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On representations of simply connected nilpotent and solvable Lie groups

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Introduction

In these notes following a course I gave during a visit at Pondicherry University in 1992, I write the main results of the theory of representations of simply connected nilpotent Lie groups (the Kirillov's theory), and some generalizations to simply connected solvable Lie groups.

After the basic properties of unitary representations of locally compact groups, especially construction of induced representations and "Mackey's machine" (Section 1), I state the classical results on Lie groups and Lie algebras (section 2).

In section 3, I give the constuction of polarizations in solvable Lie algebras and in section 4 the description of the the dual space of connected nilpotent Lie groups, using the famous orbit method of A.A. Kirillov. For general solvable Lie groups, I write in section 5 the construction of irreducible holomorphical induced representations. This shows the use of complex polarizations.

The section 6 is devoted to a computation of the Kirillov's character formula and the Plancherel formula for nilpotent Lie groups and also a generalization to some homogeneous spaces of nilpotent Lie groups.

In the last section, I write a survey (and some proofs) of the main results of L. Pukanszky about solvable Lie groups.

The results given in these notes have been discovered during the years 1960-1980. The most important contributions are due to A.A Kirillov, J. Dixmier, P. Bernat, M. Vergne, M. Duflo, L. Pukanszky, L. Auslander, B. Kostant. It is an introduction for topical research. At the present time, mathematicians are concerned with problems about algebraic groups, homogeneous spaces of Lie groups (Plancherel formulas, differential operators ...).

The results in these notes are well known and the proofs given are not new, except for a part of section 6. I have used lectures or books among others, M. Raïs [23], P. Bernat and Al. [3], A.A. Kirillov [14], P. Torasso [24], L. Corwin and F. Greenleaf [5].

I have given many examples in low dimensional Lie groups, and many results (more or less easy) are left in exercises.

I am grateful to Miss V. Gayatri, who attended my lectures, for a carefull reading of a first version of these notes and a lot of remarks and corrections.

1. Basic facts on unitary representations

In this section $\boldsymbol{G}$ is a locally compact group which is separable. We give, almost without proofs (but with references to litterature), classical results on representation theory. A good introduction for this theory is G.W. Mackey [16] or J. Dixmier [7]. For an abstract, you could read the chapter 1 of [1].

An unitary representation π of $\boldsymbol{G}$ in an Hilbert space $\mathcal{H}$ is an homomorphism from $\boldsymbol{G}$ into the group $\boldsymbol{\mathcal{U}}(\mathcal{H})$ of unitary operators on $\mathcal{H}$ and such that for every $v \in \mathcal{H}$ the map

$$\begin{array}{ccc} \boldsymbol{G} & \longrightarrow & \mathcal{H} \\ g & \rightsquigarrow & \pi(g).v \end{array}$$

is continuous. We assume in these notes that the Hilbert spaces are separable which is not a serious restriction since it is known that an irreducible unitary representation of a connected Lie group has a separable space. It is also a well known result (see [1]), that the continuity of the above map follows from the measurability of the map $g \longrightarrow < \pi(g)v, w >$ for all v and w in $\mathcal{H}$.

SOME BASIC DEFINITIONS

1.1.– Let π be an unitary representation of $\boldsymbol{G}$ in $\mathcal{H}$ and let $\mathcal{V} \subset \mathcal{H}$ a closed subspace which is invariant by every operator $\pi(x)$ for $x \in \boldsymbol{G}$. This defines a new representation $(\pi, \mathcal{V})$ in $\mathcal{V}$. We say that it is a *subrepresentation* of π. It is immediate that $\mathcal{V}^{\perp}$ is also $\pi(\boldsymbol{G})-$invariant, so it defines an other subrepresentation of π.

1.2.– A representation π is said to be *irreducible* if it has no subrepresentation other than the trivial ones. There is an other way to say this property: let π and π' two representations in $\mathcal{H}$ and $\mathcal{H}'$ respectively. Let T be a bounded linear operator from $\mathcal{H}$ into $\mathcal{H}'$. The operator T is said to be an intertwining operator between π and π' if we have the relation

$$T \circ \pi(x) = \pi'(x) \circ T$$

for all $x \in \boldsymbol{G}$. Usualy one denotes by $\mathrm{Hom}(\pi, \pi')$ the space of intertwining operators and we say that π and π' are unitarly equivalent if there is a bijective isometry in $\mathrm{Hom}(\pi, \pi')$ and this is write $\pi \simeq \pi'$. The relation with irreducibillity is described by the following lemma.

Lemma 1.1 — *(Schur Lemma* cf. [17] p. 14*). An unitary representation π in $\mathcal{H}$ is irreducible if and only if* $\mathrm{Hom}(\pi, \pi) = \mathbb{C}I_{\mathcal{H}}$.

It is easy to see that the equivalence classes of irreducible representations is a set denoted by $\widehat{\boldsymbol{G}}$.

1.3.– If $\boldsymbol{G}$ is an abelian group the Schur lemma proves that $\widehat{\boldsymbol{G}}$ is the set of characters of $\boldsymbol{G}$. In fact, if $\pi \in \widehat{\boldsymbol{G}}$, for $x \in \boldsymbol{G}$, $\pi(x) \in \mathrm{Hom}(\pi, \pi)$. So, by Schur lemma $\pi(x) = \chi(x)\mathrm{Id}$ where $\chi(x)$ is a complex number and it is clear that $\chi(x)$ is a character.

1.4.– If $\boldsymbol{K} \subset \boldsymbol{G}$ is a closed subgroup, the operators $\pi(k)$ for $k \in \boldsymbol{K}$ define a representation of $\boldsymbol{K}$ in $\mathcal{H}$. We say that it is the restriction of π to $\boldsymbol{K}$ and note $\pi|_{\boldsymbol{K}}$ this representation.

1.5.– *Direct sum.* Let $\pi_1, \ldots, \pi_n$ be representations in Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_n$ respectively. It is clear that one defines a new unitary representation of $\boldsymbol{G}$ in $\mathcal{H} = \oplus_{i=1}^n \mathcal{H}_i$ by letting

$$\pi(x)(v_1 + \cdots + v_n) = \pi_1(v_1) + \cdots + \pi_n(v_n)$$

for all $x \in \boldsymbol{G}$ and $v_i \in \mathcal{H}_i$; $i = 1, \ldots, n$. π is said to be the direct sum of the π_i.

1.6.– *Hilbert integral.* We only give a weak definition. Let X be a separable locally compact topological space, $\mathcal{H}$ an Hilbert space, $\boldsymbol{G}$ a locally compact group and for all $x \in X$, π^x a representation of $\boldsymbol{G}$ in $\mathcal{H}$. We denote by μ a Radon positive measure on X. We consider the space $V = \mathbf{L}^2(X, \mathcal{H})$ of all functions φ on X such that $\int_X \|\varphi\|^2 d\mu(x) < \infty$. We suppose that for all $g \in \boldsymbol{G},\ \varphi \in V,\ \psi \in V$, the map $x \longrightarrow < \pi^x(g)\varphi(x)\,,\, \psi(x) >$ is measurable. We define a new representation ρ by the formula

$$\big[\rho(g)f\big](x) = \pi^x(g)f(x) \qquad f \in \mathcal{H},\ x \in X$$

This representation is denoted $\rho = \int_X^{\oplus} \pi^x d\mu(x)$. If ν is a positive measure equivalent to μ, we have $\rho_\mu \simeq \rho_\nu$.

1.7.– EXAMPLES.

(a) Let $\boldsymbol{G} = \mathbb{R}$ and $\mathcal{H} = \mathbb{C}$. Then $\boldsymbol{\mathcal{U}}(\mathcal{H}) = \mathbb{U} = \{z \in \mathbb{C}\ ;\ |z| = 1\}$. So every unitary representation of $\boldsymbol{H}$ is a continuous character χ of $\boldsymbol{G}$ (see 1.3) and it is well known that there exists an $y \in \mathbb{R}$ such that $\chi(x) = e^{ixy}$, for all $x \in \boldsymbol{G}$.

(b) Let $\boldsymbol{G} = \mathcal{O}(n, \mathbb{R}), \mathcal{H} = \mathbb{C}^n$ and let π be the natural injection of $\boldsymbol{G}$ into the group $\boldsymbol{\mathcal{U}}(\mathcal{H})$. Then π is a unitary representation of $\boldsymbol{G}$.

(c) *The regular representation.* Let $\boldsymbol{G}$ be any separable locally compact group. We denote by μ its left Haar measure. Consider $\mathcal{H} = \mathbf{L}^2(\boldsymbol{G}, \mu)$. For $g \in \boldsymbol{G}$ define the operotor $\lambda(g)$ by $[\lambda(g).f](x) = f(g^{-1}x)$. We have $\|\lambda(g).f\| = \|f\|$, so $\lambda(g)$ is a unitary operator. Moreover, the map $g \longrightarrow \lambda(g).f$ of $\boldsymbol{G}$ into $\mathcal{H}$ is continuous. This is clear when f is a continuous function with compact support, from which one deduces the case of other $f \in \mathbf{L}^2(\boldsymbol{G}, \mu)$ by an easy exercise of measure theory. The representation λ is called the left regular representation. The right regular representation is defined by $[\rho(g).f](x) = \Delta(g)^{\frac{1}{2}} f(xg)$, for $f \in \boldsymbol{L}^2(\boldsymbol{G}, \mu)$ where Δ is the modular function of $\boldsymbol{G}$ and μ is the left Haar measure on $\boldsymbol{G}$.

1.8.– INDUCED REPRESENTATIONS.

We now describe the most important tool of the theory to build representations of $\boldsymbol{G}$ from representations of closed subgroups. We start with a particular case which is the only necessary for the class of groups we study in these notes.

(a) Let $\boldsymbol{G}$ be a locally compact group and consider $\boldsymbol{H}$ a closed subgroup of $\boldsymbol{G}$. Let σ be a representation of $\boldsymbol{G}$ in an Hilbert space $\mathcal{H}$. Suppose that there is a left-invariant measure ν on the locally compact quotient space $\boldsymbol{G}/\boldsymbol{H}$. We remark that this is always the case if $\boldsymbol{G}$ and $\boldsymbol{H}$ are unimodular. Then we can construct a new Hilbert space $\mathcal{H}_\sigma$: we first denote by $\mathcal{C}_\sigma$ the set of all continuous functions from $\boldsymbol{G}$ into $\mathcal{H}$ such that

(i) $f(gh) = \sigma(h)^{-1}f(g)$ for all $g \in \boldsymbol{G}$ and $h \in \boldsymbol{G}$;

(ii) $\displaystyle\int_{\boldsymbol{G}/\boldsymbol{H}} \|f(g)\|^2 d\nu(g) < +\infty$

The function $g \longrightarrow \|f(g)\|^2$ is constant on each left coset of $\boldsymbol{G}/\boldsymbol{H}$, so the integral in (ii) exists and we define an inner product on $\mathcal{C}_\sigma$ by the formula

$$< f, f' >= \int_{\boldsymbol{G}/\boldsymbol{H}} < f(g), f'(g) > d\nu(g) \tag{1}$$

We note $\mathcal{H}_\sigma$ the completion of $\mathcal{C}_\sigma$ for this inner product. Then, we define the induced representation $\pi = \mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}}\sigma$ by left action of $\boldsymbol{G}$ on functions of $\mathcal{H}_\sigma$

$$\pi(x).f(y) = f(x^{-1}y)$$

for all $(x, y) \in \boldsymbol{G} \times \boldsymbol{G}$. It is clear that $\pi(x)$ is isometric and one to one, so there is an unique unitary operator (also noted $\pi(x)$) in $\mathcal{H}_\sigma$ which is equal to $\pi(x)$ on $\mathcal{C}_\sigma$. An easy computation shows that $\pi(x)$ is an homomorphism from $\boldsymbol{G}$ into $\boldsymbol{\mathcal{U}}(\mathcal{H}_\sigma)$. We want to prove that $x \longrightarrow \pi(x)$ is continuous. Let $x \in \boldsymbol{G}$, $\varepsilon > 0$ and $\varphi \in \mathcal{C}_\sigma$. We have

$$\|\pi(x)\varphi - \varphi\|^2 = \int_{\boldsymbol{G}/\boldsymbol{H}} \|\varphi(x^{-1}y) - \varphi(y)\|^2 d\nu(y)$$

and the function $\psi : x \longrightarrow \|\varphi(x^{-1}y) - \varphi(y)\|^2$ is continuous and has a compact support S. So, for every $y \in S$ there is a neighbourhood V_y of $1 \in \boldsymbol{G}$ such that $\|\varphi(x^{-1}y) - \varphi(y)\|^2 < \varepsilon$ for $x \in V_y$. The compactness of the support of φ (in $\boldsymbol{G}/\boldsymbol{H}$) shows that there exists a neighbourhood V of $\boldsymbol{G}$ such that for all $x \in V$ and $y \in \boldsymbol{G}$, $\|\varphi(x^{-1}y) - \varphi(y)\|^2 < \varepsilon$ and the continuity follows easily.

(b) If there is no $\boldsymbol{G}-$invariant measure on $\boldsymbol{G}/\boldsymbol{H}$, the construction of $\mathcal{H}_\sigma$ is more complicated. Let us denote by Δ (*resp.* δ) the modular function of $\boldsymbol{G}$ (*resp.* $\boldsymbol{H}$). So, we have if $\mu_{\boldsymbol{G}}$ is the left Haar measure on $\boldsymbol{G}$,

$$\int_{\boldsymbol{G}} f(xyx^{-1}) d\mu_{\boldsymbol{G}}(y) = \Delta(x) \int_{\boldsymbol{G}} f(y) d\mu_{\boldsymbol{G}}(y) \tag{2}$$

$$\int_{\boldsymbol{G}} f(y) d\mu_{\boldsymbol{G}}(y) = \int_{\boldsymbol{G}} \Delta(y)^{-1} f(y^{-1}) d\mu_{\boldsymbol{G}}(y) \tag{3}$$

for all $f \in \mathcal{K}(\boldsymbol{G})$, the space of continuous functions with compact support on $\boldsymbol{G}$ and $x \in \boldsymbol{G}$. Similarly, we fix a left Haar measure on $\boldsymbol{H}$ and for $h \in \boldsymbol{H}$ let

$$\chi(h) = \Delta_{\boldsymbol{H},\boldsymbol{G}}(h) = \delta(h)/\Delta(h) \tag{4}$$

Let $\mathcal{K}^\chi(\boldsymbol{G})$ be the space of continuous functions F with compact support from $\boldsymbol{G}$ to $\mathbb{C}$ which verify

$$F(xh) = \chi(h)F(x) \tag{5}$$

For $f \in \mathcal{K}(\boldsymbol{G})$ we define $f^\chi \in \mathcal{K}^\chi(\boldsymbol{G})$ by the formula

$$f^\chi(x) = \int_{\boldsymbol{H}} f(xh)\chi(h)^{-1} d\mu_{\boldsymbol{H}}(h) \tag{6}$$

It is wellknown (see [4] §2 prop. 2 and 3) that the map $f \longrightarrow f^\chi$ is onto and if $f \in \mathcal{K}(\boldsymbol{G})$ is such that $f^\chi = 0$ then $\mu_{\boldsymbol{G}}(f) = 0$, so, there is a positive linear form $\mu_{\boldsymbol{G},\boldsymbol{H}}$ on $\mathcal{K}^\chi(\boldsymbol{G})$ such that

$$\int_{\boldsymbol{G}} f(x) d\mu_{\boldsymbol{G}}(x) = \oint_{\boldsymbol{G}/\boldsymbol{H}} \Big(\int_{\boldsymbol{H}} f(xh)\chi(h)^{-1} d\mu_{\boldsymbol{H}}(h)\Big) d\mu_{\boldsymbol{G},\boldsymbol{H}}(x) \tag{7}$$

The linear form $\mu_{\boldsymbol{G},\boldsymbol{H}}$ is $\boldsymbol{G}$-invariant and unique up to a multiplicative scalar. There is a theory of integrable functions for $\mu_{\boldsymbol{G},\boldsymbol{H}}$ (*cf.* [4] §2 or [3] Chap. V by M. Duflo). We can now define the space $\mathcal{H}_\sigma$ as follows : let $\mathcal{C}_\sigma$ the space of continuous functions from $\boldsymbol{G}$ into $\mathcal{H}$ with compact support modulo $\boldsymbol{H}$ such that

$$f(xh) = \chi(h)^{1/2}\sigma(h)^{-1}f(x) \qquad (x \in \boldsymbol{G},\ h \in \boldsymbol{H}) \tag{8}$$

On this space there is a scalar product

$$< f, f' >= \oint_{\boldsymbol{G}/\boldsymbol{H}} < f(g), f'(g) > d\mu_{\boldsymbol{G},\boldsymbol{H}}(g) \tag{9}$$

and the Hilbert completion $\mathcal{H}_\sigma$ of $\mathcal{C}_\sigma$ is the space of $\pi = \mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}}\sigma$. The representation π acts by left translation as in the previous case.

$$[\pi(x)]\varphi(y) = \varphi(x^{-1}y) \tag{10}$$

(c) Examples

(1) The left regular representation of $\boldsymbol{G}$ is the more simple example of induced representation. The subgroup $\boldsymbol{H}$ is the trivial subgroup, the linear form $\mu_{\boldsymbol{G},\boldsymbol{H}}$ is the Haar measure of $\boldsymbol{G}$: $\lambda_{\boldsymbol{G}} = \mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}}(1)$.

