A SPLITTING METHOD FOR THE CAHN-HILLIARD EQUATION WITH INERTIAL TERM

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Abstract. P. Galenko et al. proposed a Cahn-Hilliard model with inertial term in order to model spinodal decomposition caused by deep supercooling in certain glasses. Here we analyze a finite element space semidiscretization of their model, based on a scheme introduced by C. M. Elliott et al. for the Cahn-Hilliard equation. We prove that the semidiscrete solution converges weakly to the continuous solution as the discretization parameter tends to 0. We obtain optimal a priori error estimates, assuming enough regularity on the solution. We also show that the semidiscrete solution converges to an equilibrium as time goes to infinity and we give a simple finite difference version of the scheme.

Keywords: discrete negative norms, mixed finite elements, Lojasiewicz inequality, error estimates.

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1. Introduction

In this paper we consider a space semidiscretization of the modified Cahn-Hilliard equation

$$\epsilon u_{tt} + \nu u_t + \alpha \Delta^2 u - \Delta f(u) = 0, \quad \text{in } \Omega \times \mathbb{R}_+,$$

with periodic boundary conditions. Here $\Omega = \Pi_{k=1}^d (0, L_k)$ ($L_k > 0$ for $k = 1, \ldots, d$), $1 \leq d \leq 3$, $f$ is the derivative of a nonconvex potential (typically, $f(y) = y^3 - y$), $\alpha > 0$ is related to the interfacial energy, and the constant coefficients $\epsilon, \nu$ satisfy $\epsilon > 0$ and $\nu \geq 0$. The unknown $u$ is the relative concentration of one phase.

When $\epsilon = 0$ and $\nu = 1$, we recover the well-known Cahn-Hilliard equation (see [4], cf. also [28] and references therein). On the other hand, P. Galenko et al. [9, 10, 11] have proposed to add the inertial term $\epsilon u_{tt}$ in order to model non-equilibrium decompositions caused by deep supercooling in certain glasses. Equation (1.1) shows a good agreement with experiments [12].

In comparison with the Cahn-Hilliard equation, equation (1.1) presents some mathematical difficulties because the solutions do not regularize in finite time anymore. The one-dimensional case is rather well understood [2, 6, 13, 26, 27], but the two-dimensional [17] or the three-dimensional case [16, 30] is far more complicated, because the so-called energy bounded solutions (of $H^1$-type) are no longer bounded in $L^\infty$ (see Theorem 2.1 below). We refer in particular the reader to the introduction of [16] for an analysis of these issues.

In this paper we perform a numerical analysis of a space semidiscretization of (1.1) in view of computations. This analysis is justified because of the hyperbolic-like features which make it difficult to find fully discretized schemes which are stable. In this regard, a time semidiscretization and numerical simulations will possibly be presented in another paper. In the papers cited above it is always assumed that $\nu > 0$ in order to have some dissipation (or simply $\nu = 1$ by a change of the time scale). On the contrary, here we can also include the “conservative” case $\nu = 0$ since we are mostly interested in finding error estimates on finite time intervals. Moreover, allowing $\nu = 0$ can be useful for studying numerical schemes with small dissipation.
The space semidiscretization that we consider is a finite element scheme based on a splitting formulation introduced by C. M. Elliott et al. for the Cahn-Hilliard equation [7] which has already been extended to other Cahn-Hilliard type equations with regularizing properties (see for instance [1, 20]). We first prove that a solution of the semidiscrete scheme converges weakly to an energy solution, as the discretization parameter \( h \) tends to 0 (see Theorem 3.5). This result is refined in Theorem 4.6 where we prove optimal a priori error estimates, assuming enough regularity on the solution. In both cases, the main difficulty is to deal with the negative norms (\( H^{-1} \) or \( H^{-2} \)).

The paper is organized as follows. We first introduce in Section 2 some notation and assumptions. 2.1. Notation and assumptions. Let \( L^2(\Omega) \) with scalar product \((\cdot,\cdot)\) and norm \( \|\cdot\|_0 \). For each \( u \in L^2(\Omega) \), we denote by \( m(u) \) its average

\[
m(u) = \frac{1}{|\Omega|}(u,1), \quad \text{with } |\Omega| = \sum_{i=1}^{d_i} I_i.
\]

We define

\[
\hat{L}^2(\Omega) = \{ u \in L^2(\Omega), \quad m(u) = 0 \},
\]

and for every \( u \in L^2(\Omega) \), we set \( \hat{u} = u - m(u) \).

We denote \( \hat{V} = H^1_{per}(\Omega) = H^1(\mathbb{T}) \) the \( L^2 \)-Sobolev space on the torus

\[
\mathbb{T} = \prod_{k=1}^{d_i}(\mathbb{R}/L_k\mathbb{Z}).
\]

Using the inclusions \( V \subset L^2(\Omega) \subset \hat{V}' \), the semiscalar product

\[
a(u,v) = (\nabla u, \nabla v), \quad \forall u, v \in V
\]

defines a linear operator \( \mathcal{A} : D(\mathcal{A}) \to \hat{L}^2(\Omega) \) with domain

\[
D(\mathcal{A}) = H^2_{per}(\Omega) = H^2(\mathbb{T}).
\]

In fact, \( \mathcal{A} \) is the Laplace operator with periodic boundary conditions. The operator \( \hat{A} : D(\hat{A}) \to \hat{L}^2(\Omega) \) is the restriction of \( \mathcal{A} \) to \( \hat{L}^2(\Omega) \) with domain

\[
D(\hat{A}) = D(\mathcal{A}) \cap \hat{L}^2(\Omega).
\]

The operator \( \mathcal{A} \) is a nonnegative self-adjoint operator; it has an orthonormal basis of eigenvectors \( (e_i)_{i \in \mathbb{N}} \), associated to the eigenvalues \( (\lambda_i)_{i \in \mathbb{N}} \), with

\[
\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_i \leq \cdots, \quad \lambda_i \to +\infty \text{ as } i \to +\infty.
\]

In particular, \( e_0 = 1/|\Omega|^{1/2} \). For all \( i \geq 0 \), the function \( e_i \) belongs to \( D(\mathbb{T}) \), the space of functions which are of class \( C^\infty \) on \( \mathbb{T} \).

By spectral theory [19], it is possible to define the powers \( (\mathcal{A} + I)^s \), for all \( s \in \mathbb{R} \), and the spaces

\[
V_s = D((\mathcal{A} + I)^{s/2}), \quad V_s' = (V_s)' \quad \forall s \geq 0.
\]

The space \( V_s \) can be identified with the space (see for instance [23])

\[
H^s(\mathbb{T}) = \left\{ u \mid u \in D'(\mathbb{T}), \sum_{i=0}^{\infty} \lambda_i^s \langle u, e_i \rangle_{D'(\mathbb{T}), D(\mathbb{T})}^2 < +\infty \right\}.
\]
Similarly, $\hat{A}$ is a positive self-adjoint operator and it is possible to define the powers $\hat{A}^s$ and the spaces

$$\hat{V}_s = D(\hat{A}^{s/2}), \quad \hat{V}_{-s} = (\hat{V}_s)', \quad \forall s \geq 0.$$ 

In fact, when $s \geq 0$,

$$u = \sum_{i \geq 1} u_i e_i \in \hat{V}_s \iff \sum_{i \geq 1} \lambda^s_i u_i^2 < +\infty,$$

and

$$\hat{A}^s u = \sum_{i \geq 1} \lambda^s_i u_i e_i, \quad \text{for all } u = \sum_{i \geq 1} u_i e_i, \in \hat{V}_{2s}.$$ 

For all $s \in \mathbb{R}$, the operator $\hat{A}^{s/2}$ maps $\hat{V}_s$ onto $\dot{L}^2(\Omega)$, and the space $\hat{V}_s$ is endowed with the Euclidean norm $|u|_s = \|\hat{A}^{s/2} u\|_0$. It is easily seen that the map $u \mapsto (m(u), \dot{u})$, where

$$m(u) = \frac{1}{|\Omega|} \langle u, 1 \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)}, \quad \dot{u} = u - m(u)$$

defines an isomorphism from $V_s$ onto $\mathbb{R} \times \dot{V}_s$, so that any element $u \in V_s$ is identified by $(m(u), \dot{u})$ and that the norm on $V_s$ is equivalent to the Euclidean norm

$$|u|_s = (m(u)^2 + |\dot{u}|_2^2)^{1/2}.$$ 

By extension, for $u \in V_s$, we denote

$$|u|_s = |\dot{u}|_s$$

the seminorm on $V_s$. When necessary, we will denote $\dot{P} : V_s \to \dot{V}_s$ the map $u \mapsto \dot{u}$.

For all $r < s$, we have the compact inclusion $\dot{V}_s \subset \dot{V}_r$. We also have $\dot{V}_0 = \dot{L}^2(\Omega)$, $V_1 = \dot{V}$ and $V_{-1} = V'$. For all $s \in \mathbb{R}$, the operator $\hat{A}$ can be extended as an isomorphism from $\hat{V}_s$ onto $\hat{V}_{s-2}$ and we will denote $\hat{A}$ this extension. Similarly, the operator $\hat{A}$ operates from $\hat{V}_s$ into $\hat{V}_{s-2}$.

The nonlinearity $f$ is a polynomial of odd degree whose leading coefficient is positive and which vanishes at 0:

$$f(y) = \sum_{i=1}^{2p+1} a_i y^i, \quad \forall y \in \mathbb{R}, \quad a_{2p+1} > 0,$$ 

(2.2)

with any $p \in \mathbb{N}$ if $d = 1$ or $d = 2$, and with $p \in \{0, 1, 2\}$ if $d = 3$. We denote $F$ the antiderivative of $f$ which vanishes at 0, i.e.,

$$F(y) = \sum_{i=1}^{2p+1} \frac{a_i}{i+1} y^{i+1}, \quad \forall y \in \mathbb{R}.$$ 

(2.3)

We have in particular the Sobolev injection $V_1 \hookrightarrow L^{2p+2}(\Omega)$, which implies

$L^{(2p+2)/(2p+1)}(\Omega) \subset V'_1.$

The functional $u \mapsto f(u)$ is Lipschitz continuous on bounded subsets of $V_1$ with values into $(V_1)' = V_{-1}$ (see for instance [21]). Also, note that there exist constants $c_1 > 0$ and $c_2 \geq 0$ such that

$$F(y) \geq c_1 y^{2p+2} - c_2, \quad \forall y \in \mathbb{R}.$$ 

(2.4)

2.2. The continuous problem. The abstract formulation of the singular Cahn-Hilliard equation (1.1) with periodic boundary conditions reads:

$$\epsilon u_{tt} + \nu u_t + \alpha A^2 u + A f(u) = 0,$$ 

(2.5)

$$u(0) = u_0, \quad u_t(0) = u_1,$$ 

(2.6)

where $u_0$ and $u_1$ are initial data.

