A Robinson-Schensted algorithm in the geometry of flags for Classical Groups

Een Robinson-Schensted algoritme
in de meetkunde van vlaggen
bij Klassieke Groepen

(Met een samenvatting in het Nederlands)

PROEFSCHRIFT

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§1. Introduction.

1.1. *Statement of the problem.*

3.13 Problem. *Describe the bijection of 3.8 combinatorially in the case of the other classical groups.*
T. A. Springer, *[Spr2]*

We consider certain classical algebraic groups G over an algebraically closed field k. Associated with G is the set $\mathcal{B}$ of its Borel subgroups, which is endowed with the structure of an irreducible algebraic variety, on which G acts by conjugation. For a fixed unipotent element $u \in G$, let $\mathcal{B}_u$ denote the subvariety of $\mathcal{B}$ of Borel subgroups fixed by u, which is in general reducible. These varieties $\mathcal{B}_u$ arise in the study of the unipotent variety of G, and have been studied, amongst others, by Steinberg [St1], [St2], [St3] and by Spaltenstein [Spa]. Any ordered pair of elements of $\mathcal{B}$ has a relative position, which is an element of the Weyl group W of G. Similarly, an ordered pair of irreducible subsets of $\mathcal{B}$ has a generic relative position, namely the relative position of generically chosen elements of these subsets. The problem we shall focus on, is the determination of the generic relative positions of irreducible components of $\mathcal{B}_u$.

In the case $G = GL_n$, the irreducible components of $\mathcal{B}_u$ are parametrised by Young tableaux, and their relative positions are computed by the well-known Robinson-Schensted algorithm (which is described in appendix A), see [St3] and [Spa II.9.8]. We shall give analogous computations for groups G of type B_n, C_n, and D_n, when $\mathrm{char}(k) \neq 2$. A combinatorial parametrisation for the irreducible components of $\mathcal{B}_u$ in these cases is given in [Spa II.6]; it is described in our §3. The parameters used are similar to Young tableaux, but more complicated: they are built up from pairs of adjacent squares rather than from individual squares, and they also involve certain signs; we shall call these objects 'signed domino tableaux'. The Weyl group W can be explicitly represented as a hyperoctahedral group H_n (consisting of permutations with signs), or in the case of type D_n as a subgroup of index 2 thereof. Therefore we shall describe the generic relative positions of irreducible components of $\mathcal{B}_u$ in the form of an algorithm that takes as arguments a pair of signed domino tableaux, and yields a permutation with signs as result.

1.2. *Notational conventions.*

Paranotions, which designate constructs, may now contain metanotions and "hypernotions" have been introduced in order to designate protonotions.
A. van Wijngaarden et al., *[Wijn 0.4.4.a]*

The symbols '$\subset$' and '$\supset$' will always denote strict inclusions of sets; for not necessarily strict inclusions we shall write '$\subseteq$' and '$\supseteq$'. The operations '$\cup$' and '$\cap$' will only be applied to subsets of a given set; to unrelated sets we may apply the operation '$\amalg$'

forming their (external) disjoint union. When forming the (internal) union of subsets, we may indicate the fact that they are actually disjoint by writing '$\uplus$' rather than '$\cup$'. Similarly, for Abelian groups, we write '$+$' for the internal sum of subgroups, '$\times$' for the direct product of groups, and '$\oplus$' for a sum of subgroups that is a direct sum.

For any finite set S, we denote its cardinality by $\#S$. The 2-cyclic group $\{+1,-1\}$ will be denoted simply as $\mathbf{2}$. The image under a map f of a subset X of its domain will be denoted $f[X]$; square brackets will also be used to indicate the inverse image $f^{-1}[y]$ of either a single point or a subset of the codomain. The image of the whole domain of f will be denoted $\mathrm{Im}(f)$, and when applicable the kernel of f will be denoted $\mathrm{Ker}(f)$. For the logical connectives 'and' and 'or' we shall sometimes write '$\wedge$' and '$\vee$' respectively, either because they occur in a formula, or in order to indicate that they bind more strongly than other connectives that are expressed in words.

The set of natural numbers is denoted $\mathbf{N}$, the set of positive natural numbers $\mathbf{N}_{>0}$. A *partition* is an infinite weakly decreasing sequence of natural numbers ending with zeros; the terms of this sequence are called the *parts* of the partition. The ith part of a partition λ will be denoted λ_i, and if $\lambda_i = 0$ for $i > m$ then λ may be denoted as $(\lambda_1, \ldots, \lambda_m)$. In this case λ is called a partition of the number $n = \sum_{i=1}^{m} \lambda_i$. (A partition of a *set* S however, will as usual mean a set of mutually disjoint non-empty sets whose union is S.) A partition λ determines a subset $Y(\lambda)$ of $\mathbf{N}_{>0} \times \mathbf{N}_{>0}$, called its *Young diagram*, defined by $(i,j) \in Y(\lambda) \iff j \leq \lambda_i$; clearly λ is determined by $Y(\lambda)$. The elements of a Young diagram will be called its *squares*, and we may correspondingly depict the Young diagram: the square (i,j) will be drawn in row i and in column j. The *transpose* of λ will be denoted ${}^t\lambda$, and is defined by ${}^t\lambda_j = \#\{\, i \mid \lambda_i \geq j \,\}$; we have that $Y({}^t\lambda)$ is the transpose diagram of $Y(\lambda)$. We define $m_j(\lambda) = {}^t\lambda_j - {}^t\lambda_{j+1} = \#\{\, i \mid \lambda_i = j \,\}$, and put $m_0(\lambda) = \infty$.

For any algebraic variety X, $\mathrm{Con}(X)$ denotes its set of connected components, and $\mathrm{Irr}(X)$ its set of irreducible components. If G is an algebraic group, then $G^\circ \in \mathrm{Con}(G)$ is the component containing the identity $\mathbf{e}$, and $\mathrm{Con}(G)$ is identified with the quotient group G/G°. When G acts algebraically on X, then X is called a *G-space*; if the action is transitive, then X is a *homogeneous* G-space. A morphism of varieties between two G-spaces is called *G-equivariant* if it commutes with the action of any $g \in G$. For any G-space X, and $x \in X$, the stabiliser of x in G is denoted G_x. The *normaliser* of a subset $Y \subseteq X$ is defined as $N_G(Y) = \{\, g \in G \mid g \cdot Y = Y \,\}$, and the *centraliser* of Y as $C_G(Y) = \bigcap_{y \in Y} G_y$; when $Y \subseteq G$, conjugation action is implied.

For a vector space V, the identity transformation $V \to V$ is denoted $\mathbf{1}_V$ or just $\mathbf{1}$. The linear span of a set of vectors is denoted $\langle v_1, \ldots, v_n \rangle$. The projective space associated with V is denoted $\mathbf{P}(V)$; for a subspace S of V,we have $\mathbf{P}(S) \subseteq \mathbf{P}(V)$. The elements of $\mathbf{P}(V)$ are lines in V, and although we shall often switch between these two points of view, we shall always call these objects lines; on the few occasions that we consider projective lines we shall include the adjective. If b is a bilinear form on V, we shall write $\mathrm{sg}(b) = +1$ if b is symmetric, and $\mathrm{sg}(b) = -1$ if it is alternating. Also,

$S^{\perp}$ will denote the subspace $\{ v \in V \mid \forall s \in S\colon b(s,v) = 0 \}$.

1.3. *The group* G.

Let $n \in \mathbf{N}$; three distinct cases will be considered, called case B_n, C_n and D_n respectively, since G will be defined to be a specific group of that type. As the cases are very similar, it is convenient to treat them all at once, but we shall explicitly note the places where the definitions differ according to the case under consideration.

Let M be a vector space over an algebraically closed field k with $\mathrm{char}(k) \neq 2$, equipped with a basis $\{e_{-n}, \ldots, e_0, \ldots, e_n\}$ in case B_n, or $\{e_{-n}, \ldots, e_{-1}, e_1, \ldots, e_n\}$ in cases C_n and D_n. Define $\varepsilon = 1$ in the cases B_n and D_n, and $\varepsilon = -1$ in case C_n. Define a non-degenerate bilinear form b_M on M—with $\mathrm{sg}(b_M) = \varepsilon$—by

$$\begin{aligned} b_M(e_i, e_j) &= 0 \quad \text{when } i + j \neq 0 \\ b_M(e_i, e_{-i}) &= 1 = \varepsilon b_M(e_{-i}, e_i) \quad \text{for } i \geq 0. \end{aligned} \tag{1}$$

Let G be the automorphism group of the vector space M equipped with b_M; according to the case considered, G is isomorphic to either O_{2n+1}, Sp_{2n} or O_{2n}, and indeed the Dynkin type of G corresponds to the case we are in.

1.4. *Flags and Borel subgroups.*

A full isotropic flag f in M is a sequence $f_0 \subset f_1 \subset \cdots \subset f_n$ of subspaces of M, where f_n is isotropic with respect to b_M, i.e., b_M vanishes on $f_n \times f_n$. We call the spaces f_i the *parts* of f. All parts are isotropic subspaces, f_n is a maximal isotropic subspace, and $\dim(f_i) = i$. Since we shall consider no other kind of flags, in the sequel the term flag will always mean full isotropic flag. The set $\mathcal{F}$ of all such flags in M has a natural structure of a (projective) algebraic variety, see for instance [Hu 1.8]. Each map $f \mapsto f_i$ onto the set of isotropic subspaces of dimension i of M is a morphism; with the obvious G-action, $\mathcal{F}$ becomes a homogeneous G-space. The set $\mathcal{F}$ is connected in the cases B_n and C_n, while $\#\,\mathrm{Con}(\mathcal{F}) = 2$ in case D_n. Since $\mathcal{F}$ is a homogeneous G-space, we have $\mathrm{Irr}(\mathcal{F}) = \mathrm{Con}(\mathcal{F})$ in all cases. In case D_n, the component containing $f \in \mathcal{F}$ is determined by f_n. In fact, given f there is a unique flag $f' \in \mathcal{F}$, that we shall call its *companion*, with $f_i = f'_i$ for $i < n$ and $f_n \neq f'_n$; these two flags lie in different connected components of $\mathcal{F}$.

For any $f \in \mathcal{F}$, the stabiliser G°_f is a Borel subgroup of G. All Borel subgroups are obtained in this way, since they are all conjugate. We choose a particular Borel subgroup $B = G^{\circ}_F$, where $F \in \mathcal{F}$ is given by

$$F_0 = \{0\}, \quad F_1 = \langle e_n \rangle, \quad F_2 = \langle e_{n-1}, e_n \rangle, \quad \ldots \quad F_n = \langle e_1, \ldots, e_n \rangle \tag{2}$$

(note that $F_i^{\perp} = \langle e_{-n+i}, \ldots, e_n \rangle$). The variety $\mathcal{B}$ of Borel subgroups of G is in bijection with G°/B via the map $gB \mapsto {}^{g}B \stackrel{\mathrm{def}}{=} gBg^{-1}$; this makes $\mathcal{B}$ into an algebraic variety, and in fact a homogeneous G-space. The map $\mathcal{F} \to \mathcal{B}$ that maps $f \mapsto G_f^\circ$ is G-equivariant and surjective; in cases B_n and C_n it is an isomorphism, and in case D_n the fibres are pairs of companion flags. This allows us for most purposes to identify $\mathcal{B}$ with one component of $\mathcal{F}$; for this we choose the embedding of $\mathcal{B}$ into the component of $\mathcal{F}$ containing F (but note that in case D_n this embedding is only G°-equivariant).

1.5. *Relative positions.*

We choose a maximal torus $T \subseteq B$ of G, namely $T = \bigcap_{i=-n}^{n} N_G(\langle e_i \rangle)^\circ$. The Weyl group of G is defined as $W = N_{G^\circ}(T)/C_{G^\circ}(T)$. According to Bruhat's lemma, we have $G^\circ = \biguplus_{w \in W} BwB$, from which it immediately follows that the orbits of the diagonal G°-action on $(G^\circ/B) \times (G^\circ/B) \cong \mathcal{B} \times \mathcal{B}$ are in bijection with W. Therefore we define

$$O(w) = \{\, ({}^{g}B, {}^{gw}B) \in \mathcal{B} \times \mathcal{B} \mid g \in G^\circ \,\} \quad \text{for } w \in W. \tag{3}$$

The sets $O(w)$ form a partition of the set $\mathcal{B} \times \mathcal{B}$, so we define for every pair of Borel subgroups $B', B'' \in \mathcal{B}$ an element $\pi(B', B'') \in W$ by

$$w = \pi(B', B'') \iff (B', B'') \in O(w). \tag{4}$$

We call $\pi(B', B'')$ the *relative position* of the Borel subgroups B' and B''.

Likewise, one can consider the diagonal G-orbits in $\mathcal{F} \times \mathcal{F}$. Put $\widetilde{B} = G_F$ and $\widetilde{W} = N_G(T)/C_G(T)$. Now in case C_n we have $\widetilde{B} = B$, since $G = G^\circ$; in case D_n we also have $\widetilde{B} = B$, since any $g \in G \setminus G^\circ$ exchanges two elements of $\mathrm{Con}(\mathcal{F})$, and therefore cannot stabilise F. In case B_n however, we have $\widetilde{B} = \mathbf{2} \times B$. From these facts it is easily deduced that $G = \biguplus_{w \in \widetilde{W}} \widetilde{B} w \widetilde{B}$, and, since $\mathcal{F} \cong G/\widetilde{B}$, that the diagonal G-orbits in $\mathcal{F} \times \mathcal{F}$ are in bijection with $\widetilde{W}$. Define

$$\widetilde{O}(w) = \{\, (g \cdot F, gw \cdot F) \in \mathcal{F} \times \mathcal{F} \mid g \in G \,\} \quad \text{for } w \in \widetilde{W}. \tag{5}$$

Since again this forms a partition of $\mathcal{F} \times \mathcal{F}$, we define the relative position $\pi(F', F'') \in \widetilde{W}$ of $F', F'' \in \mathcal{F}$ by

$$w = \pi(F', F'') \iff (F', F'') \in \widetilde{O}(w). \tag{6}$$

One easily checks that this definition is compatible with that of the relative position of Borel subgroups via the embedding of $\mathcal{B}$ in $\mathcal{F}$.

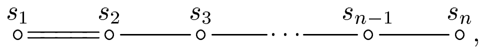
In view of these facts, we shall from this point on concern ourselves only with the relative positions of flags, and not of Borel subgroups. This is an advantage, because it allows a more uniform treatment of the three cases considered, and also because our actual computations will be expressed in terms of flags rather than of Borel subgroups.

1.6. *The hyperoctahedral group H_n.*

We can describe $\widetilde{W}$ explicitly as follows, showing that in all cases it is isomorphic to the hyperoctahedral group H_n, which is the Weyl group of type B_n or C_n. Recall that H_n equals the wreath product of S_n with the group $\mathbf{2}$. An element $w \in H_n$ can therefore be specified by a permutation $p \in S_n$, together with an n-tuple $(\delta_1, \ldots, \delta_n)$ of signs $\delta_i \in \mathbf{2}$; hence we can represent w by the n-tuple of integers $(w_1, \ldots, w_n)$, where $w_i = \delta_i p(i)$. Such sequences of integers can be characterised by

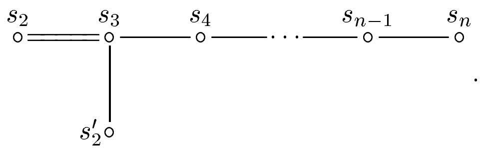
$$\{|w_1|, \ldots, |w_n|\} = \{1, \ldots, n\}, \tag{7}$$

and will be called *signed permutations* of n. For the simple reflections of H_n we take $s_1, \ldots, s_n$, where s_1 is represented by $(-1, 2, 3, \ldots, n)$, and s_i is the transposition of $i-1$ and i (no minus signs) for $i > 1$. The corresponding Dynkin diagram of H_n is



where the root lengths depend on whether H_n is viewed as Weyl group of type B_n or C_n.

Now any $w \in \widetilde{W}$ permutes the lines $\langle e_{-n}\rangle, \ldots, \langle e_{-1}\rangle, \langle e_1\rangle, \ldots, \langle e_n\rangle$, and is determined by that permutation. Since this permutation is such that whenever $w\langle e_i\rangle = \langle e_j\rangle$ then $w\langle e_{-i}\rangle = \langle e_{-j}\rangle$, it is on its turn determined by the images of $\langle e_1\rangle, \ldots, \langle e_n\rangle$, and the integers $w_1, \ldots, w_n$ defined by $w\langle e_i\rangle = \langle e_{w_i}\rangle$ satisfy (7). This defines a map $\widetilde{W} \to H_n$, which is an injective homomorphism of groups. Now we can find elements $\tilde{s}_1, \ldots, \tilde{s}_n \in N_G(T)$ as follows, whose images in $\widetilde{W}$ map to the simple reflections of H_n, proving that the map is an isomorphism. For $\tilde{s}_1$ define $\tilde{s}_1(e_1) = e_{-1}$, $\tilde{s}_1(e_{-1}) = \varepsilon e_1$, and in case B_n, $\tilde{s}_1(e_0) = -e_0$; $\tilde{s}_1$ fixes all other basis vectors e_i. For $i > 1$, $\tilde{s}_i$ exchanges each of the pairs of basis vectors $\{e_i, e_{i-1}\}$ and $\{e_{-i}, e_{1-i}\}$, while fixing all other basis vectors. In cases B_n and C_n we have that all $\tilde{s}_i$ are in $N_{G^\circ}(T)$, so $\widetilde{W} \cong W$ and we have the standard isomorphism $W \xrightarrow{\sim} H_n$. On the other hand, in case D_n, $\tilde{s}_1 \notin G^\circ$, and the simple reflections of W are the images s_i of $\tilde{s}_i$ for $i = 2, \ldots, n$, together with the image s_2' of $\tilde{s}_1\tilde{s}_2\tilde{s}_1$, according to the (somewhat unusually labeled) Dynkin diagram



Therefore, in this case, W can be viewed as a subgroup of index 2 of $\widetilde{W} \cong H_n$.

1.7. *Unipotent elements.*

Let u be a unipotent element in G. We have $u \in G^\circ$ because $\det(u) = +1$. Define $J(u)$ to be the partition of $\dim(M)$ whose parts are the sizes of the Jordan blocks of u arranged in decreasing order. This is well-defined, and determined by the conjugacy class of u in $GL(M)$; we shall call $J(u)$ the *Jordan type* of u. We have the following important fact.

1.7.1. Theorem. *Let $u, u' \in G$ be unipotent elements with $J(u) = J(u')$. Then u and u' are conjugate in G.*

Proof. This is proved for case C_n in [Spr1]; proofs for all cases can be found in [Spr-St] and in [Wall]. Note that in case D_n this becomes invalid if we replace G by $G^\circ = SO_{2n}$. □

Define signs $\varepsilon_j \in \mathbf{2}$ for $j \in \mathbf{N}$ by

$$\varepsilon_j = -\varepsilon(-1)^j. \tag{8}$$

Then for all j with $\varepsilon_j = -1$ we have that $m_j(J(u))$ is even. We shall see in the course of §2 that this condition stems from the fact that non-degenerate alternating forms only exist in even dimensions, and also, by explicit construction, that all partitions of $\dim(M)$ that satisfy it actually occur as $J(u)$ for some unipotent $u \in G$.

1.8. *Fixed point sets.*

Denote by $\mathcal{B}_u$ and $\mathcal{F}_u$ the fixed-point sets of u on the varieties $\mathcal{B}$ and $\mathcal{F}$ respectively. Since $B' = N_{G^\circ}(B')$ for any $B' \in \mathcal{B}$, the variety $\mathcal{B}_u$ may alternatively be described as the set of Borel subgroups containing u. These varieties are extensively studied in [Spa], and since they will play a central rôle in our discussion, we reproduce the facts that apply to our situation. In general $\mathcal{B}_u$ and $\mathcal{F}_u$ are reducible, so define $S_u = \mathrm{Irr}(\mathcal{F}_u)$.

Let $Z_u = C_G(\{u\}) = G_u$; this group acts on $\mathcal{B}_u$, $\mathcal{F}_u$, and S_u. Contrary to the case $G = GL_n$, this centraliser is generally not connected, so put $A_u = \mathrm{Con}(Z_u)$. Since Z_u° acts trivially on S_u, we have an action of A_u on S_u. A description of S_u and of the A_u-action on it are given in [Spa II.6]. These descriptions are quite complicated, and in view of their importance for our computations, we reproduce them in full in §3. Our version will be a slightly modified one, to suit our particular need, and is also restricted to the groups we are in fact considering (Spaltenstein treats many other ones at the same time). At this point, we just state two fundamental geometric facts about $\mathcal{B}_u$ and $\mathcal{F}_u$.

1.8.1. Proposition. *The variety $\mathcal{B}_u$ is connected.*

Proof. See [Spa II.1.7]. □

It follows that $\#\,\mathrm{Con}(\mathcal{F}_u) = \#\,\mathrm{Con}(\mathcal{F})$, which is 1 or 2 depending on the case we are in.

1.8.2. Proposition. *All irreducible components of $\mathcal{F}_u$ have the same dimension.*

Proof. The analogous statement for $\mathcal{B}_u$ is proved in [Spa II.1.12]; for $\mathcal{F}_u$ it directly follows. □

Before we proceed with a more detailed study of the situation, let us briefly indicate the way in which we shall work with $\mathcal{F}_u$. When $n > 0$, define α_u to be the Z_u-equivariant map $\mathcal{F}_u \to \mathbf{P}(\mathrm{Ker}(u - \mathbf{1}))$ given by $f \mapsto f_1$. Also, for $l \in \mathrm{Im}(\alpha_u)$ and $f \in \alpha^{-1}[l]$, define a flag $f^{\downarrow}$ in the space $l^{\perp}/l$, given by

$$f_i^{\downarrow} = f_{i+1}/l, \qquad \text{for } i = 0 \ldots, n-1, \tag{9}$$

then f is determined by $l = f_1$ and $f^{\downarrow}$. Now u induces a unipotent transformation u' of $l^{\perp}/l$, and we have $f^{\downarrow} \in \mathcal{F}_{u'}$. In fact, the map $f \mapsto f^{\downarrow}$ is an isomorphism $\alpha_u^{-1}[l] \xrightarrow{\sim} \mathcal{F}_{u'}$. This allows proofs of statements about the sets $\mathcal{F}_u$ to proceed by induction on the rank n of G; this is the technique that will be used to establish our main results.

As we shall see below, $\mathrm{Im}(\alpha_u)$ consists of finitely many orbits of the Z_u-action, which is an important fact for such inductive proofs. It follows in particular that for any $\sigma \in \mathrm{Irr}(\mathcal{F}_u)$ there is a unique orbit $U \subseteq \mathrm{Im}(\alpha_u)$, such that $\alpha_u^{-1}[U] \cap \sigma$ is dense in σ, so in proving statements for flags f in a dense subset of σ, we may choose any $l \in U$ and assume $f \in \alpha_u^{-1}[l]$. It is important to note, however, that the fibre $\alpha_u^{-1}[l]$ inherits less symmetry from the action of Z_u, than its isomorphic image $\mathcal{F}_{u'}$ posesses due to the action of $Z_{u'}$. Formulated more technically: the natural homomorphism of the stabiliser group $(Z_u)_l$ to $Z_{u'}$ is not generally surjective (nor, for that matter, injective). Due to this circumstance, it is for instance quite possible that $\mathcal{F}_u$ has infinitely many Z_u-orbits, and that some of its irreducible components do not contain any dense orbits. Therefore, despite the straightforward nature of the inductive proofs, some careful reasoning is often required.

§2. Structure of $k[u]$-modules.

2.1. *Definitions.*

Fixing u, we can make the vector space M into a module over the polynomial ring $k[X]$, by letting X act as $u - \mathbf{1}$ (and hence nilpotently). We shall encounter many similar modules, so we define a specific class of such modules.

2.1.1. Definition. *A $k[u]$-module N is a finite dimensional module over $k[X]$, that is equipped with a bilinear form b, such that*
(1) X acts nilpotently,
(2) b is fixed by the action of $X+1$, (i.e., $b((X+1)v, (X+1)v') = b(v, v')$ for all $v, v' \in N$),
(3) $\mathrm{sg}(b) = \varepsilon$.

Note that this definition depends via ε on the case we are considering, so when discussing $k[u]$-modules we always assume that ε has a fixed value, even if G nor M are mentioned. It is clear that M equipped with b_M is a $k[u]$-module. We introduce the following notations to facilitate the discussion of $k[u]$-modules. If N is a $k[u]$-module, then b_N is its bilinear form, u_N is the transformation of N given by the action of $X+1$, and $\eta_N = u_N - \mathbf{1}_N$ is given by the action of X. Clearly u_N is unipotent and η_N is nilpotent. A morphism of $k[u]$-modules $f: N \to N'$ is a morphism of $k[X]$-modules satisfying $b_N(v, v') = b_{N'}(f(v), f(v'))$. The automorphism group of the $k[u]$-module N will be denoted Z_{u_N}, which is consistent with the notation Z_u for $N = M$. The Jordan type $J(u_N)$ of u_N will also be denoted simply as $J(N)$. If b_N is a non-degenerate form, then we will also call N non-degenerate.

The most important way in which a new $k[u]$-module can be derived from M is the following. Let L be some u-stable isotropic subspace of M, then $L^\perp \subseteq M$ is a u-stable subspace containing L, and since b_M vanishes on $L \times L^\perp$, a bilinear form is induced in $L^\perp/L$, which is non-degenerate. Also, $L^\perp/L$ is a $k[X]$-module, and the conditions of 2.1.1 are satisfied, so $L^\perp/L$ is a non-degenerate $k[u]$-module. For this special case we write $u_{[L]}$, $\eta_{[L]}$ and $b_{[L]}$ as abbreviations for $u_{L^\perp/L}$, $\eta_{L^\perp/L}$ and $b_{L^\perp/L}$ respectively. We shall mostly consider this construction with L an isotropic line in $\mathrm{Ker}(\eta_M)$.

2.2. *First results.*

2.2.1. Lemma. *Let N, N' be non-degenerate $k[u]$-modules with $J(N) = J(N')$. Then N and N' are isomorphic as $k[u]$-modules.*

Proof. We obviously have $\dim(N) = \dim(N')$. Since k is algebraically closed, it is well-known that all non-degenerate bilinear forms b on the vector space N with $\mathrm{sg}(b) = \varepsilon$ are equivalent, i.e., $GL(N)$ acts transitively on the set of such forms. Therefore there exists a linear isomorphism $\phi: N \to N'$ that satisfies $b_N(n, m) = b_{N'}(\phi(n), \phi(m))$. Then $u' = \phi^{-1} \circ u_{N'} \circ \phi$ is a unipotent transformation of N preserving b_N. Since $J(u_N) = J(u')$, we get from 1.7.1 that $u' = \theta u_N \theta^{-1}$ for some linear automorphism θ of N preserving b_N; then $\phi \circ \theta$ is a $k[u]$-module isomorphism $N \xrightarrow{\sim} N'$. □

Fix a $k[u]$-module N, and write $\mu = J(N)$, $\eta = \eta_N$, and $\langle x \mid y \rangle = b_N(x, y)$. From 2.1.1(2) we get

$$\langle x \mid \eta y \rangle + \langle \eta x \mid y \rangle + \langle \eta x \mid \eta y \rangle = 0 \qquad \text{for } x, y \in N. \tag{10}$$

2.2.2. Lemma.
(a) For all $c \in \mathbf{N}$ we have $\mathrm{Ker}(\eta^c) \perp \mathrm{Im}(\eta^c)$.
(b) For $c \in \mathbf{N}_{>0}$, $x \in \mathrm{Ker}(\eta^{c+1})$ and $y \in \mathrm{Im}(\eta^{c-1})$, we have $\langle \eta x \mid y \rangle = -\langle x \mid \eta y \rangle$.

Proof. We prove part (a) by induction on c. For $c = 0$ the statement is trivial, so assume that $c > 0$ and the proposition is true for $c - 1$; let $x \in \mathrm{Ker}(\eta^c)$ and $y \in \mathrm{Im}(\eta^c)$ be given, say $y = \eta y'$ with $y' \in \mathrm{Im}(\eta^{c-1})$. Then from (10) we get $\langle x \mid y \rangle = \langle x \mid \eta y' \rangle = -\langle \eta x \mid y' \rangle - \langle \eta x \mid \eta y' \rangle = 0$ by induction, since $\eta x \in \mathrm{Ker}(\eta^{c-1})$. This proves part (a); part (b) follows immediately from (10) and (a). □

If N is non-degenerate, then it follows by considering dimensions that $\mathrm{Ker}(\eta^c)^\perp = \mathrm{Im}(\eta^c)$ and $\mathrm{Im}(\eta^c)^\perp = \mathrm{Ker}(\eta^c)$.

2.3. *Analysis of* Ker(η).

We shall be especially interested in the subspace $\mathrm{Ker}(\eta)$ of N. It has a filtration by Z_{u_N}-stable subspaces.

2.3.1. Definitions. *Define for* $j \in \mathbf{N}_{>0}$*:*
(1) $W_j(N) = \mathrm{Ker}(\eta) \cap \mathrm{Im}(\eta^{j-1})$,
(2) $U_j(N) = \mathbf{P}(W_j(N)) \setminus \mathbf{P}(W_{j+1}(N))$,
(3) $V_j(N) = W_j(N)/W_{j+1}(N)$.

It is readily checked that we have $\dim(W_j(N)) = {}^t\mu_j$ and $\dim(V_j(N)) = m_j(\mu)$; when $m_j(\mu) = 0$ then $U_j(N) = \emptyset$ and otherwise $\dim(U_j(N)) = {}^t\mu_j - 1$.

For each j there is an isomorphism $\psi_j : \mathrm{Ker}(\eta^j)/\mathrm{Ker}(\eta^{j-1}) \xrightarrow{\sim} W_j(N)$ induced by η^{j-1}. By 2.2.2(a) it follows that for $x, y \in W_j(N)$, and with y' any representative of $\psi_j^{-1}(y)$, the expression $\langle x \mid y' \rangle$ has a value depending on x and y only; we denote this value as $\langle x \mid \eta^{1-j}[y] \rangle$. Again by 2.2.2(a) we find that if $x \in W_{j+1}(N)$ then $\langle x \mid \eta^{1-j}[y] \rangle = 0$ for all y, and by repeated application of 2.2.2(b) for $c = 1, \ldots, j-1$ that $\langle x \mid \eta^{1-j}[y] \rangle = \varepsilon_j \langle y \mid \eta^{1-j}[x] \rangle$. Therefore, we may define a bilinear form $b_{j,N}$ on $V_j(N)$ by

$$b_{j,N}(\bar{x}, \bar{y}) = \langle x \mid \eta^{1-j}[y] \rangle, \tag{11}$$

where the bars denote images in $V_j(N)$; we have $\mathrm{sg}(b_{j,N}) = \varepsilon_j$.

From this point on, we assume that N is non-degenerate.

2.3.2. Theorem. *The bilinear form* $b_{j,N}$ *is non-degenerate on* $V_j(N)$.

This theorem explains why $m_j(\mu)$ is even whenever $\varepsilon_j = -1$. To prove it, we need only the first part of the following lemma. The full statement will be used below.

2.3.3. Lemma. *There exists a decomposition* $N = \bigoplus_j N_j$, *where the* N_j *are mutually orthogonal non-degenerate sub-*$k[u]$*-modules, and all parts of* $J(N_j)$ *are equal to* j. *Furthermore, if* $i > 0$ *and* $l \in U_i(N)$ *are given, the decomposition can be chosen such that* $l \subseteq N_i$.

Proof. We may assume that $\dim(N) > 0$. Let $m = \min\{\, j \mid \eta^j = 0 \,\}$, and choose a complementary subspace S to $\mathrm{Ker}(\eta^{m-1})$. If $i < m$, then $l \not\subseteq \mathrm{Im}(\eta^{m-1})$, so $\mathrm{Ker}(\eta^{m-1}) \not\subseteq l^{\perp}$, and S may be chosen such that $S \perp l$. Now we take for N_m the $k[X]$-module generated by S; one immediately sees that all parts of $J(N_m)$ are equal to m. Clearly we have $W_m(N_m) = W_m(N) = \eta^{m-1}[S]$, so if $i = m$ then $l \subseteq N_m$. From the non-degeneracy of N we get that b_{m,N_m} is non-degenerate, from which the non-degeneracy of N_m is deduced as follows. Let $x \in N_m$ be nonzero, then $0 \neq \eta^j x \in W_m(N_m)$ for certain $j \in \mathbf{N}$, so there exists a $y \in N_m$ with $\langle \eta^j x \mid y \rangle \neq 0$, and by repeated application of 2.2.2(b) for $c = 1, \ldots, j$, it follows that $\langle x \mid \eta^j y \rangle \neq 0$. We therefore have $N = N_m \oplus N_m^{\perp}$; if $i < m$ then we also have $l \subseteq N_m^{\perp}$ since $l \perp S$ and $l \subseteq \mathrm{Ker}(\eta) = \mathrm{Im}(\eta)^{\perp}$. The lemma now follows by induction, applied to $N_m^{\perp}$. □

Proof of 2.3.2. We have $W_i(N) = \bigoplus_{j\geq i} W_i(N_j)$ for any decomposition as in the lemma, and hence $V_j(M) \cong V_j(N_j)$ canonically, while $V_j(N_i) = 0$ when $i \neq j$. Using such a decomposition it is sufficient to prove the theorem for $V_j(N_j)$. We have already shown that b_{m,N_m} is non-degenerate, and by the inductive construction of the decomposition, we get the same for any b_{j,N_j}. □

Each $l \in U_j(N)$ determines a line $\bar{l} \subseteq V_j(N)$. We assign to l a *type* $\mathrm{typ}(l)$ in the following way: if $\varepsilon_j = -1$, then $\bar{l}$ is always an isotropic line, and we write $\mathrm{typ}(l) = (\mathrm{I}-)$; if $\varepsilon_j = +1$, then $\bar{l}$ may be either an isotropic or a non-isotropic line, and we write $\mathrm{typ}(l) = (\mathrm{I}+)$ in the former case, and $\mathrm{typ}(l) = (\mathrm{N})$ in the latter. Accordingly, when $\varepsilon_j = +1$, we further split up the sets $U_j(N)$ into

$$U_j^{\mathrm{I}}(N) = \{\, l \in U_j(N) \mid \mathrm{typ}(l) = (\mathrm{I}+)\,\} \tag{12}$$

and

$$U_j^{\mathrm{N}}(N) = \{\, l \in U_j(N) \mid \mathrm{typ}(l) = (\mathrm{N})\,\}. \tag{13}$$

When $m_j(\mu) = 1$ the set $U_j^{\mathrm{I}}(N)$ is empty and of course both these sets are empty if $U_j(N)$ is so; otherwise we have $\dim(U_j^{\mathrm{I}}(N)) = {}^t\mu_j - 2$ and $\dim(U_j^{\mathrm{N}}(N)) = {}^t\mu_j - 1$. When $m_j(\mu) = 2$, then $\#\,\mathrm{Con}(U_j^{\mathrm{I}}(N)) = 2$ since $V_j(N)$ has 2 isotropic lines; in all other cases these sets are irreducible (if non-empty). Clearly, the sets $U_j(N)$, $U_j^{\mathrm{I}}(N)$ and $U_j^{\mathrm{N}}(N)$ are stable under the action of Z_{u_N}. The following fact will prove to be very useful.

2.3.4. Theorem. *Any orbit of the action of* Z_{u_N} *on* $\mathbf{P}(\mathrm{Ker}(\eta))$ *is either equal to* $U_j(N)$ *for some* j *with* $\varepsilon_j = -1$*, or to* $U_j^{\mathrm{I}}(N)$ *or* $U_j^{\mathrm{N}}(N)$ *for some* j *with* $\varepsilon_j = +1$.

Proof. The point to prove is that Z_{u_N} acts transitively on the indicated sets, so let two elements l, l' in one same set be given. There exists a decomposition of N as in 2.3.3, with $l \in U_i(N)$ as given element, and another one where $l' \in U_i(N)$ is given. Since by 2.2.1 each of the summands in one decomposition is isomorphic to the corresponding summand in the other, there is an automorphism of N that transforms the first decomposition into the second. We are therefore reduced to the case that l and l' lie in the same summand N_i. Now $\mathrm{Aut}(V_i(N_i))$—where the automorphisms are those of a vector space equipped with the bilinear form b_{i,N_i}—is either a symplectic group (if $\varepsilon_i = -1$), which acts transitively on $U_i(N_i)$, or an orthogonal group (if $\varepsilon_i = +1$), in which case $U_i^{\mathrm{I}}(N_i)$ and $U_i^{\mathrm{N}}(N_i)$ are the two orbits of its action on $U_i(N_i)$. Therefore, the theorem follows from the following lemma.

2.3.5. Lemma. *Let* N *be a non-degenerate* $k[u]$*-module such that all parts of* $J(N)$ *are equal to* i*. Then the natural homomorphism of groups* $Z_{u_N} \to \mathrm{Aut}(V_i(N))$ *is surjective.*

Proof. This is a special case of the more general results formulated in [Spr-St 2.22–2.25], which show that there is in fact a subgroup C of Z_{u_N} that is isomorphically mapped to $\mathrm{Aut}(V_i(N))$; we sketch its construction for this particular case. Rather than the nilpotent η we employ $\eta' = \eta(u_N+1)^{-1}$, which is well-defined since $\mathrm{char}(k) \neq 2$ implies that u_N+1 is invertible. This nilpotent (the Cayley transform of u_N) satisfies

$$\langle \eta' x \mid y\rangle + \langle x \mid \eta' y\rangle = 0, \tag{14}$$

which can be verified by computing $\langle \eta'(u_N+1)x \mid (u_N+1)y\rangle = \langle \eta x \mid y\rangle - \langle x \mid \eta y\rangle$ with the aid of (10), and similarly for $\langle (u_N+1)x \mid \eta'(u_N+1)y\rangle$; also η' is centralised by Z_{u_N}. With some subspace S as in the proof of 2.3.3 we may decompose the vector space N as $\bigoplus_{j=0}^{i-1} \eta'^j[S]$, and we have $\eta'^{(i-1)}[S] = W_i(N) \cong V_i(N)$. Now let C be the normaliser of S in Z_{u_N}; it is clear that C stabilises the given decomposition of N, and that the action of any $z \in C$ on any of the summands determines its action on all other summands. Using (14) one now easily shows that the homomorphism $C \to \mathrm{Aut}(V_i(N))$ is bijective. □ □

We return to the case of an arbitrary non-degenerate $k[u]$-module N again, and let $l \in U_j(N)$ be an isotropic line. We consider the formation of the non-degenerate $k[u]$-module $l^\perp/l$, and the way it is related to N. Clearly this can essentially depend only on the Z_u-orbit of l, i.e., on j and $\mathrm{typ}(l)$. We denote the projection $l^\perp \to l^\perp/l$ by Π.

