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ACCELERATION OF CONVERGENCE IN DONTCHEV'S ITERATIVE METHOD FOR SOLVING VARIATIONAL INCLUSIONS

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Abstract. In this paper we investigate the existence of a sequence (x_k) satisfying $0 \in f(x_k)+\nabla f(x_k)(x_{k+1}-x_k)+\frac{1}{2}\nabla^2 f(x_k)(x_{k+1}-x_k)^2+G(x_{k+1})$ and converging to a solution x^* of the generalized equation $0 \in f(x)+G(x)$; where f is a function and G is a set-valued map acting in Banach spaces. We show that the previous sequence is locally cubic convergent to x^* whenever the set-valued map $[f(x^*)+\nabla f(x^*)(\cdot\, - x^*)+\frac{1}{2}\nabla^2 f(x^*)(\cdot\, - x^*)^2+G(\cdot)]^{-1}$ is M-pseudo-Lipschitz around $(0, x^*)$.

1. Introduction. Throughout this paper X and Y are two real or complex Banach spaces and we consider a generalized equation of the form

$$0 \in f(x)+G(x) \tag{1}$$

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where f is a function from X into Y an G is a set-valued map from X to the subsets of Y.
When $G = \partial\psi_C$ is the subdifferential of the function

$$\psi_C(x) = \begin{cases} 0 & \text{if } x \in C \\ +\infty & \text{otherwise,} \end{cases}$$

(1) has been studied by Robinson [10]. The key of his idea is to associate to (1) a linearized equation. His study concerns especially the stability of solutions of some minimization problems.

When ∇f is locally Lipschitz Dontchev [4] associates to (1) a Newton-type method based on a partial linearization which provides a local quadratic convergence. Following his work, Pietrus [9] obtains a Newton-type sequence which converges whenever ∇f satisfies a Hölder-type condition.

In this paper we associate to (1) the relation

$$(2) \qquad 0 \in f(x_k) + \nabla f(x_k)(x_{k+1} - x_k) + \frac{1}{2}\nabla^2 f(x_k)(x_{k+1} - x_k)^2 + G(x_{k+1}),$$

where $\nabla f(x)$ and $\nabla^2 f(x)$ denote respectivly the first and the second Fréchet derivative of f at x. One can note that if $x_k \longrightarrow x^*$, then x^* is a solution of (1). Let us mention that relation (2) derives from a second-degree Taylor polynomial expansion of f at x_k and that such an approximation is an extension of Dontchev's original work [3].

The paper is organized as follows: in section 2 we recall a few preliminary results and make some fundamental assumptions on f. Then, in section 3 we prove the existence of a sequence (x_k) satisfying (2) and we show that it is locally cubic convergent.

2. Preliminaries and fundamental assumptions.

Definition 2.1. *A set-valued map $\Gamma : X \longrightarrow Y$ is said to be M-pseudo-lipschitz around $(x_0, y_0) \in \operatorname{graph}\Gamma := \{(x,y) \in X \times Y \mid y \in \Gamma(x)\}$ if there exist neighbourhoods V of x_0 and U of y_0 such that*

$$(3) \qquad \sup_{y \in \Gamma(x_1) \cap U} dist(y, \Gamma(x_2)) \leq M \parallel x_1 - x_2 \parallel, \forall x_1, x_2 \in V.$$

When a multiapplication Γ is M-pseudo-Lipschitz, the constant M is called the modulus of Aubin continuity.

The Aubin continuity of Γ is equivalent to the openess with linear rate of Γ^{-1} (the covering property) and to the metric regularity of Γ^{-1} (a basic well-posedness property in optimization).

Finally, when f is a function which is strictly differentiable at some x_0, then the Aubin continuity of f^{-1} around $(f(x_0), x_0)$ is equivalent to the surjectivity of $\nabla f(x_0)$. For more details, the reader can refer to [1, 2, 8, 11, 12].

Let A and C be two subsets of X, we recall that the excess e from the set A to the set C is given by $e(C, A) = \sup_{x \in C} dist(x, A)$.

Then, we have an equivalent definition of M-pseudo-Lipschitzness in terms of excess by replacing (3) by

$$e(\Gamma(x_1) \cap U, \Gamma(x_2)) \leq M \parallel x_1 - x_2 \parallel, \forall x_1, x_2 \in V, \tag{4}$$

in the previous definition. In [6] the above property is called Aubin property and in [5] it has been used to study the problem of the inverse for set-valued maps. In the sequel, we will need the following fixed point statement which has been proved in [5].

