ASYMPTOTIC EXPANSION FOR A DELAY DIFFERENTIAL EQUATION WITH CONTINUOUS AND PIECEWISE CONSTANT ARGUMENTS

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Abstract. We introduce a family of solutions to a linear delay differential equation with continuous and piecewise constant arguments, depending on four parameters, and we give an asymptotic expansion of any solution of this equation with respect to this family. The fundamental solutions are given by the zeros, counting multiplicities, and, for particular values of the parameters, by the poles of a function meromorphic in the complex plane. This function generalizes the characteristic equation which was known to characterize the oscillatory behaviour of the differential equation for certain values of the parameters. The proof uses the Laplace transform, Fourier series, and the adjoint equation.

Key words. delay differential equation, asymptotic expansion, adjoint equation, Laplace transform
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1. Introduction

In this paper we study the linear delay differential equation

$$f'(x) + pf(x-\tau) + qf([x-\theta]) = 0 \tag{1.1}$$

on $(0,\infty)$, where $[\cdot]$ denotes the greatest integer function, and θ, τ, p, q satisfy

$$\theta \geq 0, \qquad \tau \geq 0, \qquad p \in \mathbb{C} \qquad \text{and} \qquad q \in \mathbb{C}. \tag{1.2}$$

By definition, a solution of (1.1) is a continuous function $f : \mathbb{R} \to \mathbb{C}$ for which the derivative exists on the open set $\{x > 0 : x \not\equiv \theta\ [1]\}$, and which satisfies the differential equation (1.1) on that set, or equivalently, a continuous function $f : \mathbb{R} \to \mathbb{C}$ which satisfies (1.1) in the sense of distributions on $(0,\infty)$.

It is well known (see [1, 4] for instance) that for every θ, τ, p, q which satisfy (1.2) and every continuous function $\psi : (-\infty, 0] \to \mathbb{C}$, there exists a unique solution f to (1.1) such that $f \equiv \psi$ on $(-\infty, 0]$. Moreover,

$$|f(x)| \leq Ce^{rx} \qquad (x \geq 0), \tag{1.3}$$

for some constants $C \geq 0$ and $r \in \mathbb{R}$ independent of x. The solution f is real-valued if the parameters p and q are real numbers and if ψ is real-valued.

When $q = 0$, equation (1.1) reduces to

$$f'(x) + pf(x - \tau) = 0, \tag{1.4}$$

which has been extensively studied [1]. In particular, a function $x \mapsto \mathrm{e}^{zx}$ satisfies (1.4) if and only if

$$\phi(z) = 0, \tag{1.5}$$

where

$$\phi(z) = z + p\mathrm{e}^{-\tau z} \qquad (z \in \mathbb{C}). \tag{1.6}$$

Equation (1.5) is called the characteristic equation of (1.4) and its roots, the characteristics roots. Using a Laplace transform, any solution f of (1.4) can easily be expressed in terms of its initial values over $(-\tau, 0]$ and ϕ by means of a contour integral. Under suitable assumptions [1], f can then be expanded in the form of a infinite series,

$$f(x) = \sum \mathrm{e}^{s_r x} p_r(x) \qquad (x > 0), \tag{1.7}$$

where the sum is over all characteristic roots s_r, and where $p_r(x)$ is a polynomial in x of degree less than the multiplicity of s_r.

When $p = 0$ and $\theta \in \mathbb{N}$, (1.1) reduces to

$$f'(x) + qf\big([x] - \theta\big) = 0, \tag{1.8}$$

whose study is straighforward. Indeed, every solution f is affine on every interval $[n, n+1]$, $n \in \mathbb{N}$, and thus completely defined by the sequence $\{f(n)\}_{n \in \mathbb{N}}$, which satisfies a linear difference equation of order $\theta + 1$. In this case, the characteristic equation, obtained by letting $f(n) = \lambda^n$, $n \in \mathbb{N}$, is

$$\lambda - 1 + q\lambda^{-\theta} = 0. \tag{1.9}$$

Of course, any solution f of (1.8) is a finite sum

$$f(n) = \sum \lambda_r^n p_r(n) \qquad (n \geq 0), \tag{1.10}$$

where the sum is over the roots λ_r of the polynomial (1.9), and where $p_r(n)$ a polynomial in n of degree less than the multiplicity of λ_r. When $p = 0$ and $\theta \in \mathbb{R} \setminus \mathbb{N}$, a similar analysis holds, using the values $f(\theta + n)$, $n \in \mathbb{N}$.

In view of (1.7), (1.10), it is natural to try and find an expansion of any solution of (1.1) in terms of the roots of a 'characteristic equation'. The results obtained so far on equation (1.1) relate its oscillatory behaviour to a 'characteristic equation'. Recall that when $p, q \in \mathbb{R}$, a real-valued solution of (1.1) is called oscillatory if it has arbitrarily large zeros. By extension, the differential equation is oscillatory if every real-valued solution is oscillatory. For instance, equation (1.4) is oscillatory if and only if its characteristic equation (1.5) has no real root [4]. Similarly, (1.8) oscillates if and only if (1.9) has no root in the interval $(0, +\infty)$ [4].

In 1989, K. Gopalsamy, I. Györi and G. Ladas [2] give algebraic conditions involving p, q, τ and the root $z_0 \in [-1/\tau, 0]$ of (1.5) such that equation (1.1) oscillates (see also [3]). In 1991, I. Györi and G. Ladas [4] ask the following problem: find a 'characteristic equation' of (1.1) which reduces to (1.5) when $q=0$, and to (1.9) when $p=0$ and $\theta \in \mathbb{N}$. Notice that, in order to answer such a question, one has to choose between z or $\lambda = \mathrm{e}^z$ for the unknown. In 1998, Y. Wang and J. Yan [7] give the following answer to this problem, for numbers

$$\theta \in \mathbb{N}, \qquad \tau \in \mathbb{N}, \qquad p \in \mathbb{R} \qquad \text{and} \qquad q \in \mathbb{R}. \tag{1.11}$$

By letting $f(x) = \lambda^{[x]} g(x-[x])$, $g : [0,1) \to \mathbb{R}$, they first prove that

$$\lambda \exp\left(p\lambda^{-\tau}\right) - 1 + q\lambda^{-\theta} \int_0^1 \exp\left(p\lambda^{-\tau} t\right) \mathrm{d}t = 0 \tag{1.12}$$

is a characteristic equation of (1.1). Then, using an improved Z-transform, they show that a necessary and sufficient condition for the oscillation of (1.1) is that (1.12) has no real root in $(0,\infty)$.

In this paper, by introducing a new family of complex functions depending on a complex parameter z, we first find in section 2 a characteristic equation of (1.1) for numbers θ, τ, p, q which satifie (1.2); this equation reduces to (1.12) for numbers (1.11), by letting $\lambda = \mathrm{e}^z$. In section 3, we compute the Laplace transform of any solution f of (1.1), using the adjoint equation of (1.1) defined by duality (see for instance [1, 5, 8]). By an inversion formula, we obtain an integral representation of f in terms of the values of f over $(-\infty, 0]$ and the numbers θ, τ, p, q. In section 4, we extend the family of solutions of (1.1) found in section 2 to the general case; this allows us, in the last section, to obtain an asymptotic expansion of any solution of (1.1) as a linear combination of these 'fundamental solutions'.

Notations. In the remainder of this paper, we assume that the numbers θ, τ, p, q satifie (1.2).

When necessary, we shall use the shortcut $\overline{\sum_{n\in\mathbb{Z}}} u_n$ in place of $\lim_{N\to\infty} \sum_{-N\le n\le N} u_n$ for any sequence $\{u_n\}_{n\in\mathbb{Z}}$ of complex numbers (or functions, or distributions), whenever the limit exists (in a space to be specified).

We shall use the notations $f \ll g$ and $f = O(g)$ interchangeably to mean that $|f| \le Cg$ holds for some constant C in the range under consideration.

It will be useful to consider the derivative with respect to x in the sense of distributions. For any open interval $I \subset \mathbb{R}$, $\mathcal{D}(I)$ will denote the space of smooth complex functions with compact support in I, $\mathcal{D}'(\mathbb{R})$ the space of complex distributions on $\mathbb{R}$, that is the dual space of $\mathcal{D}(\mathbb{R})$, and $\langle \cdot, \cdot \rangle$ the duality product. In particular,

$$\langle f, \varphi \rangle = \int_{\mathbb{R}} f(x)\varphi(x)\mathrm{d}x \qquad \left(f \in \mathcal{C}(\mathbb{R}),\ \varphi \in \mathcal{D}(\mathbb{R})\right),$$

where $\mathcal{C}(\mathbb{R})$ is the space of complex continuous functions on $\mathbb{R}$. As usual, $L^1_{loc}(\mathbb{R})$ will be the space of complex locally integrable functions on $\mathbb{R}$.

It will also be useful, because we deal somewhat with Fourier series, to denote $\mathcal{H}(\mathbb{R})$ the space of functions $g : \mathbb{R} \to \mathbb{C}$ such that g is continuous in $\mathbb{R}\setminus\mathbb{Z}$, and for every $n\in\mathbb{Z}$, g has a limit $g(n^-)$ to the left and $g(n^+)$ to the right which satisfie

$$g(n)=\frac{g(n^-)+g(n^+)}{2}. \tag{1.13}$$

2. Characteristic equation

Because we shall make a constant use of the entire function ϕ defined by (1.6), it is important to recall a few facts concerning its set of zeros in $\mathbb{C}$, $Z(\phi)$.

If $p=0$ or $\tau=0$, $\phi(z)=z+p$ and $Z(\phi)=\{-p\}$. Now assume $p\neq 0$ and $\tau>0$. If $z=a+ib$ ($a\in\mathbb{R}$, $b\in\mathbb{R}$) is a zero of ϕ, then $a+ib=-p\mathrm{e}^{-\tau(a+ib)}$ and taking the modulus in each member,

$$\sqrt{a^2+b^2}=|p|\mathrm{e}^{-\tau a} \iff \begin{cases}|p|^2\mathrm{e}^{-2\tau a}-a^2\geq 0,\\ b=\pm\sqrt{|p|^2\mathrm{e}^{-2\tau a}-a^2}.\end{cases} \tag{2.1}$$

This implies that for a given $a\in\mathbb{R}$ there exist at most two zeros of ϕ whose real part is a. If such two distinct zeros exist, then they are conjugate.

Of course, $Z(\phi)$ is discrete and closed in $\mathbb{C}$, and it is possible to prove that $Z(\phi)$ is infinite [1], but we shall not use this fact. The zeros of ϕ lie along the curve defined by (2.1) which is symmetric with respect to the real axis and lies entirely in a left half-plane; as $|z|\to\infty$ along the curve, the curve becomes more and more nearly parallel to the imaginary axis, and $\operatorname{Re} z\to-\infty$.

In this section, our aim is to generalize equation (1.12), but we choose the variable z instead of the variable $\lambda=\mathrm{e}^z$. Since e^z is unchanged if z is changed into $z+2\pi i$, it is natural to recover this translation invariance. We define

$$\Omega:=\{z\in\mathbb{C}\text{ such that }\phi(z+2\pi in)\neq 0\quad\forall n\in\mathbb{Z}\}. \tag{2.2}$$

In other words, $\Omega^c:=\mathbb{C}\setminus\Omega$ is the set of zeros of ϕ in $\mathbb{C}$ and their images by the translations $z\to z+2\pi in$, $n\in\mathbb{Z}$. By (2.1), for every $a\in\mathbb{R}$ the set

$$\{z\in\mathbb{C}\ :\ \operatorname{Re} z\geq a\}\cap Z(\phi)$$

is finite. Hence, $\Omega^c\cap K$ is finite for every compact $K\subset\mathbb{C}$. In other words, *Ω^c is a discrete and closed subset of* $\mathbb{C}$.

Recall that when $q=0$, the characteristic equation of (1.1) is obtained by letting $f(x)=\mathrm{e}^{xz}$. In the following lemma, whose easy proof is postponed at the end of this section, we introduce a family of functions which play a role similar when $q\neq 0$.

Lemma 2.1. *For all $(x,z)\in\mathbb{R}\times\Omega$, the series*

$$F(x,z)=\sum_{n\in\mathbb{Z}}\frac{(1-\mathrm{e}^{-z})}{(z+2\pi in)}\ \frac{\mathrm{e}^{(x-\theta)(z+2\pi in)}}{\phi(z+2\pi in)} \tag{2.3}$$

converges absolutely, and the convergence is uniform on every compact subset of $\mathbb{R} \times \Omega$.

As a consequence, the function F is continuous in $\mathbb{R} \times \Omega$, and for every $x \in \mathbb{R}$, $z \mapsto F(x,z)$ is holomorphic in Ω. Moreover, seeing F as a function of x, we have, for every $z \in \Omega$,

$$F(x,z) = \mathrm{e}^{(x-\theta)z} \tilde{F}_{p,\tau,z}(x-\theta) \qquad (x \in \mathbb{R}), \tag{2.4}$$

where $\tilde{F}_{p,\tau,z}$ is a 1-periodic function of the variable $u = x - \theta$, defined by its Fourier series. By Lemma 2.1, this Fourier series converges uniformly on $[0,1]$; it is also easily seen that the Fourier coefficients are not all equal to zero; hence, $\tilde{F}_{p,\tau,z}$ is continuous and not identically zero on $\mathbb{R}$.

Now we focus on equation (1.1). It is useful to introduce the linear operator $\mathcal{T} : \mathcal{C}(\mathbb{R}) \to \mathcal{D}'(\mathbb{R})$ given by

$$\mathcal{T} f = f' + pT_\tau f + q \sum_{n \in \mathbb{Z}} f(n) \mathbb{1}_{[n+\theta, n+\theta+1)} \qquad (f \in \mathcal{C}(\mathbb{R})) \, , \tag{2.5}$$

where, for any $f \in \mathcal{C}(\mathbb{R})$,

$$T_\tau f(x) = f(x-\tau) \qquad (x \in \mathbb{R}),$$

and $\mathbb{1}_I$ is the characteristic function of the set I ($\mathbb{1}_I(x) = 1$ if $x \in I$ and 0 otherwise). The sum in the right side of (2.5) is locally finite and thus well defined as a distribution. As pointed out in the introduction, a function $f \in \mathcal{C}(\mathbb{R})$ is a solution to (1.1) if and only

$$\langle \mathcal{T} f, \varphi \rangle = 0 \qquad \big(\varphi \in \mathcal{D}\big((0,+\infty)\big)\big) \, .$$

Define

$$P(z) := 1 + q \sum_{n \in \mathbb{Z}} \frac{1 - \mathrm{e}^{-z}}{z + 2\pi i n} \, \frac{\mathrm{e}^{-\theta(z+2\pi i n)}}{\phi(z + 2\pi i n)}. \tag{2.6}$$

Since $P(z) = 1 + qF(0,z)$, Lemma 2.1 implies that P is well defined and holomorphic in Ω. The following theorem shows that the equation

$$P(z) = 0 \tag{2.7}$$

can be considered as a characteristic equation of (1.1).