2.3.6. Lemma. *Put $j' = j-1$ if $\mathrm{typ}(l) = (\mathrm{N})$, and $j' = j$ otherwise.*

(a) *For $i \neq j$ we have $W_i(l^\perp) = W_i(N)$, and $W_j(l^\perp)$ is a subspace of codimension 1 of $W_j(N)$.*

(b) *We have $l \subseteq W_i(l^\perp)$ if and only if $i \leq j'$.*

(c) *If $i \geq j'$ then $\Pi[W_i(l^\perp)] = W_i(l^\perp/l)$, while if $i < j'$ then $\Pi[W_i(l^\perp)]$ is a subspace of $W_i(l^\perp/l)$ of codimension 1.*

(d) *If $i \leq j'$ then $\Pi^{-1}[W_i(l^\perp/l)] \cap \mathrm{Ker}(\eta_{l^\perp}) = W_i(l^\perp)$, and if $i > j'$ we have that $\Pi^{-1}[W_i(l^\perp/l)] = W_i(l^\perp) \oplus l$.*

Proof. It follows from the definition of b_{j_N} that the image of $W_j(l^\perp)$ in $V_j(N)$ is $\bar{l}^\perp$, where $\bar{l}$ is the image of l. Choose a decomposition of N as in 2.3.3 with $l \subseteq N_j$; this induces a similar decomposition of each $W_i(N)$. Since all summands except N_j are contained in $l^\perp$, and noting for the case $i = j-1$ that $W_{j-1}(N_j \cap l^\perp) = W_j(N_j)$, part (a) readily follows. Part (b) can be checked for both cases in the definition of j'. For parts (c) and (d) we consider the spaces $\mathrm{Ker}(\eta)$ and $\mathrm{Im}(\eta^{i-1})$ appearing in the definition of W_i. It is immediate that $\Pi[\mathrm{Im}(\eta^i_{l^\perp})] = \mathrm{Im}(\eta^i_{[l]})$ for any i, and that $\Pi[\mathrm{Ker}(\eta_{l^\perp})] \subseteq \mathrm{Ker}(\eta_{[l]})$. The latter inclusion is strict if $j' > 1$, since then $\Pi^{-1}[\mathrm{Ker}(\eta_{[l]})]$ is spanned by $\mathrm{Ker}(\eta_{l^\perp})$ and a line $\hat{l} \subseteq N_j$ with $\eta[\hat{l}] = l$ (check this for both cases in the definition of j'). We have $\Pi[\hat{l}] \in U_{j'-1}(l^\perp/l)$, and $\mathrm{Ker}(\eta_{[l]}) = \Pi[\hat{l}] \oplus \Pi[\mathrm{Ker}(\eta_{l^\perp})]$. It is now easy to derive parts (c) and (d). □

2.3.7. Corollary. *For $l \in U_j(N)$, the partition $J(l^\perp/l)$ is obtained form $J(N)$ by modifying it in the following way: if $\mathrm{typ}(l) = (\mathrm{N})$ then one part j is replaced by a part $j-2$, and otherwise two parts j are both replaced by a part $j-1$.*

Proof. By comparing dimensions of spaces W_i, we get from 2.3.6 (using its notation) that $J(l^\perp)$ is obtained from $J(N)$ by decreasing a part j by 1, and from this $J(l^\perp/l)$ is obtained by decreasing a part j' by 1. The corollary follows. □

The results of 2.3.6 may also be interpreted in terms of the spaces V_i. In the following, 'natural' means $(Z_u)_l$-equivariant, and the isomorphisms preserve the respective bilinear forms.

2.3.8. Corollary. *Let* j' *and* Π *be as in 2.3.6; let* $\bar{l}$ *be the image of* l *in* $V_j(N)$*, and if* $j' > 1$ *let* S *be the image of* $\Pi[W_{j'-1}(l^\perp)]$ *in* $V_{j'-1}(l^\perp/l)$*, which has codimension 1 by 2.3.6(c). The statements involving* S *tacitly assume* $j' > 1$.

(a) *The image* $V_j(l^\perp)$ *of* $W_j(l^\perp)$ *in* $V_j(N)$ *equals* $\bar{l}^\perp$.
(b) *If* $\mathrm{typ}(l) = (\mathrm{N})$*, there are natural isomorphisms* $V_j(l^\perp/l) \cong \bar{l}^\perp$*,* $V_{j'-1}(N) \cong S$*, and* $S^\perp \cong \bar{l}$.
(c) *If* $\mathrm{typ}(l) \neq (\mathrm{N})$*, there are natural isomorphisms* $V_j(l^\perp/l) \cong \bar{l}^\perp/\bar{l}$ *and* $V_{j'-1}(N) \cong S/S^\perp$.
(d) *If* $i \notin \{j, j'-1\}$*, there is a natural isomorphism* $V_i(N) \cong V_i(l^\perp/l)$.

Proof. (a) By 2.3.6(a) we have $W_{j+1}(l^\perp) = W_{j+1}(N)$, so that $V_j(l^\perp)$ is indeed a subset of $V_j(N)$; by definition it is the image of $W_j(l^\perp)$. In the proof of 2.3.6(a) we have seen that it equals $\bar{l}^\perp$. (b) By 2.3.6(c,d), Π induces an isomorphism $V_j(l^\perp) \xrightarrow{\sim} V_j(l^\perp/l)$ which is obviously natural, hence the first statement follows from (a). Since $W_i(N) = W_i(l^\perp)$ for $i = j'$, $j'-1$ and $\Pi[W_{j'}(N)] = W_{j'}(l^\perp/l)$, the second isomorphism is also induced by Π. For the third isomorphism, consider the space $q = \Pi^{-1}[\mathrm{Ker}(\eta_{[l]})]/\mathrm{Ker}(\eta_{l^\perp})$, which is represented by $\hat{l}$ in the proof of 2.3.6. Now η defines an isomorphism $q \xrightarrow{\sim} l \cong \bar{l}$, while Π defines an isomorphism $q \xrightarrow{\sim} V_{j'-1}(l^\perp/l)/S \cong S^\perp$ (S is non-degenerate), and these isomorphisms piece together to a natural isomorphism of vector spaces. Since these spaces are 1-dimensional and non-degenerate, this isomorphism can be modified by scalar multiplication in order to preserve the bilinear form (using (10) one checks that the square of this scalar should be -1). (c) We have $j' = j$, so by 2.3.6(c) Π induces a surjective map $V_j(l^\perp) \to V_j(l^\perp/l)$, whose kernel is easily seen to be $\bar{l}$; therefore the first statement follows from (a). Using the notation of (11), observe that $\langle x \mid \eta^{2-j}[y]\rangle = 0$ for all $x \in W_j(N)$ and $y \in W_{j-1}(l^\perp)$, which implies that $\langle \Pi(x) \mid \eta_{[l]}^{2-j}[\Pi(y)]\rangle = 0$, so the image p of $\Pi[W_j(N)]$ in $V_{j-1}(l^\perp/l)$ is perpendicular to S. We have $W_j(l^\perp/l) = \Pi[W_j(l^\perp)]$, so $\dim(p) = 1$ by 2.3.6(a,b), and $p = S^\perp$. Therefore Π induces an isomorphism $W_{j-1}(l^\perp)/W_j(N) \xrightarrow{\sim} S/S^\perp$, and since $W_{j-1}(l^\perp) = W_{j-1}(N)$, this establishes the second isomorphism. (d) All cases are immediate from 2.3.6, except possibly $i = j-1$ when $\mathrm{typ}(l) = (\mathrm{N})$. In that case $W_j(l^\perp) \oplus l = W_j(N) \subseteq W_i(l^\perp) = W_i(N)$, so we have a natural isomorphism $V_i(l^\perp)/\tilde{l} \xrightarrow{\sim} V_i(N)$ for the image $\tilde{l}$ of l in $V_i(l^\perp)$. On the other hand $\tilde{l}$ is the kernel of the surjective map $V_i(l^\perp) \to V_i(l^\perp/l)$ induced by Π, so we obtain an isomorphism $V_i(N) \cong V_i(l^\perp/l)$ in this case as well. □

2.4. *The group A_u.*

We give an explicit description of the component group A_u of Z_u. Define for $j \in \mathbf{N}_{>0}$ the groups $Z_{u,j} = \mathrm{Aut}(V_j(M))$ (as in 2.3.5), and $A_{u,j} = \mathrm{Con}(Z_{u,j})$. We have either $Z_{u,j} \cong Sp_{m_j(\lambda)}$ or $Z_{u,j} \cong O_{m_j(\lambda)}$, according as $\varepsilon_j = -1$ or $\varepsilon_j = +1$. Consequently, $A_{u,j}$ is trivial unless $\varepsilon_j = +1$ and $m_j(\lambda) > 0$, in which case $A_{u,j} \cong \mathbf{2}$. Now there is a natural homomorphism $\mathrm{Aut}(M) = Z_u \to \prod_j Z_{u,j}$, which by 2.3.3 and 2.3.5 is surjective. We refer again to [Spr-St 2.22–2.25], where it is shown that the kernel of this homomorphism is the unipotent radical of Z_u, and hence connected. Composing with the homomorphism $\prod_{j>0} Z_{u,j} \to \prod_{j>0} A_{u,j}$ we therefore obtain a surjective homomorphism $Z_u \to \prod_{j>0} A_{u,j}$ with kernel Z_u°. This proves

2.4.1. Proposition. *For any unipotent $u \in G$ with $J(u) = \lambda$ the group A_u is isomorphic to $\prod_{j>0} A_{u,j}$, where $A_{u,j}$ is trivial unless $\varepsilon_j = +1 \wedge m_j(\lambda) > 0$, in which case $A_{u,j} \cong \mathbf{2}$.* □

In particular, A_u is an Abelian group. Hence for $N \cong M$ we can canonically identify A_{u_N} with A_u, regardless of the choice of isomorphism $N \xrightarrow{\sim} M$. We shall not bother to distinguish between these two groups, and shall denote it as A_λ in places where no particular unipotent is specified, but only the Jordan type λ.

§3. The parametrisation of the irreducible components of $\mathcal{F}_u$.

3.1. *Fundamentals.*

We shall study $\mathcal{F}_u$ using the morphism $\alpha_u{:}\,\mathcal{F}_u \to \mathbf{P}(\mathrm{Ker}(\eta_M))$ defined in 1.8. The basic geometric fact that will be used is the following elementary

3.1.1. Lemma. *Let a connected algebraic group Z, a Z-space Y, a homogeneous Z-space X and a Z-equivariant map $\alpha{:}\,Y \to X$ be given. For any $x \in X$ there is a bijection from $\mathrm{Irr}(Y)$ to the set of Z_x orbits in $\mathrm{Irr}(\alpha^{-1}[x])$ that is given by $\phi{:}\,C \mapsto \mathrm{Irr}(C \cap \alpha^{-1}[x])$. The same is true when 'Irr' is replaced by 'Con' in all places.*

Proof. Let $C \in \mathrm{Irr}(Y)$ be given. Then it is a general property of morphisms (see [Hu 4.1]) that each irreducible component σ of $C \cap \alpha^{-1}[x]$ (i.e., each $\sigma \in \phi(C)$) has $\dim(\sigma) \geq r = \dim(C) - \dim(X)$. But since Z acts transitively on the set of fibres of α, we must have $\dim(\sigma) = r$. Therefore $Z \cdot \sigma = \{\, z \cdot y \mid z \in Z, y \in \sigma \,\} \subseteq C$ has dimension $\dim(C)$, so for any other $\sigma' \in \phi(C)$ we have $Z \cdot \sigma' = Z \cdot \sigma$. It follows that Z_x acts transitively on $\phi(C)$, and that $Z \cdot \sigma = C$. If $\sigma \subseteq \tilde\sigma \in \mathrm{Irr}(\alpha^{-1}[x])$ then we have $Z \cdot \sigma \subseteq Z \cdot \tilde\sigma$ which is irreducible, so $\sigma = \tilde\sigma$, which proves $\phi(C) \subseteq \mathrm{Irr}(\alpha^{-1}[x])$. Therefore ϕ is indeed a map to the set of Z_x-orbits in $\mathrm{Irr}(\alpha^{-1}[x])$, and it is clearly injective and surjective. The proof for connected components is entirely similar. □

Remark. A more geometric proof can be given, based on the fact that the map $Z \times \alpha^{-1}[x] \to Y$ given by $(z, y) \mapsto z \cdot y$ is smooth. This is true because it is a pull-back by α of the smooth map $Z \to X$ that maps $z \mapsto z \cdot x$.

We shall use the following common notation. Given a closed subgroup $H \subseteq Z$ and a H-space S, the quotient of $Z \times S$ by the H-action $h\cdot(z,s) = (zh^{-1}, h\cdot s)$—which exists for any reasonable S—will be denoted as $Z \times^H S$; it is a Z-space via the left action of Z on the first factor of $Z \times S$. With the natural injection $S \to Z \times^H S$ it has the following universal property: any H-equivariant map from S to a Z-space X can be uniquely extended to a Z-equivariant map $Z \times^H S \to X$. In particular it is up to isomorphism the unique Z-equivariant bundle over Z/H whose fibre at the coset H is S (as H-space).

Returning to $\mathcal{F}_u$, fix $l \in \mathrm{Im}(\alpha_u)$, and let $X = Z_u \cdot l$ be its orbit, which is as described in 2.3.4. Let $X^\circ = Z_u^\circ \cdot l$ be the connected component of X containing l, let $Y = \alpha_u^{-1}[X]$ and $Y^\circ = \alpha_u^{-1}[X^\circ]$, and put $u' = u_{[l]}$. We have $Y \cong Z_u \times^{(Z_u)_l} \alpha_u^{-1}[l]$, and we can derive a similar expression for $\mathrm{Irr}(Y)$ as A_u-space, in which all groups and sets are finite. We assume that $S_{u'} \cong \mathrm{Irr}(\alpha_u^{-1}[l])$ with its $A_{u'}$-action is known. Let K be the image of $(Z_u^\circ)_l$ in $A_{u'}$, let H be the image of $(Z_u)_l$ in A_u, and H' the image of $(Z_u)_l$ in $A_{u'}$. We have a canonical homomorphism $H \to H'/K$, via which H acts on the orbit set $K\backslash S_{u'}$.

3.1.2. Lemma. *There is an A_u-equivariant bijection* $\mathrm{Irr}(Y) \cong A_u \times^H (K\backslash S_{u'})$.

Proof. We can apply 3.1.1 to the restriction $Y^\circ \to X^\circ$ of α_u, and find that $\mathrm{Irr}(Y^\circ)$ is isomorphic to $(Z_u^\circ)_l\backslash S_{u'}$. The action of $(Z_u^\circ)_l$ on $S_{u'}$ factors through its projection onto K, so we obtain an isomorphism of sets with H-action $\mathrm{Irr}(Y^\circ) \xrightarrow{\sim} K\backslash S_{u'}$. Any $z \in Z_u$ that normalises X° (and hence also Y°) lies in the subgroup generated by Z_u° and $(Z_u)_l$, so H is the normaliser of $\mathrm{Irr}(Y^\circ)$ in the action of A_u on $\mathrm{Irr}(Y)$. It follows that $\mathrm{Irr}(Y)$ is as described in the lemma. (Admittedly this description seems more complicated than necessary in view of 2.4.1, but it does reflect the way in which it is obtained.) □

We call an orbit $X \subseteq \mathrm{Im}(\alpha_u)$ *good*, if $\dim(Y) = \dim(X) + \dim(\alpha_u^{-1}[l])$ has the maximal possible value, which is $\dim(\mathcal{F}_u)$.

3.1.3. Lemma. $S_u = \mathrm{Irr}(\mathcal{F}_u) = \biguplus_X \mathrm{Irr}(\overline{\alpha_u^{-1}[X]})$, *where X ranges over all good orbits in* $\mathrm{Im}(\alpha_u)$, *and the bar denotes Zariski closure.*

Proof. By 2.3.4 there are only finitely many orbits X in $\mathrm{Im}(\alpha_u)$, and by 1.8.2 the elements of $\mathrm{Irr}(\overline{Y})$ for $Y = \alpha_u^{-1}[X]$ are irreducible components of $\mathcal{F}_u$ if and only if $\dim(Y) = \dim(\mathcal{F}_u)$, i.e., if X is good. The disjointness of the sets $\mathrm{Irr}(\overline{Y})$ (i.e., the fact that no two sets $\overline{Y}$ share an irreducible component) is evident. □

We know the dimensions of all orbits X by 2.3.4; from these we obtain $\dim(\mathcal{F}_u)$ as follows. We may fill the Young diagram $Y(\lambda)$ with natural numbers, inserting into position (i,j) (i.e., column j of row i) the number $\lfloor \frac{i-1}{2} \rfloor$ when $\varepsilon_j = +1$, and $\lfloor \frac{i}{2} \rfloor$ when $\varepsilon_j = -1$ (the only importance of these numbers is that they make the following lemma work). Let $l \in X$ and $\lambda' = J(u_{[l]})$.

3.1.4. Lemma. *When $Y(\lambda)$ is filled in this way with numbers, then*
(a) $\dim(X)$ *equals the sum of the numbers in the squares of $Y(\lambda) \setminus Y(\lambda')$, except when $X = U_j^{\mathrm{N}}(M)$ and $m_{j-1}(\lambda) > 0$, in which case* $\dim(X)$ *is strictly less than that sum,*
(b) $\dim(\mathcal{F}_u)$ *is the sum of the numbers in all squares of $Y(\lambda)$,*
(c) X is a good orbit if and only if we have equality in (a).

Proof. Part (a) follows using 2.3.7, by separately checking the cases $X = U_j(M)$, $X = U_j^{\mathrm{I}}(M)$ and $X = U_j^{\mathrm{N}}(M)$; in the excepted case we have $\varepsilon_{j-1} = -1$, and therefore $m_{j-1}(\lambda) \geq 2$. Then (b) follows by induction on n, provided that we have equality in (a) for at least one orbit X, which is clear since in the excepted case we have $j > 1$ and $U_{j-1}(M) \neq \emptyset$. Part (c) is an immediate consequence of (a) and (b) $\square$

We denote by Q_u the union of all good orbits in $\mathrm{Im}(\alpha_u)$. By 3.1.3, $\alpha_u^{-1}[Q_u]$ is dense in $\mathcal{F}_u$. The condition given by 3.1.4 for being a good orbit can be reformulated in the following simple form.

3.1.5. Proposition. *Let $l \in X$ and $\lambda' = J(u_{[l]})$. Then X is a good orbit if and only if the two squares in $Y(\lambda) \setminus Y(\lambda')$ are adjacent (either horizontally or vertically).* $\square$

3.2. *Some combinatorics: domino shapes and unsigned tableaux.*

Inspired by the facts above, we introduce some combinatorial objects.

3.2.1. Definition. *A domino shape is a pair p of adjacent squares, that has one of the following three forms. According to the case applying, the type* $\mathrm{typ}(p) \in \{(\mathrm{I}-), (\mathrm{I}+), (\mathrm{N})\}$ *of p is also defined.*
(1) $p = \{(i-1, j), (i, j)\}$ *and* $\varepsilon_j = -1$*; then* $\mathrm{typ}(p) = (\mathrm{I}-)$.
(2) $p = \{(i-1, j), (i, j)\}$ *and* $\varepsilon_j = +1$*; then* $\mathrm{typ}(p) = (\mathrm{I}+)$.
(3) $p = \{(i, j-1), (i, j)\}$ *and* $\varepsilon_j = +1$*; then* $\mathrm{typ}(p) = (\mathrm{N})$.
In each case we write $\pi(p) = (i, j)$, indicating the lower right-hand square.

This definition is such that for any $l \in Q_u$, and with $\lambda' = J(u_{[l]})$, we have that $p = Y(\lambda) \setminus Y(\lambda')$ is a domino shape, and $\mathrm{typ}(p) = \mathrm{typ}(l)$. We call p the domino shape associated with l, or with its orbit X; conversely we call X the orbit in $\mathrm{Im}(\alpha_u)$ associated with p.

Recall that the map $f \mapsto f^{\downarrow}$ maps $\alpha_u^{-1}[l] \to \mathcal{F}_{u_{[l]}}$ isomorphically. We define inductively a dense subset $\widetilde{\mathcal{F}}_u \subseteq \mathcal{F}_u$. For $n = 0$ put $\widetilde{\mathcal{F}}_u = \mathcal{F}_u$, otherwise we have for $f \in \alpha_u^{-1}[l]$ that $f \in \widetilde{\mathcal{F}}_u$ if and only if $l \in Q_u$ and $f^{\downarrow} \in \widetilde{\mathcal{F}}_{u_{[l]}}$. So, for any $f \in \widetilde{\mathcal{F}}_u$ the lines $l = f_1$, ${f^{\downarrow}}_1$, ${f^{\downarrow\downarrow}}_1$ etc. lie in good orbits in the respective images of α_u, $\alpha_{u_{[l]}}$ etc. This information can be encoded into a combinatorial object.

3.2.2. Definition. *An unsigned domino tableau t of shape λ is an assignment of natural numbers to the squares of the Young diagram $Y(\lambda)$, such that*

(1) 0 is assigned to square $(1,1)$ in case B_n, and it is not otherwise assigned to any square;
(2) any other number is either assigned to two squares forming a domino shape or to no squares at all;
(3) the numbers are weakly increasing both along the rows and the colums of $Y(\lambda)$.
When t assigns a number m to some square, then it is said that that square is numbered m in t.

Condition (3) ensures that by omitting the two highest numbered squares we again obtain another unsigned domino tableau. For parametrising S_u, we only need unsigned domino tableaux numbered with $1, \ldots, n$, and possibly 0. We shall call such tableaux *standard* (there is no relation with the notion of standard Young tableaux used elsewhere). Given $f \in \widetilde{\mathcal{F}}_u$, we obtain a standard unsigned domino tableau of shape λ by assigning n to each of the squares of the domino shape associated with $l = f_1$ and then inductively numbering the squares of $Y(J(u_{[l]}))$ according to $f^{\downarrow} \in \widetilde{\mathcal{F}}_{u_{[l]}}$. We write $t(f)$ for the resulting tableau. For any standard unsigned domino tableau t put $\mathcal{F}_{u,t} = \{\, f \in \widetilde{\mathcal{F}}_u \mid t(f) = t \,\}$. Now 3.1.3 may be sharpened to

3.2.3. Lemma. *$S_u = \biguplus_t \operatorname{Irr}(\overline{\mathcal{F}_{u,t}})$, where t ranges over all standard unsigned domino tableaux of shape λ. Furthermore for any such t we have $\operatorname{Irr}(\mathcal{F}_{u,t}) = \operatorname{Con}(\mathcal{F}_{u,t})$.*

Proof. Using the definition of $\mathcal{F}_{u,t}$ the first part follows by induction from 3.1.2 and 3.1.3. The second part is also proved by induction: a single orbit X is involved in the induction step, and 3.1.1 can be applied to each of its connected components. □

Remarks. We have split $\mathcal{F}_u$ into $\widetilde{\mathcal{F}}_u$ and its lower-dimensional complement, and $\widetilde{\mathcal{F}}_u$ is further dissected into equal dimensional parts $\mathcal{F}_{u,t}$; a few comments on the nature of this division are in place. First of all, although all these parts are Z_u-stable, the division is rather *ad hoc* from a purely geometric viewpoint; this is due to our choice of analysing $\mathcal{F}_u$ by means of α_u. For instance, for a flag $f \in \widetilde{\mathcal{F}}_u$ that lies on the intersection of irreducible components of $\mathcal{F}_u$, the value of $t(f)$ will correspond to that of generic flags of exactly one of the intersecting components. There needn't be anything singular about flags in the complement of $\widetilde{\mathcal{F}}_u$, in fact we may have $\widetilde{\mathcal{F}}_u \neq \mathcal{F}_u$ even when $\mathcal{F}_u$ is a smooth irreducible variety; the only thing 'wrong' with such flags is that they do not fit into the scheme of assigning unsigned tableaux to flags. (In the analogous construction for Dynkin type A_n, a Young tableau can be assigned to *all* flags in $\mathcal{F}_u$, so there is no need to define $\widetilde{\mathcal{F}}_u$.)
On the other hand the division of $\widetilde{\mathcal{F}}_u$ is such that the individual parts $\mathcal{F}_{u,t}$ are more well behaved than the (union of) irreducible components they lie in. For instance we cannot prove that the irreducible components of $\mathcal{F}_u$ are always smooth (in fact we do not believe this is true), but it is not difficult to prove the smoothness of $\mathcal{F}_{u,t}$. Another example is given by the second part of 3.2.3, which would not be true if we replaced $\mathcal{F}_{u,t}$ by $\overline{\mathcal{F}_{u,t}}$, as we shall see in examples at the end of this section.

If for $\sigma \in S_u$ and an unsigned domino tableau t we have $\sigma \subseteq \overline{\mathcal{F}_{u,t}}$, then we write $t = t(\sigma)$; this is well-defined by above lemma. Also put $S_{u,t} = \{\, \sigma \in S_u \mid t(\sigma) = t \,\}$, which is in bijection with $\mathrm{Con}(\mathcal{F}_{u,t})$. Clearly $S_{u,t}$ is closed under the action of A_u; however the action need not be transitive on $S_{u,t}$. Extra data have to be given for a complete parametrisation of S_u; these will be provided by adding signs to the domino tableaux.

3.3. *More combinatorics: signed tableaux and clusters.*

In order to complete the parametrisation of S_u, we compute the groups K, H, and H' of 3.1.2. Let a line $l \in U_j(M) \cap Q_u$ be given, and put $u' = u_{[l]}$. For each non-trivial factor $A_{u,i}$ of A_u (see 2.4.1), we denote its generator by g_i, and similarly write g'_i for the generator of any non-trivial factor $A_{u',i}$ of $A_{u'}$. In case C_n we also put $g'_0 = \mathbf{e}$.

3.3.1. Lemma. *The groups H, H' and K in 3.1.2 satisfy*

(a) $H = A_u$ *unless* $\mathrm{typ}(l) = (\mathrm{I}+)$ *and* $m_j(\lambda) = 2$*, in which case H is generated by* $\{\, g_i \mid i \neq j \,\}$.

(b) $H' = A_{u'}$ *unless* $\mathrm{typ}(l) = (\mathrm{I}-)$ *and* $m_{j-1}(\lambda) = 0$*, in which case H' is generated by* $\{\, g'_i \mid i \neq j-1 \,\}$.

(c) $K = \{\mathbf{e}\}$ *unless* $\mathrm{typ}(l) = (\mathrm{N})$ *and* $m_j(\lambda) > 1$*, in which case K is the 2-cyclic subgroup of $A_{u'}$ generated by $g'_j g'_{j-2}$.*

(d) *The canonical homomorphism $H \to H'/K$ maps g_i to $g'_i K$ for all i, except for $i = j$ if* $\mathrm{typ}(l) = (\mathrm{N})$ *and $m_j(\lambda) = 1$, in which case g_j is mapped to $g'_{j-2}K$.*

Proof. Throughout the proof we use the description of A_u (and of $A_{u'}$) in 2.4. Now with the description of the orbit $Z_u \cdot l$ in 2.3.4, part (a) follows from the fact that H is the stabiliser—in the action of A_u on $\mathrm{Con}(Z_u \cdot l)$—of the component containing l. For the other parts we consider the linear transformations that some $z \in (Z_u)_l$ induces in the vector spaces $V_i(M)$ and $V_i(l^\perp/l)$; by the description of A_u the corresponding determinants determine the image of z in A_u and $A_{u'}$ respectively. Using a decomposition as in 2.3.3, and considering particular elements z that centralise N_j, we may check parts (b) and (d) in as far as generators g_i and g'_i are concerned with $i \neq j$.

For the remainder of the proof we use 2.3.8, and its notation, with $N = M$. For $i \notin \{j, j'-1\}$ the natural isomorphism of 2.3.8(d) ensures that z induces transformations with the same determinant in $V_i(M)$ and in $V_i(l^\perp/l)$. If $\mathrm{typ}(l) = (\mathrm{N})$, then z preserves the decompositions $V_j(M) = \bar{l} \oplus \bar{l}^\perp$ and (if $j \geq 3$) $V_{j-2}(l^\perp/l) = S^\perp \oplus S$, and there exists some $z \in (Z_u)_l$ that acts as $-\mathbf{1}$ on $\bar{l}$. Now using 2.3.8(b) the remaining statements for this case are easily verified. When $\mathrm{typ}(l) \neq (\mathrm{N})$ the isomorphisms of 2.3.8(c) allow us to apply the following easy fact: if V is a vector space equipped with a non-degenerate symmetric or alternating bilinear form, $\zeta \in \mathrm{Aut}(V)$ and $p \subset V$ is an isotropic line normalised by ζ, then for the induced automorphism $\zeta' \in \mathrm{Aut}(p^\perp/p)$ we have $\det(\zeta') = \det(\zeta)$ (where we put $\det(\zeta') = +1$ if $p = p^\perp$). We conclude that in this case there is a canonical isomorphism $H \xrightarrow{\sim} H'$, which completes the proof. □

Using this lemma we shall construct a combinatorial model of S_u as a set with A_u-action. We need a number of preliminary definitions. Define a *domino* d to be an object comprising the following data: a domino shape $\operatorname{supp}(d)$ called its *support*; a number $|d| \in \mathbf{N}_{>0}$ called its *entry*; and an element $\operatorname{sg}(d)$ of $\{`+', `-', `\circ'\}$ called its *sign*, with $\operatorname{sg}(d) \neq `\circ'$ if and only if $\operatorname{typ}(\operatorname{supp}(d)) = (\mathrm{I}+)$. We abbreviate $\operatorname{typ}(d) = \operatorname{typ}(\operatorname{supp}(d))$ and $\pi(d) = \pi(\operatorname{supp}(d))$. The sign '$\circ$' stands for 'absent', and $\pi(d)$ is called the *position* of d; when $\pi(d) = (i,j)$ we write $\rho_d = i$ and $\kappa_d = j$, called the row and column of d.

For the orbit in $\operatorname{Im}(\alpha_u)$ associated with $\operatorname{supp}(d)$, which is either $U_j^{\mathrm{N}}(M)$, $U_j^{\mathrm{I}}(M)$ or $U_j(M)$ according as $\operatorname{typ}(d) = (\mathrm{N})$, $(\mathrm{I}+)$ or $(\mathrm{I}-)$, we shall write $U_d(M)$ for short. Since a domino is completely determined by its position, type, entry and sign, we define an operation 'dom' by

$$d = \operatorname{dom}\bigl(\pi(d), \operatorname{typ}(d), |d|, \operatorname{sg}(d)\bigr). \tag{15}$$

A domino d is said to lie at the *periphery* of a partition λ if $\operatorname{supp}(d) \subseteq Y(\lambda)$ and $Y(\lambda) \setminus \operatorname{supp}(d)$ is a Young diagram $Y(\lambda')$ for some other partition λ'.

Dominoes can be put together into *signed domino tableaux*, that we shall mostly call just *tableaux* for short. A domino tableau T is simply a list of dominoes with disjoint supports, ordered by *decreasing* entries, such that assigning $|d|$ to the squares of $\operatorname{supp}(d)$ for every domino d occurring in T, and 0 to square $(1,1)$ in case B_n, we obtain an unsigned domino tableau of some shape λ, denoted $|T|$. We call λ the *shape* of T, and write $\operatorname{sh}(T) = \lambda$ and $\operatorname{supp}(T) = Y(\lambda)$.

The most natural way to represent a tableau is to draw a diagram, as is suggested by the terminology used. Each domino is represented by a rectangle occupying the two squares of its support, in which both its entry and its sign (unless it is '$\circ$') are written; in case B_n there is also a single square at $(1,1)$ with '0' in it. In this way the type of the dominoes and their order in the tableau are implicitly specified, and one may even deduce to which case the tableau applies. For instance '$\begin{array}{|c|c|cc|}\hline 1 & 2 & \multicolumn{2}{c|}{3} \\ \cline{3-4} & + & & \\ \cline{1-2}\end{array}$' is a tableau applying to case C_3, which consists of the list of dominoes $\bigl(\operatorname{dom}((1,4),(\mathrm{N}),3,`\circ'),\ \operatorname{dom}((2,2),(\mathrm{I}+),2,`+'),\ \operatorname{dom}((2,1),(\mathrm{I}-),1,`\circ')\bigr)$ (drawing diagrams for individual dominoes is not practical), and $\begin{array}{|c|c|}\hline 0 & 2 \\ \cline{1-1} 1 & \\ \cline{2-2} - & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$ is a tableau applying to case B_2. We draw rather tiny diagrams since we use them as formulae, not as illustrations.

For recursive definitions involving tableaux we adopt the following notation. The trivial tableau consisting of an empty list of dominoes will be denoted $\odot$, and a tableau consisting of a list starting with a domino d, followed by a remaining list T of dominoes (which itself is a tableau) will be denoted $d : T$ (so the first tableau depicted above could be written as $\operatorname{dom}((1,4),(\mathrm{N}),3,`\circ') : \begin{array}{|c|c|}\hline 1 & 2 \\ & + \\ \hline\end{array}$). The ith domino occurring in the tableau T will be denoted T_i; if $T \neq \odot$ the *leading* domino is the domino T_1, which has the highest entry in T, and lies at the periphery of $\operatorname{sh}(T)$. When $T = d : P$, we shall write $P = T^{\downarrow}$ (the down-arrow means "remove the leading domino") so we have $T = T_1 : T^{\downarrow}$; the operator ':' associates to the right, so that we may also write $T = T_1 : T_2 : T^{\downarrow\downarrow}$ when T has at least 2 dominoes. This notation will be seen to be in accordance with that defined for flags in (9). Note that in case B_n we have $\operatorname{supp}(\odot) = \{(1,1)\}$, and $\operatorname{supp}(\odot) = \emptyset$ in the other two cases; in all cases $\operatorname{supp}(d : T) = \operatorname{supp}(d) \uplus \operatorname{supp}(T)$. We define $\mathcal{T}_\lambda$ as the set of all tableaux T with $\operatorname{sh}(T) = \lambda$ and such that $|T|$ is standard.

The dominoes in a tableau T are partitioned into *clusters*, as defined presently. This is needed to express the merging of irreducible components in passing from fibres to the whole space, as specified by 3.1.2 when K is non-trivial. To simplify the definitions, we adjoin a "dummy" domino 0 to those of T in cases B_n and C_n. So let $D = \{T_1, \ldots, T_n\}$ be the set of dominoes appearing in T, and define $D_T = D$ in case D_n and $D_T = D \cup \{0\}$ otherwise. The clusters of T will be subsets of D_T. To the jth column of $\operatorname{supp}(T)$ we shall associate a cluster of T if $\varepsilon_j = +1$ and $\operatorname{supp}(T)$ has at least one row of length j; every domino whose support contains a square at the right end of such a row will appear in that cluster. For convenience we also associate a cluster to $j = 0$ in case C_n; the dummy domino 0 appears in that cluster. The set of such columns j depends only on $\lambda = \operatorname{sh}(T)$, so define

$$B_\lambda = \{\, j \in \mathbf{N} \mid \varepsilon_j = +1 \wedge m_j(\lambda) > 0 \,\} \tag{16}$$

(recall that $m_0(\lambda) = \infty$). The association of clusters to columns is expressed as a map $b_T : B_\lambda \to C_T$, where C_T is the set of clusters of T, which is a partition of the set D_T.

We now inductively define C_T and b_T. For $T = \odot$ they are defined in the only possible way. Now assume $T = d : T^{\downarrow}$, and put $\lambda' = \operatorname{sh}(T^{\downarrow})$ and $j = \kappa_d$. Define I to be the subset of $B_{\lambda'}$ of columns j for which $m_j(\lambda')$ differs from $m_j(\lambda)$; more formally

$$I = \begin{cases} \{j\} \cap B_{\lambda'} & \text{if } \operatorname{typ}(d) = (\mathrm{I}+) \\ \{j-1\} & \text{if } \operatorname{typ}(d) = (\mathrm{I}-) \\ \{j, j-2\} \cap B_{\lambda'} & \text{if } \operatorname{typ}(d) = (\mathrm{N}). \end{cases} \tag{17}$$

Clearly $\#I \leq 2$. Now adding the domino d to $T^{\downarrow}$, the clusters associated to the columns in I are merged to a cluster of T, to which d is also added; the remaining clusters are unaltered. If $I = \emptyset$ then $\{d\} \in C_T$ forms a singleton cluster. Formally we have

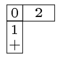
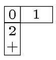
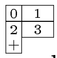
$$C_T = \Big\{ \{d\} \cup \bigcup_{i \in I} b_{T^{\downarrow}}(i) \Big\} \cup \big(C_{T^{\downarrow}} \setminus \{\, b_{T^{\downarrow}}(i) \mid i \in I \,\} \big). \tag{18}$$

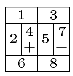
For each $x \in D_T$, the cluster containing it is denoted $\operatorname{cl}(x)$. Finally b_T is defined by

$$b_T(i) = \begin{cases} \operatorname{cl}(d) & \text{if } i \in I \text{ or } i = j \\ b_{T^{\downarrow}}(i) & \text{otherwise.} \end{cases} \tag{19}$$

Remarks. The condition $I = \emptyset$ corresponds to $H \neq A_u$ in 3.3.1(a). Two clusters of $T^{\downarrow}$ are actually merged if and only if $\#b_{T^{\downarrow}}[I] = 2$, i.e., if $I = \{j, j-2\}$ and

$b_{T^\downarrow}(j) \neq b_{T^\downarrow}(j-2)$; as we shall see this corresponds to a non-trivial group K in 3.3.1(c) whose action in 3.1.2 is also non-trivial. The map b_T is not necessarily surjective, since if $\operatorname{typ}(d) = (\mathrm{I}-)$ we might have $j-1 \notin B_\lambda$; this corresponds to $H' \neq A_{u'}$ in 3.3.1(b). Clusters that are not in the image of b_T will be called *closed* (in T), the others will be called *open*. Clearly any closed cluster in $T^\downarrow$ is also a closed cluster in T. Each cluster—with the possible exception of $\mathrm{cl}(0)$—contains at least one domino of type (I+). When clusters are drawn inside their tableau, the squares they occupy form a subset of $Y(\lambda)$ that is 'connected' via pairs of adjacent squares. In cases B_n and C_n, $\mathrm{cl}(0)$ is always an open cluster.

The definition of clusters is most easily understood by computing them for some examples. So one should verify that, of the two tableaux drawn before, the first has two clusters, one containing the dominoes numbered 2 and 3, the other containing 0 and the domino numbered 1; the second has only one cluster, which is $\mathrm{cl}(0)$. Not only the positions of the dominoes is relevant, their order is so as well. E.g., for  the only cluster is $\mathrm{cl}(0)$, but in  the domino numbered 2 forms a singleton cluster. However, by adding a third domino to the latter tableau, forming , we again obtain a tableau whose only cluster is $\mathrm{cl}(0)$, exemplifying the merging phenomenon mentioned above. All the clusters encountered so far are open, but  is a tableau (in case D_2) whose sole cluster is closed. Clusters can get pretty complex, and



is a tableau in case C_8 where $\mathrm{cl}(0)$ contains all dominoes except those numbered 4 and 5, so it actually surrounds a (necessarily closed) cluster. In this example the order of the numbers 3 and 4 and of the numbers 5 and 6 is crucial.

It is clear that the partitioning of the dominoes in a tableau T into clusters depends only on $|T|$. More precisely, when $|T| = |T'|$ there is a bijection $C_T \to C_{T'}$, such that the cluster of T_i is mapped to the cluster of T'_i for all i; we shall call the image of a cluster of T under this bijection the corresponding cluster of T'.

3.3.2. Definition. *Two tableaux T, T' are called equivalent, written $T \sim T'$, if $|T| = |T'|$, and for each of the clusters of T that is not equal to $\mathrm{cl}(0)$, the product of the signs of dominoes of type (I+) in that cluster is the same as the product of those signs in the corresponding cluster of T'. We write $[T]$ for the equivalence class of T, and define $S_\lambda = \{\, [T] \mid T \in \mathcal{T}_\lambda \,\}$ and $S_{\lambda,t} = \{\, [T] \in S_\lambda \mid |T| = t \,\}$.*

Put $\Xi_\lambda = \{\, r \mid \lambda_r \in B_\lambda \,\}$; we define for $r \in \Xi_\lambda$ and any T with $\operatorname{sh}(T) = \lambda$, a tableau $\xi_r(T)$, obtained by changing the sign of the first (i.e., highest numbered) domino of type (I+) in T whose support contains a square in row r (if such a domino exists), and leaving everything else unchanged. It is easily verified that such a domino, if it exists, must lie in the cluster $b_T(\lambda_r)$. Note that neither $|T|$ nor the signs in closed clusters are affected by ξ_r. Any ξ_r maps equivalence classes of tableaux into equivalence classes, and if $\lambda_r = \lambda_{r'}$ for $r, r' \in \Xi_\lambda$ then ξ_r and $\xi_{r'}$ act identically on equivalence classes. Since by 2.4.1 A_λ is isomorphic to the elementary 2-group generated by $B_\lambda \setminus \{0\}$ (i.e., the product $\mathbf{2} \times \cdots \times \mathbf{2}$ with one factor for each $j \in B_\lambda \setminus \{0\}$), we may define an action of A_λ on S_λ by $g_{\lambda_r} \cdot [T] = \xi_r \cdot [T]$. This action can also be characterised be the fact that g_i changes the product of signs of the cluster $b_T(i)$, and of no other cluster (in case that cluster is $\mathrm{cl}(0)$, g_i has no effect at all, since the sign in that cluster is "ignored" by 3.3.2).

