Convergence of exponential attractors for a finite element discretization of the Allen-Cahn equation

Morgan PIERRE

Laboratoire de Mathématiques et Applications
Université de Poitiers & CNRS, France

International Video-workshop on Infinite Dimensional Dynamical Systems
Donghua University, Shanghai, China
September 12th and 19th, 2020.

A key concept in the study of dissipative systems is the **global attractor**, a compact invariant set which attracts uniformly the bounded sets of the phase space

Some drawbacks of the global attractor:

- it may be sensible to perturbations (if the rate of attraction of the trajectories is small): upper semicontinuity generally holds, but lower semicontinuity can be proved only in some particular cases
- it may fail to capture important transient behaviours

Example: **[Stuart & Humphries 1996]** we consider the dynamical system on $\mathbf{R}$ defined by

$$u'(t) = -f_\varepsilon(u(t)), \quad t \geq 0,$$

where for $\varepsilon \geq 0$, $f_\varepsilon \in C^1(\mathbf{R})$ is defined by

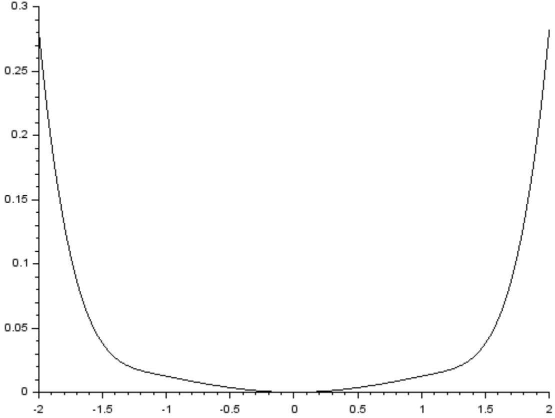
$$f_\varepsilon(u) = \begin{cases} (u+1)^3 - \varepsilon, & \text{if } u \leq -1, \\ \varepsilon(3u/2 - u^3/2), & \text{if } -1 < u < 1, \\ (u-1)^3 + \varepsilon, & \text{if } u \geq 1. \end{cases}$$

This is a gradient flow for F_ε such that $F'_\varepsilon(u) = f_\varepsilon(u)$ and $F_\varepsilon(0) = 0$. Thus, the global attractor is

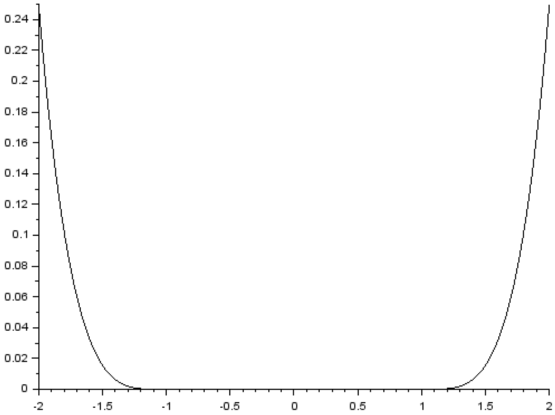
$$\mathcal{A}_\varepsilon = \{0\} \quad \text{if } \varepsilon > 0,$$

and

$$\mathcal{A}_0 = [-1, 1].$$



The potential $F_\varepsilon(u)$, $\varepsilon > 0$



The potential $F_0(u)$

The notion of **exponential attractor** has been proposed in **[Eden, Foais, Nicolaenko & Temam 1994]**: a compact positively invariant set which contains the global attractor, has finite fractal dimension and attracts exponentially the trajectories.

- More robust to perturbations
- Can capture important transient behaviours
- *But* : not necessarily unique (in contrast with the global attractor)

The continuity of exponential attractors was shown in **[Eden, Foias, Nicolaenko & Temam 1994]** for classical Galerkin approximations, but only up to a **time shift**.
see also **[Fabrie, Galunsinski & Miranville 2000], [Galusinski PhD thesis 1996]** for continuity up to a time shift, and **[Aida & Yagi 2004]** for related robustness results for finite element approximations.

