Stability of a cubically convergent method for generalized equations

Michel H. Geoffroy [1], **Saïd Hilout** [2] and **Alain Piétrus** [1]

April 5, 2006

Abstract. In [8] we showed the convergence of a cubic method for solving generalized equations of the form $0 \in f(x) + G(x)$ where f is a $\mathcal{C}^2$ function and G stands for a set-valued map. We investigate here the stability of such a method with respect to some perturbations. More precisely, we consider the perturbed equation $y \in f(x) + G(x)$ and we show that the pseudo-Lipschitzness of the map $(f + G)^{-1}$ is closely tied to the uniformity of our method in the sense that the attraction region does not depend on small perturbations of the parameter y. Finally, we provide an enhanced version of the convergence theorem established in [8].

Key words. Set-valued mapping, generalized equation, cubic convergence, pseudo-Lipschitzness.

AMS subject classification. 49J53, 47H04, 65K10

1 Introduction

Generalized equations are an abstract model of a wide variety of variational problems including linear and nonlinear complementarity problems, systems of nonlinear equations, variational inequalities (for example first-order necessary conditions for nonlinear programming) etc. In particular, they may characterize optimality or equilibrium and then have several applications

[1]Laboratoire Analyse Optimisation Contrôle, Université des Antilles et de la Guyane, Département de Mathématiques et Informatique, F–97159 Pointe–à–Pitre, France, E–mail: (Michel H. Geoffroy) michel.geoffroy@univ–ag.fr, E–mail: (Alain Piétrus) apietrus@univ–ag.fr

[2]Département de Mathématiques Appliquées et Informatique, Faculté des Sciences et Techniques, BP 523, 23000 Béni–Mellal, Maroc, E–mail: said.hilout@yahoo.fr

in engineering (analysis of elastoplastic structures, traffic equilibrium problems...) and economics (Walrasian equilibrium, Nash equilibrium). For further details on such applications one can refer to [7].
Throughout, X and Y are Banach spaces, we denote by $B_r(x)$ the closed ball centered at x with radius r. The distance between a point x and a subset A of X will be denoted by $\operatorname{dist}(x, A) = \inf\{\|x - a\| \mid a \in A\}$ while the excess e from a set B to a set C is given by $e(C, B) = \sup\{\operatorname{dist}(c, B) \mid c \in C\}$. A set-valued mapping F from X to Y is indicated by $F : X \rightrightarrows Y$ and its graph is the set $\operatorname{gph} F := \{(x, y) \in X \times Y \mid y \in F(x)\}$. From now on $f : X \to Y$ denotes a twice (Fréchet) differentiable function while $G : X \rightrightarrows Y$ stands for a set-valued mapping with closed graph. In [8], the present authors considered generalized equations of the form

$$0 \in f(x) + G(x), \tag{1}$$

and studied the following iterative method for solving (1):

$$0 \in f(x_k) + \nabla f(x_k)(x_{k+1} - x_k) + \frac{1}{2}\nabla^2 f(x_k)(x_{k+1} - x_k)^2 + G(x_{k+1}) \tag{2}$$

From now on we set

$$A(y, x) = f(x) + \nabla f(x)(y - x) + \frac{1}{2}\nabla^2 f(x)(y - x)^2, \ \forall x, y \in X. \tag{3}$$

It has been showed that this method, based on the second-degree Taylor polynomial expansion A of f, is locally cubically convergent whenever f has a Lipschitz second order derivative and provided that the set-valued mapping $[A(\cdot, x^*) + G(\cdot)]^{-1}$ is *pseudo-Lipschitz* at $(0, x^*)$ (x^* being a solution of (1)). Recall that a set-valued map F from Y to the subsets of X is pseudo-Lipschitz at $(y_0, x_0) \in$ graph F if there exist constants a, b, M such that for every $y_1, y_2 \in B_b(y_0)$ and for every $x_1 \in F(y_1) \cap B_a(x_0)$ there exists $x_2 \in F(y_2)$ with

$$\|x_1 - x_2\| \leq M\|y_1 - y_2\|.$$

The notion of pseudo-Lipschitzness, also known as Aubin continuity (see [1]), is tied to the concept of metric regularity; actually, the pseudo-Lipschitzness of a set-valued mapping F at (y_0, x_0) is equivalent to the metric regularity of the inverse F^{-1} of F at x_0 for y_0, i.e., $y_0 \in F^{-1}(x_0)$ and there exists $\kappa \in [0, \infty[$ along with neighborhoods U of x_0 and V of y_0 such that

$$\text{dist }(x, F(y)) \leq \kappa \text{dist }(y, F^{-1}(x)), \quad \forall x \in U, \ y \in V.$$

The infimum of the set of values κ for which this holds is the modulus of metric regularity. Finiteness of that modulus means that the generalized equation problem is, from a certain perspective, well-posed. For more details on these topics one can refer to [4, 6, 9, 10, 12, 13] and to the monograph [14].

