Computations on finite dimensional Lie algebras.

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ABSTRACT. This paper is concerned with computations on finite dimensional simple Lie algebras using Chevalley bases. Our work (with Patrice TAUVEL and Claude QUITTÉ) is first the construction of explicit tables of structural constants for exceptional simple Lie algebras, second the use of these tables to obtain new results on nilpotents orbits of simple exceptional Lie algebras (a work with A. ELASHVILI).

Let $\mathfrak{g}$ be finite dimensional Lie algebra over an algebraically close field ($\mathbb{C}$ in these notes). If we have a basis $B = \{e_1, \ldots, e_n\}$ of $\mathfrak{g}$, one can make computations using this basis and structural constants $C_{i,j}^k,\ 1 \leq i < j \leq n,\ 1 \leq k \leq n$ defined by

$$[e_i, e_j] = \sum_{k=1}^{n} C_{i,j}^k e_k$$

Of course, it is interesting to use a basis such that most of the constants are zero. It is well known that this is the case for semi-simple Lie algebras for which exist Chevalley bases. Semi-simple Lie algebras are direct sums of simple Lie algebras. We recall the well known classification of all simple finite dimensional Lie algebras over $\mathbb{C}$. There are four classical series

- A_n: Lie algebra of $n \times n$ matrix with trace equal zero;
- B_n: Lie algebra of $2n+1 \times 2n+1$ skew symetric complex matrix;
- C_n: Lie algebra of $2n \times 2n$ matrix M such that ${}^tMJ_{2n} = -J_{2n}M$ where

$$J_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

- D_n: Lie algebra of $2n \times 2n$ complex skew symetric matrix.

and five exceptional Lie algebras denoted by

- G_2: algebra of rank 2 and dimension 14;
- F_4: algebra of rank 4 and dimension 56;
- E_6: algebra of rank 6 and dimension 78;
- E_7: algebra of rank 7 and dimension 133;
- E_8: algebra of rank 8 and dimension 248.

Recall that the rank of a semi-simple Lie algebra is the dimension of the Cartan subalgebras (i.e. nilpotent subalgebras which are equal to their normalizer).

We denote by r the rank of $\mathfrak{g}$ an by $h_1, \ldots, h_r$ a basis of a fixed Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. The Cartan's subalgebras are abelian. The integer $(\dim \mathfrak{g} - r)$ is even, say $2p$ and it can be shown that there exists vectors $x_1, \ldots, x_p$ and $y_1, \ldots, y_p$ such that $B = \{h_1, \ldots, h_r, x_1, \ldots,\ x_p, y_1, \ldots, y_p\}$ is a basis for $\mathfrak{g}$ with the following properties:

- For $i \in 1, \ldots, r,\ j \in 1, \ldots, p$ there exists integers $c_{i,j}$ such that

$$[h_i, x_j] = c_{i,j} x_j \text{ and } [h_i, y_j] = -c_{i,j} y_j$$

- For $i \in 1, \ldots, p,\ j \in 1, \ldots, p$ that exists integers $N_{i,j},\ N'_{i,j}$ such that

$$[x_i, x_j] = N_{i,j} x_{k(i,j)} \ ;\ [y_i, y_j] = N'_{i,j} y_{k(i,j)}$$

- For $i \in 1, \ldots, p,\ j \in 1, \ldots, p,\ i \neq j$ we have $[x_i, y_j] = C_{i,j} z_{k(i,j)}$ where $z_{k(i,j)}$ is an x_k or an y_k and $C_{i,j}$ are integers;
- For $i \in 1, \ldots, p$, there exists integers m_k such that $[x_i, y_i] = \sum_{k=1}^{r} m_k h_k$;

So, we see that all the structural constants are integers and most of them are zero. In fact, for brackets of 2 basis elements, there is at most r non-zero constants.

1. Construction of Chevalley basis.

For general features concerning semi-simple Lie algebras one could consult [**1**], [**2**], [**9**] and [**10**].

