Ship hull optimization: an approach based on Michell's formula

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We use a simplified approach, where the resistance of water to the motion of a ship is represented as

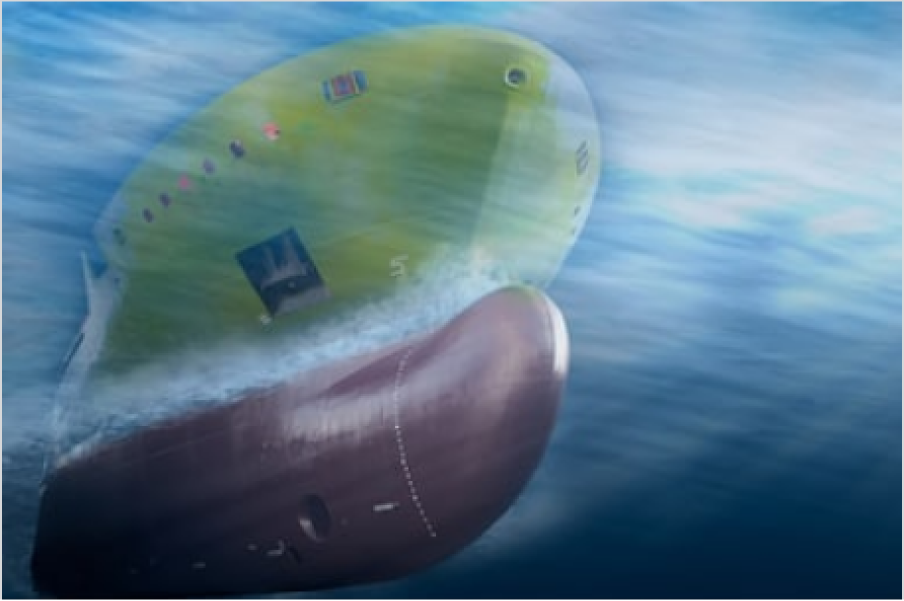
$$R_{water} = R_{viscous} + R_{wave},$$

and R_{wave} is given by **Michell's formula (1898)**.

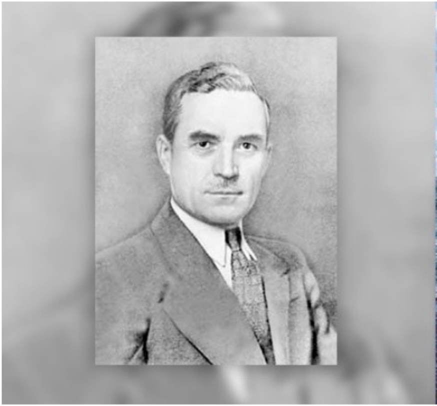
The *wave resistance* reflects the energy to push the water out of the way of the hull. This energy goes into creating the wave.

Optimization problem: minimize the resistance for a given speed U of the ship and a given volume V of the hull.

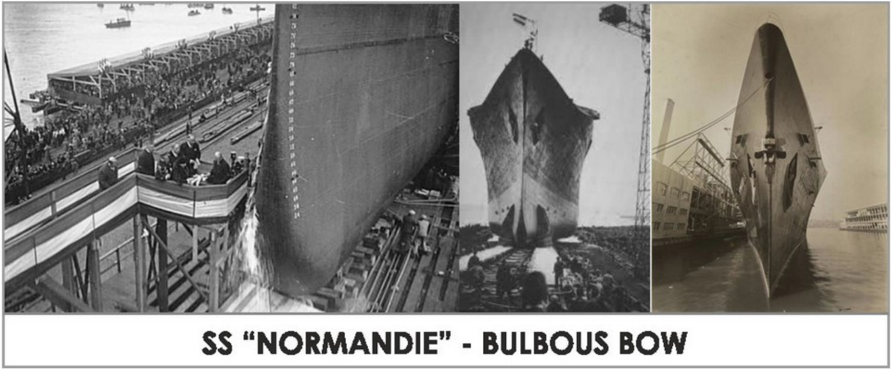
The bulbous bow of a common tanker



Another bulbous bow



Vladimir Ivanovich Yurkevich – a representative of the professional shipbuilding school of the Russian Empire who emigrated from Russia after "October" revolution of 1917. It's



the bulbous bow of the Normandie (won the “blue ruban” in 1935)

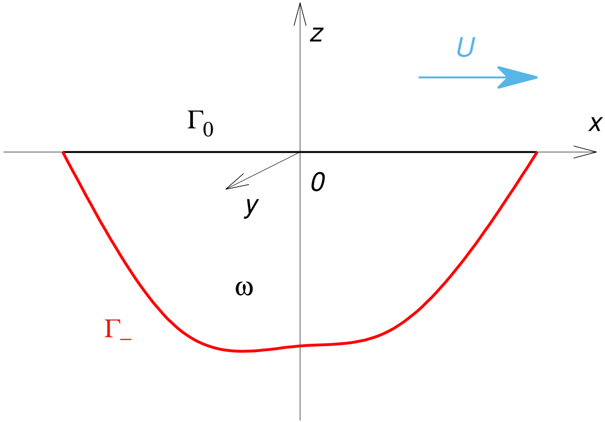
The bulbous bow of “Harmony of the Seas” (2015)
Speed : 20 knots / Length : 362m / Fr=0.17 (/T=9.1m / B=47m)

Caractéristiques techniques	
Longueur	362,12 m[1]
Maître-bau	47,42 m (flottaison)[1] 66 m (maximum)
Tirant d'eau	9,322 m[1]
Tirant d'air	70 m
Déplacement	60 000 t
Port en lourd	15 000 t
Tonnage	226 963 tjb[1]

Vitesse	20 nœuds en croisière, 23,1 nœuds maximum

JOHN HENRY MICHELL (1863 – 1940)

The domain ω of parameters (x, z)



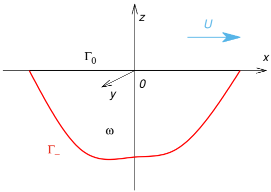
Consider a ship moving with constant velocity U on the surface of an unbounded fluid.