Here, we consider the following perturbed equation

$$y \in f(x) + G(x), \tag{4}$$

where y is a perturbation parameter and we study the stability of the method (5) below under perturbations.

$$y \in f(x_k) + \nabla f(x_k)(x_{k+1} - x_k) + \frac{1}{2}\nabla^2 f(x_k)(x_{k+1} - x_k)^2 + G(x_{k+1}) \tag{5}$$

Such a study can be of interest for example in nonlinear programming for solving optimization problems of the form

$$\begin{aligned} &\text{minimize } f_0(x) \\ &\text{subject to } \begin{cases} f_i(x) = 0, & i = 1, \cdots, m \\ f_i(x) \leq 0, & i = m+1, \cdots, p \end{cases} \end{aligned} \tag{6}$$

where $f_i : I\!R^n \to I\!R$, $i = 1, \cdots, p$ are $\mathcal{C}^3$ functions on $I\!R^n$. The Lagragian L associated with (6) is defined by

$$L : (x, \lambda) \in I\!R^n \times I\!R^p \mapsto f_0(x) + \sum_{i=1}^{p} \lambda_i f_i(x),$$

Hence, the Karush-Kuhn-Tucker first order optimality conditions read as follows:

$$\text{(KKT)} \begin{cases} \nabla_x L(x, \lambda) & = & 0 \\ \nabla_\lambda L(x, \lambda) & \in & N_\Lambda(\lambda) \end{cases}$$

where $N_\Lambda(\lambda)$ denotes the normal cone to the set $\Lambda = I\!R^m \times I\!R_+^{p-m}$ at the point λ. Then, it is easy to see the above conditions amount to

$$0 \in (\ \nabla_x L(x,\lambda), -\nabla_\lambda L(x,\lambda)\ ) + N_C(x,\lambda) \tag{7}$$

where $C = I\!R^n \times \Lambda$. Moreover, relation (7) can be reformulated in the following way:

$$0 \in f(x,\lambda) + G(x,\lambda), \tag{8}$$

where $f(x,\lambda) = (\nabla_x L(x,\lambda), -\nabla_\lambda L(x,\lambda))$ and $G(x,\lambda) = N_C(x,\lambda)$. Hence, the Karush-Kuhn-Tucker optimality system is equivalent to (8) which is a generalized equation of the form of (1) and then can be studied using the method presented in this paper.

This kind of stability problems have already be studied, in different frameworks, in [2, 11]. In this paper, following the work of Dontchev [2], we try to identify what kind of well-posedness of our method would correspond to the well-posedness represented by the property of pseudo-Lipschitzness. We show, in section 2, that the pseudo-Lipschitzness of the map $(f+G)^{-1}$ is closely tied to the uniformity of our method in the sense that the attraction region does not depend on small perturbations of the parameter y. Then, in section 3, we enhance the convergence result established in [8] by showing that the pseudo-Lipschitzness of $(f+G)^{-1}$ implies that our method is actually uniformly cubically convergent.

2 Uniform convergence

In this section, we study the behavior of the solution of the generalized equation (1) when the data input $y = 0$ is subjected to small perturbations. To this end, we need the following Lemma stating that the concept of pseudo-Lipschitzness is robust under the kind of approximation of f we use in (2).

Lemma 2.1 *Let $\varphi : X \to Y$ be a function and let $(\tilde{x}, \tilde{y}) \in graph\,(\varphi + G)$. Assume that φ is twice differentiable in an open neighborhood of $\tilde{x}$ and that its second order derivative is continuous at $\tilde{x}$. Then the following are equivalent:*

(1) *The map* $(\varphi+G)^{-1}$ *is pseudo-Lipschitz at* $(\tilde{y},\tilde{x})$*;*

(2) *The map* $P_{\tilde{x}}(\cdot)=[\varphi(\tilde{x})+\nabla\varphi(\tilde{x})(\cdot-\tilde{x})+\frac{1}{2}\nabla^2\varphi(\tilde{x})(\cdot-\tilde{x})^2+G(\cdot)]^{-1}$ *is pseudo-Lipschitz at* $(\tilde{y},\tilde{x})$*.*

Proof. The proof is a straightforward consequence of [3, corollary 2] where we set $F:=\varphi+G$ and $f(\cdot)=-\varphi(\cdot)+\varphi(\tilde{x})+\nabla\varphi(\tilde{x})(\cdot-\tilde{x})+\frac{1}{2}\nabla^2\varphi(\tilde{x})(\cdot-\tilde{x})^2$.

