A SPECIAL CONFIGURATION OF 12 CONICS AND GENERALIZED KUMMER SURFACES

DAVID KOHEL, XAVIER ROULLEAU, ALESSANDRA SARTI

Abstract. A generalized Kummer surface X obtained as the quotient of an abelian surface by a symplectic automorphism of order 3 contains a $9\mathbf{A}_2$-configuration of (-2)-curves. Such a configuration plays the role of the $16\mathbf{A}_1$-configurations for usual Kummer surfaces. In this paper we construct 9 other such $9\mathbf{A}_2$-configurations on the generalized Kummer surface associated to the double cover of the plane branched over the sextic dual curve of a cubic curve. The new $9\mathbf{A}_2$-configurations are obtained by taking the pullback of a certain configuration of 12 conics which are in special position with respect to the branch curve, plus some singular quartic curves. We then construct some automorphisms of the K3 surface sending one configuration to another. We also give various models of X and of the generic fiber of its natural elliptic pencil.

[2000]Primary: 14J28

1. Introduction

A Kummer surface $\mathrm{Km}(A)$ is the minimal desingularization of the quotient of an abelian surface A by the involution $[-1]$. It is a K3 surface containing 16 disjoint (-2)-curves, which lie over the 16 singularities of $A/\langle[-1]\rangle$. We call such set of curves a $16\mathbf{A}_1$-configuration. A well-known result of Nikulin [18] gives the converse: if a K3 surface contains a $16\mathbf{A}_1$-configuration, then it is the Kummer surface of an abelian surface A, such that the 16 (-2)-curves lie over the singularities of $A/\langle[-1]\rangle$.

Shioda [31] then asked the following question: if two abelian surfaces A and B satisfy $\mathrm{Km}(A) \simeq \mathrm{Km}(B)$, is it true that $A \simeq B$? Gritsenko and Hulek [14] gave a negative answer to that question in general. In [22], [23], we studied and constructed examples of two $16\mathbf{A}_1$-configurations on the same Kummer surface such that their associated abelian surfaces are not isomorphic.

Kummer surfaces have natural generalizations to quotients of an abelian surface A by other symplectic groups $G \subseteq \mathrm{Aut}(A)$. If $G \cong \mathbb{Z}/3\mathbb{Z}$, then the quotient surface A/G has 9 cusp singularities, in bijection with the fixed points of G. Its minimal desingularization, denoted by $\mathrm{Km}_3(A)$, is a K3 surface which contains 9 disjoint $\mathbf{A}_2$-configurations, i.e. pairs (C, C') of (-2)-curves such that $CC' = 1$. It is then natural to ask if an isomorphism

Date: July 4, 2021.

$\mathrm{Km}_3(A) \simeq \mathrm{Km}_3(B)$ between two generalized Kummer surfaces implies that A and B are isomorphic.

With this question in mind, in the present paper we construct geometrically several $9\mathbf{A}_2$-configurations on some generalized Kummer surfaces previously studied by Birkenhake and Lange [7]. Their construction is as follows. For λ generic, the dual of a cubic curve $E_\lambda : x^3 + y^3 + z^3 - 3\lambda xyz = 0$ is a sextic curve C_λ with a set $\mathcal{P}_9$ of $9\mathbf{A}_2$ singularities corresponding to the nine inflection points on E_λ. The minimal desingularization X_λ of the double cover of $\mathbb{P}^2$ branched over C_λ is a generalized Kummer surface with a natural $9\mathbf{A}_2$-configuration $\mathcal{A}_0$. The surface X_λ has a natural elliptic fibration $\varphi : X_\lambda \to \mathbb{P}^1$ for which the 18 (-2)-curves in the $9\mathbf{A}_2$-configuration are sections, and the reduced strict transform of C_λ is a fiber.

In order to find other (-2)-curves on X_λ we study the set $\mathcal{C}_{12}$ of conics that contain at least 6 points in $\mathcal{P}_9$. One has

Theorem 1. *The set $\mathcal{C}_{12}$ has cardinality 12. Each conic in $\mathcal{C}_{12}$ contains exactly 6 points in $\mathcal{P}_9$ and through each point in $\mathcal{P}_9$ there are 8 conics. The sets $(\mathcal{P}_9, \mathcal{C}_{12})$ form therefore a $(9_8, 12_6)$-configuration.*

The configuration $(\mathcal{P}_9, \mathcal{C}_{12})$ has interesting symmetries, e.g. there are 8 conics among the 12 passing through a fixed point q in $\mathcal{P}_9$ and the 8 points in $\mathcal{P}_9 \setminus \{q\}$, which form a 8_5 point-conic configuration. The freeness of the arrangement of curves $\mathcal{C}_{12}$ is studied in [20], where we learned that this configuration has been also independently discovered in [12].

The irreducible components of the curves in the K3 surface X_λ above the 12 conics are 24 (-2)-curves. This set of (-2)-curves contains nine $8\mathbf{A}_2$ sub-configurations which are the strict transform of the nine 8_5 sub-configurations of conics. Using the pullback to X_λ of some 9 special (singular) quartic curves, we are able to complete each of these $8\mathbf{A}_2$-configurations into a new $9\mathbf{A}_2$-configuration $\mathcal{A}_k$, $(k \in \{1, \dots, 9\})$.

According to [4], to a $9\mathbf{A_2}$-configuration corresponds an Abelian surface A and an order 3 symplectic group G such that $X_\lambda \simeq \mathrm{Km}_3(A)$ and the $9\mathbf{A_2}$-configuration is the exceptional divisor of the minimal desingularisation $\mathrm{Km}_3(A) \to A/G$ (this is the analog of Nikulin's result for $16\mathbf{A_1}$-configurations).

A Kummer structure on K3 surface X is an isomorphism class of Abelian surfaces A such that $X \simeq \mathrm{Km}(A)$. Kummer structures are in one-to-one correspondence with the orbits under $\mathrm{Aut}(X)$ of Nikulin's $16\mathbf{A_1}$-configurations (see e.g. [22, Proposition 21]).

Similarly, one can define a generalized Kummer structure on a K3 surface X as an isomorphism class of pairs (A, G) of abelian surfaces A and order 3 symplectic group G such that $X \simeq \mathrm{Km}_3(A)$. We show that there is again a one-to-one correspondence between generalized Kummer structures and the orbits of the $9\mathbf{A_2}$-configurations. Using the $9\mathbf{A_2}$-configurations we constructed, we obtain:

Theorem 2. *The* $9\mathbf{A_2}$*-configurations* $\mathcal{A}_1, \dots, \mathcal{A}_9$ *we obtained on the K3 surface* X_λ *are contained in the* $\mathrm{Aut}(X_\lambda)$*-orbit of* $\mathcal{A}_0$*.*

For the proof we construct automorphisms sending one configuration to another. Some of these automorphisms are obtained by using translations by torsion sections of the natural elliptic fibration $\varphi : X_\lambda \to \mathbb{P}^1$, and some other by using the Torelli Theorem for K3 surfaces.

We then continue our study of the surface X_λ by obtaining various models in projective space, in particular as a degree 8 non-complete intersection in $\mathbb{P}^5$. We construct a model of the generic fiber E_{K3} of the fibration φ.

Theorem 3. *The fibration* $\varphi : X_\lambda \to \mathbb{P}^1$ *has* 8 *singular fibers of type* $\tilde{\mathbf{A}}_2$*; the* 24 (-2)*-curves above the* 12 *conics in* $\mathcal{C}_{12}$ *are contained in these fibers. A Hessian model of the generic fiber of* φ *is*

$$E_{K3} : x^3 + y^3 + z^3 + \frac{\lambda^3(t^2+3) - 4t^2}{\lambda^2(t^2-1)} xyz = 0.$$

We also get a Weierstrass model of E_{K3}. It turns out that the Mordell-Weil group of the fibration φ has rank 1 and torsion $(\mathbb{Z}/3\mathbb{Z})^2$; we compute its generators. Using the translation maps constructed from the model E_{K3}, we can acquire other $9\mathbf{A}_2$-configurations from the previously known one. We also obtain another construction of the K3 surface X_λ as a double plane:

Theorem 4. *The surface* X_λ *is the minimal desingularization of the double cover of* $\mathbb{P}^2$ *branched over the sextic curve which is the union of the elliptic curves* E_λ *and its Hessian*

$$\mathrm{He}(\lambda) : \quad x^3 + y^3 + z^3 + \frac{(\lambda^3 - 4)}{\lambda^2} xyz = 0,$$

The strict transform on X_λ *of the* 12 *lines of the Hesse arrangement in* $\mathbb{P}^2$ *are the* 24 (-2)*-curves above the* 12 *conics.*

Finally let us mention that the conics through 6 points in $\mathcal{P}_9$ have been used by Degtyarev [10] in order to study Oka's Conjecture.

Acknowledgements The authors wish to thank Carlos Rito for sharing his program `LinSys`, Remke Kloosterman for pointing out the reference [10], and also Cédric Bonnafé, Igor Dolgachev, Antonio Laface, Ulf Persson, Piotr Pokora, Giancarlo Urzúa. Part of the computations were done using Magma software [8]. The second author thanks the Max-Planck Institute for Mathematics of Bonn for its hospitality and support. The third author is partially supported by the ANR project No. ANR-20-CE40-0026-01 (SMAGP).

2. Preliminaries

2.1. **Notations and conventions.** Let $\eta : Y \to Z$ be a dominant map between two surfaces and let $C \hookrightarrow Z$ be a curve. In this paper, the reduced pullback of C minus the irreducible components contracted by η is called the strict transform of C on Y.

Definition 5. We say that two (-2)-curves E, E' on a K3 surface form an $\mathbf{A}_2$-configuration if their intersection matrix is: $\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\cdot$ We say that the (-2)-curves $E_1, E_1', \dots, E_n, E_n'$ form an $n\mathbf{A}_2$-configuration if the curves E_j, E_j', $j \in \{1, \dots, n\}$ form n disjoint $\mathbf{A}_2$-configurations.

We recall (see [11]) that a (v_r, b_k)-configuration is the data of two sets $\mathcal{P}_v, \mathcal{L}_b$ of respective cardinality v and b, plus a subset $I \subset \mathcal{P}_v \times \mathcal{L}_b$, such that for each element p in $\mathcal{P}_v$, there are r elements l in $\mathcal{L}_b$ such that $(p, l) \in I$, and for each element l in $\mathcal{L}_b$, there are k elements p in $\mathcal{P}_v$ such that $(p, l) \in I$. One has $vr = bk$. When $v = b$ and $r = k$, we call such a configuration a v_r-configuration.

2.2. **Generalized Kummer structures.** Let X be a K3 surface. Suppose that $\mathcal{C} = A_1, A_1', \dots, A_9, A_9'$ is a $9\mathbf{A}_2$-configuration on X. By the results of Barth [4], there exist coefficients $a_j, a_j' \in \{1, 2\}$ and a divisor $D \in \mathrm{NS}(X)$ such $\sum_{j=1}^{9}(a_j A_j + a'_J A_j') = 3D$. Let $\tilde{X} \to X$ be the blow-up at the 9 intersection points $A_k A_k'$ $(k = 1, ..., 9)$. The strict transform of $\mathcal{C}$ on $\tilde{X}$ is the union of 18 disjoint (-3)-curves. The quoted results of Barth gives that there exists a unique triple cover map $\tilde{B} \to \tilde{X}$ (for the theory of cyclic covers, see [6, Chapter 1, Section 17]) branched over the 18 (-3)-curves; the reduced pull-back of these curves are (-1)-curves and the pull-back of the exceptional curves on $\tilde{X}$ are 9 (-3)-curves. The minimal model of $\tilde{B}$ is then an abelian surface B with an order 3 symplectic automorphism group G coming from the cyclic cover, and the surface X is (isomorphic to) the minimal desingularization of the quotient surface B/G.

Definition 6. A generalized Kummer structure (of order 3) on a K3 surface X is an isomorphism class of pairs (A, G) of abelian surfaces equipped with an order 3 symplectic automorphism subgroup $G \subset \mathrm{Aut}(A)$, such that $X \simeq \mathrm{Km}_3(A)$, where $\mathrm{Km}_3(A)$ is the minimal desingularization of A/G.

Proposition 7. *There is a one-to-one correspondence between Kummer structures on X and $\mathrm{Aut}(X)$-orbits of $9\mathbf{A}_2$-configurations.*

Proof. Let $\mu : X \to X$ be an automorphism of X sending the configuration $\mathcal{C}$ to the configuration $\mathcal{C}'$. Let B, B' be the abelian surfaces and let $G \subset \mathrm{Aut}(B)$, $G' \subset \mathrm{Aut}(B')$ be the order 3 automorphism groups such that X is the minimal desingularization of B/G and B'/G', and the exceptional curves of the minimal desingularization are respectively in $\mathcal{C}, \mathcal{C}'$. Let us prove that there exists an isomorphism τ between the abelian surfaces B, B' such that $G' = \tau G \tau^{-1}$.
As above let $\tilde{X}$, $\tilde{X}'$ be the blow-up of X at the 9 singular points of $\mathcal{C}$ and $\mathcal{C}'$ respectively. The automorphism μ extends to an isomorphism $\tilde{\mu} : \tilde{X} \to \tilde{X}'$. The map $\tilde{B} \to \tilde{X} \xrightarrow{\tilde{\mu}} \tilde{X}'$ is branched with order 3 over the strict transform of $\mathcal{C}'$ in $\tilde{X}'$. By uniqueness of the triple cover, there is an isomorphism $\tilde{\tau} : \tilde{B} \to \tilde{B}'$, (which gives an isomorphism to $\tau : B \to B'$ between the minimal

models). Moreover, the order 3 automorphisms group $\tilde{G}$ of transformations of the triple cover $\tilde{B} \to \tilde{X}$ is sent by transport of structures to the order 3 automorphism group $\tilde{G}'$ of transformations of the triple cover $\tilde{B}' \to \tilde{X}'$, i.e. $\tilde{G}' = \tilde{\tau}\tilde{G}\tilde{\tau}^{-1}$. This property is preserved when taking the minimal models: $G' = \tau G \tau^{-1}$.