The continuity of exponential attractors was shown in **[Eden, Foias, Nicolaenko & Temam 1994]** for classical Galerkin approximations, but only up to a **time shift**.
see also **[Fabrie, Galunsinski & Miranville 2000], [Galusinski PhD thesis 1996]** for continuity up to a time shift, and **[Aida & Yagi 2004]** for related robustness results for finite element approximations.

Efendiev, Miranville & Zelik (2004) proposed a construction of exponential attractors where **continuity holds without time shift**. It is based on a uniform **"smoothing property"** and an appropriate error estimate. It is valid in Banach spaces and it gives a **uniform bound on the fractal dimension** of the attractor.

This result has been adapted to many situations, including singular perturbations:
[Fabrie, Galusinski, Miranville & Zelik 2004], [Gatti, Grasselli & Pata 2004], [Conti, Pata & Squassina 2005], [Gatti, Grasselli, Miranville & Pata 2006], [Cavaterra & Grasselli 2006] . . .
see in particular the review paper by **[Miranville & Zelik 2008]**.

Question: can we adapt the construction of **Efendiev, Miranville & Zelik (2004)** when the perturbation is a space and/or time discretization of the PDE ?

We consider a **model problem**: the **Allen-Cahn equation** in space dimension $1 \leq d \leq 3$.
We study:

- First, a **space semidiscretization** by P^1 finite elements.
- Second, a **time semidiscretization** by the backward Euler scheme.

Some definitions

$H = L^2(\Omega)$ with norm $|\cdot|_H$ and $\mathcal{K}$ is a closed subset of H.
A **continuous-in-time semigroup** $\{S(t),\ t \in \mathbf{R}_+\}$ on $\mathcal{K}$ is a family of (nonlinear) operators such that $S(t)$ is a continuous operator from $\mathcal{K}$ into itself, for all $t \geq 0$, with $S(0) = Id$ (identity in $\mathcal{K}$) and

$$S(t+s) = S(t) \circ S(s), \quad \forall s, t \geq 0.$$

A **discrete-in-time semigroup** $\{S(t),\ t \in \mathbf{N}\}$ on H is a family of (nonlinear) operators which satisfy these properties with $\mathbf{R}_+ (= [0, +\infty))$ replaced by $\mathbf{N}$.
A discrete-in-time semigroup is usually denoted $\{S^n,\ n \in \mathbf{N}\}$, where $S (= S(1))$ is a continuous (nonlinear) operator from $\mathcal{K}$ into itself.
Remark 1: semigroup $\simeq$ dynamical system

• $dist_H$ denotes the **non-symmetric Hausdorff semidistance** in H between two subsets defined as

$$dist_H(A,B) = \sup_{a\in A} \inf_{b\in B} |a-b|_H.$$

Remark: $dist_H(A,B) = 0 \iff A \subset \overline{B}$

• Let $A \subset H$ be a subset of H. For $\varepsilon > 0$, we denote $N_\varepsilon(A,H)$ the minimum number of balls of H of radius $\varepsilon > 0$ which are necessary to cover A. The **fractal dimension of** A in H is the number

$$dim_F(A,H) = \limsup_{\varepsilon\to 0} \frac{\log_2(N_\varepsilon(A,H))}{\log_2(1/\varepsilon)} \in [0,+\infty].$$

Remark: $dim_F(A,H) < +\infty \Rightarrow$ **A is relatively compact in H.**

Definition (Exponential attractor)

Let $\{S(t),\ t \geq 0\}$ *be a continuous or discrete semigroup on* $\mathcal{K}$*. A set* $\mathcal{M} \subset \mathcal{K}$ *is an exponential attractor of the dynamical system if the following three conditions are satisfied:*

1. $\mathcal{M}$ *is compact in* H *and has finite fractal dimension;*
2. $\mathcal{M}$ *is positively invariant, i.e.* $S(t)\mathcal{M} \subset \mathcal{M}$*, for all* $t \geq 0$*;*
3. $\mathcal{M}$ *attracts exponentially the bounded subsets of* $\mathcal{K}$ *in the following sense:*

$$\forall B \subset \mathcal{K} \text{ bounded},\ dist_H(S(t)B, \mathcal{M}) \leq \mathcal{Q}(\|B\|_H)e^{-\alpha t},\ t \geq 0,$$

where the positive constant α *and the monotonic function* $\mathcal{Q}$ *are independent of* B*. Here,* $\|B\|_H = \sup_{b \in B} |b|_H$*.*

The exponential attractor, if it exists, contains the global attractor.