Proposition 2.1 *Let* $(\tilde{x},\tilde{y})\in graph\,(f+G)$ *and* f *be a function which is twice (Fréchet) differentiable in an open neighborhood* Ω *of* $\tilde{x}$ *and whose second derivative* $\nabla^2 f$ *is continuous at* $\tilde{x}$*. If we suppose that* G *has closed graph and that* $(f+G)^{-1}$ *is pseudo-Lipschitz at* $(\tilde{y},\tilde{x})$ *then there exist positive constants* r*,* s *and* M *such that*

$$e\Big(P_x(y')\cap B_r(\tilde{x}),P_x(y'')\Big)\le M\parallel y'-y''\parallel,$$

for every $x\in B_r(\tilde{x})$ *and* $y',y''\in B_s(\tilde{y})$*, where*

$$P_x(\cdot)=[A(\cdot,x)+G(\cdot)]^{-1}.$$

Proof. From Lemma 2.1 $P_{\tilde{x}}$ is pseudo-Lipschitz at $(\tilde{y},\tilde{x})$; let a, b and M' be the associated constants. For any ε such that $M'\varepsilon<2/5$ there exists $r>0$ such that $\|\nabla^2 f(x)-\nabla^2 f(\tilde{x})\|\le\varepsilon$ for every $x\in B_{4r}(\tilde{x})$. Take r smaller if necessary so that $4r\le a$ and $11r^2\le b$. Choose also $s>0$ such that

$$s+11r^2\varepsilon\le b \ \text{ and } \ \frac{10M's}{2-5M'\varepsilon}\le r. \tag{9}$$

Let $x\in B_r(\tilde{x})$, y', $y''\in B_s(\tilde{y})$ and $x_1\in P_x(y')\cap B_r(\tilde{x})$. Then

$$x_1\in P_{\tilde{x}}\Big(y'-A(x_1,x)+A(x_1,\tilde{x})\Big)\cap B_r(\tilde{x}).$$

We show that both $y'-A(x_1,x)+A(x_1,\tilde{x})$ and $y''-A(x_1,x)+A(x_1,\tilde{x})$ belong to $B_b(\tilde{y})$. Let $\Delta_{y'}=\|y'-A(x_1,x)+A(x_1,\tilde{x})-\tilde{y}\|$ then

$$\Delta_{y'}\le\|y'-\tilde{y}\|+\|f(x_1)-A(x_1,x)\|+\|f(x_1)-A(x_1,\tilde{x})\|.$$

And the continuity of $\nabla^2 f$ at $\tilde{x}$ yields $\Delta_{y'} \leq s+\varepsilon\|x_1-x\|^2+\dfrac{\varepsilon}{2}\|x_1-\tilde{x}\|^2$. Since $\|x-x_1\| \leq \|x-\tilde{x}\|+\|\tilde{x}-x_1\| \leq 2r$ we get $\Delta_{y'} \leq s+\dfrac{9}{2}\varepsilon r^2 \leq b$. Obviously, the same inequality holds for $\Delta_{y''}$. Then the pseudo-Lipschitzness of $P_{\tilde{x}}$ implies that there exists

$$x_2 \in P_{\tilde{x}}\big(y'' - A(x_1, x) + A(x_1, \tilde{x})\big) \cap B_a(\tilde{x}),$$

i.e.,

$$y'' \in A(x_1, x) - A(x_1, \tilde{x}) + A(x_2, \tilde{x}) + G(x_2),$$

such that

$$\| x_2 - x_1 \| \leq M' \| y' - y'' \| . \tag{10}$$

Proceeding by induction, we suppose that there exist an integer $n > 2$ and points $x_2, x_3, \ldots, x_n$ such that $y'' \in A(x_{i-1}, x) - A(x_{i-1}, \tilde{x}) + A(x_i, \tilde{x}) + G(x_i)$ and

$$\| x_i - x_{i-1} \| \leq \frac{5M'}{2} \| y' - y'' \| (\frac{5M'}{2}\varepsilon)^{i-2}, \quad i = 2, \ldots, n. \tag{11}$$

Then

$$\begin{aligned} \| x_n - \tilde{x} \| &\leq \sum_{j=2}^{n} \| x_j - x_{j-1} \| + \| x_1 - \tilde{x} \| \\ &\leq 5M's \sum_{j=2}^{n} (\frac{5}{2}M'\varepsilon)^{j-2} + r \\ &\leq \frac{5M's}{1-(5M'\varepsilon/2)} + r \leq 2r \leq a. \end{aligned} \tag{12}$$