We denote by $\mathfrak{g}$ a semi-simple Lie algebra over $\mathbb{C}$ with rank equal to r. With $\mathfrak{g}$ we consider

$\mathbf{G}$ the adjoint group of $\mathfrak{g}$;
K the Killing form on $\mathfrak{g}$;
$\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$;
$\mathcal{R}$ the set of roots for $(\mathfrak{g}, \mathfrak{h})$;
$\mathcal{B} = \{a_1, \ldots, a_l\}$ a basis for $\mathcal{R}$;
$\mathcal{R}_+$ (*resp.* $\mathcal{R}_-$) the set of positive (*resp.* negative) roots of $\mathcal{R}$ with respect to $\mathcal{B}$.
For $a \in \mathcal{R}$, $\mathfrak{g}^a$ is the root subspace corresponding to a, and X_a is an element of $\mathfrak{g}^a \setminus \{0\}$. We put

$$\mathfrak{n}_+ = \sum_{a \in \mathcal{R}_+} \mathfrak{g}^a \qquad , \qquad \mathfrak{n}_- = \sum_{a \in \mathcal{R}_-} \mathfrak{g}^a$$

1.1. Let be $\lambda \in \mathfrak{h}^*$. There exists an element $h_\lambda \in \mathfrak{h}$ such that

$$\lambda(h) = K(h, h_\lambda)$$

for all $h \in \mathfrak{h}$. For all $\lambda, \mu \in \mathfrak{h}^*$, we write $<\lambda, \mu>= K(h_\lambda, h_\mu)$ and, for $a \in \mathcal{R}$, we set

$$H_a = \frac{2}{<a, a>} h_a.$$

We write H_i instead of $H_{a_i}, 1 \leq i \leq r$. It is well known that $(H_1, \ldots, H_r)$ is a basis of $\mathfrak{h}$.

For $a, b \in \mathcal{R}$, the scalars

$$n_{a,b} = 2\frac{<a, b>}{<b, b>}$$

are integers, and are called Cartan's integers of $\mathfrak{g}$. The square of the length of the root a is $<a, a>$.

1.2. Let $a, b \in \mathcal{R}$ be two roots such that $a+b \neq 0$. One defines the scalar $N(a,b)$ by the following conditions:

— If $a+b \notin \mathcal{R}$, we set $N(a,b)=0$;

— If $a+b \in \mathcal{R}$, $N(a,b)$ is define by $[X_a, X_b] = N(a,b)\, X_{a+b}$.

The coefficients $N(a,b),\ a,b \in \mathcal{R}$, satisfy the relations :

(1) $N(b,a) = -N(a,b)$ for $a,b \in \mathcal{R}$ such that $a+b \neq 0$.

(2) If $a,b,c \in \mathcal{R}$ are two by two independant and such that $a+b+c=0$, then

$$\frac{N(a,b)}{<c,c>} = \frac{N(b,c)}{<a,a>} = \frac{N(c,a)}{<b,b>}$$

(3) Let $a,b,c,d,e \in \mathcal{R}$ such that $a+b=c+d=e$. Then

$$N(b,-c)\, N(a,d-a) + N(-c,a)\, N(b,d-b) - N(a,b)\, N(-c,-d)\frac{<d,d>}{<e,e>} = 0$$

(4) Let a,b be in $\mathcal{R}$ such that $a+b \in \mathcal{R}$, and let $b-ra, \ldots, b+qa$ be the a-chain of b.

$$N(a,b)\, N(-a,-b) = -(r+1)^2 = -q(r+1)\frac{<a+b,a+b>}{<b,b>}.$$

Thus, it is clear that to know the numbers $N(a,b),\ a,b \in \mathcal{R}$ with $a+b \neq 0$, it is enough to calculate the coefficients $N(a,b)$ for $a,b \in \mathcal{R}_+$ and $a+b \neq 0$.

It is possible to select the vectors $X_a, a \in \mathcal{R}$, in such a way that

(a) $[X_a, X_{-a}] = H_a$ for all $a \in \mathcal{R}$.

(b) For $a,b \in \mathcal{R}$ such that $a+b \neq 0$, one has $N(a,b) = -N(-a,-b)$.