Lemma 2.1. *Let (X, ρ) be a complete metric space, let ϕ a map from X into the closed subsets of X, let $\eta_0 \in X$ and let r and λ be such that $0 \leq \lambda < 1$ and*
a) dist $(\eta_0, \phi(\eta_0)) \leq r(1 - \lambda)$,
b) $e(\phi(x_1) \cap B_r(\eta_0), \phi(x_2)) \leq \lambda\ \rho(x_1, x_2)\ \forall x_1, x_2 \in B_r(\eta_0)$,
then ϕ has a fixed point in $B_r(\eta_0)$. That is, there exists $x \in B_r(\eta_0)$ such that $x \in \phi(x)$. If ϕ is single-valued, then x is the unique fixed point of ϕ in $B_r(\eta_0)$.

The previous lemma is a generalization of a fixed-point theorem in [7], where in (b) the excess e is replaced by the Haussdorff distance.
We suppose that $x^* \in X$ is a solution of equation (1). Before studying our problem, we make the following assumptions:

(H0) *G has closed graph;*

(H1) *f is Fréchet differentiable on some neighborhood V of x^*;*

(H2) $\nabla^2 f$ *is Lipschitz on* V *with constant* L*;*

(H3) *For all* $y \in V$*, the application*

$$[f(x^*) + \nabla f(x^*)(\cdot - x^*) + \frac{1}{2}\nabla^2 f(x^*)(\cdot - x^*)^2 + G(\cdot)]^{-1},$$

is M*-pseudo-Lipschitz around* $(0, x^*)$.

3. Convergence analysis. The main theorem of this study reads as follows:

Theorem 3.1. *Let* x^* *be a solution of* (1)*, if we suppose that assumptions* **(H0)**-**(H3)** *are satisfied, then for every* $C > \dfrac{ML}{6}$ *one can find* $\delta > 0$ *such that for every starting point* $x_0 \in B_\delta(x^*)$*, there exists a sequence* (x_k) *for* (1)*, defined by* (2)*, which satisfies*

$$\| x_{k+1} - x^* \| \leq C \| x_k - x^* \|^3 . \tag{5}$$

In other words, (2) *generates* (x_k) *with cubic order.*

Before proving Theorem 3.1, we need to introduce a few notation. First, for $k \in \mathbb{N}$ and $x_k \in X$ we define the set-valued map Q from X to the subsets of Y by

$$Q(x) = f(x^*) + \nabla f(x^*)(x - x^*) + \frac{1}{2}\nabla^2 f(x^*)(x - x^*)^2 + G(x).$$

Then we set

$$Z_k(x) := f(x^*) + \nabla f(x^*)(x - x^*) + \frac{1}{2}\nabla^2 f(x^*)(x - x^*)^2 - f(x_k) - \nabla f(x_k)(x - x_k) - \frac{1}{2}\nabla^2 f(x_k)(x - x_k)^2 .$$

Finally, we define the set-valued map $\phi_k : X \to X$ by

$$\phi_k(x) = Q^{-1}[Z_k(x)].$$

One can note that x_1 is a fixed point of ϕ_0 if and only if the following holds:

$$f(x^*) + \nabla f(x^*)(x_1 - x^*) + \frac{1}{2}\nabla^2 f(x^*)(x_1 - x^*)^2 - f(x_0) - \nabla f(x_0)(x_1 - x_0) - \frac{1}{2}\nabla^2 f(x_0)(x_1 - x_0)^2 \in Q(x_1).$$

Thus, it is easy to see that the previous assertion is equivalent to

$$0 \in f(x_0) + \nabla f(x_0)(x_1 - x_0) + \frac{1}{2}\nabla^2 f(x_0)(x_1 - x_0)^2 + G(x_1). \tag{6}$$

Once x_k is computed, we show that the function ϕ_k has a fixed point x_{k+1} in X. This process allows us to prove the existence of a sequence (x_k) satisfying (2).

Now, we state a result which is the starting point of our algorithm. It will be very usefull to prove Theorem 3.1 and reads as follows:

Proposition 3.1. *Under the hypotheses of Theorem* 3.1, *there exists* $\delta > 0$ *such that for all* $x_0 \in B_\delta(x^*)$ $(x_0 \neq x^*)$, *the map* ϕ_0 *has a fixed point* x_1 *in* $B_\delta(x^*)$ *satisfying* $\|x_1 - x^*\| \leq C\|x_0 - x^*\|^3$.