Theorem 2.2. *For all* $z \in \Omega$,

$$\mathcal{T}\left(x \mapsto F(x,z)\right) = P(z) \sum_{n \in \mathbb{Z}} \mathrm{e}^{nz} \mathbb{1}_{[n+\theta, n+\theta+1)} \qquad \text{in } \mathcal{D}'(\mathbb{R}). \tag{2.8}$$

In particular $x \mapsto F(x,z)$ *is a solution to* (1.1) *if and only if* $P(z) = 0$.

Proof. Let $z \in \Omega$. By Lemma 2.1, the function $f : x \mapsto F(x,z)$ is defined and continuous on $\mathbb{R}$, and the derivative of f in the sense of distributions is

$$f'(x) = \frac{\partial}{\partial x} \sum_{n \in \mathbb{Z}} \frac{1 - \mathrm{e}^{-z}}{z + 2\pi i n} \, \frac{\mathrm{e}^{(x-\theta)(z+2\pi i n)}}{\phi(z + 2\pi i n)} = \sum_{n \in \mathbb{Z}} \frac{1 - \mathrm{e}^{-z}}{\phi(z + 2\pi i n)} \, \mathrm{e}^{(x-\theta)(z+2\pi i n)},$$

where the limit exists in $\mathcal{D}'(\mathbb{R})$. Thus,

$$\begin{aligned}\mathcal{T}f(x) &= \overline{\sum_{n\in\mathbb{Z}}} \frac{1-\mathrm{e}^{-z}}{\phi(z+2\pi in)}\mathrm{e}^{(x-\theta)(z+2\pi in)} + p\overline{\sum_{n\in\mathbb{Z}}}\frac{1-\mathrm{e}^{-z}}{z+2\pi in}\frac{\mathrm{e}^{(x-\theta-\tau)(z+2\pi in)}}{\phi(z+2\pi in)} \\ &\qquad + qF([x-\theta],z), \\ &= \mathrm{e}^{z(x-\theta)}\overline{\sum_{n\in\mathbb{Z}}}\frac{1-\mathrm{e}^{-z}}{(z+2\pi in)}\mathrm{e}^{2\pi in(x-\theta)} + qf([x-\theta]).\end{aligned}$$

Now, by Fourier-Parseval's Theorem, we know that

$$\overline{\sum_{n\in\mathbb{Z}}}\frac{1-\mathrm{e}^{-z}}{(z+2\pi in)}\mathrm{e}^{2\pi inu} = \mathrm{e}^{z[u]-zu} \tag{2.9}$$

in $L^2\big((0,1)\big) \simeq L^2(\mathbb{R}/\mathbb{Z})$. Indeed, $(1-\mathrm{e}^{-z})/(z+2\pi in)$ is the n^{th} Fourier coefficient of the 1-periodic function $u \mapsto \mathrm{e}^{z[u]-zu}$. Equality (2.9) holds in the sense of distributions on $\mathbb{R}$, so

$$\mathcal{T}f(x) = \mathrm{e}^{z[x-\theta]} + qf([x-\theta]) \qquad \text{in } \mathcal{D}'(\mathbb{R}).$$

In other words,

$$\mathcal{T}f(x) = \sum_{n\in\mathbb{Z}}\mathrm{e}^{nz}\mathbb{1}_{[n+\theta,n+\theta+1)}(x) + q\sum_{n\in\mathbb{Z}}F(n,z)\mathbb{1}_{[n+\theta,n+\theta+1)}(x)$$

in $\mathcal{D}'(\mathbb{R})$. Now we notice that

$$\mathrm{e}^{nz} + qF(n,z) = P(z)\mathrm{e}^{nz} \qquad (n\in\mathbb{Z}),$$

and this concludes the proof. □

For numbers which satisfie (1.11), the series in (2.3) can be computed explicitely, using (2.9). Here, we shall only compute $P(z)$, in order to understand the relation between (2.7) and (1.12). First notice that by Dirichlet's theorem, equality (2.9) at the discontinuity point $u=0$ yields

$$\overline{\sum_{n\in\mathbb{Z}}}\frac{1-\mathrm{e}^{-z}}{(z+2\pi in)} = \frac{1+\mathrm{e}^{-z}}{2} \qquad (z\in\mathbb{C}). \tag{2.10}$$

Now assume that $\theta,\ \tau\in\mathbb{N}$ and $p,\ q\in\mathbb{C}^*$. Then, by (2.10)

$$\begin{aligned}P(z) &= 1+q\sum_{n\in\mathbb{Z}}\frac{1-\mathrm{e}^{-z}}{z+2\pi in}\frac{\mathrm{e}^{-\theta z}}{\phi(z)+2\pi in}, \\ &= 1+q\frac{(1-\mathrm{e}^{-z})\mathrm{e}^{-\theta z}}{\phi(z)-z}\left(\overline{\sum_{n\in\mathbb{Z}}}\frac{1}{z+2\pi in} - \overline{\sum_{n\in\mathbb{Z}}}\frac{1}{\phi(z)+2\pi in}\right), \\ &= 1+q\frac{(1-\mathrm{e}^{-z})\mathrm{e}^{-\theta z}}{\phi(z)-z}\left(\frac{1+\mathrm{e}^{-z}}{2(1-\mathrm{e}^{-z})} - \frac{1+\mathrm{e}^{-\phi(z)}}{2(1-\mathrm{e}^{-\phi(z)})}\right), \\ &= 1+q\frac{\mathrm{e}^{-\theta z}}{\phi(z)-z}\left(\frac{\mathrm{e}^{-z}-\mathrm{e}^{-\phi(z)}}{1-\mathrm{e}^{-\phi(z)}}\right).\end{aligned} \tag{2.11}$$

On the other hand, letting $\lambda = \mathrm{e}^z$, the left side $E(\lambda)$ of (1.12) becomes

$$E(\mathrm{e}^z) = \mathrm{e}^{\phi(z)} - 1 + q\mathrm{e}^{-\theta z}\ \frac{[\mathrm{e}^{\phi(z)-z} - 1]}{\phi(z) - z}.$$

Now, multiplying (2.11) by $\mathrm{e}^{\phi(z)} - 1$ yields

$$(\mathrm{e}^{\phi(z)} - 1)P(z) = E(\mathrm{e}^z). \tag{2.12}$$

The computation in (2.11) is valid for every $z \in \Omega$ such that $\mathrm{e}^{-\phi(z)} - 1 \neq 0$, that is for all $z \in \mathbb{C}$ with the exception of a countable set. By analycity, (2.12) is valid for every $z \in \Omega$. The continuity of the right side gives a meaning to the left side for every $z \in \mathbb{C}$. When $p = 0$ and $\theta \in \mathbb{N}$, a similar computation shows that (2.12) is still satisfied (or use continuity of both sides with respect to p). When $q = 0$, $P \equiv 1$ and (2.12) is obviously satisfied.

Now we turn to the proof of Lemma 2.1.

Proof of Lemma 2.1. Let $K \subset \mathbb{R} \times \Omega$ be a compact subset. The assertion will be proved by the following estimate:

$$\frac{\mathrm{e}^{-z} - 1}{(z + 2\pi i n)}\ \frac{\mathrm{e}^{(x-\theta)(z+2\pi i n)}}{\phi(z + 2\pi i n)} \ll \frac{1}{n^2 + 1} \qquad ((x, z) \in K, n \in \mathbb{Z}). \tag{2.13}$$

Let $K' = \{z \in \Omega\ :\ \exists x \in \mathbb{R},\ (x, z) \in K\}$ denote the projection of K on $\mathbb{C}$; in particular, K' is bounded.

First notice that for every $n \in \mathbb{Z}$, the function $z \mapsto (\mathrm{e}^{-z} - 1)/(z + 2\pi i n)$ is holomorphic in $\mathbb{C} \setminus \{-2\pi i n\}$, and extends to a continuous (and thus holomorphic) function in $\mathbb{C}$. In particular, for every $N \in \mathbb{N}$,

$$\frac{\mathrm{e}^{-z} - 1}{(z + 2\pi i n)} \ll 1 \qquad (z \in K',\ |n| \leq N).$$

On the other hand, $|z + 2\pi i n| \geq 2\pi|n| - |z|$, so for some N large enough which depends on K,

$$\frac{1}{|z + 2\pi i n|} \ll \frac{1}{\sqrt{n^2 + 1}} \qquad (z \in K',\ |n| \geq N).$$

Since K' is bounded, we also have

$$\frac{\mathrm{e}^{-z} - 1}{(z + 2\pi i n)} \ll \frac{1}{\sqrt{n^2 + 1}} \qquad (z \in K',\ |n| \geq N).$$

Similarly, since

$$|\phi(z + 2\pi i n)| = |z + 2\pi i n + p\mathrm{e}^{-\tau(z+2\pi i n)}| \geq 2\pi|n| - (|z| + |p\mathrm{e}^{-\tau z}|),$$

for some N large which depends on K,

$$\frac{1}{\phi(z + 2\pi i n)} \ll \frac{1}{\sqrt{n^2 + 1}} \qquad (z \in K',\ |n| \geq N),$$

and since K is compact, we also have

$$\frac{\mathrm{e}^{(x-\theta)(z+2\pi in)}}{\phi(z+2\pi in)} \ll \frac{1}{\sqrt{n^2+1}} \qquad ((x,z)\in K,\ |n|\geq N)\,.$$

Now, for fixed $n \in \mathbb{Z}$, the continuous function $z \mapsto |\phi(z+2\pi in)|$ has no root in K' by definition (2.2) of Ω, so

$$\frac{1}{\phi(z+2\pi in)} \ll 1 \qquad (z\in K').$$

Hence for every $N \in \mathbb{N}$,

$$\frac{\mathrm{e}^{(x-\theta)(z+2\pi in)}}{\phi(z+2\pi in)} \ll 1 \qquad ((x,z)\in K,\ |n|\leq N)\,.$$

Summing up, we have prove (2.13). The proof shows also that for every $n \in \mathbb{Z}$, the function

$$(x,z) \mapsto \frac{\mathrm{e}^{-z}-1}{(z+2\pi in)}\,\frac{\mathrm{e}^{(x-\theta)(z+2\pi in)}}{\phi(z+2\pi in)}$$

is continuous on $\mathbb{R}\times\Omega$, and holomorphic in Ω as a function of z for fixed $x \in \mathbb{R}$. □

With a little more work, it is easy to prove the following

Lemma 2.3. *For all $x \in \mathbb{R}$, the function $z \mapsto F(x,z)$ is meromorphic in $\mathbb{C}$.*

Proof. By Lemma 2.1, for every $x \in \mathbb{R}$, the function $z \mapsto F(x,z)$ is holomorphic in Ω. Let $(x,z) \in \mathbb{R}\times\Omega^c$ and define $I_z = \{n \in \mathbb{Z}\ :\ \phi(z+2\pi in) = 0\}$. By definition of Ω, $|I_z| \geq 1$, and by (2.1), $|I_z| \leq 2$. Since Ω^c is discrete, there exists $\varepsilon > 0$ such that $\overline{B(z,\varepsilon)} \subset \Omega \cup \{z\}$, where

$$B(z,\varepsilon) = \{w \in \mathbb{C}\ :\ |w-z| < \varepsilon\} \text{ and } \overline{B(z,\varepsilon)} = \{w \in \mathbb{C}\ :\ |w-z| \leq \varepsilon\}.$$

The same argument as in the proof of Lemma 2.1 shows that the series

$$\sum_{n\in\mathbb{Z},\ n\notin I_z} \frac{(1-\mathrm{e}^{-w})}{(w+2\pi in)}\,\frac{\mathrm{e}^{(x-\theta)(w+2\pi in)}}{\phi(w+2\pi in)}$$

converges uniformly for $w \in \overline{B(z,\varepsilon)}$. Hence, its sum is holomorphic in $B(z,\varepsilon)$. Writing

$$F(x,w) = \sum_{n\in I_z} \frac{(1-\mathrm{e}^{-w})}{(w+2\pi in)}\,\frac{\mathrm{e}^{(x-\theta)(w+2\pi in)}}{\phi(w+2\pi in)} + \sum_{n\in\mathbb{Z},\ n\notin I_z} \frac{(1-\mathrm{e}^{-w})}{(w+2\pi in)}\,\frac{\mathrm{e}^{(x-\theta)(w+2\pi in)}}{\phi(w+2\pi in)}, \tag{2.14}$$

we see that $w \mapsto F(x,w)$ is the sum of one or two meromorphic functions and one holomorphic function in $B(z,\varepsilon)$, and therefore a meromorphic function in $B(z,\varepsilon)$, and the proof is complete. □

We end this section with the following interesting

Proposition 2.4. *The zeros of P lie in a left half-plane $\{z \in \mathbb{C} \ : \ \operatorname{Re} z \leq r\}$.*

Proof. The proposition will proved by the following assertion:

$$\sup_{\operatorname{Re} z \geq r} \left| \sum_{n\in\mathbb{Z}} \frac{1-\mathrm{e}^{-z}}{z+2\pi i n} \, \frac{\mathrm{e}^{-\theta(z+2\pi i n)}}{\phi(z+2\pi i n)} \right| \to 0 \tag{2.15}$$

as $r \to +\infty$.

First, by $2\pi i$-periodicity, the supremum in (2.15) is the same as the supremum of the same expression over the band

$$\mathcal{B}(r) = \{z \in \mathbb{C} \ : \ \operatorname{Re} z \geq r, \ -\pi \leq \operatorname{Im} z \leq \pi\}.$$

Let $r_0 > |p|$. Then, for every $r \geq r_0$, $z \in \mathcal{B}(r)$ and $n \in \mathbb{Z}^*$,

$$|1-\mathrm{e}^{-z}| \leq 2, \qquad |\mathrm{e}^{-\theta(z+2\pi i n)}| = \mathrm{e}^{-\theta \operatorname{Re} z} \leq 1,$$

$$|z+2\pi i n| = \sqrt{(\operatorname{Re} z)^2 + (2\pi n + \operatorname{Im} z)^2} \geq \sqrt{r^2 + (2\pi|n|-\pi)^2},$$

$$\begin{aligned} |\phi(z+2\pi i n)| \geq |z+2\pi i n| - |p| \ &\geq \ \sqrt{r^2+(2\pi|n|-\pi)^2} - |p|, \\ &\geq \ \sqrt{r^2+(2\pi|n|-\pi)^2}(1-|p|/r_0). \end{aligned}$$

Hence,

$$\sup_{z\in\mathcal{B}(r)} \left| \frac{1-\mathrm{e}^{-z}}{z+2\pi i n} \, \frac{\mathrm{e}^{-\theta(z+2\pi i n)}}{\phi(z+2\pi i n)} \right| \ll \frac{1}{r^2+(2\pi|n|-\pi)^2} \qquad (r \geq r_0, \ n \in \mathbb{Z}^*). \tag{2.16}$$

This bound is also valid for $n = 0$, $r \geq r_0$. Hence, if we denote $N_n(r)$ the left side of (2.16), the convergence of $\sum_{n\in\mathbb{Z}} N_n(r)$ is uniform on $[r_0, +\infty)$; interverting limits, we obtain $\lim_{r\to+\infty} \sum_{n\in\mathbb{Z}} N_n(r) = 0$. Now the left side of (2.15) is $\leq \sum_{n\in\mathbb{Z}} N_n(r)$, and this concludes the proof. □

3. Laplace transform

This section is devoted to the computation of the Laplace transform of any solution of (1.1). Recall that if f is a function which is locally integrable in $(0, \infty)$, and which satisfies

$$f(x) \ll \mathrm{e}^{rx} \qquad (x > 0) \tag{3.1}$$

for some $r \in \mathbb{R}$, the Laplace transform of f is defined by

$$\mathcal{L}f(z) = \int_0^\infty f(x)\mathrm{e}^{-xz}\mathrm{d}x \qquad (z \in \mathbb{C}, \ \operatorname{Re} z \geq r). \tag{3.2}$$

The integral in (3.2) is absolutely convergent, and $\mathcal{L}f$ is holomorphic in $\{z \in \mathbb{C} \ : \ \operatorname{Re} z > r\}$. If f is absolutely continuous in the neighborhood of $x_0 > 0$, the following inversion formula holds, for $r' > r$ [1]:

$$f(x_0) = \frac{1}{2\pi i} \int_{r'-i\infty}^{r'+i\infty} \mathcal{L}f(z)\mathrm{e}^{x_0 z}\mathrm{d}z, \tag{3.3}$$

where

$$\frac{1}{2\pi i}\int_{r'-i\infty}^{r'+i\infty}\mathcal{L}f(z)\mathrm{e}^{x_0 z}\mathrm{d}z := \lim_{T\to\infty}\frac{1}{2\pi}\int_{-T}^{T}\mathcal{L}f(r'+it)\mathrm{e}^{x_0(r'+it)}\mathrm{d}t.$$

We shall need the following lemma, which is analogous to Lemma 2.1 for the adjoint equation of (1.1) (see remark below).