Definition (Exponential attractor on a bounded set)

If $\mathcal{B}$ is a closed bounded subset of H and if L is a (nonlinear) continuous operator from $\mathcal{B}$ into $\mathcal{B}$, we will say that a set $\mathcal{M}^d \subset \mathcal{B}$ is an exponential attractor for (the dynamical system generated by) the iterations of L if

1. *$\mathcal{M}^d$ is compact and has finite fractal dimension in H,*
2. *$\mathcal{M}^d$ is positively invariant, i.e. $L\mathcal{M}^d \subset \mathcal{M}^d$,*
3. *$\mathcal{M}^d$ attracts $\mathcal{B}$ exponentially, i.e.*

$$dist_H(L^n\mathcal{B}, \mathcal{M}^d) \leq Ce^{-\alpha n}, \quad n \in \mathbf{N},$$

where C and $\alpha > 0$ are independent of n.

Theorem (Efendiev, Miranville & Zelik 2000)

Let H, V be two Banach spaces such that V is compactly imbedded in H and let $\mathcal{B}$ be a closed bounded subset of H. Let $L : \mathcal{B} \to \mathcal{B}$ be a (nonlinear) continuous mapping which enjoys the **smoothing property**, *i.e.*

$$\|Lu_1 - Lu_2\|_V \leq c|u_1 - u_2|_H, \quad \forall u_1, u_2 \in \mathcal{B}. \tag{1}$$

Then the discrete dynamical system generated by the iterations of L possesses an exponential attractor $\mathcal{M}^d \subset \mathcal{B}$.

Theorem (Efendiev,Miranville & Zelik 2004)

Let H, V be two Banach spaces such that V is compactly imbedded in H and let $\mathcal{B}$ be a closed bounded subset of E. We assume that the family of continuous operators $L_\varepsilon : \mathcal{B} \to \mathcal{B}$, $\varepsilon \in [0,1]$ satisfies the following assumptions:

1. *(Uniform, with respect to ε, smoothing property)* $\forall \varepsilon \in [0,1]$, $\forall u_1, u_2 \in \mathcal{B}$,

$$\|L_\varepsilon u_1 - L_\varepsilon u_2\|_V \leq c_1 |u_1 - u_2|_H,$$

   where c_1 is independent of ε

2. *(The trajectories of the perturbed system approach those of the nonperturbed one, uniformly with respect to ε, as ε tends to 0)* $\forall \varepsilon \in [0,1]$, $\forall i \in \mathbf{N}$, $\forall u \in \mathcal{B}$,

$$|L_\varepsilon^i u - L_0^i u|_H \leq c_2^i \varepsilon \quad (c_2 \text{ independent of } \varepsilon).$$

Theorem (continued)

Then, $\forall \varepsilon \in [0,1]$, the discrete dynamical system generated by the iterations of L_ε possesses an exponential attractor $\mathcal{M}_\varepsilon^d$ on $\mathcal{B}$ such that
1. the fractal dimension of $\mathcal{M}_\varepsilon^d$ is bounded, uniformly with respect to ε,
2. $\mathcal{M}_\varepsilon^d$ attracts $\mathcal{B}$, uniformly with respect to ε,
3. the family $\{\mathcal{M}_\varepsilon^d,\ \varepsilon \in [0,1]\}$ is continuous at 0,

$$dist_{sym}(\mathcal{M}_\varepsilon^d, \mathcal{M}_0^d) \leq c\varepsilon^{c'},$$

where c and $c' \in (0,1)$ are independent of ε

$dist_{sym}$ denotes the symmetric Hausdorff distance between sets defined by

$$dist_{sym}(A,B) := \max\left(dist_H(A,B), dist_H(B,A)\right).$$

Remark: property 3 (continuity at 0) does not imply 1 (uniform bound on the fractal dimension).