- coordinates xyz are fixed to the ship
- the xy-plane is the (undisturbed) water surface, z is vertically upward

The (half-)immerged hull surface is represented by a continuous **nonnegative** function

$$y = f(x,z) \geq 0, \quad (x,z) \in \omega,$$

with $f(x,z) = 0$ **on** Γ^- (= the boundary of ω under the surface)



Michell's formula (1898)[1] for the **wave resistance** reads:

$$R_{Michell}(f) = \frac{4\rho g^2}{\pi U^2} \int_1^{\infty} (I(\lambda)^2 + J(\lambda)^2) \frac{\lambda^2}{\sqrt{\lambda^2 - 1}} d\lambda, \qquad (1)$$

with

$$I(\lambda) = \int_{\omega} \frac{\partial f(x,z)}{\partial x} \exp\left(\frac{\lambda^2 g z}{U^2}\right) \cos\left(\frac{\lambda g x}{U^2}\right) dxdz, \qquad (2)$$

$$J(\lambda) = \int_{\omega} \frac{\partial f(x,z)}{\partial x} \exp\left(\frac{\lambda^2 g z}{U^2}\right) \sin\left(\frac{\lambda g x}{U^2}\right) dxdz. \qquad (3)$$

- U (in $\mathrm{m} \cdot \mathrm{s}^{-1}$) is the speed of the ship
- ρ (in $\mathrm{kg} \cdot \mathrm{m}^{-3}$) is the (constant) density of the fluid
- g (in $\mathrm{m} \cdot \mathrm{s}^{-2}$) is the standard gravity.
- $R_{Michell}(f)$ is a force and $\lambda = 1/\cos\theta$ where θ is the angle at which the wave energy is propagating.

[1] J.H. Michell. *The wave resistance of a ship*, Phil. Mag. (1898)

- The fluid is incompressible, inviscid, the flow is irrotational
- A steady state has been reached
- Linearized theory (flow potential with linearized boundary conditions)
- Thin ship assumptions: $0 \leq f << 1$, $|\partial_x f| << 1$, $|\partial_z f| << 1$.

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Experiments starting in the 1920's (**Wigley, Weinblum**): reasonable good agreement between theory and experiment (**Gotman'02**). Typical values for Wigley: $L/B \approx 10$ and $T/B = 1.5$.

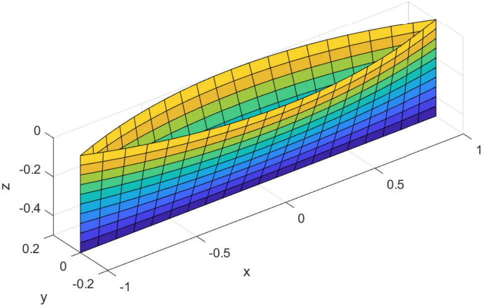
Example: the Wigley hull

For a **Wigley hull** with **beam** B and draft T, we have

$$\omega = (-L/2, L/2) \times (-T, 0) \quad \textbf{rectangle}$$

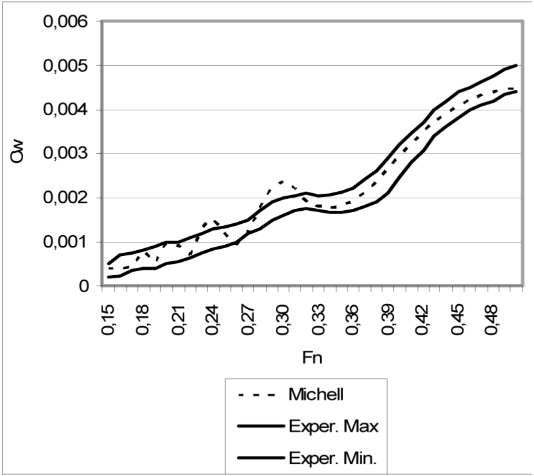
and

$$f(x,z) = (B/2)\left(1 + \frac{z}{T}\right)\left(1 - \frac{4x^2}{L^2}\right).$$



Wigley hull ($L = 2$, $B = 0.4$ and $T = 0.5$)

The following figure shows the **wave coefficient** $C_W = 2R_{wave}/(\rho U^2 A)$ (with A the wetted surface of the hull) in terms of the **Froude number** $F = U/\sqrt{gL}$.



Comparison Michell and experimental data (parabolic Wigley model, **Bai'79**)

Derivation of Michell's formula (sketch)

In the coordinates xyz fixed to the ship, we have $\overline{\mathrm{U}} = -Ue_x + \mathrm{u}$, where u is the perturbed velocity flow. We seek a **potential flow** Φ (i.e. with $\mathrm{u} = \nabla\Phi$), even with respect to y, which satisfies

$$\begin{aligned}
\Delta\Phi &= 0, \text{ in } (\mathrm{R}^2 \times \mathrm{R}_-) \setminus \overline{\omega} &(4)\\
\partial_{xx}\Phi + (g/U^2)\partial_z\Phi &= 0, \text{ for } z = 0 &(5)\\
\partial_y\Phi(x, y = 0^\pm, z) &= \mp U\partial_x f, \text{ for } (x, z) \in \omega, &(6)\\
|\nabla\Phi| \to 0 & \text{ as } x \to +\infty. &(7)
\end{aligned}$$

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|\nabla\Phi| \to & \ 0 \quad \text{as } x \to +\infty. && (7)
\end{aligned}$$

NB : $\overline{\mathrm{U}} = \nabla\tilde{\Phi} = -Ue_x + \nabla\Phi$ is the velocity field (irrotational)
$\tilde{\Phi} = -Ux + \Phi$ is the unperturbed potential
div $\overline{\mathrm{U}} = 0 = \Delta\Phi$ is the incompressibility condition
(5) is a consequence of the Bernoulli equation and the no-slip condition on the **free surface** (+ linearization)
(6) is the linearized no-slip condition on the hull

Φ can be computed explicitly by means of Green functions and Fourier transform.
Remark: radiation condition and uniqueness of Φ ?