We recall two fundamental a priori estimates. We first take the scalar product of (2.5)-(2.6) by the constant function $1/|\Omega|$; we obtain

$$\epsilon m(u_{tt}) + \nu m(u_t) = 0,$$ 

(2.7)
with initial data \( m(u(0)) = m(u_0) \) and \( m(u_1(0)) = m(u_1) \). Thus, if \( \nu > 0 \),
\[
m(u(t)) = m(u_0) + m(u_1) e^\frac{\nu}{\epsilon} \left( 1 - \exp(-\frac{\nu}{\epsilon} t) \right), \quad m(u_1(t)) = m(u_1) \exp(-\frac{\nu}{\epsilon} t),
\] (2.8)
and \( m(u(t)) \) is bounded on \([0, +\infty)\). If \( \nu = 0 \), then
\[
m(u(t)) = m(u_0) + tm(u_1), \quad m(u_1(t)) = m(u_1),
\] (2.9)
and \( m(u(t)) \) is bounded on \([0, T]\) for all \( T > 0 \).

Second, formally applying \( \dot{A}^{-1} \dot{P} \) to (2.5), and taking the scalar product by \( u_t \), we find that
\[
e^\frac{\nu}{\epsilon} \frac{d}{dt} |u_t|^2 - \frac{\alpha}{2} |u_t|^2 + (f(u), \dot{u}_t) = 0.
\] (2.10)
Moreover, we have
\[
\frac{d}{dt} \int_\Omega F(u) = (f(u), \dot{u}_t) + (f(u), m(u_t)).
\] (2.11)
In view of (2.10), we introduce the energy functional \( \mathcal{E} : V_1 \times V_- \rightarrow \mathbb{R} \) defined by
\[
\mathcal{E}(u, v) = \frac{\epsilon}{2} |u|^2 + \frac{\alpha}{2} |u_t|^2 + \int_\Omega F(u).
\]
If \( m(u_1) = 0 \), then integrating (2.10) and (2.11) from 0 to \( t \), and using (2.8)-(2.9), we get
\[
\mathcal{E}(u(t), u_1(t)) + \nu \int_0^t |u_t|^2 \leq \mathcal{E}(u_0, u_1), \quad \forall t \geq 0.
\] (2.12)
In particular, \( t \mapsto \mathcal{E}(u(t), u_1(t)) \) is nonincreasing. In the general case \( m(u_1) \neq 0 \), we use that, by the definitions (2.2) and (2.3) of \( f \) and \( F \), there exist two constants \( c_3 > 0, c_4 \geq 0 \) such that
\[
|f(s)| \leq c_3 F(s) + c_4, \quad \forall s \in \mathbb{R},
\] (2.13)
so that
\[
|(f(u), m(u_1))| \leq |m(u_1)| \int_\Omega |f(u)| \leq \|m(u_1)\|_{L^\infty(0, T)} \left( c_3 \int_\Omega F(u) + c_4|\Omega| \right).
\]
Thus, since \( \|m(u_1)\|_{L^\infty(0, T)} = \|m(u_1)\| \), we obtain
\[
e^\frac{\nu}{\epsilon} \frac{d}{dt} |u_t|^2 + \frac{\alpha}{2} |u_t|^2 + \frac{d}{dt} \int_\Omega F(u) \leq |m(u_1)| \left( c_3 \int_\Omega F(u) + c_4|\Omega| \right),
\] (2.14)
and by Gronwall’s lemma,
\[
\mathcal{E}(u(t), u_1(t)) + \nu \int_0^t |u_t|^2 \leq \mathcal{E}(u_0, u_1) e^{c_3 |m(u_1)| t} + \frac{c_4|\Omega|}{c_3} (e^{c_3 |m(u_1)| t} - 1), \quad \forall t \geq 0.
\] (2.15)
Thus, if \( \mathcal{E}(u_0, u_1) < +\infty \),
\[
u \in L^\infty(0, T; V_1), \quad u_t \in L^\infty(0, T; V_-), \quad \forall T > 0.
\]
We use here that \( F \) is bounded from below (see (2.4))
\[
\int_\Omega F(v) \geq -c_2|\Omega|, \quad \forall v \in V.
\] (2.16)

The computations above are valid if \( u \) is regular enough. In the general case, they can be justified by means of a Galerkin approximation. From these a priori estimates, we deduce

**Theorem 2.1.** Assume that \( f \) is defined by (2.2) and let \((u_0, u_1) \in V_1 \times V_-\). Then, there exists a solution \( u \) of (2.5)-(2.6) such that
\[
u \in L^\infty(0, T; V_1), \quad u_t \in L^\infty(0, T; V_-), \quad \forall T > 0,
\] (2.17)
and \( u \) satisfies the energy estimate (2.15). If \( d = 1 \) (with no restriction on \( p \)), or if \( d = 2 \) and \( p \in \{0, 1\} \), then there exists at most one solution \( u \) of (2.5)-(2.6) with regularity (2.17).
Following [17], a solution \( u \) of (2.5)-(2.6) with regularity (2.17) and which satisfies the energy estimate (2.15) is called an **energy solution**.

**Proof.** The proof of existence can be made by standard Galerkin approximation, using the basic a priori estimates (2.8) and (2.15). We refer the reader to [22] (see also [6] for the case \( d = 1 \) and [30] for the case \( d = 3 \) with Dirichlet-type boundary conditions). Also Theorem 3.5 below provides an energy solution. As far as uniqueness is concerned, the proof is standard in one dimension (see again [6]). If \( d = 2 \), uniqueness is proven in [17] by means of a more refined argument which takes advantage of the so-called Brézis-Gallouet inequality (cf. [3]) to show the whole Galerkin approximating sequence converge to (the) solution. □

**Remark 2.2.** By (2.2), the map \( u \mapsto f(u) \) is Lipschitz continuous on bounded subsets of \( V \) with values into \( V' \). Therefore, if \( u \) is a solution of (2.5) which has regularity (2.17), then \( f(u) \in L^\infty(0,T;V_{-1}) \). Hence equation (2.5) has to be understood in \( D'(0,T;V_{-3}) \), with \( u_{tt} \in L^\infty(0,T;V_{-3}) \), and \( (u,u_t) \) belongs (at least) to \( C([0,T];V_{-1} \times V_{-3}) \) [22]. The initial conditions (2.6) make sense in \( V_{-1} \times V_{-3} \).

**Remark 2.3.** Existence of an energy bounded solution can also be established in dimension one without any restriction on the growth of \( f \), thanks to the embedding \( H^1(\Omega) \hookrightarrow L^\infty(\Omega) \) (see [13]). In dimension two or three, similarly to the semilinear wave equation (cf. [22]), the existence of an energy bounded solution holds when \( f \) is a continuous function of polynomially controlled growth satisfying a suitable coercivity condition. In particular, if \( d = 3 \) then one has to require that the initial datum \( u_0 \) is such that \( F(u_0) \in \mathbb{R} \), \( F \) being an antiderivative of \( f \), unless this is ensured by \( u_0 \in V_1 \). Note that existence holds also for negative times, i.e., with \((0,T)\) replaced by \((-T,0)\).

**Remark 2.4.** The existence of more regular (global) solutions can be proven relatively easy in the case \( d = 1 \). This can also be done in dimension two for \( f \) with cubic controlled growth, though proofs are much more technical (see [17]). If \( d = 3 \), existence of smoother (global) solutions has been recently established in [16] for \( \epsilon \) small enough and taking (smooth) initial data bounded by a constant which blows up as \( \epsilon \) goes to 0. In this case, solutions are such that (at least) \( u \in L^\infty(0,T;V_2) \hookrightarrow L^\infty(0,T;C(\overline{\Omega})) \) and no restriction on the growth of \( f \) is required. Hence, uniqueness is straightforward [16]. On the contrary, in dimension three (no restriction on \( \epsilon \) for \( f \) of cubic controlled growth, uniqueness of energy bounded solutions is still an open problem as well as the existence of smoother solutions. A situation which reminds the well-known case of incompressible Navier-Stokes equations.

### 3. Weak convergence of the splitting method

For the space semidiscretization of (1.1), we introduce \((V^h)^{h>0}\), a sequence of finite dimensional subspaces of \( V \) such that

\[
\text{for all } h > 0, V^h \text{ contains the constants};  \quad (3.1)
\]

\[
\cup_{h>0} V^h \text{ is dense in } V. \quad (3.2)
\]

The space \( V^h \) is typically a space of conformal finite elements (see Section 4.1). By assumption (3.1), for every \( u^h \in V^h \), the function \( \dot{u}^h = u - m(u) \) belongs to \( V^h \). We will set

\[
\dot{V}^h := \{ u^h \in V^h : m(u^h) = 0 \}.
\]

Assumptions (3.1)-(3.2) imply that

\[
\cup_{h>0} \dot{V}^h \text{ is dense in } \dot{V}. \quad (3.3)
\]
The space discretization that we use is based on the following formulation of (1.1), which is (formally) equivalent to
\[
\varepsilon u_{tt} + \nu u_t = \Delta w, \\
w = -\alpha \Delta u + f(u).
\]
The space semidiscretization reads: let \((u^h_1, u^h_t) \in V^h \times V^h\) and find \((u^h, w^h) : [0, +\infty] \to V^h \times V^h\) which satisfies
\[
\varepsilon(u^h_{tt}, \varphi) + \nu(u^h_t, \varphi) = -\langle \nabla w^h, \nabla \varphi \rangle \quad \forall \varphi \in V^h, \tag{3.4}
\]
\[
(w^h, \chi) = \alpha \langle \nabla u^h, \nabla \chi \rangle + \langle f(u^h), \chi \rangle \quad \forall \chi \in V^h, \tag{3.5}
\]
\[
u m(u^h(t)) + \nu m(u^h(t)) = 0, \tag{3.10}
\]
so that
\[
m(u^h(t)) = m(u^h_0) + m(u^h_1) \sum_{i=1}^{N^h} \left( 1 - \exp\left(-\frac{\nu t}{\epsilon}\right) \right), \quad m(u^h(t)) = m(u^h_0) \exp\left(-\frac{\nu t}{\epsilon}\right), \tag{3.11}
\]
if \(\nu > 0\), and
\[
m(u^h(t)) = m(u^h_0) + m(u^h_1), \quad m(u^h(t)) = m(u^h_0), \tag{3.12}
\]
if \(\nu = 0\). The matrix \(A\) is not invertible, because of the constants. For this purpose, we define
\[
\hat{A} = (A_{ij})_{2 \leq i, j \leq N^h}, \tag{3.13}
\]
so that \(\hat{A}\) is a symmetric positive definite matrix. For a vector
\[
X = (x_1, \ldots, x_{N^h}) \in \mathbb{R}^{N^h},
\]
we denote \(\hat{X} = (x_2, \ldots, x_{N^h}) \in \mathbb{R}^{N^h-1}\); in particular, we have
\[
U = (u_1, \hat{U}) \quad \text{and} \quad F^h(u_1, \hat{U}) = (f(u^h), e_i^h)_{2 \leq i \leq N^h}.
\]
By the Cauchy-Lipschitz Theorem, equation (3.9) has a unique maximal solution
\[
eq 0. \tag{3.14}
\]
Multiplying (3.14) by \( \dot{U_t} \dot{A}^{-1} \), we find the identity
\[
\frac{\epsilon}{2} \frac{d}{dt}(U_t \dot{A}^{-1} U_t) + \nu U_t \dot{A}^{-1} \dot{U_t} + \alpha \frac{d}{dt}(U_t \dot{A} \dot{U_t}) + \dot{U_t} \dot{F}^h(u_1, \dot{U_t}) = 0, \tag{3.15}
\]
which is a discrete version of (2.10). In view of this, we introduce the discrete energy \( E^h : V^h \times V^h \rightarrow \mathbb{R} \) by setting
\[
E^h(u^h, v^h) = \frac{\epsilon}{2} \| u^h \|^2 + \frac{\nu}{2} \| \dot{v} \|^2_{-1,h} + \int_\Omega F(u^h), \tag{3.16}
\]
where \( \| \cdot \|_{-1,h} \) is the Euclidean norm on \( \dot{V}^h \) defined by
\[
\| \dot{v} \|^2_{-1,h} := \dot{V}^h \dot{A}^{-1} \dot{V}, \quad v^h = \sum_{i=2}^{N_h} v_i \epsilon_i, \quad \dot{V} = (v_2, \ldots, v_{N_h})^t. \tag{3.17}
\]
Notice that \( \| \dot{v} \|^2_{-1,h} \) is a discrete version of \( \| \dot{v} \|^2_{-1} \) but it is not equal, because we have not used a spectral approximation.