For the converse, let (B, G) and (B', G') be equipped abelian surfaces such that $\mathrm{Km}_3(B) = X = \mathrm{Km}_3(B')$ and there is an isomorphism $\tau : B \to B'$ with $G = \tau^{-1} G' \tau$. The map $B \to B' \to B'/G'$ is 3 to 1 and G-invariant, thus it induces an isomorphism $B/G \simeq B'/G'$. That isomorphism extends to the desingularization: $X = \mathrm{Km}_3(B) \simeq \mathrm{Km}_3(B') = X$ and sends the first $9\mathbf{A}_2$-configuration to the second. □

2.3. **The Néron-Severi and transcendental lattices.** In this Section, which can be skipped in a first reading, we compute the Néron-Severi lattice and the transcendent lattice of the generalized Kummer surface X, the results will be used in Section 3.

2.3.1. *The Néron-Severi lattice.* Let A be an abelian surface with a symplectic action of a group $G := \mathbb{Z}/3\mathbb{Z}$. The quotient A/G has $9\mathbf{A}_2$ singularities and the minimal resolution is a generalized Kummer surface $X := \mathrm{Km}_3(A)$ which carries a $9\mathbf{A}_2$-configuration $A_1, A_1', \dots, A_9, A_9'$ of (-2)-curves.

Observe that the abelian surface has Picard number at least 3, see [5, Proposition on p. 10] and the K3 surface X has generically Picard number $\rho(X)$ equal to 19. Let $\mathcal{K}_3$ denote the minimal primitive (rank 18) sub-lattice of the K3 lattice $H^2(X, \mathbb{Z})$ containing the 9 configurations $\mathbf{A}_2$. The lattice $\mathcal{K}_3$ is described as follows. By [3, Proof of Proposition 1.3], it is generated by the classes $A_1, A_1', \dots, A_9, A_9'$ and the following three classes

$$\begin{aligned} v_1 &= \tfrac{1}{3}\textstyle\sum_{i=1}^{9}(A_i - A_i') \\ v_2 &= \tfrac{1}{3}((A_2 - A_2') + 2(A_3 - A_3') + A_6 - A_6' + 2(A_7 - A_7') + A_8 - A_8' + 2(A_9 - A_9')) \\ v_3 &= \tfrac{1}{3}(A_4 - A_4' + 2(A_5 - A_5') + A_6 - A_6' + 2(A_7 - A_7') + 2(A_8 - A_8') + A_9 - A_9'). \end{aligned}$$

Then the discriminant group $\mathcal{K}_3^\vee/\mathcal{K}_3$ is generated by the classes

$$\begin{aligned} w_1 &= \tfrac{1}{3}(A_5 - A_5' + A_7 - A_7' + A_8 - A_8') \\ w_2 &= \tfrac{1}{3}(2(A_4 - A_4') + A_6 - A_6' + 2(A_7 - A_7') + A_8 - A_8') \\ w_3 &= \tfrac{1}{3}(A_3 - A_3' + A_5 - A_5' + A_6 - A_6') \end{aligned}$$

with intersection matrix (the coefficients are in $\mathbb{Q}/2\mathbb{Z}$):

$$\begin{pmatrix} 0 & 0 & -\frac{2}{3} \\ 0 & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & 0 \end{pmatrix}.$$

Assume $\rho(X) = 19$ (the minimal possible) and that D_2, a generator of $\mathcal{K}_3^\perp \subset \mathrm{NS}(X)$, has square $D_2^2 = 2$. We state here the following Lemma for later use in Sections 4.1 5.3:

Lemma 8. *The Néron-Severi lattice is* $\mathrm{NS}(X) = \mathbb{Z}D_2 \oplus \mathcal{K}_3$ *and its discriminant group has order* 54.

Proof. The lattice $\mathcal{K}_3$ is 3-elementary and $D_2^2 = 2$, thus the result. □

2.3.2. *The transcendental lattice.* Since $\mathrm{NS}(X) = \mathbb{Z}D_2 \oplus \mathcal{K}_3$, we know that the discriminant group is

$$\frac{\mathrm{NS}(X)^\vee}{\mathrm{NS}(X)} = \frac{(\mathbb{Z}D_2)^\vee}{\mathbb{Z}D_2} \oplus \frac{\mathcal{K}_3^\vee}{\mathcal{K}_3}.$$

The intersection matrix of the generators $w_1, w_2, w_3 - w_1 - w_2 + \frac{1}{2}D_2$ is

$$\mathcal{N} := \begin{pmatrix} 0 & 0 & -\frac{2}{3} \\ 0 & -\frac{2}{3} & 0 \\ -\frac{2}{3} & 0 & \frac{1}{2} \end{pmatrix}.$$

Recall that the quadratic form on the discriminant group has values in $\mathbb{Q}/2\mathbb{Z}$ and the corresponding bilinear form has values in $\mathbb{Q}/\mathbb{Z}$, see e.g. [17, Section 2]. The transcendental lattice T_X of X has rank 3 signature $(2,1)$, and since $H^2(X,\mathbb{Z})$ is unimodular $\det T_X = -2 \cdot 3^3$ with discriminant form the same as the discriminant form of $\mathrm{NS}(X)$ scaled by -1. As in [24, Definition 2.1], we define

Definition 9. We call the discriminant d of an indefinite rank 3 lattice *small* if $4 \cdot d$ is not divisible by k^3 for any non square natural number k congruent to 0 or 1 modulo 4.

The lattice T_X is small, thus by [9, Theorem 21, p. 395] and [9, Corollary 1.9.4] it is uniquely determined by its signature and its discriminant form. Consider now the rank three lattice T of signature $(2,1)$ with Gram matrix

$$\begin{pmatrix} 0 & 3 & 0 \\ 3 & 6 & -3 \\ 0 & -3 & 6 \end{pmatrix}.$$

It has determinant $-2 \cdot 3^3$ and if one calls u_1, u_2, u_3 its generators, one computes that the discriminant group is generated by

$$\tfrac{2u_1}{3},\ \tfrac{u_3}{3},\ \tfrac{3u_1+2u_2+u_3}{6}$$

and the discriminant form is the same as $-\mathcal{N}$ so that by unicity:

Proposition 10. *The transcendental lattice T_X is isometric to T.*

2.4. **The Hesse, dual Hesse, and related configurations.** Let us recall the construction and properties of the Hesse configuration before introducing a configuration with analogous properties in the next section.

Let $E \hookrightarrow \mathbb{P}^2$ be a smooth cubic curve and $\mathcal{T}_9$ its set of 9 inflection points. Fixing an inflection point as the group identity, the set $\mathcal{T}_9$ is the 3-torsion subgroup $E[3]$. The *Hesse configuration* is defined to be the pair $(\mathcal{T}_9, \mathcal{L}_{12})$, where $\mathcal{L}_{12}$ is the set of 12 lines through pairs of points in $\mathcal{T}_9$. Since each line meets $\mathcal{T}_9$ at 3 points and each point of $\mathcal{T}_9$ is contained in 4 lines, the Hesse configuration is a $(9_4, 12_3)$-configuration. Removing one point of the

Hesse configuration and its 4 incident lines, one obtains a symmetric 8_3-configuration of 8 points and 8 lines.

We note that the construction is not unique to the curve E. If $\mathcal{P}$ is the set of inflection points of any nonsingular cubic plane curve E, then there exists a pencil of elliptic curves (called the Hesse pencil) whose set of inflection points is $\mathcal{P}$.

By taking the 12 points and 9 lines in the dual space $(\mathbb{P}^2)^*$ corresponding to the 12 lines and 9 points dual to the Hesse configuration, one obtains a dual $(12_3, 9_4)$-configuration called the *dual Hesse configuration*. As above, removing one line and the 4 points it meets gives a dual symmetric 8_3-configuration.

As abstract configurations, the above configurations can be realized from the symmetric 13_4-configuration of points and lines in the plane $\mathbb{P}^2(\mathbb{F}_3)$. Removing a fixed point and the set of 4 lines which pass through it, gives the $(12_3, 9_4)$-configuration, and removing a fixed line and the 4 points it contains gives the $(9_4, 12_3)$-configuration. The $(9_4, 12_3)$-configuration is naturally identified with $\mathbb{A}^2(\mathbb{F}_3) = \mathbb{P}^2(\mathbb{F}_3) \setminus \mathbb{P}^1(\mathbb{F}_3)$ equipped with its system of affine lines, consistent with the identification $\mathcal{T}_9 = E[3] \cong \mathbb{F}_3^2$ in the Hesse configuration.

3. Nine new $9\mathbf{A}_2$-configurations

3.1. $(9_8, 12_6)$ **and** 8_5**-configurations of conics.** Let us fix $\lambda \notin \{1, \omega, \omega^2\}$, for ω such that $\omega^2 + \omega + 1 = 0$. The dual C_λ of the elliptic curve

$$E_\lambda : \; x^3 + y^3 + z^3 - 3\lambda xyz = 0,$$

is a 9-cuspidal sextic curve, (i.e. a sextic curve with 9 cusps) and conversely any 9-cuspidal sextic curve is obtained in that way. The curve C_λ is:

$$\begin{aligned} C_\lambda : \; & (x^6 + y^6 + z^6) + 2(2\lambda^3 - 1)(x^3y^3 + x^3z^3 + y^3z^3) \\ & -6\lambda^2 xyz(x^3 + y^3 + z^3) - 3\lambda(\lambda^3 - 4)x^2y^2z^2 = 0. \end{aligned}$$

The images by the dual map of the 9 inflection points of E_λ are the 9 cusps of C_λ. The set $\mathcal{P}_9$ of the 9 cusps $p_1, \dots, p_9$ is

$$\begin{array}{lll} p_1 = (\lambda : 1 : 1), & p_4 = (\lambda : \omega : \omega^2), & p_7 = (\lambda : \omega^2 : \omega) \\ p_2 = (1 : \lambda : 1), & p_5 = (\omega^2 : \lambda : \omega), & p_8 = (\omega : \lambda : \omega^2) \\ p_3 = (1 : 1 : \lambda), & p_6 = (\omega : \omega^2 : \lambda), & p_9 = (\omega^2 : \omega : \lambda). \end{array}$$

When λ varies, the closure of the set of points p_j is a line, denoted by L_j; we obtain in that way a set $\mathcal{L}_9$ of 9 lines. Dually, the points on L_j correspond to the pencil of lines meeting in the inflection point (corresponding to p_j) of the elliptic curve E_λ. One can check moreover that the line L_j is the tangent line to the cusp p_j. Let $\mathcal{P}_{12}$ be the intersection points set of the lines in $\mathcal{L}_9$.

Theorem 11. *The set $\mathcal{P}_{12}$ has cardinality* 12*. The pair $(\mathcal{P}_{12}, \mathcal{L}_9)$ forms a $(12_3, 9_4)$-configuration which is the dual Hesse configuration. The set $\mathcal{C}_{12}$ of conics containing* 6 *points in $\mathcal{P}_9$ has cardinality* 12*; each conic of $\mathcal{C}_{12}$ is*

smooth. The pair of sets $(\mathcal{P}_9, \mathcal{C}_{12})$ of points and conics form a $(9_8, 12_6)$-configuration.
More precisely, the union of the pairwise intersections of the 12 conics in $\mathcal{C}_{12}$ is $\mathcal{P}_9 \cup \mathcal{P}_{12}$. The intersections between the conics are transverse. Moreover, each point of $\mathcal{P}_{12}$ is contained in exactly two conics. A conic C in $\mathcal{C}_{12}$ meets 9 conics of $\mathcal{C}_{12}$ in 4 points of $\mathcal{P}_9$ and the two remaining conics in 3 points of $\mathcal{P}_9$ and one point of $\mathcal{P}_{12}$.

Proof. By a computer search, the 12 conics are:

$$\begin{array}{ll}
C_{1,2,3,4,5,6}: & x^2+(\lambda+1)(\omega xy+\omega^2xz+yz)+\omega^2y^2+\omega z^2=0,\\
C_{1,2,3,7,8,9}: & x^2+(\lambda+1)(\omega^2xy+\omega xz+yz)+\omega y^2+\omega^2z^2=0,\\
C_{1,2,4,5,7,8}: & xy-\lambda z^2=0,\\
C_{1,2,4,6,8,9}: & x^2+(\omega\lambda+1)(xy+\omega xz+\omega yz)+y^2+\omega^2z^2=0,\\
C_{1,2,5,6,7,9}: & x^2+(\omega^2\lambda+1)(xy+\omega^2xz+\omega^2yz)+y^2+\omega z^2=0,\\
C_{1,3,4,5,8,9}: & x^2+(\omega\lambda+\omega^2)(xy+yz+\omega xz)+\omega y^2+z^2=0,\\
C_{1,3,4,6,7,9}: & -\lambda y^2+xz=0,\\
C_{1,3,5,6,7,8}: & x^2+(\omega^2\lambda+\omega)(xy+yz+\omega^2xz)+\omega^2y^2+z^2=0,\\
C_{2,3,4,5,7,9}: & x^2+(\lambda+\omega^2)(xy+xz+\omega^2yz)+\omega y^2+\omega z^2=0,\\
C_{2,3,4,6,7,8}: & x^2+(\lambda+\omega)(xy+xz+\omega yz)+\omega^2(y^2+z^2)=0,\\
C_{2,3,5,6,8,9}: & \lambda x^2-yz=0,\\
C_{4,5,6,7,8,9}: & x^2+(\lambda+1)(xy+xz+yz)+y^2+z^2=0,
\end{array}$$

where the index $i, j, \dots, n$ of the conic $C_{i,j,\dots,n}$ means that this conic contains the 6 points p_s, $s \in \{i, j, \dots, n\}$. It is easy to see that the points in $\mathcal{P}_9$ are in general position: no line contains 3 cusps, thus the conics are smooth. From the data of the conics and the knowledge of the points in $\mathcal{P}_9$ they contain, one can check the assertions about the configuration of the 12 conics and the 9 points. If one renumbers the 12 conics by their order $C_1, \dots, C_{12}$ from the top to bottom of the above list, one obtains that the pair of (indexes of) conics which have an intersection point not in $\mathcal{P}_9$ are

$$\begin{array}{c}
(1,2),\ (1,12),\ (2,12),\ (3,7),\ (3,11),\ (4,8),\\
(4,9),\ (5,6),\ (5,10),\ (6,10),\ (7,11),\ (8,9),
\end{array}$$

and correspondingly, the 12 points are

$$\begin{array}{c}
(1:1:1),\ (\omega:\omega^2:1),\ (\omega^2:\omega:1),\ (1:0:0),\ (0:1:0),\ (\omega^2:1:1),\\
(1:\omega^2:1),\ (\omega:1:1),\ (1:\omega:1),\ (\omega^2:\omega^2:1),\ (0:0:1),\ (\omega:\omega:1).
\end{array}$$

respectively. One can check easily that these 12 points in $\mathcal{P}_{12}$ are the intersection points of the lines in $\mathcal{L}_9$, which lines form the dual Hesse arrangement (see Section 2.4). By Bézout's Theorem, the intersections between the conics are transverse. □

Let $q \in \mathcal{P}_9$ and define $\mathcal{P}_q = \mathcal{P}_9 \setminus \{q\}$.