The wave resistance reads

$$R_{wave} = -2 \int_{\omega} \delta p f_x(x,z) dxdz,$$

where δp is the difference of pressure due to the ship. (Notice that R_{wave} is the **drag force** in this linearized model).
From Φ, we derive δp so that

$$R_{wave} = -2\rho U \int_{\omega} \Phi_x(x,0,z) f_x(x,z) dxdz.$$

Computing, we obtain $R_{wave} = R_{Michell}$ as given by (1).

The optimization problem (fixed ω)

1st idea: finding a ship of **minimal wave resistance** among admissible functions $f : \omega \to \mathbb{R}_+$, for a constant speed U and a given volume V of the hull.

$f \mapsto R_{Michell}(f)$ is a positive semi-definite quadratic functional, **but** the problem above is **ill-posed** (**Sretensky'35, Krein'52**).

[2] A. A. Kostyukov, Theory of ship waves and wave resistance, 1968

[3] V. G. Sizov, *The seminar on ship hydrodynamics, organized by Professor M. G. Krein* (2000)

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Many authors proposed to add conditions and/or to work in finite dimension (**Weinblum'56, Kostyukov'68**[2]**,...**).
Another approach: add the **viscous resistance** which can be interpreted as a **regularization** and work with the **total resistance** (**Krein & Sizov'60, '00**[3]**, Hsiung'72, '81, '84, Lian-en'84, Michalski et al'87, Dambrine, P. & Rousseaux'16**).

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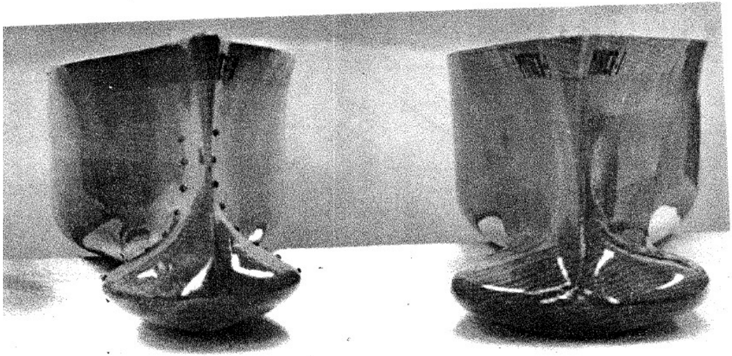
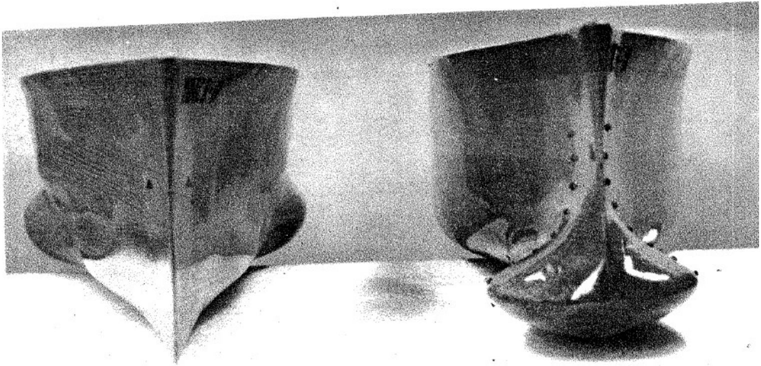


Hsiung's thesis (1972)



Hsiung's thesis (1972)

Michell's wave resistance rewritten

We define

$$\nu = g/U^2 > 0 \quad \text{and} \quad T_f(\nu,\lambda) = I(\lambda) - iJ(\lambda),$$

where I and J are given by (2)-(3). Then

$$T_f(\nu,\lambda) = \int_\omega \partial_x f(x,z) e^{\lambda^2 \nu z} e^{-i\lambda \nu x} dxdz, \tag{8}$$

and $R_{Michell}$ can be written

$$R_{Michell}(f) = \frac{4\rho g \nu}{\pi} \int_1^\infty |T_f(\nu,\lambda)|^2 \frac{\lambda^2}{\sqrt{\lambda^2 - 1}} d\lambda. \tag{9}$$

Remark: $R_{Michell}(f)$ is invariant by translation in the x-direction
Remark 2: ν is the **Kelvin wave number** ($1/\nu$ is the typical length of transverse waves)

The viscous resistance

$$R_{viscous}(f) = \frac{1}{2}\rho U^2 C_F A(f),$$

where C_F is the constant **viscous drag coefficient**, and $A(f)$ is the wetted surface area given by

$$A(f) = 2\int_\omega \sqrt{1 + |\nabla f(x,z)|^2}\,\mathrm{d}x\mathrm{d}z\,.$$

For instance, the **ITTC 1957** model-ship correlation line gives

$$C_F = 0.075/(\log_{10}(Re) - 2)^2,$$

where $Re = UL/\nu$ is the Reynolds number and ν the kinematic viscosity of water.
For small ∇f (thin ship assumption)

$$R_{viscous}(f) \approx \rho U^2\, C_F \left(\int_\omega dxdz + \frac{1}{2}\int_\omega |\nabla f(x,z)|^2\,\mathrm{d}x\mathrm{d}z \right)\,.$$

The total resistance

The **total water resistance** functional R_{total} is

$$R_{total}(f) := \frac{1}{2}\rho U^2 \, C_F \int_\omega |\nabla f(x,z)|^2 dxdz + R_{Michell}(f)$$