Indeed, consider in $H = l^2(\mathbf{N})$ the n-dimensional ball of radius $\varepsilon_n = 1/n$, namely

$$B_{1/n} = \left\{ (u_k)_{k \in \mathbf{N}} \ : \ \sum_{k=0}^{+\infty} u_k^2 < \frac{1}{n^2} \text{ and } u_k = 0 \text{ if } k \geq n \right\}.$$

Then $dim_F(B_{1/n}, H) = n \to +\infty$ but

$$dist_{sym}\left(B_{1/n}, \{0\}\right) = dist_H\left(B_{1/n}, \{0\}\right) = \frac{1}{n} \to 0$$

(note that $dist_H(\{0\}, B_{1/n}) = 0$ since $0 \in B_{1/n}$)

The continuous problem

We consider the following reaction-diffusion equation

$$\partial_t u - \Delta u + g(u) = 0 \quad \text{in } \Omega \times \mathbf{R}_+, \tag{2}$$

subject to homogeneous Dirichlet boundary conditions; Ω is a **convex** open bounded subset of $\mathbf{R}^d$ $(1 \leq d \leq 3)$ with C^2 boundary, and

$$g(s) = \sum_{j=0}^{2p-1} b_j s^j, \quad b_{2p-1} > 0, \; p \geq 1.$$

If $d = 3$, then $p \in \{1, 2\}$ (no restriction on p if $d = 1$ or 2).
When $g(s) = s^3 - s$ (then $p = 2$), equation (2) is known as the **Allen-Cahn equation**.
We supplement (2) with an initial condition

$$u(0) = u_0. \tag{3}$$

- $H = L^2(\Omega)$ with norm $|\cdot|_H$ and scalar product $(\cdot,\cdot)_H$.
- $V = H_0^1(\Omega)$ with norm $\|\cdot\|_V = |\nabla\cdot|_{L^2(\Omega)^d}$.

It is well-known that (2) defines a continuous-in-time semigroup S_0:

$$S_0(t) : u_0 \in H \mapsto u(t) \in H.$$

Proposition (Absorbing set in V, see e.g. [Temam 1997])

There exist a constant $\mathcal{R}_1 > 0$ and a monotonic function $\mathcal{T}_1(\cdot)$ such that for all $u_0 \in H$,

$$\|u(t)\|_V \leq \mathcal{R}_1, \quad \forall t \geq \mathcal{T}_1(|u_0|_H).$$

Lemma (based on the gradient flow structure of (2))

For any $R_1 > 0$, there exists a constant $C_1(R_1)$ such that for all $u_0 \in V$ with $\|u_0\|_V \leq R_1$,

$$\|u(t)\|_V^2 + \int_0^t |\partial_t u|_H^2 \, ds \leq C_1(R_1), \quad \forall t \geq 0.$$

In particular, for all $t_1, t_2 \geq 0$, we have

$$|u(t_1) - u(t_2)|_H^2 \leq C_1(R_1)|t_1 - t_2|.$$

Let u and $\hat{u}$ be two solutions of (2) and let $v(t) = u(t) - \hat{u}(t)$ be their difference, which satisfies

$$\partial_t v - \Delta v + g(u) - g(\hat{u}) = 0 \quad \text{in } \Omega \times \mathbf{R}_+. \tag{4}$$

Lemma ($S_0(t)$ is Lipschitz continuous on H)

For all $t \geq 0$,

$$|v(t)|_H^2 + 2\int_0^t \|v\|_V^2 \, ds \leq |v(0)|_H^2 \exp(2c_1' t).$$

Lemma (H-V smoothing property)

If $\|u(0)\|_V \leq R_1$ and $\|\hat{u}(0)\|_V \leq R_1$, then for all $t > 0$, we have

$$\|v(t)\|_V^2 \leq C_2(R_1, t)|v(0)|_H^2,$$

where the function $C_2 : (0, +\infty)^2 \to \mathbf{R}_+$ is continuous.