Proof. By hypothesis **(H3)** there exist positive numbers a and b such that

$$e(Q^{-1}(y') \cap B_a(x^*), Q^{-1}(y'')) \leq M \parallel y' - y'' \parallel, \ \forall y', y'' \in B_b(0). \tag{7}$$

Fix $\delta > 0$ such that

$$\delta < \min\left\{a, \left(\frac{2b}{3L}\right)^{\frac{1}{3}}, \frac{1}{\sqrt{C}}\right\}. \tag{8}$$

To prove Proposition 3.1 we intend to show that both assertions (a) and (b) of Lemma 2.1 hold; where $\eta_0 := x^*$, ϕ is the function ϕ_0 defined at the very begining of this section and where r and λ are numbers to be set.

According to the definition of the excess e, we have

$$\text{dist }(x^*, \phi_0(x^*)) \leq e\Big(Q^{-1}(0) \cap B_\delta(x^*), \phi_0(x^*)\Big). \tag{9}$$

Moreover, for all $x_0 \in B_\delta(x^*)$ such that $x_0 \neq x^*$ we have

$\|Z_0(x^*)\| = \|f(x^*) - f(x_0) - \nabla f(x_0)(x^* - x_0) - \frac{1}{2}\nabla^2 f(x_0)(x^* - x_0)^2\|$, so

$\|Z_0(x^*)\| \leq \frac{L}{6}\|x^* - x_0\|^3$.

Then (8) yields, $\|Z_0(x^*)\| < b$. Hence from (7) one has

$$e\Big(Q^{-1}(0)\cap B_\delta(x^*), \phi_0(x^*)\Big) = e\Big(Q^{-1}(0)\cap B_\delta(x^*), Q^{-1}[Z_0(x^*)]\Big) \le \frac{ML}{6}\|x^*-x_0\|^3.$$

By (9), we get

$$\text{dist }(x^*, \phi_0(x^*)) \le \frac{ML}{6}\|x^* - x_0\|^3. \tag{10}$$

Since $C > \frac{ML}{6}$ there exists $\lambda \in]0,1[$ such that $C(1-\lambda) \ge \frac{ML}{6}$. Hence,

$$\text{dist }(x^*, \phi_0(x^*)) \le C(1-\lambda)\|x^* - x_0\|^3. \tag{11}$$

By setting $\eta_0 := x^*$ and $r := r_0 = C\|x^* - x_0\|^3$ we can deduce from the last inequalities that assertion (a) in Lemma 2.1 is satisfied.

Now, we show that condition (b) of lemma 2.1 is satisfied. Since $\frac{1}{\sqrt{C}} \ge \delta$ and $\|x^* - x_0\| \le \delta$, we have $r_0 \le \delta \le a$.

Moreover for $x \in B_\delta(x^*)$,

$$\begin{aligned}\|Z_0(x)\| &\le \|f(x^*) - f(x) - \nabla f(x^*)(x - x^*) - \frac{1}{2}\nabla^2 f(x^*)(x - x^*)^2\| \\ &\quad + \|f(x) - f(x_0) - \nabla f(x_0)(x - x_0) - \frac{1}{2}\nabla^2 f(x_0)(x - x_0)^2\| \\ &\le \frac{L}{6}\|x - x^*\|^3 + \frac{L}{6}\|x - x_0\|^3 \\ &\le \frac{3L}{2}\delta^3.\end{aligned}$$

Then by (8) we deduce that for all $x \in B_\delta(x^*)$, $Z_0(x) \in B_b(0)$. Then it follows that for all $x', x'' \in B_{r_0}(x^*)$, we have

$e(\phi_0(x') \cap B_{r_0}(x^*), \phi_0(x'')) \le e(\phi_0(x') \cap B_\delta(x^*), \phi_0(x''))$, which yields by (7):

$$\begin{aligned}e(\phi_0(x') \cap B_{r_0}(x^*), \phi_0(x'')) &\le M\|Z_0(x') - Z_0(x'')\| \\ &\le M\|\nabla f(x^*)(x' - x'') - \nabla f(x_0)(x' - x'')\end{aligned}$$

$$+\frac{1}{2}\nabla^2 f(x^*)(x'-x^*)^2 - \frac{1}{2}\nabla^2 f(x^*)(x''-x^*)^2$$

$$+\frac{1}{2}\nabla^2 f(x_0)(x''-x_0)^2 - \frac{1}{2}\nabla^2 f(x_0)(x'-x_0)^2\|$$

$$\leq M\|\nabla f(x^*)(x'-x'') - \nabla f(x_0)(x'-x'')$$

$$+\frac{1}{2}\nabla^2 f(x^*)(x'-x''+x''-x^*)^2 - \frac{1}{2}\nabla^2 f(x^*)(x''-x^*)^2$$

$$+\frac{1}{2}\nabla^2 f(x_0)(x''-x_0)^2 - \frac{1}{2}\nabla^2 f(x_0)(x'-x''+x''-x_0)^2\| .$$