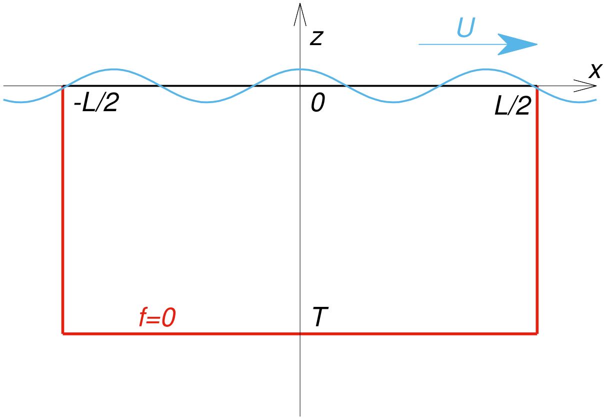
Remark: we have dropped the constant term $\rho U^2 C_F |\omega|$.

Recall that:

- ρ and g are given physical constants
- U and C_F are independent parameters and $v = g/U^2$
- the set ω is given and for simplicity we will assume

$$\omega = (-L/2, L/2) \times (-T, 0)$$

The domain ω of parameters (x, z) (fixed ω)



Functional setting

The function space is

$$H(\omega) = \{f \in H^1(\omega) \ : f(\pm L/2, \cdot) = 0 \text{ and } f(\cdot, T) = 0 \text{ a.e. } \},$$

Let $V > 0$ be the (half-)volume of an immerged hull. The set of admissible functions is

$$C_V(\omega) = \left\{ f \in H(\omega) \ : \ \int_\omega f(x,z)dxdz = V \text{ and } f \geq 0 \text{ a.e. in } \omega \right\}.$$

Notice that $C_V(\omega)$ is a **closed convex subset** of $H(\omega)$.
NB: the volume is proportional to the *displacement tonnage* of the ship.

The optimization problem (fixed ω)

Our **optimization problem** $\mathcal{P}_\omega$ reads: for a given Kelvin wave number $\nu = g/U^2$, a given drag coefficient C_F and a given volume $V > 0$, find the function $f^\star$ which minimizes the total resistance $R_{total}(f)$ among functions $f \in C_V(\omega)$.

In short, *"minimize the (total) drag for a given displacement tonnage of the ship"*.

Well-posedness

The (positive) parameters ρ, g, U (speed), ν, V (volume), and C_F are fixed (unless otherwise stated).

Theorem (Dambrine, P. & Rousseaux'15)

Problem $\mathcal{P}_\omega$ has a unique solution $f^\star$ in $C_V(\omega)$. Moreover, $f^\star$ is even with respect to x.

- Existence by a minimizing sequence
- Uniqueness by strict convexity
- Symmetry thanks to the symmetry of $R_{Michell}$ and R_{total} through $x \mapsto -x$.
- $f^\star$ depends linearly on V

Regularity of the solution

Theorem (Dambrine, P. & Rousseaux)

We have $f^\star \in W^{2,q}(\omega)$ for all $1 \leq q < 5/4$. In particular, $f^\star$ is uniformly α-Hölder continuous on $\overline{\omega}$ for all $0 < \alpha < 2/5$.

See also **Krein & Sizov'60 (unpublished)**: $f^\star \in C^0(\overline{\omega})$ (cf. the review **Sizov'00**)

Regularity of the solution (continued)

Theorem (Dambrine, P. & Rousseaux)

Let $\omega^\delta = \{(x,z) \in \omega \ : z < -\delta\}$ with $\delta > 0$ small. Then $f^\star$ belongs to $W^{2,p}(\omega^\delta)$ for all $1 \leq p < \infty$. In particular, $f^\star \in C^1(\overline{\omega^\delta})$.

Remark: Since $f_V^\star = V f_1^\star$, by letting $V \to 0^+$, we recover the thin ship assumptions in ω^δ (i.e. below the free surface $z = 0$)

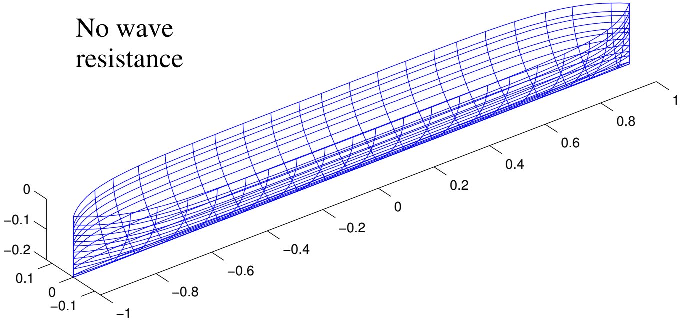
Sketch of proof (regularity)

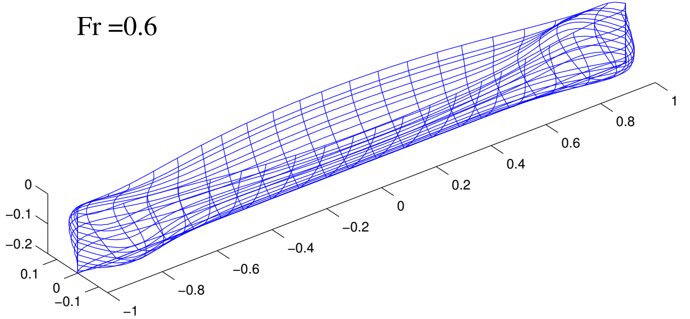
The problem is a perturbation of an **obstacle-type problem** for the Dirichlet energy

- The Euler-Lagrange equation gives a variational inequality for an obstacle-type problem
- By a standard result, the regularity of the obstacle problem is given by the regularity of the unconstrained problem
- The unconstrained problem reads $-\Delta f^\star = w$ with $w \in L^q(\omega)$, and homogeneous Dirichlet BC on 3 sides + no-flux BC on 1 side of the rectangle, hence (by symmetry) $f^\star \in W^{2,q}(\omega)$.
- $w \in L^q(\omega)$ for $1 \leq q < 5/4$ is related to the **regularity of Michell's wave resistance kernel**, which belongs to $L^{5/4-\epsilon}_{loc}$.

