EXPLICIT NIKULIN CONFIGURATIONS ON KUMMER SURFACES

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Abstract. A Nikulin configuration is the data of 16 disjoint smooth rational curves on a K3 surface. According to results of Nikulin, the existence of a Nikulin configuration means that the K3 surface is a Kummer surface, moreover the abelian surface from the Kummer structure is determined by the 16 curves. In the paper [15], we constructed explicitly non isomorphic Kummer structures on some Kummer surfaces. In this paper we generalize the construction to Kummer surfaces with a weaker restriction on the degree of the polarization and we describe some cases where the previous construction does not work.

1. Introduction

A (projective, as always in this paper) Kummer surface is obtained as the desingularization of the quotient of an abelian surface by an involution with 16 isolated fixed points. It is well known that Kummer surfaces are K3 surfaces and that their Picard number is at least 17, the rank 17 sub-group being generated by the 16 rational curves in the resolution of the 16 nodes and by the polarization. In [12], Nikulin showed the converse, i.e. that a K3 surface containing 16 disjoint smooth rational curves, or (-2)-curves, is the Kummer surface associated to an abelian surface. Let X be a K3 surface; we call a *Kummer structure* on X an abelian surface A (up to isomorphism) such that $X \simeq Km(A)$, and we call a *Nikulin configuration* a set of 16 disjoint smooth rational curves on X. By the result of Nikulin we have a bijection:

$$\{\text{Kummer structures}\} \longleftrightarrow \{\text{Nikulin configurations}\}/\mathrm{Aut(X)}$$

In 1977, see [20, Question 5], T. Shioda raised the following question :

Is it possible to have non-isomorphic abelian surfaces A and B, such that $Km(A)$ and $Km(B)$ are isomorphic?

Shioda and Mitani in [10, Theorem 5.1] answer negatively the question if $\rho(Km(A)) = 20$, where $\rho(Km(A))$ is the Picard number of $Km(A)$, i.e. the rank of the Néron-Severi group of $Km(A)$. The answer is also negative if A is a generic principally polarized abelian surface, i.e. A is the jacobian of a curve of genus 2 and $\rho(A) = 1$. Then in [7, Theorem 1.5], Gritsenko and Hulek answered positively the question. They showed that if A is a generic $(1, t)$-polarized abelian surface with $t > 1$ then the abelian surface A and its

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dual $\hat{A}$, though not isomorphic, satisfy $Km(A) \cong Km(\hat{A})$. In [9, Theorem 0.1], Hosono, Lian, Oguiso and Yau, by using lattice theory, showed that the number of Kummer structures is finite and for each integer $N \in \mathbb{N}^*$, they construct a Kummer surface of Picard number 18 with at least N Kummer structures.

In [13, Example 4.16], Orlov showed that if A is a generic abelian surface (i.e. $\rho(Km(A)) = 17$) then the number of abelian surfaces (up to isomorphism) with equivalent bounded derived categories is 2^ν, where ν is the number of prime divisors of $\frac{1}{2}M^2$, for M an ample generator of the Néron-Severi group of A. By [9, Theorem 0.1], there is a one-to-one correspondence between these equivalent bounded derived categories of A and the Kummer structures on the Kummer surface $\mathrm{Km}(A)$ associated to A. Thus, for example if A is principally polarized we have that $M^2 = 2$ so that $\nu = 0$ and we find again the fact that in this case there is only one Kummer structure on $\mathrm{Km}(A)$. Observe that ν can be also defined as the number of prime divisors of $\frac{1}{4}L^2$, where L is the polarization induced by M on $Km(A)$, (in particular L is orthogonal to the 16 rational curves ; it is easy to see that by changing the 16 rational disjoint curves, the number ν does not change).

In [15, Theorem 1], we constructed explicit examples of two Nikulin configurations $\mathcal{C}$, $\mathcal{C}'$ on some K3 surface X such that the abelian surfaces A and A' associated to these two configurations are not isomorphic. This was the first geometric construction of two distinct Kummer structures. These examples are for generic Kummer surfaces, such that the orthogonal complement of the 16 rational curves in $\mathcal{C}$ is generated by a class L such that $L^2 = 2k(k+1)$ for some integer k (we give a motivation for this restriction in the Appendix of this paper).

The main goal of this paper is to provide a generalization of that result to other Kummer surfaces. For that aim, let $t \in \mathbb{N}^*$ be an integer and let X be a general Kummer surface with a Nikulin configuration $\mathcal{C}$ such that the orthogonal complement of the 16 (-2)-curves $A_1, \dots, A_{16}$ in $\mathcal{C}$ is generated by L with $L^2 = 4t$. A class C of the form $C = \beta L - \alpha A_1$ with $\beta \in \mathbb{N}^*$ has self-intersection C^2 equals to -2 if and only if the coefficients (α, β) satisfy the Pell-Fermat equation $\alpha^2 - 2t\beta^2 = 1$. There is a non-trivial solution if and only if $2t$ is not a square. Let us suppose that this is the case. Then there exists a so-called fundamental solution which we denote by (α_0, β_0). Our main result is as follows:

Theorem 1. *Suppose that β_0 is even. Then $\beta_0 L - \alpha_0 A_1$ is the class of an irreducible (-2)-curve A_1', which curve is disjoint from $A_2, \dots, A_{16}$.*
The Nikulin configurations $\mathcal{C} = \sum_{i=1}^{16} A_i$ and $\mathcal{C}' = A_1' + \sum_{i=2}^{16} A_i$ define the same Kummer structure on the Kummer surface X if and only if the negative Pell-Fermat equation $\alpha^2 - 2t\beta^2 = -1$ has a solution.
Suppose that this is the case. Then there exists a double cover map $X \to \mathbb{P}^2$ branched over 6 lines $L_1, \dots, L_6$, contracting the 15 (-2)-curves $A_j, j \geq 2$ to the singularities of $\sum_{i=1}^6 L_i$, and such that the induced involution exchanges the curves A_1, A_1' and therefore the configurations $\mathcal{C}, \mathcal{C}'$.

The integers $t \in \mathbb{N}$ such that β_0 is even have density at least $\frac{3}{4}$; among these integers, at least $\frac{2}{3}$ are such that the negative Pell-Fermat equation have no solution, and therefore give examples of two distinct Kummer structures (see Remark 6 for a precise meaning of that affirmation, and also the table in the Appendix). As a by-product of our study, let us mention the following result (see Proposition 17), which we believe can be of independent interest: *Suppose that the equation $2\mu^2 - t\nu^2 = -1$ has a solution. Then there exists a model of the K3 surface X as a quartic surface in $\mathbb{P}^3$ with 15 nodes.*

One could also raise a weaker question than Shioda's question by asking if $Km(A) \cong Km(B)$ implies A and B must be isogenous ? The answer is positive and the result was surely known, but we could not find an explicit proof in the literature, hence we recall it in Section 2 and we show how it can be obtained as a direct consequence of a result of Stellari [21, Theorem 1.2]. In the rest of the paper, we point out why the construction in Theorem 1 can not work for β_0 odd, moreover we study examples of Nikulin configurations in the case that β_0 is odd or $2t$ is a square.

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2. Construction of Nikulin configurations

2.1. **Preliminaries: the Pell-Fermat equation and its negative.** The aim of this first sub-section is to recall results on Pell-Fermat equations. We give various criteria when the fundamental solution (α_0, β_0) of it is such that β_0 is even, and when the negative Pell-Fermat solution has no solution.

2.1.1. *The Pell-Fermat equation.* For $t \in \mathbb{N}^*$, the Pell-Fermat equation

$$\alpha^2 - 2t\beta^2 = 1 \tag{2.1}$$

has a non-trivial solution $(\alpha, \beta) \in \mathbb{Z}^2$ if and only if $2t$ is not a square. Then there exists a fundamental solution $(\alpha_0, \beta_0) \in \mathbb{N}$, such that for every other solution (α, β), there exists $k \in \mathbb{Z}$ with $\alpha + \beta\sqrt{2t} = \pm(\alpha_0 + \sqrt{2t}\beta_0)^k$.