Hence $x_n \in P_{\tilde{x}}\big(y''-A(x_{n-1}, x)+A(x_{n-1}, \tilde{x})\big) \cap B_a(\tilde{x})$ and by the same method as in the beginning of the proof we show that both $y' - A(x_n, x) + A(x_n, \tilde{x})$ and $y'' - A(x_n, x) + A(x_n, \tilde{x})$ belong to $B_b(\tilde{y})$. Then there exists

$$x_{n+1} \in P_{\tilde{x}}\left(y'' - A(x_n, x) + A(x_n, \tilde{x})\right) \cap B_a(\tilde{x}),$$

i.e.,

$$y'' \in A(x_n, x) - A(x_n, \tilde{x}) + A(x_{n+1}, \tilde{x}) + G(x_{n+1}) \tag{13}$$

such that

$\|x_{n+1} - x_n\| \leq M'\|(\nabla f(x) - \nabla f(\tilde{x}))(x_{n-1} - x_n)+$

$$\frac{1}{2}\nabla^2 f(x)((x_{n-1}-x)^2-(x_n-x)^2)-\frac{1}{2}\nabla^2 f(\tilde{x})((x_{n-1}-\tilde{x})^2-(x_n-\tilde{x})^2)\|.$$

For all $z \in \Omega,\ x \mapsto \nabla f(x)$ and $x \mapsto \nabla^2 f(x)(z-x)$ are continuous at $\tilde{x}$, thus we can choose r such that

$$\|x_{n+1}-x_n\| \leq M'\varepsilon\big(2\|x_{n-1}-x_n\|+\frac{1}{2}\|x_{n-1}-x_n\|^2\big).$$

By choosing Ω smaller if necessary

$$\|x_{n+1}-x_n\| \leq M'\varepsilon\frac{5}{2}\|x_{n-1}-x_n\|.$$

Hence

$$\|x_{n+1}-x_n\| \leq \frac{5M'}{2}\|y'-y''\|\big(\frac{5M'}{2}\varepsilon)\big)^{n-1},$$

and the induction is complete. Thus (x_k) is a Cauchy sequence, let x'' be its limit, passing to the limit in (13) yields $y'' \in A(x'',x)+G(x'')$, or equivalently $x'' \in P_x(y'')$. Since $x_1 \in P_x(y')$,

$$\| x''-x_1 \| \leq \lim_{n\to+\infty}\sup\sum_{i=2}^{n} \| x_i - x_{i-1} \|,$$

we obtain

$$\| x''-x_1 \| \leq \lim_{n\to+\infty}\sup\sum_{i=2}^{n}\frac{5M'}{2}\ \| y'-y'' \|\ \big(\frac{5M'}{2}\varepsilon\big)^{i-2} \leq \frac{5M'}{2-5M'\varepsilon}\ \| y'-y'' \|,$$

and thus Proposition 2.1 holds with $M \leq \dfrac{5M'}{2-5M'\varepsilon}$. □

Now, from Proposition 2.1 we derive the following stability result for the solution of equation (1).

Theorem 2.1 *Let x^* be a solution of* (4) *for $y=0$, if $\nabla^2 f$ is continuous on Ω then the following are equivalent:*

1. *$(f+G)^{-1}$ is pseudo-Lipschitz at $(0,x^*)$;*

2. *There exists $c > 0$ such that for every y in some neighborhood of 0 and x_0 in some neighborhood of x^* there is a sequence (x_k) satisfying (5), starting from x_0 and converging to a solution x of (4). Furthermore, $\|x - x_0\| \leq c\|y - y_0\|$ whenever x_0 is a solution of (4) for $y = y_0$.*

PROOF. **(1) $\Rightarrow$ (2).** Let r, s and M be the constants in Proposition 2.1. Let $a > 0$ and choose $\varepsilon > 0$ such that $M\varepsilon < 1$ and satisfying

$$\| f(x'') - A(x'', x')\| \leq \frac{\varepsilon}{2}\|x'' - x'\|^2, \;\; \forall x', x'' \in B_a(x^*).$$

Let $\sigma > 0$ satisfy

$$\frac{\varepsilon}{2}(\sigma^2 + \sigma) \leq s, \quad \sigma \leq r, \quad \frac{\sigma^2 + \sigma}{1 - (M\varepsilon/2)} < a \tag{14}$$

and choose $b > 0$ such that

$$b(1 + \frac{\varepsilon M}{2}) + \frac{\varepsilon}{2}(\sigma^2 + \sigma) \leq s; \tag{15}$$