A numerical test

- $\rho = 1000\,\mathrm{kg}\cdot\mathrm{m}^{-3}$, $g = 9.81\,\mathrm{m}\cdot\mathrm{s}^{-2}$, $L = 2\,\mathrm{m}$, $T = 20\,\mathrm{cm}$, $V = 0.03\,\mathrm{m}^3$.
- $N_x = 100$ and $N_z = 20$
- $\epsilon = \frac{1}{2}\rho C_F U^2$ with $C_F = 0.01$
- $\boxed{Fr = U/\sqrt{gL}}$


No wave resistance
0
−0.1
−0.2
0.1
0
−0.1
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1


Fr =0.6
0
−0.1
−0.2
0.1
0
−0.1
−1
−0.8
−0.6
−0.4
−0.2
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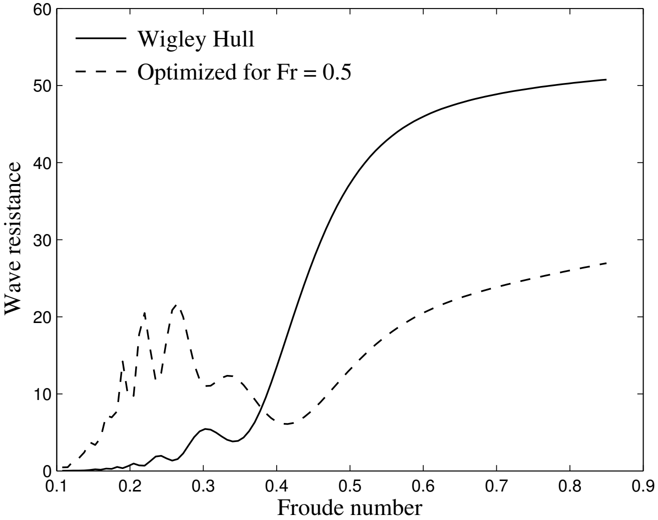
0.6
0.8
1



Optimized hull vs. Wigley hull

Froude scaling

Let $T = \alpha\overline{T}$ / $L = \alpha\overline{L}$ / $x = \alpha\overline{x}$ / $z = \alpha\overline{z}$ / $f(x,z) = \alpha\overline{f}(\overline{x},\overline{z})$.
The **wave resistance** reads

$$R_{Michell}(\nu, f) = \alpha^3 \overline{R}_{Michell}(\alpha\nu, \overline{f}),$$

where $\nu = g/U^2$. It is natural to set $\overline{\nu} = \alpha\nu$, i.e. $U = \sqrt{\alpha}\,\overline{U}$, and with this choice,

$$Fr = U/\sqrt{gL} = \overline{Fr} = \overline{U}/\sqrt{g\overline{L}} \quad \textbf{(Froude number)}.$$

The **viscous drag** reads

$$\frac{1}{2}\rho U^2 C_F \int_\omega |\nabla f(x,z)|^2 dxdz = \alpha^3 \frac{1}{2}\rho \overline{U}^2 C_F \int_{\overline{\omega}} |\nabla \overline{f}(\overline{x},\overline{z})|^2 d\overline{x}d\overline{z}.$$

Geometric shape optimization

Idea: consider also the set of parameters ω as an unknown (in order to minimize even more the total resistance)

Formal problem: For a given area a, find $\omega^\star$ which solves

$$\mathcal{J}(\omega^\star) = \min_{|\omega|=a} \mathcal{J}(\omega)$$

in the set of admissible sets ω, where

$$\mathcal{J}(\omega) = R_{total}(f_\omega^\star) = \min_{f \in C_V(\omega)} R_{total}(f).$$

Here, ω is a set under the free surface.

Some issues:

- Existence of $\omega^\star$
- Regularity of $f_{\omega^\star}^\star$ and of $\omega^\star$
- "Continuity" of $f_{\omega^\star}^\star$ with respect to ν and C_F

In order to simplify the notation, we multiply $R_{total}(f)$ by the constant $4/(\rho U^2 C_F)$. We obtain the **normalized total resistance**

$$\frac{4}{\rho U^2 C_F} R_{total}(f) = 2\int_\omega |\nabla f(x,z)|^2 dxdz + \frac{4}{\rho U^2 C_F} R_{Michell}(f).$$

Next, we consider the even symmetric of f, namely

$$u(x,z) = \begin{cases} f(x,z) & \text{if } (x,z) \in \overline{\omega}, \\ f(x,-z) & \text{if } (x,-z) \in \overline{\omega}. \end{cases}$$

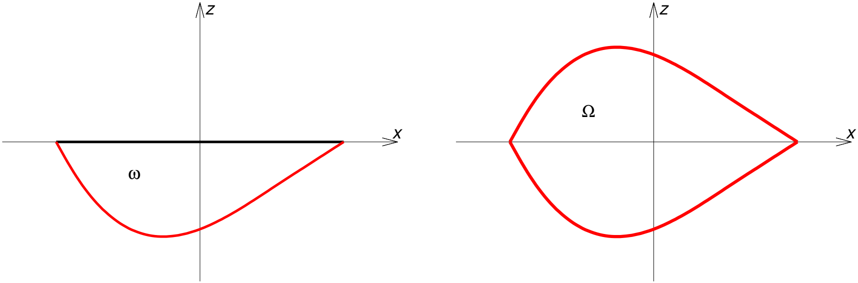


Figure: Symmetrization $z \mapsto -z$

$$f : \omega \to \mathrm{R} \text{ becomes } u : \Omega \to \mathrm{R}$$

The **normalized total resistance** is

$$J(u) = J_0(u) + \frac{1}{C_F} J^{\nu}_{wave}(u), \tag{10}$$

where

$$J_0(u) = \int_{\mathrm{R}^2} |\nabla u(x,z)|^2 dxdz \tag{11}$$

is the **normalized viscous resistance**, and

$$J^{\nu}_{wave}(u) = \frac{4\nu^4}{\pi} \int_1^{\infty} |T_u(\nu,\lambda)|^2 \frac{\lambda^4}{\sqrt{\lambda^2 - 1}} d\lambda \tag{12}$$