Remark 2. For $k \in \mathbb{Z}$, let $(x_k, y_k) \in \mathbb{Z}^2$ be such that

$$x_k + y_k\sqrt{2t} = (\alpha_0 + \sqrt{2t}\beta_0)^k.$$

Using that $\alpha_0 \geq 1, \beta_0 \geq 1$ and an induction, one can check that the sequences $(x_k)_{k\in\mathbb{N}}$, $(y_k)_{k\in\mathbb{N}}$ are strictly increasing with $k \in \mathbb{N}$. Therefore, if $(\alpha, \beta) \in \mathbb{N}^2$ is a solution different from $(1, 0)$ and with $\alpha \leq \alpha_0$ or $\beta \leq \beta_0$, then (α, β) is the fundamental solution.
We observe moreover that for a solution (α, β) of equation (2.1), the integer α is necessarily odd.

For t a positive integer such that $2t$ is not a square, we denote by (α_0, β_0) the fundamental solution of $\alpha^2 - 2t\beta^2 = 1$. Part a) of the following Lemma shows that the density of integers t such that β_0 is even is at least $\frac{3}{4}$ (we thank Joël Rivat for useful discussions on that question, and also Lemma 5 below):

Lemma 3. *a) Suppose that $t \neq 0$ mod 4. Then β_0 is even.*
b) There is an infinite number of integers s such that the fundamental solution (α_0, β_0) of $\alpha^2 - 8s^2\beta^2 = 1$ has odd β_0.
c) There is an infinite number of integers s such that the fundamental solution (α_0, β_0) of $\alpha^2 - 8s^2\beta^2 = 1$ has even β_0.

Proof. Let (α, β) be a solution of equation $\alpha^2 - 2t\beta^2 = 1$. Suppose that β is odd. Then

$$\beta = \pm 1, \pm 3 \bmod 8,$$

and one has $\beta^2 = 1 \bmod 8$. Since $\alpha^2 - 2t\beta^2 = 1$, one has $\alpha^2 = 1 + 2t$ mod 8. Since α is also odd, $\alpha^2 = 1 \bmod 8$, thus $2t = 0$ mod 8 and therefore $t = 0$ mod 4. That proves part a).

Let (x_1, y_1) be the fundamental solution of $x^2 - 2ty^2 = 1$. For $n \in \mathbb{Z}$, the integers $\pm x_n, \pm y_n$ defined by

$$x_n + y_n\sqrt{2t} = (x_1 + y_1\sqrt{2t})^n$$

are the solutions of equation $x^2 - 2ty^2 = 1$. The sequence $(y_n)_{n \geq 1}$ is strictly increasing and we see that the fundamental solution of

$$x^2 - 2ty_n^2 y^2 = 1$$

is $(x_n, 1)$. Using part a), we remark that always $ty_n^2 = 0$ mod 4. Take now $t = 4$, we therefore obtain result b). For n even, y_n is even; let z_n be such that $y_n = 2z_n$. The fundamental solution of

$$x^2 - 2tz_n^2 y^2 = 1$$

is $(x_n, 2)$; taking $t = 4$ as in the previous case, one obtains result c). □

Example 4. For $1 \leq s \leq 100$ such that $8s$ is not a square, i.e. for $s \notin \{2, 8, 18, 32, 50, 72, 98\}$, the fundamental solution (α_0, β_0) of equation $\alpha^2 - 8s\beta^2 = 1$ is such that β_0 is even if and only if s is in

$$\{7, 9, 14, 23, 30, 31, 33, 34, 46, 47, 56, 57, 62, 63, 69, 71, 73, 75, 77, 79, 81, 82, 89, 90, 94\}.$$

2.1.2. *The negative Pell-Fermat equation.* The equation

$$\alpha^2 - 2t\beta^2 = -1 \tag{2.2}$$

is called the *negative Pell-Fermat equation.* If (x, y) is a solution, then $(\alpha, \beta) = (x^2 - 2ty^2, 2xy)$ is a solution of the Pell-Fermat equation (2.1), with β even. The negative Pell-Fermat equation can be solved by the method of continued fractions and it has solutions if and only if the period of the continued fraction has odd length. A necessary (but not sufficient) condition for solvability is that t is not divisible by a prime of form $4k + 3$. The following Lemma implies that the density of integers t such that the negative Pell-Fermat equation (2.2) has no solution is at least $\frac{5}{6}$:

Lemma 5. *Suppose that the negative Pell-Fermat equation* (2.2) *has a solution. Then $t = 1 \bmod 4$ and $t \neq 0 \bmod 3$, in other words: $t = 1$ or $5 \bmod 12$.*

Proof. Suppose that (α, β) is a solution of equation (2.2). Since $\alpha^2 - 2t\beta^2 = -1$, the integer α is odd, thus $\alpha^2 = 1 \bmod 8$, and $2t\beta^2 = \alpha^2 + 1 = 2 \bmod 8$, which implies that β is odd (otherwise $2t\beta^2 = 0 \bmod 8$), thus $\beta^2 = 1 \bmod 8$. In that way $2t = 2 \bmod 8$ hence $t = 1 \bmod 4$.
Since $\alpha^2 = 0$ or $1 \bmod 3$, one has $2tb^2 = 1$ or $2 \bmod 3$, thus $t \neq 0 \bmod 3$. □

The first few numbers t for which equation (2.2) is solvable are

$$1, 5, 13, 25, 29, 37, 41, 53, 61, 65, 85, 101, 109...$$

Remark 6. From Lemmas 3 and 5, we conclude that the density of integers t such that the negative Pell-Fermat equation (2.2) has no solution and the Pell-Fermat equation (2.1) has a solution (α_0, β_0) with β_0 even is at least $\frac{7}{12}$.

2.2. **The general problem.**

2.2.1. *Isogenies.* Before to state our results about the question of Shioda [20, Question 5] recalled in the Introduction, we can generalize the problem to the following question:

Given two abelian surfaces A and B such that $Km(A) \cong Km(B)$ are then A and B isogenous ?

The answer is positive and certainly well known, in particular to people working on derived categories on abelian surfaces. For convenience we give here a short proof:

Proposition 7. *Let A and B be abelian surfaces such that the associated Kummer surfaces are isomorphic, then A and B are isogenous abelian surfaces.*

Proof. Since $Km(A) \cong Km(B)$ then the derived categories $D^b(Km(A))$ and $D^b(Km(B))$ are equivalent. Thus by [21, Theorem 1.2], the abelian surfaces are isogenous. □

2.2.2. *Notations and known results on the Néron-Severi group of a Kummer surface.* Let $t \in \mathbb{N}$ be an integer and let B be a generic Abelian surface with (primitive) polarization M such that $M^2 = 2t$. Let $X = \mathrm{Km}(B)$ be the associated Kummer surface. Let $A_1, \dots, A_{16}$ be the 16 disjoint (-2)-curves on X that are resolution of the singularities of the quotient $B/[-1]$. By [11, Proposition 3.2], [6, Proposition 2.6], corresponding to the polarization M on B, there is a primitive big and nef divisor L on $\mathrm{Km}(B)$ such that

$$L^2 = 4t$$

and $LA_i = 0$, $i \in \{1, \dots, 16\}$. The Néron-Severi group of $X = \mathrm{Km}(B)$ satisfies:

$$\mathbb{Z}L \oplus K \subset \mathrm{NS}(X),$$

where K denotes the Kummer lattice (the saturated lattice containing the 16 disjoint (-2)-curves A_i, $i = 1, \dots, 16$) which is a negative definite lattice of rank 16 and discriminant 2^6. For B generic among polarized Abelian surfaces $\mathrm{rk}(\mathrm{NS}(X)) = 17$ and $\mathrm{NS}(X)$ is an over-lattice of index two of $\mathbb{Z}L \oplus K$ which is described precisely in [6, Theorem 2.7], in particular we will repeatedly use the following result:

Lemma 8. *(*[6, Remarks 2.3 & 2.10]*) An element* $\Gamma \in \mathrm{NS}(X)$ *has the form* $\Gamma = \alpha L - \sum \beta_i A_i$ *with* $\alpha, \beta_i \in \frac{1}{2}\mathbb{Z}$*. If* α *or* β_i *for some* i *is in* $\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$*, then at least* 4 *of the* β_j*'s are in* $\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$*. If* $\alpha \in \mathbb{Z}$*, then at least* 8 *of the* β_j*'s are in* $\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ *or* $\forall j$*,* $\beta_j \in \mathbb{Z}$*.*

2.2.3. *The Pell-Fermat equation and construction of* (-2)*-classes.* We are looking for a polarization L' and a class A_1' of the form

$$\begin{array}{c} A_1' = \beta L - \alpha A_1 \\ L' = bL - aA_1 \end{array},$$

with $\alpha, \beta, a, b \in \mathbb{N} \setminus \{0\}$ such that one has $A_1'^2 = -2$, $L'A_1' = 0$ and $L'^2 = L^2 = 4t$. These three conditions are respectively

$$\begin{array}{c} \alpha^2 - 2t\beta^2 = 1 \\ 2tb\beta = a\alpha \\ a^2 = 2t(b^2-1) \end{array}, \tag{2.3}$$

the first expresses that A_1' is a (-2)-class, the second that this (-2)-class is disjoint from the polarisation L', the third that $L^2 = L'^2$. We will use the divisor L' and the property that $L'A_1' = 0$ in order to show that A_1' can be represented by an irreducible curve.

Lemma 9. *There are non-trivial solutions to the three equations* (2.3) *if and only if* $2t$ *is not a square. In that case, if* (α, β) *is a solution of the first equation in* (2.3)*, one has*

$$(a, b) = (2t\beta, \alpha).$$

Proof. In order that the Pell-Fermat equation (2.1) admits a solution, we need that $2t$ is not a square. Let us suppose that this is the case and let (α, β) be such a solution, which we can suppose with $\alpha > 0$, $\beta > 0$. By replacing $a = 2t\frac{\beta}{\alpha}b$ in the third equation, one gets

$$4t^2b^2\beta^2 = 2t\alpha^2(b^2-1),$$

which is equivalent to

$$b^2(\alpha^2 - 2t\beta^2) = \alpha^2,$$

since $\alpha^2 - 2t\beta^2 = 1$ and we search solutions with $b > 0$, we obtain $b = \alpha$. Then by the third equality, we get $a^2 = 2t(\alpha^2 - 1)$ and equality $\alpha^2 - 1 = 2t\beta^2$ implies $a = 2t\beta$. □

2.3. **The** β_0 **odd case.** Suppose $2t$ is not a square and let (α_0, β_0) be a solution of equation (2.1). Let us suppose that β_0 is odd and let us define

$$A_1' = \beta_0 L - \alpha_0 A_1,$$

which is a (-2)-class. Then

Proposition 10. *The* (-2)*-class* $A_1' = \beta_0 L - \alpha_0 A_1$ *cannot be the class of an irreducible rational curve.*

Proof. Suppose that A_1' is irreducible. Then we have two Nikulin configurations

$$\mathcal{C} = \sum_{i=1}^{16} A_i,\, \mathcal{C}' = A_1' + \sum_{i=2}^{16} A_i.$$

Since Nikulin configurations are 2-divisible (see [12]), the divisor $A_1 + A_1'$ is 2-divisible and

$$\frac{1}{2}(A_1 + A_1') = \frac{\beta_0}{2}L - \frac{\alpha_0 - 1}{2}A_1$$

is an integral class. Since β_0 is odd by assumption and α_0 must be odd by the equality $\alpha_0^2 - 2t\beta_0^2 = 1$, it follows that $\frac{L}{2} \in \mathrm{NS}(X)$, which contradicts L being a primitive class. $\square$

We will come back to this case (when β_0 is odd) in Subsection 3.1.

2.4. **The β_0 even case: $\beta_0 L - \alpha_0 A_1$ is the class of a (-2)-curve.** Assume $2t$ is not a square. Let (α_0, β_0) be the fundamental solution of the Pell-Fermat equation (2.1). We assume in this section that β_0 is even and we define as in Subsection 2.2.3 the classes:

$$A_1' = \beta_0 L - \alpha_0 A_1,\ L' = \alpha_0 L - 2t\beta_0 A_1.$$

One has $A_1'^2 = -2$, $L'A_1' = 0$, $L'^2 = L^2 = 4t$.

Proposition 11. *Suppose that β_0 is even. The class L' is big and nef and the classes $A_1', A_2 \dots, A_{16}$ are the only (-2)-classes contracted by L'.*

Proof. The class L' is nef if and only if for any (-2)-curves Γ, one has $\Gamma L' \geq 0$. Let

$$\Gamma = uL - \sum_{i=1}^{16} v_i A_i$$

be a (-2)-curve (thus $\sum v_i^2 - 2tu^2 = 1$) ; we recall that by Lemma 8, if one coefficient u or v_i is in $\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$, then at least four of the v_i's are in $\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$. Suppose that

$$\Gamma L' \leq 0,$$

this is equivalent to

$$u\alpha_0 \leq v_1\beta_0,$$

in other words $u \leq \frac{\beta_0}{\alpha_0}v_1$, thus

$$\sum_{i\geq 1} v_i^2 = 2tu^2 + 1 \leq 2t\left(\frac{\beta_0^2}{\alpha_0^2}v_1^2\right) + 1 = 2t\beta_0^2\left(\frac{v_1^2}{\alpha_0^2}\right) + 1$$

and therefore using the relation $\alpha_0^2 - 2t\beta_0^2 = 1$, one obtains

$$\sum_{i\geq 1} v_i^2 \leq (\alpha_0^2 - 1)\left(\frac{v_1^2}{\alpha_0^2}\right) + 1$$

and therefore

$$\sum_{i\geq 2} v_i^2 \leq 1 - \frac{v_1^2}{\alpha_0^2}.$$

Apart from the trivial cases of curves $\Gamma = A_i$ for $i > 1$, one can suppose $u > 0$. If $v_1 > \frac{1}{2}\alpha_0$ then $\sum_{i\geq 2} v_i^2 < \frac{3}{4}$, then by Lemma 8, one gets $v_2 = \cdots = v_{16} = 0$, $v_1 = \alpha_0$, $u = \beta_0$, so that $\Gamma = A_1'$, for which $\Gamma L' = 0$. Thus one can suppose that

$$0 < v_1 \leq \frac{1}{2}\alpha_0 \tag{2.4}$$

(if $v_1 = 0$, then $u = 0$, which we already excluded) and, up to permutation of the indices: $v_2 = v_3 = v_4 = \frac{1}{2}$ (since $\sum_{i\geq 2} v_i^2 < 1$ and by the structure of the Néron-Severi group as described in Lemma 8). The relation $\sum v_i^2 - 2tu^2 = 1$ is now $v_1^2 - 2tu^2 = \frac{1}{4}$, which is

$$(2v_1)^2 - 2t(2u)^2 = 1.$$

Defining $V = 2v_1 \in \mathbb{N}$ and $U = 2u \in \mathbb{N}$, we see that (U, V) is a solution of the Pell-Fermat equation $\beta^2 - 2t\alpha^2 = 1$. Moreover, by Equation (2.4) we know that $0 < V \leq \alpha_0$. Since by hypothesis (α_0, β_0) is the primitive solution, and $V > 0$, we have $\alpha_0 \leq V$. Therefore, by remark 2: $V = \alpha_0$, which implies that $U = \beta_0$, thus $v_1 = \frac{1}{2}\alpha_0$, $u = \frac{1}{2}\beta_0$, and thus (for $\Gamma \neq A_2, \dots, A_{16}$) we have

$$\Gamma L' \leq 0$$

if and only if $\Gamma L' = 0$ and Γ has the form $\Gamma = \frac{1}{2}(\beta_0 L - \alpha_0 A_1 - A_2 - A_3 - A_4)$. But by Lemma 8, in order for Γ to be in $\mathrm{NS}(X)$, the integer β_0 must be odd, which is impossible by our assumption on β_0.

