{Z}/3\mathbb{Z}$ a group isomorphic to $(\mathbb{Z}/3\mathbb{Z})^3$, each with support on 6 blocs of $\mathbf{A}_2$ (12 such words) or on 9 blocs (2 words) or 0 (thus a total of $2 \times 12 + 2 \times 1 + 1 = 27$ classes which are 3-divisible). It is simple to obtain the 12 sets of 6 blocs which are 3-divisible. Since ψ is an isometry it must send 3-divisible sets with support on $\mathcal{C}$ to 3-divisible sets with support on $\mathcal{C}'$, and one can check that in fact the number of permutations involved is only 432 instead of 9!, and therefore the number of possibilities is divided by 800, which makes the computation last a few minutes only.

By these computations, we obtain the following result which gives a more precise result (with an independent proof) than Theorem 15, but only for polarizations L such that L^2 is below some bound:

Theorem 19. *Let X be a generalized K3 surface polarized with L, such that $L^2 < 200$ and $L^2 \neq 0 \bmod 18$.*
There is an automorphism σ sending the generalized Nikulin configuration $\mathcal{C}$ to $\mathcal{C}'$ if and only if L^2 is not in the list (4.1).

Note that one can often choose σ of order 2, but for cases $L^2 = 42, 48$ all the automorphisms sending $\mathcal{C}$ to $\mathcal{C}'$ have infinite order. For the cases $L^2 = 36$ or 180, provided that the curve B_1' constructed in Section 3.2.2 is irreducible, there are also two generalized Kummer structures.

5. Examples

5.1. **A birational models of X with $9\mathbf{A}_2$ singularities.** We keep the notations: X is a generalized Kummer surface with Picard number 19, $A_1, \dots, B_9$ is a $9\mathbf{A}_2$-configuration, and L is the nef divisor that generates the orthogonal complement of these 18 curves (see proof of Proposition 7).

Proposition 20. *Suppose $L^2 > 2$. The linear system $|L|$ induces a morphism $\varphi_L : X \to \mathbb{P}^{\frac{L^2}{2}+1}$ which is an embedding outside the (-2)-curves $A_1, B_1, \dots, A_9, B_9$ and maps these curves to 9 cusps.*

Proof. We know that L is nef and big. If $|L|$ has base points, then

$$L = uF + \Gamma,$$

where F is an elliptic curve and Γ is an irreducible (-2)-curve such that $F\Gamma = 1$ (see [23, Section 3.8]). Moreover since L is big and nef we have

$$0 \leq L\Gamma = u - 2$$

so that $u > 0$. If $\Gamma \neq A_k, B_k$ we have $FA_k \geq 0$, $\Gamma A_k \geq 0$ and we compute

$$0 = LA_k = uFA_k + \Gamma A_k$$

Since $u > 0$ we obtain $FA_k = \Gamma A_k = 0$ similarly $FB_k = \Gamma B_k = 0$ so that Γ is in the orthogonal complement of $A_1, B_1, \dots, A_9, B_9$, which is clearly impossible. If $\Gamma = A_k$ we have $L = uF + A_k$ hence

$$0 = LA_k = u - 2$$

which gives $u = 2$, now $L = 2F + A_k$ which has $L^2 = 2$, which contradicts the assumption, similarly for B_k. In conclusion $|L|$ is base-point-free. Suppose that $|L|$ is hyperelliptic (see [28]). Since L is primitive, one cannot have $L = 2D_2$ with $D_2^2 = 2$. Suppose there is an elliptic curve F such that $FL = 2$. Write

$$F = aL - \sum \alpha_i A_i + \beta_i B_i,$$

with $a \geq 0$, $a, \alpha_i, \beta_i \in \frac{1}{3}\mathbb{Z}$, then

$$2 = FL = aL^2.$$

Since $L^2 > 2$, this is possible only if $L^2 = 6$. But in that case J. Bertin and P. Vanhaecke [4] proved that φ_L is an embedding outside the (-2)-curves $A_1, B_1, \dots, A_9, B_9$ (see the description below). $\square$

5.2. **Case** $L^2 = 2$**.** In this case the generalized Kummer surface $X = \mathrm{Km}(A, G)$ is the double cover of the plane ramified on a sextic curve with $9\mathbf{A}_2$ singularities. This double cover were first studied by Ch. Birkenhake and H. Lange in [15] and then by the two authors of this paper and D. Kohel in [14] where the authors determines several $9\mathbf{A}_2$-configurations related to a special configuration of conics in the plane. If $L^2 = 2$ by Section 3.2 we have $B_1' = 6L - (4A_1 + 7B_1)$. Since $LB_1' = 12$, the curve B_1' is sent to a singular curve of degree 12 in the plane, which passes through the cusp obtained by the contraction of A_1, B_1. Moreover as shown in [25] (see also Corollary 19), up to automorphism of the K3 surface, there is only one $9\mathbf{A}_2$-configuration, so that we have an automorphism sending the configuration $\mathcal{C}$ to the configuration $\mathcal{C}'$ observe that clearly this automorphism is not the covering involution.