with

$$T_u(\nu,\lambda) = \int_{\mathrm{R}^2} u(x,z) e^{-i\lambda\nu x} e^{-\lambda^2 \nu |z|} dxdz$$

is the **normalized wave resistance** functional.

The shape optimization problem in $\mathbb{R}^2$

Let $V > 0$ (the volume of the hull) and $a > 0$ (the area of Ω). Find an open and symmetric set $\Omega^\star$ such that

$$J(u_{\Omega^\star}) = \inf \left\{ J(u_\Omega),\ \Omega \subset \mathbb{R}^2 \text{ open and symmetric},\ |\Omega| = a \right\}, \tag{13}$$

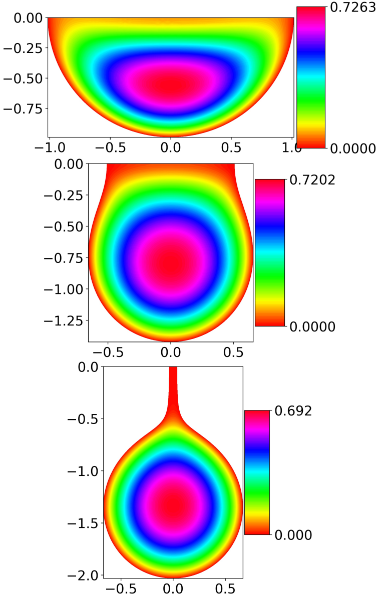
where u_Ω is uniquely defined by

$$J(u_\Omega) = \min \left\{ J(v),\ v \in H_0^1(\Omega)^+,\ \check{v} = v,\ \int_\Omega v = V \right\}. \tag{14}$$

We denote here $\check{v}(x,z) = v(x,-z)$. We introduce the **area Froude number**

$$Fr_a = \frac{1}{\sqrt{\nu\sqrt{a}}} = \frac{U}{\sqrt{g\sqrt{a}}}.$$

Two questions: *existence* **of** $\Omega^\star$ **and** *regularity* **of** $u_{\Omega^\star}$ **?**



A minimizing sequence for $Fr_a = 1.75$

The Saint-Venant inequality

The Saint-Venant problem reads: Find an open and symmetric set $\Omega^\star$ such that

$$J_0(u_{\Omega^\star}) = \inf \left\{ J_0(u_\Omega),\ \Omega \subset \mathrm{R}^2 \text{ open and symmetric},\ |\Omega| = 1 \right\},$$

where u_Ω is uniquely defined by

$$J_0(u_\Omega) = \min \left\{ J_0(v),\ v \in H^1_0(\Omega)^+,\ \check{v} = v,\ \int_\Omega v = 1 \right\}.$$

The **disc** centered at $(0,0)$ solves the St-Venant problem. It is unique up to translation (along the x-axis)[4].

Moreover, $\boxed{J_0(u_{\Omega^\star}) = 8\pi}$.

[4] L. Brasco, G. De Philippe and B. Velichkov, *Faber-Kahn inequalities in sharp quantitative form* (2015)

A non-existence result (Dambrine & P.)

The problem: find an open and symmetric set $\Omega^\star$ such that

$$J(u_{\Omega^\star}) = \inf \left\{ J(u_\Omega),\ \Omega \subset \mathrm{R} \times \mathrm{R}^\star \text{ open and symmetric, } |\Omega| = 1 \right\},$$

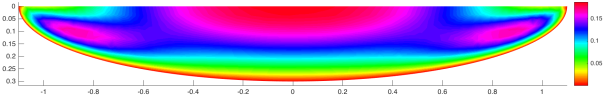
where u_Ω is uniquely defined by

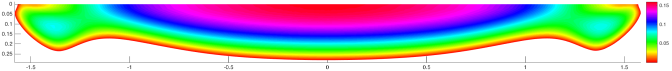
$$J(u_\Omega) = \min \left\{ J(v),\ v \in H_0^1(\Omega)^+,\ \check{v} = v,\ \int_\Omega v = 1 \right\}.$$

has **no solution**.
Proof. Recall that

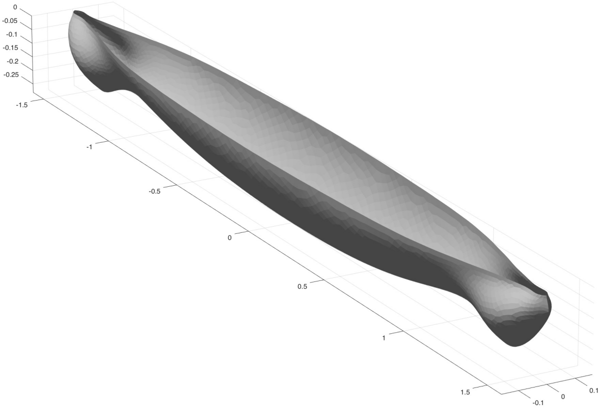
$$J(u) = J_0(u) + \frac{1}{C_F} J_{wave}(u).$$

By the St-Venant inequality, the infimum is equal to $\boxed{16\pi}$ by letting two symmetric balls going to $z = \pm\infty$. A result of **Krein** states that if Ω is bounded, then $J_{wave}(v) > 0$ for all v in the set above.