Lemma 3.1. *For all $(y,z)\in\mathbb{R}\times\Omega$, the limit*

$$G(y,z)=\overline{\sum_{n\in\mathbb{Z}}}\frac{\mathrm{e}^{-y(z+2\pi in)}}{\phi(z+2\pi in)} \tag{3.4}$$

exists in $\mathbb{C}$. The function $(y,z)\mapsto G(y,z)$ is bounded on every compact subset of $\mathbb{R}\times\Omega$, holomorphic in Ω as a function of z, for every $y\in\mathbb{R}$, and belongs to $\mathcal{H}(\mathbb{R})$ as a function of y, for every $z\in\Omega$.

Proof. Define

$$g_0(y,z)=\frac{\mathrm{e}^{-yz}}{\phi(z)}\qquad (y\in\mathbb{R}, z\in\Omega), \tag{3.5}$$

and for $n\geq 1$, let $g_n:\mathbb{R}\times\Omega\to\mathbb{C}$ be defined by

$$\frac{\mathrm{e}^{-y(z+2\pi in)}}{\phi(z+2\pi in)}+\frac{\mathrm{e}^{-y(z-2\pi in)}}{\phi(z-2\pi in)}=-\frac{\sin(2\pi ny)}{\pi n}\mathrm{e}^{-yz}+g_n(y,z), \tag{3.6}$$

where $y\in\mathbb{R}$, $z\in\Omega$. Then, for every $N\geq 1$, summing on $-N\leq n\leq N$,

$$\sum_{-N\leq n\leq N}\frac{\mathrm{e}^{-y(z+2\pi in)}}{\phi(z+2\pi in)}=-\sum_{1\leq n\leq N}\frac{\sin(2\pi ny)}{\pi n}\mathrm{e}^{-yz}+\sum_{0\leq n\leq N}g_n(y,z). \tag{3.7}$$

Obviously, for every $n\in\mathbb{N}$, g_n is continuous on $\mathbb{R}\times\Omega$, holomorphic in Ω as a function of z, for every $y\in\mathbb{R}$, and meromorphic in $\mathbb{C}$ as a function of z, for every $y\in\mathbb{R}$. We shall prove that for every compact $K\subset\mathbb{R}\times\Omega$,

$$g_n(y,z)\ll\frac{1}{n^2+1}\qquad \big(n\in\mathbb{N},\ (y,z)\in K\big). \tag{3.8}$$

As a consequence, the series $\sum_{n\geq 0}g_n(y,z)$ will converge uniformly on every compact subset of $\mathbb{R}\times\Omega$ and its sum will be continuous on $\mathbb{R}\times\Omega$, and holomorphic in Ω as a function of z, for every $y\in\mathbb{R}$. The lemma will then follow by letting $N\to+\infty$ in (3.7) and using Lemma 3.2 below.

Now let us prove estimate (3.8). Let K be a compact subset of $\mathbb{R}\times\Omega$, and denote $K'=\{z\in\Omega\ :\ \exists y\in\mathbb{R},\ (y,z)\in K\}$ the projection of K on Ω. By definition, $\phi(z+2\pi in)=z+2\pi in+p\mathrm{e}^{-\tau(z+2\pi in)}$, so for some N large enough which depends on K,

$$\frac{1}{\phi(z+2\pi in)}-\frac{1}{2\pi in}\ll\frac{1}{n^2+1}\qquad \big(|n|\geq N,\ z\in K'\big).$$

Multiplying by $\mathrm{e}^{-y(z+2\pi in)}$ which is bounded on K,

$$\frac{\mathrm{e}^{-y(z+2\pi in)}}{\phi(z+2\pi in)}-\frac{\mathrm{e}^{-y(z+2\pi in)}}{2\pi in}\ll\frac{1}{n^2+1}\qquad \big(|n|\geq N,\ (y,z)\in K\big).$$

Adding this estimate for $n \geq N$ and $-n$ yields

$$\frac{\mathrm{e}^{-y(z+2\pi in)}}{\phi(z+2\pi in)} + \frac{\mathrm{e}^{-y(z-2\pi in)}}{\phi(z-2\pi in)} + \frac{\sin(2\pi ny)}{\pi n}\mathrm{e}^{-yz} \ll \frac{1}{n^2+1}$$

for $n \geq N$, $(y,z) \in K$. On the other hand, by continuity,

$$g_n(y,z) \ll 1 \qquad \big(n < N,\ (y,z) \in K\big).$$

These last two estimates give (3.8), and conclude the proof. □

Lemma 3.2. *For all $y \in \mathbb{R}$,*

$$\lim_{N\to\infty} \sum_{n=1}^{N} \frac{\sin(2\pi ny)}{\pi n} = \beta(y), \tag{3.9}$$

where β is the unique 1-periodic function in $\mathcal{H}(\mathbb{R})$ such that

$$\beta(y) = \frac{1}{2} - y \qquad (0 < y < 1).$$

Moreover, there exists a constant $C > 0$ independent of N such that

$$\sup_{y\in\mathbb{R}} \left| \sum_{n=1}^{N} \frac{\sin(2\pi ny)}{\pi n} \right| \leq C \tag{3.10}$$

Proof. These results are well known. For (3.9), apply Dirichlet's Theorem to the function β. The uniform bound (3.10), which will only be needed later on, is more tricky: for every $0 < \varepsilon < 1/2$, the limit in (3.9) is uniform on $[\varepsilon, 1-\varepsilon]$ by the Dini-Lipschitz test; near the discontinuity point $y = 0$, the bound (3.10) is a consequence of Gibbs's phenomenon. We refer the reader to [9] for more details. □

Remark. For every $f, g \in \mathcal{D}(\mathbb{R})$, $\langle \mathcal{T} f, g\rangle = \langle f, \mathcal{T}^* g\rangle$, where

$$\mathcal{T}^* g = -g' + pT_{-\tau} g + q \sum_{n\in\mathbb{Z}} \int_{n+\theta}^{n+\theta+1} g(t)\mathrm{d}t\ \delta_n \quad \text{in } \mathcal{D}'(\mathbb{R}) \tag{3.11}$$

($\delta_n \in \mathcal{D}'(\mathbb{R})$ denotes the unit mass concentrated at $n \in \mathbb{Z}$). We can consider the adjoint equation of (1.1),

$$-g'(y) + pg(y+\tau) + q \sum_{n\in\mathbb{Z}} \int_{n+\theta}^{n+\theta+1} g(t)\mathrm{d}t\ \delta_n(y) = 0 \tag{3.12}$$

on $(-\infty, 0)$, and it is natural to see $\mathcal{T}^*$, the adjoint operator of $\mathcal{T}$, as mapping $\mathcal{H}(\mathbb{R})$ into $\mathcal{D}'(\mathbb{R})$. With these definitions, we can prove, as in Theorem 2.2, that

$$\mathcal{T}^*\big(y \mapsto G(y,z)\big) = P(z) \sum_{n\in\mathbb{Z}} \mathrm{e}^{-nz}\delta_n \qquad \text{in } \mathcal{D}'(\mathbb{R}) \tag{3.13}$$

for all $z \in \Omega$. In particular, $y \mapsto G(y,z)$ is a solution to (3.12) if and only if $P(z) = 0$.

□

Similarly to Lemma 2.3, we have

Lemma 3.3. *For all $y \in \mathbb{R}$, the function $z \mapsto G(y,z)$ is meromorphic in $\mathbb{C}$.*

Proof. Let $(y,z) \in \mathbb{R} \times \Omega^c$. We already know that $w \mapsto G(y,w)$ is holomorphic in Ω. Define

$$J_z = \{n \in \mathbb{N} \ : \ \phi(z + 2\pi i n) = 0 \text{ or } \phi(z - 2\pi i n) = 0\},$$

and let $B(z,\varepsilon)$ be as in the proof of Lemma 2.3. Using (3.7), we find

$$G(y,w) = -\Big(\lim_{N\to\infty} \sum_{n=1}^{N} \frac{\sin(2\pi n y)}{\pi n} \Big) \mathrm{e}^{-yw} + \sum_{n \in J_z} g_n(y,w) + \sum_{n \in \mathbb{Z}, n \notin J_z} g_n(y,w) \tag{3.14}$$

for $w \in B(z,\varepsilon)$. The first term on the right side of (3.14) is an entire function; the second term is holomorphic in $B(z,\varepsilon) \setminus \{z\}$ and meromorphic in $B(z,\varepsilon)$ by the definition (3.6) of g_n. By the proof of Lemma 3.1, the third sum in holomorphic in $B(z,\varepsilon)$, and this concludes the proof. □

In the next lemma, we compute the Laplace transform of any solution of (1.1). As a useful shortcut, we introduce the bilinear form $[\![\cdot,\cdot]\!]$, defined for $(f,g) \in \mathcal{C}(\mathbb{R}) \times \mathcal{H}(\mathbb{R})$ by

$$\begin{aligned}[\![f,g]\!] \ := \ & f(0)\Big(g(0) - \frac{q}{2}\int_\theta^{\theta+1} g(t)\mathrm{d}t\Big) \\ & -p\int_0^\tau f(t-\tau)g(t)\mathrm{d}t - q\int_0^\theta f\big([t-\theta]\big)g(t)\mathrm{d}t. \end{aligned} \tag{3.15}$$

Notice that $[\![f,g]\!]$ depends only on the values of f over $(-\tau,0] \cup \{-[\theta], -[\theta]+1, \ldots, 0\}$ and of g over $[0, \max(\tau,\theta))$.

Lemma 3.4. *Let $f \in \mathcal{C}(\mathbb{R})$ be a solution of* (1.1)*. Then its Laplace transform $\mathcal{L}f$ has a (unique) meromorphic extension on $\mathbb{C}$ defined by*

$$\mathcal{L}f(z) = \frac{[\![f, t \mapsto \mathrm{e}^{-tz}]\!]}{\phi(z)} - q\int_\theta^{\theta+1} \mathrm{e}^{-tz} dt \, \frac{[\![f, t \mapsto G(t,z)]\!]}{\phi(z)P(z)} \qquad (z \in \mathbb{C}). \tag{3.16}$$

Proof. Let r be large enough such that f satisfies (3.1) (recall (1.3)). Then, $\mathcal{L}f(z)$ is well defined for $\operatorname{Re} z \geq r$. Let us first prove the identity

$$\phi(z)\mathcal{L}f(z) = [\![f, t \mapsto \mathrm{e}^{-tz}]\!] - q\int_\theta^{\theta+1} \mathrm{e}^{-tz}\mathrm{d}t \; a(z) \qquad (\operatorname{Re} z > r), \tag{3.17}$$

where

$$a(z) = \frac{f(0)}{2} + \sum_{n \geq 1} f(n)\mathrm{e}^{-nz} \qquad (\operatorname{Re} z > r). \tag{3.18}$$

Notice that these series converges absolutely by (3.1). Let $z \in \mathbb{C}$ such that $\operatorname{Re} z > r$. By definition of a solution, f has a derivative in $\mathcal{O} = \{x > 0 \ :$

$x \not\equiv \theta\ [1]\}$ which satisfies (1.1), and from (3.1) we deduce

$$f'(x) \ll \mathrm{e}^{rx} \qquad (x \in \mathcal{O}). \tag{3.19}$$

Multiplying (1.1) by $x \mapsto \mathrm{e}^{-xz}$ and integrating on $(0, +\infty)$ yields

$$\int_0^\infty f'(x)\mathrm{e}^{-xz}\mathrm{d}x + p\int_0^\infty f(x-\tau)\mathrm{e}^{-xz}\mathrm{d}x + q\int_0^\infty f\big([x-\theta]\big)\mathrm{e}^{-xz}\mathrm{d}x = 0. \tag{3.20}$$

For the first integral, an integration by parts gives

$$\int_0^\infty f'(x)\mathrm{e}^{-xz}\mathrm{d}x = -f(0) + z\mathcal{L}f(z).$$

Splitting the second integral in (3.20) into an integral over $[0, \tau]$ and an integral over $[\tau, +\infty)$, we find

$$p\int_0^\infty f(x-\tau)\mathrm{e}^{-xz}\mathrm{d}x = p\int_0^\tau f(x-\tau)\mathrm{e}^{-xz}\mathrm{d}x + p\mathrm{e}^{-\tau z}\mathcal{L}f(z).$$

Similarly, the third integral in (3.20) becomes

$$q\int_0^\infty f\big([x-\theta]\big)\mathrm{e}^{-xz}\mathrm{d}x = q\int_0^\theta f\big([x-\theta]\big)\mathrm{e}^{-xz}\mathrm{d}x + q\int_0^\infty f\big([t]\big)\mathrm{e}^{-(t+\theta)z}\mathrm{d}t.$$

Summing these three integrals, equation (3.20) becomes

$$\begin{aligned}\phi(z)\mathcal{L}f(z) \quad = \quad & f(0) - p\int_0^\tau f(t-\tau)\mathrm{e}^{-tz}\mathrm{d}t - q\int_0^\theta f\big([t-\theta]\big)\mathrm{e}^{-tz}\mathrm{d}t \\ & -q\int_0^\infty f\big([t]\big)\mathrm{e}^{-(t+\theta)z}\mathrm{d}t.\end{aligned}$$

By definition of $a(z)$,

$$\int_0^\infty f\big([t]\big)\mathrm{e}^{-(t+\theta)z}\mathrm{d}t = \frac{f(0)}{2}\int_\theta^{\theta+1}\mathrm{e}^{-tz}dt + \int_\theta^{\theta+1}\mathrm{e}^{-tz}dt\ a(z),$$

and this, together with the definition (3.15) of $[\![\cdot,\cdot]\!]$, yields equation (3.17).