In conclusion, we obtain that L' is big and nef, and if β_0 is even, then the only (-2)-classes Γ such that $\Gamma L' = 0$ are $A_1', A_2, \dots, A_{16}$. □

Let us prove the following result:

Proposition 12. *Suppose that β_0 is even. The line bundle $3L'$ (where $L' = \alpha_0 L - 2t\beta_0 A_1$) defines a morphism $\phi_{3L'} : X \to \mathbb{P}^N$ which is birational onto its image and contracts exactly the divisor $A_1' = \beta_0 L - \alpha_0 A_1$ and the 15 (-2)-curves A_i, $i \geq 2$.*

Proof. By [14, Section 3.8] either $|3L'|$ has no fixed part or $3L' = aE + \Gamma$, where $|E|$ is a free pencil, and Γ is a (-2)-curve with $E\Gamma = 1$. However if $E\Gamma = 1$, then $3L'E = aE^2 + 1$, but since $E^2 = 0$ this is impossible. Thus $|3L'|$ has no fixed part; moreover by [18, Corollary 3.2], it has then no base points.

Let us prove that the morphism $\phi_{3L'}$ has degree one, i.e. that $|3L'|$ is not hyperelliptic (see [18, Section 4]). By loc. cit., $|3L'|$ is hyperelliptic only if there exists a genus 2 curve C such that $3L' = 2C$ or there exists an elliptic curve E such that $(3L')E = 2$. Suppose we are in the first case. Since $C^2 = 2$, one has $9 \cdot 4t = 8$, which is impossible. The second alternative is also readily impossible. Thus the morphism $\phi_{3L'}$ has degree one. Moreover since $3L'A_1' = 3L'A_2 = \cdots = 3L'A_{16} = 0$, the 16 divisors are contracted by $\phi_{3L'}$. □

We obtain:

Corollary 13. *Suppose that β_0 is even. The divisor A_1' is an irreducible (-2)-curve.*

Proof. Since $A_1'^2 = -2$ and $LA_1' \geq 0$, by Riemann-Roch Theorem we can assume it is effective. Let B be one of the divisors $A_1', A_2, \dots, A_{16}$. One has $3L'B = 0$, thus the linear system $|3L'|$ contracts B to a singular point. Since the Picard number of the K3 surface $X = \mathrm{Km}(B)$ is 17, that singularity must be a node and therefore A_1' is irreducible. □

2.5. **Two Kummer structures in case β_0 even and the negative Pell-Fermat equation is not solvable.** Suppose that $2t$ is not a square and let (α_0, β_0) be the fundamental solution of the Pell-Fermat equation

$$\lambda^2 - 2t\mu^2 = 1. \tag{2.5}$$

We suppose that β_0 is even, and we recall that $A_1' = \beta_0 L - \alpha_0 A_1$ in $\mathrm{NS}(X)$. The aim of this Section is to prove the following:

Theorem 14. *Suppose that $t \geq 2$ and the negative Pell-Fermat equation*

$$\lambda^2 - 2t\mu^2 = -1 \tag{2.6}$$

has no solution. There is no automorphism f of X sending the configuration $\mathcal{C} = \sum_{i=1}^{16} A_i$ to the configuration $\mathcal{C}' = A_1' + \sum_{i=2}^{16} A_i$.

Remark. We recall that the Nikulin configurations $\mathcal{C}$ and $\mathcal{C}'$ define two distinct Kummer structures if and only if there is no automorphism sending $\mathcal{C}$ to $\mathcal{C}'$ (see [9]; a proof is given in [15, Proposition 21]).

In order to prove Theorem 14, let us suppose that such an automorphism f exists. The group of translations by the 2-torsion points on B acts on $X = \mathrm{Km}(B)$ and that action is transitive on the set of curves $A_1, \dots, A_{16}$. Thus, up to changing f by $f \circ t$ (where t is such a translation), one can suppose that the image of A_1 is A_1'. Then the automorphism f induces a permutation of the curves $A_2, \dots, A_{16}$. The (-2)-curve $A_1'' = f^2(A_1) = f(A_1')$ is orthogonal to the 15 curves A_i, $i > 1$ and therefore its class is in the group generated by L and A_1. Let $\lambda, \mu \in \mathbb{Z}$ such that $A_1'' = \lambda A_1 + \mu L$, so that (λ, μ) is a solution of the Pell-Fermat equation (2.5). Let us prove:

Lemma 15. *Let $C = \lambda A_1 + \mu L$ be an effective (-2)-class. Then there exists $u, v \in \mathbb{N}$ such that $C = uA_1 + vA_1'$, in particular the only (-2)-curves in the lattice generated by L and A_1 are A_1 and A_1'.*

Proof. If (λ, μ) is a solution of equation (2.5), then so are $(\pm\lambda, \pm\mu)$. We say that a solution is positive if $\lambda \geq 0$ and $\mu \geq 0$. Let us identify $\mathbb{Z}^2$ with $\mathbb{Z}[\sqrt{2t}]$ by sending (λ, μ) to $\lambda + \mu\sqrt{2t}$. The solutions of equation 2.5 are units of the ring $\mathbb{Z}[\sqrt{2t}]$. Let $\alpha_0 + \beta_0\sqrt{2t}$ $(\alpha_0, \beta_0 \in \mathbb{N}^*)$ be the fundamental solution to equation (2.5). The solutions with positive coefficients are the elements of the form

$$\lambda_m + \mu_m\sqrt{2t} = (\alpha_0 + \beta_0\sqrt{2t})^m,\ m \in \mathbb{N}.$$

An effective (-2)-class $C = \lambda A_1 + \mu L$ either equals A_1 or satisfies $CL > 0$ and $CA_1 > 0$, therefore $\mu > 0$ and $\lambda < 0$. Thus if $C \neq A_1$, there exists m

such that $C = -\lambda_m A_1 + \mu_m L$. Since $A_1' = \beta_0 L - \alpha_0 A_1$ corresponds to the fundamental solution of equation (2.5), we have $L = \frac{1}{\beta_0}(A_1' + \alpha_0 A_1)$ and we obtain

$$C = -\lambda_m A_1 + \frac{\mu_m}{\beta_0}(A_1' + \alpha_0 A_1) = \frac{\mu_m}{\beta_0} A_1' + (\frac{\alpha_0}{\beta_0} b_m - \lambda_m) A_1$$

and the Lemma is proved if the coefficients $u_m = \frac{\mu_m}{\beta_0}$ and $v_m = \frac{\alpha_0}{\beta_0}\mu_m - \lambda_m$ are both positive and in $\mathbb{Z}$. Using the fact that

$$\lambda_{m+1} + \mu_{m+1}\sqrt{2t} = (\alpha_0 + \sqrt{2t}\beta_0)(\lambda_m + \mu_m\sqrt{2t}),$$

we obtain

$$\begin{array}{c} \lambda_{m+1} = \alpha_0 \lambda_m + 2t\beta_0 \mu_m \\ \mu_{m+1} = \alpha_0 \mu_m + \beta_0 \lambda_m \end{array}.$$

Then we compute that

$$u_{m+1} = \frac{\mu_{m+1}}{\beta_0} = \alpha_0 \frac{\mu_m}{\beta_0} + \lambda_m, \;\; v_{m+1} = \frac{\alpha_0}{\beta_0}\mu_{m+1} - \lambda_{m+1} = \frac{\mu_m}{\beta_0}$$

and by induction we conclude that u_m, v_m are in $\mathbb{N}$ for any $m \geq 1$. □

Lemma 15 implies that $A_1'' = A_1$ i.e. f permutes A_1 and A_1'. Let us continue the proof of Theorem 14:

Since the automorphism f preserves the set

$$B = \{A_1', A_1, \dots, A_{16}\},$$

it acts with finite order n_0 on B. Since B is a $\mathbb{Q}$-basis of $\mathrm{NS}(X) \otimes \mathbb{Q}$, the automorphism f^{n_0} acts trivially on $\mathrm{NS}(X)$, thus it preserves an ample class, and by [8, Proposition 5.3.3], the automorphism f^{n_0} has finite order, which proves that f has finite order. Up to taking an odd power of f, one can suppose that f has order 2^m for some $m \in \mathbb{N}^*$. Suppose $m = 1$, i.e. f is an involution. Then the integral class

$$D = \frac{1}{2}(A_1 + A_1') = \frac{\beta_0}{2} L - \frac{\alpha_0 - 1}{2} A_1$$

(recall that β_0 is even and α_0 is odd) is fixed by f. Let us define

$$d_0 = GCD(\beta_0, \alpha_0 - 1)/2,$$

then the class $D' = \frac{1}{d_0} D$ is primitive in $\mathrm{NS}(X)$. We have

$$D'^2 = \frac{\alpha_0 - 1}{d_0^2} \in \mathbb{Z},$$

and in fact, since $\mathrm{NS}(X)$ is an even lattice, D'^2 is even, so that $\frac{\alpha_0 - 1}{2d_0^2} \in \mathbb{Z}$. Let us define

$$W = \frac{2d_0^2}{\alpha_0 - 1} D'.$$

Let $\mathrm{NS}(X)^f$ be the sub-lattice of $\mathrm{NS}(X)$ fixed by f. By Lemma 8, for any class E in $\mathrm{NS}(X)^f$, there exists $a, b_2, \dots, b_{16} \in \mathbb{Z}$ such that

$$E = \frac{1}{2}(aD' + \sum_{i=2}^{16} b_i A_i).$$

Since $WD' = 2$, we get $WE = a \in \mathbb{Z}$, therefore W is an element of the dual of $\mathrm{NS}(X)^f$, and the discriminant group of $\mathrm{NS}(X)^f$ contains the sub-group isomorphic to $\mathbb{Z}/\frac{\alpha_0-1}{2d_0^2}\mathbb{Z}$ generated by the class of W.

Case when f is a non-symplectic involution. Suppose that f is non-symplectic. Then (see e.g. [1]) $\mathrm{NS}(X)^f$ is a 2-elementary lattice, which means that the discriminant group of $\mathrm{NS}(X)^f$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^h$ for some positive integer h. Since $\mathbb{Z}/\frac{\alpha_0-1}{2d_0^2}\mathbb{Z}$ is a sub-group of the discriminant group, and f is supposed to be non-symplectic, we get two cases:

$$\frac{\alpha_0-1}{2d_0^2} \in \{1,2\}.$$

Sub-case $\frac{\alpha_0-1}{2d_0^2}=1$. If $\frac{\alpha_0-1}{2d_0^2}=1$, then $\alpha_0 = 1+2d_0^2$. Since $d_0|\beta_0/2$, there exists $e_0 \in \mathbb{Z}$ such that $\beta_0 = 2e_0d_0$. From the relation $\alpha_0^2 - 2t\beta_0^2 = 1$, it follows that d_0 and e_0 satisfy the negative Pell-Fermat equation

$$d_0^2 - 2te_0^2 = -1.$$

Conversely, suppose that (d_0, e_0) is the primitive solution to the equation $d^2 - 2te^2 = -1$. The fundamental solution to the Pell-Fermat equation is

$$\alpha_0 + \beta_0\sqrt{2t} = (d_0 + e_0\sqrt{2t})^2 = 2d_0^2 + 1 - 2d_0e_0\sqrt{2t}$$

and $\frac{\alpha_0-1}{2d_0^2}=1$. Since, in the hypothesis of Theorem 14, we supposed that the negative Pell-Fermat equation has no solution, the case $\frac{\alpha_0-1}{2d_0^2}=1$ is excluded.

Remark 16. In [15], we studied the cases with $2t = k(k+1)$. Then $D = 2D' = 2L - 2kA_1$, which gives $d_0 = 1$ and $\alpha_0 = 2k+1$. The proof of [15, Theorem 19] implies that the negative Pell-Fermat equation $x^2 - k(k+1)y^2 = -1$ has no solution for $k \geq 2$.

Sub-case $\frac{\alpha_0-1}{2d_0^2}=2$. If $\frac{\alpha_0-1}{2d_0^2}=2$, then $\alpha_0 = 1+4d_0^2$. Since $d_0|\beta_0/2$, there exists $e_0 \in \mathbb{Z}$ such that $\beta_0 = 2e_0d_0$. From the relation $\alpha_0^2 - 2t\beta_0^2 = 1$, it follows that d_0 and e_0 satisfy the relation

$$2d_0^2 - te_0^2 = -1, \tag{2.7}$$

(conversely, if (d_0, e_0) is a solution of (2.7), then $\alpha_0 = 1+4d_0^2$ and $\beta_0 = 2e_0d_0$ is a solution of the Pell-Fermat equation (2.5)). We have

$$D' = \frac{1}{d_0}D = e_0L - 2d_0A_1,$$

with $D'^2 = 4, D'A_1 = D'A_1' = 4d_0, D'A_j = 0$ for $j \in \{2, \dots, 16\}$. Let us prove that

Proposition 17. *The divisor D' is nef, the linear system $|D'|$ is base point free, non hyperelliptic and defines a morphism $\varphi : X \to \mathbb{P}^3$ such that $\varphi(X) = Y$ is a quartic surface with* 15 *nodes, which are images of the disjoint curves $A_j, j \geq 2$.*

Proof. Let us prove that D' is nef. Let $\Gamma = \alpha L - \sum_{i=1}^{16} \beta_i A_i$ be a (-2)-curve:

$$2t\alpha^2 - \sum_{i=1}^{16} \beta_i^2 = -1 \tag{2.8}$$

where $\alpha, \beta_i \in \frac{1}{2}\mathbb{Z}$ are subject to the restrictions in Lemma 8. Suppose that $D'\Gamma \leq 0$, which is equivalent to

$$\frac{te_0\alpha}{d_0} \leq \beta_1.$$

Using this relation in equation (2.8), we get

$$-1 \leq 2t\alpha^2 - \left(\frac{te_0\alpha}{d_0}\right)^2 - \sum_{i=2}^{16} \beta_i^2.$$

By using the relation (2.7), this is equivalent to

$$-1 \leq -\frac{\alpha^2 t}{d_0^2} - \sum_{i=2}^{16} \beta_i^2 \iff \frac{\alpha^2 t}{d_0^2} + \sum_{i=2}^{16} \beta_i^2 \leq 1. \tag{2.9}$$

Suppose that α is an integer. Then from Lemma 8 and equation (2.9), we have either $\forall j \geq 2$, $\beta_j = 0$ or $\exists k \geq 2$, $\beta_k = 1$ and $\forall j \geq 2$, $j \neq k$, $\beta_j = 0$. In the first case $\Gamma = \alpha L - \beta_1 A_1$ with $\beta_1 \in \mathbb{Z}$, and from Lemma 15, either $\Gamma = A_1$ or $\Gamma = A_1'$. Since $D'A_1 = D'A_1' = 4d_0 > 0$, this is impossible. In the second case, $\sum_{i=2}^{16} \beta_i^2 = 1$ implies $\alpha = 0$ and $\Gamma = A_k$ for $k \geq 2$, and indeed $D'A_k = 0$. It remains the case when α is an half-integer. Then three of the β_i with $i > 1$ are equal to $\frac{1}{2}$, the others are 0, and β_1 is in $\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$. Let $a, b \in \mathbb{Z}$ be the odd integers such that $\alpha = \frac{a}{2}$, $\beta_1 = \frac{b}{2}$. Equation (2.8) becomes $b^2 - 2ta^2 = 1$. By hypothesis, the fundamental solution (α_0, β_0) of that Pell-Fermat equation $\alpha^2 - 2t\beta^2 = 1$ is such that β_0 is even. Then an easy induction shows that every solution (α, β) is also such that β is even. Hence that case is also impossible, and we obtain that D' is nef with $D'\Gamma = 0$ for a (-2)-curve Γ if and only if $\Gamma = A_k$ for $k \geq 2$.