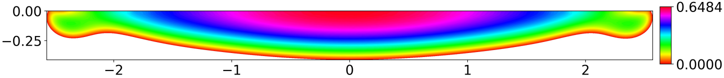




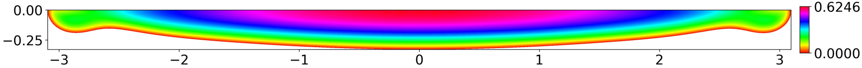
The initial and converged domain of arguments
(algorithm from **Allaire's book**)



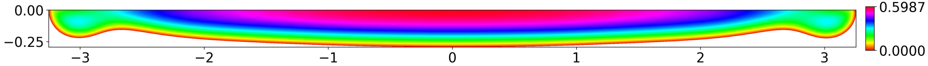
The corresponding optimized hull



Optimal domain for $Fr_a = 0.67$



Optimal domain for $Fr_a = 0.81$



Optimal domain for $Fr_a = 0.98$

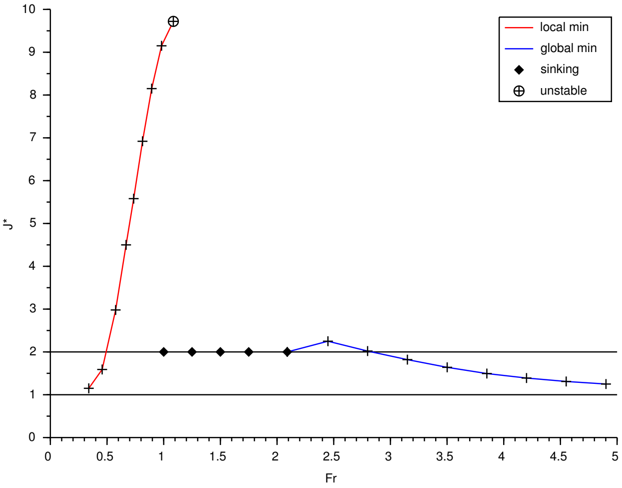


Figure: $J^{\star}_{num}$ vs Fr_a $(C_F = 0.01)$

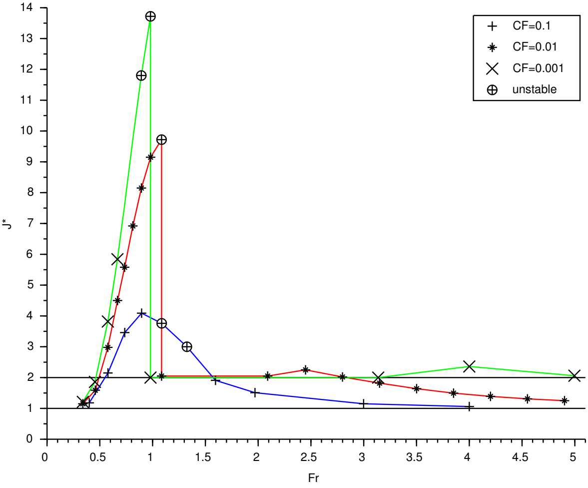


Figure: $J^{\star}_{num}$ vs Fr_a for three different drag coefficients

The shape optimization problem in a bounding box

Let D be a symmetric bounded domain of R^2 with Lipschitz boundary such that $|D| > a$.
Find an open and symmetric set $\Omega^\star \subset D$ such that

$$J(u_{\Omega^\star}) = \inf\left\{J(u_\Omega),\ \Omega \subset D \text{ open and symmetric},\ |\Omega| = a\right\},$$

where u_Ω is uniquely defined by

$$J(u_\Omega) = \min\left\{J(v),\ v \in H_0^1(\Omega)^+,\ \check{v} = v,\ \int_\Omega v = V\right\}.$$

The shape optimization problem in a bounding box

Let D be a symmetric bounded domain of R^2 with Lipschitz boundary such that $|D| > a$.
Find a **quasi**-open and symmetric set $\Omega^\star \subset D$ such that

$$J(u_{\Omega^\star}) = \inf \left\{ J(u_\Omega),\ \Omega \subset D \text{ **quasi**-open and symmetric},\ |\Omega| \leq a \right\},$$

where u_Ω is uniquely defined by

$$J(u_\Omega) = \min \left\{ J(v),\ v \in H_0^1(\Omega)^+,\ \check{v} = v,\ \int_\Omega v = V \right\}.$$

Following a standard approach (see **Henrot and Pierre's book**[5]), we work with the space

$$\check{H} = \{u \in H_0^1(D),\ \check{u} = u \text{ a.e. in } D\},$$

which is a closed subspace of $H_0^1(D)$. For a function $u \in \check{H}$

$$\Omega_u = \{(x,z) \in D \ :\ u(x,z) \neq 0\}$$

is its support, with area $|\Omega_u|$.

[5] A. Henrot and M. Pierre, Variation et optimisation de formes, 2005

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is its support, with area $|\Omega_u|$. We define

$$C_V^a = \{v \in \check{H}\ : v \geq 0 \text{ a.e. in } D, \int_D v dx dz = V,\ |\Omega_v| \leq a\},$$

and we reformulate the previous problem as follows:

$$(\mathcal{P}_V^a) \begin{cases} \text{Find } u \in C_V^a \text{ such that} \\ J(u) \leq J(v),\ \forall v \in C_V^a. \end{cases}$$

[5] A. Henrot and M. Pierre, Variation et optimisation de formes, 2005

Theorem

Problem $(\mathcal{P}_V^a)$ *has a solution* u *such that* $J(u) < +\infty$.

Existence by considering a minimizing sequence in C_V^a.

Theorem

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Existence by considering a minimizing sequence in C_V^a.