We thus have the discrete version of Theorem 2.1:

**Proposition 3.2.** Let \((u_0^h, u_1^h) \in V^h \times V^h\). There exists a unique solution \((u^h, w^h)\) of (3.4)-(3.6) such that \((u^h, w^h) \in C^2([0, +\infty); V^h \times V^h)\). Moreover,
\[
E^h(u^h(t), u_1^h(t)) + \nu \int_0^t \| u_t^h \|^2_{-1,h} \leq E^h(u_0^h, u_1^h) + c_3|m(u_1^h)|^t
+ \frac{c_4|\Omega|}{c_3}(e^{c_3|m(u_1^h)|^t} - 1), \quad \forall t \geq 0. \tag{3.18}
\]

**Proof.** By the Cauchy-Lipschitz Theorem, equation (3.9) has a unique maximal solution \( U \in C^2([0, T^+); V^h) \) which satisfies the initial conditions (3.6). The function \( u^h \) is uniquely defined by (3.8). We have the a priori estimates (3.11) and (3.12) on \( m(u^h) \) and \( m(u_1^h) \), which show that \( (m(u^h), m(u_1^h)) \) is bounded in \( L^\infty(0, T; \mathbb{R} \times \mathbb{R}) \) for all \( T \in (0, T^+) \). Next, we notice that
\[
\frac{d}{dt} \int_\Omega F(u^h) = (f(u^h), \dot{u}^h) + (f(u^h), m(u_1^h)) = \dot{U}_t \dot{F}^h(u_1, \dot{U_t}) + (f(u^h), m(u_1^h)), \tag{3.19}
\]
so that (3.15) reads
\[
\frac{d}{dt} E^h(u^h(t), u_1^h(t)) + \nu \| \dot{u}^h \|^2_{-1,h} = (f(u^h), m(u_1^h)), \quad \forall t \in [0, T^+}. \tag{3.20}
\]
By (2.13) and (3.11)-(3.12),
\[
\frac{d}{dt} E^h(u^h(t), u_1^h(t)) + \nu \| \dot{u}^h \|^2_{-1,h} \leq |m(u_1^h)| \left( c_3 \int_\Omega F(u^h) + c_4|\Omega| \right), \quad \forall t \in [0, T^+). \tag{3.21}
\]
Gronwall’s lemma gives the estimate (3.18), for all \( t \in [0, T^+] \). This implies, since \( v \mapsto \int_\Omega F(v) \) is bounded from below on \( V^h \) (recall (2.16)), that \((\dot{u}^h, \dot{u}_1^h) \in L^\infty([0, T^+), \dot{V}^h \times \dot{V}^h)\), for all \( T \in (0, T^+) \). Thus, \((u^h, u_1^h) \in L^\infty(0, T; V^h \times V^h)\), for all \( T \in (0, T^+), \) and \( T^+ = +\infty \). □

We can also state:

**Theorem 3.3.** Let \((u_0^h, u_1^h) \in V^h \times V^h\), and let \((u^h, w^h) \in C^2([0, +\infty); V^h \times V^h)\) be the solution of (3.4)-(3.6) given by Proposition 3.2. If \( \nu > 0 \), then
\[
(u^h(t), u_1^h(t), w^h(t)) \rightarrow (u^{h,*}, 0, w^*), \quad \text{as} \ t \rightarrow +\infty, \tag{3.23}
\]
where \((u^{h,*}, 0, w^{h,*}) \in V^h \times V^h \times \mathbb{R}\) is a steady state for (3.4)-(3.5), i.e., a solution of
\[
\alpha(\nabla u^{h,*}, \nabla \chi) + (f(u^{h,*}), \chi) = (w^{h,*}, \chi), \quad \forall \chi \in V^h. \tag{3.24}
\]
Remark 3.4. Notice that when $\nu = 0$, one cannot expect convergence to equilibrium, due to the energy conservation. Moreover, it is worth recalling that, because of the translation invariance of equation (1.1) and because of the periodic boundary conditions, the set of equilibria for the continuous problem (2.5)-(2.6) can be a continuum even in one dimension. For the space semidiscretized problem (3.4)-(3.6), the number of stationary states is expected to be very large in general. Thus the convergence to a single stationary state is not a trivial consequence of the dissipative nature of equation (1.1).

Proof. Convergence to equilibrium is a consequence of the Lojasiewicz inequality for analytic nonlinearities. When $m(u^h) = 0$, we can apply [18, Theorem 1.1] to the ODE (3.14), which is a second-order gradient-like system. In the general case $m(u^h) \neq 0$, the system is no longer gradient-like. Then we have to use the fact that $m(u^h(t))$ converges exponentially fast to a constant as $t \to +\infty$ and arguing as in the proofs of [18, Theorem 1.1] and [15, Theorem 4.2], we obtain a similar result. For the reader’s convenience, we sketch the proof. We assume, just for the sake of simplicity, that $\epsilon = \nu = \alpha = 1$.

Equation (3.11) reads

$$m(u^h(t)) = m(u^h_0) + m(u^h_1) - m(u^h_1) \exp(-t), \quad m(u^h_1(t)) = m(u^h_0) \exp(-t), \quad (3.22)$$

so that (3.20) and (2.13) imply

$$\frac{d}{dt} E^h(u^h(t), u^h_1(t)) + \left| \dot{u}^h_1 \right|^2_{1,h} \leq |m(u^h_1)| \exp(-t) \left( c_3 E^h(u^h(t), u^h_1(t)) + c_4 |\Omega| \right), \quad \forall t \geq 0. \quad (3.23)$$

Using Gronwall’s lemma and the estimate $\int_0^\infty \exp(-t)dt \leq 1$, we find that, for all $t \geq 0$,

$$E^h(u^h(t), u^h_1(t)) + \int_0^t \left| \dot{u}^h_1(\tau) \right|^2_{1,h} d\tau \leq |m(u^h_1)| \left( c_3 E^h(u^h_0, u^h_1) + c_4 |\Omega| \exp(c_3 |m(u^h_1)|) \right). \quad (3.24)$$

In particular, by (2.16) and (3.22), $(u^h(t), u^h_1(t)) \in L^\infty(0, +\infty; V^h \times V^h)$. Moreover, we have

$$u^h_1(t) \to 0 \text{ in } V^h, \quad \text{as } t \to +\infty. \quad (3.25)$$

Indeed, by (3.24) $\int_0^{+\infty} \left| \dot{u}^h_1(\tau) \right|^2_{1,h} < +\infty$, and we know that $t \mapsto \left| \dot{u}^h_1(t) \right|^2_{1,h}$ is uniformly continuous on $[0, +\infty)$, since

$$\frac{d}{dt} \left| \dot{u}^h_1 \right|^2_{1,h}(t) = 2 \dot{U}^t_{1}(t) \dot{A}^{-1} \dot{U}_1(t), \quad \forall t \geq 0,$$

and $\dot{U}_1(t)$, defined by (3.14), is bounded in $\mathbb{R}^{N^h-1}$, uniformly in $t \in [0, +\infty)$.

Let now $(t_n)_{n \geq 0}$ be a sequence such that $t_n \to +\infty$ and $u^h(t_n) \to u^{h,*}$. Then, letting $n \to +\infty$, for the solution of the ODE (3.9) with initial conditions $(u^h(t_n), u^h_1(t_n))$, and using (3.25), we find that $u^{h,*}$ is a steady state, i.e., there exists $w^{h,*} \in \mathbb{R}$ such that $(w^{h,*}, u^{h,*})$ satisfies (3.21).

Now, we notice that $m(u^h(t)) \to m(u^h_0) + m(u^h_1)$ (see (3.22)), so we introduce the useful change of variable

$$\tilde{f}(y) = f(m(u^h_0) + m(u^h_1) + y), \quad \text{and} \quad \tilde{F}(y) = \int_0^y \tilde{f}(\sigma)d\sigma.$$

We also define the functional

$$G(\dot{V}) = \frac{1}{2} \dot{V}^t \dot{A} \dot{V} + \int_\Omega \tilde{F} \left( \sum_{i=2}^{N^h} v_i e_i^h \right), \quad \forall \dot{V} = (v_2, \ldots, v_{N^h})^t \in \mathbb{R}^{N^h-1}, \quad (3.26)$$

and we observe that

$$\nabla_v G(\dot{V}) = \dot{A} \dot{V} + \tilde{F}^h(\dot{V}), \quad \forall \ddot{V} \in \mathbb{R}^{N^h-1},$$
where
\[ \hat{F}^h(\tilde{V}) = (\tilde{f}(\sum_{j=2}^{N^h} v_j e_j^h), e_i^h)_{2 \leq i \leq N^h}. \]