Let us now prove the following assertion:

$$a(z) = \overline{\sum_{n\in\mathbb{Z}}}\mathcal{L}f(z + 2\pi i n) \qquad (\operatorname{Re} z > r). \tag{3.21}$$

Let $z \in \mathbb{C}$ such that $\operatorname{Re} z > r$ and define

$$A_z(x) = \sum_{m\geq 0} f(m+x)\mathrm{e}^{-(m+x)z} \qquad (0 \leq x \leq 1). \tag{3.22}$$

By (3.1), the series in (3.22) converges uniformly on $[0,1]$, and by continuity of f, A_z is continuous on $[0,1]$. In fact A_z is of class $\mathcal{C}^1$ on $I_- = \big[0, \theta - [\theta]\big]$ and $I^+ = \big[\theta - [\theta], 1\big]$. Indeed, for every $m \in \mathbb{N}$, the function $x \mapsto f(x+m)$ is of class $\mathcal{C}^1$ on I_- and I_+, and by (3.1) again and (3.19), the series obtained by differentiating (3.22) with respect to x,

$$\sum_{m\geq 0}\big(f'(m+x)-zf(m+x)\big)\mathrm{e}^{-(m+x)z},$$

converges uniformly on I_- and on I_+. By definition of $a(z)$,

$$a(z)=\frac{A_z(0)+A_z(1)}{2},$$

and applying Dirichlet's theorem to the unique 1-periodic function on $\mathbb{R}$ whose restriction to $[0,1)$ is A_z,

$$a(z)=\overline{\sum_{n\in\mathbb{Z}}}\int_0^1 A_z(x)\mathrm{e}^{-2\pi inx}\mathrm{d}x. \tag{3.23}$$

For every $n\in\mathbb{Z}$, integrating term by term in (3.22) yields

$$\int_0^1 A_z(x)\mathrm{e}^{-2\pi inx}\mathrm{d}x \;=\; \sum_{m\geq 0}\int_0^1 f(m+x)\mathrm{e}^{-(m+x)z}\mathrm{e}^{-2\pi inx}\mathrm{d}x,$$

and setting $t=m+x$,

$$\int_0^1 A_z(x)\mathrm{e}^{-2\pi inx}\mathrm{d}x \;=\; \sum_{m\geq 0}\int_m^{m+1} f(t)\mathrm{e}^{-t(z+2\pi in)}\mathrm{d}t=\mathcal{L}f(z+2\pi in).$$

The last term comes from the definition (3.2) of $\mathcal{L}f$. Replacing in (3.23), we obtain (3.21).

Thirdly, let us prove

$$P(z)a(z)=[\![f,t\mapsto G(t,z)]\!]\qquad(\operatorname{Re}z>r). \tag{3.24}$$

Let $z\in\mathbb{C}$ such that $\operatorname{Re}z>r$. We may assume $r\geq|p|$ in which case, since $\tau\geq 0$, $|\phi(z)|\geq|z|-|p|>0$. By (3.17),

$$\mathcal{L}f(z)+q\int_\theta^{\theta+1}\frac{\mathrm{e}^{-tz}}{\phi(z)}\mathrm{d}t\;a(z)=\frac{[\![f,t\mapsto \mathrm{e}^{-tz}]\!]}{\phi(z)}.$$

The function a is $2\pi i$-periodic (see (3.18)), so changing z into $z+2\pi in$ yields

$$\mathcal{L}f(z{+}2\pi in){+}q\int_\theta^{\theta+1}\frac{\mathrm{e}^{-t(z+2\pi in)}}{\phi(z+2\pi in)}\mathrm{d}t\;a(z)=\frac{[\![f,t\mapsto \mathrm{e}^{-t(z+2\pi in)}]\!]}{\phi(z+2\pi in)}\qquad(n\in\mathbb{Z}). \tag{3.25}$$

An easy computation yields

$$P(z)-1=\overline{\sum_{n\in\mathbb{Z}}}q\int_\theta^{\theta+1}\frac{\mathrm{e}^{-(z+2\pi in)t}}{\phi(z+2\pi in)}\mathrm{d}t,$$

so summing (3.25) on $n \in \mathbb{Z}$ and using (3.21), we obtain

$$P(z)a(z) = \overline{\sum}_{n\in\mathbb{Z}} \frac{[\![f, t \mapsto \mathrm{e}^{-t(z+2\pi in)}]\!]}{\phi(z+2\pi in)}.$$

The definition of G yields (3.24) (recall that the series in (3.4) converges in $L^1_{loc}(\mathbb{R})$).

So far we have proved

$$P(z)\phi(z)\mathcal{L}f(z) = P(z)[\![f, t \mapsto \mathrm{e}^{-tz}]\!] - q\int_\theta^{\theta+1} \mathrm{e}^{-tz}\mathrm{d}t\,[\![f, t \mapsto G(t,z)]\!] \quad (3.26)$$

on $\{z \ : \ \operatorname{Re} z > r\}$. By continuity of f and the classical Lemma 5.3, the function $z \mapsto [\![f, t \mapsto \mathrm{e}^{-tz}]\!]$ is holomorphic in $\mathbb{C}$. Similarly, by Lemma 3.4, $z \mapsto [\![f, x \mapsto G(x,z)]\!]$ is holomorphic in Ω. If $z \in \Omega^c$, then, by the proof of Lemma 3.3, $w \mapsto [\![f, x \mapsto (w-z)^{\alpha_z} G(x,w)]\!]$ is holomorphic in a neighbourhood of z for some integer α_z independent of x. Hence, the right side of (3.26) is a meromorphic function of z in $\mathbb{C}$, which is holomorphic in Ω. Dividing both sides by $\phi(z)P(z) \not\equiv 0$, the right side defines a meromorphic function in $\mathbb{C}$ which is equal to $\mathcal{L}f$ on $\{z \in \mathbb{C} \ : \ \operatorname{Re} z > r, \ P(z) \neq 0\}$, and this concludes the proof. □

Define

$$Z(P) = \{z \in \Omega \ : \ P(z) = 0\} \quad \text{and} \quad \mathcal{U} = \Omega \setminus Z(P). \quad (3.27)$$

Of course, $Z(P)$ is discrete and closed in Ω. In fact, $Z(P) \cup \Omega^c$ is discrete and closed in $\mathbb{C}$, becauses P is meromorphic in $\mathbb{C}$. By $2\pi i$-periodicity of P, $\mathcal{U}$ is an open subset of $\mathbb{C}$ which is invariant by the translation $z \mapsto z + 2\pi i$. By Proposition 2.4, $\mathcal{U}$ contains a right half-plane. The above proof shows that $\mathcal{L}f$ is holomorphic in $\mathcal{U}$. We have:

Theorem 3.5. *Let $f \in \mathcal{C}(\mathbb{R})$ be a solution of* (1.1)*, let $r > |p|$ be large enough such that* (3.1) *holds, and let $\kappa \in \mathbb{C}$ such that*

$$\operatorname{Re}\kappa > r, \quad \kappa + i\mathbb{R} \subset \mathcal{U}, \quad \textit{and} \quad \kappa + \mathbb{R} \subset \mathcal{U}. \quad (3.28)$$

Then,

$$f(x) = \frac{1}{2\pi i}\int_\kappa^{\kappa+2\pi i} \left[\!\!\left[f, t \mapsto G(t-x,z) - q\frac{G(t,z)}{P(z)}F(x,z) \right]\!\!\right] dz \qquad (x > 0). \quad (3.29)$$

Proof. A number κ which satisfies (3.28) does exist. Indeed, by Proposition 2.4, $\kappa + i\mathbb{R} \subset \mathcal{U}$ if $\operatorname{Re}\kappa$ is large enough; the last condition on κ is equivalent to $\kappa \notin \big(\Omega^c \cup Z(P)\big) + \mathbb{R}$, and the latter is a countable number of horizontal lines in $\mathbb{C}$. Notice also that by $2\pi i$-invariance,

$$\kappa + 2\pi in + \mathbb{R} \subset \mathcal{U} \qquad (n \in \mathbb{Z}).$$

The solution f of (1.1) is absolutely continuous on $[0,+\infty)$, because f' is locally bounded on $[0,+\infty)$ and $f(x) = f(0) + \int_0^x f'(s)\mathrm{d}s$ for all

$x \in [0, +\infty)$. Hence, we can apply the inversion formula (3.3):

$$f(x) = \frac{1}{2\pi i} \int_{\kappa - i\infty}^{\kappa + i\infty} \mathcal{L}f(w) \mathrm{e}^{xw} \mathrm{d}w \qquad (x > 0).$$

Letting $w = z + 2\pi i n$, we have for every $n \in \mathbb{Z}$,

$$\int_{\kappa + 2\pi i n}^{\kappa + 2\pi i (n+1)} \mathcal{L}f(w) \mathrm{e}^{xw} \mathrm{d}w = \int_{\kappa}^{\kappa + 2\pi i} \mathcal{L}f(z + 2\pi i n) \mathrm{e}^{x(z + 2\pi i n)} \mathrm{d}z \qquad (x > 0),$$

so

$$f(x) = \frac{1}{2\pi i} \lim_{N \to \infty} \int_{\kappa}^{\kappa + 2\pi i} \left(\sum_{-N \le n \le N} \mathcal{L}f(z + 2\pi i n) \mathrm{e}^{x(z + 2\pi i n)} \right) \mathrm{d}z \qquad (x > 0). \tag{3.30}$$

On the other hand, by Lemma 3.4,

$$\mathcal{L}f(z) = \frac{[\![f, t \mapsto \mathrm{e}^{-tz}]\!]}{\phi(z)} - q \int_{\theta}^{\theta + 1} \mathrm{e}^{-tz} \mathrm{d}t \, \frac{[\![f, t \mapsto G(t, z)]\!]}{\phi(z) P(z)} \qquad (z \in \mathcal{U}).$$

A direct computation gives

$$\int_{\theta}^{\theta + 1} \mathrm{e}^{-tz} \mathrm{d}t = \frac{1 - \mathrm{e}^{-z}}{z} \mathrm{e}^{-\theta z} \qquad (z \in \mathbb{C}),$$

so $\mathcal{L}f$ satisfies

$$\mathcal{L}f(z) = \frac{[\![f, t \mapsto \mathrm{e}^{-tz}]\!]}{\phi(z)} - q \frac{[\![f, t \mapsto G(t, z)]\!]}{P(z)} \frac{1 - \mathrm{e}^{-z}}{z} \frac{\mathrm{e}^{-\theta z}}{\phi(z)} \qquad (z \in \mathcal{U}).$$

By bilinearity,

$$\mathcal{L}f(z) \mathrm{e}^{xz} = \left[\!\!\left[f, t \mapsto \frac{\mathrm{e}^{(x-t)z}}{\phi(z)} - q \frac{G(t, z)}{P(z)} \frac{1 - \mathrm{e}^{-z}}{z} \frac{\mathrm{e}^{(x - \theta)z}}{\phi(z)} \right]\!\!\right] \qquad (x \in \mathbb{R}, z \in \mathcal{U}).$$

Summing on n and using the $2\pi i$-periodicity of the functions $z \mapsto P(z)$, $z \mapsto \mathrm{e}^{-z}$ and $z \mapsto G(t, z)$, we obtain for every $N \ge 1$, $z \in \mathcal{U}$, and $x \in \mathbb{R}$,

$$\sum_{-N \le n \le N} \mathcal{L}f(z + 2\pi i n) \mathrm{e}^{x(z + 2\pi i n)} = \left[\!\!\left[f, t \mapsto G_N(t - x, z) - q \frac{G(t, z)}{P(z)} F_N(x, z) \right]\!\!\right],$$

where

$$G_N(y, z) = \sum_{-N \le n \le N} \frac{\mathrm{e}^{-y(z + 2\pi i n)}}{\phi(z + 2\pi i n)} \qquad (y \in \mathbb{R}),$$

and

$$F_N(x, z) = \sum_{-N \le n \le N} \frac{(1 - \mathrm{e}^{-z})}{(z + 2\pi i n)} \frac{\mathrm{e}^{(x - \theta)(z + 2\pi i n)}}{\phi(z + 2\pi i n)} \qquad (x \in \mathbb{R}).$$

Replacing in (3.30),

$$f(x) = \frac{1}{2\pi i} \lim_{N \to \infty} \int_{\kappa}^{\kappa + 2\pi i} \left[\!\!\left[f, t \mapsto G_N(t - x, z) - q \frac{G(t, z)}{P(z)} F_N(x, z) \right]\!\!\right] \mathrm{d}z \tag{3.31}$$

for every $x > 0$. By the definitions of F and G,

$$\lim_{N\to\infty} F_N(t,z) = F(t,z) \text{ and } \lim_{N\to\infty} G_N(t,z) = G(t,z) \qquad (t \in \mathbb{R}, z \in \Omega).$$

Moreover, for every compact $K \subset \mathbb{R} \times \Omega$,

$$F_N(t,z) \ll 1 \text{ and } G_N(t,z) \ll 1 \qquad \big((t,z) \in K,\ N \in \mathbb{N}^*\big), \tag{3.32}$$

by Lemma 2.1 for F and equation (3.7) and estimates (3.8), (3.10) for G. Hence, by Lebesgue's dominated convergence theorem, for every $(x,z) \in \mathbb{R} \times \mathcal{U}$,

$$\left[\!\!\left[f, t \mapsto G_N(t-x,z) - q\frac{G(t,z)}{P(z)} F_N(x,z) \right]\!\!\right]$$

tends to

$$\left[\!\!\left[f, t \mapsto G(t-x,z) - q\frac{G(t,z)}{P(z)} F(x,z) \right]\!\!\right]$$

as $N \to \infty$. Using (3.32) again, for every compact $K \in \mathbb{R} \times \mathcal{U}$,

$$\left[\!\!\left[f, t \mapsto G_N(t-x,z) - q\frac{G(t,z)}{P(z)} F_N(x,z) \right]\!\!\right] \ll 1 \qquad \big((x,z) \in K,\ N \in \mathbb{N}^*\big).$$

Thus, the dominated convergence theorem applies to (3.31), and yields (3.29). □

Remark. There are analogues of Lemma 3.4 and Theorem 3.5 for the adjoint equation (3.12). We leave the computation to the reader. □

4. Fundamental solutions

In section 2, we related every root of P to a solution of (1.1). Now, if $z \in \Omega$ is a root of P with multiplicity $v_z \geq 2$, then differentiating (2.8) v_z times with respect to z, we obtain similarly n_z solutions of (1.1) (at least formally). When $z \in \Omega^c$, it is also sometimes possible to associate to z a solution of (1.1). In this section, we build these additional solutions of (1.1), which will be named $\{F_{z,k}\}_{1 \leq k \leq n_z}$. As it will be made clear in the next section, the existence of those solutions depends in fact on the residues of the integrand in (3.29).