3.3.3. Lemma. *For any unipotent $u \in G$ with $J(u) = \lambda$ there exists a bijection $S_\lambda \to S_u$ that is A_u-equivariant and maps $S_{\lambda,t}$ to $S_{u,t}$ for each unsigned domino tableau t.*

Proof. By induction on the rank n, we establish the existence of appropriate bijections $S_{\lambda,t} \to S_{u,t}$ for each unsigned domino tableau t, using 3.1.2 and by comparing 3.3.1 with the definitions (17), (18) and (19). The case $n = 0$ is trivial, so assume $n > 0$. Choose a tableau $T = d : T^{\downarrow}$ with $|T| = t$, and a line $l \in U_d(M)$. Let $u' = u_{[l]}$, $\lambda' = J(u') = \mathrm{sh}(T^{\downarrow})$, $t' = |T^{\downarrow}|$ and let an $A_{u'}$-equivariant bijection $S_{\lambda',t'} \to S_{u',t'}$ be given by induction. Let H, H' and K be as given by 3.3.1, and let I be given by (17); it depends only on t.

First consider the case $I = \emptyset$, which is equivalent to $H \neq A_u$. In this case $H \cong A_{u'}$ and $A_u \cong A_{u,j} \times H$, so 3.1.2 states that $S_{u,t} \cong \mathbf{2} \times S_{u',t'}$, where $A_{u,j}$ acts on the first factor and H on the second. From the definitions it follows that also $S_{\lambda,t} \cong \mathbf{2} \times S_{\lambda',t'}$ with similar A_u-action, establishing the induction step in this case.

Next consider the case $\#I = 2$, which is equivalent to $K \neq \{\mathbf{e}\}$, so 3.1.2 gives $S_{u,t} \cong K \backslash S_{u',t'}$. On the other hand the generator $g'_j g'_{j-2}$ of K acts on $S_{\lambda',t'}$ by changing the product of signs in each of the two clusters (corresponding to) $b_{T^{\downarrow}}(j)$ and $b_{T^{\downarrow}}(j-2)$; in case those two clusters coincide, the action of K is trivial. Also the map $S_{\lambda',t'} \to S_{\lambda,t}$ given by $[T'] \mapsto [d : T']$ is a surjection, and by (18) its fibers are precisely the K-orbits in $S_{\lambda',t'}$. Therefore we get a bijection $S_{\lambda,t} \to S_{u,t}$ that is A_u-equivariant by 3.3.1(d) and (19), proving the induction step for this case.

In the remaining case $\#I = 1$ and $A_u \cong H' \subseteq A_{u'}$, so by (18) the map $[T'] \mapsto [d : T']$ is a bijection $S_{\lambda',t'} \to S_{\lambda,t}$, and by 3.1.2 there is a bijection $S_{u,t} \to S_{u',t'}$. We therefore get a bijection $S_{\lambda,t} \to S_{u,t}$, and by 3.3.1(d) and (19) it is A_u-equivariant, which completes the proof. □

3.4. *Standard $k[u]$-modules.*

The bijection established in 3.3.3 is not uniquely determined. For instance, in the case $I = \emptyset$ there are two possible bijections $S_{\lambda,t} \to S_{u,t}$ that are compatible with the given bijection $S_{\lambda',t'} \to S_{u',t'}$, and since this situation may arise at several points of the inductive construction, there is a considerable amount of freedom of choice. (It may seem that in the case mentioned there is a canonical choice for the bijection, since the same group $\mathbf{2}$ appears in the expressions for $S_{\lambda,t}$ and $S_{u,t}$, but the identification of $S_{u,t}$ with $\mathbf{2} \times S_{u',t'}$ depends on the choice of $l \in U_d(M)$.)

In fact it is possible to choose the bijection in such a way that a specific element of $S_{\lambda,t}$ (e.g., the class of the tableau containing no '$-$'-signs) is mapped to any chosen element of $S_{u,t}$. This is particularly unsatisfactory because, due to closed clusters, the action of A_u on $S_{\lambda,t}$ (or on $S_{u,t}$) need not be transitive. However, at the points in the inductive process where the choices are actually made, the action of A_u *does* interchange the two possibilities (because newly formed clusters are initially open). Consequently, there is no way to specify these individual choices during the inductive process in a way that is invariant under automorphisms of M. (Apparently, some choices contained in the induction hypothesis do have an eventual effect that is invariant under such automorphisms, but to specify this choice we would have to endow $l^\perp/l$ with more structure than that of $k[u]$-module alone, representing the specific way in which it is a subquotient of M, and such extra information would interfere with the induction.)

We shall resolve this difficulty, by constructing for each occurring Jordan-type λ a particular *standard $k[u]$-module* M_λ with $J(M_\lambda) = \lambda$, and then—with $\mathbf{u} = u_{M_\lambda}$— specifying a standard parametrisation $S_\lambda \xrightarrow{\sim} S_{\mathbf{u}}$. Then a parametrisation $S_\lambda \xrightarrow{\sim} S_u$ is obtained after choosing an isomorphism $M \xrightarrow{\sim} M_\lambda$; the remaining indeterminacy is inevitable, due to the action of A_u. In the standard $k[u]$-modules, we shall also define certain *special lines* in $\operatorname{Ker}(\eta)$.

We first construct basic $k[u]$-modules M_j with $J(M_j) = (j)$, where $\varepsilon_j = +1$. The underlying $k[X]$-module of M_j is the vector space k^j with X acting on te basis $e_1, \ldots, e_j$ by $X \cdot e_1 = 0$ and $X \cdot e_i = e_{i-1}$ for $1 < i \leq j$. It remains to define b_{M_j}, which is done inductively. The case $j = 0$ (for $\varepsilon = -1$) is trivial, while for $j = 1$ ($\varepsilon = +1$) we define $b_{M_1}(e_1, e_1) = 1$; otherwise assume that M_{j-2} has already been defined. Now b_{M_j} is determined such that (a) it makes M_j into a non-degenerate $k[u]$-module, which implies in particular that $\langle e_1\rangle^\perp = \langle e_1, \ldots, e_{j-1}\rangle$, and (b) the $k[X]$-module isomorphism $\langle e_1\rangle^\perp/\langle e_1\rangle \to M_{j-2}$ sending the coset of e_i to e_{i-1} ($1 < i < j$) becomes a $k[u]$-module isomorphism. It is easily checked that these requirements may be met—indeed they do not completely determine b_{M_j}, and we might impose further conditions like $e_j \perp e_j$ in M_j for $j > 1$—but it suffices that M_j is chosen once and for all. The special line in M_j is $\langle e_1\rangle$, which is of type (N). We also define two special lines in the $k[u]$-module $M_j \times M_j$ which, as we shall see, may be written $M_{(j,j)}$, namely the elements of $U_j^{\mathrm{I}}(M_{(j,j)})$ (so these are of type (I+)). Explicitly, these lines are $l_+ = \langle(e_1, \mathbf{i}e_1)\rangle$ and $l_- = \langle(e_1, -\mathbf{i}e_1)\rangle$, where $\mathbf{i}$ denotes one fixed solution of $x^2 = -1$ in k.

For $\varepsilon_j = -1$ we also define basic $k[u]$-modules named $M_{j,j}$, which have $J(M_{j,j}) = (j, j)$ (the minimal Jordan type possible with a part j). They are not defined directly, but in terms of the previously defined modules: $M_{j,j}$ is defined as a subquotient module of $M_{(j+1,j+1)}$, namely $l_+^\perp/l_+$. The image $(l_+ \oplus l_-)/l_+$ of $\mathrm{Ker}(\eta_{M_{(j+1,j+1)}})$ is the unique special line in $M_{j,j}$; it is of type (I$-$). Calling this special line p, note that if $j > 1$ there is a canonical isomorphism $p^\perp/p \xrightarrow{\sim} M_{(j-1,j-1)}$, under which $\mathrm{Ker}(\eta_{M_{j,j}})/p$ corresponds to the special line l_+ in $M_{(j-1,j-1)}$.

Now let λ be an arbitrary partition such that $m_j(\lambda)$ is even for all j with $\varepsilon_j = -1$. The Young diagram $Y(\lambda)$ can be partitioned in a unique way into rows of length j with $\varepsilon_j = +1$ and pairs of adjacent rows of length j with $\varepsilon_j = -1$. According to this partitioning we define M_λ as the product of the following modules: for each row of length j with $\varepsilon_j = +1$ a factor M_j, and for each pair of adjacent rows of length j with $\varepsilon_j = -1$ a factor $M_{j,j}$. For each of these summands of M_λ there is an automorphism that acts as $-\mathbf{1}$ on that summand and as $\mathbf{1}$ on all other summands. This automorphism will be denoted χ_r for a summand M_j corresponding to a part consisting of a single row r of $Y(\lambda)$, and $\chi_{r,r+1}$ for a summand $M_{j,j}$ corresponding to a part consisting of two rows r and $r+1$.

The special lines in M_λ are defined such that they correspond precisely to the dominoes at the periphery of λ, and have the same type as that domino. Let d be such a domino, and let $j = \kappa_d$, then $\mathrm{supp}(d)$ meets 1 or 2 rows of $Y(\lambda)$, of length j, and there is a corresponding summand of M_λ of the form M_j, or $M_{(j,j)}$ with $\varepsilon_j = +1$ (when $\mathrm{typ}(d) =$ (N) respectively (I+)), or $M_{j,j}$ with $\varepsilon_j = -1$ (when $\mathrm{typ}(d) =$ (I$-$)). We define the special line in M_λ *belonging* to d to be a special line of type $\mathrm{typ}(d)$ in that summand as defined above; in case $\mathrm{typ}(d) =$ (I+) this leaves us the choice between l_+ and l_-, of which we choose $l_{\mathrm{sg}(d)}$.

For the special line p in M_λ belonging to a domino d, there is a canonical isomorphism $p^\perp/p \xrightarrow{\sim} M_{\lambda'}$, where $Y(\lambda') = Y(\lambda) \setminus \mathrm{supp}(d)$. This only needs elucidation when $\mathrm{sg}(d) =$ '$-$'. In that case we use the automorphism χ_{ρ_d} of M_λ—which maps l_- to l_+—to transform the canonical isomorphism $l_+^\perp/l_+ \xrightarrow{\sim} M_{\lambda'}$ into an isomorphism $l_-^\perp/l_- \xrightarrow{\sim} M_{\lambda'}$. When a flag f in $p^\perp/p$ corresponds to $f' \in M_{\lambda'}$ under this canonical isomorphism, we shall write $f \simeq f'$. Now for any $T \in \mathcal{T}_\lambda$ a flag $F_T \in \mathcal{F}_{\mathbf{u},|T|}$ is defined as follows. If $T = \odot$ there is only one flag possible. Otherwise $\alpha_{\mathbf{u}}(F_T)$ is the special line p belonging to the domino T_1, and $F_T^\downarrow \simeq F_{T^\downarrow}$. These flags have the following properties, with ξ_r as defined below 3.3.2.

3.4.1. Proposition.

(a) The set $\{\, F_T \mid T \in \mathcal{T}_\lambda \,\}$ is stable under any of the automorphisms χ_r and $\chi_{r,r+1}$.

(b) Any $\chi_r(F_T)$ lies in the same component of $\mathcal{F}_{\mathbf{u},|T|}$ as $F_{\xi_r(T)}$, and any $\chi_{r,r+1}(F_T)$ lies in the same component as F_T.

(c) If $T \sim T'$, then F_T and $F_{T'}$ lie in the same component of $\mathcal{F}_{\mathbf{u},|T|}$.

Proof. We apply induction on $T \in \mathcal{T}_\lambda$. The case $T = \odot$ is trivial; so let $T = d : T^\downarrow$, and assume that the proposition is true for $\lambda' = \mathrm{sh}(T^\downarrow)$. Let $\chi = \chi_r$ or $\chi_{r,r+1}$ be given, and put $p = \alpha_{\mathbf{u}}(F_T)$. If $\mathrm{supp}(d)$ doesn't meet the corresponding row or rows of $Y(\lambda)$, then p is stabilised by χ, and the summand of M_λ on which χ acts as $-\mathbf{1}$ is unaffected by passing to $p^\perp/p$, whence (a) and (b) follow from the induction hypothesis. Otherwise we distinguish three cases. If $\chi = \chi_{r,r+1}$ then $\mathrm{typ}(d) = (\mathrm{I}-)$ and $\rho_d = r+1$; p is stabilised by χ, and $\chi(F_T)^\downarrow \simeq \chi_r \circ \chi_{r+1}(F_{T^\downarrow})$, which proves (a) by induction; (b) follows from $\chi \in Z^\circ_{\mathbf{u}}$. If $\mathrm{typ}(d) = (\mathrm{N})$ then $\chi = \chi_r$ stabilises p, and $\chi(F_T)^\downarrow \simeq \chi_r(F_{T^\downarrow})$, which by induction lies in the same component as $F_{\xi_r(T^\downarrow)} \simeq F_{\xi_r(T)}{}^\downarrow$, proving (a) and (b). If $\mathrm{typ}(d) = (\mathrm{I}+)$ then $\chi = \chi_r$ and $\rho_d \in \{r, r+1\}$. If $\rho_d = r$, then we get $\chi_r(F_T) = F_{\xi_r(T)}$ immediately from the definitions, while if $\rho_d = r+1$ we have $\alpha_{\mathbf{u}}(\chi_r(F_T)) = \alpha_{\mathbf{u}}(F_{\xi_r(T)})$ and $\chi_r(F_T)^\downarrow = \chi_{r,r+1}(F_{\xi_r(T)}{}^\downarrow)$, which proves (a) and (b).

To prove (c), let $T' \sim T$ be given. If $T'_1 = d$ and $T'^\downarrow \sim T^\downarrow$, then (c) follows directly from the induction hypothesis. Otherwise put $(r,j) = \pi(d)$; if $T'^\downarrow \not\sim T^\downarrow$ then we must have $\mathrm{typ}(d) = (\mathrm{N})$ and $m_j(\lambda) > 1$, while $T'_1 \neq d$ can only be due to differing signs when $\mathrm{typ}(d) = (\mathrm{I}+)$ and $m_j(\lambda) > 2$. In the first case we must have $T'^\downarrow \sim \xi_r \circ \xi_{r-1}(T^\downarrow) = \xi_r \circ \xi_{r-1}(T)^\downarrow$, so by induction we may reduce to the case $T' = \xi_r \circ \xi_{r-1}(T)$. But then by (b), $F_{T'}$ lies in the same component as $\chi_r \circ \chi_{r-1}(F_T)$, while $\chi_r \circ \chi_{r-1} \in Z^\circ_{\mathbf{u}}$, which proves (c) for this case. In the second case, we must have $T'^\downarrow \sim \xi_{r-2}(T^\downarrow) = \xi_{r-2}(T)^\downarrow$, and we may reduce to the case $T' = \xi_r \circ \xi_{r-2}(T)$. But then (c) is proved as in the previous case, with ξ_{r-2} replacing ξ_{r-1}. □

By part (c) of this proposition we may define maps $S_{\lambda,t} \to \mathrm{Con}(\mathcal{F}_{\mathbf{u},t})$ by sending $[T]$ to the component containing F_T; this component will be denoted $\mathcal{F}_{\mathbf{u},T}$. By (b) and 3.3.3 these bijections are A_u-equivariant, and by 3.2.3 they may be put together (taking closures) into an A_u-equivariant bijection $S_\lambda \tilde{\to} S_{\mathbf{u}}$, which is the promised standard parametrisation of $S_{\mathbf{u}}$. The image of $[T]$ under this bijection is $\overline{\mathcal{F}_{\mathbf{u},T}}$.

Remark. We see that 3.4.1(c) may be sharpened to: F_T and $F_{T'}$ lie in the same component of $\mathcal{F}_{\mathbf{u},|T|}$ if and only if $T \sim T'$. Therefore, by 3.4.1(a), (b), it would have been possible to define operations χ_r and $\chi_{r,r+1}$ on $\mathcal{T}_\lambda$ with $\chi_r(T) \sim \xi_r(T)$, such that $\chi_r(F_T) = F_{\chi_r(T)}$ and $\chi_{r,r+1}(F_T) = F_{\chi_{r,r+1}(T)}$. We could then have replaced ξ_r by χ_r in the definitions above with no difference in effect, but allowing 3.4.1(b) to be rendered in a more pleasing form. However, these operations χ_r are more difficult to define than ξ_r, since they may change more than one sign in a tableau. Because we are eventually interested in tableaux only up to equivalence, we have not deemed such an approach worth while.

We have completed the parametrisation of S_u; there are a few points that we would like to note specifically. The flags F_T are useful to discriminate the different irreducible components of $\mathcal{F}_{\mathbf{u},|T|}$ because these are disjoint according to the second part of 3.2.3; however, it is not generally true that F_T lies in a unique irreducible component of $\mathcal{F}_{\mathbf{u}}$. In fact the flags F_T are far from being "generic", and tend to lie on the intersection of such components. This can be seen already in the simplest cases where $\mathcal{F}_{\mathbf{u}}$ is reducible, as will be illustrated at the end of this section. The reason for this is that when a standard $k[u]$-module M_λ is viewed as subquotient of another such module N, then the special lines of M_λ often lie in the image of $\mathrm{Ker}(\eta_N)$, even if other lines in the same $Z_{\mathbf{u}}$-orbit do not; so in this respect the special lines are quite special indeed.

We would also like to note the points at which our definitions differ from the original ones given in [Spa II.6]. Our construction is entirely based on that of Spaltenstein, although the notation is often different. In particular we use the same unsigned tableaux, signs are added to the same dominoes and the same standard $k[u]$-modules are used. Our interpretation of the signs, however, differs slightly from the original one, and so does our equivalence of tableaux.

The difference can be described as follows, using our notation. In the original construction there are operations similar to ξ_r, say $\hat{\xi}_r$, that change signs in a tableau whose domino meets row r. These $\hat{\xi}_r$, however, change the signs of *all* such dominoes of type (I+), rather than just of the first one from the right. Equivalence of tableaux in [Spa], which we shall denote by $\hat{\sim}$, satisfies $T \mathrel{\hat{\sim}} \hat{\xi}_r \circ \hat{\xi}_s(T)$ when $\lambda_r = \lambda_s$, and $T \mathrel{\hat{\sim}} T'$ implies $d : T \mathrel{\hat{\sim}} d : T'$; this is similar to the situation for ξ_r and $\sim$.

These properties are achieved by defining equivalence classes of tableaux to be cosets of an inductively defined subgroup of the elementary 2-group generated by the dominoes of type (I+); the quotient group parametrises $S_{\mathbf{u},T}$. For this quotient no particular set of independent generators is specified, and the concept of clusters does not appear in [Spa]. The original description also associates a particular flag to each tableau T; it lies in $\{\, F_T \mid T \in \mathcal{T}_\lambda \,\}$ but it generally differs from F_T. Denoting it as $\hat{F}_T$, we have $\chi_r(\hat{F}_T) = \hat{F}_{\hat{\xi}_r(T)}$. A notable difference is that it is not always true that $\hat{F}_T^{\downarrow} \simeq \hat{F}_{T^{\downarrow}}$; this may fail to hold when $\mathrm{sg}(T_1) =$ '$-$', in which case we have $\hat{F}_T^{\downarrow} \simeq \hat{F}_{\hat{\xi}_r(T)^{\downarrow}}$ instead, with $r = \rho_{T_1}$.

The reason for our deviation from the orginal definitions, is that we wanted to have a simple, explicit description of the set parametrising $S_{\mathbf{u},T}$ as a freely generated 2-group such that each $g_i \in A_\lambda$ affects at most one of its factors $\mathbf{2}$. This leads naturally to our concept of clusters: it is clear from 3.3.2 that $S_{\lambda,T}$ can be described as the 2-group generated by the clusters of T except $\mathrm{cl}(0)$, and g_i affects only the cluster $b_T(i)$. We believe this approach is combinatorially more transparant than the original one; apart from this, it reduces the complications in the computations to be given. Nonetheless quite a few technicalities remain, but these seem to correspond to essential features of the geometry involved.

For compatibility with the original definitions, we indicate how tableaux can be translated from one interpretation to the other, in such a way that the same element of $S_{\mathbf{u}}$ is described in both cases. Any tableau may be built up, starting from $\odot$, by a sequence of operations that are either the addition of a leading domino d with $\mathrm{sg}(d) \neq$ '$-$' or the application of some valid ξ_r. A corresponding tableau in the setup of [Spa] is obtained by the same sequence of operations, but replacing ξ_r by $\hat{\xi}_r$.

A last remark concerns the connected components of $\mathcal{F}_{\mathbf{u}}$ in case D_n; to this end we make a few simple observations. When for two tableaux $S, T \in \mathcal{T}_\lambda$ we have the same leading domino, and $F_{S^{\downarrow}}$ and $F_{T^{\downarrow}}$ lie in the same connected component, then the same holds for F_S and F_T. From the definition of the standard $k[u]$-modules it is not difficult to find a way of assigning to each domino shape within $Y(\lambda)$ a vector in M_λ, in such a way that for any tableau T containing no dominoes d with $\mathrm{sg}(d) =$ '$-$', the space $(F_T)_i$ is spanned by the vectors assigned to the supports of the dominoes $T_1, \ldots, T_i$, and that this span depends only on the union of those supports (the main difficulty is naming the vectors, not finding them). Therefore, if S and T are two such tableaux, then F_S and F_T lie in the same connected component since the maximal isotropic spaces $(F_S)_n$ and $(F_T)_n$ coincide; we identify that connected component with $\mathcal{B}_{\mathbf{u}}$.

Finally, any automorphism χ_r exchanges the connected components of $\mathcal{F}_{\mathbf{u}}$, while ξ_r always changes exactly one sign in a tableau (since we are in case D_n), and there is no $\mathrm{cl}(0)$. It follows that the connected component of F_T is determined by the product of the signs in all the dominoes d in T with $\mathrm{typ}(d) = (\mathrm{I}+)$. When S and T differ only in the sign of their last domino (the one with support $\{(1,1),(2,1)\}$), then F_S and F_T are companion flags.

We close this subsection with a proposition that emphasises the asymmetry between '$+$' and '$-$' in the definitions.

3.4.2. Proposition. *Let d be a domino at the periphery of λ, with $\mathrm{typ}(d) = (\mathrm{I}-)$, $\kappa_d = c \geq 2$ and $m_{c-1}(\lambda) = 0$; let p be the special line in M_λ belonging to d, and let Π be as in 2.3.6 (for $l = p$). Then, identifying $p^\perp/p$ with $M_{\lambda'}$, the space $S \subseteq V_{c-1}(M_{\lambda'})$ of 2.3.8, which is the image of $\Pi[W_c(M_\lambda)] = \Pi[W_{c-1}(p^\perp)]$, coincides with the image of the special line l_+ of the summand $M_{(c-1,c-1)}$ of $M_{\lambda'}$.*

Proof. It follows from the remark following the definition of $M_{j,j}$ that the indicated line l_+ lies in $\Pi[W_c(M_\lambda)]$; since $\dim(S) = m_{c-1}(\lambda') - 1 = 1$, the proposition follows. $\square$

3.5. *Keeping track of isomorphisms with standard modules.*

As was indicated above, a particular correspondence $S_u \xrightarrow{\sim} S_\lambda$ can be obtained after the choice of an isomorphism $M \xrightarrow{\sim} M_\lambda$. There is a simply transitive action by right-composition of $Z_u = \mathrm{Aut}(M)$ on the set of such isomorphisms, and the effect of composing an isomorphism with $z \in Z_u$ on the correspondence $S_u \xrightarrow{\sim} S_\lambda$ is composition with the action of z on S_u; in particular, composition by any $z \in Z_u^\circ$ has no effect. Therefore, we shall call the Z_u°-orbits in the set of isomorphisms $M \xrightarrow{\sim} M_\lambda$ *classes*, and two isomorphisms in the same class will be called *equivalent*. When a fixed class of isomorphisms has been chosen, $\mathcal{F}_{u,T} \subseteq \widetilde{\mathcal{F}}_u$ will denote the component corresponding to $\mathcal{F}_{\mathbf{u},T}$. The freedom of replacing any isomorphism by an equivalent one is very useful, since it allows us to make any chosen $l \in Q_u$ correspond to some special line p in M_λ. Choosing such an isomorphism then induces another isomorphism $l^\perp/l \xrightarrow{\sim} p^\perp/p \cong M_{\lambda'}$.

For a fixed class of isomorphisms $M \xrightarrow{\sim} M_\lambda$ however, we do not always obtain a single class of isomorphisms $l^\perp/l \xrightarrow{\sim} M_{\lambda'}$, and this may be due to one of the following two circumstances. In the first place, there may be two special lines in M_λ, each of which corresponds to l for a suitably chosen isomorphism in the given class. This situation arises when $l \in U_j^{\mathrm{I}}(M)$ and $m_j(\lambda) > 2$, since then $U_j^{\mathrm{I}}(M)$ is connected and Z_u° acts transitively on it. In the second place, for a fixed special line p, the choice between *equivalent* isomorphisms $M \xrightarrow{\sim} M_\lambda$, each sending l to p, may result in *inequivalent* isomorphisms $l^\perp/l \xrightarrow{\sim} M_{\lambda'}$; this happens when the group $K \subseteq A_{u_{[l]}}$ of our earlier analysis is non-trivial, i.e., when $l \in U_j^{\mathrm{N}}(M)$ and $m_j(\lambda) > 1$. It may be necessary to restrict the class of the induced isomorphism in some way.

3.5.1. Proposition. *Let a class of isomorphisms $M \xrightarrow{\sim} M_\lambda$, a tableau T, and a flag $f \in \mathcal{F}_{u,T}$ with $l = \alpha_u(f)$ be given. There exists an isomorphism in that class that maps l to the special line belonging to the leading domino T_1, and such that the induced isomorphism $l^\perp/l \xrightarrow{\sim} M_{\lambda'}$ sends $f^\downarrow$ into $\mathcal{F}_{\mathbf{u}',T^\downarrow}$.*

Proof. It follows from the definition of $\mathcal{F}_{u,T}$ that $\alpha_u[\mathcal{F}_{u,T}]$ is a Z_u°-orbit containing the indicated special line, so there is an isomorphism that meets the first requirement. The flag corresponding to F_T for this isomorphism lies in $\alpha_u^{-1}[l]$, so by 3.1.1 its $(Z_u^\circ)_l$-orbit meets the component of $\alpha_u^{-1}[l] \cap \mathcal{F}_{u,T}$ in which f lies. Therefore the isomorphism may be modified by a suitable $z \in (Z_u^\circ)_l$ such that it meets the second requirement as well. □

Since an $a \in A_{u_{[l]}}$ that fixes any element of $S_{u_{[l]},|T^\downarrow|}$ will fix all such elements, the requirements of 3.5.1 determine a unique parametrisation of $S_{u_{[l]},|T^\downarrow|}$. Note however that this parametrisation depends on T and f rather than just on $[T]$ and l. It is for this reason that we shall choose to work with concrete representatives T of $[T]$; we retain the possibility to deliberately switch to another representative, if that suits us better. As regards f, as long as we consider flags individually we may adapt the choice of an isomorphism to the flag under consideration, and 3.5.1 allows us to write

$$f \in \mathcal{F}_{u,T} \Rightarrow f^\downarrow \in \mathcal{F}_{u',T^\downarrow}, \tag{20}$$

which justifies the use of the same operator '↓' for flags and tableaux, and is quite useful for induction. When considering sets of flags however, e.g., when we are proving that some condition holds on a dense subset of $\mathcal{F}_{u,T}$, we should be aware that it may not be possible to choose an isomorphism such that (20) holds for all $f \in \alpha_u^{-1}[l]$ simultaneously. Nevertheless it is clear that if it holds for one flag, then it also holds for all flags in the same component of $\alpha_u^{-1}[l] \cap \mathcal{F}_{u,T}$, so if we consider each of these components separately—there are at most two of them according to 3.3.1(c)—and choose appropriate isomorphisms, then we may still employ (20) for all flags so considered.

A converse of (20) can be used without restriction. In order to formulate it we introduce some notation: since any flag f with at least one part is completely determined by f_1 and $f^\downarrow$, we shall write $f = f_1 : f^\downarrow$ (in analogy of a similar rule for tableaux). For a given class of isomorphisms $M \xrightarrow{\sim} M_\lambda$, and a domino d we have

$$l \in U_d(M) \wedge f' \in \mathcal{F}_{u_{[l]},T'} \Rightarrow l : f' \in \mathcal{F}_{u,d:T'}, \tag{21}$$

where any isomorphism in the class that sends l to the special line belonging to d is used to obtain an induced isomorphism $l^\perp/l \xrightarrow{\sim} M_{\lambda'}$. For, if the image of f' in $M_{\lambda'}$ lies in the same component of $\mathcal{F}_{u_{[l]},|T'|}$ as $F_{T'}$, then certainly the image of $l : f'$ lies in the same component of $\mathcal{F}_{u,|d:T'|}$ as $F_{d:T'}$.

Related to these matters, there is another difficulty that has to be mentioned. Let $n \geq 2$, and consider two flags $f \in \mathcal{F}_{u,T}$ and $f' \in \mathcal{F}_{u,T'}$, which have $f_i = f'_i$ for $i \geq 2$, but $l = f_1 \neq l' = f'_1$. Then l and l' are two isotropic lines spanning an isotropic plane $L = l + l' = f_2 = f'_2$. Now $f^{\downarrow\downarrow}$ and $f'^{\downarrow\downarrow}$ are identical flags in $\mathcal{F}_{u_{[L]}}$; however, notwithstanding (20), this does not imply that we have $T^{\downarrow\downarrow} \sim T'^{\downarrow\downarrow}$. This is because adapting to f and T we may have obtained a different parametrisation of $S_{u_{[L]},|T^{\downarrow\downarrow}|}$ than adapting to f' and T'. First of all, we should note that $T^{\downarrow\downarrow} \not\sim T'^{\downarrow\downarrow}$ can be due to a "wrong" choice of T and T'; indeed it may happen even for $f = f'$, i.e., when $T \sim T'$ (if $\mathrm{sg}(T_1) \neq \mathrm{sg}(T'_1)$ or $\mathrm{sg}(T_2) \neq \mathrm{sg}(T'_2)$). Therefore we may first replace T and T' by different representatives of $[T]$ and$[T']$ respectively, but in some cases this is not sufficient to obtain $T^{\downarrow\downarrow} \sim T'^{\downarrow\downarrow}$.

We examine the situation more closely according to the position of L. Put $\bar{L} = \mathbf{P}(L) \subseteq \mathbf{P}(\mathrm{Ker}(\eta_M))$ and $d = T_1$, $d' = T'_1$. First suppose that all of $\bar{L}$ is contained in $U_d(M)$. Then we may map $\bar{L}$ into $\mathcal{F}_{u,|T|}$ by sending $p \in \bar{L}$ to the flag obtained from f by replacing f_1 by p; the image of $\bar{L}$ in $\mathcal{F}_{u,|T|}$ obtained in this way is obviously contained in one component. Therefore $f' \in \mathcal{F}_{u,T}$, so $T \sim T'$, and we can take $T' = T$ giving $T^{\downarrow\downarrow} = T'^{\downarrow\downarrow}$. Next suppose that $\bar{L} \not\subseteq U_d(M)$, but $\bar{L} \cap U_d(M)$ is connected and contains l' as well as l, then the same argument remains valid when $\bar{L}$ is replaced by $\bar{L} \cap U_d(M)$. On the other hand, suppose that $\kappa_d \neq \kappa_{d'}$, or equivalently $\mathrm{supp}(d) \cap \mathrm{supp}(d') = \emptyset$. Then with $j = \kappa_d$ and $j' = \kappa_{d'}$, and assuming $j < j'$, we have $\bar{L} \subseteq U_j(M) \cup U_{j'}(M)$ and $\bar{L} \cap U_{j'}(M) = \{l'\}$. It is not difficult to see that an isomorphism $M \xrightarrow{\sim} M_\lambda$ may be chosen that sends l and l' to the special lines in M_λ belonging to d and d' respectively, since these lie in distinct summands. In that case the images L/l and L/l' of L in the spaces $l^\perp/l$ and $l'^\perp/l'$ already correspond—under the induced isomorphisms with standard modules—to the special lines belonging to d' and d respectively. It follows that in this case we may choose T and T' such that $T^{\downarrow\downarrow} = T'^{\downarrow\downarrow}$, and the two leading dominoes of T' are obtained by interchanging those of T, except for their entries.

In the remaining case we have $\kappa_d = \kappa_{d'} = j$ with $\varepsilon_j = +1$, $\bar{L} \subseteq U_j(M)$, and $\bar{L} \cap U^{\mathrm{I}}_j(M)$ consists of either one or two points, at least one of which is l or l'; we assume l is one of them (so $\mathrm{typ}(d) = (\mathrm{I}+)$). The image $\tilde{L}$ of L in $V_j(M)$ has dimension 2, and $\bar{L} \cap U^{\mathrm{I}}_j(M) = \{l\}$ would imply that the restriction of $b_j(M)$ to $\tilde{L}$ is degenerate, and hence that $L/l \in U^{\mathrm{N}}_j(l^\perp/l)$ and $L/l \notin Q_{u_{[l]}}$; this contradicts the assumption $f \in \widetilde{\mathcal{F}}_u$. Therefore $\bar{L} \cap U^{\mathrm{I}}_j(M)$ consists of two points, and we can choose an isomorphism $M \xrightarrow{\sim} M_\lambda$ such that L corresponds to $W_j(M_{(j,j)})$, where the summand $M_{(j,j)}$ corresponds to the two rows of $Y(\lambda)$ containing $\mathrm{supp}(d)$, and such that l corresponds to the special line belonging to d. If we have $\mathrm{typ}(d') = (\mathrm{I}+)$, then—possibly after replacing T' by an equivalent tableau so as to get $\mathrm{sg}(d') \neq \mathrm{sg}(d)$—we also have that l' corresponds to the special line belonging to d'. When $\mathrm{typ}(d') = (\mathrm{N})$ we may also assume that l' corresponds to the special line belonging to d', either by a further adaption of the isomorphism $M \xrightarrow{\sim} M_\lambda$, or by applying a case already treated.

If $\mathrm{typ}(d') = (\mathrm{I}+)$ we obtain induced isomorphisms $l^\perp/l \xrightarrow{\sim} M_{\lambda'} \xleftarrow{\sim} l'^\perp/l'$, and under these isomorphisms the images L/l' and L/l' of L correspond to the same line $\hat{L} \in U_{j-1}(M_{\lambda'})$; this is the special line (of type $(\mathrm{I}-)$) of the summand $M_{j-1,j-1}$ that corresponds to the rows of $\mathrm{supp}(d)$ (but it is not a special line of $M_{\lambda'}$ unless $m_{j-1}(\lambda') = 2$). Each of these isomorphisms gives rise to an isomorphism $L^\perp/L \xrightarrow{\sim} \hat{L}^\perp/\hat{L}$, and these two differ by the automorphism $\mathbf{1} \oplus -\mathbf{1}$ of the summand $M_{(j-2,j-2)}$ of $\hat{L}^\perp/\hat{L}$ obtained from the mentioned summand $M_{j-1,j-1}$ of $M_{\lambda'}$. (This is essentially due to the automorphism χ_{ρ_d} of M_λ which was used to transform the canonical isomorphism $l_+^\perp/l_+ \xrightarrow{\sim} M_{\lambda'}$ into $l_-^\perp/l_- \xrightarrow{\sim} M_{\lambda'}$, where l_+ and l_- are the special lines beloning to d and d'.) The image of that automorphism in the group $A_{u_{[L]}}$ is the generator g_{j-2} of $A_{u_{[L]},j-2}$, so the two isomorphisms are not in the same class (unless $j = 2$); we have $\left[T^{\downarrow\downarrow}\right] = g_{j-2} \cdot \left[T'^{\downarrow\downarrow}\right]$ in this case. If $\mathrm{typ}(d') = (\mathrm{N})$ we similarly find that under the induced isomorphisms of $l^\perp/l$ and $l'^\perp/l'$ with standard modules, the images of L correspond to special lines (of type $(\mathrm{I}-)$ and (N) respectively). When moreover $\mathrm{sg}(d) =$ '+' one finds that the two induced isomorphisms $L^\perp/L \xrightarrow{\sim} M_{\lambda''}$ are identical. Hence we have $T^{\downarrow\downarrow} \sim T'^{\downarrow\downarrow}$ when $\mathrm{sg}(d) =$ '+', and combining with the previous case, we get $\left[T^{\downarrow\downarrow}\right] = g_{j-2} \cdot \left[T'^{\downarrow\downarrow}\right]$ when $\mathrm{sg}(d) =$ '$-$'. We summarise our results.

3.5.2. Lemma. *Let $f, f' \in \widetilde{\mathcal{F}}_u$ be such that $f_i = f'_i$ for $i \geq 2$, but $f_1 \neq f'_1$. There exist tableaux $T, T' \in \mathcal{T}_\lambda$ such that $f \in \mathcal{F}_{u,T}$ and $f' \in \mathcal{F}_{u,T'}$ and that moreover the following conditions hold:*

(a) If $\mathrm{supp}(T_1) \cap \mathrm{supp}(T'_1) = \emptyset$ then $T^{\downarrow\downarrow} = T'^{\downarrow\downarrow}$ and $\mathrm{sg}(T'_1) = \mathrm{sg}(T_2)$, $\mathrm{sg}(T'_2) = \mathrm{sg}(T_1)$.
(b) If $\mathrm{supp}(T_1) = \mathrm{supp}(T'_1)$ and $\mathrm{typ}(T_1) \neq (\mathrm{I}+)$ then $T = T'$.
(c) If $\mathrm{supp}(T_1) = \mathrm{supp}(T'_1)$ and $\mathrm{typ}(T_1) = (\mathrm{I}+)$ then, with $j = \kappa_{T_1}$, we have $T = T'$ if the images of f_1 and f'_1 in $V_j(M)$ are perpendicular and otherwise $\mathrm{sg}(T'_1) = -\mathrm{sg}(T_1)$, $T'_2 = T_2$ and $T'^{\downarrow\downarrow} = \xi_r(T^{\downarrow\downarrow})$ where $r = \rho_{T_2} = {}^t\lambda_{j-1}$.
(d) If $\mathrm{supp}(T_1) \cap \mathrm{supp}(T'_1) = \{(r, j)\}$ for some r, j, then if $\mathrm{sg}(T_1) =$ '$-$' or $\mathrm{sg}(T'_1) =$ '$-$' we have $T'^{\downarrow\downarrow} = \xi_r(T^{\downarrow\downarrow})$, and else $T^{\downarrow\downarrow} = T'^{\downarrow\downarrow}$. □

3.6. *Examples.*

Merely coroborative detail,
intended to give artistic verisimilitude
to an otherwise bald and unconvincing narrative.
W. S. Gilbert, *The Mikado*

We close this section with a number of examples of specific varieties $\mathcal{F}_u$ together with the parametrisation of their irreducible components. In simple cases we can give a more-or-less complete geometric description of $\mathcal{F}_u$, from which one may get an impression of the geometric structure of $\mathcal{F}_u$ in general, and of how much of it is accounted for by the corresponding tableaux.

Before we do so, we must discuss an additional structure on $\mathcal{F}_u$, that we have not introduced before since it is not needed in our computation, but which plays an important rôle in the geometric description of $\mathcal{F}_u$. Recall that $\mathcal{F} \cong G/\widetilde{B}$. For any simple reflection s of W, let P_s be the (parabolic) subgroup generated by $\widetilde{B}$ and (a representative of) s, then $P_s/\widetilde{B}$ is isomorphic to the projective line $\mathbf{P}_1$. Consequently any fiber of the projection $G/\widetilde{B} \to G/P_s$ is also isomorhic to $\mathbf{P}_1$, and the corresponding projective line in $\mathcal{F}$ is called a *line of type* s. Clearly the action of G on $\mathcal{F}$ preserves these lines of type s, and one easily verifies that two distinct flags f, f' lie on the same line of type s if and only if $\pi(f, f') = s$. For $1 \leq i \leq n$ (and $i \neq 1$ in case D_n) the line of type s_i through a given flag f consists of all flags obtained from f by varying f_{n+1-i}, while fixing all other parts of f. In case D_n, a line of type s_2' is the set of companion flags to a line of type s_2.