Theorem 12. *The set $\mathcal{C}_q$ of conics containing the point q has cardinality 8. The set of points $\mathcal{P}_q$ and the set of conics $\mathcal{C}_q$ form an 8_5-configuration: each point is on 5 conics and each conic contains 5 of the points in $\mathcal{P}_q$.*

For each conic C in $\mathcal{C}_q$ there exists a unique conic $C' \in \mathcal{C}_q$ such that there is a unique point in the intersection of C and C' which is not in $\mathcal{P}_q$.

Proof. That can be checked directly from the datas in the proof of Proposition 11. □

Let X_λ be the minimal desingularization of the double cover branched over the sextic curve C_λ with 9 cusps. We denote by $\eta : X_\lambda \to \mathbb{P}^2$ the natural map and we denote by A_j, A'_j the two (-2)-curves in X above the point p_j in $\mathcal{P}_9$. The curves A_j, A'_j, $j \in \{1, \dots, 9\}$ form a $9\mathbf{A}_2$-configuration, which we call the *natural* $9\mathbf{A}_2$*-configuration*. We have

Lemma 13. *The strict transform by the map $\eta : X_\lambda \to \mathbb{P}^2$ of a conic $C \in \mathcal{C}_{12}$ is the union of two disjoint (-2)-curves θ_C, θ'_C.*
Let C, D be two conics in $\mathcal{C}_{12}$. Suppose that C and D meet in 4 points in $\mathcal{P}_9$. Then the (-2)-curves $\theta_C, \theta'_C, \theta_D, \theta'_D$ are disjoint.
Suppose that C and D meet in 3 points in $\mathcal{P}_9$. Then, up to exchanging θ_D and θ'_D, the curves $\theta_C, \theta_D, \theta'_C, \theta'_D$ form a $2\mathbf{A}_2$-configuration.

Proof. Rather than performing a double cover and taking the resolution of surface singularities, we perform one blow-ups at each cusp $q \in \mathcal{P}_9$ of C_λ, so that the branch locus is smooth and tangent to the exceptional curve E (see also Figure 3.1). On the double cover, the reduced image inverse of the curve E is the union of two (-2)-curves on the smooth K3 surface X_λ.
By that local computation, we see that for $C \in \mathcal{C}_{12}$, the curves $\theta_C, \theta_{C'}$ are disjoint (the strict transform $\overline{C}$ of C under the blow-up map do not meet the branch locus).
Suppose C and D meet in 4 points in $\mathcal{P}_9$. The intersection being transverse, each strict transforms $\overline{C}, \overline{D}$ of the curves C, D under the 3 blow-ups at each cusps is the union of two disjoint curves not meeting the branch curve and the 4 curves in $\overline{C}, \overline{D}$ are disjoint.
If C and D meet in 3 points in $\mathcal{P}_9$ then they meet transversely at a unique point not in $\mathcal{P}_9$. Then taking the above notations, we have this time $\overline{C}\overline{D} = 1$, so that the last assertion holds. □

Let $\mathcal{P}_q$ and $\mathcal{C}_q$ as above. Using Theorem 12 and Lemma 13, we get:

Corollary 14. *The strict transform by η of the 8 conics in $\mathcal{C}_q$ forms an $8\mathbf{A}_2$-configuration.*

For each point $q = p_j$, $j \in \{1, \dots, 9\}$, we denote by $\mathcal{A}'_j$ the corresponding $8\mathbf{A}_2$-configuration on X_λ. In order to obtain new $9\mathbf{A}_2$-configurations, one needs to find other $\mathbf{A}_2$-configurations. This will be done in the next section by using singular quartics instead of conics.

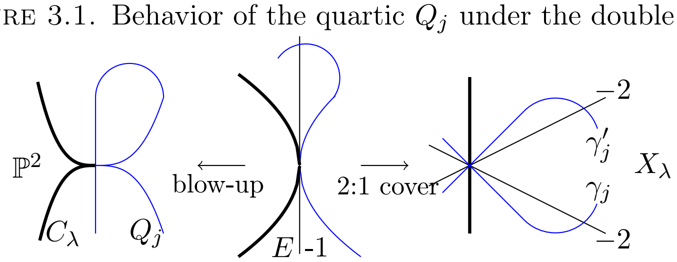
Remark 15. Using a computer, we found eight $8\mathbf{A}_2$-configurations in the set of 32 (-2)-curves which is the union of the two $8\mathbf{A}_2$-configurations $\mathcal{A}'_j$ and $\{A_1, A'_1, \dots, A_9, A'_9\} \setminus \{A_j, A'_j\}$. However one can compute that the orthogonal complement of 6 of them are lattices with no (-2)-classes, thus one cannot complete these 6 configurations into $9\mathbf{A}_2$-configurations. The 32 (-2)-curves can be realized as lines in a projective model of X_λ, see Proposition 19.

3.2. **Nine new $9\mathbf{A}_2$-configurations.** Let $p_j \in \mathcal{P}_9$ be one of the 9 cusp singularities of the sextic C_λ.

Theorem 16. *There exists a unique quartic curve Q_j passing through the 9 points in $\mathcal{P}_9$, with a unique singularity at the point p_j. The singularity has multiplicity 3, it is of type $\mathbf{D}_5$: it has two tangents, one branch is smooth while the other branch is a cuspidal singularity. The tangent at the cusp of Q_j is also the tangent to the cuspidal singularity of the sextic C_λ at p_j.*
The curve Q_j has geometric genus 0. Its strict transform denoted by η on X_λ is the union of two (-2)-curves γ_j, γ'_j which form an $\mathbf{A}_2$-configuration. The curves γ_j, γ'_j and the 16 curves in $\mathcal{A}'_j$ form a $9\mathbf{A}_2$-configuration $\mathcal{A}_j$.

Proof. We give in the Appendix the equations of the 9 curves Q_j, $j \in \{1, \dots, 9\}$. These curves have been constructed using the `LinSys` program by C. Rito which enables to find curves of given degree with prescribed singularities and given tangencies at a set of points in the plane. Conversely, one can check that the singularity of Q_j at p_j has multiplicity 3, is resolved by one blow-up, with the exceptional divisor meeting the strict transform in two points, one of multiplicity 2.
The curve Q_j has genus 0, (see e.g. [13, Chapter 4, Section 2]). By Bézout's Theorem, the intersections of the quartic Q_j with the 8 conics in $\mathcal{C}_{12}$ that contain p_j are transverse, so that the curves in $\mathcal{A}'_j$ are disjoint from γ_j, γ'_j and we thus get a $9\mathbf{A}_2$-configuration. See Figure 3.1 for the behavior of the quartic curve Q_j under the double cover. $\square$

FIGURE 3.1. Behavior of the quartic Q_j under the double cover



Remark 17. In Section 4.3, we study some automorphisms of X_λ. We obtain that the nine above $9\mathbf{A}_2$-configurations are in the same orbit under the action by 3-torsion of the Mordell-Weil group of a fibration of X_λ.

4. PROJECTIVE, HESSIAN AND WEIERSTRASS MODELS OF X_λ

4.1. **The natural elliptic fibration of X_λ.** Let D_2 be the big and nef divisor on X_λ which is the pull back of a line in $\mathbb{P}^2$. For λ generic, the divisors $D_2, A_1, A_1' \dots, A_9, A_9'$ form a $\mathbb{Q}$-basis $\mathcal{B}_{\mathbb{Q}}$ of $\mathrm{NS}(X_\lambda)_{/\mathbb{Q}}$; they generate an index 3^6 sub-lattice of $\mathrm{NS}(X_\lambda)$ (see also Lemma 8).

Let $\mu : Y_\lambda \to \mathbb{P}^2$ be the blow-up of the plane at the 9 cusps of the sextic curve C_λ and let $E_1, \dots, E_9$ be the exceptional curves over $p_1, \dots, p_9$. The strict transform by μ of the curve C_λ is the smooth genus 1 curve

$$\overline{C}_\lambda = \mu^* C_\lambda - 2\textstyle\sum_{j=1}^9 E_i, \text{ such that } \overline{C}_\lambda^2 = 0.$$

The surface X_λ is the double cover of Y_λ branched over $\bar{C}_\lambda$; we denote by

$$\eta' : X_\lambda \to Y_\lambda$$

the double cover morphism (so that $\eta'^* E_j = A_j + A_j'$) and by F_o the ramification locus, so that $2F_o = {\eta'}^* \bar{C}_\lambda$. Since $2F_o = \eta'^* \bar{C}_\lambda \equiv 6D_2 - 2\sum_{j=1}^9 (A_j + A_j')$, we get

$$F_o \equiv 3D_2 - \left(\textstyle\sum_{j=1}^9 A_j + A_j' \right).$$

Let $L \hookrightarrow \mathbb{P}^2$ be a line; the curve C_λ belongs to the linear system

$$\delta = |6L - 2\textstyle\sum_{j=1}^9 p_j|$$

of sextic curves with a double point at points in $\mathcal{P}_9$. One computes that this linear system is 1 dimensional. Moreover there exists a unique cubic curve $\mathrm{Ca}(\lambda)$ (called the Cayleyan curve, see [1]) that contains the 9 points in $\mathcal{P}_9$, which is

$$\mathrm{Ca}(\lambda): \ x^3 + y^3 + z^3 - \tfrac{(\lambda^3+2)}{\lambda} xyz = 0, \tag{4.1}$$

so that $2\mathrm{Ca}(\lambda) \in \delta$. The linear system δ lifts to a base point free linear system δ' on Y_λ with $\bar{C}_\lambda \in \delta'$. The linear system δ' defines a morphism $\varphi' : Y_\lambda \to \mathbb{P}^1$ and induces an elliptic fibration

$$\varphi : X_\lambda \to \mathbb{P}^1$$

for which F_o is a fiber, and which we call the *natural fibration*.

Let p, q be the images by φ of the strict transforms in X_λ of $\mathrm{Ca}(\lambda)$ and C_λ. In fact the surface X_λ is the fiber product of the fibration φ' and the quadratic transformation $\mathbb{P}^1 \to \mathbb{P}^1$ branched at p, q. Indeed both maps $X_\lambda \to Y_\lambda$ and $Y_\lambda \times_{\mathbb{P}^1} \mathbb{P}^1$ has the same branch locus in the rational surface Y_λ.

The curves $A_1, A_1', \dots, A_9, A_9'$ are sections of φ, and one can check that the curves $\gamma_1, \gamma_1', \dots, \gamma_9, \gamma_9'$ are also sections (see Figure 3.1).

Theorem 18. *The fibration φ contracts the 24 (-2)-curves Θ_j, $j \in \{1, \dots, 24\}$ which are above the 12 conics $\mathcal{C}_{12}$. The singular fibers of φ are 8 fibers of type $\tilde{\mathbf{A}}_2$, each singular fiber is the union of 3 curves Θ_j. For λ generic, the fibration φ has fibers with non-constant moduli and the Mordell-Weil group of the fibration φ is $\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^2$.*

Proof. The following four sextic curves

$$C_{123456} + C_{123789} + C_{456789},\ C_{124578} + C_{134679} + C_{235689},$$
$$C_{124689} + C_{135678} + C_{234579},\ C_{125679} + C_{134589} + C_{234678},$$

belong to the linear system δ of sextic curves that have multiplicity 2 at the points in $\mathcal{P}_9$; actually their singularities are nodes. By the results in the proof of Theorem 11, the strict transform to Y_λ of the above 4 sextic form 4 fibers of type $\tilde{\mathbf{A}}_2$, which lies in the étale locus of η. Their strict transform on X_λ is therefore the union of eight fibers of type $\tilde{\mathbf{A}}_2$. A fiber of type $\tilde{\mathbf{A}}_2$ contributes to 3 in the Euler characteristic of X_λ, which is equal to 24. Since there are $8\tilde{\mathbf{A}}_2$ singular fibers, the fibration has no other singular fibers. The 24 curves Θ_j above the 12 conics are in the fibers, thus are contracted by φ.

The strict transform $\mathrm{He}(\lambda)$ on X_λ of $\mathrm{Ca}(\lambda)$ is smooth, of genus 1 (we will see that this is the Hessian of the curve E_λ, thus the notation; see also Remark 25). Since $\mathrm{He}(\lambda)\cdot F_o = 0$, we have that $\mathrm{He}(\lambda)\equiv F_o$. The curve F_o is isomorphic to E_λ. For generic λ, the curves $\mathrm{Ca}(\lambda)$ and E_λ have distinct j-invariants, thus the fibers of φ have a non-constant moduli. Since the fibration is not isotrivial, results of Shioda (see [30, Corollary 1.5]) apply and tell that the Mordell-Weil group of sections of $\varphi : X_\lambda \to \mathbb{P}^1$ has rank $1 = 19-(2+8(3-1))$.

In fact, elliptic fibrations of K3 surfaces are classified by Shimada in [28]. A table with the 3278 possible cases is available in [29]. Our fibration is case number 2373 in that table, where one can find moreover that the torsion part of its Mordell-Weil group is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^2$. □

4.2. **Two polarizations and a degree 8 projective model.** The divisor

$$D_{14} = 4D_2 - \left(\textstyle\sum_{j=1}^{9} A_j + A_j'\right)$$

is linearly equivalent to D_2+F (F a fiber of φ) and is effective. Let us define $D_8 = D_{14} - (A_1 + A_1')$.

Proposition 19. *The divisors D_8 and D_{14} are ample of square $D_8^2 = 8$, $D_{14}^2 = 14$. The linear system $|D_8|$ is base point free, non-hyperelliptic, and defines an embedding $X_\lambda \hookrightarrow \mathbb{P}^5$ as a degree 8 surface. For $d\in\mathbb{N}^*$, let n_d be the number of (-2)-curves of degree d for D_8. The series $\sum n_d T^d$ begins with*

$$32T + 20T^2 + 334T^4 + 576T^5 + 880T^6 + 8640T^7 + 17784T^8 ...,$$

in particular X_λ contains 32 lines and 20 conics.