A basis $\{X_a; a \in \mathcal{R}, H_i, 1 \leq i \leq r\}$ which satisfy the previous conditions is a Chevalley basis of $\mathfrak{g}$. For such a basis we have the following properties :

(i) $[H_i, H_j] = 0, \quad 1 \leq i,j \leq r$.

(ii) $[H_i, X_a] = <a_i, a>\, X_a, \quad 1 \leq i \leq r,\ a \in \mathcal{R}$.

(iii) For all $a \in \mathcal{R}$, $H_a = [X_a, X_{-a}]$ is a linear combination with integer coefficients of the vectors $H_i, 1 \leq i \leq r$. Precisely, for

$$a = \sum_{i=1}^{r} n_i a_i$$

we have

$$<a,a>\ H_a = \sum_{i=i}^{r} n_i\ <a_i,a_i>\ H_i.$$

(iv) Let $a,b \in \mathcal{R}$ be such that $a+b \in \mathcal{R}$. If we denote $b-ra, \ldots, b+qa$ the a-chain of b, we have:

$$N(a,b)^2 = (r+1)^2 = q(r+1)\,\frac{<a+b,a+b>}{<b,b>}$$

So, we see that, knowing $\mathcal{R}$ (therefore, the length of the roots, the Cartan matrix and the Dynkin diagram), it is easy to find $N(a,b)$ *apart from the sign*. For the problem of signs of the constants of structure one could consult [**11**].

1.3. An order on $\mathcal{R}$. For each $a = \sum_{i=1}^{r} n_i a_i$ in $\mathcal{R}_+$, we put $|a| = \sum_{i=1}^{r} n_i$. Let $a, b \in \mathcal{R}_+$ be roots such that $a \neq b$. We say that $a < b$ if

- either $|a| < |b|$
- or either $|a| = |b|$ and a is smaller than b for the reverse lexicographical order define by $\mathcal{B}$. This defines a total order on $\mathcal{R}_+$ which is compatible with the addition.

Denote by n the cardinal of $\mathcal{R}_+$ and let

$$R_1 < R_2 \cdots < R_n$$

be the elements of $\mathcal{R}_+$. We select X_i (*resp.* Y_i) in $\mathfrak{g}^{R_i}$ (*resp.* $\mathfrak{g}^{-R_i}$), $1 \leq i \leq n$ such that $(X_i, Y_i,\ 1 \leq i \leq n; H_j,\ 1 \leq j \leq r)$ be a Chevalley basis of $\mathfrak{g}$. With the previous order we have $[X_i, Y_i] = H_i$ for $1 \leq i \leq r$.

We obtain the table of integers $N(a, b)$ for exceptional Lie algebras with a computer program[1].

The main part of the program is the computation of $N(a, b)$ for two positive roots a and b such that $a < b$ (for the order define in 1.3.) Other constants are easily deduced by using the formulas (1), (2), (3) in paragraph 1.2.

We use the following algorithm :

1.– The table of positive roots (and the square of the length of the roots for G_2 and F_4), and the Cartan matrix, are read from a file.

2.– The computation is inductive on the order of the positive roots. For each positive root v, we compute the list of pairs of positive roots (c, d) with $d < c$ such that $c + d = v$.

3.– Let (a, b) be a pair of roots, if it exists, which verifies $a + b = v,\ b < a$ and b minimum according to these conditions (so a is maximum). The absolute value of $N(a, b)$ being given by the property (iv) in paragraph 1.2 (equal to 1 for the algebras E_6, E_7, E_8),

we arbitrary fix the sign of $N(a, b) = -N(b, a)$(plus for our realization).