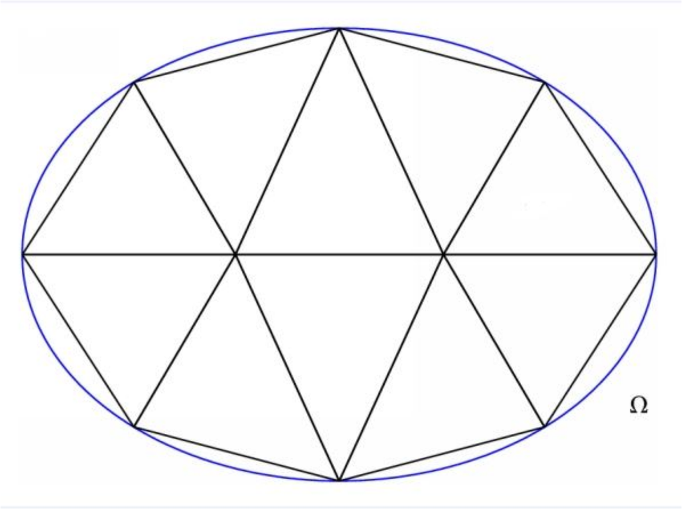
The space semidiscrete problem

We use continuous piecewise linear (P_1) finite elements.
Following **[Raviart & Thomas 1983]**, we use a **regular family of triangulations** $(\mathcal{T}_h)_{h>0}$ such that for every h, Ω is approximated by a convex d-polyhedron $\Omega_h = \bigcup_{K\in\mathcal{T}_h} K$.
For a given $\mathcal{T}_h$, the finite element space is

$$V_h = \left\{ v \in C^0(\overline{\Omega}) \;:\; v = 0 \text{ on } \overline{\Omega} \setminus \Omega_h \text{ and } \forall K \in \mathcal{T}_h,\; v|_K \in P_1 \right\},$$

and we have a **conforming approximation**, namely

$$V_h \subset H^1_0(\Omega_h) \subset H^1_0(\Omega) = V.$$



A triangulation of $\Omega_h \subset \Omega$

The space semidiscrete scheme reads: find $u_h : \mathbf{R}_+ \to V_h$ such that

$$\frac{d}{dt}(u_h(t), \varphi_h)_H + (\nabla u_h(t), \nabla \varphi_h)_0 + (g(u_h(t)), \varphi_h)_H = 0, \qquad (5)$$

for all $t \geq 0$ and for all $\varphi_h \in V_h$, with the initial condition

$$u_h(0) = u_h^0 \in V_h. \qquad (6)$$

Since V_h has finite dimension, it is easily seen that for every $u_h^0 \in V_h$, problem (5)-(6) has a unique solution $u_h \in C^1(\mathbf{R}_+, V_h)$. Thus, we have a semigroup S_h acting on V_h,

$$S_h(t) : u_h^0 \in V_h \mapsto u_h(t) \in V_h.$$

It is easy to show that for the space semidiscrete problem, we obtain a priori estimates similar to the continuous problem, and **which are uniform in** h:

- Absorbing set in V_h for the V-norm
- Gradient-flow structure (S_h is $1/2$-Hölder continuous in time)
- S_h is Lipschitz continuous in space
- H-V smoothing property

Theorem (Error estimate with H^1 data)

For all $R_1 > 0$ and for all $T > 0$, there exists a constant $C_3(R_1, T)$ independent of h such that, if $\|u_0\|_V \leq R_1$ and $u_h^0 = \Pi_h(u_0)$ where $\Pi_h : V \to V_h$ is the (linear) elliptic projection, then

$$\sup_{t\in[0,T]} |u_h(t) - u(t)|_H \leq C_3(R_1, T)h.$$

We follow the approach of **[Johsson, Larsson, Thomée & Wahlbin 1987]** with some ideas from **[Elliott & Larsson 1992]**

- Error estimates for the linear elliptic problem **[Raviart & Thomas 1983]** **also valid in dimension $d = 3$**
- Error estimates with nonsmooth data for the linear evolution problem: **book of [Thomée 2006]** **(smoothing properties of the heat equation)**

The main result

Theorem (Convergence of exponential attractors)

The continuous semigroup $\{S_0(t),\ t \in \mathbf{R}_+\}$ on H associated to (2) possesses an exponential attractor $\mathcal{M}_0$ and for every $h \in J$, the continuous semigroup $\{S_h(t),\ t \in \mathbf{R}_+\}$ on V_h associated to (5) possesses an exponential attractor $\mathcal{M}_h$ such that:

NB: $J = (0, h_{max}]$ typically

Theorem (Convergence of exponential attractors, continued)

1. *the fractal dimension of $\mathcal{M}_h$ is bounded, uniformly with respect to $h \in J$, $dim_F(\mathcal{M}_h, H) \leq c_{10}$, where c_{10} is independent of h;*
2. *$\mathcal{M}_h$ attracts the bounded sets of V_h, uniformly with respect to $h \in J$, i.e. $\forall h \in J$, $\forall B_h \subset V_h$ bounded,*

$$dist_H(S_h(t)B_h, \mathcal{M}_h) \leq \mathcal{Q}(\|B_h\|_H)e^{-c_{11}t}, \quad t \geq 0,$$

where the positive constant c_{11} and the monotonic function $\mathcal{Q}$ are independent of h;

3. *the family $\{\mathcal{M}_h,\ h \in J \cup \{0\}\}$ is continuous at 0,*

$$dist_{sym}(\mathcal{M}_h, \mathcal{M}_0) \leq c_{12}h^{\kappa'},$$

where c_{12} and $\kappa' \in (0,1)$ are independent of h.

Remark: the global attractors $\mathcal{A}_h$ satisfy $dim_F(\mathcal{A}_h, H) \leq c_{10}$

Outline of the proof

- We introduce the absorbing sets

  $$\mathcal{B}_0 = \{v \in V \ : \ \|v\|_V \leq \mathcal{R}_1\}, \quad \mathcal{B}_h = \{v_h \in V_h \ : \ \|v_h\|_V \leq \mathcal{R}_1\},$$

  for all $h \in J$. Note that $\Pi_h(\mathcal{B}_0) = \mathcal{B}_h$.
- For $T > 0$ large enough, we have

  $$S_0(T)(\mathcal{B}_0) \subset \mathcal{B}_0 \quad \text{and} \quad \forall h \in J,\ S_h(T)(\mathcal{B}_h) \subset \mathcal{B}_h.$$

- We build a robust family $(\mathcal{M}_h^d)_{h \geq 0}$ of exponential attractors for $L_h = S_h(T)$ on $\mathcal{B}_h$, $h \in J \cup \{0\}$. For this we adapt the construction of **[Efendiev, Miranville & Zelik]**, using also some ideas from singularly perturbed case **[Fabrie, Galusinski, Miranville & Zelik 2004],[Gatti, Grasselli, Miranville & Pata 2006], [Miranville, Pata & Zelik 2007]**. The essential ingredients are the H-V smoothing property, the error estimate on finite time intervals, and the error for Π_h.
- We define $\mathcal{M}_h$ by the standard formula

  $$\mathcal{M}_h = \bigcup_{t \in [0,T]} S_h(t)\mathcal{M}_h^d.$$

The time semidiscrete problem

We apply the backward Euler scheme to (2). $\tau > 0$ is the time step. Let $u^0 \in H$ and for $n = 0, 1, 2, \ldots$ let $u^{n+1} \in V$ solve

$$\frac{u^{n+1} - u^n}{\tau} - \Delta u^{n+1} + g(u^{n+1}) = 0. \tag{7}$$

The discrete semi-group $S_\tau^n u_0 = u^n$ is **well-defined:**

Proposition (Well-posedness)

Assume that $\tau \in (0, \tau_0]$ for some $\tau_0 > 0$ small enough. Then for every $u \in H$, there exists a unique $v = v_{\tau,u} \in V$ such that

$$\frac{v-u}{\tau} - \Delta v + g(v) = 0 \text{ in } V'. \tag{8}$$

Moreover, the mapping $S_\tau : u \mapsto v_{\tau,u}$ is Lipschitz continuous from H into V, with

$$\|S_\tau u - S_\tau \hat{u}\|_V \leq \frac{c_0}{\tau}|u-\hat{u}|_H, \quad \forall u, \hat{u} \in H. \tag{9}$$

NB: c_0 is the optimal constant in the Poincaré inequality.
As a consequence, S_τ is Lipschitz continuous from H into H, and from V into V. We note that the Lipschitz constant c_0/τ **blows up** as $\tau \to 0^+$.