(2) *The Heisenberg group.* Let us denote by $\boldsymbol{N}_3$ the group of 3×3 matrix with real coefficients

$$M(a,b,c) = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

We have[2]

$$M(a,b,c).M(a',b',c') = M(a+a', b+b', c+c'+ab')$$

$$M(a,b,c)^{-1} = M(-a,-b,-c+ab)$$

The center $\boldsymbol{Z}$ of $\boldsymbol{N}_3$ is the set of matrix $M(0,0,c)$ with $c \in \mathbb{R}$. There is two natural abelian subgroups of $\boldsymbol{N}_3$:

$$\boldsymbol{K} = \{M(a,0,c)\ ;\ a \in \mathbb{R}\}$$

$$\boldsymbol{H} = \{M(0,b,c)\ ;\ b \in \mathbb{R}\}$$

The subgroups $\boldsymbol{H}$ and $\boldsymbol{K}$ are invariant subgroups of $\boldsymbol{N}_3$. Let z, y be two real numbers and

$$\chi\big(M(0,b,c)\big) = e^{i(by+cz)} \qquad M(0,b,c) \in \boldsymbol{H} \tag{11}$$

For each (z,y), χ is a character of $\boldsymbol{H}$, so χ is a one dimensional representation of $\boldsymbol{H}$. The space of χ is $\mathbb{C}$. We can consider the induced representation $\rho_{z,y} = \mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}}\chi$. It is interesting to give an explicit realization of this induced representation.

First we define the space of $\rho_{z,y}$. We look at the functions $\varphi \in \mathcal{H}_\chi$. We have

$$\varphi\big(M(a,b,c).M(0,\beta,\gamma)\big) = e^{-i(\beta y+\gamma z)}\varphi\big(M(a,b,c)\big)$$

so if we notice that $M(a,b,c) = M(a,0,0).M(0,b,c-ab)$ then

$$\varphi\big(M(a,b,c)\big) = e^{-i(by+(c-ab)z)}\varphi\big(M(a,0,0)\big) \tag{12}$$

and we see that φ is completly known with its values on the subgroup $\boldsymbol{A} = \{M(a,0,0);\ a \in \mathbb{R}\}$. Conversely, every function $\phi : \mathbb{R} \longrightarrow \mathbb{C}$ gives a function $\varphi \in \mathcal{H}_\chi$ by the formula

$$\varphi\big(M(a,b,c)\big) = e^{-i(by+(c-ab)z)}\phi(a) \tag{13}$$

The action of $M \in \boldsymbol{N}_3$ in $\boldsymbol{N}_3/\boldsymbol{H} \simeq \boldsymbol{A}$ is the translation by the first parameter a, so, the Lebesgue measure on $\boldsymbol{A}$ is $\boldsymbol{N}_3-$invariant and we can identify the space $\mathcal{H}_\chi$ with $\boldsymbol{L}^2(\boldsymbol{A}, da)$. Now, we describe the action of $\mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}}\chi$ on this space.

Let ϕ be a function in $\mathcal{K}(\boldsymbol{A})$. The function φ defined by

$$\varphi\big(M(a,b,c)\big) = e^{-i(by+(c-ab)z)}\phi(a) \tag{14}$$

[2] We sometimes write (a,b,c) for $M(a,b,c)$.

is in $\mathcal{H}_\chi$ and we have

$$\begin{aligned}[\rho_{z,y}(a,b,c).\varphi](\alpha,0,0) &= \varphi\big((-a,-b,-c+ab)(\alpha,0,0)\big)\\ &= \varphi(\alpha-a,-b,-c+ab)\\ &= \varphi\big((\alpha-a,0,0)(0,-b,-c+\alpha b)\big)\\ &= e^{i(by+(c-\alpha b)z)}\phi(\alpha-a)\end{aligned}$$

Finally, the induced representation acting in $\mathbf{L}^2(\boldsymbol{A},da)$ is defined by

$$[\rho_{z,y}(a,b,c).\phi](\alpha) = e^{i(by+(c-\alpha b)z)}\phi(\alpha-a) \tag{15}$$

Proposition 1.1 — *For each $(y,z)\in\mathbb{R}\times\mathbb{R},\ z\neq 0$ the representation $\rho_{z,y}$ is irreducible.*

Proof — We want to show that $\rho_{z,y}$ has no proper subrepresentation. For this we take $\phi\in\mathbf{L}^2(\boldsymbol{A},da)=\mathcal{H}_\chi, \phi\neq 0$ and we prove that the only $\rho_{z,y}(\boldsymbol{N}_3)-$invariant space containing ϕ is $\mathcal{H}_\chi$.

Let f be any function in $\mathbf{L}^2(\boldsymbol{A},da)$ which is orthogonal to $\rho_{z,y}(\boldsymbol{N}_3).\phi$. It is enough to prove that $f=0$. Since $\rho_{z,y}(a,b,0).\phi(\alpha)=e^{-ibz\alpha}\phi(\alpha-a)$ and $z\neq 0$ we see that for all $(a,b)\in\mathbb{R}\times\mathbb{R}$

$$\int_{\mathbb{R}} e^{ib\alpha}\phi(\alpha-a)\overline{f(\alpha)}d\alpha = 0$$

This shows that the Fourier transform of the function $h_a:\alpha\longrightarrow\phi(\alpha-a)\overline{f(\alpha)}$ is zero so, h_a is zero almost everywhere for all $a\in\mathbb{R}$. By Fubini's theorem, the positive function $(a,\alpha)\longrightarrow|\phi(\alpha-a)||f(\alpha)|$ is zero almost everywhere. By Fubini's theorem again, we have

$$\begin{aligned}0 &= \int\Big(\int|f(\alpha)||\phi(\alpha-a)|da\Big)d\alpha\\ &= \int|f(\alpha)|\int|\phi(\alpha-a)|dad\alpha\\ &= \int|f(\alpha)|d\alpha\int|\phi(a)|da\end{aligned}$$

and since $\phi\neq 0$ this shows that $f=0$. ∎

1.9.– PROPERTIES OF INDUCTION.

(a) If $\sigma\simeq\sigma'$ are two equivalent representations of $\boldsymbol{H}\subset\boldsymbol{G}$ then $\mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}}\sigma\simeq\mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}}\sigma'$.

(b) If $\pi\simeq\oplus_{i=1}^n\sigma_i$ then $\mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}}\pi\simeq\oplus_{i=1}^n\mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}}\sigma_i$ (n is not necessary a finite number). So, if $\mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}}\sigma$ is irreducible, then σ is irreducible but the converse is false (look at the regular representation).

This result extends to Hilbert integrals. If $\pi = \int_X^{\oplus} \pi^x d\mu(x)$ is an Hilbert integral of representations of $\boldsymbol{H}$ then $\mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}} \pi \simeq \int_X^{\oplus} (\mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}} \pi^x)\, d\mu(x)$.

(c) (*Induction by stage*). Let $\boldsymbol{K} \subset \boldsymbol{H}$ be two subgroups of $\boldsymbol{G}$ and let σ be a representation of $\boldsymbol{K}$ in $\mathcal{H}$. Then

$$\mathrm{Ind}_{\boldsymbol{K}}^{\boldsymbol{G}} \sigma \simeq \mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}} (\mathrm{Ind}_{\boldsymbol{K}}^{\boldsymbol{H}} \sigma) \tag{16}$$

The proof of these results could be found in [16] and are good exercises to understand induction.

(d) Let σ be a representation of $\boldsymbol{H} \subset \boldsymbol{G}$ and $\gamma \in \mathrm{Aut}(\boldsymbol{G})$, ($\mathrm{Aut}(\boldsymbol{G})$ is the group of automorphisms of $\boldsymbol{G}$). It is clear that $\sigma \circ \gamma$ define a representation of $\gamma^{-1}(\boldsymbol{H})$ in the same Hilbert space $\mathcal{H}$.

Proposition 1.2 — *We have* $(\mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}} \sigma) \circ \gamma \simeq \mathrm{Ind}_{\gamma^{-1}(\boldsymbol{H})}^{\boldsymbol{G}} (\sigma \circ \gamma)$.

Proof — Let $\mathcal{H}$ be the space of σ, $\mathcal{H}_\sigma$ (resp. $\mathcal{H}_{\sigma\circ\gamma}$) the space of $\mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}} \sigma$ $\big($resp. $\mathrm{Ind}_{\gamma^{-1}(\boldsymbol{H})}^{\boldsymbol{G}} (\sigma \circ \gamma)\big)$.

For a function $\varphi \in \mathcal{H}_\sigma$ we have

$$\varphi(gh) = \sigma(h)^{-1} \varphi(g) \qquad h \in \boldsymbol{H}$$

and for a function $\psi \in \mathcal{H}_{\sigma\circ\gamma}$ we have

$$\psi(gk) = (\sigma \circ \gamma)(k)^{-1} \psi(g) \qquad k \in \gamma^{-1}(\boldsymbol{H}) \tag{17}$$

For $\varphi \in \mathcal{H}_\sigma$ we define $T\varphi(g) = \varphi\big(\gamma(g)\big)$. Then, for $k \in \gamma^{-1}(\boldsymbol{H})$ we have

$$\begin{aligned} T\varphi(gk) &= \varphi\big(\gamma(gk)\big) \\ &= \varphi\big(\gamma(g)\gamma(k)\big) \end{aligned}$$

Since $k \in \gamma^{-1}(\boldsymbol{H}),\ \gamma(k) \in \boldsymbol{H}$ and we have

$$\begin{aligned} T\varphi(gk) &= \sigma\big(\gamma(k)\big)^{-1} \varphi\big(\gamma(g)\big) \\ &= (\sigma \circ \gamma)(k)^{-1} T\varphi(g) \end{aligned}$$

so, $T\varphi$ verifies the relation (17).

If μ is a left-invariant measure on $\boldsymbol{G}/\boldsymbol{H}$, we define a left-invariant measure ν on $\boldsymbol{G}/\gamma^{-1}(\boldsymbol{H})$ by $\nu(E) = \mu\big(\gamma(E)\big)$ for each Borel set E in $\boldsymbol{G}/\gamma^{-1}(\boldsymbol{H})$, and we have

$$\begin{aligned} \|T\varphi\|^2 &= \int_{\boldsymbol{G}/\gamma^{-1}(\boldsymbol{H})} \|\varphi \circ \big(\gamma(\dot{x})\big)\|^2 d\nu(\dot{x}) \\ &= \int_{\boldsymbol{G}/\boldsymbol{H}} \|\varphi(\dot{x})\|^2 d\mu(\dot{x}) \\ &= \|\varphi\|^2 \end{aligned}$$

Finally, we have to show that T is an intertwining operator between the two representations

$$\begin{aligned} T\big[(\mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}}\sigma)\big(\gamma(x)\big)\varphi\big](y) &= \big[(\mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}}\sigma)\big(\gamma(x)\big)\varphi\big]\big(\gamma(y)\big) \\ &= \varphi\big(\gamma(x)^{-1}\gamma(y)\big) \\ &= \varphi\big(\gamma(x^{-1}y)\big) \\ &= T\varphi(x^{-1}y) \\ &= \big[\,\mathrm{Ind}_{\gamma^{-1}(\boldsymbol{H})}^{\boldsymbol{G}}(\sigma\circ\gamma)T\varphi\big](y) \end{aligned}$$

This completes the proof. ∎

Corollary 1.1 — *If $x \in \boldsymbol{G}$ and $\gamma = \gamma_x$ is the inner automorphism of $\boldsymbol{G}$ defined by x, then $\mathrm{Ind}_{x^{-1}\boldsymbol{H}x}^{\boldsymbol{G}}(\sigma\circ\gamma_x) \simeq \mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}}\sigma$.*

Proof — This is clear by proposition **1.2** because $\mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}}\sigma \simeq \big(\mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}}\sigma\big)\circ\gamma_x$ when γ_x is an inner automorphism of $\boldsymbol{G}$. ∎

(e) Let $\boldsymbol{G}$ be a locally compact group and let $\boldsymbol{Z}$ be a closed normal subgroup of $\boldsymbol{G}$. Let $\overline{\boldsymbol{G}}$ be the locally compact group $\boldsymbol{G}/\boldsymbol{Z}$ and $p : \boldsymbol{G} \longrightarrow \overline{\boldsymbol{G}}$ the canonical map. If $\overline{\boldsymbol{H}}$ is a closed subgroup of $\overline{\boldsymbol{G}}$, then $\boldsymbol{H} = p^{-1}(\overline{\boldsymbol{H}})$ is a closed subgroup of $\boldsymbol{G}$. If $\overline{\sigma}$ is a representation of $\overline{\boldsymbol{H}}$ we define a representation σ of $\boldsymbol{H}$ by $\sigma(h) = \overline{\sigma}\big(p(h)\big)$, and $\sigma(z)$ is the identical operator if $z \in \boldsymbol{Z}$. We also define $\overline{\pi} = \mathrm{Ind}_{\overline{\boldsymbol{H}}}^{\overline{\boldsymbol{G}}}\overline{\sigma}$ and $\pi(x) = \overline{\pi}\big(p(x)\big)$. It is clear that π is a representation of $\boldsymbol{G}$.

Proposition 1.3 — *We have $\pi = \mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}}\sigma$.*

Proof — We assume that there is an invariant measure on $\boldsymbol{G}/\boldsymbol{H}$. Let φ be a function in $\mathcal{H}_\sigma$, the space of $\mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}}\sigma$. We have for $g \in \boldsymbol{G}$ and $h \in \boldsymbol{H}$:

$$\begin{aligned} \varphi(gh) &= \sigma(h)^{-1}\varphi(g) \\ &= \overline{\sigma}\big(p(h)\big)^{-1}\varphi(g) \end{aligned}$$

so, φ is right-invariant by $\boldsymbol{Z}$ and there is one and only one function ψ on $\overline{\boldsymbol{G}}$ such that $\psi\circ p = \varphi$. It is easy to see that $\psi \in \mathcal{H}_{\overline{\sigma}}$. The map $\varphi \longrightarrow \psi$ is a bijective intertwining operator for the two representations (we leave the details to the reader ...)

1.10.– IRREDUCIBILLITY CRITERION ; MACKEY'S THEORY.

As we have seen before, an induced representation is not always irreducible. This is the case when the subgroup $\boldsymbol{H}$ is "*too small*". The philosophy of the "Mackey's machine" is to extend the representation to a subgroup between $\boldsymbol{H}$ and $\boldsymbol{G}$ and to induce this new representation.