Let us prove that the linear system $|D'| = |e_0 L - 2d_0 A_1|$ is base point free. Suppose that this is not the case. Then (see [18]) there exist an elliptic curve E and a (-2)-curve $\Gamma = \alpha L - \sum_{i=1}^{16} \beta_i A_i$, $\alpha, \beta_i \in \frac{1}{2}\mathbb{Z}$, such that $E\Gamma = 1$ and $D' = 3E + \Gamma$. One has $D'\Gamma = (3E + \Gamma)\Gamma = 1$ but

$$D'\Gamma = (e_0 L - 2d_0 A_1)(\alpha L - \sum_{i=1}^{16} \beta_i A_i) = 2(2\alpha e_0 t - 2\beta_1 d_0) \in 2\mathbb{Z}.$$

This is a contradiction, and we conclude that $|D'|$ is base-point free.

Let us study the degree of the morphism defined by the linear system $|D'|$. By [18], the morphism has degree 2 if and only if there exists en elliptic curve

$$E = \frac{a}{2}L - \sum_{i=1}^{16} \frac{b_i}{2} A_i, \ a, b_i \in \mathbb{Z}$$

such that $D'E = 2$. Equality $D'E = 2$ is equivalent to

$$ate_0 - b_1 d_0 = 1. \tag{2.10}$$

Since E is an elliptic curve, one has

$$2ta^2 = \sum_{i=1}^{16} b_i^2. \tag{2.11}$$

Let us define $S = \sum_{i=2}^{16} b_i^2 \in \mathbb{N}$. The equations (2.10) and (2.11) give

$$\begin{cases} 2a^2t^2 = \frac{2}{e_0^2}(1+b_1d_0)^2 \\ 2t^2a^2 = tb_1^2 + tS \end{cases},$$

therefore

$$2(1+b_1d_0)^2 = te_0^2b_1^2 + te_0^2S,$$

which is equivalent to

$$(e_0^2t - 2d_0^2)b_1^2 - 4d_0b_1 + e_0^2tS - 2 = 0. \tag{2.12}$$

Since $e_0^2t - 2d_0^2 = 1$, the reduced discriminant of equation (2.12) in b_1 is

$$\Delta = 4d_0^2 + 2 - e_0^2tS = 4d_0^2 + 2 - (1+2d_0^2)S = (1+2d_0^2)(2-S).$$

Since b_1 is an integral solution, that implies $S \in \{0,1,2\}$. Lemma 8 implies that cases $S = 1$ and $S = 2$ are not possible. The case $S = 0$ is not possible either since $2(1+2d_0^2)$ is not a square. We thus proved that the morphism defined by $|D'|$ has degree 1 and that concludes the proof of Proposition 17. $\square$

Using that model $Y \hookrightarrow \mathbb{P}^3$, we can use the same reasoning as in [15, Proof of Theorem 19] (which was made for the case $t = 3$, with $d_0 = e_0 = 1$) in order to prove that sub-case $\frac{\alpha_0 - 1}{2d_0^2} = 2$ is also impossible if we suppose that the non-symplectic automorphism f exists.

Case when f is a symplectic involution. Suppose f is a symplectic involution. It fixes $A_1 + A_1'$, it permutes the classes A_j $(j > 1)$ by pairs, thus the number s of fixed A_j is odd. A symplectic automorphism acts trivially on the transcendental lattice T_X, which in our situation has rank 5. Therefore the trace of f on $H^2(X,\mathbb{Z})$ equals $5 + \mathrm{rk}(\mathrm{NS}(X)^f) \geq 6 + s > 6$. But the trace of a symplectic involution equals 6 (see e.g. [19, Section 1.2]). This is a contradiction, thus f cannot have order 2 and the integer m (such that the order of f is 2^m) is larger than 1.

Remaining cases. We know that f has order $2^m > 1$. The automorphism $g = f^{2^{m-1}}$ has order 2 and $g(A_1) = A_1$, $g(A_1') = A_1'$, thus $g(L) = L$. There are curves A_i, $i > 1$ such that $f(A_i) = A_i$ (say s of such curves, s is odd since A_1 is fixed) and the remaining curves A_j are permuted 2 by 2 (there are $s' = \frac{1}{2}(15 - s)$ such pairs). Let $\mathcal{L}'$ be the sub-lattice generated by L, A_1 and the fix classes A_i, $A_j + g(A_j)$. It is a finite index sub-lattice of the fixed lattice $\mathrm{NS}(X)^g$ and its discriminant group is

$$\mathbb{Z}/4t\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^{s+1} \times (\mathbb{Z}/4\mathbb{Z})^{s'}.$$

By the same reasoning as before, the involution g must be symplectic as soon as $t>1$. However the trace of g is $8+s>6$, thus g cannot be symplectic either. Therefore we conclude that such an automorphism f does not exist, which conclude the proof of Theorem 14.

Remark 18. In the course of the proof of Theorem 14, using the divisor D', we obtained a model of our K3 surface as a quartic in $\mathbb{P}^3$ with 15 nodes, as soon as Equation (2.7) has a solution. This is the case for example for $t=3,9,11,19,27...$

2.6. **When the negative Pell-Fermat equation has a solution.** Suppose that the negative Pell-Fermat equation $\lambda^2-2t\mu^2=-1$ has a solution and let (d_0,e_0) be the fundamental solution. Then $(\alpha_0,\beta_0)=(1+2d_0^2,2e_0d_0)$ is the fundamental solution of the Pell-Fermat equation $\lambda^2-2t\mu^2=1$, in particular β_0 is even. Moreover we have $A_1+A_1'=2d_0D'$ (we keep the notation of the previous Section) with

$$D'=e_0L-d_0A_1$$

such that $D'^2=2$. Let us prove that

Proposition 19. *The divisor D' is nef. We have $D'\Gamma=0$ for a (-2)-curve Γ if and only if $\Gamma=A_j$, for $j\in\{2,\dots,16\}$. The linear system $|D'|$ is base point free. It defines a double cover $\varphi:X\to\mathbb{P}^2$ which contracts exactly the 15 curves $A_j, j\geq 2$ to 15 singular points of a sextic curve which is the union of 6 lines.*
The involution σ defined by the double cover φ exchanges A_1 and A_1' and the two Nikulin configurations $\mathcal{C},\mathcal{C}'$, which therefore give the same Kummer structure.

Proof. Let us prove that D' is nef. Let $\Gamma=\alpha L-\sum_{i=1}^{16}\beta_iA_i$ be a (-2)-curve: $2t\alpha^2-\sum_{i=1}^{16}\beta_i^2=-1$, where $\alpha,\beta_i\in\frac{1}{2}\mathbb{Z}$ are subject to the restrictions in Lemma 8. We suppose that $D'\Gamma\leq 0$. This is equivalent to:

$$D'\Gamma=(e_0L-d_0A_1)(\alpha L-\sum_{i=1}^{16}\beta_iA_i)=4t\alpha e_0-2\beta_1d_0\leq 0.$$

Then $\beta_1\geq\frac{2t\alpha e_0}{d_0}$, thus $-\left(\frac{2t\alpha e_0}{d_0}\right)^2\geq-\beta_1^2$. From the relation $2t\alpha^2+1-\beta_1^2=S$, where $S=\sum_{i=2}^{16}\beta_i^2$, we get

$$S\leq 1+2t\alpha^2-\left(\frac{2t\alpha e_0}{d_0}\right)^2.$$

By using the relation $d_0^2-2te_0^2=-1$, we obtain

$$S+\frac{2t\alpha^2}{d_0^2}\leq 1. \tag{2.13}$$

We can then follow the same proof as for the divisor in Proposition 17, and we conclude that D' is nef, with $D'\Gamma=0$ for a (-2)-curve Γ if and only if $\Gamma=A_k$ for $k\in\{2,\dots,16\}$.