Proof. Let B be a (-2)-curve such that $D_{14}B \le 0$. Since D_2 is effective and $D_2^2 > 0$, one has $D_2 B \ge 0$, moreover since F is a fiber, $FB \ge 0$ and we must have $D_2 B = 0 = FB$. That implies that B is an irreducible component of a singular fiber, ie $B = \Theta_j$ for some $j \in \{1, \dot{,} 24\}$. But since $D_2\Theta_j = 2$ for all j, such a curve B cannot exist, thus D_{14} is ample.

Let us prove that D_8 is ample. We have

$$\gamma_1 + \gamma_1' \equiv 4D_2 - 2(A_1 + A_1') - \textstyle\sum_{j=1}^{9}(A_j + A_j'),$$

thus $D_8 \equiv A_1 + A_1' + \gamma_1 + \gamma_1'$ and the divisor D_8 is effective. We check that $D_8A_1 = D_8A_1' = D_8\gamma_1 = D_8\gamma_1' = 1$ and $D_8^2 = 8$, therefore D_8 is nef and big.

Suppose that there is a (-2)-curve B on X_λ such that $D_8B = 0$. Then by the above expression of D_8, one has $A_1B = A_1'B = 0$. Let $L \hookrightarrow \mathbb{P}^2$ be a line. For $j \in \{2, \dots, 9\}$, let us consider the linear system

$$\delta_j = |4L - (p_1 + p_j + \textstyle\sum_{j=1}^{9} p_k)|$$

of the quartic curves that go through the points in $\mathcal{P}_9$ and with multiplicity 2 at p_1 and p_j. Using `LinSys`, one can compute that for each $j > 1$, the linear system δ_j is a pencil of curves and the base points set is $\mathcal{P}_9$. Moreover, the generic element ϑ_j of δ_j is an irreducible curve of geometric genus 1 which cuts C_λ in $\mathcal{P}_9$ and two more points. Thus we obtain that for each $j > 1$, the strict transform of ϑ_j is an irreducible curve Γ_j such that

$$D_8 \equiv \Gamma_j + A_j + A_j' \text{ and } \Gamma_j^2 = 2.$$

Since $D_8B = 0$, we obtain $A_jB = A_j'B = 0$ for all $j \in \{1, \dots, 9\}$. Since the orthogonal of the classes A_j, A_j', $j \in \{1, \dots, 9\}$ (on which B belongs) is generated by D_2, the class of B must be a multiple of D_2 and have positive square, which is absurd. Therefore D_8 is ample.

Suppose that there is a fiber F' such that $D_8F' \in \{1, 2\}$. Observe that by using the expression for D_8, we get that $F'\Gamma_j = 0, 1, 2$. If $F'\Gamma_j = 0$, then Γ_j is contained in a fiber of the fibration determined by F', but this is not possible since $\Gamma_j^2 = 2$. If $F'\Gamma_j = 1$, then Γ_j is a section of the fibration so is a rational curve, but again this is not possible. If $F'\Gamma_j = 2$ (we can assume that this holds for all j, otherwise we are in a previous case), then F' is in the orthogonal complement of the A_j, A_j' but this is not possible since this is generated by L, which is of square 2. Therefore there are no such fiber F' and using [25], we obtain that the linear system $|D_8|$ is base-point free and gives an embedding of X_λ.

With respect to the divisor D_8, the degrees of the curves $A_1, A_1', \gamma_1, \gamma_1'$ equal 2 and the degrees of curves A_i, A_i', $i \geq 2$ is 1. For the assertions on the number of rational curves of degree $d \leq 8$ we used an algorithm (see e.g. [21]), which computes the classes of (-2)-curves in $\mathrm{NS}(X_\lambda)$ of given degrees with respect to a fixed ample class. □

Proceeding in a similar way as in the proof of Proposition 19, we obtain:

Proposition 20. *Let $i, j \in \{1, \dots, 9\}$, $i \neq j$. The divisor*

$$D_{i,j} = D_{14} - (A_i + A_i' + A_j + A_j')$$

is nef of square 2 and the linear system $|D_{i,j}|$ is base point free.

One can compute that the intersection with D_{ij} is 0 for the 10 curves $\theta_{ijklmn}, \theta'_{ijklmn}$ (where $\{k, l, m, n\} \subset \{1, \dots, 9\}$ is a set of 4 elements such that the conic C_{ijklmn} exists), and for the (-2)-curve which is the strict transform on X_λ of the line through cusps p_i, p_j.

4.3. A Hessian model of the natural fibration of X_λ.

4.3.1. *The generic fiber of the elliptic fibration φ and 18 rational points.* Let f_λ be the equation of the 9 cuspidal sextic C_λ which is the dual of E_λ, and let c_λ be the equation of the Cayleyan elliptic curve $\mathrm{Ca}(\lambda)$ (see equation (4.1)), the unique cubic that goes through the 9 cusps of the sextic curve C_λ.

We recall (see Section 4.1) that Y_λ is the blow-up of the plane at the 9 points in $\mathcal{P}_9$; it has a natural elliptic fibration φ', coming from the pencil of sextic curves which have double points at the 9 points in $\mathcal{P}_9$, pencil which is generated by C_λ and $2\mathrm{Ca}(\lambda)$. A singular model of Y_λ is therefore obtained as the surface in $\mathbb{P}^1\times\mathbb{P}^2$ with equation $uf_\lambda - vc_\lambda^2 = 0$, where u, v are the coordinates of $\mathbb{P}^1$. The projection onto $\mathbb{P}^1$ induces the fibration $\varphi' : Y_\lambda \to \mathbb{P}^1$. A singular model of the $K3$ surface X_λ is the surface X_λ^{sing} in $\mathbb{P}^1\times\mathbb{P}^2$ with equation $u^2 f_\lambda - v^2 c_\lambda^2 = 0$; again the projection onto $\mathbb{P}^1$ induces the natural fibration $X_\lambda^{sing} \to \mathbb{P}^1$, where the generic fibers are 9-nodal sextic curves.

In order to obtain a smooth model of X_λ, let us consider the linear system $L_4(\mathcal{P}_9)$ of quartics that contain the 9 cusps. The linear system $L_4(\mathcal{P}_9)$ has (projective) dimension 5 and defines a rational map $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^5$ not defined on $\mathcal{P}_9$. One computes that the image of X_λ^{sing} by the rational map

$$(i_d, \phi) : \mathbb{P}^1\times\mathbb{P}^2 \dashrightarrow \mathbb{P}^1\times\mathbb{P}^5$$

is a smooth model of X_λ; from Section 4.1, the images of the cusps are the 18 (-2)-curves on X_λ forming a $9\mathbf{A}_2$-configuration. Taking the generic point over $\mathbb{P}^1$, one get a smooth genus 1 curve in $\mathbb{P}^5_{/\mathbb{Q}(t)}$ (where $t = \frac{u}{v}$). That curve E_{K3} has naturally 18 rational points, corresponding to the 18 (-2)-curves. Using Magma, we computed a Hessian model $E_{K3} \hookrightarrow \mathbb{P}^2_{/\mathbb{Q}(t)}$, which is

Theorem 21. *A model of the generic fiber of the fibration $X_\lambda \to \mathbb{P}^1$ is*

$$E_{K3} \quad x^3 + y^3 + z^3 + \frac{\lambda^3(t^2+3) - 4t^2}{\lambda^2(t^2-1)} xyz = 0. \tag{4.2}$$

The elliptic curve E_{K3} contains the 9 obvious 3-torsion points

$$\begin{array}{l} Q_1 = (0:-1:1),\ Q_2 = (-1:0:1),\ Q_3 = (-1:1:0),\\ Q_4 = (0:-\omega:1),\ Q_5 = (\omega+1:0:1),\ Q_6 = (-\omega:1:0),\\ Q_7 = (0:\omega+1:1),\ Q_8 = (-\omega:0:1),\ Q_9 = (\omega+1:1:0) \end{array}$$

(where $\omega^2+\omega+1=0$; we take Q_1 as the neutral element) and the following 9 points

$$\begin{array}{l} P_1 = (-2t : \lambda(t+1) : \lambda t + \lambda),\\ P_2 = (\lambda(t-1) : -2t : \lambda t + \lambda),\\ P_3 = (-\lambda t - \lambda : -\lambda t + \lambda : 2t),\\ P_4 = ((2\omega+2)t : \lambda(\omega t + \omega) : \lambda t + \lambda)\\ P_5 = (\lambda(\omega+1)(-t+1) : -2\omega t : \lambda t + \lambda),\\ P_6 = ((\omega+1)\lambda(t+1) : \omega\lambda(-t+1) : 2t),\\ P_7 = (-2\omega t : -\lambda(\omega+1)(t+1) : \lambda t - \lambda),\\ P_8 = (\lambda\omega(t-1) : (2\omega+2)t : \lambda t + \lambda),\\ P_9 = (-\lambda\omega(t-1) : (\omega+1)\lambda(t-1) : 2t). \end{array}$$

Together, these 18 points are the above-mentioned rational points of $E_{K3/\mathbb{Q}(\omega,t)}$ corresponding to the 18 sections of the fibration of X_λ.

Remark 22. For the neutral element of E_{K3}, let us choose Q_1. One can check that the point P_k is the translate of P_1 by the 9 torsion point Q_k ($k \in \{1, \dots, 9\}$), i.e. $P_k = P_1 + Q_k$.

4.3.2. *A smooth model of X_λ in $\mathbb{P}^1 \times \mathbb{P}^2$.* By taking the homogenization of the generic fiber E_{K3} in (4.2), we get a natural model of the K3 surface X_λ as

$$\lambda^2(u^2 - v^2)(x^3 + y^3 + z^3) + (\lambda^3(u^2 + 3v^2) - 4u^2)xyz = 0. \tag{4.3}$$

in the space $\mathbb{P}^1 \times \mathbb{P}^2$ (with coordinates $u, v; x, y, z$, where $t = \frac{u}{v}$). That model is smooth, and the generic fibers are smooth cubic curves, by contrast with the previous model X_λ^{sing}. We denote by (P) the section in X_λ corresponding to the points $P \in E_{K3}$. Using Magma, it is then possible to obtain the equations of the (-2)-curves (also sections) $A_j = (Q_j)$, resp. $A'_j = (P_i)$, which are on $X_\lambda \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^2$. We can check that:

Lemma 23. *The* 9 *curves* $A_j + A'_j$ *(*$j \in \{1, \dots, 9\}$*) form a* $9\mathbf{A}_2$*-configuration.*

Proof. We use the equations of the (-2)-curves A_j, A'_j in the model $X_\lambda \subset \mathbb{P}^1 \times \mathbb{P}^2$ to check that $A_j A'_j = 1$ and $A_j A'_k = A_j A_k = A'_j A'_k = 0$ for $k \neq j$. In fact, one already knows that 3-torsion sections are disjoint by [16, VII, Proposition 3.2] (thus the sections A'_j, being translated of the group of 3-torsion sections, are also disjoint). □

One can check moreover that the 9 intersection points of A_j with A'_j for $i = 1, \dots, 9$ are on the fiber over 0 of the fibration φ, fiber which is isomorphic to E_λ. Using the addition law on the elliptic curve E_{K3}, one can find other points of E_{K3}, and therefore sections of φ. By example the following points

$$\begin{aligned}
R_1 &= (-2t : \lambda(t-1) : \lambda t + \lambda), \\
R_2 &= (\lambda(t+1) : -2t : \lambda t - \lambda), \\
R_3 &= (-\lambda t + \lambda : -\lambda t - \lambda : 2t), \\
R_4 &= ((2\omega + 2)t : \lambda\omega(t-1) : \lambda t + \lambda), \\
R_5 &= (-\lambda(\omega+1)(t+1) : -2\omega t : \lambda t - \lambda), \\
R_6 &= ((\omega+1)\lambda(t-1) : -\omega\lambda(t+1) : 2t), \\
R_7 &= (-2\omega t : \lambda(\omega+1)(-t+1) : \lambda t + \lambda), \\
R_8 &= (\lambda\omega(t+1) : (2\omega+2)t : \lambda t - \lambda), \\
R_9 &= (-\omega\lambda t + \omega\lambda : (\omega+1)(\lambda t + \lambda) : 2t),
\end{aligned}$$

are the points $R_i = -P_1 + Q_i$, $i \in \{1, \dots, 9\}$. We have

Lemma 24. *The* 9 *curves* A_i, (R_i), $i = 1, ..., 9$, *form a* $9\mathbf{A}_2$*-configurations.*

Proof. The curves A_i (resp. (R_i)) are images of the curves A'_i (resp. A_i) by the translation by $-A'_1$ and we already know that the curves A_i, A'_i form a $9\mathbf{A}_2$-configuration. □

We already know that the fiber at 0 of the elliptic K3 surface $\varphi : X_\lambda \to \mathbb{P}^1$ is isomorphic to E_λ, moreover:

Remark 25. From equation 4.3, the fiber at ∞ of $X_\lambda \to \mathbb{P}^1$ is the elliptic curve

$$\mathrm{He}(\lambda) : \quad x^3 + y^3 + z^3 + \tfrac{(\lambda^3-4)}{\lambda^2} xyz = 0,$$

which is in fact the Hessian of the curve E_λ. The j-invariants of $\mathrm{Ca}(\lambda)$ and $\mathrm{He}(\lambda)$ are distinct, in particular these curves are not isomorphic. The Cayleyan curve $\mathrm{Ca}(\lambda)$ is the quotient of Hessian $\mathrm{He}(\lambda)$ of E_λ by a 2-torsion point; in particular the two curves are 2-isogeneous, see [1].

4.3.3. *A degree 8 non-complete intersection model in* $\mathbb{P}^5$. One can check that the map from $X_\lambda \subset \mathbb{P}^1 \times \mathbb{P}^2$ to $\mathbb{P}^5$ obtained as the product of the identity map of $\mathbb{P}^1$ with the Segre embedding composed with the projection to $\mathbb{P}^5$, is an embedding with image a degree 8 K3 surface in $\mathbb{P}^5$ defined by the following 5 equations:

$$\begin{array}{r} -U_2U_4 + U_1U_5, \quad -U_2U_3 + U_0U_5, \quad -U_1U_3 + U_0U_4, \\ \lambda^2(U_0^2U_3 - U_3^3 + U_1^2U_4 - U_4^3) + (\lambda^3-4)U_0U_1U_5 \\ +\lambda^2(U_2^2U_5 + 3\lambda U_3U_4U_5 - U_5^3), \\ \lambda^2(U_0^3 + U_1^3 + U_2^3 - U_0U_3^2) + (\lambda^3-4)U_0U_1U_2 \\ +\lambda^2(-U_1U_4^2 + 3\lambda U_0U_4U_5 - U_2U_5^2), \end{array}$$

in particular this is not a complete intersection (in fact, by using [2, Chapter VIII, Exercice 11], a K3 surface has a degree 8 smooth model which is not a complete intersection if and only if it has a smooth model in $\mathbb{P}^1 \times \mathbb{P}^2$ of bi-degree $(2,3)$).