We begin by some preliminary results.

Lemma 4.1. *Every zero of ϕ has multiplicity 1 except, when $\tau p\mathrm{e} = 1$, the real number $-p\mathrm{e}$ which has multiplicity 2.*

Proof. For every $s \in \mathbb{C}$,

$$\begin{cases} \phi(s) = 0 \\ \phi'(s) = 0 \end{cases} \Leftrightarrow \begin{cases} s + p\mathrm{e}^{-\tau s} = 0 \\ 1 - \tau p\mathrm{e}^{-\tau s} = 0 \end{cases} \Leftrightarrow \begin{cases} s + p\mathrm{e}^{-\tau s} = 0 \\ 1 + \tau s = 0 \end{cases} \Leftrightarrow \begin{cases} s = -p\mathrm{e}, \\ \tau p\mathrm{e} = 1. \end{cases}$$

If $p = 0$ or $\tau = 0$, $\phi(z) = z + p$ has a unique root in $\mathbb{C}$ which is simple. If $p \neq 0$ and $\tau \neq 0$, $\phi''(z) = p\tau^2\mathrm{e}^{-\tau z} \neq 0$ on $\mathbb{C}$, and the lemma is proved. □

For every $z \in \mathbb{C}$ let

$$E_z = \{s \in z + 2\pi i\mathbb{Z} \ :\ \phi(s) = 0\} = Z(\phi) \cap \big(\{z\} + 2\pi i\mathbb{Z}\big). \qquad (4.1)$$

By the definition of Ω, $E_z = \emptyset$ if and only if $z \in \Omega$. If $z \in \Omega^c$, $|E_z| = 1$ or 2 by (2.1). When $|E_z| = 2$, $E_z = \{s, \bar{s}\}$ for some root s of ϕ. In the following lemma, we characterize the cases for which $|E_z| = 2$.

Lemma 4.2. *For all $s \in \mathbb{C}$ and $n \in \mathbb{Z}^*$,*

$$\begin{cases} \phi(s) = 0 \\ \phi(s - 2\pi i n) = 0 \end{cases} \iff \begin{cases} n\tau \notin \mathbb{Z}, \\ s = -\pi n \cot(\pi n\tau) + i\pi n, \\ p = \dfrac{\pi n}{\sin(\pi n\tau)} \exp[-\pi n\tau \cot(\pi n\tau)]. \end{cases}$$

In particular, when $\tau \in \mathbb{N}$ or $p \in \mathbb{C} \setminus \mathbb{R}$, we have $|E_z| \leq 1$ for all $z \in \mathbb{C}$.

Proof. Let $s \in \mathbb{C}$ and $n \in \mathbb{Z}^*$. Then

$$\begin{cases} \phi(s) = 0 \\ \phi(s - 2\pi i n) = 0 \end{cases} \Leftrightarrow \begin{cases} p = -s e^{\tau s}, \\ s(1 - e^{2\pi i n\tau}) = 2\pi i n. \end{cases}$$

Since $n \neq 0$, the last equation is never satisfied if $n\tau \in \mathbb{Z}$, so the last equation is equivalent to

$$s = \frac{2\pi i n}{1 - e^{2\pi i n\tau}} = -\pi n \cot(\pi n\tau) + i\pi n,$$

with the additional condition $n\tau \notin \mathbb{Z}$. Now compute

$$\begin{aligned} p &= \big(\pi n \cot(\pi n\tau) - i\pi n\big) \exp[\tau(-\pi n \cot(\pi n\tau) + i\pi n)], \\ &= \frac{\pi n}{\sin(\pi n\tau)} \big(\cos(\pi n\tau) - i\sin(\pi n\tau)\big) \exp(i\pi n\tau) \exp[-\pi n\tau \cot(\pi n\tau)], \\ &= \frac{\pi n}{\sin(\pi n\tau)} \exp[-\pi n\tau \cot(\pi n\tau)], \end{aligned}$$

and the lemma is complete. □

Let $v(x, z)$ denote the valuation of $w \mapsto F(x, w)$ at $w = z$. In other words, $v(x, z)$ is $-m_z$ if z is a pole of order $m_z \geq 1$, and the multiplicity (possibly 0) of z as a zero of $w \mapsto F(x, w)$ if z is a removable singularity. In the next lemma, we compute this number.

Lemma 4.3. *Assume $p \neq 0$ and let $(x, z) \in \mathbb{R} \times \Omega^c$. Then,*

(1) *if $z \in 2\pi i\mathbb{Z}$ and $E_z = \{s\}$, $v(x, z) \geq 0$; more precisely,*

$$\lim_{w \to z} F(x, w) = \frac{1}{\phi(0)} + \frac{e^{(x-\theta)s}}{s\phi'(s)}.$$

(2) *if $z \in 2\pi i\mathbb{Z}$ and $E_z = \{s, \bar{s}\}$, $v(x, z) \geq 0$; more precisely,*

$$\lim_{w \to z} F(x, w) = \frac{1}{\phi(0)} + \frac{e^{(x-\theta)s}}{s\phi'(s)} + \frac{e^{(x-\theta)\bar{s}}}{\bar{s}\phi'(\bar{s})}.$$

(3) *if* $\tau p\mathrm{e} = 1$ *and* $z \in -p\mathrm{e} + 2\pi i\mathbb{Z}$, $v(x, z) = -2$; *more precisely,*

$$\lim_{w\to z}(w - z)^2 F(x, w) = \frac{1 - \mathrm{e}^{-s}}{s}\,\frac{2\mathrm{e}^{(x-\theta)s}}{\phi''(s)}.$$

(4) *if* $z \notin 2\pi i\mathbb{Z}$, $E_z = \{s\}$, *and the previous case does not hold,* $v(x, z) = -1$; *more precisely,*

$$\lim_{w\to z}(w - z) F(x, w) = \frac{1 - \mathrm{e}^{-s}}{s}\,\frac{\mathrm{e}^{(x-\theta)s}}{\phi'(s)}.$$

(5) *if* $z \notin 2\pi i\mathbb{Z}$ *and* $E_z = \{s, \bar{s}\}$, $v(x, z) \geq -1$; *more precisely,*

$$\lim_{w\to z}(w - z) F(x, w) = \frac{1 - \mathrm{e}^{-s}}{s}\,\frac{\mathrm{e}^{(x-\theta)s}}{\phi'(s)} + \frac{1 - \mathrm{e}^{-\bar{s}}}{\bar{s}}\,\frac{\mathrm{e}^{(x-\theta)\bar{s}}}{\phi'(\bar{s})}.$$

Proof. For every $x \in \mathbb{R}$, $z \in \mathbb{C}$ and $n \in \mathbb{Z}$,

$$\frac{(1 - \mathrm{e}^{-z})}{(z + 2\pi i n)}\,\mathrm{e}^{(x-\theta)(z+2\pi i n)} = 0 \iff z \in 2\pi i\mathbb{Z} \setminus \{-2\pi i n\}. \tag{4.2}$$

Notice also that $\phi(0) \neq 0$. In the first case, compute the limit as $w \to z$ of each term in the series (2.3) defining F, using Lemma 4.1 and Lemma 4.2. The result is then obtained by interverting the limits, which is valid because of the uniform convergence, by a proof similar to the one of Lemma 2.3. For the other cases, the proof is similar. □

In the remainder of this paper, we shall assume $p \neq 0$ and $q \neq 0$.

We need a few more notations. First define

$$\alpha_z = \begin{cases} 0 & \text{if } z \in \Omega \text{ or } z \in \Omega^c \cap 2\pi i\mathbb{Z}, \\ 2 & \text{if } \tau p\mathrm{e} = 1 \text{ and } z \in -p\mathrm{e} + 2\pi i\mathbb{Z}, \\ 1 & \text{otherwise.} \end{cases} \tag{4.3}$$

For every $z \in \mathbb{C}$, we denote $\{f_{z,n}\}_{n=1}^{\infty}$ the sequence of functions uniquely determined by

$$F(x, w)(w - z)^{\alpha_z} = \sum_{n\geq 1} f_{z,n}(x)(w - z)^{n-1} \qquad (x \in \mathbb{R}, w \text{ near } z). \tag{4.4}$$

By Lemma 4.3, the family $\{f_{z,n}\}_{n=1}^{\infty}$ is well defined, and if $z \in \Omega^c$, $f_{z,1} \not\equiv 0$ because the family $\{x \mapsto 1, x \mapsto \mathrm{e}^{sx}, x \mapsto \mathrm{e}^{\bar{s}x}\}$ is linearly independent for every $s \in \mathbb{C} \setminus \mathbb{R}$. If $z \in \Omega$, $f_{z,1} \not\equiv 0$ by (2.4).

For all $z \in \mathbb{C}$, $v_z \in \mathbb{Z}$ will denote the valuation of $P(w)$ (2.6) at $w = z$. We denote $\{c_{z,n}\}_{n\in\mathbb{N}}$ the sequence of complex numbers defined by

$$-\frac{q}{P(w)(w - z)^{-v_z}} = \sum_{n=0}^{\infty} c_{z,n}(w - z)^n \qquad (w \text{ near } z). \tag{4.5}$$

By definition of v_z, $c_{z,0} \neq 0$.

For every $z \in \mathbb{C}$, let

$$n_z = v_z + |E_z| \in \{-1, 0, 1, 2, 3, \ldots\}. \tag{4.6}$$

The case $n_z = -1$ happens if and only if $\tau p\mathrm{e} = 1$ and $z \in p\mathrm{e} + 2\pi i\mathbb{Z}$. In general, $n_z = 0$. Otherwise, $n_z \geq 1$ is the number of fundamental solutions associated to z. By Lemma 4.3, $n_z \geq 1$ if and only if either one of the following condition is satisfied:

i) $z \in \Omega$ and z is a root of P of multiplicity $n_z = v_z \geq 1$;
ii) $z \in \Omega^c \cap 2\pi i\mathbb{Z}$;
iii)$z \in \Omega^c \setminus 2\pi i\mathbb{Z}$ and $|E_z| = 2$.

At last, for every $z \in \mathbb{C}$ such that $n_z \geq 1$, we define the finite sequence of functions $\{F_{z,k}\}_{1\leq k\leq n_z}$ as follows, for all $x \in \mathbb{R}$:

$$F_{z,k}(x) = f_{z,k}(x) \qquad (1 \leq k \leq v_z + \alpha_z), \tag{4.7}$$

and, if $E_z = \{s\} \subset 2\pi i\mathbb{Z}$,

$$F_{z,n_z}(x) = f_{z,n_z}(x) + \frac{\mathrm{e}^{xs}}{c_{z,0}}; \tag{4.8}$$

if $E_z = \{s, \bar{s}\} \subset 2\pi i\mathbb{Z}$,

$$F_{z,n_z-1}(x) = f_{z,v_z+1}(x) + \frac{\mathrm{e}^{xs}}{c_{z,0}}, \quad F_{z,n_z}(x) = f_{z,v_z+1}(x) + \frac{\mathrm{e}^{x\bar{s}}}{c_{z,0}}; \tag{4.9}$$

if $E_z = \{s, \bar{s}\} \not\subset 2\pi i\mathbb{Z}$ and $v_z = -1$,

$$F_{z,n_z}(x) = \mathrm{e}^{xs} - \mathrm{e}^{x\bar{s}}; \tag{4.10}$$

if $E_z = \{s, \bar{s}\} \not\subset 2\pi i\mathbb{Z}$ and $v_z \geq 0$,

$$F_{z,n_z}(x) = f_{z,n_z} + \frac{\mathrm{e}^{xs} + \mathrm{e}^{x\bar{s}}}{2c_{z,0}}. \tag{4.11}$$

Remarks. Notice that if $E_z = \{s\} \not\subset 2\pi i\mathbb{Z}$, then using Lemma 4.3, we find

$$f_{z,1}(x) + \frac{\mathrm{e}^{xs}}{c_{z,0}} = 0.$$

Similarly, if $E_z = \{s, \bar{s}\} \not\subset 2\pi i\mathbb{Z}$ and $v_z = -1$, we find

$$f_{z,1}(x) + \frac{\mathrm{e}^{xs} + \mathrm{e}^{x\bar{s}}}{2c_{z,0}} = (1 - \mathrm{e}^{-z})\frac{(\mathrm{e}^{xs} - \mathrm{e}^{x\bar{s}})}{2}\left(\frac{\mathrm{e}^{-\theta s}}{s\phi'(s)} - \frac{\mathrm{e}^{-\theta\bar{s}}}{\bar{s}\phi'(\bar{s})}\right).$$

Thus, $f_{z,1}(x) + \dfrac{\mathrm{e}^{xs} + \mathrm{e}^{x\bar{s}}}{2c_{z,0}}$ is a solution of (1.1), by Theorem 4.4. However, since $\operatorname{Im} s \neq 0$, there exists $\theta \in \mathbb{R}$ such that

$$\operatorname{Im}\left(\frac{\mathrm{e}^{-\theta s}}{s\phi'(s)}\right) = \frac{\mathrm{e}^{-\theta s}}{s\phi'(s)} - \frac{\mathrm{e}^{-\theta\bar{s}}}{\bar{s}\phi'(\bar{s})} = 0,$$

so this solution is not convenient as a 'fundamental solution'. $\square$

We have

Theorem 4.4. *For every $z \in \mathbb{C}$ such that $n_z \geq 1$, and every $k \in \{1, \ldots, n_z\}$, the function $F_{z,k}$ is a solution of* (1.1).