Let us prove that the linear system $|D'| = |e_0L - d_0A_1|$ is base point free. Suppose that this is not the case. Then (see [18]) there exist an elliptic curve E and a (-2)-curve Γ such that $D' = 2E + \Gamma$ and $E\Gamma = 1$. Since we have $D'\Gamma = (2E + \Gamma)\Gamma = 0$, there exists $k \geq 2$ such that $\Gamma = A_k$. Thus $D' = 2E + A_k$, and then $e_0L - d_0A_1 - A_k = 2E$, which is impossible by Lemma 8. Therefore $|D'|$ is base point free and defines a double cover of the plane that contracts the 15 disjoint (-2)-curves $A_j, j \geq 2$ to points. Since $A_1 + A_1' = 2d_0D'$, the divisor $A_1 + A_1'$ is the pull-back of a plane rational cuspidal curve of degree $2d_0$, and the involution exchanges the two curves A_1, A_1'. It is well-known that a sextic curve with 15 singular points is the union of 6 lines in general position. □

3. Further examples

3.1. An example of a Nikulin configuration when β_0 is odd. Let us study the $t = 4$ case. Then the fundamental solution (α_0, β_0) equals $(3, 1)$. This is the first case with β_0 odd (see Table in the Appendix). We have

$$A_1' = L - 3A_1,\ L' = 3L - 8A_1$$

and we already know that A_1' is not irreducible. In order to understand better what is happening, let us define

$$\begin{array}{l} A_1'' = \frac{1}{2}(L - 3A_1 - A_2 - A_3 - A_4) \\ A_2'' = \frac{1}{2}(L - A_1 - 3A_2 - A_3 - A_4) \\ A_3'' = \frac{1}{2}(L - A_1 - A_2 - 3A_3 - A_4) \\ A_4'' = \frac{1}{2}(L - A_1 - A_2 - A_3 - 3A_4) \end{array},$$

where the classes A_2, A_3, A_4 are chosen so that the classes A_j'' exists in $\mathrm{NS}(X)$ (which is possible by [6, Theorem 2.7], since $t = 0 \bmod 2$). We compute that these are (-2)-classes i.e. $A_j''^2 = -2$. Moreover we have

$$A_1' = 2A_1'' + A_2 + A_3 + A_4 \text{ and } A_1''L' = 0,$$

(but $A_i''L' \neq 0$, for $i = 2, 3, 4$). Let us also define

$$L_1 = 3L - 4(A_1 + A_2 + A_3 + A_4).$$

We remark that $L_1^2 = L^2 = 16$, $L_1A_i'' = 0$ and $A_i''A_j'' = 0$ for $i \neq j$ in $\{1, 2, 3, 4\}$.

Lemma 20. *The class L_1 is big and nef. Moreover if Γ is an effective (-2)-class, we have $L_1\Gamma \geq 0$ and $L_1\Gamma = 0$ if and only if Γ is one of the classes $A_1'', ..., A_4'', A_5, \ldots, A_{16}$.*

Proof. Let

$$\Gamma = aL - \sum_{i=1}^{16} b_iA_i$$

be an effective (-2)-class (thus $\sum b_i^2 - 8a^2 = 1$). One has

$$L_1\Gamma \leq 0$$

if and only if

$$6a \leq b_1 + b_2 + b_3 + b_4.$$

Suppose $\Gamma \notin \{A_5, \dots, A_{16}\}$. Then $a > 0$, $b_i \geq 0$ and equation $L_1\Gamma \leq 0$ is equivalent to

$$a^2 \leq \frac{1}{36}(b_1 + \dots + b_4)^2.$$

Using

$$(b_1 + \dots + b_4)^2 \leq 4(b_1^2 + \dots + b_4^2) \leq 4(b_1^2 + \dots + b_{16}^2)$$

we get

$$a^2 \leq \frac{1}{9}(b_1^2 + \dots + b_{16}^2)$$

and since $\sum b_i^2 = 8a^2 + 1$, we have

$$a^2 \leq \frac{1}{9}(8a^2 + 1),$$

thus $a^2 \leq 1$ and $a \in \{\frac{1}{2}, 1\}$. Suppose that $a = \frac{1}{2}$. Then $\sum_{i=1}^{i=16} b_i^2 = 3$ and either there are 12 b_i's equal to $\frac{1}{2}$ or (up to permutation of the indices) $b_1 = \frac{3}{2}$, $b_2 = b_3 = b_4 = \frac{1}{2}$. The first case is impossible since $3 = 6a > b_1 + b_2 + b_3 + b_4$. The second case corresponds to $A_1'', \dots, A_4''$, and then $L_1 A_j'' = 0$.
It remains to study the case $a = 1$, then $\sum b_i^2 = 9$ and

$$6 \leq b_1 + b_2 + b_3 + b_4.$$

That implies $b_i \leq \frac{5}{2}$. Up to permutation we can suppose that the largest b_i with $i \in \{1, 2, 3, 4\}$ is b_1. Suppose $b_1 = \frac{5}{2}$, then $\sum_{i \geq 2} b_i^2 = \frac{11}{4}$ and $b_2 + b_3 + b_4 \geq \frac{7}{2}$. One can suppose that b_2 is the largest among b_2, b_3, b_4, then there are two cases : $b_2 = 2$ or $b_2 = \frac{3}{2}$. The first case is impossible since one would obtain $\sum_{i \geq 2} b_i^2 > \frac{11}{4}$. Suppose $b_2 = \frac{3}{2}$, then $b_3^2 + b_4^2 = \frac{1}{2}$ and $b_3 + b_4 \geq 2$, but this is also impossible.
Suppose that $b_1 = 2$. Then $\sum_{i \geq 2} b_i^2 = 5$ and $b_2 + b_3 + b_4 \geq 4$. The largest b_i among b_2, b_3, b_4 (say it is b_2) is 2 or $\frac{3}{2}$. If $b_2 = 2$, then $b_3^2 + b_4^2 = 1$ and $b_3 + b_4 \geq 2$, which is impossible. If $b_2 = \frac{3}{2}$, then $b_3 + b_4 \geq \frac{5}{2}$ and $b_3^2 + b_4^2 = \frac{11}{4}$,thus $b_3 = \frac{3}{2}$ and $b_4 \geq 1$ gets a contradiction.
It remains $b_1 = \frac{3}{2}$, but then $b_2 = b_3 = b_4 = \frac{3}{2}$. That implies $b_j = 0$ for $j \geq 5$. But $L - \frac{3}{2}(A_1 + A_2 + A_3 + A_4)$ is not in the Néron-Severi group of the surface (see Lemma 8).
We thus proved that the only effective (-2)-classes Γ such that $L_1\Gamma \leq 0$ are $A_1'', \dots, A_4'', A_5, \dots, A_{16}$ and moreover $L_1\Gamma = 0$ for these classes. Thus L_1 is nef and big. $\square$

As before, one can prove that the linear system $3L_1$ define a degree 1 morphism which contracts $A_1'', \dots, A_4'', A_5, \dots, A_{16}$ onto singularities. Since we assume that the Kummer surface is generic, it has Picard number 17 and we conclude that the divisors $A_1'', \dots, A_4''$ are irreducible. Therefore:

Corollary 21. *The* 16 (-2)*-curves* $A_1'',\dots,A_4'',A_5,\dots,A_{16}$ *form a Nikulin configuration* $\mathcal{C}'$ *on the K3 surface* X*. The* (-2)*-class* A_1' *is not irreducible and* $A_1' = 2A_1'' + A_2 + A_3 + A_4$*.*

Remark 22. i) One can check that the class L' is big and nef; the image of X by the linear system $|3L'|$ is a surface with 12 nodal singularities and one D_4 singularity obtained by contracting A_1'', A_2, A_3, A_4.
ii) We do not know yet if $\mathcal{C}'$ is another Kummer structure on the Kummer surface X, we intend to study that problem in a forthcoming paper.