Let $\boldsymbol{G}$ be a locally compact group and $\boldsymbol{A}$ be an abelian closed normal subgroup of $\boldsymbol{G}$. The irreducible representations of $\boldsymbol{A}$ are the characters. We define a continuous action of $\boldsymbol{G}$ on $\widehat{\boldsymbol{A}}$ by $(g,\chi) \longrightarrow \chi^g$ where $\chi^g(x) = \chi(g^{-1}xg)$. We also denote by $\boldsymbol{G}_\chi$ the stabilizer of χ in $\boldsymbol{G}$.

$$\boldsymbol{G}^\chi = \{x \in \boldsymbol{G} \mid \chi(x^{-1}ax) = \chi(a),\ \forall a \in \boldsymbol{A}\}$$

This is a closed subgroup of $\boldsymbol{G}$ and there is a bijective map $\dot{g} \longrightarrow \chi^g$ from the homogeneous space $\boldsymbol{G}/\boldsymbol{G}_\chi$ onto the orbit $\boldsymbol{G}.\chi$ in $\widehat{\boldsymbol{A}}$. This map is continuous but not always an homeomorphism. When it is an homeomorphism, we say that $\boldsymbol{A}$ is *regularly embedded* in $\boldsymbol{G}$.

The following result is difficult. The proof is in [10] or [11]

Theorem 1.1 — *The following are equivalent:*

1) The map $\boldsymbol{G}/\boldsymbol{G}_\chi \longrightarrow \boldsymbol{G}.\chi$ is an homeomorphism ;

2) The orbit $\boldsymbol{G}.\chi$ is a locally closed subset of $\widehat{\boldsymbol{A}}$;

3) The space $\widehat{\boldsymbol{A}}/\boldsymbol{G}$ is a T_0 topological space ;

4) The space $\widehat{\boldsymbol{A}}/\boldsymbol{G}$ is countably separated ;

5) For each quasi-invariant ergodic Borel measure μ there is a $\boldsymbol{G}$-orbit Ω in $\widehat{\boldsymbol{A}}$ such that $\mu(\widehat{\boldsymbol{A}}\backslash\Omega) = 0$

A quasi-invariant measure μ on a transformation group $(X, \boldsymbol{G})$ is a Borel measure such that $\mu(x.E) = 0$ if and only if $\mu(E) = 0$ for $x \in \boldsymbol{G}$ and $E \subset X$. Such a measure is said to be ergodic if every $\boldsymbol{G}$-invariant Borel set B, is a μ-null set or $\widehat{\boldsymbol{A}}\backslash B$ is a μ-null set.

It is clear that $\boldsymbol{A} \subset \boldsymbol{G}_\chi$ for all $\chi \in \widehat{\boldsymbol{A}}$, but there is not always a character $\overline{\chi} \in \widehat{\boldsymbol{G}}_\chi$ whose restriction to $\boldsymbol{A}$ is χ. We say that $\rho \in \widehat{\boldsymbol{G}}_\chi$ extend a multiple of χ if $\rho(a) = \chi(a)\mathrm{Id}$, $\forall a \in \boldsymbol{A}$. The space of ρ has a dimension greater than one.

Now we are able to state the main theorem of Mackey [15] (in a special case).

Theorem 1.2 — *Let $\boldsymbol{G}$ be a locally compact group, $\boldsymbol{A}$ a closed normal subgroup of $\boldsymbol{G}$, χ a character of $\boldsymbol{A}$ and $\boldsymbol{G}_\chi$ the stabilizer of χ in $\boldsymbol{G}$.*

1) Let σ be an irreducible representation of $\boldsymbol{G}_\chi$ whose restriction to $\boldsymbol{A}$ is a multiple of χ. Then, $\mathrm{Ind}_{\boldsymbol{G}_\chi}^{\boldsymbol{G}}\sigma$ is irreducible ;

2) Let σ_1 and σ_2 two irreducible representations of $\boldsymbol{G}_\chi$ whose restrictions to $\boldsymbol{A}$ are multiple of χ. Then, $\mathrm{Ind}_{\boldsymbol{G}_\chi}^{\boldsymbol{G}}\sigma_1 \simeq \mathrm{Ind}_{\boldsymbol{G}_\chi}^{\boldsymbol{G}}\sigma_2$ if and only if $\sigma_1 \simeq \sigma_2$;

3) Let σ be an irreducible representation of $\boldsymbol{G}_\chi$ whose restriction to $\boldsymbol{A}$ is a multiple of χ. Then, the restriction of $\mathrm{Ind}_{\boldsymbol{G}_\chi}^{\boldsymbol{G}}\sigma$ to $\boldsymbol{A}$ is an Hilbert integral over the orbit of χ in $\widehat{\boldsymbol{A}}$ for a measure which is $\boldsymbol{G}$-invariant and ergodic (a transitive quasi-orbit). Every irreducible representations the restriction of which to $\boldsymbol{A}$ is such an Hilbert integral is induced by an irreducible representation of $\boldsymbol{G}_\chi$ as in 1).

Mackey has proved that, in the previous situation, there is always an irreducible representation of $\boldsymbol{G}_\chi$ whose restriction to $\boldsymbol{A}$ is a multiple of χ and he gave a construction for all such representations of $\boldsymbol{G}_\chi$. Roughly speaking, this construction gives a bijective map between irreducible representations of $\boldsymbol{G}_\chi$ whose restrictions to $\boldsymbol{A}$ are multiple of χ and the dual space of an "extension" of the group $\boldsymbol{G}_\chi/\boldsymbol{A}$. This extension is *the Mackey obstruction* [15]. This construction is not needed in these lectures.

1.11.– TYPE OF A LOCALLY COMPACT GROUP

Let $\mathcal{H}$ be an Hilbert space (separable) and $\mathcal{L}(\mathcal{H})$ the algebra of all continuous operators on $\mathcal{H}$. Let W be a subset of $\mathcal{L}(\mathcal{H})$. Denote by W' the set

$$W' = \{T \,; T \in W \ , \ T \circ w = w \circ T \ , \ \forall w \in W\}$$

It is clear that W' is a subalgebra of $\mathcal{L}(\mathcal{H})$: it is the *commuting algebra of* W. We denote by W'' the *bicommuting algebra* $(W')'$.

Definition — *1) A subalgebra W of $\mathcal{L}(\mathcal{H})$ is said to be a Von Neumann algebra if it is invariant by adjoint involution and if $W = W''$.*

2) We say that a Von Neuman algebra is a factor if $W \cap W' = \mathbb{C}\, Id$.

If W is any subset of $\mathcal{L}(\mathcal{H})$, W' is a Von Neuman algebra because for a Von Neumann algebra W we have $W'' = W$.

If π is a representation of a locally compact group $\boldsymbol{G}$, $\pi(\boldsymbol{G})''$ is a Von Neumann algebra. The representation π is a *factorial representation* if $\pi(\boldsymbol{G})'$ $\big($or $\pi(\boldsymbol{G})''\big)$ is a factor. This means that $\pi(\boldsymbol{G})' \cap \pi(\boldsymbol{G})'' = \mathbb{C}\,\mathrm{Id}$.

The Von Neumann algebra $\pi(\boldsymbol{G})''$ contains $\pi(\boldsymbol{G})$ and is exactly the smallest Von Neumann algebra which contains $\pi(\boldsymbol{G})$. It is clear from definitions that π is irreducible if and only if $\pi(\boldsymbol{G})' = \mathbb{C}\,\mathrm{Id}$ so, π is of course a factorial representation. A multiple of a factorial representation is also a factorial representation.

On a factor we can define *traces*. A trace t on a factor W is a map defined only on positive elements W^+ of W and with values in $[0, \infty]$ such that:

1) If $x \in W^+,\ y \in W^+$ then $t(x + y) = t(x) + t(y)$;

2) If $x \in W^+$ and $\lambda > 0$ then $t(\lambda x) = \lambda t(x)$;

3) If $z \in W$ then $t(zz^*) = t(z^*z)$.

The trace t is *finite* if $t(x) < \infty$ for every $x \in W^+$ and *semi finite* if $t(x) = \sup\{t(y)\,;\ y \le x\ ;\ t(y) < \infty\}$. We say that t is *faithful* if for $x \in W^+$, $t(x) = 0$ implies $x = 0$.

The trace t is *normal* if for every set $\mathcal{F} \subset W$ which is a "upper filtering set" with upper bound T, then $t(T)$ is the upper bound of $t(\mathcal{F})$.

We are now able to state the classification of factorial representations.

Definition — *Let π be a factorial representation of a locally compact group $\boldsymbol{G}$.*

- *π is type I $\Longleftrightarrow$ π is a multiple of an irreducible representation of $\boldsymbol{G}$;*
- *π is type II $\Longleftrightarrow$ π is not type I and there exists a semi-finite (or finite) normal faithful trace on $\pi(\boldsymbol{G})'$;*
- *π is type III $\Longleftrightarrow$ π is not type I or type II.*

Definition — *A locally compact group $\boldsymbol{G}$ is type I if every factor representation of $\boldsymbol{G}$ is type I. This is equivalent to say that every factorial representation is a multiple of an irreducible one.*

It is known that if a group is not type I, it has factorial representations of type II and type III.

There are large classes of type I groups : abelian, compact, semi-simple connected Lie groups, nilpotent and completely solvable connected Lie groups but, there exists connected solvable Lie groups which are not type I. Only type I groups have computable dual space. The structure of the dual space of non type I locally compact groups is very bad ! We will study (as a survey) the Pukanszky theory for non type I solvable groups where only a kind of "smooth" factorial representations is used instead of irreducible one's.

1.12.– EXERCISES.

Exercise 1.1 – Let λ be the left regular representation of a nontrivial locally compact group $\boldsymbol{G}$ and let ρ be the right regular representation.

Prove that for all $x \in \boldsymbol{G}$ the operator $\lambda(x) \in \mathrm{Hom}(\rho, \rho)$ and that ρ is not irreducible.

Exercise 1.2 – By using the Fourier transform on $\mathbb{R}^n$, show that the regular representation of $\mathbb{R}^n$ is equivalent to $\int_{\widehat{\mathbb{R}^n}}^{\oplus} \chi d\chi$

Exercise 1.3 – *The "$ax+b$" group and the Mackey's Machine.*

We denote by $\boldsymbol{G}$ the "$ax+b$" group which is the group of 2×2 matrix $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ with $a > 0$ and $b \in \mathbb{R}$.

We denote by $\boldsymbol{H}$ the normal abelian subgroup of matrix such that $a = 1$.

a) Compute the orbits of $\boldsymbol{G}$ in $\boldsymbol{H}$ and $\widehat{\boldsymbol{H}}$ for the adjoint representation.

b) Let $\chi \in \widehat{\boldsymbol{H}}$, $\chi \neq 1$. Prove that the stabilizer $\boldsymbol{G}_\chi$ of χ in $\boldsymbol{G}$ is $\boldsymbol{H}$.

c) Apply the Mackey theorem and prove that there are two irreducible inequivalent representations induced from $\boldsymbol{H}$.

d) Compute the space $\widehat{\boldsymbol{G}}$.

2. Nilpotent Lie algebras and Lie groups

2.1.– LIE ALGEBRAS.

A Lie algebra $\mathfrak{g}$ is a finite dimensional vector space over a field $\Bbbk$ on which there is a bilinear form named *the bracket* and denoted by $[\ ,\ ]$ with the following properties

$$[x, y] = -[y, x], \quad \forall x \in \mathfrak{g}, \forall y \in \mathfrak{g}$$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \tag{18}$$

for all (x, y, z) in $\mathfrak{g}$. The equality 18 is the *Jacobi identity*.

A subspace $\mathfrak{a}$ of $\mathfrak{g}$ is said to be an ideal (*resp.* a subalgebra), if $[x, a] \in \mathfrak{a}$ for all $x \in \mathfrak{g}$ and all $a \in \mathfrak{a}$ (*resp.* $a \in \mathfrak{g}$).
In these notes, the field $\Bbbk$ is $\mathbb{R}$ or $\mathbb{C}$.

An important example of Lie algebra is the Lie algebra of all $n \times n$ square matrix (or endomorphisms of a vector space) with the bracket $[M, N] = MN - NM$.

Exercise 2.1 – Let $\{e_1, e_2, \ldots, e_n\}$ a basis of a Lie algebra $\mathfrak{g}$. For $i \leq n,\ j \leq n$ we write $[e_i, e_j] = \sum_{k=1}^{n} C_{i,j}^k\, e_k$. The scalars $C_{i,j}^k$ are the structural constants of the Lie algebra $\mathfrak{g}$. Write the equations between the $C_{i,j}^k$ equivalent to the Jacobi identity.

2.2.– NILPOTENT LIE ALGEBRAS.

Let $\mathfrak{g}$ be a Lie algebra. We define inductively the descending central series by

$$C^0\mathfrak{g} = \mathfrak{g} \text{ and } C^{k+1}\mathfrak{g} = [\mathfrak{g}, C^k\mathfrak{g}] \text{ for } k \in \mathbb{N}$$

and the ascending central series by $Z^0\mathfrak{g} = 0, \quad Z^{k+1}\mathfrak{g}$ is the inverse image in $\mathfrak{g}$ of the center of $\mathfrak{g}/Z^k\mathfrak{g}$.

Definition — *A Lie algebra is said to be nilpotent if there exists n such that $C^n\mathfrak{g} = 0$.*

Let n be the smallest integer such that $C^n\mathfrak{g} = 0$. Then, $C^{n-1}\mathfrak{g}$ is central in $\mathfrak{g}$, so the center of a nilpotent Lie algebra is never zero.

A linear map $\sigma : \mathfrak{g} \longrightarrow \mathfrak{g}'$ is a Lie algebra homomorphism if for all x, y in $\mathfrak{g}$, we have $\sigma([x, y]) = [\sigma(x), \sigma(y)]$.

Proposition 2.1 — *Let $\mathfrak{g}$ be a Lie algebra and let $r \in \mathbb{N}$. The following conditions are equivalent:*

a) $C^r\mathfrak{g} = 0$;

b) There exists a sequence of ideals

$$0 = \mathfrak{a}_r \subset \mathfrak{a}_{r-1} \subset \cdots \subset \mathfrak{a}_1 \subset \mathfrak{a}_0 = \mathfrak{g}$$

such that $[\mathfrak{g}, \mathfrak{a}_k] \subset \mathfrak{a}_{k+1},\ 0 \leq k \leq r-1$ *;*
c) $Z^r\mathfrak{g} = \mathfrak{g}$.

Proof — Almost obvious (exercise). ∎

If $\mathfrak{g}$ and $\mathfrak{g}'$ are two Lie algebras we define obviously a direct product Lie algebra $\mathfrak{g}\times\mathfrak{g}'$, and if $\mathfrak{a}$ is an ideal of $\mathfrak{g}$ the quotient $\mathfrak{g}/\mathfrak{a}$ has a natural structure of Lie algebra such that the map $\mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{a}$ is a Lie algebra homomorphism: $[\overline{x}, \overline{y}] = \overline{[x, y]}$. It is easy to show that if $\mathfrak{g}$ and $\mathfrak{g}'$ are nilpotent then $\mathfrak{g}\times\mathfrak{g}'$ is nilpotent, every subalgebra and quotient of nilpotent Lie algebra is nilpotent (an exercise for the reader !).

2.3.– EXAMPLES.

a) An abelian Lie algebra is nilpotent.

b) The Heisenberg Lie algebra is nilpotent.

c) *The list of nilpotent Lie algebras of dimension lower than 5.* This list of nilpotent Lie algebras (up to isomorphism) has been established by Dixmier [6]. We give this list because it is very useful for examples. We don't write the direct products and give the nonzero brackets $[e_i, e_j]$ for a basis $(e_1, e_2, \ldots, e_n)$ and for $i < j$.