5.3. **Case** $L^2 = 6$**.** The polarization $L^2 = 6$ exhibits the K3 surface as a complete intersection of a quadric and a cubic in $\mathbb{P}^4$ with $9\mathbf{A}_2$ singularities, see the paper by J. Bertin and P. Vanhaecke [4] for more details and the equations. Observe that in this case the Pell–Fermat equation $x^2 - 4y^2 = 1$ has no solution so that we can not apply our construction. By [7] in this case there is only one generalized Nikulin configuration.

5.4. **Case** $L^2 = 20$. This is the first example for which our construction gives two non–equivalent Kummer structures (see Example 4.1), so we study it more in detail. As before let L be the class such that $L^2 = 20$, let $A_1, B_1, \dots, A_9, B_9$ be the $9\mathbf{A}_2$-configuration orthogonal to L. In this case the class B_1' is

$$B_1' = 3L - (6A_1 + 11B_1)$$

and we have shown that $\mathcal{C} = \{A_1, B_1, \dots, A_9, B_9\}$ and $\mathcal{C}' = \{A_1, B_1', \dots, A_9, B_9\}$ are not equivalent. The projective model determined by L is a surface in $\mathbb{P}^{11}$, we describe here another projective model as a double plane ramified on a special sextic curve. By Proposition 7, the class

$$D_2 = L - \sum_{k=1}^{9} (A_k + B_k)$$

is ample with $D_2^2 = 2$ and $D_2 A_k = D_2 B_k = 1$, $k \in \{1, \dots, 9\}$. The (-2)-classes

$$E_k = D_2 - A_k,\ F_k = D_2 - B_k,\ k \in \{1, \dots, 9\}$$

are also of degree 1 for D_2, hence are classes of (-2)-curves. Moreover one can check easily that:

Proposition 21. *The* 18 (-2)*-curves* $E_1, F_1, \dots, E_9, F_9$ *form a* $9\mathbf{A}_2$*-configuration.*

Suppose that D_2 has base points. Then there exist an elliptic curve F and an irreducible (-2)-curve Γ, [23, Section 3.8] such that $D_2 = 2F + \Gamma$ and $F\Gamma = 1$. But then $\Gamma D_2 = 0$, which is a contradiction since D_2 is ample. Therefore, with the previous notations:

Proposition 22. *The generalized Kummer surface is a double cover of* $\mathbb{P}^2$ *branched over a smooth sextic curve* C_6 *which has* 18 *tritangent lines, which are the images of the* 18 *couples of curves* (E_k, A_k) *and* (F_k, B_k) *for* $k \in \{1, \dots, 9\}$.

Plane sextic curves with several tritangents were studied by A. Degtyarev in [9]. Using the Néron-Severi lattice and Vinberg's algorithm [31], one can compute moreover, that there are 1728 6-tangent conics to C_6 and 67212 rational cuspidal curves which are tangent to C_6, with a cusp on C_6. Also we have:

Proposition 23. *The automorphism group* G_{36} *preserving the polarization* D_2 *is isomorphic to*

$$\mathbb{Z}_2 \times (\mathbb{Z}_3 \rtimes S_3),$$

it has order 36. *It is generated by the involution* σ *of the double cover and a symplectic group of automorphisms* G_{18} *of order* 18*;* σ *generates the center of* G_{36}.

Proof. This is obtained by a computation as explained in Section 4.2. $\square$

The involution σ is such that $\sigma(A_k) = E_k$, $\sigma(B_k) = F_k$, so that the two $9\mathbf{A}_2$-configurations $E_1, F_1, \dots, E_9, F_9$ and $\mathcal{C} = \{A_1, B_1, \dots, A_9, B_9\}$ are equivalent.

The orbit of A_1 under G_{36} is $\{A_k, E_k \,|\, 1 \leq k \leq 9\}$ and the orbit of B_1 is $\{B_k, F_k \,|\, 1 \leq k \leq 9\}$. Let L_k (respectively L'_k) be the image of A_k (respectively B_k) by the double cover map $X \to \mathbb{P}^2$. The group $\mathbb{Z}_3 \rtimes S_3$ acts on the plane and the orbit of L_1 (resp. L'_1) is $\{L_k \,|\, 1 \leq k \leq 9\}$ (resp. $\{L'_k \,|\, 1 \leq k \leq 9\}$).

The general abelian surfaces A such that $X = \mathrm{Km}_3(A)$ are simple [25].

References

[1] Barbeau E., Pell's Equation, Problem Books in Mathematics. Springer-Verlag, New York, 2003. xii+212 pp

[2] Barth W., On the classification of K3 surfaces with nine cusps, Geom. Dedicata 72 (1998), no. 2, 171–178.

[3] Bertin J., Réseaux de Kummer et surfaces K3 Invent. Math. 93 (1988), no. 2, 267–284.