$$Mb + \frac{M\varepsilon}{2}\sigma^2 + \sigma \leq 1; \tag{16}$$

$$\frac{Mb + (\sigma^2/2) + 2\sigma}{1 - (M\varepsilon/2)} \leq a. \tag{17}$$

Let $x_0 \in B_\sigma(x^*)$ and $y' = -f(x^*) + A(x^*, x_0)$. Since x^* is a solution of (1) we have $y' \in A(x^*, x_0) + G(x^*)$ then $x^* \in P_{x_0}(y')$.
Moreover $\|y'\| = \|f(x^*) - A(x^*, x_0)\| \leq \frac{\varepsilon}{2}\|x^* - x_0\|^2 \leq \frac{\varepsilon\sigma^2}{2}$. For $y \in B_b(0)$, Proposition 2.1 ensures that there exists $x_1 \in P_{x_0}(y)$ such that
$\|x_1 - x^*\| \leq M(\|y\| + \|f(x^*) - A(x^*, x_0)\|)$
$\|x_1 - x^*\| \leq M(\|y\| + \frac{\varepsilon\sigma^2}{2}) \leq Mb + \frac{M\varepsilon\sigma^2}{2}$.
Therefore

$$\|x_1 - x_0\| \leq \|x_1 - x^*\| + \|x^* - x_0\| \leq Mb + \frac{M\varepsilon\sigma^2}{2} + \sigma. \tag{18}$$

Since $x_1 \in P_{x_0}(y)$ we have $y + f(x_1) - A(x_1, x_0) \in A(x_1, x_1) + G(x_1)$ which is equivalent to $x_1 \in P_{x_1}(y')$ where $y' = y + f(x_1) - A(x_1, x_0)$ is such that

$$\begin{aligned}\|y'\| &\leq \|y\| + \|f(x_1) - A(x_1, x_0)\| \\ &\leq b + \frac{\varepsilon}{2}\|x_1 - x_0\|^2 \\ &\leq b + \frac{\varepsilon}{2}(Mb + \frac{M\varepsilon}{2}\sigma^2 + \sigma).\end{aligned}$$

Then from Proposition 2.1 there exists $x_2 \in P_{x_1}(y)$ such that

$$\begin{aligned}\|x_2 - x_1\| &\leq M\|y' - y\| = M\|f(x_1) - A(x_1, x_0)\| \\ &\leq \frac{M\varepsilon}{2}(Mb + \frac{M\varepsilon}{2}\sigma^2 + \sigma)\|x_1 - x_0\| \\ &\leq \frac{M\varepsilon}{2}\|x_1 - x_0\|.\end{aligned} \tag{19}$$

Hence

$$\begin{aligned}\|x_2 - x^*\| &\leq \|x_2 - x_1\| + \|x_1 - x_0\| + \|x_0 - x^*\| \\ &\leq (1 + \frac{M\varepsilon}{2})\|x_1 - x_0\| + \sigma \\ &\leq \frac{Mb + (M\varepsilon\sigma^2/2) + \sigma}{1 - (M\varepsilon/2)} + \sigma \\ &= \frac{Mb + (M\varepsilon/2)(\sigma^2 - \sigma) + 2\sigma}{1 - (M\varepsilon/2)}.\end{aligned} \tag{20}$$

Applying (17), we get

$$\|x_2 - x^*\| \leq a.$$

We suppose now that for $n \in I\!N$ $(n > 2)$, there exist $x_2, \ldots, x_n$ such that for $i = 1, \ldots, n$ we have

$$x_i \in P_{x_{i-1}}(y), \tag{21}$$

and

$$\|x_i - x_{i-1}\| \leq (M\varepsilon/2)^{i-1}\|x_1 - x_0\| \tag{22}$$

One can note that relations (16) and (22) imply that $\|x_i - x_{i-1}\| \leq 1$. By (21), we have

$$y \in A(x_n, x_{n-1}) + G(x_n), \tag{23}$$

i.e., $x_n \in P_{x_n}(y')$, where $y' = y + f(x_n) - A(x_n, x_{n-1})$. To apply Proposition 2.1 we show that $x_n \in B_a(x^*)$ and $y' \in B_s(0)$. Using (22), (18) and (17) we get

$$\begin{aligned} \|x_n - x^*\| &\leq \sum_{i=1}^{n} \|x_i - x_{i-1}\| + \|x_0 - x^*\| \\ &= \left(\sum_{i=1}^{n} (M\varepsilon/2)^i\right) \|x_1 - x_0\| + \|x_0 - x^*\| \\ &\leq \frac{1}{1-(M\varepsilon/2)} \|x_1 - x_0\| + \|x_0 - x^*\| \\ &\leq \frac{Mb + (\sigma^2/2) + 2\sigma}{1-(M\varepsilon/2)} \leq a. \end{aligned}$$

An easy computation yields

$$\|y'\| \leq b + \frac{\varepsilon}{2}\|x_n - x_{n-1}\|^2 \leq b + \frac{\varepsilon}{2}\|x_1 - x_0\|^2 \leq s.$$