Dimension 2 : only the abelian one ;

Dimension 3 : only the Heisenberg Lie algebra ;

Dimension 4 : one class denoted $\mathfrak{g}_4$ with the brackets

$$[e_1, e_2] = e_3 \quad ; \quad [e_1, e_3] = e_4$$

Dimension 5 : there is six algebras

$\mathfrak{g}_{5,1}$

$$[e_1, e_2] = e_5 \quad ; \quad [e_3, e_4] = e_5$$

$\mathfrak{g}_{5,2}$

$$[e_1, e_2] = e_4 \quad ; \quad [e_1, e_3] = e_5$$

$\mathfrak{g}_{5,3}$

$$[e_1, e_2] = e_4 \quad ; \quad [e_1, e_4] = e_5$$
$$[e_2, e_3] = e_5$$

$\mathfrak{g}_{5,4}$

$$[e_1, e_2] = e_3 \quad ; \quad [e_1, e_3] = e_4$$
$$[e_2, e_3] = e_5$$

$\mathfrak{g}_{5,5}$

$$[e_1, e_2] = e_3 \quad ; \quad [e_1, e_3] = e_4$$
$$[e_1, e_4] = e_5$$

$\mathfrak{g}_{5,6}$

$$[e_1, e_2] = e_3 \quad ; \quad [e_1, e_3] = e_4$$
$$[e_1, e_4] = e_5 \quad ; \quad [e_2, e_3] = e_5$$

Lists of nilpotent Lie algebras for dimension 6 and 7 can be found in several works (many examples are studied in [18]).

d) Denote by $\mathfrak{n}_+$ the set of $n \times n$ square matrix $(a_{i,j})$ such that $a_{i,j} = 0$ if $j \leq i$. Then $\mathfrak{n}_+$ is a subalgebra of the Lie algebra of all $n \times n$ square matrix. For every $k \in \mathbb{N}$ such that $0 \leq k \leq n$ we denote by $\mathfrak{n}_{n,k}$ the set of matrix $X = (x_{i,j})_{1 \leq i,j \leq n}$ such that $x_{i,j} = 0$ if $i \geq j - k$. We see that $\mathfrak{n}_{n,k+1} \subset \mathfrak{n}_{n,k}$ for $k \leq n-2$, $\mathfrak{n}_{n,0} = \mathfrak{n}_+, \mathfrak{n}_{n,n-1} = 0$ and $[\mathfrak{n}_+, \mathfrak{n}_{n,k}] = \mathfrak{n}_{n,k+1}$ and now it is clear that $\mathfrak{n}_+$ is nilpotent.

e) *An example of an algebra which is not nilpotent.* The set of $n \times n$ matrix m such that trace(m)= 0 (*i.e.* $\Sigma_i a_{i,i} = 0$) is a Lie algebra because trace$([m, m'])$= trace$(mm' - m'm) = 0$, so it is a subalgebra of the Lie algebra of $n \times n$ matrix. For $n = 2$ this Lie algebra is denoted $\mathfrak{sl}(2, \mathbb{R})$. Let

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

By an obvious computation we see that $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$ so, $[\mathfrak{g}, \mathfrak{g}] = C^1\mathfrak{g} = \mathfrak{g}$ and $C^k\mathfrak{g} = \mathfrak{g}$ for all $k \in \mathbb{N}$. This shows that $\mathfrak{sl}(2, \mathbb{R})$ is not a nilpotent Lie algebra.

2.4.– EXERCISES.

Exercise 2.2 – Let $\mathfrak{g}$ be a nilpotent Lie algebra the dimension of which is equal to n and let $\mathfrak{a}$ be a subalgebra of $\mathfrak{g}$ which dimension is $n - 1$. Show that $\mathfrak{a}$ is an ideal of $\mathfrak{g}$.

Exercise 2.3 – Show that the set of $n \times n$ matrix

$$\begin{pmatrix} 0 & x & 0 & \dots & 0 & x_n \\ 0 & 0 & x & 0 & 0 & x_{n-1} \\ & \vdots & \vdots & \vdots & & \\ 0 & 0 & \cdots & \cdots & x & x_2 \\ 0 & 0 & \cdots & \cdots & 0 & x_1 \\ 0 & 0 & \cdots & \cdot & 0 & 0 \end{pmatrix}$$

is a nilpotent Lie subalgebra of $\mathcal{L}(\mathbb{R}^n)$.

Exercise 2.4 – Let $\mathfrak{g}$ be the Lie algebra of matrix

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \qquad a \in \mathbb{R},\ b \in \mathbb{R}$$

and let $\mathfrak{h}$ be the subalgebra of $\mathfrak{g}$ defined by the matrix above with $a = 0$. Then $\mathfrak{h}$ is nilpotent, $\mathfrak{g}/\mathfrak{h}$ is nilpotent but $\mathfrak{g}$ is not nilpotent.

2.5.– THE ENGEL THEOREM.

We give this theorem without proof (cf. [24] or books on Lie Algebras).

Theorem 2.1 — *Let $\mathfrak{g}$ be a finite dimensional Lie algebra, V a finite dimensional vector space and $\varphi : \mathfrak{g} \longrightarrow \mathcal{L}(V)$ an homomorphism of Lie algebras such that $\varphi(X)$ is nilpotent for every $X \in \mathfrak{g}$. Then there exists a flag*

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$$

such that $\varphi(X).V_j \subset V_{j-1}$ for each $X \in \mathfrak{g}$, so, $\varphi(\mathfrak{g})$ is a nilpotent Lie algebra.

Corollary 2.1 — *If $\mathfrak{g}$ is a finite dimensional Lie algebra such that* $\mathrm{ad}X$ *is nilpotent for all $X \in \mathfrak{g}$ then $\mathfrak{g}$ is a nilpotent Lie algebra.*

Proof — By Engel theorem we have a flag

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathfrak{g}$$

such that $\mathrm{ad}\,\mathfrak{g}.V_i \subset V_{i-1}$, so, the vector spaces V_i are ideals of $\mathfrak{g}$ and this is exactly the nilpotency of $\mathfrak{g}$.

Definition — *Let $\mathfrak{g}$ be a nilpotent Lie algebra and $\mathfrak{h}$ a subalgebra of $\mathfrak{g}$. We say that a sequence of linearly independant (modulo $\mathfrak{h}$) elements $X_1, \ldots, X_k$ of $\mathfrak{g}$ is a Malcev complementary basis of $\mathfrak{h}$ in $\mathfrak{g}$ if*

$$\mathfrak{h}_0 = \mathfrak{h} \qquad \mathfrak{h}_j = \mathfrak{h} \oplus \mathbb{R}X_1 \oplus \cdots \oplus \mathbb{R}X_j \quad 1 \leq j \leq k$$

and $\mathfrak{h}_j$ are subalgebras of $\mathfrak{g}$.

This Malcev complementary basis is said to be a *strong* Malcev basis if each $\mathfrak{h}_j$ is an ideal in $\mathfrak{g}$.

Theorem 2.2 — *Let $\mathfrak{g}$ be a finite dimensional nilpotent Lie algebra and let $\mathfrak{h}$ be a subalgebra (resp. an ideal) of $\mathfrak{g}$, then, $\mathfrak{h}$ admits a Malcev (resp. strong complementary basis).*

Proof — If $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, then the adjoint representation of $\mathfrak{g}$ induces an action of $\mathfrak{g}$ in $\mathfrak{g}/\mathfrak{h}$ by nilpotent operators $\mathrm{ad}_{\mathfrak{g}/\mathfrak{h}}X$. By Engel's theorem, we find vectors $X_1, \ldots, X_k \in \mathfrak{g}$ such that $X_1 + \mathfrak{h}, \ldots, X_k + \mathfrak{h}$ is a basis of $\mathfrak{g}/\mathfrak{h}$ such that the matrix of every $\mathrm{ad}_{\mathfrak{g}/\mathfrak{h}}$ is strictly upper triangular. So it is clear that $X_1, \ldots, X_k$ is a strong Malcev complementary basis of $\mathfrak{h}$ in $\mathfrak{g}$.

If $\mathfrak{h}$ is a proper subalgebra and not an ideal of $\mathfrak{g}$, we prove the result by induction on the codimension of $\mathfrak{h}$. By Engel's theorem apply to the action of $\mathfrak{h}$ in $\mathfrak{g}/\mathfrak{h}$, we get $X \in \mathfrak{g}$ such that $[X_1, \mathfrak{h}] \subset \mathfrak{h}$. The subalgebra $\mathfrak{h}$ is an ideal in the subalgebra $\mathfrak{h}_1 = \mathbb{R}X_1 \oplus \mathfrak{h}$ and the codimension of $\mathfrak{h}_1$ is lower than the codimension of $\mathfrak{h}$ in $\mathfrak{g}$, so, we apply the induction hypothesis to find the rest of the Malcev complementary basis. ∎

2.6.– LIE GROUPS.

Definition — *A Lie group is a group $\boldsymbol{G}$ which is an analytic manifold such that the mapping $(x, y) \longrightarrow xy^{-1}$ of $\boldsymbol{G} \times \boldsymbol{G}$ (product manifold) is analytic.*

The dimension of a Lie group is its dimension as a manifold.

EXAMPLES.

1.– If V is a finite dimensional vector space, the group $GL(n,V)$ of linear automorphisms of V is a Lie group. It is an open subset of $\mathcal{L}(V)$ and it is well known that the maps $(u,v) \longrightarrow u \circ v$ and $u \longrightarrow u^{-1}$ are analytic.

2.– The group $\boldsymbol{G}_2$ of 2×2 matrix

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \qquad a > 0,\ b \in \mathbb{R}$$

is a Lie group. As a manifold, it is isomorphic to $\mathbb{R}_+^* \times \mathbb{R}$ and it is clear that the product and inverse maps are analytic.

An homomorphism of a Lie group $\boldsymbol{G}$ into a Lie group $\boldsymbol{G}'$ is an analytic homomorphism of groups.

For a Lie group $\boldsymbol{G}$ we denote by $\boldsymbol{G}_0$ the connected component of the neutral element e of $\boldsymbol{G}$ and by $\boldsymbol{G}_e$ the tangent space of $\boldsymbol{G}$ at e.

2.7.– THE LIE ALGEBRA OF A LIE GROUP

Let $\boldsymbol{G}$ be a Lie group. For $x \in \boldsymbol{G}$, the left translation $L_x : y \longrightarrow L_x(y) = xy$ is an analytic map from $\boldsymbol{G}$ onto $\boldsymbol{G}$. A vector field Z on $\boldsymbol{G}$ is said to be left invariant if for all $x \in \boldsymbol{G}, dL_x Z = Z$. Given a tangent vector $X \in \boldsymbol{G}_e$ there exists one and only one left invariant vector field $\widetilde{X}$ defined by

$$\big[\widetilde{X} f\big](x) = \Big[\frac{d}{dt} f\big(x(\gamma(t))\big)\Big]_{t=0}$$

where $f \in \mathcal{C}^\infty(\boldsymbol{G}),\ x \in \boldsymbol{G}$, and $t \longrightarrow \gamma(t)$ is any curve on $\boldsymbol{G}$ with tangent vector X for $t = 0$ and $\gamma(0) = e$.

For two vectors fields we can define a bracket (cf. Exercise 2.5) and it is easy to see that the bracket of two left invariant vector fields is also left invariant. So the tangent vector space at $e \in \boldsymbol{G}$ has a structure of Lie algebra. It is the Lie algebra $\mathfrak{g}$ of $\boldsymbol{G}$.

Now, we give general results on Lie groups without proofs because these results are easier for nilpotent groups.

1) Given $X \in \mathfrak{g}$ there is one and only one analytic homomorphism θ_X from $\mathbb{R}$ into $\boldsymbol{G}$ such that $d\theta_X(0) = X$ and we put $\exp X = \theta_X(1)$.

2) We have $\exp(t+s)X = \exp tX \exp sX$ for all s and t in $\mathbb{R}$ and $X \in \mathfrak{g}$. So we have $\theta_X(t) = \exp tX$ and for a function f on $\boldsymbol{G}$, $\dfrac{d}{dt} f(\exp tX)_{|t=0} = \tilde{X} f$ where $\tilde{X}$ is the left invariant vector field corresponding to $X \in \mathfrak{g}$.

3) There is a neigbourhood $V \subset \mathfrak{g}$ of 0 such that if $X \in V$ and $Y \in V$ then we have $\exp X \exp Y = \exp \eta(X,Y)$ where $\eta(X,Y)$ is in the Lie algebra generated by X and Y. We have $\eta(X,Y) = X + Y + \frac{1}{2}[X,Y] + \dots$, the other terms of this formulas are expressions like $[Y[X[\dots[Y,X]]\dots]]$. This is the *Campbell-Hausdorff formula*.

4) Let $\boldsymbol{G}$ be a Lie group and $\mathfrak{g}$ its Lie algebra. Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$. There exists one and only one connected Lie subgroup $\boldsymbol{H}$ (analytic subgroup) of $\boldsymbol{G}$ whose Lie algebra is $\mathfrak{h}$. This subgroup is not always closed and not always a submanifold of $\boldsymbol{G}$.

5) If $\mathfrak{g}$ is a Lie algebra there is one and only one connected and simply connected Lie group $\boldsymbol{G}$ (up to an isomorphism) with Lie algebra $\mathfrak{g}$. It is clear from the Campbell-Hausdorff formula that if two Lie groups have the same Lie algebra then they are localy isomorphic. If $\boldsymbol{G}$ is connected and simply connected and if $\boldsymbol{G}'$ is a connected Lie group with the same Lie algebra the local isomorphism from $\boldsymbol{G}$ into $\boldsymbol{G}'$ expand to a global homomorphism φ. The kernel of φ is a central discret subgroup of $\boldsymbol{G}$. So, every connected Lie group is a quotient of $\boldsymbol{G}$ by a central discret subgroup.

6) Let $GL(\mathbb{R}^n)$ be the set of all linear automorphisms of $\mathbb{R}^n$ (or equivalently $n \times n$ non singular real square matrix). It is a Lie group which is an open set in the space $M(n, \mathbb{R})$ of all real $n \times n$ square matrix, so $\mathcal{L}(\mathbb{R}^n)$ is identical to the tangent space of $GL(\mathbb{R}^n)$ at identity. For $X \in \mathcal{L}(\mathbb{R}^n)$ and $t \in \mathbb{R}$ the map $t \longrightarrow \operatorname{Exp} tX = \displaystyle\sum_{k=0}^{\infty} \frac{t^k}{k!} X^k$ is an analytic homomorphism of $\mathbb{R}$ into $GL(\mathbb{R}^n)$, so we have, with the above notations $d\theta_X(0) = X$ and we can see that the bracket $[X, Y]$ as left invariant vector fields is $XY - YX$ in $\mathcal{L}(\mathbb{R}^n)$ (Exercise 2.5). So $\mathcal{L}(\mathbb{R}^n)$is the Lie algebra of $GL(n, \mathbb{R})$.

7) Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$, and as above denote by $GL(\mathfrak{g})$ the group of non singular endomorphisms of $\mathfrak{g}$. We denote by ad the mapping $X \longrightarrow \operatorname{ad} X$ for $X \in \mathfrak{g}$ where $\operatorname{ad} X(Y) = [X, Y]$ for all Y in $\mathfrak{g}$. The set $\operatorname{ad}(\mathfrak{g})$ is a subalgebra of $\mathcal{L}(\mathfrak{g})$. Let $\operatorname{Int}(\mathfrak{g})$ denote the analytic subgroup of $GL(\mathfrak{g})$ whose Lie algebra is $\operatorname{ad}(\mathfrak{g})$; $\operatorname{Int}(\mathfrak{g})$ is called the *adjoint group* of $\mathfrak{g}$.