With these notations, equation (3.14) can be rewritten into the form
\[ \dot{U}_{tt} + \dot{U}_t + \dot{A}^2 \dot{U} + \dot{A} \hat{F}^h(\dot{U}) = \dot{A}H(t), \]
with
\[ H(t) = \left( \tilde{f}(\dot{u}^h(t)) - \tilde{f}(\dot{u}^h(t) - m(\dot{u}_1^h) \exp(-t)), e_i^h \right)_{2 \leq i \leq N^h}, \quad \forall t \geq 0. \]
Since all norms are equivalent in finite dimension, for each \[ \hat{M} = \frac{1}{2} \dot{U}_t \hat{A}^{-1} \dot{U}_t + G(\dot{U}(t)) - G(\dot{U}^*) + \beta |\dot{A}^{-1} \dot{U}_t^i| |\dot{A} \dot{U}(t) + \hat{F}^h(\dot{U}(t))| + c_6 \exp(-t), \]
with \[ \beta > 0 \] small and \( c_6 > 0 \) large, to be chosen later on. By computation, we find
\[ \mathcal{M}'(t) = \dot{U}_t^i \dot{A}^{-1} \dot{U}_t^i + \dot{U}_t^i [\dot{A} \dot{U}(t) + \hat{F}^h(\dot{U}(t))] + \beta |\dot{A}^{-1} \dot{U}_t^i| |\dot{A} \dot{U}(t) + \hat{F}^h(\dot{U}(t))| \]
\[ + \beta |\dot{A}^{-1} \dot{U}_t^i| |\dot{A} \dot{U}(t) + \nabla \hat{F}^h(\dot{U}(t)) \cdot \dot{U}_t^i| - c_6 \exp(-t). \]
Replacing \( \dot{U}_{tt} \) by its value from (3.27), we obtain
\[ \mathcal{M}'(t) = -\dot{U}_t^i \dot{A}^{-1} \dot{U}_t^i + \dot{U}_t^i H(t) - \beta |\dot{A} \dot{U}(t) + \hat{F}^h(\dot{U}(t))|^2 \]
\[ - \beta |\dot{A}^{-1} \dot{U}_t^i| |\dot{A} \dot{U}(t) + \hat{F}^h(\dot{U}(t))| + \beta H(t)|\dot{A} \dot{U}(t) + \hat{F}^h(\dot{U}(t))| \]
\[ + \beta |\dot{A}^{-1} \dot{U}_t^i| |\dot{A} \dot{U}(t) + \nabla \hat{F}^h(\dot{U}(t)) \cdot \dot{U}_t^i| - c_6 \exp(-t). \]
Now, we use the fact that \( \dot{U}(t) \) and \( \dot{U}_t(t) \) are uniformly bounded on \( [0, +\infty) \), the estimate (3.28) and Young’s inequality, in order to obtain, for \( c_6 > 0 \) large enough,
\[ -\mathcal{M}'(t) \geq \dot{U}_t^i \dot{A}^{-1} \dot{U}_t^i + \frac{\beta}{2} |\dot{A} \dot{U}(t) + \hat{F}^h(\dot{U}(t))|^2 - \frac{\beta}{2} |\dot{A}^{-1} \dot{U}_t^i|^2 \]
\[ - \beta |\dot{A}^{-1} \dot{U}_t^i||\dot{U}_t^i| \left( |\dot{A}| + \sup_{t \geq 0} \|\nabla \hat{F}^h(\dot{U}(t))\| \right). \]
Since all norms are equivalent in finite dimension, for \( \beta > 0 \) small enough, we have
\[ -\mathcal{M}'(t) \geq \frac{\beta}{2} \mathcal{N}^2(t), \quad \forall t \geq 0. \]
where
\[ \mathcal{N}^2(t) := |\dot{U}_t|^2 + |\dot{A} \dot{U}(t) + \hat{F}^h(\dot{U}(t))|^2, \quad \forall t \geq 0. \]
Let \( U^* = (u^*_1, \dot{U}^*) \in \mathbb{R}^{N^h} \) denote the vector associated to the limiting state \( u_{h,*}^t \). By (3.25), \( \mathcal{M}(t_n) \to 0 \) as \( n \to +\infty \). By (3.29), \( t \mapsto \mathcal{M}(t) \) is nonincreasing, so \( \mathcal{M}(t) \geq 0 \) for all \( t \geq 0 \).
If \( \mathcal{M}(t) = 0 \) for some \( t \geq 0 \), then \( \mathcal{M}(t) = 0 \) for all \( t \geq \bar{t} \), and \( \dot{U} \) is constant, by (3.29). Thus, we may assume that \( \mathcal{M}(t) > 0 \) for all \( t \geq 0 \).

Now, we notice that the function \( G \) is a polynomial of the variables \( (v_2, \ldots, v_{N^h}) \), so that we can apply the Lojasiewicz inequality in its original form [24, 25]: there exist \( \rho \in (0, 1/2) \) and \( \eta > 0 \) such that
\[ \forall \tilde{V} \in \mathbb{R}^{N^h-1}, \quad |\tilde{V} - \tilde{U}^*| \leq \eta \Rightarrow |G(\tilde{V}) - G(\tilde{U}^*)|^{1-\rho} \leq |\tilde{A} \tilde{V} + \hat{F}^h(\tilde{V})|. \]
This inequality implies that if \( t \geq 0 \) satisfies
\[ |\dot{U}(t) - \dot{U}^*| \leq \eta, \quad |\dot{U}_t(t)| \leq 1 \quad \text{and} \quad \mathcal{N}(t) > \exp(-(1-\rho)t), \]
then
\[ \mathcal{M}(t)^{1-p} \leq c_{7} \mathcal{N}(t), \]
for some constant \( c_{7} > 0 \) independent of \( t \). As a consequence, we deduce
\[ \frac{d}{dt} \mathcal{M}(t)^{p} = -\rho \mathcal{M}(t) \mathcal{M}(t)^{p-1} \geq \frac{\rho \beta}{2 c_{7}} \mathcal{N}(t). \]
Arguing as in [15, p. 55], we infer from this estimate that, for \( n \) large enough,
\[ \int_{n}^{\infty} |\dot{U}_{i}(t)| dt \leq \int_{n}^{\infty} \mathcal{N}(t) dt < +\infty. \]
Thus, \( \dot{U}(t) \) has a limit as \( t \to +\infty \), which is necessarily \( \dot{U}^{*} \), and this concludes the proof. \( \Box \)

The energy estimate (3.18) implies convergence of the discrete solution to a solution of the continuous problem, as \( h \to 0 \).

**Theorem 3.5.** Let \( (u_{0}^{h}, u_{1}^{h})_{h>0} \) in \( V^{h} \times V^{h} \) such that
\[ (m(u_{0}^{h}), m(u_{1}^{h}))_{h>0} \text{ and } (E^{h}(u_{0}^{h}, u_{1}^{h}))_{h>0} \text{ are bounded}, \]
and let \( u^{h} \in C^{2}([0, +\infty); V^{h}) \) be the solution of (3.4)-(3.6) given by Proposition 3.2. Then, up to a subsequence,
\[ u^{h} \to u \text{ weakly in } L^{\infty}(0, T; V), \quad \forall T > 0, \]
\[ u^{h} \to u \text{ strongly in } C([0, T]; H), \quad \forall T > 0, \]
where \( u \) is an energy solution of (2.5)-(2.6).

**Proof.** Let \( T > 0 \). By (3.11) or (3.12), \( (m(u^{h}))_{h>0} \) and \( (m(u_{0}^{h}))_{h>0} \) are bounded in \( L^{\infty}(0, T) \). By (3.18), \( (\dot{u}^{h})_{h>0} \) is bounded in \( L^{\infty}(0, T; V) \), and \( (|\dot{u}^{h}|_{-1, h})_{h>0} \) is bounded in \( L^{\infty}(0, T) \). Let \( \dot{z}^{h} : [0, T] \to V^{h} \) be defined by
\[ \langle \nabla \dot{z}^{h}(t), \nabla \varphi^{h} \rangle = \langle u_{1}^{h}, \varphi^{h} \rangle, \quad \forall \varphi^{h} \in \dot{V}^{h}. \] (3.30)
With the notations introduced above, we have
\[ \dot{Z}(t) = \dot{A}^{-1} \dot{U}(t), \quad t \geq 0, \]
with \( \dot{Z}(t) = \sum_{i=2}^{N_{h}} \dot{z}_{i}(t) e_{i}^{h} \) and \( \dot{Z} = (z_{2}, \ldots, z_{N_{h}}) \). In particular, we have
\[ |\dot{z}^{h} |^{2}_{1} = \dot{Z}^{t} \dot{A} \dot{Z} = \dot{U}_{1}^{t} \dot{A}^{-1} \dot{U}_{t} = |\dot{u}^{h} |^{2}_{-1, h}, \]
and \( (\dot{z}^{h})_{h>0} \) is bounded in \( L^{\infty}(0, T; V) \). This implies that \( (u^{h})_{h>0} \) is precompact in the space \( C([0, T]; H) \). Indeed, \( (u^{h})_{h>0} \) is uniformly bounded from \([0, T] \) with values in \( V_{1} \), and \( V_{1} \) is precompact in \( H \). Moreover, for all \( 0 \leq \tau \leq t \leq T \),
\[ |\dot{u}^{h}(t) - \dot{u}^{h}(\tau)|^{2}_{0} = 2 \int_{\tau}^{t} \langle \dot{u}^{h}(\sigma), \dot{u}^{h}(\sigma) - \dot{u}^{h}(\tau) \rangle d\sigma, \]
\[ = 2 \int_{\tau}^{t} \langle \nabla \dot{z}^{h}(\sigma), \nabla [\dot{u}^{h}(\sigma) - \dot{u}^{h}(\tau)] \rangle d\sigma, \]
\[ \leq 4 \| \dot{z}^{h} \|_{L^{\infty}(0, T; V)} \| \dot{u}^{h} \|_{L^{\infty}(0, T; V)} |t - \tau|. \]
We also have, by (3.11)-(3.12),
\[ |m(u^{h}(t)) - m(u^{h}(\tau))| \leq |m(u^{h}_{1})||t - \tau|. \]
Thus, the sequence \((u^h)_{h>0}\) is uniformly equicontinuous from \([0,T]\) with values in \(H\). By Ascoli-Arzela’s Theorem, \((u^h)_{h>0}\) is precompact in \(C([0,T];H)\), as claimed. Thus, up to a subsequence, we have
\[
\begin{align*}
    u^h &\to u \quad \text{weakly } * \text{ in } L^\infty(0,T;V), \\
    u^h &\to u \quad \text{strongly in } C([0,T];H), \\
    f(u^h) &\to f(u) \quad \text{a.e. in } \Omega \times (0,T), \\
    f(u^h) &\to f(u) \quad \text{weakly in } L^q(0,T;L^q(\Omega)), \\
    \dot{z}^h &\to \dot{z} \quad \text{weakly } * \text{ in } L^\infty(0,T;V),
\end{align*}
\]
where \(q = (2p+2)/(2p+1) > 1\). Now, let \(\dot{\varphi} \in \dot{V}\), and let \(\dot{\varphi}^h \in V^h\) such that \(\dot{\varphi}^h \to \dot{\varphi}\) strongly in \(\dot{V}\) (this is possible by assumptions (3.1)-(3.2)). For every \(h > 0\), let \(\dot{\Phi} = (\varphi_2, \ldots, \varphi_{N^h})\) with \(\varphi_i^h = \sum_{i=2}^{N^h} \varphi_i e_i^h\). Multiplying (3.14) to the left by \(\dot{\Phi}^t\dot{A}^{-1}\), we find
\[
\epsilon \dot{\Phi}^t\dot{A}^{-1}\dot{U}_{tt} + \nu \dot{\Phi}^t\dot{A}^{-1}\dot{U}_t + \alpha \dot{\Phi}^t\dot{A}\dot{U} + \dot{\Phi}^t\dot{F}^h(u_1, \dot{U}) = 0,
\]
or, equivalently,
\[
\epsilon (\dot{z}^h, \dot{\varphi}) + \nu (\dot{z}^h, \dot{\varphi}) + \alpha (\nabla u^h, \nabla \dot{\varphi}^h) + (f(u^h), \dot{\varphi}^h) = 0. \tag{3.31}
\]
Letting \(h \to 0\) in (3.31), we find that
\[
\epsilon (\dot{z}, \dot{\varphi}) + \nu (\dot{z}, \dot{\varphi}) + \alpha (\nabla u, \nabla \dot{\varphi}) + (f(u), \dot{\varphi}) = 0,
\]
in \(D'(0,T)\), i.e., in the sense of distributions in \((0,T)\). Letting \(h \to 0\) in (3.30), we see that \((\nabla \dot{z}, \nabla \dot{\varphi}) = (\dot{u}_t, \dot{\varphi})\) in \(D'(0,T)\). Thus, \(\dot{u}_t = \dot{A}\dot{z}\) belongs to \(L^\infty(0,T;\dot{V}_{-1})\), and
\[
\epsilon (\dot{A}^{-1}\dot{u}_{tt}, \dot{\varphi}) + \nu (\dot{A}^{-1}\dot{u}_t, \dot{\varphi}) + \alpha (\nabla \dot{u}, \nabla \dot{\varphi}) + (f(u), \dot{\varphi}) = 0,
\]
in \(D'(0,T)\). This is true for all \(\dot{\varphi} \in \dot{V}\), so
\[
\epsilon \dot{A}^{-1}\dot{u}_{tt} + \nu \dot{A}^{-1}\dot{u}_t + \alpha \dot{A}\dot{u} + \dot{\Phi}(f(u)) = 0, \quad \text{in } D'(0,T).
\]
Applying \(\dot{A}\), we find
\[
\epsilon \dot{u}_{tt} + \nu \dot{u}_t + \alpha \dot{A}^2\dot{u} + \dot{A} \circ \dot{\Phi}(f(u)) = 0, \quad \text{in } D'(0,T).
\]
Moreover, letting \(h \to 0\) in (3.10), we obtain
\[
\epsilon m(u_{tt}) + \nu m(u_t) = 0, \quad \text{in } D'(0,T).
\]
These two relations show that the function \(u\) is a solution of (2.5). The energy estimate (2.15) is obtained by letting \(h \to 0\) in (3.18). \(\square\)