Using the images of the known (-2)-curves on X_λ, one finds that this surface in $\mathbb{P}^5$ contains at least 33 lines.

The involution σ' defined by $u \to (-u_0 : -u_1 : -u_2 : u_3 : u_4 : u_5)$ acts on $X_\lambda \hookrightarrow \mathbb{P}^5$. Also one can check that the order 3 automorphisms

$$\begin{array}{l} \alpha_1 : u \to (u_1 : u_2 : u_0 : u_4 : u_5 : u_3), \\ \alpha_2 : u \to (u_0 : \omega u_1 : \omega^2 u_2 : u_3 : \omega u_4 : \omega^2 u_5) \end{array}$$

act on X_λ and so does the involution

$$\beta : u \to (u_0 : u_2 : u_1 : u_3 : u_5 : u_4).$$

The fixed point set of α_1 is the union of 6 points, the fixed point set of β is a genus 2 curve. The involution $\sigma'\beta$ is symplectic. Using the above equations of X_λ and the equations of curves $A_1, \dots, A_9'$ in $\mathbb{P}^5$, one can check that the automorphism group $G_{\mathcal{C}}$ that preserve globally the $9\mathbf{A}_2$-configuration $A_1, \dots, A_9'$ contains the group G_{18} isomorphic to $\mathbb{Z}_3 \rtimes S_3$ generated by $\alpha_1, \alpha_2, \sigma'\beta$. We prove in Section 5.3 that G_{18} has index 2 in the group $G_{\mathcal{C}}$ preserving the configuration $A_1, \dots, A_9'$.

4.4. **A Weierstrass equation.** We recall that ω is such that $\omega^2+\omega+1=0$. Let us prove the following result

Theorem 26. *A minimal Weierstrass model of the elliptic K3 surface X_λ is the elliptic curve*

$$E_{1/\mathbb{Q}(\omega,t)}: \;\; y^2 = x^3 - \frac{1}{48}Ax + \frac{1}{864}B,$$

where the three polynomials A, B, D in $\mathbb{Q}(\omega)(t)$ (of respective degree 8, 12 and 8) are defined in the Appendix. The 8 singular fibers $\tilde{\mathbf{A}}_2$ of X_λ are over the 8 zeros of D.

Proof. A direct computations gives that the j-invariant of the elliptic curve E_{K3} is

$$j = -\frac{A^3}{(\lambda^2(\lambda^3-1)D)^3},$$

where the formulas for the polynomials A and D in t are given in the Appendix. For any $j \notin \{0, 1728\}$, the elliptic curve

$$E_0(j): \;\; y^2 = x^3 - \frac{1}{48}\frac{j}{j-1728}x + \frac{1}{864}\frac{j}{j-1728}$$

has j-invariant equal to j. In our case, we compute that we have

$$\frac{j}{j-1728} = \frac{A^3}{B^2},$$

where the polynomial B is also defined in the Appendix. By taking the change of variables

$$x' = u^2x, \; y' = u^3y$$

with $u = (B/A)^{1/2}$ in the equation of $E_0(j)$, we obtain the elliptic curve E_1. The curve E_1 has also its j-invariant equals to j, is also in Weierstrass form, but its coefficients are coprime degree 8 and 12 polynomials in t. The discriminant of the equation of E_1 is

$$\Delta = -(\lambda^2(\lambda^3-1)D)^3,$$

where D is a product of 8 degree 1 polynomials in t. According to [16, Table IV.3.1], the associated elliptic surface is a K3 surface with 8 singular fibers of type $\tilde{\mathbf{A}}_2$.

Using Magma, we finally obtain an isomorphism defined over $\mathbb{Q}(\omega, t)$ between the Hesse model E_{K3} and the Weierstrass model E_1. $\square$

5. On automorphisms of the K3 surface X_λ

5.1. **On the Mordell-Weil lattice of the elliptic fibration φ.** For a point $P \in E_{K3}(\mathbb{Q}(\omega,t))$, let us denote by $(P) \hookrightarrow X_\lambda$ the corresponding section of $\varphi : X_\lambda \to \mathbb{P}^1$. We denote by $\tau \in \mathrm{Aut}(X_\lambda)$ the automorphism which is the translation by A_1'. We have

Theorem 27. *Modulo torsion, the section $A_1' = (P_1)$ generates the Mordell-Weil lattice $MWL(X_\lambda)$ of sections.*

Remark 28. One can compute the classes in $\mathrm{NS}(X_\lambda)$ of the curves (R_i) (which are the translate by $(-P_1)$ of the curves A_i); we give these classes in the Appendix. Using that knowledge, we get the matrix representation on $\mathrm{NS}(X_\lambda)$ of the action of the automorphism τ. The characteristic polynomial of τ is $(T-1)^3(T^2+T+1)^8$.

Proof. Let $O = A_1$ be the zero section, and let F be a fiber of $\varphi : X_\lambda \to \mathbb{P}^1$. Using the knowledge of the action of the automorphism τ (which is the translation by (P_1)) on $\mathrm{NS}(X_\lambda)$, we get that

$$(6P_1) - 2(3P_1) + O \equiv 6F$$

in $\mathrm{NS}(X_\lambda)$, thus (see e.g. [27, Chapter III, Theorem 9.5]) $\langle 3P_1, 3P_1\rangle = 6$ and $\langle P_1, P_1\rangle = \frac{2}{3}$, where $\langle\cdot,\cdot\rangle$ is the bilinear pairing on $\mathrm{MWL}(X_\lambda)$ associated to the canonical height.

Let $\mathrm{Triv}(X_\lambda)$ be the lattice generated the zero section and the fibers components of the fibration. The determinant formula [26, Corollary 6.39] is

$$|\det \mathrm{NS}(X_\lambda)| = |\det \mathrm{Triv}(X_\lambda)| \cdot \det \mathrm{MWL}(X_\lambda)/|\mathrm{MWL}(X_\lambda)|^2.$$

By Lemma 8, we know that $|\det \mathrm{NS}(X_\lambda)| = 54$. We have moreover $\det \mathrm{Triv}(X_\lambda) = -3^8$ and $|\mathrm{MWL}(X_\lambda)|^2 = 3^4$, thus we obtain that

$$\det \mathrm{MWL}(X_\lambda) = \tfrac{2}{3}.$$

By Theorem 18, the group $\mathrm{MWL}(X_\lambda)$ has rank 1; since $\langle P_1, P_1\rangle = \frac{2}{3}$, we conclude that P_1 generates $\mathrm{MWL}(X_\lambda)$ modulo torsion. □

Using the action of τ and its powers, we can obtain more classes in $\mathrm{NS}(X_\lambda)$ of the sections on the K3 surface $X_\lambda \to \mathbb{P}^1$.

Remark 29. We searched the $9\mathbf{A}_2$-configurations among a set of 45 sections, but we obtained only the expected ones, i.e. the $9\mathbf{A}_2$-configuration that are translate of the configuration A_i, A_i', $i \in \{1, \dots, 9\}$. Since these configurations are images of one configuration by an automorphism (the translation by A_1' and its multiples), these $9\mathbf{A}_2$-configurations give the same generalized Kummer structure.

5.2. **More elements of the automorphism group and another double plane model.** The K3 surface X_λ is constructed as the minimal desingularization of the double cover of the plane branched over the sextic curve C_λ. Let $\sigma \in \mathrm{Aut}(X_\lambda)$ be the corresponding involution. By construction $\sigma(A_j) = A_j'$, $\sigma(A_j') = A_j$ and σ preserves the fiber of the fibration $X_\lambda \to \mathbb{P}^1$. From these facts, we know the action of σ on the Néron-Severi lattice, since we know also the action of τ, and one can compute that

Lemma 30. *We have $\sigma = \tau\sigma\tau$.*

Let us recall that we denoted by (R_i) the sections corresponding to the points $R_i = -P_1 + Q_i$ in E_{K3}. We also have a model $X_\lambda \subset \mathbb{P}^1 \times \mathbb{P}^2$. From the equations of the curves involved, we obtain that:
a) the natural map $\pi_2 : X_\lambda \to \mathbb{P}^2$ induced by the projection $\mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2$ is a 2 to 1 map ,
b) it contracts the (-2)-curves $A_1, \dots, A_9$ to the 9 torsion base points $\mathcal{T}_9$ of the Hesse pencil $x^3 + y^3 + z^3 - 3\mu xyz = 0$, $\mu \in \mathbb{P}^1$.
c) for $j \in \{1, \dots, 9\}$, the curves A'_j and (R_j) are mapped to a line L'_j that contains exactly one point of $\mathcal{T}_9$, that line is therefore tangent to the sextic curve at its other intersection points. The 9 lines are in general position.

We can therefore compute the action of the involution $\sigma' \in \mathrm{Aut}(X_\lambda)$ corresponding to the double cover π_2 on the Néron-Severi lattice (the curves A'_j and (R_j) are exchanged and one can check that the fiber is preserved; the classes of (R_j) in base F, A_j, A'_j are in the appendix). The action of σ' on $X_\lambda \hookrightarrow \mathbb{P}^5$ is given in sub-section 4.3.3. We get:

Lemma 31. *We have $\tau = \sigma\sigma'$.*

We compute moreover that the pull-back by π_2 of the 9 points of $\mathcal{T}_9$ are the irreducible curves $A_1, \dots, A_9$, in fact we have

Theorem 32. *The surface X_λ is the minimal desingularization of the double cover of the plane branched over the sextic curve C_6 which is the union of the elliptic curves*

$$E_\lambda : \quad x^3 + y^3 + z^3 - 3\lambda xyz = 0$$

and the Hessian of E_λ:

$$\mathrm{He}(\lambda) : \quad x^3 + y^3 + z^3 + \tfrac{(\lambda^3-4)}{\lambda^2} xyz = 0.$$

Proof. Let F_0 and F_∞ be the two fibers of the fibration $X_\lambda \to \mathbb{P}^1$ at 0 and infinity. One computes that $\pi_2(F_0) = E_\lambda$, $\pi_2(F_\infty) = \mathrm{He}(\lambda)$, and moreover

$$\pi_2^*(E_\lambda) = 2F_0 + \textstyle\sum_{j=1}^9 A_j,\ \ \pi_2^*(\mathrm{He}(\lambda)) = 2F_\infty + \textstyle\sum_{j=1}^9 A_j,$$

thus $E_\lambda + \mathrm{He}(\lambda)$ is the branch locus of the map π_2. By Bézout Theorem, the singularities of $E_\lambda + \mathrm{He}(\lambda)$ are nodal since the curves E_λ and $\mathrm{He}(\lambda)$ meet at $\mathcal{T}_9$. □

Remark 33. For each $j = 1, \dots, 9$, the images by π_2 of the two sections $(-2P_1 + Q_j)$, $(2P_1 + Q_j)$ is the same quartic curve, which is nodal with 3 nodes. The images of $(-P_1 + Q_j)$, $(P_1 + Q_j)$ are the 9 lines, their coordinates in the dual plane with basis x, y, z are the same as the points in $\mathcal{P}_9$. These 9 lines are the inflection lines of the curve E_λ.

We recall that the Hesse configuration is the point-line configuration $(9_4, 12_3)$ of the 9 points in $\mathcal{T}_9$ and the 12 lines $\mathcal{L}_{12}$ such that each line contains 3 points in $\mathcal{T}_9$ and each point is on 4 lines.

Proposition 34. *The images by π_2 in $\mathbb{P}^2$ of the 24 irreducible components of the singular fibers of the fibration $\varphi : X_\lambda \to \mathbb{P}^1$ are the 12 lines of the Hesse configuration.*

Proof. We give in Theorem 26 the 8 points $p \in \mathbb{P}^1$ such that the fiber F_p over p is singular. We are then able to compute these singular fibers in $X_\lambda \subset \mathbb{P}^1 \times \mathbb{P}^2$ and their images in $\mathbb{P}^2$. $\square$

Remark 35. Using the elliptic curve E_{K3}, we obtain that the sub-group Tor_3 of order 3 elements in the Mordell-Weil lattice $\mathrm{MWL}(X_\lambda)$ is generated by two order 3 elements t_1, t_2 which acts on $\mathrm{NS}(X_\lambda)$ via

$$t_1(A_j) = A_{\sigma j}, t_1(A'_j) = A'_{\sigma j},\ \ t_2(A_j) = A_{\mu j}, t_2(A'_j) = A'_{\mu j}$$

where $\sigma = (1,2,3)(4,5,6)(7,8,9)$ and $\mu = (1,4,7)(2,5,8)(3,6,9)$. The elements of Tor_3 commute with σ (and of course with τ). The action of Tor_3 is transitive on the nine $9\mathbf{A}_2$-configurations we found in section 3.2.