The group $\operatorname{Aut}(\mathfrak{g})$ of all automorphisms of $\mathfrak{g}$ is a closed subgroup of $GL(\mathfrak{g})$ Thus, it is a Lie subgroup of $GL(\mathfrak{g})$. Let $\delta(\mathfrak{g})$ be the Lie algebra of $\operatorname{Aut}(\mathfrak{g})$. The exponential mapping from $\delta(\mathfrak{g})$ into $\operatorname{Aut}(\mathfrak{g})$ is the restriction of Exp thus, we have $\operatorname{Exp}(t\boldsymbol{a}) \in \operatorname{Aut}(\mathfrak{g})$ for each $\boldsymbol{a} \in \delta(\mathfrak{g})$ and $t \in \mathbb{R}$. This means that for all $X, Y \in \mathfrak{g}$ we have $\operatorname{Exp}(t\boldsymbol{a})([X, Y]) = [\operatorname{Exp}(t\boldsymbol{a})X, \operatorname{Exp}(t\boldsymbol{a})Y]$ for all $t \in \mathbb{R}$. We differentiate this relation and for $t = 0$ we get

$$\boldsymbol{a}([X, Y]) = [\boldsymbol{a}(X), Y] + [X, \boldsymbol{a}(Y)] \tag{19}$$

thus, $\boldsymbol{a}$ is a derivation of $\mathfrak{g}$. Conversely, if $\boldsymbol{a}$ is a derivation of $\mathfrak{g}$ we see, by computing the operator $\boldsymbol{a}^k$ for $k \in \mathbb{N}$ with the formula (19) that $\operatorname{Exp}(t\boldsymbol{a})([X, Y]) = [\operatorname{Exp}(t\boldsymbol{a})X, \operatorname{Exp}(t\boldsymbol{a})Y]$ for all $t \in \mathbb{R}$ and thus $\delta(\mathfrak{g})$ consists of all derivations of $\mathfrak{g}$. By using the Jacobi identity we see that $\operatorname{ad}(\mathfrak{g}) \subset \delta(\mathfrak{g})$ and therefore $\operatorname{Int}(\mathfrak{g}) \subset \operatorname{Aut}(\mathfrak{g})$.

For $x \in \boldsymbol{G}$ we denote by $\operatorname{Ad} x$ or $\operatorname{Ad}_{\boldsymbol{G}}(x)$ the differential of the inner automorphism of $\boldsymbol{G}$ defined by $u \longrightarrow xux^{-1}$ at the identity. It is clear that Ad is an homomorphism from $\boldsymbol{G}$ into $GL(\mathfrak{g})$ because we have for $x, y \in \boldsymbol{G}$ and

$X \in \boldsymbol{G}$

$$\begin{aligned}\mathrm{Ad}(xy)X &= \Big[\frac{d}{dt}\big(xy\exp tXy^{-1}x^{-1}\big)\Big]_{t=0}\\ &= \Big[\frac{d}{dt}\big(x\exp(t(\mathrm{Ad}\,y)X)x^{-1}\big)\Big]_{t=0}\\ &= \mathrm{Ad}(x)\big(\,\mathrm{Ad}(y)X\big)\end{aligned}$$

This mapping is called the *adjoint representation* of $\boldsymbol{G}$.

Proposition 2.2 — $\mathrm{Int}(\mathfrak{g})$ *is a normal subgroup of* $\mathrm{Aut}(\mathfrak{g})$ *and for every* $x \in \boldsymbol{G}$ *we have*

$$\exp(\mathrm{Ad}(x).X) = x.\exp(X).x^{-1} \tag{20}$$

The differential of Ad_e *is the adjoint representation* ad *of the Lie algebra* $\mathfrak{g}$ *and we have the formulas:*

$$\mathrm{Ad}(\exp X) = \mathrm{Exp}(\mathrm{ad}\,X) \quad ; \ \exp(\mathrm{Exp}(\mathrm{ad}\,X)Y) = \exp X \exp Y \exp(-X) \tag{21}$$

Definition — *A Lie group* $\boldsymbol{G}$ *is said to be nilpotent if its Lie algebra is nilpotent.*

Proposition 2.3 — *If* $\boldsymbol{G}$ *is a nilpotent Lie group, all the connected Lie subgroups of* $\boldsymbol{G}$ *are nilpotent Lie groups and a quotient of* $\boldsymbol{G}$ *by a closed normal subgroup is also a nilpotent Lie group.*

Proof — If $\boldsymbol{H}$ is a subgroup of $\boldsymbol{G}$ its Lie algebra $\mathfrak{h}$ is the set of $X \in \mathfrak{g}$ such that $\exp tX \in \boldsymbol{H}$ for all $t \in \mathbb{R}$, thus $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$ and is nilpotent. If $\boldsymbol{H}$ is closed normal, the mapping $\varphi : \boldsymbol{G} \longrightarrow \boldsymbol{G}/\boldsymbol{H}$ is continuous (and analytic), thus its differential is Lie algebra homomorphism and onto. This shows that the Lie algebra of $\boldsymbol{G}/\boldsymbol{H}$ is a quotient of $\mathfrak{g}$ and thus is nilpotent. ∎

There is also a group definition of nilpotent groups for *connected* Lie groups. We define the *descending central series* for the group $\boldsymbol{G}$ to be $\mathcal{C}^1\boldsymbol{G} = \boldsymbol{G}$, $\mathcal{C}^{j+1}\boldsymbol{G} = [\boldsymbol{G}, \mathcal{C}^j\boldsymbol{G}]$, where the notation $[\boldsymbol{H}, \boldsymbol{K}]$ means the subgroup generated by all commutators $hkh^{-1}k^{-1}, h \in \boldsymbol{H}, k \in \boldsymbol{K}$. Then we say that $\boldsymbol{G}$ is nilpotent if $\mathcal{C}^j\boldsymbol{G} = \{e\}$ for some $j \in \mathbb{N}$. One can show that the group $\mathcal{C}^j\boldsymbol{G}$ is a Lie subgroup of $\boldsymbol{G}$ for every j and its Lie algebra is $\mathcal{C}^j\mathfrak{g} = [\mathfrak{g}, \mathcal{C}^{j-1}\mathfrak{g}]$.

For nilpotent Lie groups, the relations between the group and its Lie algebra are simple because we have the following results.

We denote by $\boldsymbol{U}_n$, $n \in \mathbb{N}$ the closed subgroup of matrix $g = (g_{i,j}) \in GL(n, \mathbb{R})$ such that $g_{i,i} = 1$ for $1 \leq i \leq n$ and $g_{i,j} = 0$ for $i > j$. The Lie algebra of $\boldsymbol{U}_n$ is $\mathfrak{n}_n$ the set of strictly upper triangular matrix (cf. Exercise 2.6). The group $\boldsymbol{U}_n$ is called the standard unipotent group of order n.

We see that the exponential map of $\mathfrak{n}_+$ into $\boldsymbol{U}_n$ is a diffeomorphism: in fact, it is one to one because the inverse is the “logarithm” defined by

$$\log x = \sum_{k\geq 1} \frac{(-1)^{k-1}}{k}(x - \mathrm{Id})^k = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}(x - \mathrm{Id})^k$$

where $x \in \boldsymbol{U}_n$ and Id is the identity operator (note: $\boldsymbol{U}_n = \mathrm{Id} + \mathfrak{n}_+$). For the nilpotent Lie group $\boldsymbol{U}_n$, the vector $\eta(X,Y)$ in the Campbell-Hausdorff formula is clearly $\eta(X,Y) = \log(\exp X \exp Y)$ and we see that η is polynomial in the coordinates of X and Y.

Theorem 2.3 — *1) Every analytic subgroup of $\boldsymbol{U}_n$ is a simply connected closed subgroup.*

2) Conversely, if $\boldsymbol{H}$ is a simply connected nilpotent Lie group, there exists $n \in \mathbb{N}$ and an injective homomorphism of Lie groups of $\boldsymbol{H}$ in $\boldsymbol{U}_n$.

Proof — 1) Let $\boldsymbol{H}$ be an analytic subgroup of $\boldsymbol{U}_n$ and let $\mathfrak{h}$ be the Lie algebra of $\boldsymbol{H}$. $\mathfrak{h}$ is included in $\mathfrak{n}_+$ and by the Campbell-Hausdorff formula $\exp \mathfrak{h}$ is a subgroup of $\boldsymbol{U}_n$ which is closed because the map exp of $\mathfrak{n}_+$ onto $\boldsymbol{U}_n$ is a diffeomorphism. This also shows that exp maps a neighbourhood of 0 in $\mathfrak{h}$ onto a neighbourhood of 1 in $\boldsymbol{H}$ and that $\exp \mathfrak{h}$ is an open subgroup of $\boldsymbol{H}$ (and closed). But $\boldsymbol{H}$ is connected so, $\boldsymbol{H} = \exp \mathfrak{h}$.

2) Let $\mathfrak{h}$ be the Lie algebra of $\boldsymbol{H}$. This Lie algebra is isomorphic to a subalgebra $\mathfrak{h}'$ of $\mathfrak{n}_+ = \mathfrak{n}_n$ for an integer n. The set $\mathrm{Exp}\, \mathfrak{h}'$ is a simply connected Lie subgroup of $\boldsymbol{U}_n$ (Campbell-Hausdorff formula) with Lie algebra $\mathfrak{h}$, so $\boldsymbol{H}$ and $\mathrm{Exp}\, \mathfrak{h}'$ have isomorphic Lie algebras. They are locally isomorphic and because the groups are simply connected they are isomorphic as Lie groups. ∎

Corollary 2.2 — *For every simply connected nilpotent Lie group $\boldsymbol{G}$ with Lie algebra $\mathfrak{g}$, we have*

(i) $\exp : \mathfrak{g} \longrightarrow \boldsymbol{G}$ *is a diffeomorphism.*

(ii) There exists a unique map $(X,Y) \longrightarrow \eta(X,Y)$ from $\mathfrak{g} \times \mathfrak{g}$ into $\mathfrak{g}$ which is polynomial into the coordinates of X and Y and such that $\exp \eta(X,Y) = \exp X \exp Y$ *for all X and Y in $\mathfrak{g}$.*

(iii) Every analytic subgroup of $\boldsymbol{G}$ is a simply connected closed subgroup.

Proof — If $\boldsymbol{G}$ is simply connected and connected, the proof of the theorem shows that there is a diffeomorphism of $\boldsymbol{G}$ onto a closed subgroup of $\mathfrak{n}_n = \mathfrak{n}_+$ for an $n \in \mathbb{N}$. All things follows from this. ∎

Proposition 2.4 — *Let $\boldsymbol{G}$ be a connected nilpotent Lie group with center $\boldsymbol{Z}$. Then*

1) $\boldsymbol{G}/\boldsymbol{Z} = \mathrm{Ad}(\boldsymbol{G})$ *is simply connected ;*

2) $\boldsymbol{Z}$ is connected.

Proof — 1) The mapping $\mathrm{Ad} : \boldsymbol{G} \longrightarrow \mathrm{Ad}(\boldsymbol{G}) \subset GL(\mathfrak{g})$ verifies $x \exp X x^{-1} = \exp\big(\mathrm{Ad}(x)X\big)$ for every $x \in \boldsymbol{G}$ and $X \in \mathfrak{g}$. Thus, $\mathrm{Ad}\, x = \mathrm{Id}$ if and only if $x \exp X x^{-1} = \exp X$ for every $X \in \mathfrak{g}$. This means that $x.\exp X = \exp X.x$ for every X. But $\boldsymbol{G}$ is connected and so it is generated by $\exp \mathfrak{g}$ (here $\boldsymbol{G}$ is nilpotent so we have $\boldsymbol{G} = \exp \mathfrak{g}$). This shows that $\ker(\mathrm{Ad}) = \boldsymbol{Z}$ and $\boldsymbol{G}/\boldsymbol{Z} = \mathrm{Ad}(\boldsymbol{G})$. The group $\mathrm{Ad}(\boldsymbol{G})$ is an analytic subgroup of $GL(\mathfrak{g})$ whose Lie algebra is $\mathrm{ad}(\mathfrak{g})$. We choose a basis of $\mathfrak{g}$ such that the matrix of all the operators $\mathrm{ad}\, X$ are strictly upper triangular, thus $\mathrm{Ad}(\boldsymbol{G}) = \mathrm{Exp}\,\mathrm{ad}(\mathfrak{g})$ is isomorphic to a subgroup of $\mathfrak{n}_n$ where n is the dimension of $\mathfrak{g}$. The theorem above shows that $\mathrm{Ad}(\boldsymbol{G})$ is simply connected.

2) Let $\boldsymbol{Z}^0$ be the connected component of the neutral element of $\boldsymbol{G}$. The mapping $\theta : \boldsymbol{G}/\boldsymbol{Z}^0 \longrightarrow \boldsymbol{G}/\boldsymbol{Z}$ is a covering ($\boldsymbol{Z}/\boldsymbol{Z}^0$ is discret and central in $\boldsymbol{G}/\boldsymbol{Z}^0$), but $\boldsymbol{G}/\boldsymbol{Z}$ is simply connected, so θ is an isomorphism and $\boldsymbol{Z} = \boldsymbol{Z}^0$ is connected. ∎

Proposition 2.5 — *Let $\boldsymbol{G}$ be a simply connected Lie group and let $\boldsymbol{H}$ an invariant analytic subgroup of $\boldsymbol{G}$. Then $\boldsymbol{H}$ is closed and $\boldsymbol{G}/\boldsymbol{H}$ is simply connected.*

Proof — The Lie algebra $\mathfrak{h}$ of $\boldsymbol{H}$ is an ideal in $\mathfrak{g}$, the Lie algebra of $\boldsymbol{G}$. Thus $\mathfrak{g}/\mathfrak{h}$ is a Lie algebra and we consider $\boldsymbol{G}_1$ the simply connected Lie group with Lie algebra $\mathfrak{g}/\mathfrak{h}$. Since $\boldsymbol{G}$ is simply connected, there is a unique Lie group morphism $p : \boldsymbol{G} \longrightarrow \boldsymbol{G}/\boldsymbol{H}$ whose differential is the Lie algebra homomorphism $\mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{h}$. The kernel $\boldsymbol{H}_1$ of p is a closed Lie group having $\mathfrak{h}$ for Lie algebra. Since $\boldsymbol{G}_1$ is simply connected, the group $\boldsymbol{H}_1$ is connected and is equal to $\boldsymbol{H}$. Moreover, $\boldsymbol{G}/\boldsymbol{H} = \boldsymbol{G}_1$ is simply connected. ∎

2.8.– MALCEV OR COEXPONENTIAL BASES IN NILPOTENT LIE GROUPS.

Proposition 2.6 — *Let $\boldsymbol{G}$ be a simply connected nilpotent Lie group and $\mathfrak{h}$ a subalgebra of the Lie algebra $\mathfrak{g}$ of $\boldsymbol{G}$. Then there exists a basis $\{X_1, \dots, X_p\}$ of a supplementary subspace of $\mathfrak{h}$ such that if $g_j(t) = \exp tX_j$, the mapping*

$$(t_1, \dots, t_p, X) \longrightarrow g_p(t_p) \cdots g_1(t_1) \exp X$$

is a diffeomorphism from $\mathbb{R}^p \times \mathfrak{h}$ onto $\boldsymbol{G}$.

Such a basis is a Malcev basis or a coexponential basis for $\mathfrak{h}$ in $\mathfrak{g}$ (cf. Theorem **2.2**.

Proof — Since $\mathfrak{g}$ is nilpotent there is a subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$ whose dimension is $(\dim \mathfrak{g} - 1)$ and which contains $\mathfrak{h}$. The subalgebra $\mathfrak{g}_0$ is an ideal. Let $X \in \mathfrak{g}$, $X \notin \mathfrak{g}_0$, then the map $(t, Y) \longrightarrow \exp tX . \exp Y$ is a diffeomorphism of $\mathbb{R} \times \mathfrak{g}_0$ into $\boldsymbol{G}$ and its image is a connected Lie subgroup of $\boldsymbol{G}$ which strictly contains $\boldsymbol{G}_0$ so it is $\boldsymbol{G}$. It is clear that we obtain the Malcev basis by iteration of this case. ∎

2.9.– COADJOINT ORBITS OF NILPOTENT LIE GROUPS.

Let $\boldsymbol{G}$ be a connected Lie group and $\mathfrak{g}$ its Lie algebra. Above we have defined the adjoint representation of $\boldsymbol{G}$. Another important object for representation theory of Lie groups is the *coadjoint representation* of $\boldsymbol{G}$. It is the finite dimensional representation Ad^* of $\boldsymbol{G}$ in the dual space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ and is defined by the formula

$$f \in \mathfrak{g}^*,\ x \in \boldsymbol{G},\ Y \in \mathfrak{g} \qquad <\mathrm{Ad}^* x.f\,,\, Y> = <f\,,\, \mathrm{Ad}\, x^{-1}.Y>$$

The famous “orbit's method” of A.A. Kirillov is the description of the space $\widehat{\boldsymbol{G}}$ by the orbits of the representation Ad^* in $\mathfrak{g}^*$. For many simply connected Lie groups the orbits of Ad^* in $\mathfrak{g}^*$ have “good” geometric properties. If $\boldsymbol{G}$ is a nilpotent connected Lie group we have :

Theorem 2.4 — *The orbits of a nilpotent connected Lie group* $\boldsymbol{G}$ *under the action of the coadjoint representation in* $\mathfrak{g}^*$ *are closed submanifolds of* $\mathfrak{g}^*$.