It is easily shown that whenever two distinct points of a line of type s are stabilised by some unipotent u, then so is the whole line. Now it is known that for $(B', B'') \in O(w) \subseteq \mathcal{B} \times \mathcal{B}$ and any reduced expression $w = s_{i_1} \cdots s_{i_r}$ there is a unique sequence $B_0 = B', B_1, \ldots, B_r = B''$ in $\mathcal{B}$, such that $\pi(B_{j-1}, B_j) = s_{i_j}$ for $j = 1, \ldots, r$ (see [Spa 0.12]). A similar statement holds for flags $p, q \in \mathcal{F}$ with $\pi(p, q) = w \in W$ (which condition implies that p and q lie in the same component of $\mathcal{F}$). Therefore p and q are linked by a unique sequence of lines of types $s_{i_1}, \ldots, s_{i_r}$ respectively, and from the uniqueness of the intermediate flags it follows that if $p, q \in \mathcal{F}_u$, then the whole sequence of lines is contained in $\mathcal{F}_u$ (this, incidentally, is the way 1.8.1 is proved).

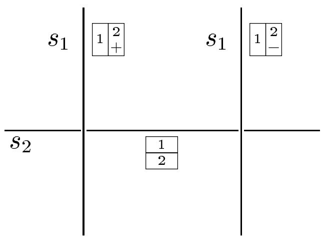
If, together with the geometric structure of $\mathcal{F}_u$, we indicate the lines of all types s lying in $\mathcal{F}_u$, then the relative position of any pair of flags in $\mathcal{F}_u$ can be immediately read off, provided that the situation is simple enough to find the shortest sequence of such lines that connect the pair of flags. (In case D_n we cannot connect two flags in different connected components, but companion flags have relative position $s_1 \in \widetilde{W}$.) Unless $\mathcal{F}_u$ is reduced to a point, every irreducible component of $\mathcal{F}_u$ is a union of lines of type s for at least one simple reflection s.

We now discuss some specific cases. For B_0, C_0 and D_1 the Dynkin diagram is empty, and $\mathbf{e} = \mathbf{1}_M$ is the sole unipotent and $\dim(\mathcal{F}) = 0$. These cases form the starting point for any induction. While B_0 and C_0 are entirely trivial, there are two flags (i.e., two isotropic lines in M) for D_1; they are companions and correspond to the two tableaux $\boxed{\begin{smallmatrix}1\\+\end{smallmatrix}}$ and $\boxed{\begin{smallmatrix}1\\-\end{smallmatrix}}$, which are interchanged by the generator of $A_{\mathbf{e}} \cong \mathbf{2}$. For cases B_1 and C_1 there are two classes of unipotents: the class of regular unipotents (with one Jordan block), and $\{\mathbf{e}\}$. When u is a regular unipotent, $\mathcal{F}_u$ consists of a single flag, and the corresponding tableau is $\boxed{0}\boxed{\;1\;}$ respectively $\boxed{\;1\;}$. The same holds for regular unipotents in all cases B_n or C_n, except that there are more dominoes (of type (N)) in row 1. For regular unipotents in case D_n there are two tableaux, obtained by extending the D_1-tableaux by type (N) dominoes in row 1, and the situation is like in case D_1, except that $A_u \cong \mathbf{2} \times \mathbf{2}$.

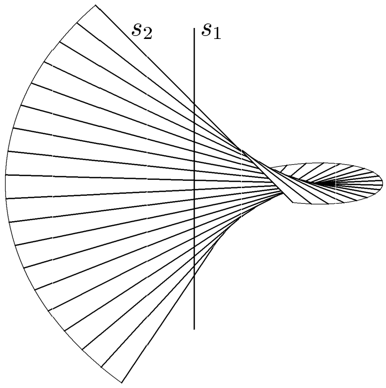
In cases B_1 and C_1, the varieties $\mathcal{F}_{\mathbf{e}} = \mathcal{F}$ consist of a single line of type s_1, and are parametrised by $\begin{array}{|c|}\hline 0\\\hline 1\\ +\\\hline\end{array} \sim \begin{array}{|c|}\hline 0\\\hline 1\\ -\\\hline\end{array}$ respectively by $\boxed{1}$. In the latter case, the line is just $\mathbf{P}(M)$, while in case B_n it is the quadric of isotropic lines in $\mathbf{P}(M) \cong \mathbf{P}_2$. In general, lines of type s_1 will be embedded as "straight" lines in case C_n, while they are embedded as "circles" in case B_n. Lines of type s_i with $i > 1$ are always "straight", since variation takes place within a fixed isotropic subspace. In all cases $\mathcal{F}_{\mathbf{e}}$ is parametrised by tableaux supported entirely in the first column; the constituent dominoes form a single cluster. This cluster is cl(0) in cases B_n and C_n, so that all such tableaux are equivalent, which is in agreement with the irreducibility of $\mathcal{F}$, but in case D_n the tableaux fall into two classes characterised by the product of signs in this cluster, which correspond to the two components of $\mathcal{F}$.

Regular unipotents and $\mathbf{e}$ are opposite extremes, and rather uninteresting from our point of view. The simplest case where u is neither regular nor identity, is when $J(u) = (2, 2)$ in case D_2. Then the two connected components of $\mathcal{F}_u$ are lines of type s_2 and s_2', that correspond respectively to the tableaux $\begin{array}{|c|c|}\hline 1 & \\ + & 2\\\hline\end{array}$ and $\begin{array}{|c|c|}\hline 1 & \\ - & 2\\\hline\end{array}$. When moving a flag f along the line of type s_2, f_1 varies within the fixed space $f_2 = W_1(M)$, but moving along the second line f_2 varies with f_1. This case exemplifies the fact that $\mathcal{F}_u$ as an abstract variety may exhibit a symmetry (viz. the interchange of companion flags) that does not come from the action of Z_u; the latter preserves lines of any given type. In fact Z_u is connected in this case, so it stabilises each of the connected components of $\mathcal{F}_u$; the two mentioned tableaux each have a single, closed, cluster. Because $G = O_4$ has two connected components, so has the conjugacy class of u.

The examples so far have been atypical in the fact that the connected components of $\mathcal{F}_u$ were irreducible. When $J(u) = (2, 2)$ in case C_2, however, $\mathcal{F}_u$ consists of a line of type s_2 parametrised by $\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}$, that is intersected in two distinct points by lines of type s_1, that are parametrised by $\begin{array}{|c|c|}\hline 1 & 2\\ & +\\\hline\end{array}$ and $\begin{array}{|c|c|}\hline 1 & 2\\ & -\\\hline\end{array}$. So S_u is the set of these three lines, of which the last two are interchanged by the generator of $A_u \cong \mathbf{2}$. The variety $\mathcal{F}_u$ may be depicted schematically as follows:



The map α_u can be viewed as the vertical projection in this picture. The intersection points of two lines belong to $\mathcal{F}_{u,T}$ for $T = \begin{array}{|c|c|}\hline 1 & 2\\ & +\\\hline\end{array}$ respectively $T = \begin{array}{|c|c|}\hline 1 & 2\\ & -\\\hline\end{array}$, not for $T = \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}$, which is why they are set apart slightly from the horizontal line. In fact these intersection points are the special flags F_T—for the former two values of T—that are used to discriminate between the two vertical lines (the third special flag F_T is somewhere else on the horizontal line).



There is one other class of unipotents for C_2, with $J(u) = (2,1,1)$. This provides the simplest case where not all or Z_u-orbits in $\mathrm{Im}(\alpha_u)$ are good. There are two such orbits: $U_2(M)$, which is a single point, and $U_1(M)$, which is a $\mathbf{P}_2$ minus that point. For $l \in U_1(M)$, the fibre $\alpha_u^{-1}[l]$ consists of just one flag since $u_{[l]}$ is regular, but $\alpha_u^{-1}[l] \cong \mathbf{P}_1$ for $l \in U_2(M)$, since in that case $u_{[l]} = \mathbf{1}_{l^\perp/l}$. It follows that $U_1(M)$ is a good orbit, whereas $U_2(M)$ is not, and $\mathcal{F}_u$ has one irreducible component, parametrised by $\begin{smallmatrix}1\\2\end{smallmatrix}$, which is isomorphic to $\mathbf{P}_2$ with one point blown up to a $\mathbf{P}_1$. This variety is a union of lines of type s_2, namely the lines in $\mathbf{P}_2$ meeting the blown-up point, which itself is the unique line of type s_1 contained in $\mathcal{F}_u$. We see that the relative position of a generic pair of flags in $\mathcal{F}_u$ is $s_2 s_1 s_2$, and that the sequence of 3 projective lines linking the pair always contains the blown up point—which is not contained in $\widetilde{\mathcal{F}}_u$—as its middle segment. (In view of this example it is not at all clear whether $\widetilde{\mathcal{F}}_u$ is connected in all cases (for D_n: whether $\#\,\mathrm{Con}(\widetilde{\mathcal{F}}_u) = 2$), since the proof for $\mathcal{F}_u$ evidently fails for $\widetilde{\mathcal{F}}_u$.)

In cases like this, when $J(u)$ has a relatively large number of small parts, and hence $\dim(\mathcal{F}_u)$ is high, $\mathcal{F}_u$ may often be more easily described—in a direct translation of its definition—as a set of (projective images of) flags subject to one or more restrictions relating to specific points, lines etc. In this case $\mathcal{F}_u$ can be viewed as the set of incident point–line pairs in $\mathbf{P}_2$ (namely in $\mathbf{P}(W_1(M))$) where the line is required to contain a given point (namely $\mathbf{P}(W_2(M))$). Similarly, in the general C_n-case, if $J(u) = (2,1,1,\ldots)$ then $\mathcal{F}_u$ is the (irreducible) set of isotropic flags whose highest dimensional part contains $p = W_2(M)$, and these flags are all contained in $p^\perp = W_1(M)$.

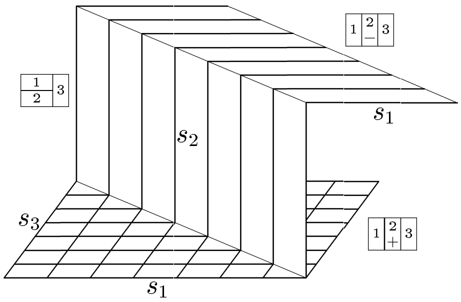
It is instructive to compare the cases B_2 and C_2. For an appropriate pairing of the unipotents we get isomorphic varieties $\mathcal{F}_u$—which is not surprising since the Dynkin diagrams of types B_2 and C_2 are isomorphic, interchanging the simple reflections s_1 and s_2—and we get a pairing between tableaux as well. However, for two corresponding pairs (u,T), we do not always have that the two associated sets $\mathcal{F}_{u,T}$ match precisely, although their closures do. In fact, even $\widetilde{\mathcal{F}}_u$ may differ between the two cases. Identity and regular unipotents pair up of course; the triple $\begin{smallmatrix}1 & 2\\ & +\end{smallmatrix}$, $\begin{smallmatrix}1\\2\end{smallmatrix}$, $\begin{smallmatrix}1 & 2\\ & -\end{smallmatrix}$ for case C_2 matches the triple $\begin{smallmatrix}0 & 1\\2 & \\+ & \end{smallmatrix}$, $\begin{smallmatrix}0 & 2\\1 & \\+ & \end{smallmatrix} \sim \begin{smallmatrix}0 & 2\\1 & \\- & \end{smallmatrix}$, $\begin{smallmatrix}0 & 1\\2 & \\- & \end{smallmatrix}$ for B_2, and the final tableau $\begin{smallmatrix}1\\2\end{smallmatrix}$ matches $\begin{smallmatrix}0 & 2\\1 & \\+ & \end{smallmatrix} \sim \begin{smallmatrix}0 & 2\\1 & \\- & \end{smallmatrix}$.

So for $J(u) = (3,1,1)$ the variety $\mathcal{F}_u$ is similar to the picture drawn above, but now $\mathrm{Im}(\alpha_u)$ consists of two intersecting lines, and for the point p of intersection, $\alpha^{-1}[p]$ is the horizontal line, which is of type s_1 (and hence "round"). The intersection points in the picture now are part of $\mathcal{F}_{u,T}$ for $[T]$ parametrising the *horizontal* line; indeed each one is equal to F_T for a representative T. For $J(u) = (2,2,1)$ we have

$\mathrm{Im}(\alpha_u) = U_2(M) \cong \mathbf{P}_1$, and for any $l \in U_2(M)$ we have that $\alpha_u^{-1}[l] \cong \mathbf{P}_1$ is a line of type s_1; there is one line of type s_2 contained in $\mathcal{F}_u$, namely $\{\, f \in \mathcal{F}_u \mid f_2 = W_2(M)\,\}$, which intersects every fibre $\alpha_u^{-1}[l]$. Although $\mathcal{F}_u$ can be shown to be isomorphic to the corresponding variety for C_2 (lines of types s_1 and s_2 being interchanged), this is not true for $\widetilde{\mathcal{F}}_u$, since in this case it equals $\mathcal{F}_u$.

The cases with $n = 3$ offer a wide variety of examples, all of which can—with a sufficient amount of effort—be geometrically understood, at least up to the point that one can deduce the relative position of generic pairs of flags in specified components of $\mathcal{F}_u$. We shall mention only a few cases that illustrate phenomena not obvious from the simpler examples.

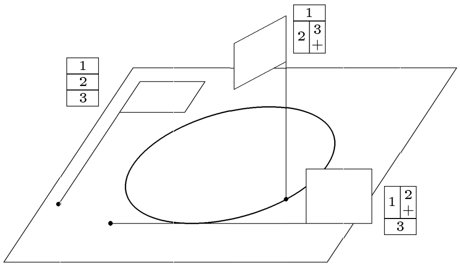
Consider in case C_3 a unipotent u with $J(u) = (3,3)$. Then $\mathrm{Im}(\alpha_u) = U_3(M) \cong \mathbf{P}_1$ is a single Z_u-orbit, and for any $l \in U_3(M)$ the fibre $\alpha_u^{-1}[l]$ is isomorphic to the three intersecting lines (of types s_1, s_2 and s_1) depicted above. Since the domino with entry 3 added to the corresponding tableaux is of type (I$-$) and therefore doesn't give rise to creation or fusion of clusters, S_u has three elements, parametrised as $\boxed{1}\boxed{\substack{2\\+}}\boxed{3}$, $\boxed{\substack{1\\2}}\boxed{3}$, and $\boxed{1}\boxed{\substack{2\\-}}\boxed{3}$.



Now $f \in \alpha_u^{-1}[l]$ lies on a line of type s_3 contained in $\mathcal{F}_u$ if and only if $f_2 = W_1(M)$, and it is readily checked that in that case $\mathrm{typ}(f_2/l) = $ (I+) in $l^\perp/l$, and in fact $f \in \mathcal{F}_{u,T}$ for $T = \boxed{1}\boxed{\substack{2\\+}}\boxed{3}$, as this is the only tableau of the three for which F_T satisfies the condition. Because $\dim(\mathcal{F}_u) = 2$, this component $\mathcal{F}_{u,T}$, being a union of lines both of type s_1 and of type s_3, must be isomorphic to $\mathbf{P}_1 \times \mathbf{P}_1$. Next, for $T = \boxed{\substack{1\\2}}\boxed{3}$ we have $f \in \overline{\mathcal{F}_{u,T}}$ if and only if $f_3 = \eta_M^{-1}[f_1]$, so this component, which is a surface ruled by lines of type s_2, intersects the previous component in a line that is "diagonal" with respect to the lines of types s_1 and s_3. The third component, $\mathcal{F}_{u,T}$ for $T = \boxed{1}\boxed{\substack{2\\-}}\boxed{3}$, is characterised by $\mathrm{typ}(f_2/l) = $ (I+) but $f_2 \neq W_1(M)$, and is ruled by lines of type s_1 only; it intersects the second component in a line transversal to the lines of type s_2, and is disjoint from the first component.

We see in this example that the symmetry present in the individual fibres $\alpha_u^{-1}[l]$ disappears globally. This phenomenon—visible also in case D_2 with $J(u) = (2,2)$, although less dramatically—is due to the transformation of open clusters to closed ones by adding a domino of type (I$-$). In the present case the components parametrised by $\boxed{1}\boxed{\substack{2\\+}}\boxed{3}$ and $\boxed{1}\boxed{\substack{2\\-}}\boxed{3}$ are not even isomorphic geometrically: the first is $\mathbf{P}_1 \times \mathbf{P}_1$, while the second proves to be isomorphic to a cone (over a circle) blown up at its apex, (as, incidentally, does the remaining component).

Another thing that can happen with clusters is that by adding a domino two clusters merge into one; when this happens, certain pairs of symmetric components of fibres turn out to be part of one irreducible component globally. An example occurs in case C_3 when $J(u) = (2, 2, 2)$. We have $\dim(\mathcal{F}_u) = 3$, and S_u has 3 elements, that are parametrised as $\begin{smallmatrix}1\\2\\3\end{smallmatrix}$, $\begin{smallmatrix}1&2\\&+\\3&\end{smallmatrix} \sim \begin{smallmatrix}1&2\\&-\\3&\end{smallmatrix}$, and $\begin{smallmatrix}1&\\2&3\\&+\end{smallmatrix} \sim \begin{smallmatrix}1&\\2&3\\&-\end{smallmatrix}$. Note that in all three cases there is only one cluster, $\mathrm{cl}(0)$, whence the two stated equivalences hold, but the first of these two equivalences becomes invalid if the dominoes with entry 3 are removed. Because its dimension is 3, it is rather difficult to give a complete geometric description of $\mathcal{F}_u$, but we may describe its components as sets of flags as follows.

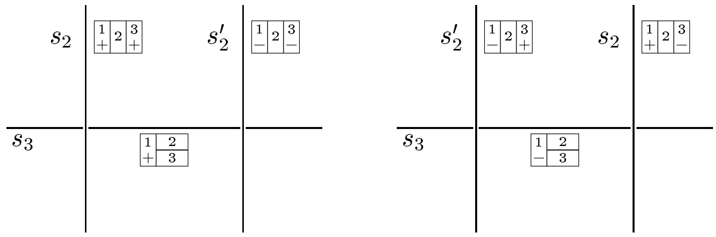


Let $P = U_2(M) \cong \mathbf{P}_2$ and $Q = U_2^{\mathrm{I}}(M)$, a smooth quadric in P. Then $\overline{\mathcal{F}_{u,T}}$ for $T = \begin{smallmatrix}1\\2\\3\end{smallmatrix}$ is the flag manifold of P (i.e., the set of all incident point–line–plane triples in P), while $\mathcal{F}_{u,T}$ is the open subset of flags whose point does not lie on Q and whose line is not tangent to Q. The flags of $\mathcal{F}_{u,T}$ for $T = \begin{smallmatrix}1&2\\&+\\3&\end{smallmatrix}$ are not all contained in P, but their line-part is, and it is tangent to Q; the point does not lie on Q, and there is one dimension of freedom "rotating" the plane around the line, outside of P. The irreducibility of this set illustrates the effect of fusion of clusters: whereas for any point $p \in P \setminus Q$, the fibre $\alpha_u^{-1}[p]$ intersects this set in two disjoint components—corresponding to the two tangent lines to Q through p—the whole set is irreducible, since this is true for the set of all incident pairs point–tangent line. Note that the line of type s_2 contained in $\alpha_u^{-1}[p]$—which lies in the first mentioned irreducible component, the flag manifold of P—intersects this second irreducible component $\overline{\mathcal{F}_{u,T}}$ in precisely two points. The third set $\mathcal{F}_{u,T}$, for $T = \begin{smallmatrix}1&\\2&3\\&+\end{smallmatrix}$, consists of flags whose point-part lies on Q; the plane may be rotated around the tangent to Q at this point, and the line is arbitrary in that plane, except that it must not be equal to the tangent line, since such flags are not in $\widetilde{\mathcal{F}}_u$.

In case B_3 the unipotents with $J(u) = (3, 3, 1)$ and $J(u) = (3, 2, 2)$ respectively illustrate the same phenomena related to cluster fusion repectively closure as the cases treated for C_3, but with the dimensions of $\mathcal{F}_u$ (which are 2 resp. 3) switched. In the first of these cases the "fused" component is parametrised by $\begin{smallmatrix}0&1&\\2&3&\\+&&\end{smallmatrix}$, and is isomorphic to the blown-up cone mentioned earlier. Its intersections with fibres $\alpha_u^{-1}[l]$ are derived from the intersections with planes taken from a pencil about a line through the apex of the cone; indeed this gives two disjoint lines, except in two instances where it gives only one line.

The second of these cases illustrates yet another fact. The two non-symmetric irreducible components parametrised by $\begin{smallmatrix}0&1&\\2&3&\\+&&\end{smallmatrix}$ and $\begin{smallmatrix}0&1&\\2&3&\\-&&\end{smallmatrix}$ intersect each fibre $\alpha_u^{-1}[l]$ for $l \in Q_u = U_2(M)$ in disjoint lines. However, for $l \in U_3(M)$ these components (i.e., the closures $\overline{\mathcal{F}_{u,T}}$) intersect the fibre in the *same* line (flags $f \in \mathcal{F}_u$ in the union of the two components are characterised by the condition $\eta_M[f_3] \subseteq f_1$). Hence these two irreducible components are not disjoint, which shows that we may not take closures in the second part of 3.2.3. A similar phenomenon can be seen in case D_3 for $J(u) = (2, 2, 1, 1)$, in which case the intersection of two irreducible components with the same unsigned tableau takes place within $\widetilde{\mathcal{F}}_u$.

A last example will illustrate that the difficulties treated in 3.5 involve an essential aspect of the geometry, and are not just an idiosyncrasy of our particular parametrisation. Consider in case D_3 a unipotent with $J(u) = (3, 3)$. Then $\mathcal{F}_u$ is 1-dimensional, and can be drawn as follows:



The horizontal lines are of type s_3, the outer vertical lines are of type s_2 and the inner ones are of type s_2'. Translating the left half of the picture so that it covers the right half puts companion flags on top of each other; as can be seen this affects the sign of the domino with entry 1 only. We have $A_u \cong \mathbf{2}$, and (some representative of) its generator acts by reflection in the vertical axis of the picture; this affects the sign in the rightmost cluster of the tableaux only, which contains one domino for the vertical lines and all three dominoes for the horizontal lines. As in the case depicted for C_2, the special flags F_T for the tableaux on the vertical lines lie on the intersection points with the horizontal lines.

The set $U_3^{\mathrm{I}}(M)$ has two points, and for $l \in U_3^{\mathrm{I}}(M)$ the fibre $\alpha_u^{-1}[l]$ is the pair of the leftmost vertical lines of each connected component, respectively the pair of rightmost ones. These two fibres are isomorphic by the action of A_u (isomorphisms identifying the vertical lines of the same connected component are *not* suitable, since they don't respect the types of lines), and our parametrisation is such that corresponding components give the same reduced tableau $T^{\downarrow}$, that can be used in inductive procedures. We may reduce one step further, taking fibres $\alpha_{u_{[l]}}^{-1}[L/l]$ of this fibre (for appropriate planes L), and find that these consist of two isolated points, each one representing a possible value of f_3.

However, while we know that f_3 determines the connected component of f, we have already identified the fibres $\alpha_u^{-1}[l]$ by an isomorphism that does not respect connected components, so we cannot expect a consistent connection between f_3 and $T^{\downarrow\downarrow}$. In particular, for $L = W_1(M)$—which can be used for $l \in U_3^{\mathrm{N}}(M)$ as well as for $l \in U_3^{\mathrm{I}}(M)$—we find as (repeated) fibres pairs of companion flags on the horizontal lines in the picture, including the intersections. Although the same pair of values for f_3 occurs in all these fibres, we see that the correspondence of this pair with the values of $T^{\downarrow\downarrow}$ flips at the intersection with the rightmost vertical line.

Clearly we might have chosen a parametrisation for which this doesn't happen—indeed Spaltensteins original parametrisation has this property—but this would be of little help, since in order to describe fibres of α_u we would have to replace $T \mapsto T^{\downarrow}$ by another operation, and we would end up in similar complications. Those who find the simplicity of this example misleading, or the fact that $\mathcal{F}_u$ is not connected, should study the 2-dimensional example with $J(u) = (4,4)$ in case C_4 ($\#S_u = 10$ in this case, but the most interesting information is obtained by studying the component parametrised by $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}$, which is isomorphic to $\mathbf{P}_1 \times \mathbf{P}_1$, and its intersections with other components). This shows even more convincingly that it is not possible to identify fibres of the projection $\alpha_u\colon f \mapsto f_1$ in a way that simultaneously respects the projection $f \mapsto f_3$.

§4. Computing relative positions.

4.1. *Relative positions of flags revisited.*

In 1.5 we have defined the relative position of a pair of flags, and in 3.6 we have seen that reduced expressions for this relative position can be interpreted in terms of sequences of projective lines in $\mathcal{F}$ connecting the pair. There is however another interpretation, which relates the signed permutation representing the relative position directly to the parts of the flags, and this interpretation will prove to be very useful.

In order to express this relationship, it is convenient to consider flags equipped with some additional information, namely a sequence of numbers that will be used to label its parts. We shall still call these objects 'flags', and define a standard sequence, to be used with flags for which no sequence is explicitly specified. These sequences of numbers consist of 0, followed by an increasing sequence of n distinct positive integers. We may use a number from the sequence attached to a flag f to select a particular part of f; if we do so, we write that number as a superscript to f (recall that subscripts were used to select a part by its dimension). This labeling is by decreasing dimension:

$$f_n = f^0 \supset f^{i_1} \supset \cdots \supset f^{i_n} = 0 \quad \text{where} \quad 0 < i_1 < \cdots < i_n; \tag{22}$$

the dimension decreases by 1 at every inclusion. We write $I(f)$ for the set $\{0, i_1, \ldots, i_n\}$ of labels attached to f. If f has the standard labeling, then $I(f) = \{0, 1, \ldots, n\}$, so that $f^i = f_{n-i}$. We define $I(f^{\downarrow}) = I(f) \setminus \{\max(I(f))\}$; so if f has standard labeling, then so has $f^{\downarrow}$. We can now get around the index-shift in the definition (9) appearing in 1.8, and instead write

$$f^{\downarrow i} = f^i / f_1 \qquad \text{for } i \in I(f^{\downarrow}). \tag{23}$$

Non-standard labeling of flags may be defined implicitly by the following convention. Recall that the values occurring as entries of dominoes in a tableau may be any set of distinct positive integral numbers. We define for any tableau T with $\operatorname{sh}(T) = J(u)$ a set $\mathcal{F}_{u,T}$ of flags, such that for all $f \in \mathcal{F}_{u,T}$ we have $I(f) = \{0\} \cup \Sigma$, where Σ is the set of entries of dominoes of T. The underlying set of (unlabeled) flags of $\mathcal{F}_{u,T}$ is $\mathcal{F}_{u,T'}$, where $T' \in \mathcal{T}_\lambda$ is obtained from T by renumbering its entries using the unique monotonic bijection $\Sigma \to \{1, \ldots, n\}$. We extend the superscript notation to negative values, so put $-I(f) = \{\, -i \mid i \in I(f) \,\}$, where the element $-0 \in -I(f)$ should be considered to be distinct from $0 \in I(f)$ (this will not bring us into any trouble). Now we define f^i for $i \in -I(f)$ by requiring

$$f^{-i} = f^{i\perp} \qquad \text{for } i \in I(f) \amalg -I(f). \tag{24}$$

Remark. The order reversal with respect to dimension introduced here compensates for the fact that the highest numbered dominoes in tableaux correspond to the lowest dimensional parts of flags. However, one may feel that we are still off by 1, since for $f \in \mathcal{F}_{u,T}$ the leading domino of T, whose entry equals the highest label of f, is related to the position of $f_1 = f^i$ where i is the highest label *but one*. On the other hand, for any i the position of the flag $f^{\downarrow\cdots\downarrow}$ induced in f^{-i}/f^i is related, by repeated application of rule (20) in 3.5, to the subtableau of T of dominoes with entries $\leq i$; these entries are the ones occurring in $I(f^{\downarrow\cdots\downarrow})$ (but 0 is absent).

We now first define a relative position $\pi(f, l)$ of a flag f and an isotropic line l, which is a non-zero integer i with $|i| \in I(f)$.

$$\pi(f,l) = \begin{cases} \min\{\, i \in I(f) \mid l \not\subseteq f^i \,\} & \text{if } l \subseteq f^0 \\ \max\{\, i \in -I(f) \mid l \subseteq f^i \,\} & \text{if } l \not\subseteq f^0 \end{cases} \tag{25}$$

Note that, since l is isotropic and f^0 is an isotropic subspace of maximal dimension, $l \not\subseteq f^0$ implies $l \not\subseteq f^{-0}$. We also define a flag $f_{[l]}$ in the space $l^\perp/l$. The parts of $f_{[l]}$ are the "images" of those of f selected by the same label:

$$I(f_{[l]}) = I(f) \setminus \{|\pi(f,l)|\} \qquad \text{and} \qquad f^i_{[l]} = (f^i \cap l^\perp + l)/l \quad \text{for } |i| \in I(f_{[l]}). \tag{26}$$

Here '$\cap$' takes priority over '$+$'. Note that this formula is consistent with (24), and would give the same results for $i = |\pi(f,l)|$ as it gives for i equal to the immediately preceding element of $I(f)$; therefore, by the omission of $|\pi(f,l)|$ from $I(f_{[l]})$, each part of $f_{[l]}$ is selected by a unique superscript. Note also that the operation '$+\,l$' is redundant when $i < \pi(f,l)$, and that '$\cap\, l^\perp$' is redundant when $i > -\pi(f,l)$. One readily verifies that the flag $f_{[l]}$, like f, is a full isotropic flag. As a special case, we have $f^\downarrow = f_{[f_1]}$.

The relative position $\pi(f,l)$ defined in (25) can also be computed by a recursive procedure π_1, where $\pi_1(f,l)$ is defined as follows. Let $m = \max\bigl(I(f)\bigr)$ and put $h = f_1$. If $h = l$ we have $\pi_1(f,l) = m$, and unless $h \perp l$ we have $\pi_1(f,l) = -m$; in the remaining case we have $\pi_1(f,l) = \pi_1(f^\downarrow, l')$, where $l' = (l+h)/h \subseteq h^\perp/h$.

4.1.1. Proposition. *Let $f \in \mathcal{F}$ and let l be an isotropic line. Then $\pi_1(f,l) = \pi(f,l)$.*

Proof. The simple inductive proof is left as an exercise to the reader. $\square$

In the same fashion we can now define a procedure π_2 that computes the relative position of two flags, expressed as a signed permutation. So $\pi_2(f,f')$ yields a sequence $(w_1,\ldots,w_n)$, which is defined as follows. If $I(f) = \{0\}$ (i.e., if $n = 0$), then the sequence is empty; otherwise assume that π_2 has been defined for flags in $l^\perp/l$, where $l = f'_1$. Then $w_n = \pi_1(f,l)$, and $(w_1,\ldots,w_{n-1}) = \pi_2(f_{[l]}, f'^{\downarrow})$. It is easy to see that if f has the standard labeling, then (7) is satisfied and $\pi_2(f,f')$ is a signed permutation. The following proposition also follows immediately from the definitions.

4.1.2. Proposition. *Let $f, \tilde{f}$ be two flags that differ only by a monotonic renumbering of their labels. Then for any flag f', the sequences $\pi_2(f,f')$ and $\pi_2(\tilde{f},f')$ have the same signs at corresponding positions, while the absolute values are related by the same renumbering as $I(f)$ and $I(\tilde{f})$ are.* $\square$

4.1.3. Proposition. *Let $f, f' \in \mathcal{F}$ and $w = \pi(f,f')$. Then $\pi_2(f,f') = (w_1,\ldots,w_n)$ is the signed permutation representing w.*

Proof. We first show by an easy computation that $\pi_2(F, w\cdot F) = (w_1,\ldots,w_n)$. Assuming $n > 0$ and using (2), we have $(w\cdot F)_1 = w\cdot\langle e_n\rangle = \langle e_{w_n}\rangle$, and $\pi(F,\langle e_{w_n}\rangle) = w_n$ by (25). Putting $l = \langle e_{w_n}\rangle$, we get from (26) that the underlying flag of $F_{[l]}$ is the standard flag F in $l^\perp/l$, but $I(F_{[l]}) = \{0,\ldots,n\}\setminus\{|w_n|\}$. Now applying induction and 4.1.2 gives $\pi_2(F_{[l]}, (w\cdot F)^{\downarrow}) = (w_1,\ldots,w_{n-1})$, whence $\pi_2(F, w\cdot F)$ is as claimed. But it is clear from the definition that $\pi_2(g\cdot f, g\cdot f') = \pi_2(f,f')$ for every $g \in G$, so that π_2 gives the same value on the whole orbit $\widetilde{O}(w) \subseteq \mathcal{F}\times\mathcal{F}$, establishing the proposition. $\square$

Conway called this number "zero",
and said that it shall be a sign
to seperate positive numbers from negative numbers.
D. E. Knuth, *Surreal Numbers*

4.2. *Generic relative positions.*

Because the varieties $\widetilde{O}(w)$ form a finite partition of $\mathcal{F}\times\mathcal{F}$, it follows that for any irreducible subset X of $\mathcal{F}\times\mathcal{F}$ there is a unique $w \in \widetilde{W}$ such that $X \cap \widetilde{O}(w)$ is dense in X. This element w will be denoted $\gamma(X)$. If X is the direct product $X_1 \times X_2$ of (irreducible) subsets of $\mathcal{F}$, we call $\gamma(X)$ the *generic relative position* of X_1 and X_2, and we shall write $\gamma(X_1,X_2) = \gamma(X)$. Our goal, that was already mentioned in the introduction, can now be more specifically formulated as determining, for any pair T,T' of tableaux with $\mathrm{sh}(T) = \mathrm{sh}(T') = J(u)$, the relative position $\gamma(\overline{\mathcal{F}_{u,T}}, \overline{\mathcal{F}_{u,T'}})$.

It is clear that, for $\sigma,\tau \in S_u$ and $a \in A_u$, we have the following properties for γ:

$$\gamma(\tau,\sigma) = \gamma(\sigma,\tau)^{-1} \tag{27}$$

$$\gamma(a\cdot\sigma, a\cdot\tau) = \gamma(\sigma,\tau) \tag{28}$$

The interest of $\gamma(\sigma,\tau)$ is demonstrated by the following fact.

4.2.1. Theorem. *Let $\bar{U}$ be a set of representatives of unipotent conjugacy classes in G°, and for any unipotent $u \in G^\circ$ let $A_u^\circ = \mathrm{Con}((G^\circ)_u)$ act diagonally on $\mathrm{Irr}(\mathcal{B}_u) \times \mathrm{Irr}(\mathcal{B}_u)$, then γ gives rise to a bijection*

$$\bar{\gamma} : \coprod_{u \in \bar{U}} A_u^\circ \backslash (\mathrm{Irr}(\mathcal{B}_u) \times \mathrm{Irr}(\mathcal{B}_u)) \xrightarrow{\sim} W,$$

where the terms of the left-hand side are sets of orbits for A_u°.

Proof. This has been proved for reductive groups G in general, see [Spr2 3.8] and [Spr3 4.4.1], or in somewhat disguised formulations [Spa II 2.11] and [St2 3.4]. The general proof is complicated, as one needs to prove that $\dim(Z_u) = 2\dim(\mathcal{B}_u) + \mathrm{rk}(G)$ [Spa II 10.15]. In the case of classical groups, however, this equality can easily be established by explicit calculation. □

4.2.2. Corollary. *Let $\tilde{U}$ be a set of representatives of unipotent conjugacy classes in G, then γ also induces a bijection*

$$\tilde{\gamma} : \coprod_{u \in \tilde{U}} A_u \backslash (S_u \times S_u) \xrightarrow{\sim} \widetilde{W}.$$

Proof. In cases C_n and B_n this is immediate from 4.2.1, so assume we are in case D_n. We first prove the following claim: $\tilde{\gamma}$ maps the subset of orbits consisting of pairs $(\sigma, \tau) \in S_u \times S_u$ where σ and τ are contained in the same connected component, bijectively to W. We examine each $u \in \tilde{U}$ separately, and distinguish two cases, namely whether $Z_u \subseteq G^\circ$ or not. Consider first the case that there exists a $g \in Z_u \setminus G^\circ$ (this happens whenever $\#A_u > 1$). Then the G-conjugacy class of u coincides with its G°-conjugacy class, and the action of g gives an isomorphism from one connected component of $\mathcal{F}_u$ to the other. Since we have identified one of these components with $\mathcal{B}_u$, it follows that each orbit in the specified subset of $A_u \backslash (S_u \times S_u)$ contains a unique orbit in $A_u^\circ \backslash (\mathrm{Irr}(\mathcal{B}_u) \times \mathrm{Irr}(\mathcal{B}_u))$, and the two are mapped to the same element of W by $\tilde{\gamma}$ and $\bar{\gamma}$ respectively. Next consider the case $Z_u \subseteq G^\circ$. Then the G-conjugacy class of u has two connected components, each of which is a G°-conjugacy class, and conjugacy by any $g \in G \setminus G^\circ$ exchanges the two. Also, the action of Z_u stabilises each of the connected components of $\mathcal{F}_u$, and, with $u' = gug^{-1}$ a representative of the component not containing u, the action of g gives an isomorphism of the component $\mathcal{F}_u \setminus \mathcal{B}_u$ with $\mathcal{B}_{u'}$. Therefore, the set S_u is in bijection with $\mathrm{Irr}(\mathcal{B}_u) \uplus \mathrm{Irr}(\mathcal{B}_{u'})$, and $A_u \cong A_u^\circ \cong A_{u'}^\circ$ stabilises both subsets (in fact A_u is trivial), from which the claim easily follows. To finish the proof, it suffices to verify that for a pair $f, \tilde{f}$ of companion flags and for all flags f', one has $\pi(f, f') = s_1 \pi(\tilde{f}, f')$. □

Clearly, for any irreducible $X \subseteq \mathcal{F} \times \mathcal{F}$, we have $\gamma(\overline{X}) = \gamma(X)$. Our computation of $\gamma(\sigma, \tau)$ will proceed essentially by repeatedly replacing $\sigma \times \tau$ by suitable dense subsets until we have reached a subset on which $\pi(f, f')$ is constant. Here, and elsewhere, 'dense' may be read as 'open and not empty', since we only exclude closed sets, and X is irreducible. We shall use the procedures π_1 and π_2 of 4.1 to compute $\pi(f, f')$, so an intermediate subset to be sought for is for instance one on which $\pi_1(f, f_1')$ is constant. In view of the definitions of π_1 and π_2, the things to be studied are the conditions $h = l$ and $h \perp l$ for $h = f_1$ and $l = f_1'$, and suitable reductions towards $h^\perp/h$ and $l^\perp/l$ in order to apply induction (at two levels). The following sections discuss the issues encountered in this—apparently straightforward—approach.

4.3. *Fundamental computations.*

As the very first reduction we replace the irreducible components $\overline{\mathcal{F}_{u,T}}$ by their dense subsets $\mathcal{F}_{u,T}$. So let two tableaux T, T' be given with $\mathrm{sh}(T) = \mathrm{sh}(T') = \lambda = J(u)$, and let a class of isomorphisms $M \xrightarrow{\sim} M_\lambda$ be chosen, so that $\mathcal{F}_{u,T}$ and $\mathcal{F}_{u,T'}$ denote well-defined components of $\mathcal{F}_u$. Until further notice, f will denote a flag in $\mathcal{F}_{u,T}$, possibly subject to further restrictions, and f' similarly denotes a flag in $\mathcal{F}_{u,T'}$. Let $d = T_1$ and $e = T_1'$ be the leading dominoes of T and T' respectively, put $(r, c) = \pi(d)$, $(r', c') = \pi(e)$, and let $h = f_1$ and $l = f_1'$ denote the 1-dimensional parts of f and f'. As a consequence of the definitions in 3.3 and 3.4 we have $h \in U_d(M)$ and $l \in U_e(M)$. Since $\alpha_u[\mathcal{F}_{u,T}]$ is connected (it is a Z_u°-orbit) we have even more precise information about h when $U_d(M)$ is not connected: in that case we know in which of the two connected components h lies. This situation arises in the exceptional case of 3.3.1(a), i.e., when $\mathrm{typ}(d) = (\mathrm{I}+)$ and $m_c(\lambda) = 2$; then $\mathrm{sg}(d)$ determines the component of h. This special case can also be characterised by the condition $\{d\} \in C_T$ (see the remark in 3.3), and since we need to refer to it quite often, we shall do so by the latter terse form; note however that it depends only on d and $\lambda = \mathrm{sh}(T)$. Similar statements hold for l. We shall write $d \approx e$ to indicate that d and e are equal up to their entries, i.e., that $\pi(d) = \pi(e)$ and $\mathrm{typ}(d) = \mathrm{typ}(e)$ and $\mathrm{sg}(d) = \mathrm{sg}(e)$.