Then from Proposition 2.1 there exists

$$x_{n+1} \in P_{x_n}(y) \tag{24}$$

such that $\|x_{n+1} - x_n\| \leq M\|y' - y\| = M\|f(x_n) - A(x_n, x_{n-1})\|$, i.e.,

$$\|x_{n+1} - x_n\| \leq (M\varepsilon/2)\|x_n - x_{n-1}\| \leq (M\varepsilon/2)^n \|x_1 - x_0\|. \tag{25}$$

Thus the induction step is complete and there exists a Cauchy sequence (x_k) converging to some $x \in X$ and satisfying (24). Passing to the limit in relation (24), we get

$$x \in P_x(y) = \left[f(x) + \nabla f(x)(\cdot - x) + \frac{1}{2}\nabla^2 f(x)(\cdot - x)^2 + G(\cdot)\right]^{-1}(y) \tag{26}$$

which is equivalent to

$$x \in (f+G)^{-1}(y). \tag{27}$$

It remains to show that $\|x - x_0\| \leq c\|y - y_0\|$ whenever x_0 is a solution of (4) for $y = y_0$. Let $y_0 \in B_b(0)$ and $x_0 \in (f+G)^{-1}(y_0) \cap B_\sigma(x^*)$. Then $x_0 \in P_{x_0}(y_0) \cap B_r(x_0)$. From proposition 2.1, there exist $x_1 \in P_{x_0}(y)$ such that $\|x_1 - x_0\| \leq M\|y - y_0\|$. By repeating the argument (18)–(25), there exists a sequence (x_k) converging to a solution x of (4) and satisfying (24) and (25). We have also

$$\begin{aligned} \|x_n - x_0\| &\leq \sum_{i=1}^{n} \|x_i - x_{i-1}\| \\ &\leq \left(\sum_{i=1}^{n} (M\varepsilon/2)^i \right) M\|y - y_0\| \\ &\leq \frac{2M}{2 - M\varepsilon} \|y - y_0\| \end{aligned} \tag{28}$$

Passing to the limit and setting $c = \dfrac{2M}{2 - M\varepsilon}$, we obtain

$$\|x - x_0\| \leq c\|y - y_0\|. \tag{29}$$

(2) ⇒ (1). From the second assertion of Theorem 2.1 there exist positive numbers a and b such that for all $y \in B_b(0)$ and $x_0 \in B_a(x^*)$ there exists a sequence (x_k) satisfying (5) starting from x_0 and converging to a solution x of (4).

Let $y_1, y_2 \in B_b(0)$ and $x_1 \in (f + G)^{-1}(y_1) \cap B_a(x^*)$ then there exists a sequence (x_k) starting from x_1 and such that $x_k \to x_2 \in (f + G)^{-1}(y_2)$, i.e., $y_2 \in f(x_2) + G(x_2)$. Since x_1 is a solution of (4) for $y = y_1$, assertion (2) yields:

$$\|x_2 - x_1\| \leq c\|y_2 - y_1\|,$$

and the mapping $(f + G)^{-1}$ is pseudo-Lipschitz at $(0, x^*)$.

□

These results can be partially extended to parameterized generalized equations. More precisely, when the function f depends also on a parameter $w \in X$ the generalized equation (1) becomes

$$0 \in f(w, x) + G(x). \tag{30}$$

Given $y \in Y$, we associate to (30) the following perturbed equation

$$y \in f(w, x) + G(x),$$

which can be rewritten

$$0 \in f_1(w, y, x) + G(x), \tag{31}$$

where $f_1(w,y,x) = f(w,x) - y$. We denote by

$$S : (w,y) \mapsto S((w,y)) = \{x \in X \mid 0 \in f_1(w,y,x) + G(x)\},$$

the solution mapping of (31) and we assume that there exists $x^* \in S((w^*,y^*))$. Now, let f_* denote any smooth first-order approximation to $f_1(w^*,y^*,\cdot)$ at x^* in the sense that

$$f_*(x^*) = f_1(w^*,y^*,x^*) \qquad \text{and} \qquad \nabla f_*(x_*) = \nabla_x f_1(w^*,y^*,x^*).$$

Here, we define f_* as the following linearization of $f_1(w^*,y^*,\cdot)$:

$$f_*(x) = f_1(w^*,y^*,x^*) + \nabla_x f_1(w^*,y^*,x^*)(x-x^*).$$

Then, we introduce the set-valued mapping S_* defined by

$$S_* : y \mapsto S_*(y) = \{x \in X \mid y \in f_*(x) + G(x)\},$$

and by Theorem 3.2 in [5], the mapping S is pseudo-Lipschitz at $((w^*,y^*),x^*)$ whenever the mapping S_* is pseudo-Lipschitz at $(0,x^*)$. According to Dontchev and Rockafellar [5], these two conditions are equivalent under some *ample parameterization* hypothesis. Moreover we know that the pseudo-Lipschitzness of S_* at $(0,x^*)$ is equivalent to the pseudo-Lipschitzness of $(f_1(w^*,y^*,\cdot) + F(\cdot))^{-1}$ at $(0,x^*)$ (see [3]). Then, to apply our stability results in Theorem 2.1 to the parameterized problem (30), it suffices to assume the pseudo-Lipschitzness of the mapping $(f_1(w^*,y^*,\cdot) + F(\cdot))^{-1}$ at $(0,x^*)$ which is a standard and natural hypothesis.