Proof — We write ρ for the representation Ad^*. We prove this result by a description of the coadjoint orbits by polynomial coordinates (cf. [3] Chapter I p. 7).

Let $f \in \mathfrak{g}^*$ and $\Omega = \rho(\boldsymbol{G}).f$ the orbit of f. Since ρ is a unipotent representation, $d\rho$ is a nilpotent representation of $\mathfrak{g}$ in $\mathfrak{g}^*$ $(d\rho(X).f(Y) = f([Y,X])$. We choose a Jordan-Holder sequence for the action of ρ in $\mathfrak{g}^*$:

$$\{0\} = V_n \subset V_{n-1} \subset \cdots \subset V_0 = \mathfrak{g}^*$$

with $\dim V_i = n - i$. Let π_j be the natural projection $V \longrightarrow V/V_j$ and $\mathfrak{g}_j = \{X \in \mathfrak{g};\ d\rho(X).f \in V_j\}$. The stabilizer of $\pi_j(f)$ is $\exp \mathfrak{g}_j$. We have $\mathfrak{g}_j \subset \mathfrak{g}_{j+1}$ and $\mathfrak{g}_n = \mathfrak{g}(f)$. Let $1 \leq j_1 < j_2 < \cdots < j_d \leq n$ be the index such that $\mathfrak{g}_{j_k} \neq \mathfrak{g}_{j_{k-1}}$ and let $e_{j_k} \in V_{j_{k-1}} - V_{j_k}$, $X_k \in \mathfrak{g}_{j_{k-1}}$ such that $X_k.f = e_{j_k}(mod.\ V_{j_k})$. Clearly $X_1, X_2, \ldots, X_d$ is a Malcev complementary basis of $\mathfrak{g}(f)$. We complete e_{j_k} in a suitable basis of $\mathfrak{g}^*$. We put

$$\exp(t_1X_1)\ldots\exp(t_dX_d).f = \sum_{j=1}^{n} P_j(t_1,\ldots,t_d)\, e_j$$

where the P_j are polynomials. We have

a) P_j is a polynomial in $t_1,\ldots,t_k$ only, where k is the greatest integer such that $j_k \leq j$

b) $P_{j_k}(t_1,\ldots,t_d) = t_k + Q_k(t_1,\ldots,t_{k-1})$.

This facts result of the following remarks : if $j_k > j$, then $X_k \in \mathfrak{g}_j$, so $\exp(tX_k).f = f$ $(mod.\ V_j)$ and we have

$$\begin{aligned}\pi_j \exp(t_1X_1)\ldots\exp(t_dX_d).f &= \pi_j \exp(t_1X_1)\ldots\exp(t_kX_k).f \\ &= \pi_j \sum_{i=1}^{j} P_i(t_1,\ldots,t_d)e_i\end{aligned}$$

We have also $\exp(tX_k).f = f + te_{j_k}\ mod\, V_{j_k}$ thus,

$$\begin{aligned}\pi_{j_k} \exp(t_1X_1)\ldots\exp(t_dX_d).f &= \pi_{j_k}\Big(\exp(t_1X_1)\exp(t_{k-1}X_{k-1}).(f + t_ke_{j_k})\Big) \\ &= \pi_{j_k} \sum_{1}^{j_k} P_i(t_1,\ldots,t_d)e_i\end{aligned}$$

It is clear that the map $(t_1,\ldots,t_d) \longrightarrow \big(P_{j_1}(t_1,\ldots,t_d),\ldots,P_{j_d}(t_1,\ldots,t_d)\big)$ is a manifold structure on Ω and Ω is isomorphic to $\mathbb{R}^d$.

Now let $g = \lim_{n\to\infty} \exp(t_1^{(n)}X_1)\ldots\exp(t_d^{(n)}X_d).f$. If the sequence $(t_1^{(n)},\ldots,t_d^{(n)})$ is bounded in $\mathbb{R}^n$ then it is clear that $g \in \Omega$ else, let k the first index such that $t_k^{(n)}$ is not bounded. Taking if it is needed a subsequence, the sequences $t_1^{(n)},\ldots,t_{k-1}^{(n)}$ have finite limits. Then, we see by the proof above that P_{j_k} cannot have a limit. ∎

2.10.– NOTES ON SOLVABLE LIE ALGEBRAS AND LIE GROUPS.

We define the derivated series of a Lie algebra by

$$\mathcal{D}^1\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]\ ;\ \mathcal{D}^k\mathfrak{g} = [\mathcal{D}^{k-1}\mathfrak{g},\mathcal{D}^{k-1}\mathfrak{g}]$$

Definition — *A Lie algebra is said to be solvable if there exists $k \in \mathbb{N}$ such that $\mathcal{D}^k\mathfrak{g} = \{0\}$.*

It is clear that a nilpotent Lie algebra is solvable, but the converse is false (look at the "$ax+b$" group).

A connected Lie group is solvable if its Lie algebra is solvable. If $\boldsymbol{G}$ is a simply connected solvable Lie group its exponential mapping is not always bijective. A simply connected Lie group such that the exponential mapping is bijective is said to be *an exponential group*. We have a theorem of Dixmier (cf. [3] Chapter 1, page 2)

Theorem 2.5 — *Let $\boldsymbol{G}$ be a connected and simply connected Lie group. The following conditions are equivalent:*
1) For every $X \in \mathfrak{g}$, $\operatorname{ad} X$ has no eigenvalue of the form $i\alpha$, ($\alpha \in \mathbb{R}^*$, $i^2 = -1$) ;
2) exp *is an injective mapping ;*
3) exp *is onto ;*
4) exp *is a bijective mapping ;*
5) exp *is a diffeomorphism ;*
6) The roots of $\mathfrak{g}$ are $\psi(x)(1+i\alpha)$, where $\alpha \in \mathbb{R}$ and ψ is a real linear form on $\mathfrak{g}$.

The representation theory for exponential groups is almost as complete as the nilpotent one. In these notes we sometimes give the proofs for exponential groups when this is not more difficult and we give the results otherwise.

We state below some useful results on solvable Lie groups. When we omit the proof it may be found in [13].

Proposition 2.7 — *Let $\boldsymbol{G}$ be a simply connected solvable Lie group and let $\boldsymbol{H}$ an analytic subgroup of $\boldsymbol{G}$. Then $\boldsymbol{H}$ is closed and simply connected. Furthermore, if $\mathfrak{h}$ is the Lie algebra of $\boldsymbol{H}$, there exists a basis $(\varepsilon_1,\ldots,\varepsilon_m)$ of $\mathfrak{g}$ containing a basis $(\varepsilon_{i_1},\ldots,\varepsilon_{i_m})$ of $\mathfrak{h}$ such that the map*

$$\mu\Big(\sum_{j=1}^{m} t_j\varepsilon_{i_j}\Big) = \exp_{\boldsymbol{G}}(t_1e_{i_1})\exp_{\boldsymbol{G}}(t_2e_{i_2})\ \ldots,\exp_{\boldsymbol{G}}(t_me_{i_m})$$

is a diffeomorphism from $\mathfrak{h}$ to $\boldsymbol{H}$.

Proposition 2.8 — *Let $\boldsymbol{G}$ be a simply connected solvable Lie group. Then $\boldsymbol{G}$ has no non-trivial compact subgroup.*

Proof — We proceed by induction on the dimension of $\boldsymbol{G}$. Let $\boldsymbol{K}$ be a compact subgroup of $\boldsymbol{G}$. We denote by $\boldsymbol{G}'$ the connected subgroup of $\boldsymbol{G}$ whose Lie algebra is $[\mathfrak{g},\mathfrak{g}]$. We may assume that $\boldsymbol{G}' \neq 0$ since otherwise $\boldsymbol{G}$ is a vector group and

$\boldsymbol{K} = \{e\}$. Since $\boldsymbol{G}$ is solvable, $\boldsymbol{G}' \neq \boldsymbol{G}$ and the compact subgroup $\boldsymbol{K}\boldsymbol{G}'/\boldsymbol{G}'$ of the simply connected abelian group $\boldsymbol{G}/\boldsymbol{G}'$ is trivial by the above argument. So, $\boldsymbol{K} \subset \boldsymbol{G}'$ and the dimension of $\boldsymbol{G}'$ is lower than the dimension of $\boldsymbol{G}$. We apply the inductive hypothesis to prove that $\boldsymbol{K}$ is trivial. ∎

Proposition 2.9 — *Let* $\mathfrak{g}$ *be a solvable Lie algebra (on* $\mathbb{R}$ *or* $\mathbb{C}$*). Then* $\mathfrak{a} = [\mathfrak{g}, \mathfrak{g}]$ *is a nilpotent ideal of* $\mathfrak{g}$*.*

Proof — This is a by-product of the Lie theorem asserting that every solvable Lie algebra is isomorphic to a subalgebra of triangular matrix. Then, $\mathfrak{d} = [\mathfrak{g}, \mathfrak{g}]$ is isomorphic to a subalgebra of $\mathfrak{n}_n$ for some $n \in \mathbb{N}$. ∎

2.11.– EXERCISES.

Exercise 2.5 – Let $\boldsymbol{G}$ be a Lie group and T_e its tangent space at $e \in \boldsymbol{G}$ the neutral element. Let X be a vector field on $\boldsymbol{G}$. We say that X is left invariant if, for all $x \in \boldsymbol{G}$, $y \in \boldsymbol{G}$ we have $dL_x(X_y) = X_{xy}$ where L_x is the left product by x and $dL_x(X_y)$ is the image of the tangent vector X_y by the tangent mapping dL_x at y.

a) Let X and Y two vector fields on $\boldsymbol{G}$ and φ an analytic function on $\boldsymbol{G}$. Show that if we define $[X, Y]$ by

$$[X, Y]\varphi = X(Y\varphi) - Y(X\varphi)$$

then $[X, Y]$ is a bracket on the space of vector fields on $\boldsymbol{G}$ which is a Lie algebra (not finite dimensional).

b) The bracket of two left invariant vector fields is left invariant.

c) Show that the map $X \longrightarrow X_1$ from the space of left invariant vector fields into T_e is one to one. By definition the Lie algebra of $\boldsymbol{G}$ is T_e with the bracket corresponding to the bracket on the left invariant vector fields.

d) Let A be a finite dimensional algebra (with unit element 1) on $\mathbb{R}$ and A^* the set of nonsingular element of A. It is wellknown that A^* is an open set in A so, A^* is a Lie algebra on $\mathbb{R}$. Show that the Lie algebra of A^* is A with the bracket $[a, b] = ab - ba$.

Exercise 2.6 – Show that the Lie algebra of the group $\boldsymbol{U}_n$ is $\mathfrak{n}_n$.

Exercise 2.7 – Let $\boldsymbol{N}_3$ be the Heisenberg group. Let

$$X = \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}$$

X is an element of the Heisenberg Lie algebra. Show that the matrix $\operatorname{Exp} tX$ is

$$\operatorname{Exp} tX = \begin{pmatrix} 1 & tx & tz + \frac{t^2}{2}xy \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{pmatrix}$$

Deduce from this that the Lie algebra of $\boldsymbol{N}_3$ is the three dimensional Heisenberg Lie algebra.

Exercise 2.8 – Let $\boldsymbol{G}$ be the "$ax+b$" group. We set

$$X = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$$

show that

$$\operatorname{Exp} tX = \begin{pmatrix} e^{tx} & \frac{(e^{tx}-1)}{x} y \\ 0 & 1 \end{pmatrix}$$

Thus, the Lie algebra $\mathfrak{g}_2$ of this group is the space of 2×2 line matrix.

Show that the mapping Exp is bijective.

Exercise 2.9 – 1) The group "$ax+b$" is simply connected.

2) The group $SL(2,\mathbb{R})$ of 2×2 real matrix with determinant equal 1 is not simply connected.

3) The group $SL(2,\mathbb{C})$ (or $SL(n,\mathbb{C})$) is simply connected.

Exercise 2.10 – Every subalgebra and every quotient of a solvable Lie algebra is solvable.

Exercise 2.11 – Let $\boldsymbol{G}$ be a connected nilpotent Lie group and $\mathfrak{g}$ its Lie algebra. Let $\mathfrak{z}$ be the center of $\mathfrak{g}$. We denote $D = \{X \in \mathfrak{g}\ ;\ \exp X = 1\}$.

Show that D is a discret subgroup of the vector group $\mathfrak{g}$ and that the exponential mapping induces a bijective mapping from $\mathfrak{g}/D$ onto $\boldsymbol{G}$. *[Use the universal covering of $\boldsymbol{G}$.]*

Exercise 2.12 – Show that if a solvable analytic group is not simply connected, it has a non-trivial compact subgroup.

Exercise 2.13 – Let $\boldsymbol{H}$ be a nilpotent analytic group whose underlying manifold is $\mathbb{R}^4$ and where the multiplication is given by

$$(x_1, y_1, z_1, t_1)(x_2, y_2, z_2, t_2) = (x_1 + x_2 + z_1 t_2,\ y_1 + y_2 + \alpha z_1 t_2,\ z_1 + z_2,\ t_1 + t_2)$$

where α is a fixed real number. Let $\boldsymbol{D}$ be the discret central subgroup of $\boldsymbol{H}$ consisting of the elements $(p, q, 0, 0)$, with arbitrary integers p and q. Let $\boldsymbol{G} = \boldsymbol{H}/\boldsymbol{D}$. Show that if α is irrational then $[\boldsymbol{G}, \boldsymbol{G}]$ is not closed in $\boldsymbol{G}$.

Exercise 2.14 – Show that there is also coexponential bases in simply connected exponential groups ([3], Chapter I 3.6 p. 5).

Exercise 2.15 – Let $\boldsymbol{G}$ be the simply connected solvable Lie group which Lie algebra is defined by the basis $\{e_1, e_2, e_3\}$ and the brackets

$$[e_1\,,\,e_2] = -e_3 \qquad [e_1\,,\,e_3] = e_2$$

Show that the exponential mapping is not injective. [remark that the center of $\mathfrak{g}$ is 0 so $\mathfrak{g} \simeq \operatorname{ad}\mathfrak{g}$. Compute the group $\operatorname{Ad}\boldsymbol{G}$,which is not simply connected and find a simply connected covering of $\operatorname{Ad}\boldsymbol{G}$].

Show that the center of $\boldsymbol{G}$ is not connected.

3. Polarizations in solvable Lie algebras

Let $\boldsymbol{G}$ be a connected and simply connected Lie group (we say *simply connected Lie group* below). The almost universal method for building representations of $\boldsymbol{G}$ is the following. Let $f \in \mathfrak{g}^*$ be a linear form on the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$. Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$ such that $[\mathfrak{h}, \mathfrak{h}] \subset \ker f$. Let $\boldsymbol{H}$ be the analytic subgroup of $\boldsymbol{G}$ whose Lie algebra is $\mathfrak{h}$. As $f([H, H']) = 0$, for all H, H' in $\mathfrak{h}$, it is clear that the map $f_{|\mathfrak{h}}$ is a Lie algebra homomorphism from $\mathfrak{h}$ into $\mathbb{R}$. If $\boldsymbol{H}$ is simply connected, there is one and only one homomorphism $\chi_f : \boldsymbol{H} \longrightarrow \mathbb{U} = \{u \in \mathbb{C} \; ; \; |u| = 1\}$ such that $\chi_f(\exp H) = e^{if(H)}$ for every $H \in \boldsymbol{H}$. Thus χ_f is a one dimensional representation of $\boldsymbol{H}$ and we can consider the representation $\rho = \mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}} \chi_f$ of $\boldsymbol{G}$. We say that ρ is a monomial representation and $\mathfrak{h}$ is a (real) polarization at f. For nilpotent (and exponential groups), all irreducible unitary representations are monomial. For more general solvable Lie groups it is necessary to consider polarizations in complexification of $\mathfrak{g}$. As it is not more difficult to study this situation I give the proofs of this section are given for solvable Lie algebras.