Assumption **(H2)** ensures the existence of $L_1 > 0$ such that $\|\nabla^2 f\| \leq L_1$ on $B_\delta(x^*)$. Then an easy computation yields:

$$e(\phi_0(x') \cap B_{r_0}(x^*), \phi_0(x'')) \leq 5ML_1\delta\|x'-x''\|. \tag{12}$$

Without loss of generality we may assume that $\delta < \dfrac{\lambda}{5ML_1}$ thus condition (b) of Lemma 2.1 is satisfied. Since both conditions of Lemma 2.1 are fulfilled, we can deduce the existence of a fixed point $x_1 \in B_{r_0}(x^*)$ for the map ϕ_0. Then the proof of Proposition 3.1 is complete. □

Now that we proved Proposition 3.1, the proof of Theorem 3.1 is straightforward as it is shown below.

P r o o f o f T h e o r e m 3.1. Proceeding by induction, keeping $\eta_0 = x^*$ and setting $r_k = C\|x_k - x^*\|^3$, the application of proposition 3.1 to the map ϕ_k gives the existence of a fixed point x_{k+1} for ϕ_k, which is an element of $B_{r_k}(x^*)$. This last fact implies that :

$$\| x_{k+1} - x^* \| \leq C \| x_k - x^* \|^3 . \tag{13}$$

In others words, (2) generates a sequence (x_k) with cubic order and the proof of theorem 3.1 is complete. □

Corollary 3.1. *Let x^* be an isolated solution of* (1), *if assumptions* **(H0)**-**(H3)** *are satisfied, then for every $C > \dfrac{ML}{6}$ one can find $\delta > 0$ such that any sequence (x_k) generated by* (2) *with $x_k \in B_\delta(x^*)$ satisfies* (5).

Proof. As we recalled it in the proof of Proposition 3.1, there exists $L_1 > 0$ such that $\|\nabla^2 f(x)\| \leq L_1$. Then, we fix δ satisfying both relation (8) and the following:

$$\delta < \min\left\{\frac{1}{3ML_1}, \frac{6C - ML}{18CML_1}\right\}. \tag{14}$$

Without loss of generality we may assume that the solution of (1) is unique in $B_{4\delta}(x^*)$. Let (x_k) be a sequence generated by (2) with $x_k \in B_\delta(x^*)$, then x^* is the only point in $B_{4\delta}(x^*)$ satisfying (1), i.e., $x^* = Q^{-1}(0) \cap B_{4\delta}(x^*)$. Moreover, for all $k \in \mathbb{N}$, by Theorem 3.1 we have:

$$x_{k+1} \in Q^{-1}[Z_k(x_{k+1})].$$

Hence,

$$\|x_{k+1} - x^*\| = \text{dist } (x_{k+1}, Q^{-1}(0)) \text{ then,}$$

$$\|x_{k+1} - x^*\| \leq e\Big(Q^{-1}[Z_k(x_{k+1})] \cap B_\delta(x^*), Q^{-1}(0)\Big),$$

$$\|x_{k+1} - x^*\| \leq M\|Z_k(x_{k+1})\|,$$

$$\|x_{k+1} - x^*\| \leq M\|f(x^*) + \nabla f(x^*)(x_{k+1} - x^*) + \frac{1}{2}\nabla^2 f(x^*)(x_{k+1} - x^*)^2$$
$$-f(x_k) - \nabla f(x_k)(x_{k+1} - x_k) - \frac{1}{2}\nabla^2 f(x_k)(x_{k+1} - x_k)^2 \|.$$

Then, an easy computation shows that

$$\|x_{k+1} - x^*\| \leq M\left(\frac{L}{6}\|x^* - x_k\|^3 + 3L_1\delta\|x_{k+1} - x^*\|\right).$$

Thus,

$$\|x_{k+1} - x^*\| \leq \frac{ML}{6(1 - 3ML_1\delta)}\|x_k - x^*\|^3.$$

Thanks to (14), we have $C > \dfrac{ML}{6(1 - 3ML_1\delta)}$ so $\|x_{k+1} - x^*\| \leq C\,\|x_k - x^*\|^3$ and then the proof is complete. □

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