4.3.1. Proposition.

(a) *We have $h = l$ for all choices of $f \in \mathcal{F}_{u,T}$ and $f' \in \mathcal{F}_{u,T'}$ if and only if either $r = r' = 1$ or $r = r' = 2 \wedge d \approx e \wedge \{d\} \in C_T$.*

(b) *We have $h \perp l$ for all choices of $f \in \mathcal{F}_{u,T}$ and $f' \in \mathcal{F}_{u,T'}$ if and only if either $c \neq 1$ or $c' \neq 1$ or $d \approx e \wedge \{d\} \in C_T$.*

Proof. (a) We have $h = l$ for all f, f' if and only if $\alpha_u[\mathcal{F}_{u,T}] = \alpha_u[\mathcal{F}_{u,T'}] = \{p\}$ for some point p. Now, as we have seen in 2.3, $\dim(U_d) = 0$ holds only if $r = 1$ or $r = 2 \wedge \mathrm{typ}(d) = (\mathrm{I}+)$, where the latter implies $\{d\} \in C_T$. When $r = r' = 1$ we necessarily have $d \approx e$ and $h = l$. When $r = r' = 2$ and $\mathrm{typ}(d) = \mathrm{typ}(e) = (\mathrm{I}+)$ we have $\alpha_u[\mathcal{F}_{u,T}] = \alpha_u[\mathcal{F}_{u,T'}]$ only if $\mathrm{sg}(d) = \mathrm{sg}(e)$, i.e., if $d \approx e$; this also implies $h = l$. (b) When $c \neq 1$ we have $h \subseteq W_c(M) \subseteq \mathrm{Im}(\eta_M) \perp \mathrm{Ker}(\eta_M) \supseteq l$ implying $h \perp l$, which follows similarly if $c' \neq 1$. Now assume $c = c' = 1$, and let $\bar{h}$ and $\bar{l}$ be the respective images of h and l in $V_1(M)$; we have $h \perp l$ if and only if $\bar{h} \perp \bar{l}$ with respect to $b_{1,M}$. Since $b_{1,M}$ is non-degenerate, and $\bar{h}$ and $\bar{l}$ are isotropic lines (so $\dim(V_1(M)) \geq 2$), we only have $\bar{h} \perp \bar{l}$ for all choices if we are forced to have $\bar{h} = \bar{l}$; as in (a) this is when $d \approx e \wedge \{d\} \in C_T$. $\square$

This proposition indicates in which cases $\pi_1(f, f_1') = \pm|d|$ for generic f, f' (recall that $|d| = \max(I(f))$), i.e., when $\gamma(\mathcal{F}_{u,T}, \mathcal{F}_{u,T'}) = w$ for some w with $|w_n| = |d|$. In the remaining cases—when we have $h \perp l$, but not $h = l$ for all choices—we want to apply induction in order to compute $\pi_1(f, l)$. Consider a fixed value of h; since the set $\alpha_u[\mathcal{F}_{u,T}]$ is a Z_u°-orbit, any choice will do equally well. By the definition of π_1, we should now consider the flag $f^\downarrow$ and the line $l' = (l + h)/h$ in $h^\perp/h$. Putting $\lambda' = J(u_{[h]}) = \mathrm{sh}(T^\downarrow)$, then by 3.5.1 there exists for any $f \in \alpha_u^{-1}[h]$, a suitable induced isomorphism $h^\perp/h \xrightarrow{\sim} M_{\lambda'}$ for which $f^\downarrow \in \mathcal{F}_{u_{[h]},T^\downarrow}$. Moreover (21) implies that, given such an isomorphism, all flags in $\mathcal{F}_{u_{[h]},T^\downarrow}$ can be obtained as $\tilde{f}^\downarrow$ for some $\tilde{f} \in \alpha_u^{-1}[h] \cap \mathcal{F}_{u,T}$. Now in order to proceed with the computation of w_n, we shall have to determine the $Z_{u_{[h]}}$-orbit of l'. Also, once w_n is determined, the definition of π_2 calls for a reduction towards $l^\perp/l$, and calculation of $f_{[l]}$, whose 1-dimensional part is $h' = h + l/l$; therefore we shall want to determine its $Z_{u_{[l]}}$-orbit as well. If these orbits of h' and l' are good they can be described, as stated in 3.3, as $U_{d'}(l^\perp/l)$ and $U_{e'}(h^\perp/h)$ respectively, for certain dominoes d' and e' (in fact only their supports are needed to specify an orbit). Our next step will be to determine such dominoes; fortunately it turns out that for any T, T' they exist under a dense condition on (h, l). Evidently there is a certain symmetry between the problems of determining d' and e'.

We first consider the simplest case. Note that, since both d and e lie at the periphery of λ, the conditions $r = r'$, $c = c'$, $\pi(d) = \pi(e)$, and $\mathrm{supp}(d) \cap \mathrm{supp}(e) \neq \emptyset$ are all equivalent.

4.3.2. Proposition. *Assume that $c \neq c'$. Then for all choices of f and f' we have $h \perp l$ but $h \neq l$; furthermore with h' and l' as defined above we have $h' \in U_d(l^\perp/l)$ and $l' \in U_e(h^\perp/h)$.*

Proof. The first statement is immediate from 4.3.1. Now using 2.3.6, we get from (a) that $h \in U_c(l^\perp)$ and from (c) and (d) that $h' = \Pi[h] \in U_c(l^\perp/l)$. Now to prove $h' \in U_d(l^\perp/l)$ it remains to show $\mathrm{typ}(h') = \mathrm{typ}(d)$, where $\mathrm{typ}(d) = \mathrm{typ}(h)$ by definition. Consider an $x \in h \setminus \{0\}$ and $y \in l^\perp$ with $\eta_M^{c-1}(y) = x$; then $b_{c,l^\perp/l}(\overline{\Pi(x)}, \overline{\Pi(x)}) = b_{[l]}(\Pi(x), \Pi(y)) = b_M(x, y) = b_{c,M}(\overline{x}, \overline{x})$, where the barred expressions denote images in the spaces $V_c(l^\perp/l)$ and $V_c(M)$ respectively. This proves the statement about h', we similarly get $l' \in U_e(h^\perp/h)$. $\square$

We now consider the remaining case $c = c'$, with moreover $h \perp l$ but $h \neq l$. Then let $\bar{h}$ and $\bar{l}$ denote the (nonzero) images of h and l in $V_c(M)$; as we have seen in 2.3.8(a), the images of $W_c(h^\perp)$ and $W_c(l^\perp)$ in that space are $\bar{h}^\perp$ and $\bar{l}^\perp$ respectively. We may formulate the questions pertaining to the orbits of h' and l' in terms of $\bar{h}$ and $\bar{l}$.

4.3.3. Proposition. *Let h, l and related symbols be as specified above.*
(a) *The three conditions $h' \subseteq W_{c+1}(l^\perp/l)$, $l' \subseteq W_{c+1}(h^\perp/h)$ and $\bar{h} = \bar{l}$ are equivalent.*
(b) *We have $h' \subseteq W_c(l^\perp/l)$ if and only if either $\mathrm{typ}(l) = (\mathrm{N})$ or $\bar{h} \perp \bar{l}$. Similarly $l' \subseteq W_c(h^\perp/h)$ if and only if either $\mathrm{typ}(h) = (\mathrm{N})$ or $\bar{h} \perp \bar{l}$.*

Proof. (a) The image of the plane $h + l$ in $V_c(M)$ is $\bar{h} + \bar{l}$. The condition $\bar{h} = \bar{l}$ is equivalent to $\dim(\bar{h} + \bar{l}) = 1$, which is equivalent to $h + l$ having a non-trivial intersection with $W_{c+1}(M)$. Using 2.3.6(a,c,d) for $i = c + 1$ and the fact that $h' = \Pi[h + l]$, we find that this condition is equivalent to $h' \subseteq W_{c+1}(l^\perp/l)$. The other equivalence is entirely similar. (b) It is sufficient to prove the first statement. As $h' = \Pi[h + l]$ we have $h' \subseteq W_c(l^\perp/l)$ if and only if $h + l \subseteq \Pi^{-1}[W_c(l^\perp/l)]$. Now if $\mathrm{typ}(l) = (\mathrm{N})$ we have by 2.3.6(d,a) that $\Pi^{-1}[W_c(l^\perp/l)] = W_c(l^\perp) \oplus l = W_c(M)$, whence this condition is always satisfied. On the other hand, if $\mathrm{typ}(l) \neq (\mathrm{N})$ we have $\Pi^{-1}[W_c(l^\perp/l)] = W_c(l^\perp) \supseteq l$, whence by 2.3.8(a) the condition is equivalent to $\bar{h} \perp \bar{l}$. □

Note that we are confronted with questions very similar to the ones dealt with in 4.3.1, but with V_i and its bilinear form replacing M, and with additional possibility of $\bar{h}$ or $\bar{l}$ being non-isotropic lines, which is not possible for h or l. We consider $\mathrm{typ}(h')$ and $\mathrm{typ}(l')$ next.

4.3.4. Lemma. *Let h, l and related symbols be as specified above.*
(a) *If $\mathrm{typ}(l) \neq (\mathrm{N})$ and $\bar{h} \not\perp \bar{l}$ then $h' \in U_{c-1}(l^\perp/l)$, and $\mathrm{typ}(h') \neq (\mathrm{N})$. Similarly if $\mathrm{typ}(h) \neq (\mathrm{N})$ and $\bar{h} \not\perp \bar{l}$ then $l' \in U_{c-1}(h^\perp/h)$, and $\mathrm{typ}(l') \neq (\mathrm{N})$.*
(b) *Assume that $\bar{h} \neq \bar{l}$. If $\mathrm{typ}(l) = (\mathrm{N})$, then $\tilde{h} = (h+l) \cap W_j(l^\perp)$ is a line in $W_j(M)$, and $\mathrm{typ}(h') = \mathrm{typ}(\tilde{h})$. Similarly, if $\mathrm{typ}(h) = (\mathrm{N})$, then $\tilde{l} = (h + l) \cap W_j(h^\perp)$ is a line in $W_j(M)$, and $\mathrm{typ}(l') = \mathrm{typ}(\tilde{l})$. Moreover, in either of the two cases, the condition $\mathrm{typ}(\tilde{h}) = (\mathrm{I+})$ respectively $\mathrm{typ}(\tilde{l}) = (\mathrm{I+})$ is equivalent to the degeneracy of $b_{c,M}$ when restricted to $\bar{h} + \bar{l}$.*

Proof. We prove the statements concerning h'. (a) From 4.3.3 we get $h' \not\subseteq W_c(l^\perp/l)$, while $h' \subseteq \Pi[W_{c-1}(M)] = W_{c-1}(l^\perp/l)$ by 2.3.6. Since the image of h in $V_{c-1}(M)$ is zero, the image of h' in $V_{c-1}(l^\perp/l)$ is isotropic. (One may check that the latter image equals $S^\perp \subseteq S$, where S is as in 2.3.8(c)). (b) We have seen in the proof of 4.3.3 that $\mathrm{typ}(l) = (\mathrm{N})$ implies $W_c(M) = W_c(l^\perp) \oplus l$, and $\tilde{h}$ is the corresponding projection of h on $W_j(l^\perp)$, which is indeed a line. The proof of $\mathrm{typ}(h') = \mathrm{typ}(\tilde{h})$ is identical to the proof of $\mathrm{typ}(h') = \mathrm{typ}(h)$ in 4.3.2. Now if $\mathrm{typ}(\tilde{h}) = (\mathrm{I+})$, then the image of $\tilde{h}$ in $V_c(M)$ is both isotropic and perpendicular to $\bar{l}$, and therefore perpendicular to all of $\bar{h} + \bar{l}$, whence $b_{c,M}$ is degenerate on $\bar{h} + \bar{l}$. On the other hand, if $\mathrm{typ}(\tilde{h}) = (\mathrm{N})$, then $\bar{h} + \bar{l}$ contains two perpendicular non-isotropic lines, and therefore $b_{c,M}$ is non-degenerate on it. □

Note that this lemma, in conjuction with 4.3.3, completely determines $\mathrm{typ}(h')$ and $\mathrm{typ}(l')$ in the cases treated, despite the negative formulations. This is because given $h' \in U_j(l^\perp/l)$ at most two possibilities for $\mathrm{typ}(h')$ remain.

4.3.5. Proposition. *The map* $\{ h \in U_c(M) \mid \bar{h} = \bar{l} \wedge h \neq l \} \to \mathbf{P}\big(W_{c+1}(l^\perp/l)\big)$ *given by* $h \mapsto h'$ *is a (Zariski-) continuous surjection.*

Proof. The map is well-defined by 4.3.3(a), and the set describing its domain can also be written as $\mathbf{P}\big((W_{c+1}(M) \oplus l) \setminus (W_{c+1}(M) \cup l)\big)$; since 2.3.6 shows that $W_{c+1}(l^\perp/l) \cong W_{c+1}(M)$ canonically, the map corresponds to projection on the first factor of $W_{c+1}(M) \oplus l$, and is evidently a continuous surjection. □

4.4. *Numerous cases.*

Il y a beaucoup de cas à considérer.
N. Spaltenstein, *[Spa II.6.25]*

We have now collected sufficient information to determine, in all cases under consideration, dominoes d' and e' such that, for (f, f') in a dense subset of $\mathcal{F}_{u,T} \times \mathcal{F}_{u,T'}$, we have $h' \in U_{d'}(l^\perp/l)$ and $l' \in U_{e'}(h^\perp/h)$. There are quite a few cases to be distinguished, depending on the dominoes d and e. To facilitate reference to these cases, we provide them with a fixed numbering.

4.4.1. Definition. *Let* d *and* e *be two dominoes at the periphery of* λ, *and let* $(r, c) = \pi(d)$ *and* $(r', c') = \pi(e)$. *We say that* (d, e) *falls into one of the following numbered cases if the specified condition holds. The negation of condition 0 is implicitly assumed in all other cases.*

0. $\mathrm{supp}(d) \cap \mathrm{supp}(e) = \emptyset$ *(or equivalently* $c \neq c'$*).*
1. $\mathrm{typ}(d) = \mathrm{typ}(e) = (\mathrm{N})$ *and* $m_c(\lambda) > 1$.
2. $m_c(\lambda) = 1$ *(implying* $\mathrm{typ}(d) = \mathrm{typ}(e) = (\mathrm{N})$*) and* $r > 1$.
3. $\mathrm{typ}(d) = \mathrm{typ}(e) = (\mathrm{I}+)$, $m_c(\lambda) > 2$ *and* $c > 1$.
4. $\mathrm{typ}(d) = \mathrm{typ}(e) = (\mathrm{I}+)$, $m_c(\lambda) = 2$ *(implying* $\{d\} \in C_T$*), and either*
   4a. $\mathrm{sg}(d) = \mathrm{sg}(e)$ *and* $r > 2$*; or*
   4b. $\mathrm{sg}(d) \neq \mathrm{sg}(e)$ *and* $c > 1$.
5. $\mathrm{typ}(d) = \mathrm{typ}(e) = (\mathrm{I}-)$ *and* $c > 1$.
6. $\mathrm{typ}(d) = (\mathrm{I}+)$ *and* $\mathrm{typ}(e) = (\mathrm{N})$.
7. $\mathrm{typ}(d) = (\mathrm{N})$ *and* $\mathrm{typ}(e) = (\mathrm{I}+)$.

This list excludes all cases where 4.3.1(a) applies, or where 4.3.1(b) does not apply. We give those cases a 'primed' numbering.

2'. $m_c(\lambda) = 1$ *and* $r = 1$.
3'. $\mathrm{typ}(d) = \mathrm{typ}(e) = (\mathrm{I}+)$, $m_c(\lambda) > 2$ *and* $c = 1$.
4'. $\mathrm{typ}(d) = \mathrm{typ}(e) = (\mathrm{I}+)$, $m_c(\lambda) = 2$ *and either*
   4'a. $\mathrm{sg}(d) = \mathrm{sg}(e)$ *and* $r = 2$*; or*
   4'b. $\mathrm{sg}(d) \neq \mathrm{sg}(e)$ *and* $c = 1$.
5'. $\mathrm{typ}(d) = \mathrm{typ}(e) = (\mathrm{I}-)$ *and* $c = 1$.

The results so far can be summarised in the following table. The first three columns recapitulate the main characteristics of the cases (and in cases *0* and *4b* the fourth column serves this purpose as well), the fourth column gives a dense condition on h and l under which condition the orbits of h' and l' can be uniquely determined, and the remaining columns specify the dominoes that describe those orbits. When these dominoes coincide, two adjacent entries in the table are merged.

case	typ(h)	typ(l)	$m_c(\lambda)$	*condition*	$\pi(d')$	$\pi(e')$	typ(d')	typ(e')
0	$*$	$*$	$*$	$c \neq c'$	$\pi(d)$	$\pi(e)$	typ(d)	typ(e)
1	(N)	(N)	>1	$\star$	$(r-1, c)$		(N)	
2	(N)	(N)	1	$\dagger$	$(r-1, \lambda_{r-1})$		$\ddagger$	
3	(I+)	(I+)	>2	$\bar{h} \not\perp \bar{l}$	$({}^t\lambda_{c-1}, c-1)$		(I−)	
4a	(I+)	(I+)	2	$\dagger$	$(r-1, \lambda_{r-1})$		$\ddagger$	
4b	(I+)	(I+)	2	$\bar{h} \neq \bar{l}$	$({}^t\lambda_{c-1}, c-1)$		(I−)	
5	(I−)	(I−)	≥ 2	$\bar{h} \not\perp \bar{l}$	$({}^t\lambda_{c-1}, c-1)$		(I+)	
6	(I+)	(N)	≥ 2	$\bar{h} \not\perp \bar{l}$	$(r-1, c)$	$(r, c-1)$	(N)	(I−)
7	(N)	(I+)	≥ 2	$\bar{h} \not\perp \bar{l}$	$(r, c-1)$	$(r-1, c)$	(I−)	(N)

$*$ = any value

$\dagger$: $\mathbf{P}(h+l)$ intersects $\mathbf{P}\big(W_{c+1}(M)\big)$ in its dense Z_u-orbit

$\star$: $\bar{h} \neq \bar{l}$ and $b_{c,M}$ non-degenerate on $\bar{h} + \bar{l}$

$\ddagger$: if $\varepsilon_{\lambda_{r-1}} = +1$ then (N) else (I−)

4.4.2. Lemma. *Assume that (d, e) falls into one of the unprimed cases of 4.4.1. The condition specified in the table above defines in each case a dense subset of all possible pairs (h, l), and for all pairs in that subset the lines h' and l' are well-defined. For such h' and l', and for dominoes d', e' whose attributes are as given in the table, we have $h' \in U_{d'}(l^\perp/l)$ and $l' \in U_{e'}(h^\perp/h)$.*

Proof. Note that since $J(u_{[l]})$ and $J(u_{[h]})$ are known by virtue of 2.3.7, and d' and e' lie at their respective peripheries, either coordinate of $\pi(d')$ or $\pi(e')$ determines the other. Now case *0* is treated in 4.3.2, case *1* is treated in 4.3.4(b) and cases *3* and *5* are treated by 4.3.4(a), which also treats one half of cases *6* and *7*. The other half of those cases is treated by 4.3.4(b) because in those cases $\bar{h} \not\perp \bar{l}$ is equivalent to the condition marked '$\star$': given an isotropic line and any other line in $\bar{h} + \bar{l}$, that plane is degenerate if and only if the two lines are perpendicular. (Incidentally this shows that the conditions in *3* and *5* could also have been rendered as '$\star$'.) In case *4* the signs $\mathrm{sg}(d)$ and $\mathrm{sg}(e)$ determine for $\bar{h}$ and $\bar{l}$ respectively which of the two isotropic lines of $V_c(M)$ they are; therefore $\bar{h} = \bar{l}$ if and only if case *4a* applies. Case *4b* is like case *3*, except that $\bar{h} \not\perp \bar{l}$ holds always, rather than just on a dense subset. In cases *2* and *4a*, where we always have $\bar{h} = \bar{l}$, we may apply 4.3.5 to see that for each l the set of $h \neq l$ for which h' lies in the dense $Z_{u_{[l]}}$-orbit in $\mathbf{P}\big(W_{c+1}(l^\perp/l)\big)$ is dense. In the canonical identification $\mathbf{P}\big(W_{c+1}(l^\perp/l)\big) \cong \mathbf{P}\big(W_{c+1}(M)\big)$—which maps $Z_{u_{[l]}}$-orbits to Z_u orbits—h' corresponds to the unique element of $\mathbf{P}(h+l) \cap \mathbf{P}\big(W_{c+1}(M)\big)$, which appears in the condition '$\dagger$' of the table; this also shows that the same condition works for l' as well. Since $W_{c+1}(l^\perp/l) = W_{\lambda_{r-1}}(l^\perp/l) \supset W_{\lambda_{r-1}+1}(l^\perp/l)$, the dense

$Z_{u_{[l]}}$-orbit in $\mathbf{P}(W_{c+1}(l^\perp/l))$ is contained in $U_{\lambda_{r-1}}(l^\perp/l)$, and if $\varepsilon_{\lambda_{r-1}} = +1$, it is $U^{\mathrm{N}}_{\lambda_{r-1}}(l^\perp/l)$. This completes the proof for cases *2* and *4a*. □

Note that this lemma implies in particular that, under the stated condition, the orbits of h' and l' are good. The given table almost, but not quite, allows us to compute w_n by induction, using π_1. Recall that in cases *4* and *4'*, where the Z_u-orbit of l has two connected components, we needed to know in which of these components l lies in order to distinguish the two subcases. Similarly we need to know in which component of its $Z_{u_{[h]}}$-orbit l' lies, in case that orbit is not connected, i.e., when $\mathrm{typ}(e') = (\mathrm{I}+)$ and $m_{\kappa_{e'}}(\lambda') = 2$. Even though, at this point, e' does not belong to any tableau, it is natural to specify the connected component by giving $\mathrm{sg}(e')$, because then for any tableau $e' : S$ that orbit equals $\alpha_{u'}[\mathcal{F}_{u',e':S}]$, where $u' = u_{[h]}$. Inspection of the table shows that the only cases in which we need to specify $\mathrm{sg}(e')$ are *0* and *5*.

If in case *0* the orbit of l' is not connected, then neither is the orbit of l, and there is a canonical isomorphism $V_{c'}(M) \xrightarrow{\sim} V_{c'}(h^\perp/h)$. This isomorphism maps $\bar{l}$ to the image $\bar{l}'$ of l'; therefore we should take $\mathrm{sg}(e') = \mathrm{sg}(e)$ in this case. In case *5*, if the orbit of l' is not connected, i.e., when $\lambda_{c-1} = 0$, then it follows from 3.4.2 that for any induced isomorphism $h^\perp/h \xrightarrow{\sim} M_{\lambda'}$ the image of l' in $V_{c-1}(h^\perp/h)$ corresponds to the image of the special line l_+ of the summand $M_{(c-1,c-1)}$; this line belongs to the domino e' provided that we put $\mathrm{sg}(e') =$ '+'. For definiteness we always put $\mathrm{sg}(e') = \mathrm{sg}(e)$ in case *0* and $\mathrm{sg}(e') =$ '+' in case *5*, whether or not the orbit of l' is connected. We can now formulate a partial result.

4.4.3. Lemma. *Given the tableau T and the domino $e = T'_1$, the final integer w_n of the signed permutation representing $w = \gamma(\mathcal{F}_{u,T}, \mathcal{F}_{u,T'})$ can be computed as follows. Put $d = T_1$, and determine the case of 4.4.1 that (d, e) falls into. If this is a primed case then $w_n = |d|$ in cases 2′ and 4′a, and $w_n = -|d|$ in cases 3′, 4′b, and 5′. Otherwise let e' be a domino whose position and type are given by the table for the case applying, and whose sign is given by the rule stated above; then repeat this computation replacing the pair (T, e) by $(T^\downarrow, e')$.*

Proof. In view of 4.1.3 and the definition of π_2, the claim can be reformulated as follows: for (f, l) in a dense subset of $\mathcal{F}_{u,T} \times \alpha_u[\mathcal{F}_{u,T'}]$ the computed value w_n equals $\pi_1(f, l)$. We shall prove for any chosen $l \in \alpha_u[\mathcal{F}_{u,T'}]$, that there is a dense subset $\mathcal{D}_l$ of $\mathcal{F}_{u,T}$ such that $\pi_1(f, l) = w_n$ for all $f \in \mathcal{D}_l$. Now if one of the primed cases applies then we take $\mathcal{D}_l = \mathcal{F}_{u,T}$ if $w_n = |d|$, or $\mathcal{D}_l = \bigcup_{h \not\perp l} \alpha_u^{-1}[h]$ if $w_n = -|d|$, and the claim follows from 4.3.1 and the definition of π_1. Otherwise the table specifies a dense condition on $h \in \alpha_u[\mathcal{F}_{u,T}]$ (for fixed l), under which by 4.4.2 l' is defined and $l' \in U_{e'}(h^\perp/h)$. It suffices to prove that $\pi_1(f, l) = w_n$ for any such h and f in a dense subset of $\alpha_u^{-1}[h] \cap \mathcal{F}_{u,T}$, for then we can take $\mathcal{D}_l$ to be the union over h of these subsets. If $\alpha_u^{-1}[h] \cap \mathcal{F}_{u,T}$ has two connected components we prove this separately for each of them. We choose an isomorphism that maps h to the special line belonging to d and such that $f^\downarrow \in \mathcal{F}_{u',T^\downarrow}$ with $u' = u_{[h]}$ for all f in the considered component. This is possible by 3.5.1, and the map $f \mapsto f^{\downarrow}$ is an isomorphism of that component to $\mathcal{F}_{u',T^{\downarrow}}$. From the discussion of $\mathrm{sg}(e')$ it follows that l' lies in $\alpha_{u'}[\mathcal{F}_{u',e':S}]$ for any tableau S of appropriate shape. We may now apply induction to $T^{\downarrow}$ and e', which gives a dense subset $\mathcal{D}_{l'}$ of $\mathcal{F}_{u',T^{\downarrow}}$ such that $\pi_1(f^{\downarrow}, l') = w_n$ for all $f^{\downarrow} \in \mathcal{D}_{l'}$. Since $\pi_1(f,l) = \pi_1(f^{\downarrow}, l')$ by definition of π_1 this establishes the claim. Note that by construction $\mathcal{D}_l$ is $(Z_u^\circ)_l$-stable. □

This lemma and its proof illustrate the line of reasoning we shall use, but in itself the lemma is of limited use, since it fails to give information about the flag $f_{[l]}$ in $l^{\perp}/l$. In order to proceed according to the definition of π_2, we would like to find a tableau $T_{[e]}$ such that for all l, and for f in a dense subset of $\mathcal{F}_{u,T}$, we have $f_{[l]} \in \mathcal{F}_{u_{[l]},T_{[e]}}$. Now such a tableau exists if and only if we have $f_{[l]} \in \widetilde{\mathcal{F}}_{u_{[l]}}$ on a dense set, and as we shall see presently this is in fact the case, following by induction from the fact that—for f in a dense set—h' lies in a good orbit. The condition $f_{[l]} \in \mathcal{F}_{u_{[l]},T_{[e]}}$ assumes the choice of some induced isomorphism $l^{\perp}/l \xrightarrow{\sim} M_\mu$ with $\mu = J(u_{[l]})$, and we require that it is such that also $f'^{\downarrow} \in \mathcal{F}_{u_{[l]},T'^{\downarrow}}$. The latter requirement however, is impractical since it involves the flag f', while at this point we are dealing with f and l only. But since $\mathrm{sg}(e)$ is given, and thereby the special line to which l should be mapped, the class of induced isomorphisms $l^{\perp}/l \xrightarrow{\sim} M_\mu$ is determined up to the action of the group K of 3.1.2. Moreover, assuming the existence of $T_{[e]}$, K will act trivially on $[T_{[e]}]$ because the dense subset of $\mathcal{F}_{u,T}$ in which f should lie may be taken to be $(Z_u^\circ)_l$-stable, in which case the same holds for the set of flags $f_{[l]}$. In view of this fact the condition that $T_{[e]}$ must satisfy can be formulated simpler: for f in a dense subset of $\mathcal{F}_{u,T}$ we must have $l : f_{[l]} \in \mathcal{F}_{u,e:T_{[e]}}$.

We shall now determine $T_{[e]}$ (which also proves its existence). Let $l \in \alpha_u[\mathcal{F}_{u,T'}]$ be chosen arbitrarily, and let f lie in the subset $\mathcal{D}_l$ of the proof of 4.4.3; it will not be necessary to make any further restrictions. If we have $w_n = |d|$ then $h = l$ and $f_{[l]} = f^{\downarrow}$, and therefore we may take $T_{[e]} = T^{\downarrow}$. If we have $w_n = -|d|$ then $h \not\perp l$ and we have canonical isomorphisms $h^{\perp}/h \xleftarrow{\sim} (h+l)^{\perp} \xrightarrow{\sim} l^{\perp}/l$. It is easily verified that via these isomorphisms one and the same flag corresponds to $f^{\downarrow}$ and to $f_{[l]}$. Also the plane $h + l$ is an orthogonal direct summand of M, and there exists an automorphism of $h+l$ that interchanges h and l; it extends to an automorphism of M that interchanges $f = h : f^{\downarrow}$ and $l : f_{[l]}$. In case *5′* this automorphism appears in Z_u°, so we may take $T_{[e]} = T^{\downarrow}$ in this case too. In cases *3′* and *4′b* this automorphism represents $g_1 \in A_u$, whose action on $[T]$ affects the sign of $\mathrm{cl}(d)$, so we may take $T_{[e]} = T^{\downarrow}$ provided that $\mathrm{sg}(d) \neq \mathrm{sg}(e)$. Now in case *4′b* this always holds, and if in case *3′* we have $\mathrm{sg}(d) = \mathrm{sg}(e)$, then we can replace T by another representative $\tilde{T}$ of $[T]$ for which $\mathrm{sg}(\tilde{T}_1) \neq \mathrm{sg}(e)$—which is possible since $\{d\} \notin C_T$—and then take $T_{[e]} = \tilde{T}^{\downarrow}$.

We have now determined $T_{[e]}$ in the terminating cases of the recursive computation in 4.4.3, but the other cases are more difficult. It turns out that we can build up $T_{[e]}$ on the "way back" of that recursion. As remarked before, h' is the 1 dimensional part of $f_{[l]}$. The other parts of $f_{[l]}$ are determined by the relation

$$f_{[l]}{}^{\downarrow} = f^{\downarrow}{}_{[l']}, \tag{29}$$

where both sides are considered as flags in $L^{\perp}/L$ with $L = h + l$. This relation is a special case of the more general equality $f_{[l][h']} = f_{[h][l']}$ that holds for any $f \in \mathcal{F}$, whether or not $h = f_1$: for the part labeled by i we have $f_{[l][h']}{}^{i} = ((f^i \cap l^{\perp} + l) \cap h^{\perp} + h)/L = (f^i \cap L^{\perp} + L)/L$ and similarly $f_{[h][l']}{}^{i} = (f^i \cap L^{\perp} + L)/L$. Since we have already determined the domino d' that describes the $Z^{\circ}_{u_{[l]}}$-orbit of h', it follows that we can determine the position of $f_{[l]}$ once we have done so for $f^{\downarrow}{}_{[l']}$.

Assume a tableau S is determined such that $l' : f^{\downarrow}{}_{[l']} \in \mathcal{F}_{u_{[h]}, e':S}$ for the chosen class of isomorphisms $h^{\perp}/h \xrightarrow{\sim} M_{\lambda'}$ (the 'systematic' name for S would be $T^{\downarrow}{}_{[e']}$). Then by (21) we have $h : l' : f^{\downarrow}{}_{[l']} \in \mathcal{F}_{u,d:e':S}$. We shall determine a tableau $\tilde{S}$ and $\operatorname{sg}(d')$ such that $l : f_{[l]} = l : h' : f_{[l]}{}^{\downarrow} \in \mathcal{F}_{u,e:d':\tilde{S}}$. This is achieved by applying 3.5.2 to the pair $(h : l' : f^{\downarrow}{}_{[l']}, l : h' : f_{[l]}{}^{\downarrow})$ of flags. In order to do so we must have $\operatorname{sg}(d) \neq \operatorname{sg}(e)$ in case *3*, which can be achieved, as in case *3'*, by modifying T within its class if necessary. We assume that such an appropriate T has been chosen, and that all dependent objects (in particular S) are adapted to this choice. Then using 3.5.2 we find the following result. The sign $\operatorname{sg}(d')$ is defined analogously to $\operatorname{sg}(e')$, namely $\operatorname{sg}(d') = \operatorname{sg}(d)$ in case *0*, $\operatorname{sg}(d') =$ '$+$' in case *5*, and $\operatorname{sg}(d') =$ '$\circ$' otherwise; also put $|d'| = |d|$. In case *3* put $\tilde{S} = \xi_r(S)$ where $r = {}^t\lambda_{c-1}$, and likewise in case *6* if $\operatorname{sg}(d) =$ '$-$' and in case *7* if $\operatorname{sg}(e) =$ '$-$'; in the other cases put $\tilde{S} = S$. Then indeed $l : f_{[l]} \in \mathcal{F}_{u,e:d':\tilde{S}}$, and therefore $T_{[e]} = d' : \tilde{S}$ is as required. We have proved the following statement.

4.4.4. Lemma. *Let $l \in \alpha_u[\mathcal{F}_{u,T'}]$ and $f \in \mathcal{D}_l \subseteq \mathcal{F}_{u,T}$ as in the proof of 4.4.3. Let $T_{[e]}$ be computed from T and $e = T'_1$ as described above, where in the applicable cases S is computed recursively by the same procedure with $(T^{\downarrow}, e')$ replacing (T, e). Then for any isomorphism $l^{\perp}/l \xrightarrow{\sim} M_{\mu}$ induced by an isomorphism $M \xrightarrow{\sim} M_{\lambda}$ sending l to the special line belonging to e, we have $f_{[l]} \in \mathcal{F}_{u_{[l]}, T_{[e]}}$.* □

4.5. *Recursive algorithms.*

"Mine is a long and sad tale!" said the Mouse,
turning to Alice and sighing.
"It is a long tail, certainly," said Alice,
looking down with wonder at the Mouse's tail;
"But why do you call it sad?"
Lewis Carroll, *Alice in Wonderland*

Up to this point we have been rather informal in our description of recursive computations. The level of complexity that has been reached, however, justifies a more formal approach. We shall use a formalism that bears resemblance to the kind of algorithmic languages that are used in computer science, in particular to those called 'functional programming languages', but we shall keep as close as possible to the conventional mathematical notation. (See [Ba-vL] for an introduction to functional programming, and for a complete and formal description of such a language, "TALE", after which our formalism is modelled; however, we assume no acquaintance with this subject.)

Algorithms will be specified as *functions*, by giving for each one an equation whose left-hand side is the name of the function applied to a formal parameter (or a tuple of formal parameters), and whose right-hand side is an expression giving the function value expressed in terms of that parameter. The algorithmic nature of the definition lies in the fact that the right-hand side may contain (recursive) applications of the function being defined. Such applications must occur at a position that is needed only conditionally, and moreover it is necessary that the arguments of those applications are simpler in some respect than the initial one, lest the computation would fail to terminate. The algorithm may then be performed, for some given argument, by evaluating the right hand side, and performing any recursive applications separately when needed.

We evidently need to have conditional expressions. The usual way of specifying these, cf. (25), becomes unwieldy when it is embedded in another expression. Therefore we shall use the following notation instead: '**if** C **then** E_1 **else** E_2 **fi**', where C is a condition and E_1 and E_2 are expressions denoting values of the same kind. The meaning of this expression is simple: the condition is evaluated; if it is true then the value of the conditional expression is that of E_1, and else it is that of E_2 (the '**fi**' indicates the end of the conditional part). Another kind of expression that will be very useful is the local definition. Recall that we have frequently used variables like h and l that depend on others (viz. f and f'), where their relationship is explained in the accompanying text. However, when we need variables in the right-hand side of a function definition that depend on its parameter, then we need a formal notation to express this relationship. In such a circumstance we shall write '**let** $V = E_1$ **in** E_2'. Here V is the newly defined variable, whose value is given by the expression E_1, which may depend on any other variables that are defined at that point, and the expression E_2, in which V may be used, gives the final value to be yielded. A local definition has no closing symbol, but we shall take for E_2 the longest single expression that follows '**in**'; it is not uncommon that E_2 itself starts with '**let**...', in which case we contract the sequence '**in let**' to ';'. Using these expressions we can now give a formal definition of π_1, that was defined informally in 4.1:

$$\begin{aligned}\pi_1(f,l) = {}&\textbf{let } m = \max\bigl(I(f)\bigr);\ h = f_1;\ l' = l + h/h \textbf{ in}\\ &\textbf{if } h = l \textbf{ then } m \textbf{ else if } h \perp l \textbf{ then } \pi_1(f^{\downarrow}, l') \textbf{ else } -m \textbf{ fi fi}.\end{aligned} \tag{30}$$

Note that we freely included the usual notations into the new formalism. We can similarly formalise the definition of π_2, but since it yields a sequence of numbers, we need expressions that construct such sequences. For this purpose we extend the notation adopted for tableaux to lists or sequences in general: '$\odot$' denotes the sequence without any terms, while '$x : t$' denotes the sequence $(x, t_1, \ldots, t_m)$ with the term x

as its "head", and the (possibly empty) sequence $t = (t_1, \ldots, t_m)$ as its "tail". We shall also use an operator '&' for extension of a sequence at the right-hand side: $t \mathbin{\&} x$ means $(t_1, \ldots, t_m, x)$. The definition of π_2 is now

$$\pi_2(f, f') = \textbf{if } I(f) = \{0\} \textbf{ then } \odot \textbf{ else let } l = f'_1 \textbf{ in } \pi_2(f_{[l]}, f'^{\downarrow}) \mathbin{\&} \pi_1(f, l) \textbf{ fi}. \quad (31)$$

Just as an example of a recursive definition dealing with sequences, taking them apart as well as putting them together, we write '&' in terms of the other operations:

$$t \mathbin{\&} x = \textbf{if } t = \odot \textbf{ then } x : \odot \textbf{ else let } (y : t') = t \textbf{ in } y : (t' \mathbin{\&} x) \textbf{ fi}. \quad (32)$$

This also illustrates a new use of the local definition: when the expression at the right of the '=' denotes a structured object, the variable being defined may be replaced by an expression containing newly defined variables (in this case y and t') and displaying (part of) that structure; the new variables are defined to be equal to the corresponding component of the expression at the right. We shall only use expressions of the form '$(x : t)$' (for a non-empty sequence) or of a form like '(x, y, z)' (for an element of a cartesian product) to replace the variable (the form $(t \& x)$ is also used in appendix A). A similar rule applies to variables being introduced as a formal parameter of a function; in this case any actual argument of the function must be similarly structured, and the new variables represent its components. Because of this rule there is no need to make a formal distinction between functions with single and multiple parameters.

There are two more constructions that will prove to be quite useful, even though they are not indispensible. Both have a bearing upon the distinction of several cases, an activity that we have been performing rather frequently, and shall continue to do in the sequel. If the cases are numerous, the conditional expression discussed above may become cumbersome, and therefore we generalise it in the following way. We may write '**case** $C_1{:}E_1, \ldots, C_m{:}E_m$ **esac**', where $C_1, \ldots, C_m$ are complementary conditions (exactly one of them is true), and $E_1, \ldots, E_m$ are expressions denoting values of one and the same kind. When condition C_i is true then the value of the case-expression is that of E_i.

The other construction is used when, in different cases, values of different kinds are to be yielded. For instance, the basic step of the procedure in 4.4.3 yields a different kind of value in the primed cases (namely a number w_n) than in the unprimed ones (namely a domino e'). In such cases, we may label the values with a 'tag' indicating which case applies; the meaning of each of the tags will be explained in the text. A tag is a short string (set in a different typeface than variables are) that is written before the expression it applies to, as if it denoted a map (indeed it may be viewed as an injection map into a disjoint union of sets). For instance, $\mathit{stop}(w_n)$ is a tagged value, where the value w_n is tagged by *stop* in order to indicate that a terminating case has been reached. If in a certain case there is no further value to be yielded, the tag may be written all by itself, simply indicating the case that applies. In order to unravel these tagged values again, we use a special variant of the case-expression, which has the form '**case** E **of** $t_1(V_1){:}\,E_1,\ldots,t_m(V_m){:}\,E_m$ **esac**'; the keyword '**of**' distinguishes this variant from the ordinary case-expression. Here E and the E_i are expressions, the t_i are distinct tags, and the V_i are defining positions of variables (like the left-hand side of a local definition), whose scope is the corresponding E_i. The expression E should yield a tagged value, where the tag matches one of the tags $t_1,\ldots,t_n$; when that value is $t_i(x)$, say, then the value of the whole case-expression equals that of '**let** $V_i = x$ **in** E_i'.