**Remark 3.6.** If the energy solution \(u\) with initial data \((u_0, u_1) \in V \times V^\prime\) is unique, \(u^h_0 \to u_0\) weakly in \(V\), \((m(u^h_t), |u^h|^\alpha_{-1,h})_{h>0}\) is bounded in \(\mathbb{R}^2\) and
\[
(u^h_t, \varphi^h) \to (u_1, \varphi) \quad \text{when } \varphi^h \to \varphi \text{ strongly in } V,
\]
then in Theorem 3.5, the whole sequence \(u^h\) tends to \(u\) as \(h \to 0\).

### 4. Error estimates

We have seen that a solution \(u^h\) of the semidiscrete scheme (3.4)-(3.5) converges to an energy solution \(u\) of (3.4), in the sense specified in Theorem 2.1. In this section, we want to be more specific by proving error estimates when \(u\) is assumed to be regular enough.
4.1. Further notation and some useful estimates. For this purpose, we let \( \{T^h\}_{h>0} \) be a quasiuniform family of conforming polygonal decompositions of \( \overline{\Omega} \). Every decomposition \( T^h \) is composed of \( d \)-simplices uniquely (i.e., triangles if \( d = 2 \) and tetrahedrons if \( d = 3 \)) or of \( d \)-parallelepipeds uniquely (i.e., rectangles if \( d = 2 \) and parallelepipeds if \( d = 3 \)) (see \([5, 29, 8]\) for details); the decomposition takes into account the periodic boundary conditions, so that \( T^h \) is in fact a triangulation of \( \mathbb{T} \).

We associate to \( T^h = \bigcup_{T \in T^h} T \) the conformal finite element space of lowest order, \( \mathbb{P}_1 \) or \( \mathbb{Q}_1 \) (see for instance \([8]\)): if the reference element of \( T \) is in fact a triangulation of \( \overline{\Omega} \), let \( \mathbb{P}_1 \) or \( \mathbb{Q}_1 \) coincide.

Obviously, we have

\[
\|u^h\|_1 \leq C_0 h^{-1} \|u\|_0, \quad \forall u^h \in V^h,
\]

\[
\|u^h\|_{L^\infty(\Omega)} \leq C_0 h^{-d/2} \|u\|_0, \quad \forall u^h \in V^h,
\]

where \( C_0 \) is independent of \( h \). In fact, \( C_0 \) depends only on the family \( \{T^h\}_{h>0} \).

We now introduce the operator of orthogonal projection

\[
P^h : L^2(\Omega) \to V^h
\]

for the \( L^2(\Omega) \) scalar product, and the operator of orthogonal projection

\[
\Pi^h : V_1 \to V^h
\]

for the \( V_1 \) scalar product, i.e., for \( u \in V \), \( \Pi^h u \in V^h \) is uniquely defined by

\[
m(\Pi^h u) = m(u) \quad \text{and} \quad (\nabla \Pi^h u, \nabla \varphi^h) = (\nabla u, \nabla \varphi^h), \quad \forall \varphi^h \in V^h.
\]

Obviously, we have

\[
\|P^h v\|_0 \leq \|v\|_0, \quad \forall v \in V_0,
\]

\[
\|\Pi^h v\|_1 \leq \|v\|_1, \quad \forall v \in V_1.
\]

The standard \( L^2 \) and \( H^1 \) error estimates for \( \Pi^h \) (see, e.g., \([5]\)) state that there exists a constant \( C_1 > 0 \) which depends only on the family \( \{T^h\}_{h>0} \) such that, for \( r \in \{1, 2\} \),

\[
\|\Pi^h u - u\|_0 + h \|\Pi^h u - u\|_1 \leq C_1 h^r \|u\|_r, \quad \forall u \in V_r.
\]

Estimates (4.8) are obtained via Cea’s Lemma and duality arguments from similar estimates for the nodal interpolate \( I^h u \) of \( u \). More precisely, for every \( u \in C(\overline{\Omega}) \), let \( I^h u \) denote the unique function in \( V^h \) such that

\[
I^h u(x_i) = u(x_i) \quad \text{for every node } x_i \text{ of the triangulation } T^h.
\]

Since \( 1 \leq d \leq 3 \), the Sobolev inequality yields \( V_2 \subset C(\overline{T}) \) and there exists a constant \( C'_1 \) which depends only on the family \( \{T^h\}_{h>0} \) such that

\[
\|u - I^h u\|_0 + h \|u - I^h u\|_1 \leq C'_1 h^2 \|u\|_2, \quad \forall u \in V_2.
\]

We will also use the inclusion \( V_2 \subset C^{0, \gamma}(\overline{T}) \) for some \( \gamma \in (0, 1] \), where \( C^{0, \gamma}(\overline{T}) \) denotes the usual Hölder space. We can choose \( \gamma = 1 \) if \( d = 1 \), \( \gamma \in (0, 1) \) if \( d = 2 \), and \( \gamma = 1/2 \) if \( d = 3 \). In particular, we have

\[
\|u - I^h u\|_{L^\infty(\Omega)} \leq C_2 h^\gamma \|u\|_2, \quad \forall u \in V_2,
\]

for some constant \( C_2 = C_2(\Omega, d) \).
4.2. Discrete negative seminorms. It will be useful to have a more general definition of the negative norm used in Section 3. Following [32], let $T^h : L^2(\Omega) \to V^h$ be the linear operator defined by $T^h f = u^h$ where, for $f \in L^2(\Omega)$, $u^h \in V^h$ solves

$$\langle \nabla u^h, \nabla v^h \rangle = \langle f, v^h \rangle, \quad \forall v^h \in V^h. \quad (4.11)$$

In terms of the operators introduced above, $T^h = \Pi^h \hat{A}^{-1}$ so that $T^h$ is a discrete version of $\Delta^{-1}$. We note that $T^h$ is selfadjoint and positive semidefinite on $L^2(\Omega)$ since

$$\langle g, T^h f \rangle = \langle \nabla T^h g, \nabla T^h f \rangle = \langle f, T^h g \rangle, \quad \forall f, g \in L^2(\Omega),$$

and

$$\langle f, T^h f \rangle = |\nabla T^h f|^2_0 \geq 0, \quad \forall f \in L^2(\Omega).$$

Moreover, $T^h$ is positive definite on $V^h$ for the $L^2$-scalar product: if $(e_i^h)_{1 \leq i \leq N^h}$ is the orthonormal basis of $V^h$ introduced above, with $e_i^h$ a constant, then $\hat{A}^{-1}$ (recall (3.13)) is the matrix of $T^h_{|V^h}$ in the basis $(e_i^h)_{2 \leq i \leq N^h}$ of $V^h$.

By spectral theory, it is possible to define, for any $s > 0$, the selfadjoint operator $(T^h)^s$. We define for $s \in \mathbb{N}$ the discrete negative seminorm

$$|v|_{-s,h} = \| (T^h)^{-s/2} v \|^2_0 = \| (T^h)^s v, v \|^{1/2}, \quad \forall v \in L^2(\Omega), \quad (4.12)$$

which is an Euclidean norm on $V^h$ for the $L^2$-scalar product: if $(e_i^h)_{1 \leq i \leq N^h}$ is the orthonormal basis of $V^h$ introduced above, with $e_i^h$ a constant, then $\hat{A}^{-1}$ (recall (3.13)) is the matrix of $T^h_{|V^h}$ in the basis $(e_i^h)_{2 \leq i \leq N^h}$ of $V^h$.

For $s = 1$, we recover the norm $| \cdot |_{-1,h}$ defined previously (see (3.17)).

As a useful shortcut, we also define the seminorm

$$\|v\|_{-s,h} = (m(v)^2 + |v|_{-s,h}^2)^{1/2}, \quad \forall v = (m(v), \hat{v}) \in L^2(\Omega), \quad (4.13)$$

which is an Euclidean norm on $V^h$. The discrete seminorm $\| \cdot \|_{-s,h}$ is equivalent to the corresponding continuous negative norm $\| \cdot \|_{-s}$, modulo a small error:

**Lemma 4.1.** There exists a constant $C_3$ independent of $h$ such that, for $s \in \{1, 2\}$,

$$\|v\|_{-s,h} \leq C_3 (|v|_{-s+h}^h + |v|_0) \quad \text{and} \quad \|v\|_{-s} \leq C_3 (|v|_{-s,h} + h^s |v|_0), \quad \forall v \in L^2(\Omega).$$

**Proof.** We follow the proof of Lemma 5.3 in [32]. It is sufficient to prove the assertion for all $v \in L^2(\Omega)$. Let $T = \hat{A}^{-1}$. For $s = 1$ and $v \in L^2(\Omega)$, we have

$$|v|^2_{-1,h} = (T^h v, v) = (Tv, v) + (T^h v - Tv, v) \leq |v|^2_{-1} + C_1 h^2 |Tv|^2_2 |v|_0,$$

using $T^h v = \Pi^h Tv$ and (4.8). By elliptic regularity, $|Tv|^2_2 \leq C_4 |v|^2_2$ for some constant $C_4$ independent of $v$, so

$$|v|^2_{-1,h} \leq |v|^2_{-1} + C_1 C_4 h^2 |v|^2_0.$$

This proves the first inequality for $s = 1$. For $s = 2$ and for $v \in L^2(\Omega)$, we have

$$|v|^2_{-2,h} = |T^h v|^2_0 \leq |Tv|^2_0 + |T^h v - Tv|^2_0 \leq |v|^2_{-2} + C_1 C_4 h^2 |v|^2_0.$$

The second inequality is obtained by interchanging the roles of $T^h$ and $T$. \hfill $\Box$

We point out some other inequalities related to these negative seminorms. First, recall the Poincaré inequality:

$$|\hat{v}|_0 \leq c_P |\hat{v}|_1, \quad \forall \hat{v} \in \hat{V}, \quad (4.14)$$

where $c_P = \lambda_1^{-1/2}$. By induction, we have

**Lemma 4.2.** Let $s \in \mathbb{N}$. Then

$$|\hat{v}|_{-s-1,h} \leq c_P |\hat{v}|_{-s,h}, \quad \forall \hat{v} \in L^2(\Omega). \quad (4.15)$$
These estimates imply in particular that
\[ \|v\|_{-1,h} \leq c_P \|v\|_0 \quad \text{and} \quad \|v\|_{-2,h} \leq (c_P)^2 \|v\|_0, \quad \forall v \in L^2(\Omega), \] (4.16)
where \(c_P = \max(1, c_P)\).