5.3. **On the stabilizer group of natural $9\mathbf{A}_2$-configuration.** Recall that X_λ is the minimal desingularization of the double cover of $\mathbb{P}^2$ ramified over the 9-cuspidal sextic C_λ. The strict transform on X_λ of C_λ is a smooth elliptic curve isomorphic to E_λ. The linear system defined by that elliptic curve defines an elliptic fibration which we denote $\varphi : X_\lambda \to \mathbb{P}^1$, and which we called the *natural* fibration. The curves A_j, A'_j (above the cusps) are sections of φ, the 24 (-2)-curves θ_J, θ'_J (for some set $J \subset \{1,\dots,9\}$ of order 6) above the 12 conics are contained in the fibers (see Proposition 18), their classes are given in the Appendix, under a simpler labelling Θ_j, $j \in \{1,\dots,24\}$. We have

Proposition 36. *An integral basis $\mathcal{B}$ of the Néron-Severi lattice of X_λ is*

$$F, A_1, A'_1, A_2, A'_2, A_3, A'_3, A_4, A'_4, A_5, A'_5, A_6, A_7, A'_7,$$
$$\Theta_5, \Theta_{14}, \Theta_{22}, \Theta_{23}, \Theta_{20},$$

where F is a fiber of the fibration φ. The discriminant group $A_{\mathrm{NS}(X_\lambda)} \simeq \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^3$ is generated by

$$\begin{aligned}
w_0 &= \tfrac{1}{2}(0,1,1,0,0,0,0,1,1,0,0,0,1,1,1,1,1,0,0)_{\mathcal{B}},\\
v'_1 &= \tfrac{1}{3}(A_3 + 2A'_3 + A_5 + 2A'_5 + A_7 + 2A'_7),\\
v'_2 &= \tfrac{1}{3}(A_2 + 2A'_2 + 2A_4 + A'_4 + 2A_5 + A'_5 + A_7 + 2A'_7),\\
v'_3 &= \tfrac{1}{3}(A_1 + 2A'_1 + A_4 + 2A'_5 + A_7 + 2A'_7).
\end{aligned}$$

Proof. By Lemma 8, the discriminant group of the Néron-Severi lattice has order 54. The lattice generated by the elements in $\mathcal{B}$ has rank 19 and discriminant equal to 54, thus these elements generate $\mathrm{NS}(X_\lambda)$. By taking the inverse of the intersection matrix of the vectors in $\mathcal{B}$, we get the generators of the discriminant group. $\square$

The non-immediate part of the Gram matrix of the basis $\mathcal{B}$ is the intersection matrix of the 5 curves Θ_j, $j = 5, 14, 22, 23, 20$ with the curves in $\mathcal{B}$, whose matrix is:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}.$$

Let $G_{\mathcal{C}}$ be the automorphism sub-group of $\mathrm{Aut}(X_\lambda)$ preserving globally the configuration $\mathcal{C} = A_1, A'_1, \dots, A_9, A'_9$. The proof of the following Proposition will also serve as a preliminary for the proof of Theorem 39 in the next section:

Proposition 37. *The group $G_{\mathcal{C}}$ is isomorphic to $\mathbb{Z}_2 \times (\mathbb{Z}_3 \rtimes S_3)$, where $\mathbb{Z}_3 \rtimes S_3$ acts symplectically on X_λ. The center of $G_{\mathcal{C}}$ is generated by the non-symplectic involution σ associated to the double cover map $X_\lambda \to \mathbb{P}^2$ branched over the cuspidal sextic curve C_λ.*

Proof. Let ϕ be an element of $G_{\mathcal{C}}$. Since it preserves globally the configuration $\mathcal{C}$, it must map its orthogonal complement (generated by D_2, the pull-back of a line) to itself. There are $2^9 9!$ bijective maps

$$\mu : \{A_1, A'_1, \dots, A_9, A'_9\} \to \{A_1, A'_1, \dots, A_9, A'_9\}$$

which preserves the incidence relations of $\mathcal{C}$. Since a linear map is defined by the images of the vectors of a basis, each map μ extends to a linear automorphism $\phi_\mu : \mathrm{NS}(X_\lambda) \otimes \mathbb{Q} \to \mathrm{NS}(X_\lambda) \otimes \mathbb{Q}$ sending D_2 to D_2 and the configuration $\mathcal{C}$ to itself. The action of ϕ on $\mathrm{NS}(X_\lambda)$ must be one of these maps. Using the integral basis $\mathcal{B}$ of Proposition 36, we obtain that among all the possibilities, only 864 matrices in basis $\mathcal{B}$ of such ϕ_μ are in $GL_{19}(\mathbb{Z})$. These 864 matrices are the elements of a group G_{864} isomorphic to the product of $\mathbb{Z}/2\mathbb{Z}$ with $AGL_2(\mathbb{F}_3)$, the affine linear group of the space $\mathbb{F}_3^2$. The center of G_{864} has order 2 and is generated by the matrix of the non-symplectic involution σ defined in section 5.2. In the appendix, we give two generators g_1, g_2 (of respective order 8 and 6) of the group $G_{432} \subset G_{864}$ isomorphic to $AGL_2(\mathbb{F}_3)$. Their action on the 2-torsion part of the discriminant group $A_{\mathrm{NS}(X_\lambda)}$ is trivial, and one computes that their action on the 3-torsion part of the discriminant group $A_{\mathrm{NS}(X_\lambda)}$ is by the matrices

$$\bar{g}_1 = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ \bar{g}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in basis v_1, v_2, v_3. The group G_{Disc} generated by $\bar{g}_1$, $\bar{g}_2$ is isomorphic to the symmetric group S_4. The kernel of the map $G_{432} \to G_{\mathrm{Disc}}$ is a group G_{18} isomorphic to $\mathbb{Z}_3 \rtimes S_3$. Let $\mathrm{Tran}(X_\lambda)$ be the orthogonal complement of $\mathrm{NS}(X_\lambda)$ in $H^2(X_\lambda, \mathbb{Z})$; it is a rank 3 signature $(2, 1)$ lattice. There exists an isomorphism

$$\gamma : A_{\mathrm{Tran}(X_\lambda)} \to A_{\mathrm{NS}(X_\lambda)}$$

between the discriminant groups, such that the quadratic forms of the groups satisfy $q_{\mathrm{Tran}(X_\lambda)} \circ \gamma = -q_{\mathrm{NS}(X_\lambda)}$. Since an element $\phi \in G_{18}$ acts trivially on $A_{\mathrm{NS}(X_\lambda)}$, we can extend it to an isometry Γ of $H^2(X_\lambda, \mathbb{Z})$ by gluing it with the identity map on $\mathrm{Tran}(X_\lambda)$ (see e.g. [31, Theorem 12] and references therein for an example of such construction). Since Γ is the identity on the space $\mathrm{Tran}(X_\lambda) \otimes \mathbb{C}$ containing the period, this is a Hodge isometry. Since Γ naturally preserves the polarization $D_{14} = 4D_2 - \sum_j (A_j + A_j')$ (studied in Proposition 19), it is effective. Therefore we can apply the Torelli Theorem for K3 surfaces (see [6, Chap. VIII, Theorem 11.1]) and conclude that there exists a unique $g \in \mathrm{Aut}(X_\lambda)$ such that $g^* = \Gamma$ on $\mathrm{NS}(X_\lambda)$.

By [15, Corollary 3.3.5], since the Picard number of X_λ is odd, the only Hodge isometry on $\mathrm{Tran}(X_\lambda)$ is $\pm I_d$. Suppose that an element g of G_{432} not contained in G_{18} comes from an automorphism of X_λ. From the definition of g, its action $\bar{g}$ on $A_{\mathrm{NS}(X_\lambda)}$ is non trivial, thus its action on $\mathrm{Tran}(X_\lambda)$ is also non-trivial. The automorphism g is therefore non symplectic and it acts by $-Id$ on $\mathrm{Tran}(X_\lambda)$. Thus $\bar{g} \in G_{\mathrm{Disc}}$ acts on the discriminant group $A_{\mathrm{Tran}(X_\lambda)}$ by $-Id$. However $-Id$ is not contained in $G_{\mathrm{Disc}} \simeq S_4$: a contradiction and g cannot come from an automorphism of X_λ.

Since σ commutes with the elements of G_{432}, the automorphism group of the configuration is the direct product of G_{18} and $\langle \sigma \rangle \simeq \mathbb{Z}_2$. □

Remark 38. We give a geometric interpretation of the some elements of the group $G_{\mathcal{C}} \simeq \mathbb{Z}_2 \times (\mathbb{Z}_3 \rtimes S_3)$ in sub-section 4.3.3: the group $G_{\mathcal{C}}$ contains the 9 translations by 3-torsion automorphisms (from the fibration $\varphi : X_\lambda \to \mathbb{P}^1$), and the involution σ from the double cover $X_\lambda \to \mathbb{P}^2$. One can recover the automorphisms in the subgroup $\mathbb{Z}_3 \rtimes S_3$ of $G_{\mathcal{C}}$ as follows:
Let A be the abelian surface and let J_A be the order 3 symplectic automorphism acting on A such that $X_\lambda = \mathrm{Km}_3(A)$ is the minimal resolution of A/J_A, the exceptional locus being $\mathcal{C}$. The automorphism J_A fixes 9 points $0 = s_1, \dots, s_9$ which form a group isomorphic to $(\mathbb{Z}_3)^2$. The group generated by the translation by the s_k and the involution $[-1] : z \in A \to -z \in A$ is isomorphic to $\mathbb{Z}_3 \rtimes S_3$. These isomorphisms induce automorphisms on A/J_A, hence automorphisms on K3 surface X_λ; these automorphisms preserve $\mathcal{C}$.

5.4. **An automorphism sending a $9\mathbf{A}_2$-configuration to another.** Let

$$\mathcal{C}' = B_1, B_1', \dots, B_9, B_9'$$

be one of the nine $9\mathbf{A}_2$-configurations found in section 3.2 (so that $B_j B_j' = 1$, $B_j B_k = 0$ for $j \neq k$), namely we choose the $9\mathbf{A}_2$-configuration

$$\mathcal{C}' = \Theta_{12}, \Theta_9, \Theta_{14}, \Theta_5, \Theta_{16}, \Theta_7, \Theta_2, \Theta_3, \gamma_1, \gamma_1', \Theta_4, \Theta_1, \Theta_8, \Theta_{15}, \Theta_6, \Theta_{13}, \Theta_{10}, \Theta_{11},$$

where the classes of the (-2)-curves Θ_j (which are the curves $\theta_{i,j,k,m,n,o}$ more conveniently labelled and numbered) and γ_1, γ_1' are in the Appendix. The eight $\mathbf{A}_2$-configurations B_j, B_j', $j \in \{1, \dots, 4, 6, \dots, 9\}$, are contained in the fibers of φ and the $\mathbf{A}_2$-configuration B_5, B_5' is the strict transform under the double cover map of a the quartic curve Q_1.

Theorem 39. *There exists an automorphism f of X_λ sending the curve A_j (resp. A_j') to the curve B_j (resp. B_j') for $j \in \{1, \dots, 9\}$. In particular, the* *configuration $\mathcal{C} = A_1, A_1', \dots, A_9, A_9'$ is sent by f to the configuration $\mathcal{C}'$, and therefore the configuration $\mathcal{C}'$ is on the* $\mathrm{Aut}(X_\lambda)$*-orbit of* $\mathcal{C}$.

Proof. For finding the automorphism f we proceeded as in Proposition 37: Let $\phi \in \mathrm{Aut}(X_\lambda)$ be an automorphism sending $\mathcal{C}$ to $\mathcal{C}'$. The orthogonal complement of $\mathcal{C}$ is generated by D_2, the pull-back of a line in $\mathbb{P}^2$ and the orthogonal complements of $\mathcal{C}'$ is generated by the divisor

$$D_2' = 7D_2 - 2\textstyle\sum_{j=1}^{j=9}(A_j + A_j') - 2(A_1 + A_1'),$$

of square 2. Since ϕ must preserve the Néron-Severi lattice, we have $\phi(D_2) = D_2'$. There are $2^9 9!$ bijective maps

$$\mu : \{A_1, A_1', \dots, A_9, A_9'\} \to \{B_1, B_1', \dots, B_9, B_9'\}$$

which preserves the incidence relations of $\mathcal{C}$ and $\mathcal{C}'$ and which extend uniquely to isometries $\tilde{\phi}_\mu$ of $\mathrm{NS}(X_\lambda) \otimes \mathbb{Q}$. If two of these maps $\tilde{\phi}_1, \tilde{\phi}_2$ preserve the lattice $\mathrm{NS}(X_\lambda)$, then $\tilde{\phi}_2^{-1}\tilde{\phi}_1$ also preserves that lattice, moreover it also preserves the configuration $\mathcal{C}$. It is therefore an element of the group G_{864} defined in the proof of Proposition 37. Thus the set of maps $\tilde{\phi}$ (among the maps $\tilde{\phi}_\mu$) that preserve $\mathrm{NS}(X_\lambda)$ is the orbit of $\tilde{\phi}_1$ under the action of G_{864} on the left. The element $\tilde{\phi}_1$ defined by sending A_j to B_j and A_j' to B_j' preserves the lattice $\mathrm{NS}(X_\lambda)$ (its matrix $f_{\mathcal{B}}$ in basis $\mathcal{B}$ is given in the Appendix). Its action on the discriminant group is trivial. Then, as in the proof of Proposition 37, we can extend $\tilde{\phi}_1$ to an Hodge isometry of $H^2(X_\lambda, \mathbb{Z})$.

Let us prove that this isometry is effective. The polarization (see Proposition 19) $D_{14} = 4D_2 - \sum(A_j + A_j')$ is sent to $D_{14}' = 4D_2' - \sum(B_j + B_j') = D_2' + F_1$ where F_1 is (the class of) a fiber of a fibration f_1 by Remark 40. Using the classes of the curves in $\mathrm{NS}(X_\lambda)$, it is easy to check that the curves B_j, B_j' are 18 sections of f_1.

The degree 7 curve $Q \hookrightarrow \mathbb{P}^2$ defined by

$$\begin{gathered}
\lambda x^7 - 2\lambda^2 x^6 y + 3\lambda^3 x^5 y^2 - (2\lambda^4 + \lambda)x^4 y^3 + \lambda^5 x^3 y^4 + 3\lambda^3 x^2 y^5 \\
+\lambda^2 y^7 - 2\lambda^2 x^6 z - 1x^5 yz + 4\lambda x^4 y^2 z - (2\lambda^5 + 9\lambda^2)x^3 y^3 z + (\lambda^3 + 1)x^2 \\
y^4 z - (3\lambda^4 + 2\lambda)xy^5 z - 2\lambda^2 y^6 z + 3\lambda^3 x^5 z^2 + 4\lambda x^4 yz^2 + (3\lambda^5 + 4\lambda^2)x^3 y^2 z^2 \\
+(4\lambda^3 - 1)x^2 y^3 z^2 + 4\lambda xy^4 z^2 + \lambda^2 y^5 z^2 - (2\lambda^4 + \lambda)x^4 z^3 - (2\lambda^5 + 9\lambda^2)x^3 yz^3 \\
+(4\lambda^3 - 1)x^2 y^2 z^3 - (3\lambda^4 + 4\lambda)xy^3 z^3 + \lambda^5 y^4 z^3 + \lambda^5 x^3 z^4 + (\lambda^3 + 1)x^2 yz^4 \\
+4\lambda xy^2 z^4 + \lambda^5 y^3 z^4 + 3\lambda^3 x^2 z^5 - (3\lambda^4 + 2\lambda)xyz^5 + \lambda^2 y^2 z^5 - 2\lambda^2 yz^6 + \lambda^2 z^7 = 0
\end{gathered}$$

is irreducible with singularities of multiplicity 4 at the point p_1 in $\mathcal{P}_9$, and with multiplicity 2 at the eight remaining points (the curve Q was found by using `LinSys`). The geometric genus of Q is 1 and Q meets the branch locus at two other points, so that its strict transform on X_λ is smooth of genus 2 in the linear system $|D_2'|$. Thus D_2' is effective, nef and $|D_2'|$ is base point free.