Proof. First, we prove that for all $z \in \mathbb{C}$,

$$\mathcal{T}[f_{z,k+1}] = \begin{cases} 0 & \text{if } 0 \le k < v_z + \alpha_z, \\ -\dfrac{q}{c_{z,0}} \sum_{n\in\mathbb{Z}} \mathrm{e}^{nz} \mathbb{1}_{[n+\theta,n+\theta+1)} & \text{if } k = v_z + \alpha_z. \end{cases} \tag{4.12}$$

Let $z \in \mathbb{C}$ and consider the functions

$$\tilde{F}(x,w) = F(x,w)(w-z)^{\alpha_z} \qquad \big(x \in \mathbb{R}, w \in \Omega\big), \tag{4.13}$$

$$\tilde{P}(w) = P(w)(w-z)^{\alpha_z} \qquad \big(w \in \Omega\big). \tag{4.14}$$

By (2.1), there exists $\varepsilon > 0$ such that $\{w \in \mathbb{C} \ : \ 0 < |w - z| \le \varepsilon\} \subset \Omega$. Multiplying (2.8) by $(w-z)^{\alpha_z}$ yields

$$\mathcal{T}\big[x \mapsto \tilde{F}(x,w)\big] = \tilde{P}(w) \sum_{n\in\mathbb{Z}} \mathrm{e}^{nw} \mathbb{1}_{[n+\theta,n+\theta+1)}(x) \qquad \big(0 < |w-z| \le \epsilon\big). \tag{4.15}$$

If $z \in \Omega$, (4.15) is also satisfied at $w = z$ and we obtain

$$\mathcal{T}\big(x \mapsto \tilde{F}(x,w)\big) = \tilde{P}(w) \sum_{n\in\mathbb{Z}} \mathrm{e}^{nw} \mathbb{1}_{[n+\theta,n+\theta+1)}(x) \qquad \big(|w-z| \le \epsilon\big). \tag{4.16}$$

If not, recall that $\tilde{P}$ is continuous at $w = z$ by Lemma 4.3, so the right side of (4.15) tends to

$$\tilde{P}(z) \sum_{n\in\mathbb{Z}} \mathrm{e}^{nz} \mathbb{1}_{[n+\theta,n+\theta+1)}(x)$$

in $\mathcal{D}'(\mathbb{R})$, as $w \to z$. On the other hand, by imitating the proofs of lemmas 4.3 and 2.1, it is easy to see that $(x,w) \to \tilde{F}(x,w)$ is continuous on $\mathbb{R} \times \{w \in \mathbb{C} \ : \ |w-z| \le \varepsilon\}$. Hence, the function $x \mapsto \tilde{F}(x,w)$ tends to $x \mapsto \tilde{F}(x,z)$ in $\mathcal{D}'(\mathbb{R})$ as $w \to z$. As a consequence,

$$\lim_{w\to z} \mathcal{T}\big[x \mapsto \tilde{F}(x,w)\big] = \mathcal{T}\big[x \mapsto \tilde{F}(x,z)\big] \quad \text{in } \mathcal{D}'(\mathbb{R}).$$

Thus, we can pass to the limit in (4.15) and we obtain (4.16) again.

Let now $\varphi \in \mathcal{D}(\mathbb{R})$. We have

$$\partial_z^k \langle \mathcal{T}\big[x \mapsto \tilde{F}(x,z)\big], \varphi\rangle = \langle \mathcal{T}\big[x \mapsto \partial_z^k \tilde{F}(x,z)\big], \varphi\rangle \qquad (k \in \mathbb{N}^*). \tag{4.17}$$

Indeed, by definition (2.5),

$$\begin{aligned} \langle \mathcal{T}\big[x \mapsto \tilde{F}(x,z)\big], \varphi\rangle &= -\int_{\mathbb{R}} \tilde{F}(x,z)\varphi'(x)\ x + p\int_{\mathbb{R}} \tilde{F}(x-\tau,z)\varphi(x)\mathrm{d}x \\ &\quad + q\int_{\mathbb{R}} \tilde{F}([x-\theta],z)\mathrm{d}x. \end{aligned}$$

Differentiating this equation k times with respect to z, which is legal by the classical Lemma 5.3, we obtain

$$\begin{aligned}\partial_z^k\langle\mathcal{T}\big[x\mapsto\tilde{F}(x,z)\big],\varphi\rangle &= -\int_{\mathbb{R}}\partial_z^k\tilde{F}(x,z)\varphi'(x)+p\int_{\mathbb{R}}\partial_z^k\tilde{F}(x-\tau,z)\varphi(x)\mathrm{d}x\\ &\quad +q\int_{\mathbb{R}}\partial_z^k\tilde{F}([x-\theta],z)\mathrm{d}x,\end{aligned}$$

that is (4.17). Notice also that for every $k\in\mathbb{N}^*$, $\partial_z^k\tilde{F}(\cdot,z):\mathbb{R}\to\mathbb{C}$ is continuous, by the Cauchy formula.

Now, by (4.4) and (4.13), we know that

$$f_{z,k+1}(x)=\frac{1}{k!}\partial_z^k\tilde{F}(x,z)\qquad(k\in\mathbb{N},x\in\mathbb{R}),$$

so, by (4.17), for all $\varphi\in\mathcal{D}(\mathbb{R})$,

$$\langle\mathcal{T}[f_{z,k+1}],\varphi\rangle=\frac{1}{k!}\partial_z^k\langle\mathcal{T}\big[x\mapsto\tilde{F}(x,z)\big],\varphi\rangle\qquad(k\in\mathbb{N}).$$

Thus, by (4.16) and by Lemma 5.3, for every $k\in\mathbb{N}$,

$$\mathcal{T}[f_{z,k+1}]=\frac{1}{k!}\left(\tilde{P}(z)\sum_{n\in\mathbb{Z}}\mathrm{e}^{nz}1\!\!1_{[n+\theta,n+\theta+1)}\right)^{(k)}(z)\qquad\text{in }\mathcal{D}'(\mathbb{R}).$$

Applying Leibniz' formula,

$$\mathcal{T}[f_{z,k+1}]=\frac{1}{k!}\sum_{j=0}^{k}\frac{k!}{j!(k-j)!}\tilde{P}^{(j)}(z)\sum_{n\in\mathbb{Z}}n^{k-j}\mathrm{e}^{nz}1\!\!1_{[n+\theta,n+\theta+1)}\quad\text{in }\mathcal{D}'(\mathbb{R}),$$

for every $k\in\mathbb{N}$. From (4.5) and (4.14), we have

$$-\frac{q}{\tilde{P}(w)}=\sum_{n\in\mathbb{N}}c_{z,n}(w-z)^{n-(v_z+\alpha_z)},$$

and thus

$$\frac{\tilde{P}^{(j)}(z)}{j!}=\begin{cases}0 & (0\le j<v_z+\alpha_z),\\ -\dfrac{q}{c_{z,0}} & (j=v_z+\alpha_z),\end{cases}$$

which in turn implies (4.12).

The second equality we shall need is obvious:

$$\mathcal{T}[x\mapsto\mathrm{e}^{xs}]=\phi(s)\mathrm{e}^{xs}+q\sum_{n\in\mathbb{Z}}\mathrm{e}^{ns}1\!\!1_{[n+\theta,n+\theta+1)}\qquad(s\in\mathbb{C}).\tag{4.18}$$

Now assume $n_z\ge 1$.

If $E_z=\emptyset$, then $n_z=v_z=v_z+\alpha_z\ge 1$, and the proof is complete by (4.12).

If $E_z = \{s\}$, then $z \in \Omega^c \cap 2\pi i\mathbb{Z}$ and $n_z = v_z + 1 = v_z + 1 + \alpha_z$ by Lemma 4.3. For every $1 \leq k \leq v_z$, $\mathcal{T}[F_{z,k}] = 0$ by (4.12) and (4.7). For F_{z,n_z} defined by (4.8), we have, by (4.12),

$$\mathcal{T}[F_{z,n_z}] = \frac{\mathcal{T}[x \mapsto \mathrm{e}^{xs}]}{c_{z,0}} + \mathcal{T}[f_{z,n_z}] = \frac{\mathcal{T}[x \mapsto \mathrm{e}^{xs}]}{c_{z,0}} - \frac{q}{c_{z,0}} \sum_{n\in\mathbb{Z}} \mathrm{e}^{nz} \mathbb{1}_{[n+\theta,n+\theta+1)}.$$

Using (4.18), we find (recall that $z, s \in 2\pi i\mathbb{Z}$):

$$\mathcal{T}[F_{z,n_z}] = \frac{\phi(s)\mathrm{e}^{xs}}{c_{z,0}} = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}),$$

and this concludes the case $|E_z| = 1$.

If $E_z = \{s, \bar{s}\} \subset 2\pi i\mathbb{Z}$, then $n_z = v_z + 2 = v_z + 2 + \alpha_z \geq 2$. For every $1 \leq k \leq v_z$, $\mathcal{T}[F_{z,k}] = 0$ by (4.12) and (4.7). By the same computation as in the previous case,

$$\mathcal{T}[F_{z,n_z-1}] = \frac{\mathcal{T}[x \mapsto \mathrm{e}^{xs}]}{c_{z,0}} + \mathcal{T}[f_{z,v_z+1}] = \frac{\mathcal{T}[x \mapsto \mathrm{e}^{x\bar{s}}]}{c_{z,0}} + \mathcal{T}[f_{z,v_z+1}] = 0.$$

If $E_z = \{s, \bar{s}\} \not\subset 2\pi i\mathbb{Z}$ and $v_z = -1$, by (4.18),

$$\mathcal{T}[x \mapsto (\mathrm{e}^{xs} - \mathrm{e}^{x\bar{s}})] = \mathcal{T}[x \mapsto \mathrm{e}^{xs}] - \mathcal{T}[x \mapsto \mathrm{e}^{x\bar{s}}] = 0.$$

If $E_z = \{s, \bar{s}\} \not\subset 2\pi i\mathbb{Z}$ and $v_z \geq 0$, then $n_z = v_z + 2 = v_z + \alpha_z + 1$. For every $1 \leq k \leq v_z + \alpha_z$, $\mathcal{T}[F_{z,k}] = 0$ by (4.12) and (4.7). From (4.11), (4.18) and (4.12),

$$\begin{aligned}
\mathcal{T}[F_{z,n_z}] &= \frac{\mathcal{T}[x \mapsto \mathrm{e}^{xs}] + \mathcal{T}[x \mapsto \mathrm{e}^{x\bar{s}}]}{2c_{z,0}} + \mathcal{T}[f_{z,n_z}], \\
&= \frac{\phi(s)\mathrm{e}^{xs} + \phi(\bar{s})\mathrm{e}^{x\bar{s}}}{2c_{z,0}} + q \sum_{n\in\mathbb{Z}} \Big(\frac{\mathrm{e}^{ns} + \mathrm{e}^{n\bar{s}}}{2} - \mathrm{e}^{nz}\Big) \mathbb{1}_{[n+\theta,n+\theta+1)}, \\
&= 0.
\end{aligned}$$

For the last equality we used $s, \bar{s} \in Z(\phi)$ and $s, \bar{s} \in \{z\} + 2\pi i\mathbb{Z}$. $\square$

Similar results apply if we replace F by G and $\mathcal{T}$ by $\mathcal{T}^*$. We shall need them in the next section.

Lemma 4.5. *Assume $p \neq 0$ and let $(y, z) \in \mathbb{R} \times \Omega^c$. Then,*

(1) *if $\tau p\mathrm{e} = 1$ and $z \in -p\mathrm{e} + 2\pi i\mathbb{Z}$, the valuation of $G(y, w)$ at $w = z$ is -2; more precisely,*

$$\lim_{w\to z} G(y, w)(w - z)^2 = 2\frac{\mathrm{e}^{-ys}}{\phi''(s)}.$$

(2) *if $E_z = \{s\}$ and the previous case does not hold, the valuation of $G(y, w)$ at $w = z$ is -1; more precisely,*

$$\lim_{w\to z} G(y, w)(w - z) = \frac{\mathrm{e}^{-ys}}{\phi'(s)}.$$

(3) *if* $E_z = \{s, \bar{s}\}$ *with* $s \notin \mathbb{R}$, *the valuation of* $G(y,w)$ *at* $w = z$ *is* ≥ -1; *more precisely,*

$$\lim_{w \to z} G(y,w)(w-z) = \frac{\mathrm{e}^{-ys}}{\phi'(s)} + \frac{\mathrm{e}^{-y\bar{s}}}{\phi'(\bar{s})}.$$

Proof. The proof is straightforward, using the definition (3.4) of G and the proof of Lemma 3.3. □

The analogue of α_z is the number

$$\beta_z = \begin{cases} 0 & \text{if } |E_z| = 0, \\ 2 & \text{if } \tau p\mathrm{e} = 1 \text{ and } z \in -p\mathrm{e} + 2\pi i\mathbb{Z}, \\ 1 & \text{if } |E_z| \geq 1 \text{ and the previous case does not hold.} \end{cases} \tag{4.19}$$

For every $z \in \mathbb{C}$, let $\{g_{z,n}\}_{n=1}^{\infty}$ be the sequence of functions uniquely defined by

$$G(y,w)(w-z)^{\beta_z} = \sum_{n \geq 1} g_{z,n}(y)(w-z)^{n-1} \qquad (y \in \mathbb{R},\ w \text{ near } z). \tag{4.20}$$

By Lemma 4.5, the family $\{g_{z,n}\}_{n=1}^{\infty}$ is well defined and $g_{z,1} \not\equiv 0$.

For every $z \in \mathbb{C}$ such that $n_z \geq 1$, we define the finite sequence of functions $\{G_{z,k}\}_{1 \leq k \leq n_z}$ as follows, for all $y \in \mathbb{R}$:
if $E_z = \emptyset$ or $E_z = \{s\} \subset 2\pi i\mathbb{Z}$,

$$G_{z,k}(y) = \sum_{m+n = n_z - k} c_{z,m}\ g_{z,n+1}(y), \qquad k = 1, \ldots, n_z; \tag{4.21}$$

if $E_z = \{s, \bar{s}\} \subset 2\pi i\mathbb{Z}$,

$$\begin{aligned} G_{z,k}(y) &= \sum_{m+n = v_z + 1 - k} c_{z,m}\ g_{z,n+1}(y), \qquad k = 1, \ldots, v_z, \\ G_{z,n_z - 1}(y) &= c_{z,0}\frac{\mathrm{e}^{-ys}}{\phi'(s)}, \quad G_{z,n_z}(y) = c_{z,0}\frac{\mathrm{e}^{-y\bar{s}}}{\phi'(\bar{s})}; \end{aligned} \tag{4.22}$$

if $E_z = \{s, \bar{s}\} \not\subset 2\pi i\mathbb{Z}$ and $v_z = -1$,

$$G_{z,n_z}(y) = \frac{s\mathrm{e}^{(\theta - y)s} - \bar{s}\mathrm{e}^{(\theta - y)\bar{s}}}{s\mathrm{e}^{\theta s}\phi'(s) + \bar{s}\mathrm{e}^{\theta\bar{s}}\phi'(\bar{s})}; \tag{4.23}$$

if $E_z = \{s, \bar{s}\} \not\subset 2\pi i\mathbb{Z}$ and $v_z \geq 0$,

$$\begin{aligned} G_{z,1}(y) &= \frac{s\mathrm{e}^{(\theta - y)s} + \bar{s}\mathrm{e}^{(\theta - y)\bar{s}}}{2(1 - \mathrm{e}^{-z})} + \sum_{m+n = n_z - 1} c_{z,m}\ g_{z,n+1}(y), \\ G_{z,k}(y) &= \sum_{m+n = n_z - k} c_{z,m}\ g_{z,n+1}(y), \qquad k = 2, \ldots, n_z. \end{aligned} \tag{4.24}$$

Starting from (3.13), and imitating the proof of Theorem 4.4, it is possible to show that for every $z \in \mathbb{C}$ such that $n_z \geq 1$, and every $k \in \{1, \ldots, n_z\}$, $\mathcal{T}^*[G_{z,k}] = 0$.

5. Asymptotic expansion

Now, we are ready to compute the residues of the integrand in (3.29) in every situation. This will allow us to find an asymptotic expansion of any solution of (1.1).