We are now ready to formalise the procedures described earlier. In the first place we define the function ξ, that performs the action of the operations ξ_r that were introduced below 3.3.2; the subscript r now appears as the first parameter. The second parameter is a tableau; it should have a shape λ for which $\varepsilon_{\lambda_r} = +1$ since ξ should represent the action of an element of A_u, but this condition is not tested.

$$
\begin{aligned}
\xi(r,T) = {}&\mathbf{if}\ T = \odot\ \mathbf{then}\ \odot \\
&\mathbf{else}\ \mathbf{let}\ (d : T^{\downarrow}) = T;\ i = \rho_d\ \mathbf{in} \\
&\qquad \mathbf{if}\ \mathrm{typ}(d) = (I{+}) \wedge r \in \{i-1, i\} \\
&\qquad \mathbf{then}\ \mathrm{dom}\big(\pi(d), (I{+}), |d|, -\,\mathrm{sg}(d)\big) : T^{\downarrow} \\
&\qquad \mathbf{else}\ d : \xi(r, T^{\downarrow}) \\
&\qquad \mathbf{fi} \\
&\mathbf{fi}
\end{aligned}
\tag{33}
$$

Note that we use $T^{\downarrow}$ as an ordinary variable rather than as an operator '$\downarrow$' applied to T, but that its value agrees with the outcome of that operation; in choosing variable names we shall use the convention that all variables with names containing '$\downarrow$' will be related similarly to the ones without.

Next we come to some auxiliary functions for the main case analysis as described in 4.4. These will be used to determine appropriate dominoes in either a given row or a given column of a diagram $Y(\lambda)$. The first is needed when we are moving upwards in the diagram, as in cases *1*, *2* and *4a*, and this function will be called Υ (Upsilon for 'up'). The second is needed when we are moving to the left, as in cases *3*, *4b* and *5*, and this function will be called Λ (for 'left'). Both functions return a tagged value, which is either of the form '$\mathit{stop}(w_n)$' indicating that w_n is directly determined, or of the form '$\mathit{cont}(d', e', \omega)$' otherwise (the tag abbreviating 'continue'). Here d' and e' are dominoes as before—in fact the entry $|e'|$ has no significance, and since for the rest $d' \approx e'$ in these cases, we have simply put $e' = d'$—and ω tells whether a modification is to be made in passing from $T^{\downarrow}{}_{[e']}$ to $T_{[e]}{}^{\downarrow}$ in the procedure of 4.4.4 (where these tableaux are called S and $\tilde{S}$ respectively). It is another kind of tagged value, that is either '$\mathit{flip}(r)$' indicating that ξ_r should be applied, or 'none' if no action is required. The parameters of Υ and Λ are triples, consisting of a shape λ, a row- or column-number (i or j), and an entry x to put into the domino d'. The construction and inspection of the $\mathit{flip}/\mathit{none}$-tagged values is facilitated by two more auxiliary functions g, g'.

$$
\begin{aligned}
g(C,i) &= \textbf{if } C \textbf{ then } \mathit{flip}(i) \textbf{ else } \mathit{none} \textbf{ fi} && (34)\\
g'(\omega,S) &= \textbf{case } \omega \textbf{ of } \mathit{none}{:}\, S,\ \ \mathit{flip}(r){:}\, \xi(r,S) \textbf{ esac} && (35)\\
\Upsilon(\lambda,i,x) &= \textbf{if } i=0 \textbf{ then } \mathit{stop}(+x) && (36)\\
&\quad\ \textbf{else let } j=\lambda_i\\
&\quad\quad\ ;\ d'=\mathrm{dom}\big((i,j),\ \textbf{if } \varepsilon_j=+1 \textbf{ then } (\mathrm{N}) \textbf{ else } (\mathrm{I}-) \textbf{ fi}, x, `\circ'\big)\\
&\quad\quad\ \textbf{in } \mathit{cont}(d',d',\mathit{none})\\
&\quad\ \textbf{fi}\\
\Lambda(\lambda,j,x) &= \textbf{if } j=0 \textbf{ then } \mathit{stop}(-x) && (37)\\
&\quad\ \textbf{else let } i={}^t\lambda_j\\
&\quad\quad\ ;\ (\alpha,\delta)=\textbf{if } \varepsilon_j=+1 \textbf{ then } ((\mathrm{I}+),`+') \textbf{ else } ((\mathrm{I}-),`\circ') \textbf{ fi}\\
&\quad\quad\ ;\ d'=\mathrm{dom}\big((i,j),\alpha,x,\delta\big) \textbf{ in } \mathit{cont}\big(d',d',g(\varepsilon_j=-1,i)\big)\\
&\quad\ \textbf{fi}
\end{aligned}
$$

Now we come to the main case analysis itself; it is given as the function ϕ_0 (here the subscript counts the levels of recursion). It takes as parameters a triple of a shape and two dominoes, and returns a pair, of which the first component is the same sort of value as returned by Υ and Λ. The second component of the pair is another value tagged by *flip* or *none*, and it indicates whether a different representative of T should be chosen (in cases *3*, *3'*); in fact it will be used to modify $T^{\downarrow}$ before recursion is applied. The interpretation of the parameters of ϕ_0 is as suggested by their names. Enclosed in square brackets we have added as comments the numbers of the corresponding cases according to 4.4.1; the primed cases are not included, but they take the same branch as the corresponding unprimed ones.

$$
\begin{aligned}
\phi_0(\lambda,d,e) &= \textbf{if } \pi(d)\neq\pi(e) \textbf{ then } \big((d,e,\mathit{none}),\mathit{none}\big)\ [\text{case } \mathit{0}] && (38)\\
&\quad \textbf{else let } (r,c)=\pi(d) \textbf{ in}\\
&\quad\quad \textbf{case } \mathrm{typ}(d)=\mathrm{typ}(e)=(\mathrm{N}){:}\ \big(\Upsilon(\lambda,r-1,|d|),\mathit{none}\big)\ [\text{cases } \mathit{1,2}]\\
&\quad\quad ,\ \mathrm{typ}(d)=\mathrm{typ}(e)=(\mathrm{I}+){:}\\
&\quad\quad\quad \textbf{if } m_c(\lambda)=2\wedge \mathrm{sg}(d)=\mathrm{sg}(e)\\
&\quad\quad\quad \textbf{then } \big(\Upsilon(\lambda,r-2,|d|),\mathit{none}\big)\ [\mathit{4a}]\\
&\quad\quad\quad \textbf{else } \big(\Lambda(\lambda,c-1,|d|),\ g(\mathrm{sg}(d)=\mathrm{sg}(e),r-2)\big)\ [\mathit{3,4b}]\\
&\quad\quad\quad \textbf{fi}\\
&\quad\quad ,\ \mathrm{typ}(d)=\mathrm{typ}(e)=(\mathrm{I}-){:}\ \big(\Lambda(\lambda,c-1,|d|),\mathit{none}\big)\quad [\mathit{5}]\\
&\quad\quad ,\ \{\mathrm{typ}(d),\mathrm{typ}(e)\}=\{(\mathrm{I}+),(\mathrm{N})\}{:}\\
&\quad\quad\quad \textbf{let } a=\mathrm{dom}\big((r-1,c),(\mathrm{N}),\ |d|,`\circ'\big)\\
&\quad\quad\quad ;\quad b=\mathrm{dom}\big((r,c-1),(\mathrm{I}-),|d|,`\circ'\big)\textbf{in}\\
&\quad\quad\quad \textbf{if } \mathrm{typ}(d)=(\mathrm{I}+)\\
&\quad\quad\quad \textbf{then } \big(\mathit{cont}(a,b,g(\mathrm{sg}(d)=`-',r)),\mathit{none}\big)\ [\mathit{6}]\\
&\quad\quad\quad \textbf{else } \ \big(\mathit{cont}(b,a,g(\mathrm{sg}(e)=`-',r)),\mathit{none}\big)\ [\mathit{7}]\\
&\quad\quad\quad \textbf{fi}\\
&\quad\quad \textbf{esac}\\
&\quad \textbf{fi}
\end{aligned}
$$

Now we are ready to formalise the recursive procedures of 4.4.3 and 4.4.4; they are combined into a single function ϕ_1. Its first argument (T) must not be $\odot$.

$$\begin{aligned}\phi_1(T,e) = {} & \mathbf{let}\ \lambda = \mathrm{sh}(T);\ (d : T^{\downarrow}) = T \\ & ;\ (\alpha,\omega_1) = \phi_0(\lambda,d,e);\ \tilde{T}^{\downarrow} = g'(\omega_1,T^{\downarrow})\ \mathbf{in} \\ & \mathbf{case}\ \alpha\ \mathbf{of}\ \mathit{stop}(w_n)\colon (\tilde{T}^{\downarrow},w_n) \\ & ,\ \mathit{cont}(d',e',\omega_2)\colon\ \mathbf{let}\ (S,w_n) = \phi_1(\tilde{T}^{\downarrow},e');\ \tilde{S} = g'(\omega_2,S) \\ & \qquad\qquad\qquad\quad \mathbf{in}\ (d' : \tilde{S},w_n) \\ & \mathbf{esac}\end{aligned} \tag{39}$$

Finally we define the function ϕ_2, that is intended to compute $\gamma(\overline{\mathcal{F}_{u,T}},\overline{\mathcal{F}_{u,T'}})$. At this point its definition is a rather obvious guess, but we shall see that it is more difficult to prove that this guess is correct.

$$\begin{aligned}\phi_2(T,T') = {} & \mathbf{if}\ T = \odot\ \mathbf{then}\ \odot \\ & \mathbf{else\ let}\ (e : T'^{\downarrow}) = T';\ (T_{[e]},w_n) = \phi_1(T,e)\ \mathbf{in}\ \phi_2(T_{[e]},T'^{\downarrow})\ \&\ w_n \\ & \mathbf{fi}\end{aligned} \tag{40}$$

One should check that the definitions up to (39) formalise the computations of 4.4. We note a few points that have taken a slightly different form. As already indicated, a number of cases have been merged, because their results can be given by the same expression. When T had to be modified to obtain $\mathrm{sg}(d) \neq \mathrm{sg}(e)$ (cases *3*, *3′*), we have used that applying $\xi_r \circ \xi_{r-2}$ is certainly a valid way to achieve such a modification, and furthermore that $\bigl(\xi_r \circ \xi_{r-2}(T)\bigr)^{\downarrow} = \xi_{r-2}(T^{\downarrow})$ in these cases. Having checked the combinatorial correctness of the definition of ϕ_1, we are ready to give a formal restatement of 4.4.3 and 4.4.4, which gives the geometric significance of ϕ_1.

4.5.1. Theorem. *Let $n > 0$ and $\lambda = J(u)$ for a unipotent element $u \in G$, and let a class of $k[u]$-module isomorphisms $M \xrightarrow{\sim} M_\lambda$ be given. Let T, T' be tableaux with* $\mathrm{sh}(T) = \mathrm{sh}(T') = \lambda$*, put $d = T_1$, $e = T'_1$, and let $(T_{[e]},w_n) = \phi_1(T,e)$. Choose any $l \in \alpha_u[\mathcal{F}_{u,T'}]$, and any isomorphism in the given class that maps l to the special line belonging to e; we use the class of isomorphisms $l^{\perp}/l \xrightarrow{\sim} M_\mu$ induced by it, where $\mu = \mathrm{sh}(T'^{\downarrow}) = \mathrm{sh}(T_{[e]})$. There exists a dense $(Z_u^\circ)_l$-stable subset $\mathcal{D}_l$ of $\mathcal{F}_{u,T}$ such that $\pi(f,l) = w_n$ and $f_{[l]} \in \mathcal{F}_{u_{[l]},T_{[e]}}$ for all $f \in \mathcal{D}_l$.*

Proof. The set $\mathcal{D}_l$ is the same as the one given in the proof of 4.4.3. Since this is also the set occurring in 4.4.4, the theorem follows directly from those two lemmas, granted that ϕ_1 correctly formalises their computations. □

Remark. It is not generally true that the union of all sets $\mathcal{D}_l$—for l traversing $\alpha_u[\mathcal{F}_{u,T'}]$—equals $\mathcal{F}_{u,T}$; e.g., in case C_2 with $T = \boxed{1}\boxed{\substack{2\\+}}$ and $T' = \boxed{1}\boxed{\substack{2\\-}}$, we have $F_T \notin \mathcal{D}_l$ for the unique choice of l, since $\pi(F_T,l) = 1$ while $w_2 = -1$ in this case. Therefore it would be wrong to fix f instead of l.

We define $\mathcal{D}_{[l]} = \{\, f_{[l]} \mid f \in \mathcal{D}_l \,\}$, then by definition $\mathcal{D}_{[l]} \subseteq \mathcal{F}_{u_{[l]},T_{[e]}}$. However, we have the following

4.5.2. Disappointment. *In general, $\mathcal{D}_{[l]}$ is not dense in $\mathcal{F}_{u_{[l]},T_{[e]}}$.*

Proof. It is sufficient to show that in the unprimed cases the 1-dimensional part h' of $f_{[l]}$ does not range over a dense subset of $U_{d'}(l^{\perp}/l)$ in all cases. For instance, if in case *0* we have $\kappa_d < \kappa_e$ then $h' = \Pi[h] =\subseteq \Pi[W_c(M)]$, where according to 2.3.6(a,c) the latter space has codimension 1 in $W_c(l^{\perp}/l)$. Cases *3* and *4b* are even worse, since the codimension of $\Pi[W_c(M)]$ in $W_{c-1}(l^{\perp}/l)$ is $1 + m_{c-1}(\lambda)$, which can be arbitrarily large. □

Due to this fact, there is no easy way to prove our main theorem that states—as formulated below—that ϕ_2 computes what it is meant to. There are (at least) two ways to complete such a proof. One is to use 4.2.2, in combination with the fact that generic relative positions are *maximal*, i.e., that for any $x \in X$ and $y \in Y$ we have $\pi(x,y) \leq \gamma(X,Y)$, where '$\leq$' denotes the Bruhat order. One then has to prove that ϕ_2 is surjective onto $\widetilde{W}$, and that it is constant on diagonal A_u-orbits, from which one concludes that ϕ_2 differs from the correspondence defined by γ, by left-composition with a permutation p of $\widetilde{W}$, which has $p(w) \leq w$ for all $w \in \widetilde{W}$, so that p can only be identity. The surjectivity of ϕ_2 can be proved by constructing a right-inverse of it by combinatorial means. A similar proof is given in [St3] for the A_n-case (where one uses the well-known bijectivity of the Robinson-Schensted correspondence), but we shall choose to use a different approach. This does not use 4.2.2, and gives a better *a priori* understanding of why things go well in spite of 4.5.2; apart from this the first approach is less attractive than in the A_n-case, because it is combinatorially much more complicated (cf. the remarks in appendix C).

The other approach, which we shall follow, is based on the following observation. Call a subset of any set associated with a given $k[u]$-module N *characteristic* if it is stable under $\mathrm{Aut}(N)^{\circ}$; e.g., $\mathcal{F}_{u,T}$ and the union of all $\mathcal{D}_l$ are characteristic in $\mathcal{F}_u$. Now the main reason for our disappointment is that $l^{\perp}/l$ is endowed with many non-characteristic subsets—such as $\Pi[W_1(l^{\perp})]$ and its relatives described in 2.3.6—in terms of which closed conditions holding on $\mathcal{D}_{[l]}$ can be formulated; consequently $\mathcal{D}_{[l]}$ itself is not characteristic in $\mathcal{F}_{u_{[l]},T_{[e]}}$. But the set of flags in $\mathcal{F}_{u_{[l]},T_{[e]}}$ that are bad (i.e., non-generic with respect to π_2) for *all* choices of $f'^{\downarrow} \in \alpha_u^{-1}[l]$ is certainly characteristic and closed, and it would be sufficient to prove that $\mathcal{D}_{[l]}$ cannot be contained in any such set. Therefore we state the following theorem.

4.5.3. Theorem. *Any characteristic subset of $\mathcal{F}_{u_{[l]},T_{[e]}}$ that contains $\mathcal{D}_{[l]}$ is dense.*

The proof of this theorem is somewhat involved—although not much combinatorially so—and it is deferred to the next section; essentially it deals with a way of reconstructing, up to isomorphism, M from $l^{\perp}/l$, by methods that are more-or-less dual to the ones used above. At this point we show how it implies our

4.5.4. Main Theorem. *Let T and T' be two tableaux of shape $\lambda = J(u)$, and let the signed permutation $(w_1, \ldots, w_n) = \phi_2(T, T')$, as defined in (40), represent $w \in \widetilde{W}$. Then for any isomorphism $M \xrightarrow{\sim} M_\lambda$ we have $\gamma(\overline{\mathcal{F}_{u,T}}, \overline{\mathcal{F}_{u,T'}}) = w$.*

Proof. We perform induction on n; the case $n = 0$ is trivial. It will be sufficient to show for each $l \in \alpha_u[\mathcal{F}_{u,T'}]$ that $\pi_2(f, f') = \phi_2(T, T')$ holds for (f, f') in a dense subset of $\mathcal{F}_{u,T} \times (\mathcal{F}_{u,T'} \cap \alpha_u^{-1}[l])$. In case $\#\operatorname{Con}(\mathcal{F}_{u,T'} \cap \alpha_u^{-1}[l]) = 2$ we prove this separately for each of the connected components, so let C be one of these components. Choose an induced class of isomorphisms $l^\perp/l \xrightarrow{\sim} M_\mu$ such that $f'^{\downarrow} \in \mathcal{F}_{u',T'^{\downarrow}}$ for all $f' \in C$, where $u' = u_{[l]}$. We apply induction to u', $l^\perp/l$ and the tableaux $T_{[e]}$ and $T'^{\downarrow}$; we obtain the existence of a dense open subset Δ of $\mathcal{F}_{u',T_{[e]}} \times \mathcal{F}_{u',T'^{\downarrow}}$ such that for all $(\tilde{f}, f'^{\downarrow}) \in \Delta$ we have $\pi_2(\tilde{f}, f'^{\downarrow}) = (w_1, \ldots, w_{n-1})$. Now using 4.5.1, putting $D = \mathcal{D}_l \times C$, and denoting by β the map $D \to \mathcal{F}_{u',T_{[e]}} \times \mathcal{F}_{u',T'^{\downarrow}}$ sending $(f, f') \mapsto (f_{[l]}, f'^{\downarrow})$, it will be sufficient to prove that $\beta^{-1}[\Delta]$ is dense in D. Since Δ is open and D is irreducible, this will follow if $\beta^{-1}[\Delta]$ is non-empty. The image of the projection of Δ on the first factor $\mathcal{F}_{u',T_{[e]}}$ is a characteristic dense open set, and it must therefore meet $\mathcal{D}_{[l]}$ by 4.5.3 (its complement cannot contain $\mathcal{D}_{[l]}$). Any element of Δ that projects to a point in the intersection lies in $\operatorname{Im}(\beta) \cap \Delta$, whence $\beta^{-1}[\Delta] \neq \emptyset$, which completes the proof. □

Conversely to the suggested first approach to the proof, we may now use 4.2.2 to *conclude* the basic combinatorial properties of ϕ_2.

4.5.5. Corollary. *Let $P, P', Q, Q' \in \mathcal{T}_\lambda$, and $a \in A_u$.*
- *(a) If $P \sim P'$ and $Q \sim Q'$ then $\phi_2(P, Q) = \phi_2(P', Q')$. We write $\phi_2([P], [Q])$ for this value.*
- *(b) $\phi_2(a \cdot [P], a \cdot [Q]) = \phi_2([P], [Q])$.*
- *(c) $\phi_2(Q, P) = \phi_2(P, Q)^{-1}$.*
- *(d) The disjoint union over all occurring Jordan types λ of the set of diagonal A_λ-orbits of equivalence classes of tableaux in $\mathcal{T}_\lambda$ is in bijection via ϕ_2 with $\widetilde{W} \cong H_n$.*

Proof. This immediate from 4.2.2 and 4.5.4, using (27) for part (c). □

It follows from 4.5.5(d) that there exists a right-inverse to ϕ_2, i.e., a map ψ_2 say, such that $\phi_2 \circ \psi_2$ is the identity on H_n. In fact ψ_2 can be given algorithmically. We shall not define ψ_2 formally as we have done for ϕ_2 however, but appendix C contains a computer program with a routine that performs such an algorithm, and a discussion of how it has been obtained.

4.6. *Proof of 4.5.3.*

For the proof of 4.5.3, we shall need a number of lemma's. In what follows we have $(T_{[e]}, w_n) = \phi_1(T, e)$, and we consider a non-degenerate $k[u]$-module N with $J(N) = \mu = \operatorname{sh}(T_{[e]})$. Certain variables will denote other objects than before, in particular we shall study flags $f \in \mathcal{F}_{u_N, T_{[e]}}$. Let $x \in I(f) \amalg -I(f)$ be such that

$|x| = \max\{\, i \in I(f) \mid i < |w_n| \,\}$ and $\mathrm{sg}(x) = \mathrm{sg}(w_n)$ (there is no f^{w_n}, so f^x is the part that comes closest to it). Recall that κ_d is the rightmost column meeting the domino d; we shall frequently need to consider also the leftmost column meeting d, which is related to κ_d as j' is to j in 2.3.6, and we shall therefore denote it as κ'_d.

4.6.1. Lemma. *Let $c = \kappa'_e$ and $f \in \mathcal{F}_{u_N, T_{[e]}}$ for a class of isomorphisms $N \xrightarrow{\sim} M_\mu$.*
(a) $f^x \supseteq W_c(N)$.
(b) If $\mathrm{typ}(e) = (\mathrm{I}-)$, $c > 1$, and $m_{c-1}(\mu) = 2$ then, for any isomorphism in the given class, f^x contains the line corresponding to the special line l_+ in the (unique) summand $M_{(c-1,c-1)}$ of M_μ.

Proof. (a) We use induction on the definition of ϕ_1; let the variables occurring in (39) be as defined there, and consider the case that (d, e) falls into. If this is a primed case, then we have either $w_n = |d|$, in which case $c = \mu_1 + 1$ so $W_c(N) = 0$ $(= f^x)$, or $w_n = -|d|$, in which case $f^x = N$; in both cases (a) is trivially true. In the unprimed cases put $p = f_1$, then $p \in U_{d'}(N)$ and $f^\downarrow \in \mathcal{F}_{u_{[p]}, \tilde{S}}$ by (39). Part (a) of the induction hypothesis gives $f^{\downarrow x} \supseteq W_{c'}(p^\perp/p)$, where $c' = \kappa'_{e'}$ (we use here that part (a) is invariant under automorphisms of N applied to f, hence the transition from S to $\tilde{S}$ has no effect). Now $f^x = \Pi^{-1}[f^{\downarrow x}]$, where Π is as in 2.3.6 for $l = p$, so (a) will follow from the same part of the induction hypothesis if $W_c(N) \subseteq \Pi^{-1}[W_{c'}(p^\perp/p)]$, or equivalently $\Pi[W_c(N)] \subseteq W_{c'}(p^\perp/p)]$. We can apply 2.3.6 (with $j = \kappa_{d'}$ since $l = p$) to obtain this inclusion in the following cases. In case *0* we have $c = c' \neq j$, so $\Pi[W_c(N)] = \Pi[W_c(p^\perp)] \subseteq W_{c'}(p^\perp/p)$; in cases *3*, *4b*, *5* and *7* we have $c' = c - 1 = j$, so $\Pi[W_c(N)] = \Pi[W_c(p^\perp)] = W_c(p^\perp/p) \subseteq W_{c'}(p^\perp/p)$; in case *6* we have $c' = c = j - 1$ so $\Pi[W_c(N)] = \Pi[W_c(p^\perp)] = W_{c'}(p^\perp/p)$; and finally in cases *1*, *2* and *4a* if $\mathrm{typ}(d') = (\mathrm{N})$ then $c \leq c' = j - 1$ and ${}^t\mu_c = {}^t\mu_{c'}$, so $\Pi[W_c(N)] = \Pi[W_{c'}(N)] = \Pi[W_{c'}(p^\perp)] = W_{c'}(p^\perp/p)$. So we are left with cases *2* and *4a* with $\mathrm{typ}(d') = (\mathrm{I}-)$. In those cases $c < c' = j$ and ${}^t\mu_c = {}^t\mu_{c'}$, and since $\tilde{S} = S$ in (39) we may apply 3.4.2 to d' and μ, which gives, in combination with part (b) of the induction hypothesis, that $g^{\downarrow x} \supseteq \Pi[W_{c'}(N)] = \Pi[W_c(N)]$, proving (a).

(b) We proceed as above, but the only relevant cases are *0* and *5*. In case *0*, because $m_{c-1}(\lambda) = 0$ we have $j \notin \{c, c-1\}$, and hence by 2.3.8(d) there is a natural isomorphism $V_{c-1}(N) \cong V_{c-1}(p^\perp/p)$, so (b) follows by induction in this case. In case *5* we have $\mathrm{sg}(d') =$ '+' from (37), so the image of p in $V_{c-1}(N)$ coincides with that of the line l_+, which directly implies (b). □

When $\mathrm{typ}(e) = (\mathrm{I}-)$ we can say even more about f^x. The next lemma supplements the previous one.

4.6.2. Lemma. *Assume that, in the situation of 4.6.1, $\mathrm{typ}(e) = (\mathrm{I}-)$.*
(a) If $c = 1$, then for every $q \in N$ with $q \perp \eta_N[f^x]$ we have $q \perp \eta_N(q)$.
(b) If $c > 1$, then let P denote the image of f^x in $V_{c-1}(M)$; the subspace $P^\perp \subset V_{c-1}(M)$ is isotropic.

Proof. We use the same notation as in the previous proof, and let $\bar{p}$ be the image of p in $V_j(N)$; if $j = c-1$ we have $\bar{p} \subseteq P$. (a) We are in case *0* or *5′*. In case *5′* we have $f^x = N$, so $q \in \mathrm{Im}(\eta_N)^\perp = \mathrm{Ker}(\eta_N)$, whence (a) is trivially satisfied. In case *0* we have $\kappa'_{d'} \geq 2$ so by 2.3.6(b) $p \subseteq W_2(p^\perp) \subseteq \eta_N[p^\perp]$, and for any $r \in \eta_N^{-1}[p] \cap p^\perp$ we have $\Pi(r) \in W_1(p^\perp/p) \subseteq f^{\downarrow x}$ by 4.6.1(a), so that $r \in f^x$ and $p \subseteq \eta_N[f^x]$. Therefore $q \perp \eta_N[f^x]$ implies $q \in p^\perp$, and we may apply Π, which gives $\Pi(q) \perp \eta_{[p]}[f^{\downarrow x}]$, and (a) follows by induction. (b) Here we are in case *0* or *5*. In case *5* we have $c' = c-1 = j$ and $f^x = \Pi^{-1}[f^{\downarrow x}] \supseteq \Pi^{-1}[W_j(p^\perp/p)] = W_j(p^\perp)$; by 2.3.8(a) the image in $V_j(N)$ of the latter space, which is contained in P, equals $\bar{p}^\perp$, establishing (b). In case *0* let $P' \subseteq V_{c-1}(p^\perp/p)$ be the image of $f^{\downarrow x}$, so that $P'^\perp$ is isotropic by induction. Assume first that $j \neq c-1$, then by 2.3.8(d) there is a natural isomorphism $V_{c-1}(N) \cong V_{c-1}(p^\perp/p)$ for which P corresponds to P', giving (b). If $j = c-1$ and $\mathrm{typ}(d') = (\mathrm{N})$, we have from 2.3.8(b) that $V_j(p^\perp/p) \cong \bar{p}^\perp \supseteq P^\perp$, and $P^\perp$ corresponds to $P'^\perp$. Finally if $j = c-1$ and $\mathrm{typ}(d') = (\mathrm{I}+)$, we have from 2.3.8(c) that $V_j(p^\perp/p) \cong \bar{p}^\perp/\bar{p}$, and the image of $P^\perp$ in $\bar{p}^\perp/\bar{p}$ corresponds to $P'^\perp$, so that (b) is established in this last case as well. □

Next, we shall consider extensions of the $k[u]$-module N, i.e., we shall construct $k[u]$-modules $\widetilde{M}$ such that $N \cong l_1^\perp/l_1$ for some isotropic line $l_1 \subseteq \widetilde{M}$. As underlying vector space of $\widetilde{M}$ we take $k \times N \times k$, where the lines corresponding to the first and last factors are called l_1 and l_2 respectively. The bilinear form $b_{\widetilde{M}}$ is determined by the requirements that its restriction to N is b_N, that $l_1 \oplus l_2 \perp N$, that l_1 and l_2 are isotropic, and that for the basis vectors v_1, v_2 in l_1 and l_2 respectively we have $b_{\widetilde{M}}(v_1, v_2) = 1 = \varepsilon b_{\widetilde{M}}(v_2, v_1)$. It follows that $l_1^\perp = l_1 \oplus N$, and we identify $l_1^\perp/l_1$ via the canonical isomorphism with N. We require that the unipotent transformation $\tilde{u} = u_{\widetilde{M}}$ stabilises l_1, and induces u_N in N, whence it must be of the general form expressed by the block-matrix

$$\tilde{u} = \begin{pmatrix} 1 & \nu & a \\ 0 & u_N & p \\ 0 & 0 & 1 \end{pmatrix}, \tag{41}$$

with $\nu \in N^*$, $a \in k$, and $p \in N$. The fact that $b_{\widetilde{M}}$ is preserved by $\tilde{u}$ is expressed by the relations

$$\nu(y) + b_N(u_N(y), p) = 0 \quad \text{for all } y \in N, \tag{42}$$

and

$$(1+\varepsilon)a + b_N(p,p) = 0, \tag{43}$$

where the latter equation is trivially satisfied if $\varepsilon = -1$. We wish to express the relationship between $J(\tilde{u})$ and $J(u_N)$ as given by 2.3.7 in terms of ν, a, and p. There are two cases that require different symbols to be introduced. First, if $\mathrm{Ker}(\eta_N) \subseteq \mathrm{Ker}(\nu)$ then there exists a unique $\hat{\nu} \in \mathrm{Im}(\eta_N)^*$ such that $\nu = \hat{\nu} \circ \eta_N$, and by (42) we have $p \in \mathrm{Ker}(\eta_N)^\perp = \mathrm{Im}(\eta_N)$. Otherwise, if $\mathrm{Ker}(\eta_N) \not\subseteq \mathrm{Ker}(\nu)$, we shall see that only the restriction of ν to $\mathrm{Ker}(\eta_N)$ matters, or equivalently the image of p in $N/\mathrm{Im}(\eta_N)$. Let $m = \max\{\, i \mid W_i(N) \not\subseteq \mathrm{Ker}(\nu)\,\}$, then ν induces a non-zero element $\bar\nu \in V_m(N)^*$. By (42) we also have $p \in \mathrm{Ker}(\eta_N^m) + \mathrm{Im}(\eta_N)$, so that its image in $N/\mathrm{Im}(\eta_N)$ lies in the image of $\mathrm{Ker}(\eta_N^m)$, and η_N^{m-1} induces an isomorphism from the latter image to $V_m(N)$; let $\bar p \in V_m(N)$ be the image of p obtained in this way.

4.6.3. Lemma. *In the situation above, $J(\tilde u)$ is obtained from $J(u_N)$ by first replacing a part $j'-1$ by a part j', and then replacing a part $j-1$ by a part j, where j', j are as follows.*

(a) If $\mathrm{Ker}(\eta_N) \subseteq \mathrm{Ker}(\nu)$ then $j' = 1$. In this case if $a = \hat\nu(p)$ then $j = 1$ and else $j = 2$. Also, if $\varepsilon = -1$ we have for any $q \in \eta_N^{-1}[p]$ that $\hat\nu(p) = b_N(p,q)$, and $q \perp \mathrm{Ker}(\hat\nu) = \eta_N[\mathrm{Ker}(\nu)]$.

(b) If $\mathrm{Ker}(\eta_N) \not\subseteq \mathrm{Ker}(\nu)$ then $j' = m+1$. In this case if $\bar\nu(\bar p) = 0$ then $j = j'$ and else $j = j'+1$. Also $\mathrm{Ker}(\bar\nu) = \bar p^\perp$, so $j = j'$ if and only if $\bar p$ is isotropic.

Proof. We apply 2.3.7 to $\widetilde M$ and l_1, and use 2.3.6 to determine j', and j. We get from 2.3.6(b) that $j' = \max\{\, i \mid l_1 \subseteq W_i(l_1^\perp)\,\}$, and this can be rewritten as $j' = \max\{\, i \mid i = 0 \vee W_i(N) \not\subseteq \mathrm{Ker}(\nu)\,\} + 1$, which is in accordance with the values given in both parts. We have $j \in \{j', j'+1\}$ where $j = j'+1$ if and only if $l_1 \subseteq \mathrm{Im}(\tilde\eta^{j'})$, where $\tilde\eta = \eta_{\widetilde M}$. We proceed separately for both cases. (a) Here $l_1 \subseteq \mathrm{Im}(\tilde\eta)$ if and only if there is some $q \in N$ such that $\tilde\eta(v_2 - q)$ is a non-zero vector in l_1. For such q we must have $\eta_N(q) = p$ so that $\nu(q) = \hat\nu(p)$, and $\tilde\eta(v_2 - q) = (a - \hat\nu(p))v_1$, independently of the choice of q; this leads to the given condition. By (42) we have $\nu(q) = -b_N(u_N(q), p)$, which rewrites to $b_N(p,q)$ if $\varepsilon = -1$. Again by (42) we have for any $y \in \mathrm{Ker}(\nu)$ that $p \perp u_N(y)$, so $b_N(\eta_N(q), u_N(y)) = 0$ whence $b_N(u_N(q), u_N(y)) = b_N(q, u_N(y))$; since u_N preserves b_N it follows that $b_N(q,y) = b_N(q, u_N(y))$ and hence $b_N(q, \eta_N(y)) = 0$, proving the relation. (b) Put $\hat\eta = \eta_{l\oplus N}$; by definition of j' we have $l_1 \not\subseteq \mathrm{Im}(\hat\eta^{j'})$, so $j = j'$ will hold if and only if $\binom{a}{p} \in \mathrm{Ker}(\hat\eta^m) + \mathrm{Im}(\hat\eta)$. Since $l_1 \subseteq \mathrm{Im}(\hat\eta)$, this question can be reduced modulo l_1 to $p \in \mathrm{Ker}(\nu \circ \eta_N^{m-1}) + \mathrm{Im}(\eta_N)$, which is easily seen to be equivalent to $\bar\nu(\bar p) = 0$. Using (42) and the fact that u_N acts as $\mathbf{1}$ on $W_{c-1}(N)$, it follows from the definitions that $\mathrm{Ker}(\bar\nu) = \bar p^\perp$. □

Now, given the construction of $\widetilde M$ for given $\tilde u$, consider flags $f \in \mathcal{F}_{u_N, T_{[e]}}$ once again. We "extend" f to a flag $\hat f$ in $\widetilde M$, by defining $I(\hat f) = I(f) \cup \{|w_n|\}$, and the parts of $\hat f$ are defined for $i \in I(f)$ by

$$\hat f^i = \begin{cases} f^i & \text{if } i > |w_n| \\ f^i \oplus l_1 & \text{if } -w_n < i < w_n \\ f^i \oplus l_2 & \text{if } w_n < i < -w_n \\ f^i \oplus l_1 \oplus l_2 & \text{if } i < -|w_n| \end{cases} \tag{44}$$

and for $|i| = |w_n|$ by

$$\hat f^{|w_n|} = f^{|x|}, \qquad \hat f^{-|w_n|} = f^{-|x|} \oplus l_1 \oplus l_2. \tag{45}$$

In this way it is assured that $\pi(\hat{f}, l_1) = w_n$ and $\hat{f}_{[l_1]} = f$. However, we have not yet assured that $\hat{f}$ is $\tilde{u}$-stable; to obtain this it is sufficient to require $\tilde{u}$-stability for the highest dimensional part not containing l_1, and for the lowest dimensional part containing l_2, i.e., for $\hat{f}^{\pm w_n}$ if $w_n > 0$, or for $\hat{f}^{\pm x}$ if $w_n < 0$. This gives us the relations

$$f^x \subseteq \mathrm{Ker}(\nu), \qquad p \in f^{-x}, \qquad \text{and} \qquad a = 0 \text{ if } w_n < 0, \qquad\qquad (46\mathrm{a,b,c})$$

of which the first two are equivalent by (42), since f^x is u_N-stable.

Now let $\mathcal{V}$ be the set of pairs $(f, \tilde{u})$, with $f \in \mathcal{F}_{u_N, T_{[e]}}$ and $\tilde{u}$ as in (41), satisfying (42), (43) and (46). Since $\tilde{u}$ is effectively specified by the vector $\binom{a}{p}$, or by p alone if $\varepsilon = +1$, this is a vector bundle over the irreducible variety $\mathcal{F}_{u_N, T_{[e]}}$, and therefore irreducible itself. (It might seem that (43) together with (46c) could give a non-linear condition on p, but if $w_n < 0$ then also $x < 0$ so that p is isotropic by (46b), so that (43) is always satisfied.) Also recall the situation of 4.5.3: a line $l \in U_e(M)$ is chosen, so we have $J(l^\perp/l) = \mu = J(N)$, which implies $l^\perp/l \cong N$ by 2.2.1.

4.6.4. Lemma. *There exist $(f, \tilde{u}) \in \mathcal{V}$, such that $J(\tilde{u}) = \lambda = J(u)$. For such $(f, \tilde{u})$ the values of j' and j of 4.6.3 attain their maximum on $\mathcal{V}$.*

Proof. The existence part is based on the observation that for any $\tilde{f} \in \mathcal{D}_l$ as in 4.5.1 we can decompose M as $k \times N \times k$ as in the construction above, in such a way that for the image f of $\tilde{f}_{[l]}$ in N we have $\tilde{f} = \hat{f}$. We must take $l_1 = l$ of course, and l_2 is chosen as follows: if $w_n > 0$ we choose l_2 such that $l_2 \subseteq \tilde{f}^{-w_n}$ and $l_2 \not\subseteq \tilde{f}^{-x}$ and that l_2 is isotropic (that this last condition may be met can be seen by projecting onto the non-degenerate space $\tilde{f}^{-w_n}/\tilde{f}^{w_n}$); if $w_n < 0$ we choose l_2 such that $l_2 \subseteq \tilde{f}^{-x}$ and $l_2 \not\subseteq \tilde{f}^{-w_n}$, which is automatically isotropic. It is easily verified that $l_2 \not\perp l$, so by choosing appropriate basis vectors v_1, v_2 in l and l_2, and putting $N = (l \oplus l_2)^\perp$ we obtain a decomposition of M as required, and b_M corresponds to $b_{\widetilde{M}}$. With this decomposition u has the form (41), and (42) and (43) are satisfied since u preserves b_M. Let f be the image of $\tilde{f}_{[l]}$ under the canonical isomorphism $l^\perp/l \xrightarrow{\sim} N$, then $f \in \mathcal{F}_{u_N, T_{[e]}}$ according to 4.5.1, and by the choice of l_2 we have $\hat{f} = \tilde{f}$. Finally (46) is immediate since $\tilde{f}$ is u-stable; this completes the existence part.

We have $Y(\lambda) = Y(\mu) \uplus \mathrm{supp}(e)$, so the values of j' and j that 4.6.3 gives for this situation are κ'_e and κ_e respectively. Now the maximality statement about j' follows from 4.6.1(a) and (46a). As $j \in \{j', j'{+}1\}$ and we cannot have $j = j'{+}1$ if $\varepsilon_{j'} = +1$, the maximality of j follows directly, unless $\mathrm{typ}(e) = (\mathrm{I}-)$. In the latter case we may assume that j' is maximal, i.e., that $j' = c$ in 4.6.1. If $c = 1$ then by a trivial induction it follows from (39) that $w_n < 0$, so $a = 0$ by (46c), and the lemma follows by (46a) from 4.6.3(a) and 4.6.2(a). If $c > 1$ then by (46a) the lemma follows from 4.6.3(b) and 4.6.2(b). □

It is clear from 4.6.3 that subsets of $\mathcal{V}$ given by conditions of the form $j \le j_0 \wedge j' \le j'_0$ for fixed j_0, j'_0 are always closed. Therefore 4.6.4 is equivalent to the statement that $J(\tilde{u}) = J(u)$ holds on a dense subset of $\mathcal{V}$.