For the time semidiscrete problem, we obtain **a priori estimates** similar to the continuous problem, and which are **uniform in τ**:

- Absorbing set in V for the V-norm
- S_τ is Lipschitz continuous in H
- bound on bounded sets of H and finite time intervals for S_τ^n (a weak discrete version of the 1/2-Hölder continuity)
- H-V smoothing property

Error estimate with H^1 data

We set

$$u_\tau(t) = u^n + \frac{t - n\tau}{\tau}(u^{n+1} - u^n), \quad t \in [n\tau, (n+1)\tau).$$

Following the methodology of **[X. Wang 2010]**, we obtain the following error estimate

Theorem (Finite time uniform error estimate)

For all $T > 0$ and $R_1 > 0$, there is a constant $C(T, R_1)$ independent of τ such that $u^0 = u_0$ and $\|u^0\|_V \leq R_1$ imply

$$\sup_{t \in [0, [T/\tau]\tau]} |u_\tau(t) - u(t)|_H \leq C(T, R_1)\tau^{1/2}.$$

Theorem (Convergence of exponential attractors)

For each $\tau \in (0, \tau_0]$*, the discrete dynamical system associated to* $\{S_\tau^n,\ n \in \mathbf{N}\}$ *possesses an exponential attractor* $\mathcal{M}_\tau$ *on* H*, and the continuous dynamical system* $\{S_0(t),\ t \in \mathbf{R}_+\}$ *possesses an exponential attractor* $\mathcal{M}_0$ *such that:*

1. *the fractal dimension of* $\mathcal{M}_\tau$ *is bounded, uniformly with respect to* $\tau \in [0, \tau_0]$*,* $dim_F \mathcal{M}_\tau \leq c_7$ *(*c_7 *independent of* τ*);*
2. $\mathcal{M}_\tau$ *attracts the bounded sets of* H*, uniformly with respect to* $\tau \in (0, \tau_0]$*;*
3. *the family* $\{\mathcal{M}_\tau,\ \tau \in [0, \tau_0']\}$ *is continuous at* 0*,*

$$dist_{sym}(\mathcal{M}_\tau, \mathcal{M}_0) \leq c_8 \tau^{c_9} \ (c_8, c_9 \in (0,1) \text{ independent of } \tau)$$

Remark: the global attractors $\mathcal{A}_\tau$ **satisfy** $dim_F(\mathcal{A}_\tau, H) \leq c_7$

Remark: The upper bound on the dimension of the exponential attractors is **explicit**, but it is quite crude. Namely,

$$dim_F(\mathcal{M}_\tau) \leq 2 + \log_2[N_{1/(4c_4')}(B(0,1;V),H)], \qquad (10)$$

where $N_\varepsilon(B(0,1;V),H)$ is the number of ball of radius ε in H which are necessary to cover the unit ball centered at 0 in V and c_4' (very large) is explicitly derived from the a priori estimates.

Conclusion and perspectives

- The time semidiscrete case: **abstract formulation** for parabolic problems (based on **[EMZ 2004]**)
- **[Batangouna and P. 2018]** applied it to a **time splitting** discretization of the Caginalp phase-field system.
- The fully discrete case for the Allen-Cahn equation: ok
- Generalization to other (parabolic) problems and/or other types of discretization : Navier-Stokes, Cahn-Hilliard, finite difference, finite volume, . . . ?
- Can we find a general construction of a robust family of exponential attractors as **[X. Wang 2016]** did for the space and time discretization of the global attractor ?

Conclusion and perspectives

- The time semidiscrete case: **abstract formulation** for parabolic problems (based on **[EMZ 2004]**)
- **[Batangouna and P. 2018]** applied it to a **time splitting** discretization of the Caginalp phase-field system.
- The fully discrete case for the Allen-Cahn equation: ok
- Generalization to other (parabolic) problems and/or other types of discretization : Navier-Stokes, Cahn-Hilliard, finite difference, finite volume, . . . ?
- Can we find a general construction of a robust family of exponential attractors as **[X. Wang 2016]** did for the space and time discretization of the global attractor ?

Thank you !