4.– Now, the other constants $N(d, c)$ for $c + d = v,\ d < c$ are easely deduced from $N(a, b)$. Indeed, let (d, c) be such two roots. We have $b < d < c < a$ and since $a+b = c+d$, we obtain $x = a - c = d - b$, $y = c - b = a - d$, and one deduces from formulas (1), and (3) in paragraph 1.2

$$N(d, c) = \frac{< a+b, a+b >}{N(a, b) < a, a >}\Big(\frac{< x, x >}{< d, d >} N(x, b) N(x, c) + \frac{< y, y >}{< c, c >} N(y, b) N(d, y)\Big)$$

(but x and y are not always roots: in this case, one puts $N(x, u) = 0$ for every root $u \in \mathcal{R}_+$). We can calculate $N(c, d)$ since all the integers $N(i, j)$ in the right hand of the previous formula are already known because $x + b = d - b + b = d < d + c = v,\ x + c = a - c + c = a < a + b = v$, *etc* ...

5.– The constants $N(u, v)$ (for u and v, not two of them positive), are deduced from formulas (1) and (2) of 1.2. These formulas imply

$$N(-u, v) = \frac{< v-u, v-u >}{< v, v >} N(v-u, u) \qquad 0 < u < v, \quad (v-u) \in \mathcal{R}_+$$

6.– The computation of brackets $[X_a, X_{-a}]$ and $[H_i, X_a]$ is a straightforward calculation using the Cartan matrix.

[1] In fact the method works for any semi-simple Lie algebras.

1.4. Programming problems. For the realization of the program, the more important difficulties were: the large dimension of the algebras (248 for E_8) and the delay for running these programs when one tries to use array structures. The output of the program is a file with lines of 4 integers

$$i \qquad j \qquad k \qquad C_{i,j}^k$$

The file is used later for computations on the Lie algebras.

With PC computers and programming languages such that Pascal or Ada, the size of a variable is limited to 64 K-bytes, so it is impossible to use a table with constants of structure for a Lie algebra of dimension greater than 40 with an array of integers. Moreover, for example, for dimension 35 the computation time for to verify the Jacobi identity is about 20 minutes. All the vectors used having little much nonzero components, only one in almost all case, we have chosen to memorize them with dynamical variables (pointers, lists). However, for E_8 a two dimensional array of pointers on brackets $[X_i, X_j]$, $1 \leq i, j \leq 248$ is greater than the maximal possible size for a variable. We have opted for a more complex structure: a one dimensional array of pointers on "lines" which are themselves arrays of pointers on "the brackets" $[X_i, X_j]$. Actually, this structure is almost easy to use than a standard array, when the usual operations have been programmed. The time for the Jacobi identity verification for the dimension 35 is 2 seconds instead of 20 minutes and about one minute for E_8.

2. New results obtained with Chevalley bases.

2.1. $\mathfrak{sl}(2)$-triples. A sequence (h, e, f) of nonzero elements of $\mathfrak{g}$ such that

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h$$

is called a $\mathfrak{sl}(2)$-triple of $\mathfrak{g}$.

It is well known that there is a one to one map between $\mathfrak{sl}(2)$-triple and nilpotent orbits (i.e. The **G**-orbit (cf. section 1 of a nilpotent element of $\mathfrak{g}$)

Let $\mathfrak{g}$ be an exceptional simple Lie algebra. In [**4**], Dynkin describes nilpotent orbits Ω with some sequences $C_\Omega = (n_1, \ldots, n_r)$ named characteristics. All the characteristics of nilpotent orbits have been computed by Dynkin in [**4**]. The list of characteristics could also be found in [**3**] and [**7**]. Giving such a characteristic $(n_1, \ldots, n_r)$, it is easy to find $h \in \mathfrak{h}$ such that $a_i(h) = n_i$ for $1 \leq i \leq r$; it is sufficient to apply the reverse of the Cartan matrix of $\mathfrak{g}$ to the column vector having $n_1, \ldots, n_r$ for components. Infortunately, there is no general method to deduce $e \in \mathfrak{n}_+$, $f \in \mathfrak{n}_-$ such that (h, e, f) is a $\mathfrak{sl}(2)$-triple (these elements are not unique).

Using the tables of Chevalley bases, P. Tauvel computed tables of $\mathfrak{sl}(2)$-triples for the 5 exceptional Lie algebras and we verified the results with a computer program. The lists of $\mathfrak{sl}(2)$-triples can be found in [**8**]. In the same paper we give the lists of brackets of Chevalley bases for the five exceptional Lie algebras, the lists of roots, Dynkin diagrams, and the listings of programs written in Turbo-Pascal. It is easy to modify these programs and to get the same results for any given simple classical Lie algebra: it is enough to change the rank, the list of roots and the Cartan matrix.