4.6.5. Lemma. *For $(f, \tilde{u})$ in a dense subset $\widetilde{\mathcal{V}}$ of $\mathcal{V}$ we have $\hat{f} \in \mathcal{F}_{\tilde{u},|T|}$, where $\hat{f}$ is given by (44) and (45).*

Proof. Once again we proceed by induction on the definition of ϕ_1. If $|w_n| = |d|$ then we have $|T_{[e]}| = |T^{\downarrow}|$, so $f \in \mathcal{F}_{u_N,|T^{\downarrow}|}$, and by 4.6.4 we have $J(\tilde{u}) = \lambda$ on a dense subset of $\mathcal{V}$, and we can take that subset as $\widetilde{\mathcal{V}}$. If $|w_n| \neq |d|$ then $h = \hat{f}_1$ lies in N by (44), so we can form the subquotient $N' = h^{\perp}/h$ of N; by (46a,b) ν and p determine $\nu' \in N'^*$ and $p' \in N'$ respectively. Now let $\mathcal{V}'$ be the variety analogous to $\mathcal{V}$ for N', then by a construction similar to that of $\hat{f}$ any element of $\mathcal{V}'$ determines a flag in $k \times N' \times k$. The flag so determined by $(f^{\downarrow}, \tilde{u}')$ with

$$\tilde{u}' = \begin{pmatrix} 1 & \nu' & a \\ 0 & u_{N'} & p' \\ 0 & 0 & 1 \end{pmatrix}, \tag{47}$$

is easily seen to be $\hat{f}^{\downarrow}$. The map $\beta\colon \{ (f, \tilde{u}) \in \mathcal{V} \mid f \in \alpha_{u_N}^{-1}[h] \} \to \mathcal{V}'$, given by $(f, \tilde{u}) \mapsto (f^{\downarrow}, \tilde{u}')$, is surjective, and by induction there is a dense subset $\widetilde{\mathcal{V}}'$ of $\mathcal{V}'$ on which $\hat{f}^{\downarrow} \in \mathcal{F}_{\tilde{u}',|T^{\downarrow}|}$ holds. We can now take for $\widetilde{\mathcal{V}}$ the intersection of $\beta^{-1}[\widetilde{\mathcal{V}}']$ with the dense subset of $\mathcal{V}$ on which $J(\tilde{u}) = \lambda$. □

4.6.6. Theorem. *Let the situation be as in 4.5.3. There exists a dense subset $\mathcal{D}'_{[l]}$ of $\mathcal{F}_{u_{[l]},T_{[e]}}$ such that for all $f \in \mathcal{D}'_{[l]}$ there exist a flag $\tilde{f} \in \mathcal{D}_{[l]}$ and $z \in \mathrm{Aut}(l^{\perp}/l)^{\circ}$ such that $f = z \cdot \tilde{f}_{[l]}$.*

Proof. We choose a decomposition $M \cong k \times N \times k$ as in the proof of 4.6.4 for some flag $\check{f}$ in $\mathcal{D}_{[l]}$, and identify $l^{\perp}/l$ with N; for $\mathcal{D}'_{[l]}$ we take the image of the projection $\widetilde{\mathcal{V}} \to \mathcal{F}_{u_{[l]},T_{[e]}}$. Now let any $f \in \mathcal{D}'_{[l]}$ be given, we choose a corresponding $\hat{u}$ such that $\hat{f} \in \mathcal{F}_{\hat{u},|T|}$. Since in particular $J(\hat{u}) = \lambda$ we have by 1.7.1 that u and $\hat{u}$ are conjugate in G, say $\hat{u} = gug^{-1}$. Since $l \in U_e(M)$ and also $l \in U_e(\widetilde{M})$ we have that l and $g^{-1} \cdot l$ are in the same Z_u-orbit by 2.3.4, say $g^{-1} \cdot l = h \cdot l$ with $h \in Z_u$. Therefore u and $\hat{u}$ are even conjugate by an element $gh \in G_l$. Let C be the conjugacy class in G_l of u, then $(f, \tilde{u}) \mapsto (\hat{f}, \tilde{u})$ defines a continuous map $\theta\colon \widetilde{\mathcal{V}} \to \{ (\tilde{f}, \tilde{u}) \in \mathcal{F} \times C \mid \tilde{f} \in \mathcal{F}_{\tilde{u},|T|} \}$. Now by 3.2.3 we have that $\{ (z \cdot \tilde{f}, zuz^{-1}) \mid \tilde{f} \in \mathcal{F}_{u,T} \wedge z \in (G_l)^{\circ} \}$ is a connected component of the latter set. It meets $\mathrm{Im}(\theta)$ in $(\check{f}, u)$, and since $\widetilde{\mathcal{V}}$ is connected, it contains all of $\mathrm{Im}(\theta)$. In particular it contains $(\hat{f}, \hat{u})$, so there exist $\tilde{f} \in \mathcal{F}_{u,T}$ and $\hat{z} \in (G_l)^{\circ}$ such that $\hat{f} = \hat{z} \cdot \tilde{f}$ and $\hat{u} = \hat{z}u\hat{z}^{-1}$. Choosing such $\tilde{f}$ and $\hat{z}$, we have for the automorphism z of $l^{\perp}/l$ induced by $\hat{z}$ that $f = z \cdot \tilde{f}_{[l]}$ and $\tilde{f} \in \mathcal{D}_{[l]}$, which proves the theorem. □

Finally we shall show that 4.6.6 is equivalent to 4.5.3. Clearly $\mathcal{D}'_{[l]}$ can be taken to be the union of all $\mathrm{Aut}(l^{\perp}/l)^{\circ}$-orbits in $\mathcal{F}_{u_{[l]},T_{[e]}}$ that meet $\mathcal{D}_{[l]}$, which is the smallest characteristic subset of $\mathcal{F}_{u_{[l]},T_{[e]}}$ containing $\mathcal{D}_{[l]}$. This establishes the equivalence.

4.7. ***Examples.***

Having completed the proof of our main theorem, we shall now give a number of examples of the kind of computations involved. One may turn back to 3.6 for geometric descriptions and illustrations of $\mathcal{F}_u$ in a number of cases that we shall consider.

First we consider a very simple case: $\lambda = (2,2)$ in case D_2. Recall that $S_\lambda = \bigl\{\boxed{\begin{smallmatrix}1&2\\+&\end{smallmatrix}}, \boxed{\begin{smallmatrix}1&2\\-&\end{smallmatrix}}\bigr\}$, which two tableaux parametrise disjoint lines of types s_2 and s_2' respectively. We compute $\phi_2\bigl(\boxed{\begin{smallmatrix}1&2\\+&\end{smallmatrix}}, \boxed{\begin{smallmatrix}1&2\\+&\end{smallmatrix}}\bigr)$ which by (40) requires computing $\phi_1\bigl(\boxed{\begin{smallmatrix}1&2\\+&\end{smallmatrix}}, e\bigr)$ where $e = \mathrm{dom}\bigl((2,2),(\mathrm{I}-),\text{‘}\circ\text{’}\bigr)$, and by (39) this on its turn calls $\phi_0((2,2),d,e)$ with $d = e$. Case *5* of (38) is selected, giving $\bigl(\Lambda((2,2),1,2), \mathit{none}\bigr)$, which evaluates to $\bigl(\mathit{cont}(d',d',\mathit{none}),\mathit{none}\bigr)$ where $d' = \mathrm{dom}((2,1),(\mathrm{I}+),2,\text{‘}+\text{’})$, so this is the outcome of the application of ϕ_0. Returning to (39) we find that $\alpha = \mathit{cont}(d',d',\mathit{none})$, $\omega_1 = \mathit{none}$ and $\tilde{T}^{\downarrow} = \boxed{\begin{smallmatrix}1\\+\end{smallmatrix}}$; the second branch of the **case-of**-expression is taken, and we should equate (S, w_n) to the result of the recursive application $\phi_1\bigl(\boxed{\begin{smallmatrix}1\\+\end{smallmatrix}}, e'\bigr)$ where $e' = d'$ as given above. That application calls ϕ_0, which returns $(\mathit{stop}(+1),\mathit{none})$—case *4′a*—and now the first branch of the **case-of**-expression is taken, so $\phi_1\bigl(\boxed{\begin{smallmatrix}1\\+\end{smallmatrix}}, e'\bigr)$ yields $(\odot,+1)$. Consequently we get $\tilde{S} = S = \odot$ and $w_n = +1$ in the original application of ϕ_1, and this application returns $(d' : \odot, +1)$, which equals $\bigl(\boxed{\begin{smallmatrix}2\\+\end{smallmatrix}}, +1\bigr)$. Back in (40), $T_{[e]}$ is equated to $\boxed{\begin{smallmatrix}2\\+\end{smallmatrix}}$, and the final result is given as $\phi_2\bigl(\boxed{\begin{smallmatrix}2\\+\end{smallmatrix}}, \boxed{\begin{smallmatrix}1\\+\end{smallmatrix}}\bigr)\ \&\ {+1}$. The remaining application of ϕ_2 leads to another instance of case *4′a*, and yields $(+2)$, so we have finally computed $\phi_2\bigl(\boxed{\begin{smallmatrix}1&2\\+&\end{smallmatrix}}, \boxed{\begin{smallmatrix}1&2\\+&\end{smallmatrix}}\bigr) = (+2)\&{+1} = (+2,+1)$. Indeed this is the signed permutation representing the simple reflection s_2, which is the generic relative position of a pair of flags on a line of type s_2. This may seem an overwhelmingly cumbersome way of computing such an easy result, but with a little practice one gets a feeling for the general structure of the algorithm, and most of the steps can be performed mentally. The computation of $\phi\bigl(\boxed{\begin{smallmatrix}1&2\\-&\end{smallmatrix}}, \boxed{\begin{smallmatrix}1&2\\-&\end{smallmatrix}}\bigr)$ proceeds similarly to the one spelled out above, but both cases *4′a* are replaced by *4′b*, so the result is $(-2,-1)$, which represents s_2'.

Next we consider $\lambda = (2,2)$ for case C_2, wherethe clusters are open and we have $S_\lambda = \bigl\{\boxed{\begin{smallmatrix}1&2\\&+\end{smallmatrix}}, \boxed{\begin{smallmatrix}1\\2\end{smallmatrix}}, \boxed{\begin{smallmatrix}1&2\\&-\end{smallmatrix}}\bigr\}$. Computing $\phi_2\bigl(\boxed{\begin{smallmatrix}1\\2\end{smallmatrix}}, \boxed{\begin{smallmatrix}1\\2\end{smallmatrix}}\bigr)$ only involves cases *1*, *2′* and yields $(+2,+1)$, which represents s_2. Computing $\phi_2(T,T)$ for one of the other two tableaux involves cases *4′a* and *5′* and yields $(-1,+2)$, which represents s_1. That this result is the same for both tableaux is in agreement with the fact that $A_u \cong \mathbf{2}$ interchanges the two; however, $\phi_2\bigl(\boxed{\begin{smallmatrix}1&2\\&+\end{smallmatrix}}, \boxed{\begin{smallmatrix}1&2\\&-\end{smallmatrix}}\bigr)$ should yield a different value. Indeed the first step leads to case *4b* instead of *4′a*, and we then proceed quite differently, resulting in the value $(-2,-1)$ which represents $s_1s_2s_1$. We encounter case *6* for the first time in the computation of $\phi_2\bigl(\boxed{\begin{smallmatrix}1&2\\&+\end{smallmatrix}}, \boxed{\begin{smallmatrix}1\\2\end{smallmatrix}}\bigr)$, and similarly $\phi_2\bigl(\boxed{\begin{smallmatrix}1\\2\end{smallmatrix}}, \boxed{\begin{smallmatrix}1&2\\&+\end{smallmatrix}}\bigr)$ involves case *7*. In the former computation the first application of ϕ_0 yields $\bigl(\mathit{cont}(a,b,\mathit{none}),\mathit{none}\bigr)$, with $a = \mathrm{dom}((1,2),(\mathrm{N}),2,\text{‘}\circ\text{’})$ and $b = \mathrm{dom}((2,1),(\mathrm{I}-),2,\text{‘}\circ\text{’})$. Then ϕ_1 recursively calls $\phi_1\bigl(\boxed{1}, \mathrm{dom}((2,1),(\mathrm{I}-),2,\text{‘}\circ\text{’})\bigr)$ and eventually yields $(\boxed{\;2\;}, -1)$; we finally obtain the signed permutation $(+2,-1)$, which represents s_1s_2. The latter computation is analogous and yields $(-2,1)$ representing s_2s_1. Replacing ‘$\boxed{\begin{smallmatrix}1&2\\&+\end{smallmatrix}}$’ by ‘$\boxed{\begin{smallmatrix}1&2\\&-\end{smallmatrix}}$’ in these last two computations changes ω_2 in (39) from *none* to *flip*(2) during the first call of ϕ_1, but since $\xi(2,\odot) = \odot$, this does not affect the final result.

A case that is similar, but where $\omega_2 = \mathit{flip}(r)$ does have an eventual effect is $\lambda = (3,3)$ in case C_3. So put $T = \boxed{1_{+}}\boxed{2}\boxed{3_{+}}$, $T' = \boxed{1_{+}}\boxed{\begin{smallmatrix}2\\3\end{smallmatrix}}$, and $T'' = \boxed{1_{-}}\boxed{2}\boxed{3_{-}}$; these lie in distinct A_λ-orbits and parametrise the three irreducible components of $\mathcal{B}_u \subset \mathcal{F}_u$. In the computations of $\phi_2(T,T)$, $\phi_2(T',T')$ and $\phi_2(T'',T'')$ we do not encounter any values $\mathit{flip}(r)$, and the results are $(+2,+1,+3)$, $(+1,+3,+2)$, and $(-2,-1,+3)$ respectively, representing s_2, s_3 and s_2'. We do get $\omega_2 = \mathit{flip}(2)$ during the evaluation of applications of ϕ_2 where exactly one of the arguments is T''. For instance we have $\phi_1(T,T''_1) = \big(\boxed{2_{-}}\boxed{3},+1\big)$ where the '$-$' in the resulting tableau stems from the application $\xi\big(2,\boxed{2_{+}}\big)$; we finally get $\phi_2(T,T'') = (-3,-2,+1)$ which represents $s_2 s_3 s_2'$. In the case of $\phi_2(T',T'')$ we similarly get $\big(\boxed{1_{-}}\boxed{3},+2\big)$ as yield of ϕ_1, and $(-3,-1,+2)$ representing $s_2 s_3$ as final result. A table with the complete results for this case can be found in appendix B.

We have not met cases *3* or *3'* yet, and consequently no instances where $\omega_1 \neq \mathit{none}$ in (39), since this requires λ to have at least three non-zero parts. We do get case *3'* when $\lambda = (1,1,1)$ in case B_1 but then the application of ξ resulting form $\omega_1 = \mathit{flip}(1)$ has no effect. The same is true for the instance of case *3* that we get when $\lambda = (2,2,2)$ in case C_3. The reader is encouraged to perform some computations for this case, and to verify that none of the signs present in the tableaux affect the outcome; this is as it should be, because all dominoes appear in cl(0). The simplest case where a non-trivial value for ω_1 does affect the result is when $\lambda = (1,1,1,1)$ in case D_2. This corresponds to $u = \mathbf{e}$, so for all applicable tableaux T we should have $\phi_2(T,T) = (-1,-2)$ which represents the longest element $s_2 s_2'$ of W. Indeed we have for instance

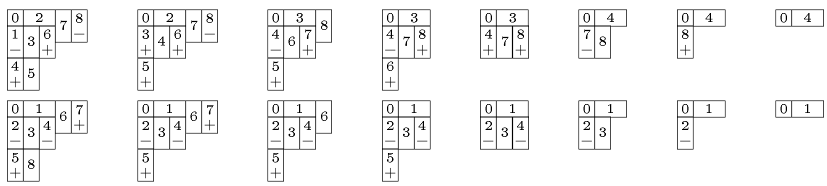
$$\phi_1\left(\begin{array}{|c|}\hline 1_{+}\\ \hline 2_{+}\\ \hline\end{array}, \mathrm{dom}((4,1),(\mathrm{I}+),2,\text{'}+\text{'})\right) = \big(\boxed{1_{-}},-2\big),$$

as a consequence of $\omega_1 = \mathit{flip}(2)$, and this gives the desired answer.

We have illustrated all the essential aspects of the algorithm, and shall now give a more substantial example, where all these aspects are involved. We show a number of intermediate stages of the computation, namely the pairs of tableaux that occur as argument to (recursive) applications of ϕ_2, and the value of w_n computed at that step.

n	8	7	6	5	4	3	2	1
w_n	-1	$+2$	-5	-6	$+3$	-7	-8	$+4$

T, T':



We do not comment on the details; these may be checked by the reader. Note in particular that during the third step the sign of the domino numbered 4 in T changes *twice*.

4.8. *Conclusion.*

We have derived an algorithm ϕ_2, defined by (33)–(40), that computes the generic relative positions of irreducible components of either $\mathcal{F}_u$ or $\mathcal{B}_u$ for classical groups G of types B_n, C_n and D_n in any characteristic other than 2, and any $u \in G$. The parametrisation that is used is defined in §3. The basic combinatorial properties of the correspondence so defined are stated in 4.5.5. The algorithm is analogous to the Robinson-Schensted algorithm—which performs these computations for groups of type A_n—but it is rather more complicated; nevertheless it can be performed (with some effort) by hand, and very efficiently by an electronic computer (see appendix C). The possibility to calculate generic relative positions can be useful in the geometric study of the varieties $\mathcal{B}_u$. The algorithm also raises new questions as to its further combinatorial properties (see appendix A).

We have not treated the characteristic 2 case, but it seems that a similar approach might work for that case, using the parametrisation given in [Spa II.6], and that this would result in a analogous, but probably more complicated, algorithm. On the other hand the exceptional groups would present more fundamental difficulties, since we lack systematic parametrisations of $\mathrm{Irr}(\mathcal{B}_u)$, as well as explicit permutation representations of the Weyl groups of the kind employed by our method.

Appendix A. Some related algorithms.

In this appendix we present two algorithms that are related to ϕ_2, the main algorithm of this thesis. The first one is the Robinson-Schensted algorithm mentioned in the title. The second is an algorithm due to D. Garfinkle, that bears a striking resemblance to our algorithm (we learned about her work by private communication, and know no published account of it to date). It involves the same Weyl groups of types B_n, C_n and D_n, and plays a rôle in the determination of their Kazhdan-Lusztig cells.

The Robinson-Schensted algorithm is a combinatorial procedure that establishes a bijective correspondence between the symmetric group S_n and the set of all ordered pairs of Young tableaux of equal shape with entries 1, …, n. It has many useful interpretations, of which the fact that it describes for groups of type A_n the generic relative positions of irreducible components of $\mathcal{B}_u$, is only a comparatively recent one. Since our algorithm was strongly inspired by it, it seems appropriate that we should reproduce it here for comparison, in a form that resembles the way our algorithm is formulated. Now the Robinson-Schensted algorithm is usually presented in a way rather different from our the description of ϕ_2, but it is not difficult to transform it in such a way that the resemblance becomes apparent.

A Young tableau is a numbering of the squares of a Young diagram with distinct numbers, such that they are increasing along rows and columns. Formulated differently, it is a numbering such that, if the diagram is non-empty, there is a unique highest numbered square, and omitting it gives another Young tableau. The Robinson-Schensted algorithm constructs a pair of Young tableaux with the same shape and both with entries 1, …, n, from a permutation of those numbers, i.e., from a sequence of those numbers in permuted order. One starts with a pair of empty tableaux—one on the left and one on the right—and then successively 'inserts' the numbers from the sequence into the left tableau, meanwhile adding the number i to the right tableau after the ith insertion. In the usual formulation, a number is inserted according to the following rule: the number is placed in the first row of the tableau, where it replaces the first larger number present if there are any. Then that number is moved to the second row, again replacing the next larger number, and so on, until a number is moved to a row containing only smaller numbers; it is then simply appended to the end of that row. Hereby the insertion is completed, and the number added to the right tableau is placed in the same position as the number that was moved last in the left tableau, so that both tableaux have the same shape after each major step of the algorithm. It is not immediately clear that the columns of the left tableau will always be increasing, but this can be proved by a simple induction; moreover this property will be quite obvious from the alternative—recursive—formulation that we shall now give.

That formulation is based on observation of the part that the highest number of the left tableau plays in the insertion process. If it is moved at all, this happens at the last step of the insertion, and the move doesn't affect the position of any other number. So if we remove the highest number from the left tableau after an insertion, and assuming that it is not the inserted number, then the result will be the same as when we had removed the number *before* the insertion. We can now easily derive the following description of the insertion process. If the number to be inserted is higher than any one present in the tableau, simply add it to the first row. Otherwise remove the highest number from the tableau, remembering its position, and recursively perform insertion of the same number into the smaller tableau. Then add the highest number again in its old place if that square has not been occupied by the insertion, but if it has been occupied, add the highest number at the end of the next row. One readily checks that in all cases the numbers will fill a Young diagram after the insertion; in the case that the highest number is moved this follows because the row it is moved to is at least one shorter then the row above it, since a new square has just been occupied in the latter. This description, although not to be recommended when performing the calculations by hand, is quite useful in mathematical considerations.

We can express this definition using the formalism introduced in 4.5. Young tableaux are represented as a sequence of 'squares', where each square s has a position $\pi(s)$ and an entry $|s|$; the squares are ordered by decreasing entries. A position is just a pair (r, c) of a row and column number, while a square with position (r, c) and entry x may be specified as $\mathrm{sq}((r, c), x)$. The shape of a Young tableau T is written as $\mathrm{sh}(T)$, and is a partition whose parts are the lengths of the rows of T; also let $|T|$ denote the highest entry of a square in T, or 0 if T is empty. The insertion step may now be specified as a function R_1, taking as arguments a Young tableau and a number, and returning a pair of a modified tableau, and the position that is occupied by that tableau but not by the original one.

$$
\begin{aligned}
R_1(T,x) = {} & \mathbf{let}\ \lambda = \mathrm{sh}(T)\ \mathbf{in} \\
& \mathbf{if}\ x > |T|\ \mathbf{then\ let}\ p = (1, \lambda_1 + 1)\ \mathbf{in}\ \big(\mathrm{sq}(p,x) : T,\ p\big) \\
& \mathbf{else\ let}\ (s : T^{\downarrow}) = T;\ \ (T', p) = R_1(T^{\downarrow}, x)\ \mathbf{in} \\
& \qquad \mathbf{if}\ \pi(s) \neq p\ \mathbf{then}\ (s : T', p) \\
& \qquad \mathbf{else\ let}\ (r,c) = p;\ \ p' = (r+1, \lambda_{r+1} + 1)\ \mathbf{in}\ \big(\mathrm{sq}(p', |s|) : T',\ p'\big) \\
& \qquad \mathbf{fi} \\
& \mathbf{fi}
\end{aligned}
\tag{48}
$$

It is now easy to express the complete algorithm, called R_2.

$$
\begin{aligned}
R_2(w) = {} & \mathbf{if}\ w = \odot\ \mathbf{then}\ (\odot, \odot) \\
& \mathbf{else\ let}\ (w' \,\&\, x) = w;\ \ (P,Q) = R_2(w');\ \ (T,p) = R_1(P,x) \\
& \qquad \mathbf{in}\ \big(T,\ \mathrm{sq}(p, |Q|+1) : Q\big) \\
& \mathbf{fi}
\end{aligned}
\tag{49}
$$

Note that the definition of R_1 could have been considerably simpler if we would have defined squares to contain only a row number and an entry; this would still allow the determination of the column numbers in the context of a Young tableau. For a proper comparison with ϕ_2 we should also give the inverse Robinson-Schensted algorithm. It is similar to the ordinary one, and we specify it by two functions R'_1 and R'_2.

$$\begin{aligned} R'_1(s:T,p) = {} & \mathbf{if}\ \pi(s)\neq p\ \mathbf{then\ let}\ (T',x)=R'_1(T,p)\ \mathbf{in}\ (s:T',x) \\ & \mathbf{else\ let}\ \lambda=\mathrm{sh}(T);\ \ (r,c)=p\ \mathbf{in} \\ & \qquad \mathbf{if}\ r=1\ \mathbf{then}\ (T,|s|) \\ & \qquad \mathbf{else\ let}\ p'=(r-1,\lambda_{r-1});\ \ (T',x)=R'_1(T,p') \\ & \qquad\qquad \mathbf{in}\ (\mathrm{sq}(p',|s|):T',\ x) \\ & \qquad \mathbf{fi} \\ & \mathbf{fi} \end{aligned} \tag{50}$$

$$\begin{aligned} R'_2(P,Q) = {} & \mathbf{if}\ Q=\odot\ \mathbf{then}\ \odot \\ & \mathbf{else\ let}\ (q:Q^{\downarrow})=Q;\ \ (P',x)=R'_1(P,\pi(q))\ \mathbf{in}\ R'_2(P',Q^{\downarrow})\ \&\ x \\ & \mathbf{fi} \end{aligned} \tag{51}$$

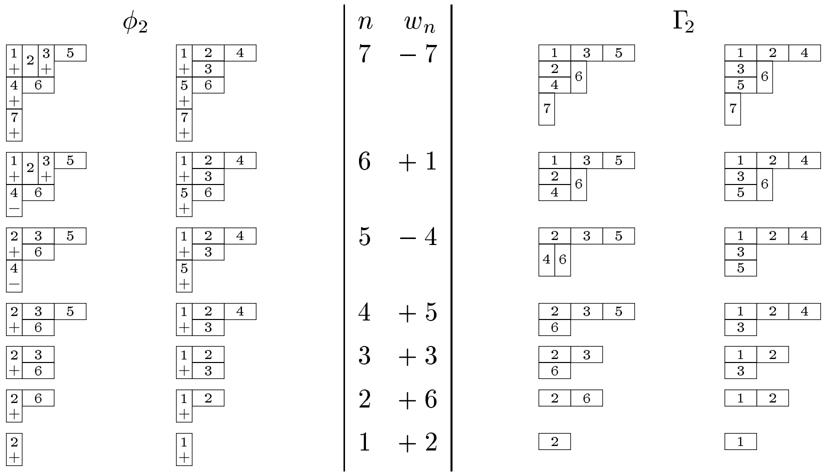
Comparing these definitions to (33)–(40) we see that the algorithm of this thesis has indeed a similar structure to the (inverse) Robinson-Schensted algorithm, but it has so many more cases that it was practical to split off a number of auxiliary functions, and to encode their results as tagged values. There is even a (slight) formal connection between the algorithms ϕ_2 and R'_2: if in case C_n a pair T,T' of signed domino tableaux has dominoes of type (N) only, then $\phi_2(T,T')$ contains only positive numbers, and can therefore be interpreted as a permutation of n, and this permutation can also be computed as $R'_2(\tilde{T},\tilde{T}')$ for Young tableaux $\tilde{T},\tilde{T}'$ obtained by "compressing" T,T' horizontally in the obvious way. This is because in the evaluation of $\phi_2(T,T')$ only cases *1* ,*2* and *2'* occur, and their effect is precisely mirrored by (50).

We now turn to the other algorithm. That one resembles our algorithm in a much more detailed way, but it doesn't involve the subtleties associated with signs in tableaux. Indeed it may justly be considered as the combinatorial archetype of our algorithm. It deals with dominoes and signed permutations like our algorithm, but here the dominoes carry no signs, nor is there a restriction on the position of their support: any two adjacent squares may form the support of a domino. Therefore we need to consider only two types of dominoes: horizontal (H) and vertical (V) ones. Such a domino is determined by its position (i.e., that of its lower righthand square), type and entry; to distinguish them from our other dominoes we write $d = \mathrm{dom}'(\pi(d),\mathrm{typ}(d),|d|)$. Domino tableaux formed from these dominoes are defined by exact analogy to the definitions in 3.3; a fixed value for supp($\odot$) is chosen, which can be $\emptyset$ (cases C_n and D_n) or $\{(1,1)\}$ (case B_n). There is however an interesting generalisation, that was suggested by I. G. Macdonald: we might also fix supp($\odot$) to the set $Y(\lambda)$ for any of the "staircase" partitions $\lambda = (n,n-1,\ldots,2,1)$, which are precisely the partitions at whose periphery no domino (in the current sense) can lie. The definition of the algorithm, that we shall call Γ_2, is identical in all cases; we present it in a form that maximally emphasises the similarity with (36)–(40).

$$
\begin{aligned}
\Upsilon'(\lambda,i,x) = {} & \textbf{if } i=0 \textbf{ then } \mathit{stop}(+x) && (52)\\
& \textbf{else let } j=\lambda_i;\ d'=\mathrm{dom}'\big((i,j),(\mathrm{H}),x\big) \textbf{ in } \mathit{cont}(d',d')\\
& \textbf{fi}\\
\Lambda'(\lambda,j,x) = {} & \textbf{if } j=0 \textbf{ then } \mathit{stop}(-x) && (53)\\
& \textbf{else let } i={}^t\lambda_j;\ d'=\mathrm{dom}'\big((i,j),(\mathrm{V}),x\big) \textbf{ in } \mathit{cont}\big(d',d'\big)\\
& \textbf{fi}\\
\Gamma_0(\lambda,d,e) = {} & \textbf{if } \pi(d)\neq\pi(e) \textbf{ then } \mathit{cont}(d,e) && (54)\\
& \textbf{else let } (r,c)=\pi(d) \textbf{ in}\\
& \qquad \textbf{case } \mathrm{typ}(d)=\mathrm{typ}(e)=(\mathrm{H})\colon \Upsilon(\lambda,r-1,|d|)\\
& \qquad ,\ \mathrm{typ}(d)=\mathrm{typ}(e)=(\mathrm{V})\colon \Lambda(\lambda,c-1,|d|)\\
& \qquad ,\ \{\mathrm{typ}(d),\mathrm{typ}(e)\}=\{(\mathrm{V}),(\mathrm{H})\}\colon\\
& \qquad\qquad \textbf{let } a=\mathrm{dom}'\big((r-1,c),(\mathrm{H}),|d|\big)\\
& \qquad\qquad ;\quad b=\mathrm{dom}'\big((r,c-1),(\mathrm{V}),|d|\big) \textbf{ in}\\
& \qquad\qquad \textbf{if } \mathrm{typ}(d)=(\mathrm{V}) \textbf{ then } \mathit{cont}(a,b) \textbf{ else } \mathit{cont}(b,a) \textbf{ fi}\\
& \qquad \textbf{esac}\\
& \textbf{fi}\\
\Gamma_1(T,e) = {} & \textbf{let } \lambda=\mathrm{sh}(T);\ (d:T^{\downarrow})=T \textbf{ in} && (55)\\
& \textbf{case } \Gamma_0(\lambda,d,e) \textbf{ of } \mathit{stop}(w_n)\colon (T^{\downarrow},w_n)\\
& ,\ \mathit{cont}(d',e')\colon \textbf{ let } (S,w_n)=\Gamma_1(T^{\downarrow},e') \textbf{ in } (d':S,w_n)\\
& \textbf{esac}\\
\Gamma_2(T,T') = {} & \textbf{if } T=\odot \textbf{ then } \odot && (56)\\
& \textbf{else let } (e:T'^{\downarrow})=T';\ (T_{[e]},w_n)=\Gamma_1(T,e) \textbf{ in } \Gamma_2(T_{[e]},T'^{\downarrow})\ \&\ w_n\\
& \textbf{fi}
\end{aligned}
$$

For Γ_2 we have more pleasing combinatorial properties than for ϕ_2: it defines a bijection from the set of pairs of domino tableaux of the same shape to H_n; no equivalence classes or group actions are involved as in 4.5.5. In fact inverses of both Γ_1 and Γ_2 can be defined in a straightforward way. This is unlike ϕ_2 which has only a right-inverse while ϕ_1 has no inverse at all in the strict sense; see appendix C for details on this matter.

Despite these differences, ϕ_2 and Γ_2 seem to be very closely related, closer even than would follow from the similarity of their definitions. One would expect that the small differences in their definitions could lead to a major divergence as the algorithm proceeds, but if we compare a pair of signed domino tableaux that is mapped by ϕ_2 to a particular $w \in H_n$ with the pair of domino tableaux that are mapped to w by Γ_2, then we generally find that they are quite similar. Experimental evidence suggests that, ignoring signs in the tableaux, the differences are limited to chains of dominoes that are "cycled around" by one square (i.e., half a domino), and that these chains lie within a single cluster, the chain being open or closed as the cluster is. As an example here are two such pairs and their succesive modifications at each major step of the algorithm; the tableaux at the left correspond to case D_n. The differences remain present up to the last step, but both pairs correspond to the same signed permutation $(+2, +6, +3, +5, -4, +1, -7)$.



Incidentally, the computation of ϕ_2 in case C_n that leads to the same signed permutation is almost identical to the one shown on the *right*; only a single '$-$'-sign has to be added in the third line. It is clear that some interesting conjectures may be made about the relationship between ϕ_2 and Γ_2, but they will have to be made very precise before they can be proved.

Appendix B. Tabulated results.

We tabulate the values computed by ϕ_2 in all cases with $n = 2$ and with $n = 3$. The first argument of ϕ_2 appears along the left border of the tables, the second argument along the top border; only a single representative of each equivalence class of tableaux is listed. In the cases D_n we only tabulate pairs of tableaux corresponding to the same connected component of $\mathcal{F}_u$ (i.e., with the same product of signs), and if these components are interchanged by the action of A_u only those corresponding to one of the components. Instead of 1×1 tables we simply give a tableau and the corresponding signed permutation. For sake of compactness we omit the parentheses from signed permutations, and write the minus-signs *above* the number they apply to while plus-signs are omitted. We give the cases in the order of increasing complexity, namely D_2, C_2, B_2, D_3, C_3, B_3.

D_2: $\begin{smallmatrix}1&2\\+&\end{smallmatrix}$: $1, 2$ $\begin{smallmatrix}\begin{smallmatrix}1\\+\end{smallmatrix}&2\end{smallmatrix}$: $2, 1$ $\begin{smallmatrix}\begin{smallmatrix}1\\-\end{smallmatrix}&2\end{smallmatrix}$: $\overline{2}, \overline{1}$ $\begin{smallmatrix}1\\+\\2\\+\end{smallmatrix}$: $\overline{1}, \overline{2}$

C_2: $\begin{smallmatrix}1&2\end{smallmatrix}$: $1, 2$

	$\begin{smallmatrix}1&\begin{smallmatrix}2\\+\end{smallmatrix}\end{smallmatrix}$	$\begin{smallmatrix}1\\2\end{smallmatrix}$	$\begin{smallmatrix}1&\begin{smallmatrix}2\\-\end{smallmatrix}\end{smallmatrix}$
$\begin{smallmatrix}1&\begin{smallmatrix}2\\+\end{smallmatrix}\end{smallmatrix}$	$\overline{1}, 2$	$2, \overline{1}$	$\overline{2}, \overline{1}$
$\begin{smallmatrix}1\\2\end{smallmatrix}$	$\overline{2}, 1$	$2, 1$	$\overline{2}, 1$
$\begin{smallmatrix}1&\begin{smallmatrix}2\\-\end{smallmatrix}\end{smallmatrix}$	$\overline{2}, \overline{1}$	$2, \overline{1}$	$\overline{1}, 2$

$\begin{smallmatrix}1&\\2&\end{smallmatrix}$: $1, \overline{2}$ $\begin{smallmatrix}1\\2\end{smallmatrix}$: $\overline{1}, \overline{2}$

B_2: $\begin{smallmatrix}0&1&2\end{smallmatrix}$: $1, 2$

	$\begin{smallmatrix}0&1\\2&\\+&\end{smallmatrix}$	$\begin{smallmatrix}0&2\\1&\\+&\end{smallmatrix}$	$\begin{smallmatrix}0&1\\2&\\-&\end{smallmatrix}$
$\begin{smallmatrix}0&1\\2&\\+&\end{smallmatrix}$	$2, 1$	$\overline{2}, 1$	$1, \overline{2}$
$\begin{smallmatrix}0&2\\1&\\+&\end{smallmatrix}$	$2, \overline{1}$	$\overline{1}, 2$	$2, \overline{1}$
$\begin{smallmatrix}0&1\\2&\\-&\end{smallmatrix}$	$1, \overline{2}$	$\overline{2}, 1$	$2, 1$

$\begin{smallmatrix}\begin{smallmatrix}0\\1\\+\end{smallmatrix}&2\end{smallmatrix}$: $\overline{2}, \overline{1}$ $\begin{smallmatrix}0\\1\\+\\2\\+\end{smallmatrix}$: $\overline{1}, \overline{2}$

D_3: $\begin{smallmatrix}1&2&3\\+&&\end{smallmatrix}$: $1, 2, 3$

	$\begin{smallmatrix}1&3\\+&\\2&\\+&\end{smallmatrix}$	$\begin{smallmatrix}1&2\\+&\\3&\\+&\end{smallmatrix}$
$\begin{smallmatrix}1&3\\+&\\2&\\+&\end{smallmatrix}$	$\overline{1}, \overline{2}, 3$	$\overline{1}, 3, \overline{2}$
$\begin{smallmatrix}1&2\\+&\\3&\\+&\end{smallmatrix}$	$\overline{1}, \overline{3}, 2$	$\overline{1}, 2, \overline{3}$

	$\begin{smallmatrix}\begin{smallmatrix}1\\+\end{smallmatrix}&2\\3&\\+&\end{smallmatrix}$	$\begin{smallmatrix}\begin{smallmatrix}1\\-\end{smallmatrix}&2\\3&\\-&\end{smallmatrix}$	$\begin{smallmatrix}\begin{smallmatrix}1\\+\end{smallmatrix}&3\\2&\\+&\end{smallmatrix}$
$\begin{smallmatrix}\begin{smallmatrix}1\\+\end{smallmatrix}&2\\3&\\+&\end{smallmatrix}$	$3, 2, 1$	$\overline{2}, 1, \overline{3}$	$\overline{2}, \overline{3}, 1$
$\begin{smallmatrix}\begin{smallmatrix}1\\-\end{smallmatrix}&2\\3&\\-&\end{smallmatrix}$	$2, \overline{1}, \overline{3}$	$\overline{3}, 2, \overline{1}$	$2, \overline{3}, \overline{1}$
$\begin{smallmatrix}\begin{smallmatrix}1\\+\end{smallmatrix}&3\\2&\\+&\end{smallmatrix}$	$3, \overline{1}, \overline{2}$	$\overline{3}, 1, \overline{2}$	$1, \overline{3}, \overline{2}$

	$\begin{smallmatrix}\begin{smallmatrix}1\\+\end{smallmatrix}&2&\begin{smallmatrix}3\\+\end{smallmatrix}\end{smallmatrix}$	$\begin{smallmatrix}\begin{smallmatrix}1\\+\end{smallmatrix}&\begin{smallmatrix}2\\3\end{smallmatrix}\end{smallmatrix}$	$\begin{smallmatrix}\begin{smallmatrix}1\\-\end{smallmatrix}&2&\begin{smallmatrix}3\\-\end{smallmatrix}\end{smallmatrix}$
$\begin{smallmatrix}\begin{smallmatrix}1\\+\end{smallmatrix}&2&\begin{smallmatrix}3\\+\end{smallmatrix}\end{smallmatrix}$	$2, 1, 3$	$2, 3, 1$	$\overline{3}, \overline{2}, 1$
$\begin{smallmatrix}\begin{smallmatrix}1\\+\end{smallmatrix}&\begin{smallmatrix}2\\3\end{smallmatrix}\end{smallmatrix}$	$3, 1, 2$	$1, 3, 2$	$\overline{3}, \overline{1}, 2$
$\begin{smallmatrix}\begin{smallmatrix}1\\-\end{smallmatrix}&2&\begin{smallmatrix}3\\-\end{smallmatrix}\end{smallmatrix}$	$3, \overline{2}, \overline{1}$	$\overline{2}, 3, \overline{1}$	$\overline{2}, \overline{1}, 3$