**Proof.** Let \(f \in L^2(\Omega)\). Then, by definition (4.11) of \(T^h\) and by the Poincaré inequality (4.14),
\[ (T^h f, f) = (\nabla T^h f, \nabla T^h f) = |T^h f|^2_1 \geq c_P^{-2} |T^h f|^2_0. \]

On the other hand, \((T^h f, f) \leq |T^h f|_0 |f|_0\) so that eliminating \(|T^h f|_0\), we have
\[ |T^h f|_0 = |f|_{-2,h} \leq c_P^2 |f|_0. \] (4.17)

Similarly, we get
\[ |f|_{-1,h} = (T^h)^{1/2} |f|_0^2 = (T^h f, f) \leq |T^h f|_0 |f|_0. \]

Using (4.17), we obtain
\[ |f|_{-1,h} \leq c_P |f|_0, \] (4.18)
which is estimate (4.15) for \(s = 0\). Now, let \(\dot{v} \in L^2(\Omega)\), and choose \(f = (T^h)^{s/2} \dot{v}\) in (4.18). This reads
\[ |(T^h)^{s/2} \dot{v}|_{-1,h} = |\dot{v}|_{-s-1,h} \leq c_P |(T^h)^{s/2} \dot{v}|_0 = c_P |\dot{v}|_{-s-1,h}, \]
and the proof is complete. \(\square\)

Analogously, we can deduce from the inverse estimate (4.1) some inverse estimates for the discrete seminorms.

**Lemma 4.3.** Let \(s \in \mathbb{N}\). Then there holds
\[ \|v^h\|_{-s,h} \leq C_0 h^{-1} \|v^h\|_{-s-1,h}, \quad \forall v^h \in V^h. \]

**Proof.** Let \((e^h_i)_{1 \leq i \leq N^h}\) denote as previously an orthonormal basis of \(V^h\) for the \(L^2(\Omega)\) scalar product, with \(e^h_1\) a constant, and let \(\hat{A}\) be as previously. Let \(\dot{v} \in \hat{V}^h\), \(\dot{v} = \sum_{i=2}^{N^h} v_i e^h_i\) and \(\hat{V} = (v_2, \ldots, v_{N^h})^t\). The inverse estimate (4.1) reads
\[ \hat{V}^t \hat{A} \hat{V} \leq C_0^2 h^{-2} \hat{V}^t \hat{V}. \]
Replacing \(\hat{V}\) by \(\hat{A}^{-s+1/2} \hat{V}\) in this estimate, we have
\[ \hat{V}^t \hat{A}^{-s/2} \hat{V} \leq C_0^2 h^{-2} \hat{V}^t \hat{A}^{-(s+1)} \hat{V}, \]
that is,
\[ |\dot{v}|_{-s,h} \leq C_0 h^{-1} |\dot{v}|_{-s-1,h}. \]
The result follows immediately. \(\square\)

### 4.3. Error estimates.
We are now ready to derive error estimates on a finite time interval \([0, T]\) \((T > 0)\). Assume that \(u\) is a solution of (2.5)-(2.6) on \([0, T]\) which is regular enough, and let \(u^h\) be a solution of the discretized scheme (3.4)-(3.6) on \([0, T]\). In order to estimate the error \(\|u^h - u\|\), we define, following [7, 32]
\[ u^h - u = \theta^u + \rho^u, \quad \text{with} \quad \theta^u = u^h - \Pi^h u, \quad \rho^u = \Pi^h - u, \] (4.19)
\[ w^h - w = \theta^w + \rho^w, \quad \text{with} \quad \theta^w = w^h - \Pi^h w, \quad \rho^w = \Pi^h w - w, \] (4.20)
The estimates on \(\rho^u\) and \(\rho^w\) follow from (4.8). In the next lemma, we estimate \(\theta^u\) and \(\theta^w\).

**Lemma 4.4.** Let \(u\) be a solution of (2.5) such that
\[ u \in L^\infty(0, T; L^\infty(\Omega)), \quad u \in L^2(0, T; V_3), \quad u_t, u_{tt} \in L^2(0, T; V_1), \] (4.21)
and let \(u^h \in C^2([0, T]; \hat{V}^h)\) be a solution of (3.4)-(3.5). Then the following inequality holds
\[ \frac{d}{dt} \left( \epsilon |\theta_t^u|_{-2,h}^2 + \alpha |\theta_u^u|_{0}^2 \right) + \nu |\theta_t^u|_{-2,h}^2 \leq 3 |\theta_t^u|_{-2,h}^2 + c^2 |\theta_{tt}^u|_{-2,h}^2 + \nu |\theta_t^u|_{-2,h}^2, \] (4.22)
Let this reads associated with $L$

We multiply (4.31) by $\dot{\Theta}$, where $\Theta$ is a solution of (2.5) which is regular enough. Then, introducing the so-called “chemical potential”

$$w = \alpha Au + f(u),$$

we have

$$\epsilon(u_{tt}, \varphi) + \nu(u_t, \varphi) = -\langle \nabla w, \nabla \varphi \rangle, \quad \forall \varphi \in V, \quad (4.25)$$

$$\langle w, \chi \rangle = \alpha \langle \nabla u, \nabla \chi \rangle + (f(u), \chi), \quad \forall \chi \in V. \quad (4.26)$$

Subtracting (4.25) from (3.4), and using the definitions (4.19) and (4.20), we obtain

$$\epsilon(\theta_{tt}, \varphi^h) + \nu(\theta_t, \varphi^h) = -\langle \nabla \theta^w, \nabla \varphi^h \rangle - \epsilon(\rho_{tt}, \varphi^h) - \nu(\rho_t, \varphi^h), \quad \forall \varphi^h \in V^h. \quad (4.27)$$

Choosing $\varphi^h = 1$ in (4.27), and using $(\rho^h(t), 1) = 0$ for all $t \geq 0$, we obtain (4.23). Subtracting (4.26) from (3.5), we obtain similarly

$$(\theta^w, \chi^h) = \alpha \langle \nabla \theta^w, \nabla \chi^h \rangle + (f(u^h) - f(u), \chi^h) - \langle \rho^w, \chi^h \rangle, \quad \forall \chi^h \in V^h. \quad (4.28)$$

Definition (4.3) of $P^h$ shows that, for all $\varphi^h, \chi^h \in V^h$

$$(\rho^h, \varphi^h) = (P^h(\rho^w), \varphi^h) \text{ and } (f(u^h) - f(u), \chi^h) = (P^h(f(u^h) - f(u)), \chi^h).$$

Using the matrix notation introduced in Section 3, equations (4.27)-(4.28) are equivalent to

$$\begin{align*}
\epsilon \Theta_{tt} + \nu \Theta_t &= -A \Theta^w - \epsilon R_{tt}^w - \nu R_t^w, \\
\Theta^w &= \alpha A \Theta^u + NL^u - R^w,
\end{align*}$$

where $\Theta^u$ and $\Theta^w$ are the vectors associated with $\theta^u$ and $\theta^w$, respectively, $R^w$ is the vector associated with $P^h(\rho^u)$, $R^w$ is the vector associated with $P^h(\rho^u)$, and $NL^u$ is the vector associated with $P^h(f(u^h) - f(u))$. Keeping only the components corresponding to $\epsilon_2, \ldots, \epsilon_N$, this reads

$$\begin{align*}
\epsilon \dot{\Theta}_{tt} + \nu \dot{\Theta}_t &= -A \dot{\Theta}^w - \epsilon \ddot{R}_{tt}^w - \nu \ddot{R}_t^w, \\
\dot{\Theta}^w &= \alpha A \dot{\Theta}^u + NL^u - \ddot{R}^w.
\end{align*} \quad (4.29)$$

Next, we multiply (4.30) by $-\dot{A}$ and we add (4.30), so that

$$\epsilon \Theta_{tt} + \nu \Theta_t + \alpha \dot{A} \dot{\Theta}^u = -\epsilon \ddot{R}_{tt}^w - \nu \ddot{R}_t^w - \dot{A} NL^u + \dot{A} \ddot{R}^w. \quad (4.31)$$

We multiply (4.31) by $(\dot{\Theta}^u)^t \dot{A}^{-2}$ and we use the Cauchy-Schwarz inequality. This yields

$$\begin{align*}
\frac{1}{2} \frac{d}{dt} \left( \epsilon |\dot{\theta}_{-2,h}|^2 + \nu |\dot{\theta}_{-2,h}|^2 \right) &\leq \epsilon |\dot{\theta}_{-2,h}| P^h(\dot{\rho}) |\dot{\theta}_{-2,h}| + \nu |\dot{\theta}_{-2,h}| P^h(\dot{\rho}) |\dot{\theta}_{-2,h}| + |\dot{\theta}_{-2,h}| P^h(f(u^h) - f(u)) |\dot{\theta}_{-2,h}| + |\dot{\theta}_{-2,h}| P^h(\dot{\rho}) |\dot{\theta}_{-2,h}|
\end{align*}$$

Finally, we note that

$$|P^h \dot{v}|_{-2,h} = |\dot{v}|_{-2,h}, \quad \forall \dot{v} \in L^2(\Omega),$$

since $T^h(P^h \dot{v}) = T^h \dot{v}$, we use (4.6) and we apply the Cauchy-Schwarz inequality. This gives the expected estimate (4.22).