3.1.– POLARIZATIONS.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{R}$, $f \in \mathfrak{g}^*$ and $\mathfrak{g}_{\mathbb{C}}$ the Lie algebra over $\mathbb{C}$ which is the complexified of $\mathfrak{g}$, $\sigma(v) = \overline{v}$, the antilinear involution of $\mathfrak{g}_{\mathbb{C}}$. Denote also by f the linear form on $\mathfrak{g}_{\mathbb{C}}$ extending f on $\mathfrak{g}$. We define a skew bilinear form B_f on $\mathfrak{g}_{\mathbb{C}}$ by $B_f(X, Y) = f([X, Y])$. The kernel of B_f in $\mathfrak{g}$ is denoted $\mathfrak{g}(f)$ or $\mathfrak{g}^f$ (and $\mathfrak{g}(f)_{\mathbb{C}}$ or $\mathfrak{g}_{\mathbb{C}}^f$ in $\mathfrak{g}_{\mathbb{C}}$).

Let $\boldsymbol{G}$ be a connected Lie group whose Lie algebra is $\mathfrak{g}$. We have defined previously the coadjoint representation of $\boldsymbol{G}$ in $\mathfrak{g}^*$. It is denoted by Ad^* and we have for $f \in \mathfrak{g}^*, x \in \boldsymbol{G}$ and $Y \in \mathfrak{g}$

$$< \mathrm{Ad}^*(x)f \, , \, Y >=< f \, , \, \mathrm{Ad}(x^{-1})Y >$$

We shall see below that the orbits of Ad^* in $\mathfrak{g}^*$ give the parametrization of $\widehat{\boldsymbol{G}}$ when $\boldsymbol{G}$ is nilpotent (or exponential).

Definition — *We say that a subalgebra $\mathfrak{h} \subset \mathfrak{g}_{\mathbb{C}}$ is a polarization at f if*

1) $B_f(\mathfrak{h}, \mathfrak{h}) = 0$ and $\mathfrak{h}$ has the maximal dimension of such subalgebras ;

2) $\mathfrak{h} + \overline{\mathfrak{h}}$ is a subalgebra of $\mathfrak{g}_{\mathbb{C}}$.

A polarization is said to be real if $\mathfrak{h} = \overline{\mathfrak{h}}$.

The set of all polarizations at f is denoted by $\mathrm{Pol}(f)$ *or* $\mathrm{Pol}(f, \mathfrak{g})$.

Definition — *A polarization $\mathfrak{h}$ is positive if the hermitian form H defined by $H(X, Y) = if([X, \overline{Y}])$ is positive on $\mathfrak{h}$ (i.e. $if([X, \overline{Y}]) \geq 0 \; \forall X \in \mathfrak{h}, \; \forall Y \in \mathfrak{h}$).*

We can remark that if $\mathfrak{h}$ is real, it is positive.

Example — Consider the Heisenberg Lie algebra $\mathfrak{n}_3 = \mathbb{R}X \oplus \mathbb{R}Y \oplus \mathbb{R}Z$ with $[X, Y] = Z$. Let $f = \lambda Z^*, \; \lambda \in \mathbb{R}^*$, $\mathfrak{h}_1 = \mathbb{C}Z \oplus \mathbb{C}(X + iY)$, $\mathfrak{h}_2 = \mathbb{C}Z \oplus \mathbb{C}(X - iY)$.

Then $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are polarizations at f. furthemore, $H_f(X+iY, X+iY) = 2\lambda$ so, $\mathfrak{h}$ is positive if an only if $\lambda > 0$.

Every polarization $\mathfrak{h}$ at f generates several other objects.

$$\mathfrak{e}_{\mathbb{C}} = \mathfrak{h} + \overline{\mathfrak{h}} \qquad \mathfrak{d}_{\mathbb{C}} = \mathfrak{h} \cap \overline{\mathfrak{h}}$$
$$\mathfrak{e} = (\mathfrak{h} + \overline{\mathfrak{h}}) \cap \mathfrak{g} \qquad \mathfrak{d} = \mathfrak{h} \cap \mathfrak{g}$$

If $\mathfrak{h}$ is real then $\mathfrak{e}$=$\mathfrak{d}$ and $\mathfrak{h} \cap \mathfrak{g}$ is a maximal isotropic subalgebra for B_f in $\mathfrak{g}$. We see that $\mathfrak{d}_{\mathbb{C}}$ is the orthogonal of $\mathfrak{e}_{\mathbb{C}}$ for B_f, this means that B_f is non-degenerated on $\mathfrak{e}_{\mathbb{C}}/\mathfrak{d}_{\mathbb{C}}$ or on $\mathfrak{e}/\mathfrak{d}$. The dimension of the space $\mathfrak{e}/\mathfrak{d}$ is even.

Now we consider $\boldsymbol{G}$ the simply connected group whose Lie algebra is $\mathfrak{g}$. Let $\boldsymbol{D}^0$ (*resp.* $\boldsymbol{E}^0$) the analytic subgroup whose Lie algebra is $\mathfrak{d}$ (*resp.* $\mathfrak{e}$). Let $\boldsymbol{G}(f)$ be the stabilizer of f in $\boldsymbol{G}$ for the coadjoint representation : it is not always a connected subgroup of $\boldsymbol{G}$. Its Lie algebra is $\mathfrak{g}(f)$ so, $\boldsymbol{G}(f)^0$, the connected component of $\boldsymbol{G}(f)$, is contained in $\boldsymbol{D}^0$. If $\mathfrak{h}$ is invariant by $\mathrm{Ad}\big(\boldsymbol{G}(f)\big)$ then it can be shown that $\boldsymbol{D} = \boldsymbol{D}^0\boldsymbol{G}(f)$ is a closed subgroup and its Lie algebra is $\mathfrak{d}$ (cf. [3] chap. 4). If $\boldsymbol{G}$ is nilpotent or exponential this fact is not needed because we have the following lemma.

Lemma 3.1 — *Let $\boldsymbol{G}$ be an exponential group (simply connected) then the stabilizer $\boldsymbol{G}(f)$ of $f \in \mathfrak{g}^*$ for the coadjoint representation is connected.*

Proof — Let $X \in \mathfrak{g}$ such that $\mathrm{Ad}^*(\exp X).f = f$. The set

$$S = \{t \in \mathbb{R};\ \mathrm{Ad}^*(\exp tX).f = f\}$$

is a closed subgroup of $\mathbb{R}$. If S is a discret subgroup, let t_0 be its lowest positive element. We have

$$\mathrm{Ad}^*(\exp \frac{t_0}{2}X).\big(\mathrm{Ad}^*(\exp \frac{t_0}{2}X).f - f\big) = -\big(\mathrm{Ad}^*(\exp \frac{t_0}{2}X).f - f\big) \neq 0$$

So, $\mathrm{ad}^*(\frac{t_0}{2}X)$ has an eigenvalue of the form $i\pi n$ with $n \neq 0$ and it is a contradiction with $\boldsymbol{G}$ exponential, so $S = \mathbb{R}$ and this shows that $\boldsymbol{G}(f)$ is connected. ∎

Definition — *If $\mathfrak{a}$ is an ideal of $\mathfrak{g}$ a polarization $\mathfrak{h}$ at f is admissible for $\mathfrak{a}$ if $\mathfrak{h} \cap \mathfrak{a}_{\mathbb{C}}$ is a polarization at $f|_{\mathfrak{a}}$.*

Definition — *We say that a polarization $\mathfrak{h}$ at f satisfies the Pukanszky condition if $\mathrm{Ad}^*(\boldsymbol{D}).f = f + \mathfrak{e}^{\perp}$.*

Definition — *Let γ be an automorphism of $\mathfrak{g}$. A polarization $\mathfrak{h}$ at f is γ-invariant if $\gamma(\mathfrak{h}) = \mathfrak{h}$.*

The aim of this section is to build polarizations at $f \in \mathfrak{g}^*$ which are positive, admissible for an ideal $\mathfrak{a}$, which verify the Pukanszky condition and are $\mathrm{Ad}(x)$-invariant for $x \in \boldsymbol{G}_f$. This is a result of L. Auslander and B. Kostant but we give here the simpler constructive proof of M. Vergne.

Remark – with the previous notations, we have $\mathfrak{h}^f = \mathfrak{h}$; $\overline{\mathfrak{h}^f} = \overline{\mathfrak{h}}$, so $\mathfrak{e}^f = \mathfrak{g} \cap (\mathfrak{h} + \overline{\mathfrak{h}})^f = (\mathfrak{h} \cap \overline{\mathfrak{h}}) \cap \mathfrak{g} = \mathfrak{d}$.

3.2.– CONSTRUCTION OF ISOTROPIC SUBALGEBRAS.

Let $\mathfrak{g}$ be a solvable Lie algebra on $\mathbb{k}(=\mathbb{R}$ or $\mathbb{C})$. We say that $\mathfrak{g}$ is completely solvable if there exists a sequence of ideals $\mathcal{S}=(\mathfrak{g}_i)_{i=1\ldots n}$ such that

$$\{0\}=\mathfrak{g}_0\subset\mathfrak{g}_1\subset\cdots\subset\mathfrak{g}_i\subset\mathfrak{g}_{i+1}\subset\cdots\subset\mathfrak{g}_n=\mathfrak{g}$$

with $\dim\mathfrak{g}_i=i$ for $0\le i\le n$. Every nilpotent Lie algebra over $\mathbb{k}$ is completely solvable and if $\mathbb{k}=\mathbb{C}$ (or if $\mathbb{k}$ is an algebraicly closed field of characteristic zero), every solvable Lie algebra is completely solvable but the following example shows that this is not true if $\mathbb{k}=\mathbb{R}$.

Example — The solvable Lie algebra defined by

$$[e_1,e_2]=-e_3\quad[e_1,e_3]=e_2$$

is not completely solvable.

The main result of this section is a construction of (complex) polarization at $f\in\mathfrak{g}^*$ for every solvable Lie algebra over $\mathbb{R}$. This construction is due to M. Vergne (cf. [3], Chapter IV).

Let $\mathcal{S}=(\mathfrak{g}_i)$ be a sequence of ideals in $\mathfrak{g}$ a completely solvable Lie algebra and $\mathcal{S}_j=(\mathfrak{g}_i)_{0\le i\le j}$. Let γ be an automorphism of $\mathfrak{g}$ such that $\gamma(\mathfrak{g}_i)=\mathfrak{g}_i$ for every $i=1\ldots n$. Let f be a linear form on $\mathfrak{g}^*$, and for every $i\le n$ we put $f_i=f|_{\mathfrak{g}_i}$ and $\mathfrak{g}_i(f_i)$ the kernel of B_{f_i} on $\mathfrak{g}_i$. Finally, we put

$$P(f,\mathcal{S})=\sum_{i=0}^{n}\mathfrak{g}_i(f_i)$$

Theorem 3.1 — *1) $P(f,\mathcal{S})$ is a subalgebra and maximal isotropic subspace for B_f ;*

2) $P(f,\mathcal{S})\cap\mathfrak{g}_j=P(f_j,\mathcal{S}_j)$ for every $j\le n$;

3) For all $\varphi\in\mathfrak{g}^*$ such that $\varphi\big(P(f,\mathcal{S})\big)=0$ we have $P(f+\varphi,\mathcal{S})=P(f,\mathcal{S})$;

4) If $\gamma\in\mathrm{Aut}(\mathfrak{g})$ is such that $\gamma(\mathfrak{g}_i)=\mathfrak{g}_i$, $i=0\ldots n$, then $P(f,\mathcal{S})$ is $\gamma-$invariant.

We need some lemmas about bilinear forms on vector spaces.

Let V be a finite dimensional vector space on a field $\mathbb{k}$ and B a skew bilinear form on $\mathbb{k}$. The kernel of B is $N(B)=\{X\ ;\ B(X,Y)=0\ \forall Y\in V\}$. The form B defines a nondegenerate form on $V/N(B)$ so this space has an even dimension. If W^B denotes the orthogonal of a space W, we remember that W is totally isotropic if and only if $W=W^B$ or if and only if $\dim W=\frac{1}{2}\big(\dim V+\dim N(B)\big)$ and $B(W,W)=0$.

Lemma 3.2 — *(J. Dixmier [3] lemme 1.1.1 p. 49) Let V' be a subspace of V whose codimension is one. Let $B'=B|_{V'}$ and $N(B')$ the kernel of B'.*

- *If $N(B) \subset V'$, then $N(B) \subset N(B')$ and the codimension of $N(B)$ in $N(B')$ is one ;*
- *If $N(B) \not\subset V'$, then $N(B') \subset N(B)$ and the codimension of $N(B')$ in $N(B)$ is one.*

Proof — If $N(B) \subset V'$ then $N(B) \subset N(B')$. Let X be a vector of V which is not in V'. $N(B)$ is the kernel of the linear form on $N(B') : Y \to B(X,Y)$. So the dimension of $N(B')$ is greater or equal than $\dim N(B) - 1$. But the integers $\dim V - \dim N(B)$ and $\dim V' - \dim N(B')$ are even, so $\dim N(B) = \dim N(B') - 1$.

If $N(B) \not\subset V'$, we get a vector $X \in N(B)$ not belonging to V' and let $u \in N(B')$. We have $B(u,Y) = 0$ $\forall Y \in V'$ because $u \in N(B')$ and $B(u,X) = 0$ because $X \in N(B)$. This shows that $N(B') \subset N(B)$, $N(B) \cap V' = N(B')$ has codimension one in $N(B)$. ∎

Lemma 3.3 — *Let V be as above and let $\mathcal{S} = (V_i)_{0 \le i \le n}$ be a sequence of subspaces*

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_i \subset V_{i+1} \subset \cdots \subset V_n = V$$

such that $\dim V_i = i$ for every $i \le n$. If $B_i = B|_{V_i}$ and $N(B_i)$ is the kernel of B_i in V_i then

1) $P(B,\mathcal{S}) = \sum_{i=1}^{n} N(B_i)$ is a maximal isotropic subspace for B ;

2) $P(B,\mathcal{S}) \cap V_j = P(B_j, \mathcal{S}_j)$ where $\mathcal{S}_j = (V_i)_{0 \le i \le j}$.

Proof — We prove this lemma by induction on the dimension of V.

By the previous lemma, we have two cases :

1) $N(B_n) \subset V_{n-1}$ and $N(B_n) \subset N(B_{n-1})$.

We have $P(B_n, \mathcal{S}_n) \cap V_{n-1} = P(B_{n-1}, \mathcal{S}_{n-1})$ and $P(B_n, \mathcal{S}_n)$ is isotropic for B. furthermore, by using the induction hypothesis

$$\begin{aligned}\dim P(B_n,\mathcal{S}_n) = \dim P(B_{n-1},\mathcal{S}_{n-1}) &= \frac{n-1+\dim\big(N(B_{n-1})\big)}{2}\\ &= \frac{n+\dim\big(N(B_n)\big)}{2}\end{aligned}$$

this proves the first case.

2) $N(B_n) \not\subset V_{n-1}$ and $N(B_{n-1}) \subset N(B_n)$.