$\begin{smallmatrix}1\\+\\2\\+\\3\\+\end{smallmatrix}$: $1, \overline{2}, \overline{3}$

C_3: $\begin{smallmatrix}1&2&3\end{smallmatrix}$: $1, 2, 3$ $\begin{smallmatrix}1\\2\\3\end{smallmatrix}$: $\overline{1}, \overline{2}, \overline{3}$

	$\begin{smallmatrix}1&\begin{smallmatrix}2\\+\end{smallmatrix}&3\end{smallmatrix}$	$\begin{smallmatrix}1&3\\2&\end{smallmatrix}$	$\begin{smallmatrix}1&2\\3&\end{smallmatrix}$	$\begin{smallmatrix}1&\begin{smallmatrix}2\\-\end{smallmatrix}&3\end{smallmatrix}$
$\begin{smallmatrix}1&\begin{smallmatrix}2\\+\end{smallmatrix}&3\end{smallmatrix}$	$\overline{1}, 2, 3$	$2, \overline{1}, 3$	$2, 3, \overline{1}$	$\overline{2}, \overline{1}, 3$
$\begin{smallmatrix}1&3\\2&\end{smallmatrix}$	$\overline{2}, 1, 3$	$2, 1, 3$	$2, 3, 1$	$\overline{2}, 1, 3$
$\begin{smallmatrix}1&2\\3&\end{smallmatrix}$	$\overline{3}, 1, 2$	$3, 1, 2$	$1, 3, 2$	$\overline{3}, 1, 2$
$\begin{smallmatrix}1&\begin{smallmatrix}2\\-\end{smallmatrix}&3\end{smallmatrix}$	$\overline{2}, \overline{1}, 3$	$2, \overline{1}, 3$	$2, 3, \overline{1}$	$\overline{1}, 2, 3$

	$\begin{smallmatrix}1&2\\3&\end{smallmatrix}$	$\begin{smallmatrix}1&3\\2&\end{smallmatrix}$
$\begin{smallmatrix}1&2\\3&\end{smallmatrix}$	$1,2,\overline{3}$	$1,\overline{3},2$
$\begin{smallmatrix}1&3\\2&\end{smallmatrix}$	$1,3,\overline{2}$	$1,\overline{2},3$

	$\begin{smallmatrix}1&2&3\\&+&\end{smallmatrix}$	$\begin{smallmatrix}1&3\\2&\end{smallmatrix}$	$\begin{smallmatrix}1&2&3\\&-&\end{smallmatrix}$
$\begin{smallmatrix}1&2&3\\&+&\end{smallmatrix}$	$\overline{1},3,2$	$3,\overline{1},2$	$\overline{3},\overline{1},2$
$\begin{smallmatrix}1&3\\2&\end{smallmatrix}$	$\overline{2},3,1$	$3,\overline{2},1$	$\overline{3},\overline{2},1$
$\begin{smallmatrix}1&2&3\\&-&\end{smallmatrix}$	$\overline{2},3,\overline{1}$	$3,\overline{2},\overline{1}$	$\overline{3},\overline{2},\overline{1}$

$\begin{smallmatrix}1\\2\\3\end{smallmatrix}: 1,\overline{2},\overline{3}$

	$\begin{smallmatrix}1\\2\\3\end{smallmatrix}$	$\begin{smallmatrix}1&\\2&3\\&+\end{smallmatrix}$	$\begin{smallmatrix}1&2\\&+\\3&\end{smallmatrix}$
$\begin{smallmatrix}1\\2\\3\end{smallmatrix}$	$3,2,1$	$2,\overline{3},1$	$\overline{3},2,1$
$\begin{smallmatrix}1&\\2&3\\&+\end{smallmatrix}$	$3,1,\overline{2}$	$1,\overline{3},\overline{2}$	$\overline{3},1,\overline{2}$
$\begin{smallmatrix}1&2\\&+\\3&\end{smallmatrix}$	$3,2,\overline{1}$	$2,\overline{3},\overline{1}$	$\overline{3},2,\overline{1}$

	$\begin{smallmatrix}1&2\\&+\\3&\end{smallmatrix}$	$\begin{smallmatrix}1&3\\&+\\2&\end{smallmatrix}$	$\begin{smallmatrix}1\\2\\3\end{smallmatrix}$	$\begin{smallmatrix}1&3\\&-\\2&\end{smallmatrix}$	$\begin{smallmatrix}1&2\\&-\\3&\end{smallmatrix}$
$\begin{smallmatrix}1&2\\&+\\3&\end{smallmatrix}$	$\overline{1},2,\overline{3}$	$\overline{1},\overline{3},2$	$2,\overline{1},\overline{3}$	$\overline{2},\overline{3},\overline{1}$	$\overline{2},\overline{1},\overline{3}$
$\begin{smallmatrix}1&3\\&+\\2&\end{smallmatrix}$	$\overline{1},3,\overline{2}$	$\overline{1},\overline{2},3$	$3,\overline{1},\overline{2}$	$\overline{1},\overline{3},\overline{2}$	$\overline{3},\overline{1},\overline{2}$
$\begin{smallmatrix}1\\2\\3\end{smallmatrix}$	$\overline{2},1,\overline{3}$	$\overline{2},\overline{3},1$	$2,1,\overline{3}$	$\overline{2},\overline{3},1$	$\overline{2},1,\overline{3}$
$\begin{smallmatrix}1&3\\&-\\2&\end{smallmatrix}$	$\overline{3},\overline{1},\overline{2}$	$\overline{1},\overline{3},\overline{2}$	$3,\overline{1},\overline{2}$	$\overline{1},\overline{2},3$	$\overline{1},3,\overline{2}$
$\begin{smallmatrix}1&2\\&-\\3&\end{smallmatrix}$	$\overline{2},\overline{1},\overline{3}$	$\overline{2},\overline{3},\overline{1}$	$2,\overline{1},\overline{3}$	$\overline{1},\overline{3},2$	$\overline{1},2,\overline{3}$

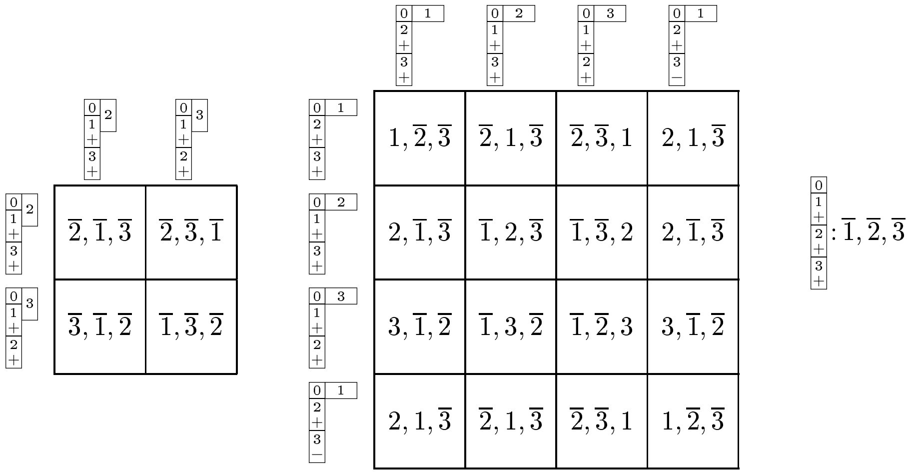
B_3:

$\begin{smallmatrix}0&1&2&3\end{smallmatrix}: 1,2,3$

	$\begin{smallmatrix}0&1&2\\3&&\\+&&\end{smallmatrix}$	$\begin{smallmatrix}0&1&3\\2&&\\+&&\end{smallmatrix}$	$\begin{smallmatrix}0&2&3\\1&&\\+&&\end{smallmatrix}$	$\begin{smallmatrix}0&1&3\\2&&\\-&&\end{smallmatrix}$	$\begin{smallmatrix}0&1&2\\3&&\\-&&\end{smallmatrix}$
$\begin{smallmatrix}0&1&2\\3&&\\+&&\end{smallmatrix}$	$1,3,2$	$3,1,2$	$\overline{3},1,2$	$1,\overline{3},2$	$1,2,\overline{3}$
$\begin{smallmatrix}0&1&3\\2&&\\+&&\end{smallmatrix}$	$2,3,1$	$2,1,3$	$\overline{2},1,3$	$1,\overline{2},3$	$1,3,\overline{2}$
$\begin{smallmatrix}0&2&3\\1&&\\+&&\end{smallmatrix}$	$2,3,\overline{1}$	$2,\overline{1},3$	$\overline{1},2,3$	$2,\overline{1},3$	$2,3,\overline{1}$
$\begin{smallmatrix}0&1&3\\2&&\\-&&\end{smallmatrix}$	$1,3,\overline{2}$	$1,\overline{2},3$	$\overline{2},1,3$	$2,1,3$	$2,3,1$
$\begin{smallmatrix}0&1&2\\3&&\\-&&\end{smallmatrix}$	$1,2,\overline{3}$	$1,\overline{3},2$	$\overline{3},1,2$	$3,1,2$	$1,3,2$

	$\begin{smallmatrix}0&2&3\\1&&+\\+&&\end{smallmatrix}$	$\begin{smallmatrix}0&2\\1&3\\+&\end{smallmatrix}$	$\begin{smallmatrix}0&1\\2&3\\+&\end{smallmatrix}$	$\begin{smallmatrix}0&2&3\\1&&-\\+&&\end{smallmatrix}$
$\begin{smallmatrix}0&2&3\\1&&+\\+&&\end{smallmatrix}$	$\overline{2},\overline{1},3$	$\overline{2},3,\overline{1}$	$3,\overline{2},\overline{1}$	$\overline{3},\overline{2},\overline{1}$
$\begin{smallmatrix}0&2\\1&3\\+&\end{smallmatrix}$	$\overline{3},\overline{1},2$	$\overline{1},3,2$	$3,\overline{1},2$	$\overline{3},\overline{1},2$
$\begin{smallmatrix}0&1\\2&3\\+&\end{smallmatrix}$	$\overline{3},\overline{2},1$	$\overline{2},3,1$	$3,\overline{2},1$	$\overline{3},\overline{2},1$
$\begin{smallmatrix}0&2&3\\1&&-\\+&&\end{smallmatrix}$	$\overline{3},\overline{2},\overline{1}$	$\overline{2},3,\overline{1}$	$3,\overline{2},\overline{1}$	$\overline{2},\overline{1},3$

	$\begin{smallmatrix}0&1\\2&3\\+&\end{smallmatrix}$	$\begin{smallmatrix}0&1\\2&3\\-&\end{smallmatrix}$	$\begin{smallmatrix}0&2\\1&3\\+&\end{smallmatrix}$
$\begin{smallmatrix}0&1\\2&3\\+&\end{smallmatrix}$	$3,2,1$	$2,\overline{3},1$	$\overline{3},2,1$
$\begin{smallmatrix}0&1\\2&3\\-&\end{smallmatrix}$	$3,1,\overline{2}$	$1,\overline{3},\overline{2}$	$\overline{3},1,\overline{2}$
$\begin{smallmatrix}0&2\\1&3\\+&\end{smallmatrix}$	$3,2,\overline{1}$	$2,\overline{3},\overline{1}$	$\overline{3},2,\overline{1}$



Appendix C. Computer program.

ALGOL 68 is a language in which algorithms may be formulated for computers, i.e., for automata and for human beings.
A. van Wijngaarden et al., *[Wijn 1.1.1.a]*

We present a computer program with routines that perform the main algorithm ϕ_2 of this thesis, and a right-inverse of ϕ_2. In contrast to the formalism used to define ϕ_2, the program is written in a "real" programming language, Algol 68, that is particularly suited for this form of communicating algorithms. This language was chosen despite its nowadays limited availability, because—contrary to its competitors—it is well defined (see [Wijn]), succinct and elegant.

The program is listed at the end of this appendix; here we give comments that may help to understand the way its works. A number of transformations have been made to the definition of ϕ_2, eliminating the auxiliary functions ϕ_0 and ϕ_1, and resulting in an iterative rather than recursive algorithm. First, we note that in (39) the domino e' never has $\mathrm{sg}(e') =$ '$-$' unless $e' = e$. Therefore, if the domino e that is a argument to the initial call of ϕ_1 has $\mathrm{sg}(e) \neq$ '$-$', then so will all the corresponding dominoes in recursive calls; moreover, this initial condition can always be satisfied by applying the diagonal action of some $g_i \in A_\lambda$ to the pair of tableaux if necessary. When this has been done, conditions as $\mathrm{sg}(d) = \mathrm{sg}(e)$ can be rewritten as $\mathrm{sg}(d) =$ '+', and more importantly, case *7* never causes $\omega_2 = \mathit{flip}(r)$ in (39). If follows that $\omega_2 = \mathit{flip}(r)$, which can be caused only by cases *3*, *4b* or *6*, implies $\mathrm{typ}(e') = (\mathrm{I}-)$ and $\rho_{e'} = r$, and its effect can be predicted as follows.

In the subsequent calls of ϕ_0, possibly after a number of instances of case *0* whose dominoes d do not meet row r, we encouter case *5* or *5′* with e equal to e' above. If it is case *5′* then row r will be empty in the modified tableau returned by this innermost call of ϕ_1, and this will remain true when returning from the intermediate calls of ϕ_1 (with case *0*); therefore ξ_r will be applied to a tableau whose row r is empty, and have no effect. If we have case *5*, then a modified tableau is constructed whose leading domino d' has $\mathrm{typ}(d') = (\mathrm{I}+)$, $\mathrm{sg}(d') =$ '+' and $\kappa_{d'} = \kappa_e - 1$. As in the previous case no dominoes intervene in row r, so ξ_r will effectively be applied to this modified tableau, and we see that this affects the sign of the cluster $\mathrm{cl}(d')$ (although possibly not of d' itself), and an *equivalent* effect can be achieved by putting $\mathrm{sg}(d') =$ '−' instead of $\mathrm{sg}(d') =$ '+'. Therefore if we record the fact that an application of ξ_r is 'pending', it can be executed during the next instance of case *5* (or simply forgotten in case *5′*); note that case *5* can also occur after cases *2* and *4a*, in which case we should put $\mathrm{sg}(d') =$ '+' as in the original formulation.

In this way each domino of the modified tableau can get its eventual value right away, and need not be reconsidered until the next major step of ϕ_2. The fact that new dominoes are computed while we are still taking apart the original tableau is no real problem: they can be simply put aside and be incorporated in the tableau at the appropriate time. In this way we have effectively transformed the recursion to iteration.

We now discuss the actual program. First come some variables that determine the considered case, and some mode declarations. A fairly straightforward representation of dominoes and signed permutations is chosen, but note that dominoes don't contain entries; these are implied by the context when relevant. In order to achieve this, tableaux are represented as linear arrays of dominoes, the selecting index being the entry of the domino. Since the sequence of entries present at any moment need not be consecutive, a union is in fact taken with the singleton mode **void**, whose sole value **empty** indicates that the entry in question is absent (other encodings may be proposed, but this is the most natural one). This will necessitate conformity (**case**) clauses with a single unit—for the mode **domino**—in several places. For efficiency reasons tableaux also contain two partitions, giving their shape and transpose shape, which are incrementally updated.

Putting aside dominoes and re-incorporating them is done just by modifying the shape, leaving the dominoes themselves in place. The first few routines are rather trivial, dealing mainly with shapes; they are followed by the routines *xi*, …, *phi*2 that implement ξ, …, ϕ_2, where the three functions ϕ are merged together to *phi*2.

Whether an application of ξ is pending is recorded in *saved sign*, that is also transferred to *up* and *left*; it is set whenever we assign '*tp* **of** $e := Imin$', and it is used by *left* in the case corresponding to *5*. The abbreviated form '(| |)' of conditional clauses is occasionally used. For clarity we have given '*sg* **of** e' its proper value in all cases, but it is never used; therefore some optimisations can still be made.

After *phi*2 we proceed to its right-inverse, *psi*2. The largest part of its definition is guided by simply "tracing back the history of *phi*2", but there is a difficulty since ϕ_1 is not strictly invertible: pairs like $\big(\boxed{\substack{2\\-}}, -1\big)$ or $\big(\boxed{\substack{2\\-}}, +1\big)$ are never yielded, although $\big(\boxed{\substack{2\\-}}\boxed{3}, -1\big)$ is. The solution is to apply the diagonal action of a suitable $g_i \in A_\lambda$ on the pair of tableaux when necessary, to eliminate the problematic left tableau. It can however not easily be predicted beforehand whether this is necessary, or which g_i should be used, so we reason as follows, on the basis of the algorithm *phi*2. Each time we encounter a domino of type (I+) in the left tableau (except in case *0*) we initially assume that its sign is due to *saved sign*, and record it. We then continue the trace back with a domino of type (I−) as candidate for e', and one of the following situations may arise.

If we next meet a domino of type (N), then we have case *6*, and the recorded sign is inserted in the new domino d of type (I+) in the left tableau. If the recorded sign is '−', and we meet a domino of type (I−) with the same support as e', then this must have been case *3*, accounting for the recorded sign, and we proceed correspondingly. If we meet a domino of type (I−) when the recorded sign is '+', then we assume that this is case *2* or *4a* if such is possible, i.e., if the next row is sufficiently short (this is similar to the distinction between cases *3* and *4a*). If this is the case, then we can proceed without difficulty. If however the next row is too long to allow cases *2* or *4a*, then it can only have been case *3*, so our assumption about *saved sign* has to be revised. Fortunately it can be verified that the domino whose sign was recorded cannot have been the only one in its cluster in this case, so the situation can be remedied by a redistribution of signs. Although this redistribution should conceptually be performed in the left tableau *before* our back-tracing modification, it can be derived from 4.5.5 that in this case applying ξ_{r+1} commutes, up to equivalence, with the modification of the tableau, so there is no need to start anew with tracing back *phi*2.

Finally there is the possibility that none of the above cases occurs, i.e., that possibly after a sequence of cases *0* we have traced back to point corresponding to the initial call of ϕ_1. If in this case the recorded sign is '+'—the value to which *saved sign* is initialised in *phi*2—there is no difficulty and a domino (of type (I−)) can simply be added to the right tableau. If the recorded sign is '−', we are in the case where ϕ_1 cannot be strictly inverted, and the diagonal action of $g_r \in A_\lambda$ has to be invoked, where r is the row of the domino to be added on the right. Conceptually this should be done before the back-tracing process, so its effect on the left tableau may be taken to be the correcting of the "wrong" sign recorded, and no explicit action should be taken; on the right tableau we apply ξ_r before adding the domino.

And now finally: here is the program itself.

```
begin loc int n, case # These define the case considered #
,  int Bn = 2, Cn = 3, Dn = 4
,  proc eps = (int j) bool: odd(j + case) # whether or not ε_j = +1 #
,  mode type = int, type Non = 1, Iplus = 2, Imin = 3
,  mode sign  = int # , sign plus = +1, minus = −1, absent = 0 #
   , domino    = struct(int row, col, type tp, sign sg)
   , partition = [2 ∗ n + abs(case = Bn)] int
   , tableau   = struct([n] union(domino, void) dom, partition sh, tr)
   , signperm = [n] int
,  op lc = (domino d) int: col of d − abs(tp of d = Non) # left column #
,  proc clear = (ref partition p) void:
   begin for i to upb p do p[i] := 0 od; if case = Bn then p[1] := 1 fi end
,  proc extend = (ref tableau T, domino d) void: # extend shape of T #
   begin int r = row of d, c = col of d, ref partition sh = sh of T, tr = tr of T
   ; sh[r] := c; tr[c] := r; if tp of d = Non then tr[c − 1] := r else sh[r − 1] := c fi
   end
,  proc set shape = (ref tableau T) void:
   begin clear(sh of T); clear(tr of T)
   ; for i to n do case (dom of T)[i] in (domino d): extend(T, d) esac od
   end
,  op ↓ = (ref tableau T, int x) void: # inverse of extend(T, (dom of T)[x]) #
   case (dom of T)[x] in (domino d):
     if int r = row of d, c = col of d, ref partition sh = sh of T, tr = tr of T
     ; tp of d = Non then sh[r] −:= 2; tr[c − 1] −:= 1; tr[c] −:= 1
                     else  tr[c] −:= 2; sh[r − 1] −:= 1; sh[r] −:= 1
     fi
   esac
,  proc xi = (int r, ref tableau T, int last) void: # T[≤ last] := ξ_r(T[≤ last]) #
   begin loc bool busy := true, loc domino dd
   ;  for x from last to 1 while busy do case (dom of T)[x] in (domino d):
        if tp of d = Iplus and (r = row of d − 1 or r = row of d)
        then dd := d; sg of dd := −sg of d; (dom of T)[x] := dd; busy := false
        fi
     esac od
   end
,  proc up =
   (ref tableau T, int i, ref domino e, ref sign sg) union(sign, domino):
   if i = 0 then +1 # sign for in permutation #
   else int j = (sh of T)[i]; e := (i, j, (eps(j) | Non | sg := +1; Imin), 0)
   fi
```

```
, proc left =
  (ref tableau T, int j, ref domino e, ref sign sg) union(sign, domino):
  if j = 0 then −1 # sign for in permutation #
  elif int i = (tr of T)[j]; eps(j) then e := (i, j, Iplus, +1); domino(i, j, Iplus, sg)
  else sg := −1; e := (i, j, Imin, 0)
  fi
, [ , ] int branch table 1 = ((1, 5, 0), (4, 2, 0), (0, 0, 3))
, proc phi2 = (ref tableau T, Tprime) signperm:
  begin n := upb dom of T; loc signperm w; set shape(Tprime)
  ; for y from n to 1 do case (dom of Tprime)[y] in (domino e0):
    begin sh of T := sh of Tprime; tr of T := tr of Tprime; Tprime ↓ y
    ; loc domino e := e0, loc sign saved sign := +1, loc bool busy := true
    ; (tp of e = Iplus and sg of e = −1 | xi(row of e, T, n); sg of e := +1)
    ; for x from n to 1 while busy do case (dom of T)[x] in (domino d):
      begin int r = row of d, c = col of d; T ↓ x
      ; if c = col of e then case union(sign, domino) # cast next clause: #
            case branch table 1[tp of d, tp of e]
            in # (N),(N) #   up(T, r − 1, e, saved sign)
            , # (I+), (I+) # if r − (tr of T)[c + 1] = 2 and sg of d = +1
                             then up(T, r − 2, e, saved sign)
                             else (sg of d = +1 | xi(r − 2, T, x − 1))
                             ; left(T, c − 1, e, saved sign)
                             fi
            , # (I−), (I−) # left(T, c − 1, e, saved sign)
            , # (I+), (N) #  (saved sign := sg of d
                             ; e := (r, c − 1, Imin, 0); domino(r − 1, c, Non, 0))
            , # (N), (I+) #  (e := (r − 1, c, Non, 0); domino(r, c − 1, Imin, 0))
            esac
          in (sign s): (w[y] := s ∗ x; (dom of T)[x] := empty; busy := false)
          , (domino dd): (dom of T)[x] := dd
          esac fi
      end esac od
    end esac od; w
  end
, proc down = (ref tableau T, int i, ref domino e) domino:
  e := (int j = (sh of T)[i]; eps(j) | (i, j + 2, Non, 0) | (i + 1, j + 1, Iplus, +1))
, proc right = (ref tableau T, int j, ref domino e) domino:
  if int i = (tr of T)[j]; eps(j)
  then e := (i + 2, j, Iplus, +1); (i + 2, j, Iplus, −1) else e := (i + 2, j, Imin, 0)
  fi
```

```
, [,] int branch table 2 = ((1,0,4),(0,3,0),(5,0,2))
, proc psi2 = (signperm w) struct(tableau left, right):
  begin n := upb w; loc tableau T, Tprime
  ; for i to n do (dom of T)[i] := empty od; clear(sh of T); clear(tr of T)
  ; for y to n
    do int x0 = abs w[y]; for x from n by −1 to x0 + 1 do T ↓ x od
    ; loc domino e; (dom of T)[x0] := (w[y] > 0 | down | right)(T, 1, e)
    ; loc sign saved sign := −1
    ; for x from x0 + 1 to n do case (dom of T)[x] in (domino d):
      begin int r = row of d, c = col of d; extend(T, d)
      ; if lc d = lc e then (dom of T)[x] := domino # body of cast follows: #
          case branch table 2[tp of d, tp of e]
          in # (N),(N) #  down(T, r + 1, e)
          , # (I−),(I−) # if (sh of T)[r + 1] < c − 1 and saved sign = +1
                          then down(T, r + 1, e)
                          else (saved sign = +1 | xi(r + 1, T, x − 1))
                          ; right(T, c + 1, e)
                          fi
          , # (I+),(I+) # (saved sign := sg of d; right(T, c + 1, e))
          , # (N),(I−) #  (e := (r, c + 1, Non, 0); (r, c + 1, Iplus, saved sign))
          , # (I−),(N) #  (e := (r + 1, c − 1, Iplus, +1); (r + 1, c, Non, 0))
          esac
        fi
      end esac od
    ; (tp of e = Imin and saved sign = −1 | xi(row of e, Tprime, x − 1))
    ; (dom of Tprime)[x] := e
    od
  ; sh of Tprime := sh of T; tr of Tprime := tr of T; (T, Tprime)
  end
 # Example of use #
; for c from Bn to Dn
  do n := 7; case := c
  ; loc struct(tableau left, right)pair := psi2((4, −3, 7, −2, −1, −5, 6))
  ; print((phi2(left of pair, right of pair), newline))
  od
end
```

"Begin at the beginning," the King said, gravely,
"and go on till you come to the end:
then stop."
Lewis Carroll, *Alice in Wonderland*

List of symbols.

We list the symbols that either have a fixed meaning, or denote a variable object that assumes a fixed value throughout a significant part of this thesis. In some cases, such as $b_{j,N}$, part of the symbol is in fact variable; notations such as $T^{\downarrow}$ where the alphabetic part is variable are listed at the end.

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Samenvatting.

We beschouwen klassieke algebraïsche groepen, dat wil zeggen groepen bestaande uit symmetrieën van een vectorruimte, die in het door ons beschouwde geval een gegeven bilineaire vorm (inprodukt) behouden. In de betreffende vectorruimte beschouwen we vlaggen: ketens van een lijn, vlak, 3-dimensionale deelruimte enzovoort, die elk hun voorganger bevatten (en de lijn bevat de oorsprong). Twee vlaggen hebben een onderlinge ligging, welke bepaald is door vergelijking van delen van de ene vlag met ieder van de delen van de andere, waarbij slechts gekeken wordt naar de dimensie van de doorsnede van de delen, en bij aanwezigheid van een bilineaire vorm naar loodrechtheid. Er zijn slechts eindig veel mogelijkheden voor de onderlinge ligging; de verzameling hiervan vormt de Weyl-groep.

Voor een vast element u uit de groep, waarvoor we een unipotent zullen nemen, noemen we $\mathcal{F}_u$ de deelverzameling van die vlaggen die door u in zichzelf worden overgevoerd. Omdat een unipotent iedere vector slechts in een andere dan diens eigen richting kan bewegen, betekent dit dat de vectoren op de lijn uit de vlag op hun plaats blijven, die uit het vlak slechts evenwijdig aan de lijn kunnen bewegen, die uit de 3-dimensionale deelruimte slechts evenwijdig aan het vlak, enzovoorts. Nu blijkt $\mathcal{F}_u$ vaak uit een aantal irreducibele komponenten te bestaan, stukken die elkaar nog in een lager dimensionaal stuk kunnen snijden. Neem bijvoorbeeld (de bilineaire vorm even achterwege latend) voor u de unipotente transformatie van een 3-dimensionale vectorruimte die iedere vector in de z-richting verschuift over een afstand gelijk aan zijn x-coördinaat. Vlaggen—in dit geval bestaande uit een lijn en een vlak—die invariant zijn onder u moeten ofwel als lijn de z-as hebben (alle verschuivingen zijn hieraan evenwijdig), ofwel als vlak het y-z-vlak (deze vectoren worden niet verplaatst); er is één vlag die aan beide kondities tegelijk voldoet.

De vraag die in dit proefschrift beschouwd wordt, is die naar de generieke onderlinge ligging van een paar irreducibele komponenten van $\mathcal{F}_u$. Dat wil zeggen de onderlinge ligging van vlaggen uit de gegeven komponenten waarvoor geen toevallige incidenties plaatsvinden. Omdat samenvallen van twee vlaggen zeker een toevallige incidentie is—tenzij de gegeven komponenten uit slechts één vlag bestaan—is deze informatie ook interessant als we tweemaal dezelfde komponent kiezen. In het gegeven voorbeeld is de generieke onderlinge ligging van de eerstgenoemde komponent met zichzelf zodanig dat de lijnen samenvallen (beide de z-as) en derhalve ook in het vlak van de andere vlag liggen, maar de vlakken niet samenvallen; de generieke onderling ligging van de eerstgenoemde komponent met de tweede is zo dat de lijn van de eerste vlag in het vlak van de tweede vlag ligt maar niet andersom, en lijnen noch vlakken samenvallen.

Om de gestelde vraag konkreet te kunnen beantwoorden is het nodig dat we over parametriseringen, dat wil zeggen systematische benamingen, beschikken voor de irreducibele componenten van $\mathcal{F}_u$, en van de Weyl-groep. In het geval van een groep van zogenaamd type A_n (waarbij er geen bilineaire vorm in het spel is),

gebeurt de eerste parametrisering met behulp van 'Young tableaux', en de tweede met permutaties; de gestelde vraag wordt dan beantwoord door het Robinson-Schensted algoritme, dat paren Young tableaux koppelt aan permutaties, en welk algoritme al veel eerder bekend was dan dat de genoemde vraag bestudeerd is.

Voor de groepen van typen B_n, C_n en D_n die in dit proefschrift centraal staan is de parametrisering van de Weyl-groep nog redelijk eenvoudig (permutaties met tekens), maar die van de irreducibele komponenten van $\mathcal{F}_u$ heeft veel voeten in de aarde: onze hele §3 is eraan gewijd, en dit vereist ook nog een zorgvuldige analyse in §2 van de lineaire algebra van de vectorruimte met de gekozen unipotente transformatie. Het resulteert erin dat de komponenten kunnen worden benoemd door middel van 'domino-tableaux met tekens': kleine diagrammen met domino-vormige vakjes met daarin gehele getallen en soms ook tekens, zoals men ze in groten getale kunt aantreffen in 3.6 en appendix B. In §4 gebeurt het eigenlijke werk: er wordt een algoritme afgeleid dat bij paren van dergelijke domino-tableaux de permutaties met tekens berekent die de betreffende generieke onderlinge ligging beschrijft. Het algoritme is analoog aan, maar veel gekompliceerder dan, het Robinson-Schensted algoritme. In appendix C wordt een computer programma gegeven dat deze berekening, en ook een omkering ervan, kan uitvoeren.

Curriculum Vitae.

De schrijver van dit proefschrift werd geboren op 1 mei 1960 in Castricum. Hij bezocht het Erasmus College te Zoetermeer, waar hij in 1978 het diploma ongedeeld V.W.O. behaalde. Gedurende de laatste twee jaren van deze opleiding nam hij deel aan de Nederlandse wiskunde olympiade, waarbij hij de achtste respectievelijk tweede plaats behaalde; in die jaren nam hij ook deel aan de Internationale Wiskunde Olympiades in Belgrado en Boekarest, waar hem beide keren een tweede prijs werd toegekend en in Boekarest ook twee speciale prijzen. Vervolgens begon hij een studie natuurkunde aan de Rijksuniversiteit te Utrecht, waar hij na enige jaren overstapte naar wiskunde. In 1982 behaalde hij het kandidaatsexamen natuurkunde, en in 1984 het doctoraalexamen wiskunde; beide diploma's werden 'cum laude' toegekend. Na zijn doctoraalexamen was hij als wetenschappelijk assistent werkzaam bij de Rijksuniversiteit te Utrecht, waar hij wetenschappelijk onderzoek verrichtte, resulterende in dit proefschrift. Bovendien participeerde hij van 1984 tot 1986 in het project "Parallelle Reductie Machines", waarin hij onderzoek deed op het gebied van functionele programmeertalen.

to the uninitiated reader

Preface.

This thesis is devoted to a single problem, that is expressed most succinctly in the quotation heading 1.1, and is more fully explained in that subsection. The problem concerns the computation of a geometrically defined correspondence that arises in the study of algebraic groups. As such the problem is clearly delimited, and a satisfactory solution—for the cases considered—is actually given in this thesis, in the form of an algorithm that performs the computation. Before taking up the treatment of that problem, I would like to make a few remarks about the thesis itself.

When first I had found the algorithm, some two years ago now, I would not have guessed, although I was very pleased at finding a result I had been searching for for some time, that this single result would fill a thesis, nor that it would take so long to get all the details written down properly. At that time my solution of the problem consisted of an informal description of the algorithm (but clear enough to write a computer program) and a mental picture of how it was derived: arguments that now appear mostly in 4.3 and 4.4, but not then worked out in so much detail. In the process of formulating a precise description and proof, the text has expanded from just a few pages to its current size. The details of the algorithm had originally been deduced by careful inspection of particular cases where they were necessary—some of which had in fact been found by detecting by computer the failure of certain properties for premature versions of the algorithm—and combining the arguments that applied to these individual cases into a single, general proof, made that proof rather complicated. Also the nature of the arguments required the basic concepts used to be developed in considerable detail.

The resulting structure of this thesis is as follows. The main text is divided into four sections (chapters did not seem appropriate). The main result is 4.5.4, and the rest serves either the formulation, proof or clarification of that result. In §4 the actual derivation of the algorithm is given. This requires a parametrisation of irreducible components of $\mathcal{F}_u$, which is given in §3. Although the fundamental aspects of this parametrisation can be found in [Spa II.6], the algorithm is so intimately related to it that a complete description is in its place; meanwhile the opportunity is taken for a slight adaption, and the investigation of some additional aspects that are important for the computation. Both these sections use a number of facts of a linear algebraic nature, which are derived in §2. Finally, in §1 the problem and basic objects of study are introduced.

The arguments used in the proof—apart from an occasional reference to a result in the literature—are mostly of a very explicit nature, and can been be derived by elementary means, using only the basic theory of linear algebra and linear algebraic groups. If the dissertation is in any way difficult, this is because of the multitude of details and cases that have to be considered (the word 'case' occurs 428 times in all), not the 'depth' of the individual arguments. I have taken considerable effort to mention all relevant facts explicitly, and to present them in a coherent way; I hope that this has made the text accessible, despite its technical nature, to interested, not necessarily specialised, mathematicians. (The dedication of this thesis is therefore not merely a joke.) Since a lot of terms and notations are introduced, an index and a list of symbols are provided. Also an ample supply of examples are given at the end of §3 and §4, to provide an illustration of the various cases that occur.

A number of incidental matters, that did not fit into the structure of the main text, are treated in appendices. Appendix A describes the proper Robinson-Schensted algorithm (readers not acquainted with this algorithm may wish to turn to this appendix first), together with another algorithm that is closely related to the algorithm of the main text. In appendix B the outcomes of the main algorithm are tabulated for the groups of rank 2 and 3. Appendix C gives a computer program that performs the algorithm, and also provides an inverse algorithm.

In the remainder of the preface I would like to briefly sketch the setting in which the problem considered in this thesis has occurred. For a detailed discussion of these matters see [Spr2] and [Spr3], but the following may give a first indication of the geometric context. Note that the meaning of certain symbols differs from that in the main text.

Let G be a connected reductive group, W its Weyl group, B a Borel subgroup and $\mathcal{B} \cong G/B$ the variety of all Borel subgroups. Consider the variety $\widetilde{G} = G \times^B B$ (the notation is as in 3.1), where B acts on itself by conjugation; there is a G-equivariant projection $\pi: \widetilde{G} \to \mathcal{B}$ whose fibre at any $B' \in \mathcal{B}$ is canonically isomorphic to B'. These isomorphisms combine to a morphism $\varphi: \widetilde{G} \to G$ that is surjective and G-equivariant, where G acts on itself by conjugation. Any fibre $\varphi^{-1}[g]$ is isomorphic via π to the set of Borel subgroups containing g. The map φ, and more in particular its fibres $\varphi^{-1}[u]$ at unipotent elements, are of considerable interest in the detailed study of conjugacy classes of G. These fibres $\varphi^{-1}[u]$ are isomorphic to the varieties $\mathcal{B}_u$ in terms of which the central problem of this thesis is formulated.

Now let $V \subset G$ be the set of unipotent elements, let $U = V \cap B$ be the unipotent radical of B, and $\widetilde{V} = G \times^B U \subset \widetilde{G}$. Consider the fibre product $Y = \widetilde{V} \times_V \widetilde{V}$ of $\widetilde{V}$ with itself with respect to φ: there is a G-equivariant morphism $\Phi: Y \to V$ with $\Phi^{-1}[u] \cong \varphi^{-1}[u] \times \varphi^{-1}[u]$ for any $u \in V$. There is also a G-equivariant projection $\Pi: Y \to \mathcal{B} \times \mathcal{B}$ that is induced by π. The fibre of Π at the point (B', B'') is isomorphic via Φ to the set $V \cap B' \cap B''$, which is a connected subgroup of codimension $\mathrm{rk}(G)$ of the stabiliser $B' \cap B''$ of that point. It follows that $\dim(\Pi^{-1}[X]) = \dim(G) - \mathrm{rk}(G)$ for any G-orbit $X \subseteq \mathcal{B} \times \mathcal{B}$. Now the set of these orbits is in bijection with W via the notion of relative position (see 1.5); denote the orbit corresponding to $w \in W$ by $O(w)$. Since W is finite it follows that $\{\, \overline{\Pi^{-1}[O(w)]} \mid w \in W \,\}$ is the set of irreducible components of Y, which all have the same dimension.

On the other hand these components may be considered in relation to Φ. The following two facts are quite non-trivial in their generality, but they have been proved (by various people) in all cases: (a) there are finitely many G-orbits in V

(i.e., unipotent conjugacy classes in G) (b) for each such orbit $G \cdot u$, one has $\dim(G \cdot u) + \dim(\Phi^{-1}[u]) = \dim(Y)$. It follows from these facts that the irreducible components of Y can also be described as the closures of the irreducible components of $\Phi^{-1}[G \cdot u]$, where u ranges over the unipotent classes of G. Intersecting an irreducible component of $\Phi^{-1}[G \cdot u]$ with $\Phi^{-1}[u]$ yields an A_u-orbit of irreducible components of $\Phi^{-1}[u]$—where A_u is the group of (connected) components of the centraliser of u in G—and the irreducible components of $\Phi^{-1}[u]$ are the cartesian products of pairs of irreducible components of $\varphi^{-1}[u] \cong \mathcal{B}_u$. When G is simple of type A_n, the groups A_u are always trivial. The given two descriptions of the set of irreducible components of Y form the background of the correspondence that this thesis aims at computing for classical groups; the properties that this correspondence must have according to the above line of reasoning are formally expressed in 4.2.1.

It is recognised, however,
that this method may be difficult
for the uninitiated reader.
A. van Wijngaarden et al., *[Wijn 0.1.1]*

Acknowledgements.

Writing this thesis has been a rather solitary labour. Nonetheless I would like to thank a number op people who have contributed to its quality. First of all I thank prof. Springer who has guided and stimulated my mathematical work, and has given me very useful references at crucial points. I thank Wilberd van der Kallen for his meticulous reading of early versions of the typescript; his complaints and comments have revealed numerous unclear or incorrect formulations, and have led to a significant increase in length and—I hope—readability of the text. I thank Bram Broer for the interest he has always shown in my work, and more in particular for drawing my attention to references [St3] and [Spr2]. I wish to thank Bas Edixhoven and Jaap Top, who were willing, on numerous occasions, to discuss my rather trivial geometric questions. I thank Emmy Schaling (to whom the dedication of this thesis definitely applies) for indicating 32 errors in the penultimate version of the text. Finally I wish to express my gratitude to D. E. Knuth, who has had, without any personal involvement, an essential influence on this thesis in two very dissimilar ways: his very readable account [Kn] formed my introduction to the Robinson-Schensted algorithm, and the typesetting of this thesis has been done entirely with the TeX and METAFONT typesetting systems created by him. So without these two contributions, neither form nor content of this thesis would have been as they are.

... I would like to thank Andy Tanenbaum
for his fast and accurate typing
of innumerable versions of the manuscript.
Andrew S. Tanenbaum, *Computer Networks*

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