Let C be an irreducible (-2)-curve on X_λ. If $CF' = CD_2' = 0$, then C is contained in a fiber of f_1 and is contracted by $|D_2'|$. The second point implies that C is one of the curves B_j, B_j' (otherwise the Picard number of

X_λ would be > 19), but then one has $CF' = 1$, which is a contradiction. Thus D'_{14} is ample and the isometry is effective.

We then conclude as in the proof of Proposition 37 that there exists an automorphism of X_λ such that its action on the curves A_j, A'_j is as described. □

We recall that for $k = 1, ..., 9$, the point $p_k \in \mathcal{P}_9$ is the point over which $A_k + A'_k$ is contracted by the double cover map $\eta : X_\lambda \to \mathbb{P}^2$.

Remark 40. For any point p_k, the pencil of lines through p_k induces an elliptic fibration $f_k : X_\lambda \to \mathbb{P}^1$ such that the curves A_k, A'_k are sections and the curves A_j, A'_j for $j \neq k$ are contained in the fibers. The singular fibers of f_k are also $8\tilde{\mathbf{A}}_2$; from the known intersection numbers of these fibers with the elements of basis $D_2, A_1, \dots, A'_9$, we obtain that the class of a fiber is

$$F_k = D_2-(A_k+A'_k) = \tfrac{1}{3}(F+\textstyle\sum_{j=1}^{9}(A_j+A'_j))-(A_k+A'_k) = 3D'_2-\textstyle\sum_{j=1}^{9}(B_j+B'_j)$$

where F is a fiber of φ and the B_j, B'_j are the curve in the $9\mathbf{A}_9$-configuration found in Section 3.2, D'_2 being the generator of the orthogonal complement of the B_j, B'_j's. The curves B_j, B'_j are sections of f_k.

So the geometric situation for the configuration $A_1, A'_1, \dots, A_9, A'_9$ and fibration f_k is very similar to the situation for $\mathcal{C}'$ and fibration φ for which 8 of the $\mathbf{A}_2$-configurations in $B_1, B'_1, \dots, B_9, B'_9$ are in the 8 singular fibers and the remaining one are sections.

5.4.1. *Aligned singularities of the union of the* 12 *conics.* The fibration φ has the remarkable property that the 18 sections A_j, A'_j meet on the same fiber, which is isomorphic to E_λ. It can be instructive to understand how the similar result holds for the fibration f_1 (see Remark 40) and the curves B_j, B'_j, this is the aim of this subsection, which also gives an explanation why each line of the dual Hesse configuration contains 4 double points of the curve $\sum_{C\in\mathcal{C}_{12}} C$ and these double points form the set $\mathcal{P}_{12}$.

We recall that the 12 conics in $\mathcal{C}_{12}$ meet in either 4 or 3 points in $\mathcal{P}_9$. If two conics meet in 3 points in $\mathcal{P}_9$ then the fourth intersection point is an ordinary singularity of the union of the 12 conics. Above the 8 conics that contain p_1 are the 16 (-2)-curves that gives a $8\mathbf{A}_8$-configuration, which one can complete to a $9\mathbf{A}_2$-configuration according to section 3.2.

The 8 conics containing p_1 have the property that they meet by pairs into 4 points $q_1, \dots, q_4$ not in $\mathcal{P}_9$ and that these 4 points are on a line containing p_1. That line L_1 is the tangent line to the cusp p_1 (it is one of the lines of the dual Hesse configuration defined in Section 2.4). It meets the cuspidal sextic in p_1 (with multiplicity 3) and at points r_1, r_2, r_3. One can check that the fiber F_0 of f_1 which is the strict transform on X_λ of L_1 is isomorphic to E_λ (since we know the branch locus $F_0 \to L_1$).

The 8 points in X_λ above $q_1, \dots, q_4$ are the meeting points of the curves in the $8\mathbf{A}_2$ configuration obtained by taking the strict transform of the 8 conics trough p_1. Following the Figure 3.2, the 9^{th} $\mathbf{A}_2$-configuration also has its intersection point on that fiber F_0. In fact, the fiber F_0 is the image by an automorphism of X_λ of the fiber over 0 of φ.

The union of the dual Hesse configuration and the 12 conics in $\mathcal{C}_{12}$ is a line-conic arrangement with 9 points of multiplicity 9, 12 points of multiplicity 5 and 72 double points, and with other remarkable properties studied in [20].

6. Appendix

Let D_2 be the pull-back of a line by the double cover map $\eta : X_\lambda \to \mathbb{P}^1$. Let us define the following classes in the $\mathbb{Q}$-basis $\mathcal{B}_0 = (D_2, A_1, A'_1, \dots, A_9, A'_9)$:

$$B_1 = 2D_2 - \tfrac{1}{3}\left(\textstyle\sum_{j=1}^{9} 2A_j + A'_j\right),\ B_2 = 2D_2 - \tfrac{1}{3}\left(\textstyle\sum_{j=1}^{9} A_j + 2A'_j\right).$$

We remark that $B_1^2 = B_2^2 = 2$, $B_1B_2 = 5$ and $B_1 + B_2 = D_{14}$. We have $D_{14}A_j = D_{14}A'_j = 1$, therefore $B_iA_j \in \{0,1\}$, $B_iA'_j \in \{0,1\}$. Using algorithms described in [21], we find that for $j \in \{1,\dots,9\}$, the classes of the curves γ_j, γ'_j above the quartic Q_j are

$$\gamma_j = B_1 - (A_j + A'_j),\ \gamma'_j = B_2 - (A_j + A'_j).$$

It is easy to check that $\gamma_j^2 = \gamma_j'^2 = -2$, $\gamma_j\gamma'_j = 1$, and for $1 \leq i \neq j \leq 9$, we have $\gamma_i\gamma_j = \gamma'_i\gamma'_j = 0$ and $\gamma_i\gamma'_j = 3$. In fact, using that the image in $\mathbb{P}^2$ of γ_j, γ'_j is a quartic curve that goes through the points in $\mathcal{P}_9$ with a multiplicity 3 at p_j, one gets

$$4D_2 \equiv \gamma_j + \gamma'_j + 2(A_j + A'_j) + \textstyle\sum_{j=1}^{9}(A_j + A'_j).$$

The classes in the $\mathbb{Q}$-basis $\mathcal{B}_0 = (L, A_1, A'_1, \dots, A_9, A'_9)$ of the 24 (-2)-curves $\theta_{i,\dots,n}, \theta'_{i,\dots,n}$ above the 12 conics $C_{i,\dots,n}$ in $\mathcal{C}_{12}$ are

$$\begin{aligned}
\theta_{123456} &= \tfrac{1}{3}(3,-2,-1,-2,-1,-2,-1,-1,-2,-1,-2,-1,-2,0,0,0,0,0,0),\\
\theta'_{123456} &= \tfrac{1}{3}(3,-1,-2,-1,-2,-1,-2,-2,-1,-2,-1,-2,-1,0,0,0,0,0,0),\\
\theta_{123789} &= \tfrac{1}{3}(3,-2,-1,-2,-1,-2,-1,0,0,0,0,0,0,-1,-2,-1,-2,-1,-2),\\
\theta'_{123789} &= \tfrac{1}{3}(3,-1,-2,-1,-2,-1,-2,0,0,0,0,0,0,-2,-1,-2,-1,-2,-1),\\
\theta_{124578} &= \tfrac{1}{3}(3,-2,-1,-1,-2,0,0,-2,-1,-1,-2,0,0,-2,-1,-1,-2,0,0),\\
\theta'_{124578} &= \tfrac{1}{3}(3,-1,-2,-2,-1,0,0,-1,-2,-2,-1,0,0,-1,-2,-2,-1,0,0),\\
\theta_{124689} &= \tfrac{1}{3}(3,-2,-1,-1,-2,0,0,-1,-2,0,0,-2,-1,0,0,-2,-1,-1,-2),\\
\theta'_{124689} &= \tfrac{1}{3}(3,-1,-2,-2,-1,0,0,-2,-1,0,0,-1,-2,0,0,-1,-2,-2,-1),\\
\theta_{125679} &= \tfrac{1}{3}(3,-2,-1,-1,-2,0,0,0,0,-2,-1,-1,-2,-1,-2,0,0,-2,-1),\\
\theta'_{125679} &= \tfrac{1}{3}(3,-1,-2,-2,-1,0,0,0,0,-1,-2,-2,-1,-2,-1,0,0,-1,-2),\\
\theta_{134589} &= \tfrac{1}{3}(3,-2,-1,0,0,-1,-2,-1,-2,-2,-1,0,0,0,0,-1,-2,-2,-1),\\
\theta'_{134589} &= \tfrac{1}{3}(3,-1,-2,0,0,-2,-1,-2,-1,-1,-2,0,0,0,0,-2,-1,-1,-2),\\
\theta_{134679} &= \tfrac{1}{3}(3,-2,-1,0,0,-1,-2,-2,-1,0,0,-1,-2,-2,-1,0,0,-1,-2),\\
\theta'_{134679} &= \tfrac{1}{3}(3,-1,-2,0,0,-2,-1,-1,-2,0,0,-2,-1,-1,-2,0,0,-2,-1),\\
\theta_{135678} &= \tfrac{1}{3}(3,-2,-1,0,0,-1,-2,0,0,-1,-2,-2,-1,-1,-2,-2,-1,0,0),\\
\theta'_{135678} &= \tfrac{1}{3}(3,-1,-2,0,0,-2,-1,0,0,-2,-1,-1,-2,-2,-1,-1,-2,0,0),\\
\theta_{234579} &= \tfrac{1}{3}(3,0,0,-2,-1,-1,-2,-2,-1,-1,-2,0,0,-1,-2,0,0,-2,-1),\\
\theta'_{234579} &= \tfrac{1}{3}(3,0,0,-1,-2,-2,-1,-1,-2,-2,-1,0,0,-2,-1,0,0,-1,-2),\\
\theta_{234678} &= \tfrac{1}{3}(3,0,0,-2,-1,-1,-2,-1,-2,0,0,-2,-1,-2,-1,-1,-2,0,0),\\
\theta'_{234678} &= \tfrac{1}{3}(3,0,0,-1,-2,-2,-1,-2,-1,0,0,-1,-2,-1,-2,-2,-1,0,0),\\
\theta_{235689} &= \tfrac{1}{3}(3,0,0,-2,-1,-1,-2,0,0,-2,-1,-1,-2,0,0,-2,-1,-1,-2),\\
\theta'_{235689} &= \tfrac{1}{3}(3,0,0,-1,-2,-2,-1,0,0,-1,-2,-2,-1,0,0,-1,-2,-2,-1),\\
\theta_{456789} &= \tfrac{1}{3}(3,0,0,0,0,0,0,-1,-2,-1,-2,-1,-2,-2,-1,-2,-1,-2,-1),\\
\theta'_{456789} &= \tfrac{1}{3}(3,0,0,0,0,0,0,-2,-1,-2,-1,-2,-1,-1,-2,-1,-2,-1,-2).
\end{aligned}$$

We also denote by Θ_j, $j = 1,\dots,24$ these curves in the order of the above list.

For $k = 1, \dots, 9$, the quartic curves Q_k through $\mathcal{P}_9$ that have a multiplicity 3 singular point at p_k are:

$$
\begin{aligned}
Q_1:\ & x^4-2\lambda x^3y+3\lambda^2x^2y^2-(\lambda^3+1)xy^3+\lambda y^4-2\lambda x^3z+(-\lambda^3+1)xy^2z-2\lambda y^3z\\
&+3\lambda^2x^2z^2+(-\lambda^3+1)xyz^2+(\lambda^4+2\lambda)y^2z^2-(\lambda^3+1)xz^3-2\lambda yz^3+\lambda z^4=0,\\
Q_2:\ & x^4-(\lambda^3+1)/\lambda x^3y+3\lambda x^2y^2-2xy^3+1/\lambda y^4-2x^3z+(-\lambda^3+1)/\lambda x^2yz-2y^3z\\
&+(\lambda^3+2)x^2z^2+(1-\lambda^3)/\lambda xyz^2+3\lambda y^2z^2-2xz^3-(\lambda^3+1)/\lambda yz^3+z^4=0,\\
Q_3:\ & x^4-2x^3y+(\lambda^3+2)x^2y^2-2xy^3+y^4-(\lambda^3+1)/\lambda x^3z+(1-\lambda^3)/\lambda x^2yz\\
&+(1-\lambda^3)/\lambda xy^2z-(\lambda^3+1)/\lambda y^3z+3\lambda x^2z^2+3\lambda y^2z^2-2xz^3-2yz^3+1/\lambda z^4=0,\\
Q_4:\ & x^4+(2\omega+2)\lambda x^3y+3\omega\lambda^2x^2y^2-(\lambda^3+1)xy^3-(\omega+1)\lambda y^4-2\omega\lambda x^3z\\
&-(\omega^2\lambda^3+\omega+1)xy^2z-2\omega\lambda y^3z-(3\omega+3)\lambda^2x^2z^2+(\omega-\omega\lambda^3)xyz^2\\
&+(\lambda^4+2\lambda)y^2z^2-(\lambda^3+1)xz^3+(2\omega+2)\lambda yz^3+\omega\lambda z^4=0,\\
Q_5:\ & x^4-(\omega^2\lambda^3+\omega^2)/\lambda x^3y+3\omega\lambda x^2y^2-2xy^3-(\omega+1)/\lambda y^4-2\omega x^3z\\
&+(1-\lambda^3)/\lambda x^2yz-2\omega y^3z-((\omega+1)\lambda^3+2\omega+2)x^2z^2+(\omega-\omega\lambda^3)/\lambda xyz^2\\
&+3\lambda y^2z^2-2xz^3-\omega^2(\lambda^3+1)/\lambda yz^3+\omega z^4=0,\\
Q_6:\ & x^4+(2\omega+2)x^3y+(\omega\lambda^3+2\omega)x^2y^2-2xy^3+\omega^2y^4-(\omega\lambda^3+\omega)/\lambda x^3z\\
&+(1-\lambda^3)/\lambda x^2yz-(\omega^2\lambda^3-\omega^2)/\lambda xy^2z-(\omega\lambda^3+\omega)/\lambda y^3z\\
&-(3\omega+3)\lambda x^2z^2+3\lambda y^2z^2-2xz^3+(2\omega+2)yz^3+\omega/\lambda z^4=0,\\
Q_7:\ & x^4-2\omega\lambda x^3y-(3\omega+3)\lambda^2x^2y^2-(\lambda^3+1)xy^3+\omega\lambda y^4+(2\omega+2)\lambda x^3z\\
&+(\omega-\omega\lambda^3)xy^2z+(2\omega+2)\lambda y^3z+3\omega\lambda^2x^2z^2-(\omega^2\lambda^3-\omega^2)xyz^2\\
&+(\lambda^4+2\lambda)y^2z^2-(\lambda^3+1)xz^3-2\omega\lambda yz^3+\omega^2\lambda z^4=0,\\
Q_8:\ & x^4-(\omega\lambda^3+\omega)/\lambda x^3y-(3\omega+3)\lambda x^2y^2-2xy^3+\omega/\lambda y^4+(2\omega+2)x^3z\\
&+(1-\lambda^3)/\lambda x^2yz+(2\omega+2)y^3z+(\omega\lambda^3+2\omega)x^2z^2-(\omega^2\lambda^3-\omega^2)/\lambda xyz^2\\
&+3\lambda y^2z^2-2xz^3-(\omega\lambda^3+\omega)/\lambda yz^3+\omega^2z^4=0,\\
Q_9:\ & x^4-2\omega x^3y+(\omega^2\lambda^3-2\omega-2)x^2y^2-2xy^3+\omega y^4-(\omega^2\lambda^3+\omega^2)/\lambda x^3z\\
&+(1-\lambda^3)/\lambda x^2yz+(-\omega\lambda^3+\omega)/\lambda xy^2z-(\omega^2\lambda^3+\omega^2)/\lambda y^3z\\
&+3\omega\lambda x^2z^2+3\lambda y^2z^2-2xz^3-2\omega yz^3+\omega^2/\lambda z^4=0,
\end{aligned}
$$