Here we write an example of $\mathfrak{sl}(2)$-triple for the Lie-algebra E_8. It corresponds to the characteristic $(2, 2, 2, 2, 2, 2, 2, 2)$

$$\begin{aligned}
h &= 92H_1 + 136H_2 + 182H_3 + 270H_4 + 220H_5 + 168H_6 + 114H_7 + 58H_8 \\
e &= X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7 + X_8 \\
f &= 92Y_1 + 136Y_2 + 182Y_3 + 270Y_4 + 220Y_5 + 168Y_6 + 114Y_7 + 58Y_8
\end{aligned}$$

2.2. "Compact nilpotent orbits". Let e be a nilpotent element and $\mathfrak{g}_e$ its centralizer in $\mathfrak{g}$. One interesting property concerning the nilpotent orbit Ω_e is the codimension of $\overline{\Omega}_e \setminus \Omega_e$: What are the nilpotent orbits such that this codimension is greater than 2? (VOGAN property). Such an orbit is said to be a compact nilpotent orbit. For classical Lie algebras it is possible to solve this problem by theorical usual methods and this was done by A. ELASHVILI (see [**5**], [**6**]) who discovered that it is true if and only if $e \in [\mathfrak{g}_e, \mathfrak{g}_e]$, also equivalent to another more technical property, and he gives the list of these orbits.

For exceptional Lie algebras, only case by case computation seems possible to us and for E_7 and E_8 it is not possible to acheive the work "by hand"! Using the tables of Chevalley bases and new programs in Pascal and Axiom, we obtain the complete list of compact nilpotent orbits for exceptional Lie algebras [**5**].

References

1. BOURBAKI N., *Groupes et algébres de Lie*, CHAP. 7, 8, MASSON, DIFFUSION C.C.L.S., 1975.
2. ______, *Groupes et algébres de Lie*, CHAP. 4, 5, 6, MASSON, DIFFUSION C.C.L.S., 1981.
3. CARTER R.W., *Finite groups of Lie type: conjugacy classes and complex characters*, CHAP. 4, 5, 6, INTERSCIENCE PUBLISHERS, JOHN WILEY & SONS, 1985.
4. DYNKIN E.B., *Semi-simple Lie subalgebras of semi-simple Lie algebras*, AMER. MATH. SOC. TRANS. **6** (1957), 111–243.
5. ELASHVILI A.A. AND GRÉLAUD G., *Classification des éléments nilpotents compacts des algèbres de lie simples exceptionnelles*, COMPTES-RENDUS DE L'ACADÉMIE DES SCIENCES **117** (1993), 445–447.
6. ELASHVILI A.G., *Centralizers of nilpotent elements in semisimple lie algebras*, SAKHARTH. SSR MECN. ACAD. MATH. INST. SROM. **46** (1975), 109–132.
7. ELKINGTON G.B., *Centralizers of unipotent elements in semi-simple Lie algebraic groups*, JOURNAL OF ALGEBRA **23** (1972), 137–163.
8. GRÉLAUD G., QUITTÉ CL., AND TAUVEL P., *Bases de Chevalley et* $\mathfrak{sl}(2)$*-triplets des algèbres de lie simples exceptionnelles*, PRÉPUBLICATIONS DU DÉPARTEMENT DE MATHÉMATIQUES DE L'UNIVERSITÉ DE POITIERS, FRANCE **52** (1990).
9. HUMPREYS J.E., *Introduction to Lie algebras and representation theory*, SPRINGER-VERLAG, 1980.
10. SAMELSON H., 23, VON NOSTRAND REINHOLD MATHEMATICAL STUDIES, 1969.
11. TITS J., *Sur les constantes de structure et le théorème d'existence des algèbres de Lie semi-simples*, INST. HAUTES ÉTUDES SC. PUBL. MATH. **31** (1966), 25–58.

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