3 Local cubic convergence

When $y = 0$ in (4), we have showed in [8] that if x^* is a solution of (1), the pseudo-Lipschitzness of $(f+G)^{-1}$ at $(0,x^*)$ ensures the existence of a sequence which is cubically convergent to x^*. We intend to prove now that this result remains true if we replace 0 by some small y.

Theorem 3.1 *Let x^* be a solution of* (1)*, assume that f is twice Fréchet differentiable in an open neighborhood Ω of x^* and that $\nabla^2 f$ is Lipschitz on Ω with constant L. If the map $(f+G)^{-1}$ is pseudo-Lipschitz at $(0, x^*)$ then there exist positive constants σ and b such that for every $y \in B_b(0)$ and $x_0 \in B_\sigma(x^*)$ there exists a sequence (x_k) defined by* (5)*, starting from x_0 and which converges to a solution x of* (4)*. Furthermore, there exists a constant γ (which does not depend on small variations of y) such that*

$$\|x_{k+1} - x\| \leq \gamma \|x_k - x\|^3, \tag{32}$$

that is (x_k) is cubically convergent to x.

Proof. First, let us remark that the pseudo-Lipschitzness of $(f+G)^{-1}$ at $(0, x^*)$ with constants l, m and c implies that for all y_1 and $y_2 \in B_m(0)$ and for all $x_1 \in (f+G)^{-1}(y_1) \cap B_l(x^*)$ there exists $x_2 \in (f+G)^{-1}(y_2)$ satisfying $\|x_1 - x_2\| \leq c\|y_1 - y_2\|$.
Taking $\delta = m$, $y_1 = 0$, $y_2 = y$, $x_1 = x^*$ and $x_2 = x$ in the above assertion, we obtain the existence of $\delta > 0$ such that for every $y \in B_\delta(0)$ there exists $x \in (f+G)^{-1}(y) \cap B_{c\|y\|}(x^*)$. Now, let us assume that σ and b satisfy the following:

(i) $\sigma \leq \dfrac{r}{2}$;

(ii) $b \leq \min\{\dfrac{s}{2}, \delta, \dfrac{r}{2c}\}$;

(iii) $cb + \sigma \leq \min\left\{ \left(\dfrac{6s}{2L}\right)^{\frac{1}{3}}, \left(\dfrac{6r}{2ML}\right)^{\frac{1}{3}}, \left(\dfrac{6}{ML}\right)^{\frac{1}{2}} \right\}$;

where r, s and M are given by Proposition 2.1 with $\tilde{x} = x^*$ and $\tilde{y} = 0$. For every $y \in B_b(0)$, we have to prove the existence of a sequence (x_n) satisfying (5) and converging to x which is solution of (4). We proceed by induction for the rest of the proof. More precisely, we are going to show that starting with a suitable x_0, we can build a sequence (x_k) satisfying relation (5). Let $x_0 \in B_\sigma(x^*)$, $y \in B_b(0)$ and $x \in (f+G)^{-1}(y) \cap B_{c\|y\|}(x^*)$ then

$\|x - x^*\| \le cb \le r$. Let us also remark that $y \in f(x) + G(x)$ is equivalent to

$$x \in P_{x_0}\big(y - f(x) + A(x, x_0)\big) \cap B_r(x^*).$$

We also have,

$$\|y - f(x) + A(x, x_0)\| \le \frac{L}{6}\|x - x_0\|^3 + b \le \frac{L}{6}(cb + \sigma)^3 + b.$$