Lemma 5.1. *For every* $(x,y) \in \mathbb{R}^2$*, let* $\Delta_{x,y}$ *be the meromorphic function*

$$\Delta_{x,y}(w) = G(y-x,w) - q\frac{G(y,w)F(x,w)}{P(w)} \qquad (w \in \mathbb{C}). \tag{5.1}$$

Then, for every $z \in \mathbb{C}$*,*

$$\mathrm{Res}_z\big(\Delta_{x,y}\big) = \sum_{1\le k\le n_z} F_{z,k}(x)G_{z,k}(y) \qquad (x \in \mathbb{R}, y \in \mathbb{R}) \tag{5.2}$$

if $n_z \geq 1$*; otherwise,* $\mathrm{Res}_z(\Delta_{x,y}) = 0$.

The symbol Res_z denotes the residue at z.

Proof. Let $(x,y) \in \mathbb{R}^2$, $z \in \mathbb{C}$ and consider the functions

$$\begin{aligned} Q(w) &= P(w)(w-z)^{-v_z} && (w \in \Omega), \\ R(w) &= F(x,w)(w-z)^{\alpha_z} && (w \in \Omega), \\ S(w) &= G(y,w)(w-z)^{\beta_z} && (w \in \Omega), \end{aligned}$$

where α_z and β_z are defined by (4.3) and (4.19) respectively. In particular, Q, R and S are holomorphic in a neighbourhood of z, and $Q(z) \neq 0$. We have

$$\Delta_{x,y}(w) = G(y-x,w) - q\frac{1}{(w-z)^{v_z+\alpha_z+\beta_z}}\frac{R(w)S(w)}{Q(w)} \qquad (w \text{ near } z),$$

so

$$\mathrm{Res}_z\big(\Delta_{x,y}\big) = \mathrm{Res}_z\big(G(y-x,w)\big) - q\mathrm{Res}_z\left(\frac{1}{(w-z)^{v_z+\alpha_z+\beta_z}}\frac{R(w)S(w)}{Q(w)}\right). \tag{5.3}$$

The function RS/Q is holomorphic near z. If $v_z + \alpha_z + \beta_z = 0$,

$$-q\mathrm{Res}_z\left(\frac{1}{(w-z)^{v_z+\alpha_z+\beta_z}}\frac{R(w)S(w)}{Q(w)}\right) = 0; \tag{5.4}$$

otherwise, $v_z + \alpha_z + \beta_z \geq 1$ and by Taylor expansion,

$$\begin{aligned} &-q\mathrm{Res}_z\left(\frac{1}{(w-z)^{v_z+\alpha_z+\beta_z}}\frac{R(w)S(w)}{Q(w)}\right) \\ &= -\frac{q}{(v_z+\alpha_z+\beta_z-1)!}\left(\frac{RS}{Q}\right)^{(v_z+\alpha_z+\beta_z-1)}(z), \\ &= \sum_{k+m+n=v_z+\alpha_z+\beta_z-1} \frac{R^{(k)}(z)}{k!}\,\frac{(-q/Q)^{(m)}(z)}{m!}\,\frac{S^{(n)}(z)}{n!}, \qquad (5.5) \\ &= \sum_{k+m+n=v_z+\alpha_z+\beta_z-1} f_{z,k+1}(x)\; c_{z,m}\; g_{z,n+1}(y). \qquad (5.6) \end{aligned}$$

In (5.5) we used Leibniz' formula, and in (5.6) we used definitions (4.4), (4.5) and (4.20). Let $Z(P)$ be the set of zeros of P in Ω.

1st case: $z \in \Omega \setminus Z(P)$. Then $v_z = \alpha_z = \beta_z = 0$ and $w \mapsto G(y-x,w)$ is holomorphic at z so $\mathrm{Res}_z\big(\Delta_{x,y}\big) = 0$.

2nd case: $z \in \Omega \cap Z(P)$. Then $v_z + \alpha_z + \beta_z = v_z \geq 1$ and $w \mapsto G(y-x,w)$ is holomorphic at z. By (5.3) and (5.6)

$$\begin{aligned}
\mathrm{Res}_z\big(\Delta_{x,y}\big) &= \sum_{k+m+n=v_z-1} f_{z,k+1}(x)\ c_{z,m}\ g_{z,n+1}(y) \\
&= \sum_{1\leq k\leq v_z} f_{z,k}(x) \sum_{m+n=v_z-k} c_{z,m}\ g_{z,n+1}(y), \\
&= \sum_{1\leq k\leq v_z} F_{z,k}(x) G_{z,k}(y).
\end{aligned}$$

3rd case: $E_z = \{s\} \not\subset 2\pi i\mathbb{Z}$ (this case includes the only situation where $v_z = -2$). We use (5.2) for the computation. By $2\pi i$-periodicity of $\Delta_{x,y}$,

$$\mathrm{Res}_z\big(\Delta_{x,y}\big) = \mathrm{Res}_s\big(\Delta_{x,y}\big).$$

By definition of F and P,

$$F(x,w) = \frac{1-\mathrm{e}^{-w}}{w}\frac{\mathrm{e}^{(x-\theta)w}}{\phi(w)} + O(1) \qquad (w \to s)$$

and

$$P(w) = q\frac{1-\mathrm{e}^{-w}}{w}\frac{\mathrm{e}^{-\theta w}}{\phi(w)} + O(1) \qquad (w \to s),$$

so

$$q\frac{F(x,w)}{P(w)} = \mathrm{e}^{xw} + O\big(\phi(w)\big) \qquad (w \to s).$$

On the other hand, by definition of G,

$$G(y,w) = \frac{\mathrm{e}^{-yw}}{\phi(w)} + O(1) \quad \text{and} \quad G(y-x,w) = \frac{\mathrm{e}^{(x-y)w}}{\phi(w)} + O(1) \qquad (w \to s).$$

Hence,

$$\Delta_{x,y}(w) = O(1) \qquad (w \to s),$$

and we find (as expected, since $n_z = 0$) $\mathrm{Res}_z\big(\Delta_{x,y}\big) = 0$.

4th case: $E_z = \{s\} \subset 2\pi i\mathbb{Z}$. Then, $v_z \geq 0$, $\alpha_z = 0$ and $\beta_z = 1$. By (5.3), (5.6) and Lemma 4.5,

$$\mathrm{Res}_z\big(\Delta_{x,y}\big) = \frac{\mathrm{e}^{(x-y)s}}{\phi'(s)} + \sum_{k+m+n=v_z} f_{z,k+1}(x) c_{z,m}\ g_{z,n+1}(y).$$

Using $g_{z,1}(y) = \dfrac{\mathrm{e}^{-ys}}{\phi'(s)}$, we find that $\mathrm{Res}_z\big(\Delta_{x,y}\big)$ equals

$$\sum_{1\le k\le v_z} f_{z,k}(x) \sum_{m+n=v_z+1-k} c_{z,m}\ g_{z,n+1}(y) + \left(f_{z,v_z+1}(x) + \frac{\mathrm{e}^{xs}}{c_{z,0}}\right) c_{z,0} g_{z,1}(y),$$

Thus,

$$\mathrm{Res}_z\big(\Delta_{x,y}\big) = \sum_{1\le k\le n_z} F_{z,k}(x) G_{z,k}(y),$$

and this concludes the 4th case.

5th case: $E_z = \{s,\bar{s}\} \subset 2\pi i\mathbb{Z}$. Then, $v_z \geq 0$, $\alpha_z = 0$ and $\beta_z = 1$. By (5.3), (5.6) and Lemma 4.5,

$$\mathrm{Res}_z\big(\Delta_{x,y}\big) \;=\; \frac{\mathrm{e}^{(x-y)s}}{\phi'(s)} + \frac{\mathrm{e}^{(x-y)\bar{s}}}{\phi'(\bar{s})} + \sum_{k+m+n=v_z} f_{z,k+1}(x) c_{z,m}\ g_{z,n+1}(y),$$

Using $g_{z,1}(y) = \dfrac{\mathrm{e}^{-ys}}{\phi'(s)} + \dfrac{\mathrm{e}^{-y\bar{s}}}{\phi'(\bar{s})}$, we find that $\mathrm{Res}_z\big(\Delta_{x,y}\big)$ equals

$$\sum_{1\le k\le v_z} f_{z,k}(x) \sum_{m+n=v_z+1-k} c_{z,m}\ g_{z,n+1}(y) + \left(f_{z,v_z+1}(x) + \frac{\mathrm{e}^{xs}}{c_{z,0}}\right) c_{z,0} \frac{\mathrm{e}^{-ys}}{\phi'(s)}$$
$$+\left(f_{z,v_z+1}(x) + \frac{\mathrm{e}^{x\bar{s}}}{c_{z,0}}\right) c_{z,0} \frac{\mathrm{e}^{-y\bar{s}}}{\phi'(\bar{s})},$$

Thus,

$$\mathrm{Res}_z\big(\Delta_{x,y}\big) \;=\; \sum_{1\le k\le n_z} F_{z,k}(x) G_{z,k}(y),$$

and this case is complete.

6th case: $E_z = \{s,\bar{s}\} \not\subset 2\pi i\mathbb{Z}$ and $v_z = -1$. Then, $\alpha_z = 1$ and $\beta_z = 1$. By (5.3), (5.6) and Lemma 4.5,

$$\mathrm{Res}_z\big(\Delta_{x,y}\big) \;=\; \frac{\mathrm{e}^{(x-y)s}}{\phi'(s)} + \frac{\mathrm{e}^{(x-y)\bar{s}}}{\phi'(\bar{s})} + f_{z,1}(x) c_{z,0} g_{z,1}(y)$$

with

$$g_{z,1}(y) = \frac{\mathrm{e}^{-ys}}{\phi'(s)} + \frac{\mathrm{e}^{-y\bar{s}}}{\phi'(\bar{s})}. \tag{5.7}$$

By Lemma 4.3,

$$f_{z,1}(x) = (1-\mathrm{e}^{-z})\left(\frac{\mathrm{e}^{(x-\theta)s}}{s\phi'(s)} + \frac{\mathrm{e}^{(x-\theta)\bar{s}}}{\bar{s}\phi'(\bar{s})}\right), \tag{5.8}$$

and

$$\frac{1}{c_{z,0}} = -(1-\mathrm{e}^{-z})\left(\frac{\mathrm{e}^{-\theta s}}{s\phi'(s)} + \frac{\mathrm{e}^{-\theta\bar{s}}}{\bar{s}\phi'(\bar{s})}\right). \tag{5.9}$$

Thus, $\mathrm{Res}_z\big(\Delta_{x,y}\big)/c_{z,0}(1-\mathrm{e}^{-z})$ equals

$$\left(\frac{\mathrm{e}^{(x-\theta)s}}{s\phi'(s)}+\frac{\mathrm{e}^{(x-\theta)\bar{s}}}{\bar{s}\phi'(\bar{s})}\right)\left(\frac{\mathrm{e}^{-ys}}{\phi'(s)}+\frac{\mathrm{e}^{-y\bar{s}}}{\phi'(\bar{s})}\right)-\left(\frac{\mathrm{e}^{(x-y)s}}{\phi'(s)}+\frac{\mathrm{e}^{(x-y)\bar{s}}}{\phi'(\bar{s})}\right)\left(\frac{\mathrm{e}^{-\theta s}}{s\phi'(s)}+\frac{\mathrm{e}^{-\theta\bar{s}}}{\bar{s}\phi'(\bar{s})}\right).$$

Expanding this expression, we find that $\mathrm{Res}_z\big(\Delta_{x,y}\big)$ equals

$$\frac{c_{z,0}(1-\mathrm{e}^{-z})}{\phi'(s)\phi'(\bar{s})}\left(\frac{\mathrm{e}^{(x-\theta)s-y\bar{s}}}{s}+\frac{\mathrm{e}^{(x-\theta)\bar{s}-ys}}{\bar{s}}-\frac{\mathrm{e}^{(x-y)s-\theta\bar{s}}}{\bar{s}}-\frac{\mathrm{e}^{(x-y)\bar{s}-\theta s}}{s}\right),$$

that is

$$\frac{c_{z,0}(1-\mathrm{e}^{-z})}{\phi'(s)\phi'(\bar{s})}\left(\frac{\mathrm{e}^{-\theta s-y\bar{s}}}{s}-\frac{\mathrm{e}^{-\theta\bar{s}-ys}}{\bar{s}}\right)(\mathrm{e}^{xs}-\mathrm{e}^{x\bar{s}}).$$

Finally, multiplying numerator and denominator by $s\bar{s}\mathrm{e}^{\theta s+\theta\bar{s}}$, and using (5.9), we find as expected

$$\mathrm{Res}_z\big(\Delta_{x,y}\big)=\frac{s\mathrm{e}^{(\theta-y)s}-\bar{s}\mathrm{e}^{(\theta-y)\bar{s}}}{s\mathrm{e}^{\theta s}\phi'(s)+\bar{s}\mathrm{e}^{\theta\bar{s}}\phi'(\bar{s})}(\mathrm{e}^{xs}-\mathrm{e}^{x\bar{s}})=F_{z,1}(x)G_{z,1}(y).$$

7th case: $E_z=\{s,\bar{s}\}\not\subset 2\pi i\mathbb{Z}$ and $v_z\geq 0$. Then, $n_z=v_z+2$, $\alpha_z=1$ and $\beta_z=1$. By (5.3), (5.6) and Lemma 4.5,

$$\mathrm{Res}_z\big(\Delta_{x,y}\big)\;=\;\frac{\mathrm{e}^{(x-y)s}}{\phi'(s)}+\frac{\mathrm{e}^{(x-y)\bar{s}}}{\phi'(\bar{s})}+\sum_{k+m+n=v_z+1}f_{z,k+1}(x)\;c_{z,m}\;g_{z,n+1}(y).$$

In other words, $\mathrm{Res}_z\big(\Delta_{x,y}\big)$ equals

$$\begin{aligned}&\frac{\mathrm{e}^{(x-y)s}}{\phi'(s)}+\frac{\mathrm{e}^{(x-y)\bar{s}}}{\phi'(\bar{s})}+f_{z,1}(x)\sum_{m+n=v_z+1}c_{z,m}\;g_{z,n+1}(y)\\&+\sum_{2\leq k\leq v_z+1}f_{z,k}(x)\sum_{m+n=v_z+2-k}c_{z,m}g_{z,n+1}(y)+f_{z,v_z+2}(x)c_{z,0}g_{z,1}(y).\end{aligned}$$