The regularity (4.21) that is assumed on $u$ implies that the computations used above are valid. In particular, we have $f(u) \in L^\infty(0, T; V_1)$, so that, by definition (4.24), $w \in L^\infty(0, T; V_1)$. In fact, by (4.25) and by elliptic regularity, $w \in L^2(0, T; V_3)$; using (4.24) and elliptic regularity again [14], we even have $u \in L^2(0, T; V_3)$. The proof is complete. \qed
Remark 4.5. Let \((u_0, u_1) \in V_2 \times V_0\). Then, taking advantage of the improved regularity, it is easily seen (see [16]) that there exists at most one solution \(u\) of (2.5)-(2.6) such that

\[
u \in C^0([0, T]; V_2), \quad u_t \in C^0([0, T]; V_0).
\]

On the other hand, if \(u\) is a solution of (2.5) with regularity (4.21), then standard regularity results [22, 31] show that \(u \in C^0([0, T]; V_2)\) (recall that \(V_3 \Subset V_2 \Subset V_1\) and \(u_t \in C^0([0, T]; V_1)\). Thus, uniqueness is not an issue in Lemma 4.4 (cf. also Remark 2.4).

In the following, \(C\) denotes a constant which is independent of \(h\) (but which may depend on the other parameters of the problem).

Theorem 4.6. Let \(u\) be a solution of (2.5) such that

\[
u \in L^2(0, T; V_3), \quad u_t, u_{tt} \in L^2(0, T; V_2), \quad (4.32)
\]

and let \(u^h \in C^2([0, T]; V^h)\) be a solution of (3.4)-(3.5). If

\[
\|u^h(0) - u(0)\|_0 \leq Ch^2 \quad \text{and} \quad \|u^h_t(0) - u_t(0)\|_{-2, h} \leq Ch^2, \quad (4.33)
\]

then

\[
\|u^h - u\|_{L^2(0, T; V_0)} + \|u^h - u\|_{L^\infty(0, T; V_{-2})} \leq Ch^2, \quad (4.34)
\]

\[
\|u^h - u\|_{L^2(0, T; V_1)} + \|u^h_t - u_t\|_{L^\infty(0, T; V_{-1})} \leq Ch. \quad (4.35)
\]

Remark 4.7. Note that the existence of a (unique) solution satisfying (4.32) can be guaranteed (at least in the cases \(d = 1, 2\)) taking \((u_0, u_1) \in V_0 \times V_1\). If \(d = 3\), then \(\epsilon\) must be taken small enough and the norm of the initial data should be bounded by a constant depending on \(\epsilon\) as specified in [16].

Proof. The regularity assumptions imply that \(u \in C^1([0, T]; V_2)\), and in particular, \(u \in L^\infty(0, T; L^\infty(\Omega))\), since \(V_2 \subset L^\infty(\Omega)\). Define

\[
M = \|u\|_{L^\infty(0, T; L^\infty(\Omega))} < +\infty.
\]

We have

\[
\|u^h(0) - u(0)\|_{L^\infty} \leq \|u^h(0) - I^h u(0)\|_{L^\infty} + \|I^h u(0) - u(0)\|_{L^\infty},
\]

\[
\leq C_0 h^{-d/2} \|u^h(0) - I^h u(0)\|_0 + C_2 h^\gamma \|u(0)\|_2,
\]

\[
\leq C_0 h^{-d/2} \left(\|u^h(0) - u(0)\|_0 + \|u(0) - I^h u(0)\|_0\right) + C_2 h^\gamma \|u(0)\|_2,
\]

where, in the second line, we have used the estimates (4.2) and (4.10). In particular, by assumption (4.33) and estimate (4.9), we find

\[
\|u^h(0) - u(0)\|_{L^\infty(\Omega)} \leq Ch^{2-d/2} + Ch^\gamma. \quad (4.36)
\]

Thus, for \(h > 0\) small enough, \(\|u^h(0) - u(0)\|_{L^\infty(\Omega)} \leq 1/2\) and \(\|u^h(0)\| \leq M + 1/2\).

Now, let \(T^h \in (0, T)\) be the maximal time such that \(\|u^h(t)\|_{L^\infty(\Omega)} \leq M + 1\) for all \(t \in [0, T^h]\), and let \(L_f\) denote the Lipschitz constant of \(f\) on the interval \([- (M + 1), M + 1]\). Then, we have

\[
\|f(u^h(t)) - f(u(t))\|_0 \leq L_f \|u^h(t) - u(t)\|_0, \quad \forall t \in [0, T^h].
\]

Thus, using (4.22) and (4.15), we find

\[
\frac{d}{dt} \left(\epsilon \|\partial_t u^h\|_{-2, h}^2 + \alpha \|\partial_t u^h\|_{0}^2\right) + \nu \|\partial_t u^h\|_{-2, h}^2 \leq 3\|\partial_t u^h\|_{-2, h}^2 + \epsilon^2 c^2_1 \|\rho^u_{t, 0}\|_0^2 + \epsilon c_1 \|\rho^u_{t, 0}\|_0^2 + L_f^2 \|u^h - u\|_0^2 + \|\rho^u\|_0^2,
\]

in \(D'(0, T^h)\). Using now

\[
\|u - u^h\|_0^2 \leq 2 \left(\|\partial_u\|_{0}^2 + |m(\theta^n)|^2\right) + 2\|\rho^u\|_0^2,
\]

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together with the standard estimate (4.8), we obtain
\[
\frac{d}{dt} \left( \epsilon |\dot{\theta}_t|^2_{-2,h} + \alpha |\theta_t|^2_{0} \right) \leq 3 |\dot{\theta}_t|^2_{-2,h} + 2L_f^2 |\theta_t|^2_0 + \epsilon^2 c_1 C_2^2 h^4 |u_t|^2 + \nu c_1 C_1^2 h^4 |u_t|^2 + 2L_f^2 |m(\theta_t)|^2 + 2L_f C_2^2 h^4 |u|_2^2 + C_1^2 h^4 |u|_2^2,
\]
still in \( \mathcal{D}'(0,T^h) \). Gronwall’s lemma yields
\[
|\dot{\theta}_t^h(t)|^2_{-2,h} + |\dot{\theta}_t(t)|^2_0 \leq C \left( |\dot{\theta}_t^h(0)|^2_{-2,h} + |\dot{\theta}_t(t)|^2_0 \right) + Ch^4 \|u\|^2_{W^{2,2}(0,T;V^2)} + C \int_0^t |m(\theta_\tau)|^2 d\tau,
\tag{4.37}
\]
for all \( t \in [0,T^h] \), for some constant \( C \) which depends on \( T, \alpha, \epsilon, \nu, C_1, L_f \) and \( c_P \), but which is independent of \( h \) and \( u \). Here, we have denoted
\[
\|u\|^2_{W^{2,2}(0,T;V^2)} = \|u\|^2_{L^2(0,T;V^2)} + \|u_t\|^2_{L^2(0,T;V^2)} + \|u_{tt}\|^2_{L^2(0,T;V^2)}.
\]
Notice that, using elliptic regularity and equation (4.25), the regularity which is assumed on \( u \) implies that \( \dot{w} \in L^2(0,T;V^2) \), with
\[
\|\dot{u}\|^2_{L^2(0,T;V^2)} \leq C \left( \|u\|^2_{L^2(0,T;V^2)} + \|u_t\|^2_{L^2(0,T;V^2)} \right).
\]
Assumptions (4.33) on the initial condition, and the regularity \( u(0), u_t(0) \in V_2 \) imply
\[
\|\theta^u(0)||_0 \leq Ch^2 \quad \text{and} \quad |\theta^u(t)||_{-2,h} \leq Ch^2.
\tag{4.38}
\]
Indeed, we have
\[
\|\theta^u(0)||_0 \leq \|h^0(0) - u(0)||_0 + \|u(0) - \Pi h^0(0)||_0 \leq Ch^2,
\]
by (4.8), and
\[
|\theta^u(t)||_{-2,h} \leq \|h^0(t) - u_0(0)||_{-2,h} + \|u(0) - \Pi h^0(0)||_{-2,h} \leq Ch^2,
\]
by (4.16) and (4.8).

We now solve (4.23) (see (2.8)-(2.9)) and we use
\[
\frac{\epsilon}{\nu} \left( 1 - \exp \left( -\frac{\nu t}{\epsilon} \right) \right) \leq t, \quad \forall t \geq 0,
\]
to see that
\[
|m(\theta^u(t))| \leq |m(\theta^u(0))| + t|m(\theta^u(0))|, \quad |m(\theta^u(t))| \leq |m(\theta^u(0))|.
\tag{4.39}
\]
Thanks to (4.38), this gives
\[
|m(\theta^u(t))| \leq Ch^2 \quad \text{and} \quad |m(\theta^u(t))| \leq Ch^2, \quad \forall t \in [0,T].
\tag{4.40}
\]
Thus, (4.37) becomes
\[
|\dot{\theta}_t^h(t)|^2_{-2,h} + |\dot{\theta}_t(t)|^2_0 \leq Ch^4, \quad \forall t \in [0,T^h],
\tag{4.41}
\]
for some constant independent of \( h \). By an argument similar to the one which gave (4.36), estimates (4.40) and (4.41) imply that
\[
\|u^h(t) - u(t)\|_{L^\infty} \leq Ch^{2-d/2} + Ch^7, \quad \forall t \in [0,T^h],
\]
so that for \( h > 0 \) small enough, \( \|u^h(t) - u(t)\|_{L^\infty} \leq 1 \), and \( T^h = 1 \). Summing up, we have proved that, for \( h > 0 \) small enough,
\[
\sup_{0 \leq t \leq T} \|\theta^u(t)||_0 + \sup_{0 \leq t \leq T} |\theta^u(t)||_{-2,h} \leq Ch^2.
\tag{4.42}
\]
We now notice that Lemma 4.1 implies
\[
\sup_{0 \leq t \leq T} |\theta^u(t)||_{-2} \leq C_3 \left( \sup_{0 \leq t \leq T} |\theta^u(t)||_{-2,h} + h^2 \sup_{0 \leq t \leq T} |\theta^u(t)||_0 \right),
\]
and that the inverse estimate of Lemma 4.3 implies

$$\|\theta^h_1(t)\|_0 \leq C_d h^{-2} \|\theta^h_1(t)\|_{-2,h} \leq C, \quad \forall \ t \in [0, T],$$

so that (4.42) implies

$$\sup_{0 \leq t \leq T} \|\theta^h_1(t)\|_{-2,h} \leq C h^2. \quad (4.43)$$

On the other hand, the standard estimate (4.8) yields

$$\sup_{0 \leq t \leq T} \|\rho^h(t)\|_0 + \sup_{0 \leq t \leq T} \|\rho^h(t)\|_0 \leq C_1 h^2 \left( \|u\|_{L^\infty(0,T;V_2)} + \|u_t\|_{L^\infty(0,T;V_2)} \right). \quad (4.44)$$

Observe that the regularity assumptions on $u$ imply that $\|u\|_{L^\infty(0,T;V_2)}$ and $\|u_t\|_{L^\infty(0,T;V_2)}$ are bounded, because the space $W^{1,2}(0,T;V_2)$, to which both $u$ and $u_t$ belong, is continuously embedded in $C^0(\Omega \times [0, T])$. Estimate (4.34) is now a consequence of the triangular inequality, estimates (4.42), (4.43) and (4.16).