Then from hypotheses (ii) and (iii) we get

$$\|y - f(x) + A(x, x_0)\| \le \frac{s}{2} + \frac{s}{2} = s. \tag{33}$$

From inequality (33), $z = y - f(x) + A(x, x_0) \in B_s(0)$. Since $x_0 \in B_\sigma(x^*) \subset B_r(x^*)$ and $(f + G)^{-1}$ is pseudo-Lipschitz at $(0, x^*)$ Proposition 2.1 yields

$$e\Bigg(P_{x_0}(z) \cap B_r(x^*), P_{x_0}(y)\Bigg) \le M\|A(x, x_0) - f(x)\|.$$

Thus, there exists $x_1 \in P_{x_0}(y)$ such that

$$\|x - x_1\| \le M\|A(x, x_0) - f(x)\| \le \frac{1}{6}ML\|x - x_0\|^3.$$

Since $x \in B_{cb}(x^*)$ and $\|x_1 - x^*\| \le \|x - x_1\| + \|x - x^*\|$ we get

$$\|x^* - x_1\| \le \frac{ML}{6}(cb + \sigma)^3 + cb \le \frac{r}{2} + \frac{r}{2} = r,$$

thus $x_1 \in B_r(x^*)$. Let us suppose that we have proved the existence of x_1, $x_2, \ldots x_k$ (all of them in $B_r(x^*)$) satisfying relation (5). We are going to show that we can find x_{k+1} with the same property. First, using hypothesis (ii), it is easy to see that

$$\|x - x_l\| \le \frac{ML}{6}(cb + \sigma)^3,\ \forall\ 2 \le l \le k.$$

Starting with x_k we have

$$x \in P_{x_k}\big(y - f(x) + A(x, x_k)\big) \cap B_r(x^*).$$

Furthermore, thanks to (iii) and the induction relation, we obtain

$$\begin{aligned}\|y - f(x) + A(x, x_k)\| &\leq (L/6)\|x - x_k\|^3 + b \\ &\leq (L/6)\left((ML/6)(cb+\sigma)^3\right)^3 + b \\ &\leq (L/6)\left(cb+\sigma\right)^3 + b \\ &\leq s/2 + s/2 = s.\end{aligned}$$

By Lemma 2.1 there exists an $x_{k+1} \in P_{x_k}(y)$ such that

$$\begin{aligned}\|x - x_{k+1}\| &\leq M\| - f(x) + A(x, x_k)\| \\ &\leq \frac{ML}{6}\|x - x_k\|^3.\end{aligned}$$

That gives the inequality of the theorem at the step $k+1$ and assertion (32) is satisfied for any $\gamma \geq ML/6$. To complete the proof let us note that since M is the constant of pseudo-Lipschitzness of $(f+G)^{-1}$ at $(0, x^*)$, γ doesn't depend on small variations of y. □

Acknowledgement. The authors thank the referee for his valuable comments and suggestions on this work.

References

[1] J.P. Aubin, H. Frankowska, Set–valued analysis, Birkhäuser, Boston, 1990.

[2] A.L. Dontchev, Uniform convergence of the Newton method for Aubin continuous maps, Serdica Math. J., 22, (1996), p. 385–398.

[3] A.L. Dontchev, W.W. Hager, An inverse function theorem for set–valued maps, Proc. Amer. Math. Soc, 121, 1994, p. 481–489.

[4] A.L. Dontchev, R.T. Rockafellar, Characterizations of strong regularity for variational inequalities over polyhedral convex sets, SIAM J. optim, 6, 4, (1996), p. 108–1105.

[5] A.L. Dontchev, R.T. Rockafellar, Ample parameterization of variational inclusions, SIAM J. Optim. 12, No.1, 170–187 (2001).

[6] A.L. Dontchev and R.T. Rockafellar, Regularity and conditioning of solution mappings in variational analysis. Set-Valued Anal. 12 (2004), no. 1-2, 79–109.

[7] M.C. Ferris and J.S. Pang, Engineering and economic applications of complementarity problems, SIAM Rev., 39, No.4, 1997, pp 669–713.

[8] M. Geoffroy, S. Hilout and A. Piétrus, Acceleration of convergence in Dontchev's iterative method for solving variational inclusions, Serdica Math. J., 29, (2003), p. 45–54.

[9] B.S. Mordukhovich, Complete characterization of openess metric regularity and Lipschitzian properties of multifunctions, Trans. Amer. Math. Soc., 340, (1993), p. 1–36.

[10] B.S. Mordukhovich, Coderivatives of set–valued mappings: calculs and applications, Nonlinear Anal. Theo. Meth. and Appl., 30, (1997), p. 3059–3070.

[11] A. Piétrus, Does Newton's method converges uniformly in mild differentiability context? Revista Colombiana de Matematicas, 32 (2), (2000), p. 49-56.

[12] S.M. Robinson, Strong regular generalized equations, Math. of Oper. Res., 5, (1980), p. 43–62.

[13] R.T. Rockafellar, Lipschitzian properties of multifunctions, Nonlinear Analysis, 9, (1984), p. 867–885.

[14] R.T. Rockafellar, R. J–B. Wets, Variational analysis, A Series of Comprehentives Studies in Mathematics, Springer, 317, (1998).