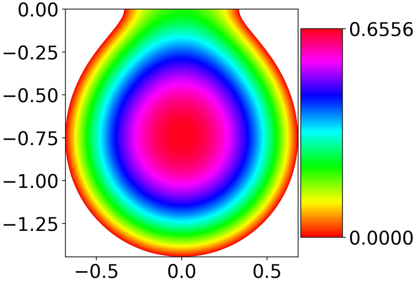
NB: Dambrine & P.'20: Hölder regularity of u was proved if the nonnegativity of u is an assumption instead of a constraint (method of **Alt and Cafarelli'81, Briançon, Hayouni & Pierre'04**).

Continuity with respect to the speed (Dambrine & P.'21

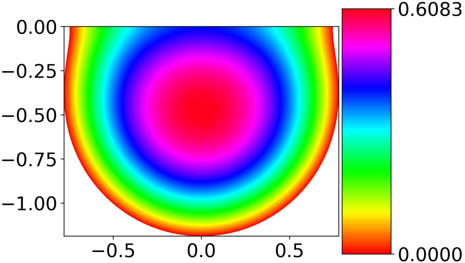
Using a Γ-convergence approach, we also proved that:

- "the" solution u_{ν,C_F} of $(\mathcal{P}_V^a)$ depends continuously (up to a subsequence) on ν and C_F for the strong $H_0^1(D)$ topology.
- As $\nu \to 0$ (i.e. $U \to +\infty$) with C_F constant, u_{ν,C_F} converges strongly in $H_0^1(D)$ (up to a subsequence) to the solution u_0 of the Dirichlet energy functional with area and volume constraint.

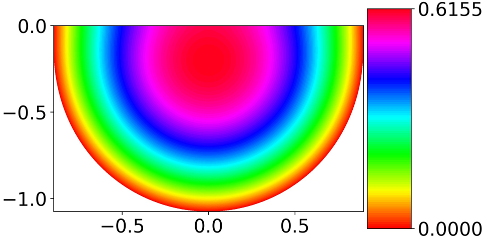
NB: The function u_0 can be computed by symmetrization (if D is large enough) and its support is a disc by the **St-Venant inequality**. u_0 is unique up to translation along the x-axis



Optimal domain for $Fr_a = 2.45$



Optimal domain for $Fr_a = 3.15$



Optimal domain for $Fr_a = 4.90$

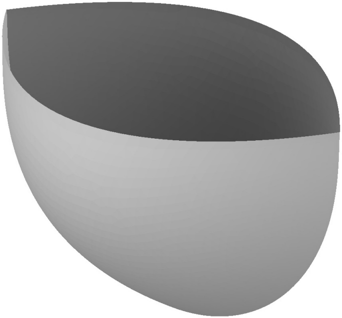


Figure: Optimal hull for $Fr_a = 4.90$

Perspectives

- Compute a hull which is optimal for U random in a range $[U_{min}, U_{max}]$ (with **S. Zerrouq**)
- Existence/non-existence in R^2 ?
- Regularity of u and of the optimal domain with nonnegativity constraint ?

Publications

- J. Dambrine, M. P. and G. Rousseaux, *A theoretical and numerical determination of optimal ship forms based on Michell's wave resistance*, ESAIM COCV (2016)
- J. Dambrine and M. P., *Regularity of optimal ship forms based on Michell's wave resistance*, Appl. Math. Optim. (2018)
- E. Noviani, *PhD thesis* (2018)
- J. Dambrine, E. Noviani and M. P., *Rankine-type cylinders having zero wave resistance*, IMA J. Appl. Math. (2020)
- J. Dambrine and M. P., *Continuity with respect to the speed for optimal ship forms based on Michell's formula*, Math. Control Relat. Fields (2021)

Selected references

- E.O. Tuck, “The wave resistance formula of J.H. Michell (1898) and its significance to recent research in ship hydrodynamics” (1989)
- V.G. Sizov, “The seminar on ship hydrodynamics, organized by Professor M.G. Krein” (2000)
- A. Gotman, “Navigating the wake of past efforts” (2007)
- L. D. Ferreiro, “The social history of the bulbous bow” (2011)

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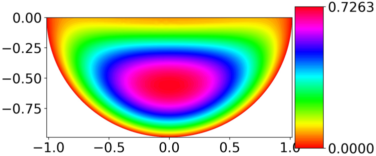
Thank you for your attention !



Optimal hull based on a half-disc for $Fr_a = 1.75$

Figure: Optimal hull based on a half-disc for $Fr_a = 1.75$

$$J_{wave}(u) = \int_D \int_D k(x,z,x',z')u(x,z)u(x',z')dxdzdx'dz' \geq 0 \quad (15)$$

is the **normalized wave resistance** functional. Here, $k : D \times D \to \mathrm{R}$ belongs to $L^q(D \times D)$ for some $q \in (1, +\infty]$ and satisfies the following symmetry assumptions:

$$k(x,z,x',z') = k(x',z',x,z) \quad (x,z,x',z') \in D \times D,$$

$$k(x,-z,x',z') = k(x,z,x',z') \quad (x,z,x',z') \in D \times D.$$

Michell's wave resistance kernel reads

$$k_\nu(x, z, x', z') = \frac{4\nu^4}{\pi C_F(\nu)} K(\nu(x - x'), \nu(|z| + |z'|)), \tag{16}$$

with $\nu = g/U^2$ (g=gravity and U=speed of ship), and

$$K(X, Z) = \int_1^\infty e^{-\lambda^2 Z} \cos(\lambda X) \frac{\lambda^4}{\sqrt{\lambda^2 - 1}} d\lambda. \tag{17}$$

Proposition

Michell's normalized wave resistance kernel k_ν (16) belongs to $L^q(D \times D)$ for all $1 \leq q < 5/4$. Moreover, if D contains an open disc centered on the x-axis, then k_ν does not belong to $L^{5/4}(D \times D)$.