This last expression is equal to

$$\begin{aligned}&f_{z,1}(x)\left(\frac{s\mathrm{e}^{(\theta-y)s}+\bar{s}\mathrm{e}^{(\theta-y)\bar{s}}}{2(1-\mathrm{e}^{-z})}+\sum_{m+n=v_z+1}c_{z,m}\;g_{z,n+1}(y)\right)\\&+\sum_{2\leq k\leq v_z+1}F_{z,k}(x)G_{z,k}(y)+\left(\frac{\mathrm{e}^{xs}+\mathrm{e}^{x\bar{s}}}{2c_{z,0}}+f_{z,n_z}(x)\right)c_{z,0}g_{z,1}(y),\end{aligned}$$

as expected. Indeed, by (5.8) which is still valid,

$$\begin{aligned}
&f_{z,1}(x)\frac{s\mathrm{e}^{(\theta-y)s}+\bar{s}\mathrm{e}^{(\theta-y)\bar{s}}}{2(1-\mathrm{e}^{-z})}+\frac{\mathrm{e}^{xs}+\mathrm{e}^{x\bar{s}}}{2}g_{z,1}(y)\\
&\quad=(1-\mathrm{e}^{-z})\left(\frac{\mathrm{e}^{(x-\theta)s}}{s\phi'(s)}+\frac{\mathrm{e}^{(x-\theta)\bar{s}}}{\bar{s}\phi'(\bar{s})}\right)\frac{s\mathrm{e}^{(\theta-y)s}+\bar{s}\mathrm{e}^{(\theta-y)\bar{s}}}{2(1-\mathrm{e}^{-z})}\\
&\quad+\left(\frac{\mathrm{e}^{xs}+\mathrm{e}^{x\bar{s}}}{2}\right)\left(\frac{\mathrm{e}^{-ys}}{\phi'(s)}+\frac{\mathrm{e}^{-y\bar{s}}}{\phi'(\bar{s})}\right).
\end{aligned}$$

Moreover, $p\in\mathbb{R}$ by Lemma 4.2, so the right side of this equality is

$$\frac{\mathrm{e}^{(x-y)s}}{\phi'(s)}+\frac{\mathrm{e}^{(x-y)\bar{s}}}{\phi'(\bar{s})}+\operatorname{Re}\left(\frac{\mathrm{e}^{(x-\theta)s}\bar{s}\mathrm{e}^{(\theta-y)\bar{s}}}{s\phi'(s)}\right)+\operatorname{Re}\left(\frac{\mathrm{e}^{xs}\mathrm{e}^{-y\bar{s}}}{\phi'(\bar{s})}\right). \tag{5.10}$$

Now recall $v_z\geq 0$, so by Lemma 4.3,

$$\frac{\mathrm{e}^{-\theta s}}{s\phi'(s)}+\frac{\mathrm{e}^{-\theta\bar{s}}}{\bar{s}\phi'(\bar{s})}=0,$$

and (5.10) is finally equal to $\dfrac{\mathrm{e}^{(x-y)s}}{\phi'(s)}+\dfrac{\mathrm{e}^{(x-y)\bar{s}}}{\phi'(\bar{s})}$, and this concludes the proof. □

We can now state our final

Theorem 5.2. *Let $f\in\mathcal{C}(\mathbb{R})$ be a solution of equation* (1.1)*. Then, for every $\sigma\in\mathbb{R}$, the function $R_{f,\sigma}$ defined by*

$$f(x)=\sum_{\substack{z\in\mathcal{P}(\sigma)\\1\leq k\leq n_z}}[\![f,G_{z,k}]\!]F_{z,k}(x)+R_{f,\sigma}(x)\qquad(x>0), \tag{5.11}$$

where

$$\mathcal{P}(\sigma)=\{z\in\mathbb{C}\ :\ \operatorname{Re}z\geq\sigma\ \textit{and}\ 0\leq\operatorname{Im}z<2\pi\}, \tag{5.12}$$

satisfies the estimate

$$R_{f,\sigma}(x)=o(\mathrm{e}^{\sigma x})\qquad(x>0). \tag{5.13}$$

As usual, the notation (5.13) means that $\mathrm{e}^{-\sigma x}R_{f,\sigma}(x)\to 0$ as $x\to+\infty$.

Proof. The idea is to apply the residue theorem to (3.29). Let $\sigma\in\mathbb{R}$, let $r>|p|$ be large enough such that (3.1) holds, and choose $\kappa\in\mathbb{C}$ which satisfies (3.28) and $\operatorname{Re}\kappa>\sigma$. By Theorem 3.5,

$$f(x)=\int_{\kappa}^{\kappa+2\pi i}\Delta_x(z)\mathrm{d}z, \tag{5.14}$$

where for all $x\in\mathbb{R}$,

$$\Delta_x(z)=\left[\!\!\left[f,t\mapsto G(t-x,z)-q\frac{G(t,z)}{P(z)}F(x,z)\right]\!\!\right]. \tag{5.15}$$

Notice that Δ_x is $2\pi i$-periodic.

Since $\mathcal{U}^c = Z(P) \cup \Omega^c$ is discrete, closed in $\mathbb{C}$, and $2\pi i$-invariant, there exists $\sigma' < \sigma$ such that $x + i\mathbb{R} \subset \mathcal{U}$ for all $\sigma' \leq x < \sigma$. With these choices, σ is in the open rectangle $R(\sigma', \kappa)$ with vertices $\sigma' + i\mathrm{Im}\,\kappa$, κ, $\kappa + 2\pi i$, $\sigma' + i\mathrm{Im}\,\kappa + 2\pi i$, whose boundary $\partial R(\sigma', \kappa)$ is included in $\mathcal{U}$. By Lemma 2.1, Lemma 3.1 and Lemma 5.3, Δ_x is holomorphic in $\mathcal{U}$. Since $\mathcal{U}^c := \mathbb{C} \setminus \mathcal{U}$ is discrete, the residue theorem yields

$$\frac{1}{2\pi i} \oint_{\partial R(\sigma', \kappa)} \Delta_x(z)\mathrm{d}z = \sum_{z \in R(\sigma', \kappa) \cap \mathcal{U}^c} \mathrm{Res}_z(\Delta_x).$$

By $2\pi i$-periodicity of Δ_x, the horizontal contributions on the left side add to nothing, and we are left with the vertical contributions:

$$\int_{\kappa}^{\kappa + 2\pi i} \Delta_x(z)\mathrm{d}z + \int_{\sigma' + i\mathrm{Im}\,\kappa + 2\pi i}^{\sigma' + i\mathrm{Im}\,\kappa} \Delta_x(z)\mathrm{d}z = 2\pi i \sum_{z \in R(\sigma', \kappa) \cap \mathcal{U}^c} \mathrm{Res}_z(\Delta_x).$$

By $2\pi i$-periodicity, the contribution on the segment $[\sigma' + i\mathrm{Im}\,\kappa, \sigma' + i\mathrm{Im}\,\kappa + 2\pi i]$ is equal to the contribution on the segment $[\sigma', \sigma' + 2\pi]$, so

$$\int_{\kappa}^{\kappa + 2\pi i} \Delta_x(z)\mathrm{d}z - \int_{\sigma'}^{\sigma' + 2\pi i} \Delta_x(z)\mathrm{d}z = 2\pi i \sum_{z \in R(\sigma', \kappa) \cap \mathcal{U}^c} \mathrm{Res}_z(\Delta_x). \tag{5.16}$$

In other words (recall (5.14)),

$$f(x) = \sum_{z \in R(\sigma', \kappa)} \mathrm{Res}_z(\Delta_x) + r_{\sigma'}(x) \qquad (x > 0), \tag{5.17}$$

where

$$r_{\sigma'}(x) = \frac{1}{2\pi i} \int_{\sigma'}^{\sigma' + 2\pi i} \Delta_x(z)\mathrm{d}z. \tag{5.18}$$

Let us now compute the right side of (5.16). Let $z \in R(\sigma', \kappa) \cap \mathcal{U}^c$ and let $\epsilon > 0$ be small enough such that $\{w \in \mathbb{C} \ : \ 0 < |w - z| \leq \epsilon\} \subset \mathcal{U}$. We have

$$\mathrm{Res}_z(\Delta_x) = \frac{1}{2\pi i} \oint_{|w-z|=\epsilon} \frac{\Delta_x(w)}{w - z}\mathrm{d}w \qquad (x > 0).$$

Writing

$$\Delta_x(w) = [\![f, y \mapsto \Delta_{x,y}(w)]\!] \qquad (x > 0,\ w \in \mathcal{U}),$$

where $\Delta_{x,y}$ is defined by (5.1), and using bilinearity of $[\![\cdot, \cdot]\!]$, we find

$$\mathrm{Res}_z(\Delta_x) = \frac{1}{2\pi i} \oint_{|w-z|=\epsilon} \left[\!\!\left[f, y \mapsto \frac{\Delta_{x,y}(w)}{w - z} \right]\!\!\right] \mathrm{d}w \qquad (x > 0).$$

Now $(w, y) \mapsto \Delta_{x,y}(w)/(w - z)$ is continuous on $\{w \in \mathbb{C} \ : \ |w - z| = \epsilon\} \times \mathbb{R}$, so we can apply Fubini's Theorem, which yields

$$\begin{aligned} \mathrm{Res}_z(\Delta_x) &= \left[\!\!\left[f, y \mapsto \frac{1}{2\pi i} \oint_{|w-z|=\epsilon} \frac{\Delta_{x,y}(w)}{w - z}\mathrm{d}w \right]\!\!\right] \qquad (x > 0), \\ &= [\![f, y \mapsto \mathrm{Res}_z(\Delta_{x,y})]\!] \qquad (x > 0). \end{aligned} \tag{5.19}$$

Consider Lemma 5.1. If $n_z \le 0$, the right side of (5.19) is 0. Now assume $n_z \ge 1$. In this case, (5.19) becomes

$$\operatorname{Res}_z(\Delta_x) = \left[\!\!\left[f, y \mapsto \sum_{1\le k\le n_z} F_{z,k}(x) G_{z,k}(y) \right]\!\!\right] \qquad (x > 0),$$

and by bilinearity,

$$\operatorname{Res}_z(\Delta_x) = \sum_{1\le k\le n_z} [\![f, G_{z,k}]\!] F_{z,k}(x) \qquad (x > 0).$$

Replacing in (5.17):

$$f(x) = \sum_{z\in R(\sigma',\kappa)\cap\mathcal{U}^c} \sum_{1\le k\le n_z} [\![f, G_{z,k}]\!] F_{z,k}(x) + r_{\sigma'}(x) \qquad (x > 0).$$

Letting $\operatorname{Re}\kappa \to +\infty$ (recall Proposition 2.4) yields

$$f(x) = \sum_{z} \sum_{1\le k\le n_z} [\![f, G_{z,k}]\!] F_{z,k}(x) + r_{\sigma'}(x) \qquad (x > 0),$$

where the first sum is over all $z \in \mathcal{U}^c$ such that $\operatorname{Re} z > \sigma$ and $\operatorname{Im}\kappa < \operatorname{Im} z \le \operatorname{Im}\kappa + 2\pi$. By $2\pi i$-periodicity, this sum is the same if we change this latter set into $\mathcal{P}(\sigma)$ (5.12):

$$f(x) = \sum_{z\in\mathcal{P}(\sigma)} \sum_{1\le k\le n_z} [\![f, G_{z,k}]\!] F_{z,k}(x) + r_{\sigma'}(x) \qquad (x > 0).$$

This shows in particular that $r_{\sigma'}(x) = R_{f,\sigma}(x)$, where the latter is defined by (5.13).

Now, let $K := \{z = \sigma' + ib \ : \ 0 \le b \le 2\pi\} \subset \mathcal{U}$. By (5.18),

$$r_{\sigma'}(x) \ll \sup_{z\in K} |\Delta_z(x)| \qquad (x > 0). \tag{5.20}$$

The functions $z \mapsto P(z)$ and $z \mapsto [\![f, t \mapsto G(t-y,z)]\!]$ (which do not depend on x) are continuous on K and P has no zero in K, so

$$\Delta_z(x) \ll |[\![f, t \mapsto G(t-x,z)]\!]| + |F(x,z)| \qquad (x > 0,\ z \in K). \tag{5.21}$$

Imitating the proof of Lemma 2.1, and using $|\mathrm{e}^{x(z+2\pi in)}| = \mathrm{e}^{\sigma' x}$ for all $z \in K$, it is easy to see that

$$\frac{1-\mathrm{e}^{-z}}{z+2\pi in} \, \frac{\mathrm{e}^{(x-\theta)(z+2\pi in)}}{\phi(z+2\pi in)} \ll \frac{\mathrm{e}^{\sigma' x}}{n^2+1} \qquad (x > 0,\ z \in K),$$

and consequently that

$$F(x,z) \ll \mathrm{e}^{\sigma' x} \qquad (x > 0,\ z \in K). \tag{5.22}$$

Similarly, imitating the proof of Lemma 3.1, and using $|\mathrm{e}^{-yz)}| = \mathrm{e}^{-\sigma' y}$ for all $z \in K$, we find

$$\frac{\mathrm{e}^{-y(z+2\pi in)}}{\phi(z+2\pi in)} + \frac{\mathrm{e}^{-y(z-2\pi in)}}{\phi(z-2\pi in)} + \frac{\sin(2\pi ny)}{\pi n}\mathrm{e}^{-yz} \ll \frac{\mathrm{e}^{-\sigma' y}}{n^2+1} \qquad (y \in \mathbb{R},\ z \in K),$$

and consequently,

$$G(y,z)+\mathrm{e}^{-yz}\sum_{n=1}^{\infty}\frac{\sin(2\pi ny)}{\pi n}\ll \mathrm{e}^{-\sigma' y}\qquad (y\in\mathbb{R},\ z\in K).$$

Using in addition Lemma 3.2,

$$G(t-x,z)\ll \mathrm{e}^{\sigma' x}\qquad (0\le t\le \tau+\theta,\ x>0,\ z\in K). \tag{5.23}$$

Putting together (5.20), (5.21), (5.22) and (5.23), we obtain

$$r_{\sigma'}(x)\ll \mathrm{e}^{\sigma' x}\qquad (x>0),$$

and this concludes the proof. □

As previously, Theorem 5.2 has an analogue for equation (3.12). We leave it to the reader to verify this. Theorem 5.2 raises (at least) two questions:
1) It is possible to obtain a series expansion rather than a asymtotic expansion, for every solution of (1.1) ?
2) It is possible to relate the characterictic equation (2.7) to the oscillatory behaviour of the delay differential equation (1.1), as was done previously for certain values of the parameters [7] ?

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Appendix

We recall here a classical lemma concerning the holomorphy of an integral. It was useful in sections 3 and 4 (for the proof, see [6] for instance).

Lemma 5.3. *Let $I\subset\mathbb{R}$ be an interval, let Ω be an open subset of $\mathbb{C}$, and assume that $F: I\times\Omega\to\mathbb{C}$ satisfies the following conditions:*
i) for every $x\in I$, the function $F(x,\cdot):\Omega\to\mathbb{C}$ is holomorphic;
ii) for every $z_0\in\Omega$, the function $F(\cdot,z_0): I\to\mathbb{C}$ is measurable, and there exist a neighbourhood V_{z_0} of z_0 in Ω and a measurable function $l: I\to[0,+\infty)$ such that

$$\int_I l(x)\,dx<\infty\qquad \textit{and}\qquad \forall z\in V_{z_0},\ \forall x\in I,\ |F(x,z)|\le l(x).$$

Then, the function $f:\Omega\to\mathbb{C}$ defined by $f(z)=\displaystyle\int_I F(x,z)\,dx$, is holomorphic in Ω. Moreover, for every $k\in\mathbb{N}$, the function $\partial_z^k F(x,z)$ satisfies the conditions i) and ii), and we have $f^{(k)}(z)=\displaystyle\int_I \partial_z^k F(x,z)\,dx$.

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