In order to obtain estimate (4.35), we apply the inverse estimate (4.1) and Lemma 4.3 to (4.42), which reads

$$\sup_{0 \leq t \leq T} \|\theta^h_1(t)\|_1 + \sup_{0 \leq t \leq T} \|\theta^h_1(t)\|_{-1,h} \leq C h. \quad (4.45)$$

From this, we proceed as previously with a triangular inequality to obtain (4.35). The proof is complete.

**Remark 4.8.** The $L^2$ and $H^1$ estimates (4.34)-(4.35) for $u - u_h$ are optimal in the sense that the error $\|u - I^h u\|$ for the interpolate $I^h u$ of $u$ has the same order. Similarly, the $H^{-2}$ error for $u_t - u^h_t$ is not better than $O(h^2)$ in general; in contrast, the $H^{-1}$ estimate for $u_t - I^h u_t$ is $O(h^2)$, so that one could expect $\|u_t - u^h_t\|_{-1} = O(h^2)$ instead of $O(h)$. But in our problem, we work with the product space $H^1 \times H^{-1}$, so that the $H^{-1}$ estimate is dominated by the $H^1$ estimate. We can say that the error estimates in Theorem 4.6 are optimal for the $L^2 \times H^{-2}$ and $H^1 \times H^{-1}$ norms.

**Remark 4.9 (Homogeneous Dirichlet boundary conditions).** The weak convergence result of Theorem 3.5 and the error estimates in Theorem 4.6 can be adapted for the singularly perturbed Cahn-Hilliard equation (1.1) with homogeneous Dirichlet boundary conditions on a convex bounded domain of $\mathbb{R}^d$ ($1 \leq d \leq 3$) with smooth boundary. In this case, $V_1 = H^1_0(\Omega)$, and since $-\Delta$ is positive definite on $V_1$, there is no need to introduce $V_1$'s, so that proofs are slightly easier. Choosing $\Omega$ as a convex domain and $P^1$ conforming elements allows to consider a triangulation $\Omega^h$ such that $\Omega^h \subset \Omega$, and consequently to build $V^h \subset V$ with optimal error estimates [32]. However, it is more difficult in this case to find solutions $u$ which satisfy regularity assumptions similar to (4.32), because of the compatibility conditions on the boundary.

**Remark 4.10 (Homogeneous no-flux boundary conditions).** For equation (1.1) endowed with homogeneous no-flux boundary conditions, the situation is more delicate. The space $V = H^1_{per}(\Omega)$ is replaced by $V = H^1(\Omega)$, where $\Omega$ is a bounded domain of $\mathbb{R}^d$ with smooth boundary ($1 \leq d \leq 3$), and the spaces $V_n, V_n$ are defined accordingly. The weak convergence stated in Theorem 4.6 still holds, but it is more difficult in this case to build a finite element space $V^h \subset V$ which can actually be computed. In fact, since $V^h \not\subset V$ in general, nonconforming methods or other approaches (see [32]) should be considered in order to adapt the error estimates of Theorem 4.6 in this case.

5. A finite difference version of the splitting scheme

Equation (1.1) can easily be discretized in space by a finite difference method; this is important for numerical simulations because it allows the use of the Fast Fourier Transform, which in turn can make 3D computations possible. Our purpose here is to explain how the finite
difference scheme can be derived from our finite element scheme (3.4)-(3.5) via a quadrature formula.

For this purpose, let \( h = (h_k)_{1 \leq k \leq d} \) with \( h_k = L_k / N_k \) where \( N_k \in \mathbb{N}^* \) for \( 1 \leq k \leq d \). We work with the grid \( G^h = \Pi_{k=1}^d (\mathbb{Z} / N_k \mathbb{Z}) \); in particular, an index \((i_1, \ldots, i_d) \in G^h\) can always be chosen such that \( 1 \leq i_1 < N_1, \ldots, 1 \leq i_d < N_d \). The value \( u(x_{i_1}, \ldots, x_{i_d}) \) of \( u \) at a grid point \( x_{i_1,\ldots,i_d} = (i_1h_1, \ldots, i_dh_d) \) with \((i_1, \ldots, i_d) \in G^h\) is approximated by \( u_{i_1,\ldots,i_d} \), so that \( u \) is approximated by \( U = (u_{i_1,\ldots,i_d})_{(i_1,\ldots,i_d) \in G^h} \in \mathbb{R}^{G^h} \) (for sake of simplicity, we omit the index \( h \) on \( U \)). The discrete finite difference Laplace operator \( \Delta^h : \mathbb{R}^{G^h} \rightarrow \mathbb{R}^{G^h} \) is defined as usually for \( U = (u_{i_1,\ldots,i_d})_{(i_1,\ldots,i_d) \in G^h} \in \mathbb{R}^{G^h} \) by the \( 2d + 1 \) stencil

\[
(\Delta^h U)_{i_1,\ldots,i_d} = \sum_{k=1}^{d} \frac{u_{i_1,\ldots,i_k+1,\ldots,i_d} - 2u_{i_1,\ldots,i_d} + u_{i_1,\ldots,i_{k-1},\ldots,i_d}}{h_k^2}, \quad \forall (i_1, \ldots, i_d) \in G^h.
\]

The most natural finite difference semidiscretization of equation (1.1) with periodic boundary conditions reads

\[
\epsilon (u_{i_1,\ldots,i_d})_{tt} + \nu (u_{i_1,\ldots,i_d})_t + \alpha ((\Delta^h)^2 U)_{i_1,\ldots,i_d} - (\Delta^h (f(U)))_{i_1,\ldots,i_d} = 0, \tag{5.1}
\]

for all \((i_1, \ldots, i_d) \in G^h\), where \( U : [0, T] \rightarrow \mathbb{R}^{G^h} \) is the unknown function, and \( f : \mathbb{R}^{G^h} \rightarrow \mathbb{R}^{G^h} \) is obviously defined by

\[
(f(U))_{i_1,\ldots,i_d} = f(u_{i_1,\ldots,i_d}), \quad \forall U = (u_{i_1,\ldots,i_d})_{(i_1,\ldots,i_d) \in G^h} \in \mathbb{R}^{G^h}.
\]

Equation (5.1) is a second order ordinary differential system which is completed with initial conditions \( U(0) \in \mathbb{R}^{G^h} \) and \( U_t(0) \in \mathbb{R}^{G^h} \). By the Cauchy-Lipschitz Theorem, given initial conditions \( U(0) \) and \( U_t(0) \), there exists a unique maximal solution \( U : [0, T^+] \rightarrow \mathbb{R}^{G^h} \) which solves (5.1). Proceeding as in the proof of Proposition 3.2, and replacing the \( L^2 \)-scalar product \( (, , ) \) by the approximate \( L^2 \)-scalar product \( (, , )^h \) with trapezoidal rule (see below), it is easily seen that the discrete energy of a solution remains bounded on finite time intervals, and consequently that \( T^+ = +\infty \). If \( \nu > 0 \), then it is possible to prove, as in Theorem 3.3, that \( U(t) \) tends to a discrete steady state as \( t \) tends to \( +\infty \).

On the other hand, the grid \( G^h \) gives a decomposition \( T^h \) of \( T \) into \( N_1 \times \cdots \times N_d \) d-parallelepipeds. Let \( V^h \) be the conforming \( Q^1 \) finite element space associated to this decomposition \( T^h \), as described in Section 4.1. We can consider the finite element scheme (3.4)\((3.5)\) for this space \( V^h \). For the actual computation of (3.4)\((3.5)\), we introduce the nodal basis \((\varphi_{i_1,\ldots,i_d})_{(i_1,\ldots,i_d) \in G^h} \) associated to \( G^h \simeq T^h \), i.e., for all \((i_1, \ldots, i_d) \in G^h\), \( \varphi_{i_1,\ldots,i_d} \) is the unique function in \( V^h \) such that

\[
\varphi_{i_1,\ldots,i_d}(x_{j_1}, \ldots, x_{j_d}) = \begin{cases} 1 & \text{if } (j_1, \ldots, j_d) = (i_1, \ldots, i_d), \\ 0 & \text{otherwise,} \end{cases} \quad \forall (j_1, \ldots, j_d) \in G^h.
\]

Letting \( w^h = \sum_{i \in G^h} u_i \varphi_i \), \( w^h = \sum_{i \in G^h} w_i \varphi_i \), the scheme (3.4)\((3.5)\) reads

\[
\epsilon M^h U_{tt} + \nu M^h U_t = -A^h W, \quad M^h W = \alpha A^h U + F^h(U), \tag{5.2}
\]

where, for \( i = (i_1, \ldots, i_d) \in G^h \) and \( j = (j_1, \ldots, j_d) \in G^h \),

\[
A^h_{ij} = (\nabla \varphi_i, \nabla \varphi_j), \quad M^h_{ij} = (\varphi_i, \varphi_j) \quad \text{and} \quad (F^h(U))_i = (f(u^h), \varphi_i). \tag{5.4}
\]

Now, assume that the integrals in (5.4) are computed with the trapezoidal rule, and let \( M^h, \bar{A}^h \) denote the resulting matrices and \( F^h(U) \) the resulting vector. Recall that for a parallelepiped \( P \) of \( T^h \) with sides \( h_1, \ldots, h_d \), and for a function \( g : T \rightarrow \mathbb{R} \), the trapezoidal rule reads

\[
\int_P g(x_1, \ldots, x_d) dx_1 \ldots dx_d \approx \frac{\Pi_{k=1}^d h_k}{2^d} \sum g(S), \tag{5.5}
\]
where the sum is over all the $2^d$ vertices $S$ of $P$. The integral on $\Omega$ is computed through

$$\int_\Omega g(x_1, \ldots, x_d)dx_1 \ldots dx_d = \sum_{P \in T^h} \int_P g(x_1, \ldots, x_d),$$

where the integral on every $d$-parallelepiped $P$ is approximated by (5.5). For two functions $u, v \in L^2(\Omega)$, the $L^2$-scalar with trapezoidal rule $(u, v)^h$ is obtained by replacing $g(x)$ with $u(x)v(x)$ in the integral above, and using the trapezoidal rule on each parallelepiped.

Then, a straightforward computation shows that

$$\tilde{M}_{ij} = (\Pi_{k=1}^d h_k) \delta_{ij} \quad \text{and} \quad (\tilde{F}^h(U))_i = (\Pi_{k=1}^d h_k) f(u_i),$$

for all $i, j \in \mathcal{G}^h$, and $\delta_{ij}$ is the Kronecker symbol. Another computation entails that

$$\tilde{A}^h \simeq - (\Pi_{k=1}^d h_k) \Delta^h.$$

The scheme (5.2)-(5.3) becomes then

$$\epsilon U_{tt} + \nu U_t = \Delta^h W,$$

$$W = - \alpha \Delta^h + f(U).$$

Eliminating $W$, this system is equivalent to

$$\epsilon U_{tt} + \nu U_t + \alpha(\Delta^h)^2 U - \Delta^h (f(U)) = 0,$$

and we recover the finite difference scheme (5.1), as claimed.

Notice that, because of the quadrature formula, it is not clear whether the convergence result of Theorem 3.5 or whether the error estimates of Theorem 4.6 still apply to the scheme (5.8). This is an interesting open question related to the numerical analysis of nonconforming methods for this equation (see Remark 4.10).

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REFERENCES


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