[4] Bertin J., Vanhaecke P., The even master system and generalized Kummer surfaces. Math. Proc. Cambridge Philos. Soc. 116 (1994), no. 1, 131–142.

[5] Birkenhake C., Lange H., Complex abelian varieties, Sec. ed. Grun. der Math. Wis., 302. Springer-Verlag, Berlin, 2004. xii+635 pp

[6] Bridgeland T., Maciocia A., Complex surfaces with equivalent derived categories. Math. Z. 236 (2001), no. 4, 677–697.

[7] Cartwright D., Roulleau X., A refinement of Bézout's Lemma and order 3 elements in some quaternion algebras over $\mathbb{Q}$, preprint arXiv 2107.02414.

[8] Conway J.H., Sloane N.J.A., Sphere packings, lattices and groups, Third edition. Grund. der Math. Wiss., 290. Springer-Verlag, New York, 1999.

[9] Degtyarev A., Tritangents to smooth sextic curves. Annales de l'Institut Fourier (to appear).

[10] Hosono S., Lian B.H., Oguiso K., Yau S.-T., Kummer structures on K3 surface: an old question of T. Shioda. Duke Math. J. 120 (2003), no. 3, 635–647.

[11] Hosono S., Lian B.H., Oguiso K., Yau S.-T., Fourier-Mukai number of a K3 surface, in "Algebraic structures and moduli spaces", 177–192, CRM Proc. Lecture Notes, 38, Amer. Math. Soc., Providence, RI, 2004.

[12] Gritsenko V., Hulek K., Minimal Siegel modular threefolds, Math. Proc. Cambridge Philos. Soc. 123 (1998), 461–485.

[13] Inose I., On certain Kummer surfaces which can be realized as non-singular quartic surfaces in $\mathbb{P}^3$. J. Fac. Sci. Univ. Tokyo Sect. IA Math. 23 (1976), no. 3, 545–560.

[14] Kohel D., Roulleau X., Sarti A., A special configuration of 12 conics and generalized Kummer surfaces, accepted for publication in Manuscripta Mathematica.

[15] Birkenhake Ch., Lange H., A family of Abelian surfaces and curves of genus four, Manuscripta Math. 85 (1994), no. 3-4, 393–407.

[16] Garbagnati A., On $K3$ surface quotients of $K3$ or Abelian surfaces, Canad. J. Math. 69 (2017), no. 2, 338–372

[17] Garbagnati A., Sarti A., Projective models of $K3$ surfaces with an even set, Adv. Geom. 8 (2008), no. 3, 413–440.

[18] Morrison D. R., On K3 surfaces with large Picard number, Invent. Math. 75 (1984), no. 1, 105–121.

[19] Mukai, S. On the moduli space of bundles on K3 surfaces. I., in "Vector bundles on algebraic varieties" (Bombay, 1984), 341–413

[20] Mukai S., Finite groups of automorphisms of $K3$ surfaces and the Mathieu group, Invent. Math. 94 (1988), no. 1, 183–221

[21] Nikulin V.V., Integer symmetric bilinear forms and some of their geometric applications, Izv. Akad. Nauk SSSR Ser. Mat. 43 (1979), no. 1, 111–177, 238.
[22] Nikulin V.V., On Kummer surfaces, Izv. Akad. Nauk SSSR Ser. Mat. 39 (1975), 278293. English translation: Math. USSR. Izv, 9 (1975), 261–275.
[23] Reid M., Chapters on algebraic surfaces, in Complex algebraic geometry, Lecture notes from the Sum- mer School held in Park City, UT, 1993. Edited by János Kollár. IAS/Park City Mathematics Series, 3. American Mathematical Society, Providence, RI; Institute for Advanced Study (IAS), Princeton, NJ, 1997.
[24] Roulleau X., An atlas of K3 surfaces with finite automorphism group, preprint arXiv 2003.08985.
[25] Roulleau X., Number of Kummer structures and Moduli spaces of generalized Kummer surfaces, preprint arXiv 2106.05541.
[26] Roulleau X., Sarti A., Construction of Nikulin configurations on some Kummer surfaces and applications, Math. Annalen 373 (2019), no 11, 7651–7668.
[27] Roulleau X., Sarti A., Explicit Nikulin configurations on Kummer surfaces, to appear in Advances in Geometry.
[28] Saint-Donat B., Projective Models of K3 surfaces, Amer. J. of Math. 96 (1974) 602–639.
[29] Shimada I., An algorithm to compute automorphism groups of K3 surfaces and an application to singular K3 surfaces, Int. Math. Res. Not. IMRN 2015, no. 22, 11961–12014.
[30] Shioda T., Some remarks on abelian varieties, J. Fac. Sci. Univ. Tokyo Sect. IA 24 (1977),11–21.
[31] Vinberg E. B., Some arithmetical discrete groups in Lobacevskii spaces, Discrete subgroups of Lie groups and applications to moduli (Internat. Colloq. Bombay, 1973), 1975, 323–348.

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