where $\omega^2+\omega+1=0$.

The matrices in basis $\mathcal{B}$ of the generators of the group $G_{432}\simeq AGL_2(\mathbb{F}_3)$ preserving the natural $9\mathbf{A}_2$-configuration $A_1,A_1',\ldots,A_9,A_9'$ are

$$
g_1=\begin{pmatrix}
1&0&0&0&0&0&0&1&-2&0&0&0&0&0&0&1&0&1&0\\
0&0&0&0&0&-1&1&0&0&0&0&0&0&0&1&0&0&-1&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&0\\
0&1&0&0&0&1&-1&1&-1&0&0&0&0&0&-1&0&0&1&0\\
0&0&1&0&0&2&-2&1&0&0&0&0&0&0&-1&-1&0&1&0\\
0&0&0&0&0&4&-2&1&0&1&0&0&0&0&-1&-1&-1&2&0\\
0&0&0&0&0&2&-1&1&-1&0&1&0&0&0&-1&0&-1&1&0\\
0&0&0&0&0&-1&1&0&1&0&0&0&1&0&0&0&1&-1&0\\
0&0&0&0&0&-2&2&-1&1&0&0&0&0&1&1&0&1&-1&0\\
0&0&0&1&0&-1&0&0&-1&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&1&-2&0&-1&0&0&0&0&0&0&0&1&0&-1&0\\
0&0&0&0&0&2&-1&0&0&0&0&1&0&0&0&-1&-1&1&0\\
0&0&0&0&0&-3&2&-1&1&0&0&0&0&0&1&1&1&-2&0\\
0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&-2&1&-1&2&0&0&0&0&0&1&0&1&-2&0\\
0&0&0&0&0&1&1&-1&2&0&0&0&0&0&1&-1&0&0&0\\
0&0&0&0&0&3&-3&0&0&0&0&0&0&0&-1&-1&-1&1&0\\
0&0&0&0&0&-4&2&-2&1&0&0&0&0&0&1&1&1&-2&1\\
0&0&0&0&0&2&-1&1&1&0&0&0&0&0&-1&-1&0&1&0
\end{pmatrix},
$$

$$g_2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 4 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 & -2 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 2 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -2 & 2 & -1 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -3 & 2 & -1 & 1 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 & 1 & -1 & 2 & 0 & 0 & 0 & 0 & 3 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 3 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -4 & 2 & -2 & 1 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 2 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 2 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & -1 & 0 & 0
\end{pmatrix},$$

The automorphism f of Theorem 39 acts on the Néron-Severi lattice by

$$f_{\mathcal{B}} = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 1 & -1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & -3 & 2 & 0 & 0 & 2 & -1 & 1 & -1 & 2 & -2 & -3 & 2 & -1 & 0 & 0 & 0 & 0 & -1 \\
0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & -2 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 2 & 2 & -1 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & -1 & 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & -1 & 0 & 0 & -1 & 1 & -1 & 1 & -1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & -1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & -1 & 0 & 0 & -2 & 1 & 0 & 0 & -1 & 2 & 2 & -1 & 1 & 0 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 2 & 2 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & -1 & 1 & 1 & 0 & 0 & -1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & -2 & 1 & 0 & 0 & 2 & -1 & 0 & 0 & 1 & -2 & -2 & 2 & -1 & 0 & -1 & 0 & 0 & -1 \\
0 & 2 & -1 & 0 & 0 & -2 & 1 & -1 & 0 & -2 & 2 & 3 & -1 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & -2 & 1 & 0 & 0 & 2 & -1 & 1 & -1 & 1 & -1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & -2
\end{pmatrix}.$$

Let us define

$$S = \textstyle\sum_{j=1}^{9} A_i,\ S' = \sum_{j=1}^{9} A'_i$$

The classes of the (-2)-curves (R_i) defined in Section 4.3 are

$$(R_i) = 2L - \tfrac{1}{3}(S + 2S' + 3A_i + 3A'_i),$$

where L is the pullback of a line by the double cover map $X_\lambda \to \mathbb{P}^2$. The translation automorphism τ defined in Section 4.3 sends L to the class

$$L' = 7L - \tfrac{4}{3}(S + 2S').$$

The three polynomials A, B, D in $\mathbb{Q}(\omega)(t)$ of Section 4.4 are defined as follows:

$$A = (\lambda^3t^2 + 3\lambda^3 - 4t^2)(\lambda^3t^2 + 3\lambda^3 + (6\omega + 6)\lambda^2t^2 + (-6\omega - 6)\lambda^2 - 4t^2)$$
$$\cdot(\lambda^3t^2 + 3\lambda^3 - 6\lambda^2t^2 + 6\lambda^2 - 4t^2)(\lambda^3t^2 + 3\lambda^3 - 6\omega\lambda^2t^2 + 6\omega\lambda^2 - 4t^2),$$

$$B = (\lambda^6t^4 + 6\lambda^6t^2 + 9\lambda^6 + 6\lambda^5t^4 + 12\lambda^5t^2 - 18\lambda^5 - 18\lambda^4t^4 + 36\lambda^4t^2 - 18\lambda^4$$
$$-8\lambda^3t^4 - 24\lambda^3t^2 - 24\lambda^2t^4 + 24\lambda^2t^2 + 16t^4)$$
$$\cdot(\lambda^6t^4 + 6\lambda^6t^2 + 9\lambda^6 + (-6\omega - 6)\lambda^5t^4 + (-12\omega - 12)\lambda^5t^2 + (18\omega + 18)\lambda^5 - 18\omega\lambda^4t^4$$
$$+36\omega\lambda^4t^2 - 18\omega\lambda^4 - 8\lambda^3t^4 - 24\lambda^3t^2 + (24\omega + 24)\lambda^2t^4 + (-24\omega - 24)\lambda^2t^2 + 16t^4)$$
$$\cdot(\lambda^6t^4 + 6\lambda^6t^2 + 9\lambda^6 + 6\omega\lambda^5t^4 + 12\omega\lambda^5t^2 - 18\omega\lambda^5 + (18\omega + 18)\lambda^4t^4$$
$$+(-36\omega - 36)\lambda^4t^2 + (18\omega + 18)\lambda^4 - 8\lambda^3t^4 - 24\lambda^3t^2 - 24\omega\lambda^2t^4 + 24\omega\lambda^2t^2 + 16t^4),$$

$$D = ((\lambda + 2)t - (2\omega + 1)\lambda)((\lambda - 2\omega - 2)t - (2\omega + 1)\lambda)((\lambda + 2\omega)t - (2\omega + 1)\lambda)$$
$$\cdot(t^2 - 1)((\lambda + 2)t + (2\omega + 1)\lambda)((\lambda - 2\omega - 2)t + (2\omega + 1)\lambda)((\lambda + 2\omega)t + (2\omega + 1)\lambda).$$

References

[1] Artebani M., Dolgachev I., The Hesse pencil of plane cubic curves, Ens. Math. (2) 55 (2009), no. 3-4, 235–273.

[2] Beauville A., Complex algebraic surfaces, 2nd ed. Lond. Math. Soc., 34. Cambridge University Press, 1996. x+132 pp.

[3] J. Bertin, Réseaux de Kummer et surfaces K3, Invent. Math. 93 (1988), no. 2, 267–284.

[4] Barth W., K3 surfaces with nine cusps. Geom. Dedicata 72 (1998), no. 2, 171–178.

[5] Barth W., On the classification of K3 surfaces with nine cusps, Complex analysis and algebraic geometry, de Gruyter, Berlin, 2000, pp. 41–59. MR 1760871.

[6] Barth W., Hulek K., Peters C.A.M., Van de Ven A., Compact complex surfaces. Sd ed. Erg Math Grenz. 3. Folge., 4. Springer-Verlag, Berlin, 2004. xii+436 pp.

[7] Birkenhake C., Lange H., A family of Abelian surfaces and curves of genus four, Manus. Math. 85, (1994), 393–407.

[8] Bosma W,. Cannon J., Playoust C., The Magma algebra system. I. The user language, Computational algebra and number theory (London, 1993), J. Symbolic Comput. 24, 1997, 3-4, 235--265.

[9] Conway J. H., Sloane N. J. A., Sphere packings, lattices and groups, Sde. ed., Grund. der Math. Wiss 290. Springer-Verlag, New York, 1993. xliv+679 pp.

[10] Degtyarev A., Oka's conjecture on irreducible plane sextics, J. Lond. Math. Soc. (2) 78 (2008), no. 2, 329–351

[11] Dolgachev I., Abstract configurations in algebraic geometry, The Fano Conference, 423–462, Univ. Torino, Turin, 2004.

[12] Dolgachev I., Laface A., Persson U., Urzúa G, Chilean configuration of conics, lines and points, preprint.

[13] Griffiths P., Harris J., Principles of Algebraic Geometry, Wiley-Interscience, New York, 1978. xii+813 pp.

[14] Gritsenko V., Hulek K., Minimal Siegel modular threefolds, Math. Proc. Cambridge Philos. Soc. 123 (1998), 461–485.

[15] Huybrecht D., Lectures on K3 surfaces, Camb. Stud. Adv. Math., 158, Cambridge University Press, 2016. xi+485 pp.

[16] Miranda R., The Basic Theory of Elliptic Surfaces, Dottorato di Ricerca in Matematica, ETS Editrice, Pisa, 1989. vi+108 pp.

[17] Morrison D. R., On K3 surfaces with large Picard number. Invent. Math. 75 (1984), no. 1, 105–121.

[18] Nikulin V.V., On Kummer surfaces, Izv. Akad. Nauk SSSR Ser. Mat. 39 (1975), 278–293. English translation: Math. USSR. Izv, 9 (1975), 261–275.
[19] Nikulin V.V., Integer symmetric bilinear forms and some of their geometric applications, Izv. Akad. Nauk SSSR Ser. Mat., 43, 1979, no 1, 111–177.
[20] Pokora P., Szemberg T., Conic-line arrangements in the complex projective plane, preprint, ArXiv 2002.01760
[21] Roulleau X., On the geometry of K3 surfaces with finite automorphism group: the compact case, preprint, ArXiv 1909.01909
[22] Roulleau X., Sarti A., Construction of Nikulin configurations on some Kummer surfaces and applications, Math. Annalen 373 (2019), no 11, 7651–7668.
[23] Roulleau X., Sarti A., Explicit Nikulin configurations on Kummer surfaces, preprint, ArXiv 1907.12215
[24] Sarti A., Transcendental lattices of some K3-surfaces, Math. Nachr. 281 (2008), no. 7, 1031–1046.
[25] Saint Donat B., Projective models of K-3 surfaces. Amer. J. Math. 96 (1974), 602–639.
[26] Schütt M., Shioda T., Mordell-Weil lattices, Erg. der Math. 3. Folge. A Series of Modern Surveys in Mathematics 70. Springer, 2019. xvi+431 pp.
[27] Silverman J., Advanced topics in the arithmetic of elliptic curves, Graduate Texts in Mathematics, 151. Springer-Verlag, New York, 1994. xiv+525 pp.
[28] Shimada I., On elliptic K3 surfaces, Michigan Math. J. 47 (2000), no. 3, 423–446.
[29] Shimada I., On elliptic K3 surfaces, arXiv:math/0505140
[30] Shioda T., On elliptic modular surfaces. J. Math. Soc. Japan 24 (1972), 20–59.
[31] Shioda T., Some remarks on abelian varieties, J. Fac. Sci. Univ. Tokyo Sect. IA 24 (1977),11–21.

David Kohel, Xavier Roulleau,
Aix-Marseille Université, CNRS, Centrale Marseille,
I2M UMR 7373,
13453 Marseille, France
David.Kohel@univ-amu.fr
Xavier.Roulleau@univ-amu.fr

Alessandra Sarti,
Laboratoire de Mathématiques et Applications, UMR CNRS 7348,
Université de Poitiers, Téléport 2,
Boulevard Marie et Pierre Curie,
86962 Futuroscope Chasseneuil, France
sarti@math.univ-poitiers.fr

http://www-math.sp2mi.univ-poitiers.fr/ sarti/