3.2. **An example of a Nikulin configuration when $2t$ is a square.** Let us consider the case $t = 2$ i.e. A is a $(1, 2)$-polarized abelian surface. Then $2t$ is a square and the method in Section 2.4 does not apply. We start by recalling the following

Remark 23. Since t is even, by [6, Theorem 2.7 and Remark 2.10], up to re-labelling the 16 curves (-2)-curves A_j, we can suppose that the classes

$$F_1 = \tfrac{1}{2}(L - A_1 - A_2 - A_3 - A_4), \qquad F_2 = \tfrac{1}{2}(L - A_5 - A_6 - A_7 - A_8),$$
$$F_3 = \tfrac{1}{2}(L - A_9 - A_{10} - A_{11} - A_{12}), \quad F_4 = \tfrac{1}{2}(L - A_{13} - A_{14} - A_{15} - A_{16})$$

are contained in $\mathrm{NS}(X)$. For $j \in \{1, ..., 4\}$, we define

$$B_j = F_2 - A_j$$

and for $j \in \{5, ..., 8\}$, we define

$$B_j = F_1 - A_j.$$

These are (-2)-classes; they are effective since $LB_j > 0$. We check moreover that

$$B_j B_k = -2\delta_{jk},$$

where δ_{jk} is the Kronecker symbol. Let us prove the following result:

Proposition 24. *The classes* $B_1,\dots,B_8,A_9,\dots,A_{16}$ *are* 16 *disjoint* (-2)*-curves.*

Proof. We have $B_k A_j = 0$ for $k \in \{1,\dots,8\}$ and $j \in \{9,\dots,16\}$. It remains to prove that $B_1,\dots,B_8$ are irreducible. We compute that $F_1^2 = 0 = F_2^2$, $F_1F_2 = 2$. Since $LF_1 = 4$, the divisor F_1 is effective. The fact that F_1 is nef can be found in [6, Proposition 4.6], but for completeness, let us prove it here. Suppose that F_1 is not nef. Then there exist a (-2)-curve Γ such that $F_1\Gamma = -a < 0$. By [8, Remark 8.2.13], the divisor $E = F_1 - a\Gamma$ is effective, moreover

$$E^2 = 0,\ E\Gamma = a,\ (\text{and } F_1 = E + a\Gamma).$$

Let us write $E = \alpha L - \sum \beta_i A_i$ with $\alpha \in \frac{1}{2}\mathbb{Z}$, and $\alpha > 0$ since E is effective and the lattice $L^\perp$ is negative definite. Since L is nef and

$$4 = F_1 L = 8\alpha + aL\Gamma$$

with $a > 0$, that forces $\alpha = \frac{1}{2}$ and $L\Gamma = 0$. Thus Γ is one of the curves A_j. But for these curves $F_1 A_j \in \{0, 1\}$ is not negative, a contradiction. Therefore F_1 is nef and the linear system $|F_1|$ has no base points. We prove similarly that all the linear system $|F_k|$, $k \in \{1, 2, 3, 4\}$ have no base points, and then defines a fibration $\psi_k : X \to \mathbb{P}^1$. Since $F_1 A_j = 1$ for $j \in \{1, ..., 4\}$, the fibration ψ_1 has connected fibers. By the same kind of argument so is ψ_2. For $k \in \{5, ..., 16\}$, let us define

$$C_k = F_1 - A_k$$

(so that in fact $B_k = C_k$ for $k \in \{5, ..., 8\}$). The divisor C_k is an effective (-2)-class and the 12 divisors

$$C_k + A_k,\ k \in \{5, \dots, 16\}$$

are distinct singular fibers of ψ_1, with $A_k C_k = 2$. We now use [3, Proposition 11.4, Chapter III]: the Euler characteristic of X (equal to 24) is the sum $\sum_s e(f_s)$ of the Euler numbers of all the singular fibers. By the Kodaira classification of singular fibers of elliptic fibrations (see e.g. [3, Table 3, Chapter V, Section 7]), the reducible fibers $f_s = C_k + A_k$ for $k \geq 5$ satisfy $e(f_s) \geq 2$. Moreover, by the above cited Table, a singular fiber f_s containing a smooth rational curve satisfies $e(f_s) = 2$ if and only if it is the union of two (-2)-curves D_1, D_2 with $D_1 D_2 = 2$ and meeting transversally. Computing the Euler characteristic of X, we see that necessarily $e(C_k + A_k) = 2$, for $k \in \{5, \dots, 16\}$ and therefore the curves $B_k = C_k$, $k \in \{5, ..., 8\}$ are irreducible (-2)-curves. We proceed in a similar way with ψ_2 for the curves B_k with $k \in \{1, ..., 4\}$, and we thus obtain the claimed result. □

Remark 25. i) By the Proposition 24, we see that the elliptic fibration defined by F_1 contains 12 fibers of type I_2. By general results on elliptic K3 surfaces, the rank ρ of the Néron-Severi group is $14 = 12 + 2$ plus the rank of the Mordell-Weil group, which is the group generated by the zero section (we can take A_1 as zero section) and the sections of infinite order. Since we know that $\rho = 17$ we get that the rank of the Mordell-Weil group is three. That group contains the disjoint sections A_2, A_3 and A_4. The remark is similar for the fibration defined by F_2.
ii) On the K3 surface X we have two Nikulin configurations

$$\mathcal{C} = \sum_{i=1}^{16} A_i,\ \mathcal{C}' = \sum_{i=1}^{8} B_i + \sum_{i=9}^{16} A_i.$$

We do not know if these configurations define two Kummer structures on X. We intend to come back on the subject later.
iii) It is also possible to check that the divisor $L' = 3L - 2(A_1 + \cdots + A_8)$ is big and nef, $L'^2 = 8$ and $L'\Gamma = 0$ for an effective (-2)-class Γ if and only if Γ is in $\{B_1, \dots, B_8, A_9, \dots, A_{16}\}$.

Appendix

Why it was natural to study the case $t = \frac{1}{2}k(k+1)$ in the paper [15]. Since $\alpha^2 = 1 + 2t\beta^2$, the integer α is odd. Let $k \in \mathbb{N}$ be such that $\alpha = 2k+1$ (then one has $A_1 A_1' = 4k+2$). The integer β is then solution of the equation

$$(2k+1)^2 - 2t\beta^2 = 1,$$

which is equivalent to

$$t\beta^2 = 2k(k+1).$$

Then

$$a = 2t\beta,\ b = 2k+1$$

are solutions of the three conditions in (2.3). Since $a^2 = 2t(b^2-1)$, one gets

$$a^2 = 2t \cdot 4k(k+1). \tag{3.1}$$

Thus $2t\cdot 4k(k+1)$ must be the square of an integer and it is therefore natural to define

$$t = \frac{1}{2}k(k+1).$$

Then one computes easily that $a = 2k(k+1)$ and $\beta = 2$. Then one has

$$GCD(\beta, \alpha_0 - 1) = GCD(2, 2k) = 2,$$

thus as soon as $\alpha_0 > 5$, i.e. $k > 2$, one can apply Theorem 14. That were the cases we studied in [15].

A table. We resume in the following table the fundamental solutions (α_0, β_0) of the Pell-Fermat equation $\alpha^2 - 2t\beta^2 = 1$ for $2t \leq 60$. Recall that there are non-trivial solutions if and only if $2t$ is not a square. Observe that when $2t = k(k+1)$ the minimal solution is $(2k+1, 2)$, these correspond to Nikulin configurations studied in the paper [15], we put a $*$ close to these cases. We put a box around the cases with β_0 odd, and a prime $'$ when β_0 is even but such that the negative Pell-Fermat equation has a solution: these cases are left out in this paper.

Table 1. Fundamental solutions of the Pell-Fermat equations

$2t$	2*	4	6*	$\boxed{8}$	10'	12*	14	16	18	20*	22	$\boxed{24}$	26'	28	30*
α_0	3	-	5	3	19	7	15	-	17	9	197	5	51	127	11
β_0	2	-	2	1	6	2	4	-	4	2	42	1	10	24	2

$2t$	$\boxed{32}$	34	36	38	$\boxed{40}$	42*	44	46	$\boxed{48}$	50'	52	54	56*	58'	60
α_0	17	35	-	37	19	13	199	24335	7	99	649	485	15	19603	31
β_0	3	6	-	6	3	2	30	3588	1	14	90	66	2	2574	4

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