We have $\dim N(B_n) = \dim N(B_{n-1}) + 1$ and $P(B_n, \mathcal{S}_n) = P(B_{n-1}, \mathcal{S}_{n-1}) + N(B_n)$. Thus $P(B_n, \mathcal{S}_n)$ is isotropic for B and

$$\begin{aligned}\dim P(B_n,\mathcal{S}_n) = \dim P(B_{n-1},\mathcal{S}_{n-1}) + 1 &= \frac{n-1+\dim\big(N(B_{n-1})\big)}{2} + 1\\ &= \frac{n+\dim N(B_n)}{2}\end{aligned}$$

clearly we also have $P(B_n, \mathcal{S}_n) \cap V_{n-1} = P(B_{n-1}, \mathcal{S}_{n-1})$. This completes the proof of the lemma. ∎

We now prove the theorem **3.1**. By the above lemma $P(f,\mathcal{S})$ is maximal isotropic and $P(f,\mathcal{S})\cap\mathfrak{g}_j = P(f_j,\mathcal{S}_j)$. We prove that $P(f,\mathcal{S})$ is a subalgebra. Take $X_i \in \mathfrak{g}_i(f_i)$ and $X_j \in \mathfrak{g}_j(f_j)$ for $i \geq j$ (so $\mathfrak{g}_j \subset \mathfrak{g}_i$). Because $\mathfrak{g}_j$ is an ideal of $\mathfrak{g}$, $[X_i, X_j] \in \mathfrak{g}_j$ and for $Y \in \mathfrak{g}_j$ we have by the Jacobi identity

$$f\big(\big[[X_i,X_j],Y\big]\big) + f\big(\big[[X_j,Y],X_i\big]\big) + f\big(\big[[Y,X_i],X_j\big]\big) = 0$$

but $f\big(\big[[X_j,Y],X_i\big]\big) = 0$ because $[X_j,Y] \in \mathfrak{g}_j \subset \mathfrak{g}_i$ and $X_i \in \mathfrak{g}_i(f_i)$, and $f\big(\big[[Y,X_i],X_j\big]\big) = 0$ because $[Y,X_i] \in \mathfrak{g}_j$ and $X_j \in \mathfrak{g}_j(f_j)$.

We prove 3) by induction on the dimension of $\mathfrak{g}$. We first remark that if $\varphi_n \in \mathfrak{g}_n^*$ and $\varphi_n\big(P(f_n,\mathcal{S}_n)\big) = 0$ then $P(f_n,\mathcal{S}_n)$ is isotropic for $\varphi_n + f_n$, so $\dim P(f_n,\mathcal{S}_n) \leq \dim P(f_n+\varphi_n,\mathcal{S}_n)$. If $\mathfrak{g}_n(f_n+\varphi_n) \subset P(f_n,\mathcal{S}_n)$ we have $P(f_n+\varphi_n,\mathcal{S}_n) = P(f_{n-1}+\varphi_{n-1},\mathcal{S}_{n-1}) + \mathfrak{g}_n(f_n+\varphi_n)$ and by the induction hypothesis, $P(f_{n-1}+\varphi_{n-1},\mathcal{S}_{n-1}) = P(f_{n-1},\mathcal{S}_{n-1}) \subset P(f_n,\mathcal{S}_n)$. Thus, $P(f_n + \varphi_n,\mathcal{S}_n) \subset P(f_n,\mathcal{S}_n)$ and by the above inequality on the dimensions we see that $P(f_n+\varphi_n,\mathcal{S}_n) = P(f_n,\mathcal{S}_n)$. Thus the only thing we have to show is

$$\mathfrak{g}_n(f_n+\varphi_n) \subset P(f_n,\mathcal{S}_n).$$

- If $\mathfrak{g}_n(f_n) \subset \mathfrak{g}_{n-1}$, we have $P(f_n,\mathcal{S}_n) = P(f_{n-1},\mathcal{S}_{n-1}) = P(f_{n-1} + \varphi_{n-1},\mathcal{S}_{n-1}) \subset P(f_n+\varphi_n,\mathcal{S}_n)$. Let $X_n \in \mathfrak{g}_n(f_n+\varphi_n)$, $Y_n \in P(f_n,\mathcal{S}_n) \subset P(f_n + \varphi_n,\mathcal{S}_n)\cap\mathfrak{g}_{n-1}$. Then, $[X_n,Y_n] \in \mathfrak{g}_{n-1}\cap P(f_n+\varphi_n,\mathcal{S}_n) = P(f_{n-1}+\varphi_{n-1},\mathcal{S}_{n-1})$ so,
$$0 = (f_n+\varphi_n)([X_n,Y_n]) = f_n([X_n,Y_n])$$
- If $\mathfrak{g}_n(f_n) \not\subset \mathfrak{g}_{n-1}$, then
$$\begin{aligned}\dim P(f_n,\mathcal{S}_n) &= \dim P(f_{n-1},\mathcal{S}_{n-1}) + 1\\ &= \dim P(f_{n-1}+\varphi_{n-1},\mathcal{S}_{n-1}) + 1\\ &\geq \dim P(f_n+\varphi_n,\mathcal{S}_n)\end{aligned}$$

and we have the equality by the begining remark.

The last assertion is clear because $\gamma(\mathfrak{g}_i) = \mathfrak{g}_i$ and $\gamma.f_i = f_i$ so, $\gamma\big(\mathfrak{g}_i(f_i)\big) = \mathfrak{g}_i(f_i)$ and $P(f,\mathcal{S})$ is $\gamma-$invariant. ∎

3.3.– ON THE PUKANSZKY CONDITION.

Proposition 3.1 — *Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}_\mathbb{C}$ such that $f([\mathfrak{h},\mathfrak{h}]) = 0$ and $\mathfrak{g}(f)_\mathbb{C} \subset \mathfrak{h}$. Then, the following conditions are equivalent :*

1) $\mathrm{Ad}^*\,\boldsymbol{D}^0.f = f + \mathfrak{e}^\perp$ *;*

2) $f + \mathfrak{e}^\perp \subset \mathrm{Ad}^*\,\boldsymbol{G}.f = \Omega(f)$ *and $\mathfrak{h}$ has maximal dimension ;*

3) $\forall\varphi \in \mathfrak{e}^\perp$, *$\mathfrak{h}$ is maximal isotropic for $f+\varphi$.*

Proof — First we suppose 1). To show 2) it is enough to verify $\dim_\mathbb{C}\mathfrak{h} = \frac{1}{2}\big(\dim\mathfrak{g} + \dim\mathfrak{g}(f)\big)$. But by construction of $\mathfrak{e}$ and $\mathfrak{d}$ we have $\dim_\mathbb{C}\mathfrak{h} = \frac{1}{2}\big(\dim_\mathbb{R}\mathfrak{e} + \dim_\mathbb{R}\mathfrak{d}\big)$. We must show that

$$\dim_\mathbb{R}(\mathfrak{g}) + \dim_\mathbb{R}\mathfrak{g}(f) = \dim_\mathbb{R}\mathfrak{e} + \dim_\mathbb{R}\mathfrak{d}$$

The mapping $\theta : x \longrightarrow \mathrm{Ad}^* x.f$ from $\boldsymbol{D}^0$ in $f + \mathfrak{e}^\perp$ is onto (by 1)). By the Sard theorem, there is a point $x_0 \in \boldsymbol{D}^0$ such that the differential of θ is onto. Thus, $X \longrightarrow \mathrm{ad}^* X(\mathrm{Ad}^* x_0.f)$ from $\mathfrak{d}$ to $\mathfrak{e}^\perp$ is onto. So, $\mathfrak{d}^{\mathrm{Ad}^* x_0.f} = \{X; \mathrm{Ad}^* x_0.f([X, \mathfrak{d}]) = 0\} = \mathfrak{e}$. This implies that $\mathfrak{d}^f = \mathfrak{e}$ because $\mathfrak{d}$ and $\mathfrak{e}$ are $\boldsymbol{D}^0-$invariant. Recall that $\mathfrak{g}(f) \subset \mathfrak{d}$ and we have $\dim \mathfrak{e} + \dim \mathfrak{d} = \dim \mathfrak{g} + \dim \mathfrak{g}(f)$.

We now show 2) $\Longrightarrow$ 3). Since we have $\varphi \in \mathfrak{e}^\perp$, $\mathfrak{h}$ is isotropic for $f + \varphi$. So it is enough to show that $\mathfrak{g}(f+\varphi) = \mathfrak{g}(f)$ for every $\varphi \in \mathfrak{e}^\perp$. But $f + \varphi = \mathrm{Ad}^* x.f$ for an $x \in \boldsymbol{G}$ and we have $\mathfrak{g}(\mathrm{Ad}^* x.f) = \mathrm{Ad}^* x.\mathfrak{g}(f)$ thus, $\dim \mathfrak{g}(f) = \dim \big(\mathrm{Ad}^* x.\mathfrak{g}(f)\big)$.

If we assume 3), by a previous result the orbits $\mathrm{Ad}^* \boldsymbol{D}^0.g$ are closed for every $g \in f + \mathfrak{e}^\perp$ so they are also closed and of course $\mathrm{Ad}^* \boldsymbol{D}^0.f = f + \mathfrak{e}^\perp$. ∎

Corollary 3.1 — *Let $\mathfrak{h}$ be a real subalgebra of $\mathfrak{g}$ such that $f([\mathfrak{h}, \mathfrak{h}]) = 0$ and $\boldsymbol{H}$ the analytic subgroup with Lie algebra $\mathfrak{h}$. Then, the following conditions are equivalent :*

1) $\mathrm{Ad}^* \boldsymbol{H}^0.f = f + \mathfrak{h}^\perp$ *;*

2) $f + \mathfrak{h}^\perp \subset \mathrm{Ad}^* \boldsymbol{G}.f = \Omega(f)$ *and $\mathfrak{h}$ has maximal dimension ;*

3) $\forall \varphi \in \mathfrak{h}^\perp$, $\mathfrak{h}$ is maximal isotropic for $f + \varphi$.

Proof — We only remark that the condition $\mathfrak{g}(f)_{\mathbb{C}} \subset \mathfrak{h}$ is not necessary because in this situation, $\mathfrak{e} = \mathfrak{d}^f = \mathfrak{d}$ so $\mathfrak{g}(f) \subset \mathfrak{d} \subset \mathfrak{h}$. ∎

Corollary 3.2 — *1) The polarization $P(f, \mathcal{S})$ of the theorem* **31** *satisfies the Pukanszky condition ;*

2) If $\boldsymbol{G}$ is a nilpotent connected Lie group, all the polarizations satisfy the Pukanszky condition.

Proof — 1) is clear. If $\boldsymbol{G}$ is a nilpotent the orbits of $\boldsymbol{D}^0$ in $f + \mathfrak{e}^\perp$ are closed (by Theorem **2.4**) and they are always open. ∎

3.4.– POSITIVE POLARIZATIONS.

In this paragraph we consider a real solvable Lie algebra $\mathfrak{g}$ and $\mathfrak{g}_{\mathbb{C}}$ its complexified Lie algebra. we will show that it is possible to choose the flag $\mathcal{S} = (\mathfrak{g}_i)_{i=1,\dots,n}$ in $\mathfrak{g}_{\mathbb{C}}$ such that $P(f, \mathcal{S})$ is a positive polarization at $f \in \mathfrak{g}^*$.

Recall that we denote by σ the conjugaison $X \longrightarrow \overline{X}$ in $\mathfrak{g}_{\mathbb{C}}$.

Definition — *We say that $\mathcal{S} = (\mathfrak{g}_i)_{i=1,\dots,n}$ is a "good" sequence of ideals of $\mathfrak{g}_{\mathbb{C}}$ if we have the following property:*

If $\mathfrak{g}_i$ is not σ-invariant, then $\mathfrak{g}_{i-1}$ and $\mathfrak{g}_{i+1}$ are σ-invariant.

Proposition 3.2 — *If $\mathfrak{g}$ is a solvable real Lie algebra, there exists "good" sequences in $\mathfrak{g}_{\mathbb{C}}$.*

Proof — We consider a Jordan-Holder sequence in $\mathfrak{g}$:

$$0 = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \ \cdots \ \subset \mathfrak{g}_n = \mathfrak{g}$$

with $\dim \mathfrak{g}_i/\mathfrak{g}_{i-1}$ =1 or 2 and the action of $\mathfrak{g}$ in $\mathfrak{g}_i/\mathfrak{g}_{i-1}$ is 0 or irreducible. It is enough to consider the sequence

$$0 = \mathfrak{g}_{0\mathbb{C}} \subset \mathfrak{g}_{1\mathbb{C}} \subset \ \cdots \ \subset \mathfrak{g}_{n\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}$$

and if $\dim \mathfrak{g}_{i\mathbb{C}}/\mathfrak{g}_{i-1\mathbb{C}} = 2$ we choose a complex $\tilde{\mathfrak{g}_i}$ such that $\tilde{\mathfrak{g}_i} \in \mathfrak{g}_{i\mathbb{C}}$ with $\dim \mathfrak{g}_{i\mathbb{C}}/\tilde{\mathfrak{g}_i} = 1$.

We recall that the hermitian form H_f is defined by $\boldsymbol{H}_f(X,Y) = if([X,\overline{Y}])$.

Lemma 3.4 — *Let $\mathcal{S} = (\mathfrak{g}_i)_{i=1\ldots n}$ be a good sequence in $\mathfrak{g}_{\mathbb{C}}$.*

1) If $\mathfrak{g}_i$ is an ideal which is non σ-invariant, then $[\mathfrak{g}_{i+1}\,,\,\mathfrak{g}_{i+1}] \subset \mathfrak{g}_{i-1}$;

2) If $i \neq j$ or if $\mathfrak{g}_i$ is σ-invariant, then $H_f\big(\mathfrak{g}_i(f_i), \mathfrak{g}_j(f_j)\big) = 0$.

Proof — 1) Since $\mathfrak{g}$ is solvable $[\mathfrak{g}_{i+1}, \mathfrak{g}_{i+1}]$ is contained in $\mathfrak{g}_i$. We remark that $\mathfrak{g}_{i-1} + [\mathfrak{g}_{i+1}, \mathfrak{g}_{i+1}]$ is a σ-invariant ideal contained in $\mathfrak{g}_i$, so it is $\mathfrak{g}_{i-1}$ and $[\mathfrak{g}_{i+1}, \mathfrak{g}_{i+1}] \subset \mathfrak{g}_{i-1}$.

2) We suppose $i \geq j$ so $\overline{\mathfrak{g}}_j \subset \mathfrak{g}_i$ for the two cases. If $X_i \in \mathfrak{g}_i(f_i),\ X_j \in \mathfrak{g}_j(f_j) \subset \mathfrak{g}_j \subset \mathfrak{g}_i$, we have $\overline{X}_j \subset \mathfrak{g}_i$ so $f_i([X_i, \overline{X}_j]) = 0$. ∎

Proposition 3.3 — *If $\mathcal{S}$ is a good sequence of ideals of $\mathfrak{g}_{\mathbb{C}}$ then $P(f,\mathcal{S})$ is a polarization at f.*

Proof — By the previous results, it is enough to prove that $P(f,\mathcal{S}) + \overline{P(f,\mathcal{S})}$ is a subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Let $i \geq j,\ X_i \in \mathfrak{g}_i(f_i),\ X_j \in \mathfrak{g}_j(f_j)$. It is enough to show that $[\overline{X}_i,\ X_j] \in P(f,\mathcal{S})$.

• If $i > j$ or if $\sigma(\mathfrak{g}_i) = \mathfrak{g}_i$, then $\overline{\mathfrak{g}}_j \subset \mathfrak{g}_i$. If $u \in \mathfrak{g}_j$

$$f([[\overline{X}_i, X_j], u]) + f([[X_j, u], \overline{X}_i]) + f([[u, \overline{X}_i], X_j]) = 0$$

But, $[X_j, u] \subset \mathfrak{g}_j \subset \mathfrak{g}_i$ and $\overline{[X_j, u]} \subset \mathfrak{g}_i$ so, $f([\overline{[X_j, u]}, X_i]) = 0$ because $X_i \in \mathfrak{g}_i(f_i)$. Furthermore, $[u, \overline{X}_i] \in \mathfrak{g}_j$ so, $f([u, \overline{X}_i], X_j]) = 0$. Thus, we have $[\overline{X}_i, X_j] \in \mathfrak{g}_j(f_j) \subset P(f,\mathcal{S})$.

• If $i = j$ and $\mathfrak{g}_i \neq \overline{\mathfrak{g}}_i$, we have $\overline{\mathfrak{g}}_{i-1} = \mathfrak{g}_{i-1}$ and $\overline{\mathfrak{g}}_{i+1} = \mathfrak{g}_{i+1},\ [\mathfrak{g}_{i+1}, \mathfrak{g}_{i+1}] \subset \mathfrak{g}_{i-1}$ by the lemma. So, $[\overline{X}_i, X_j] \in \mathfrak{g}_{i-1}$ and by the Jacobi identity like before, we see that $[\overline{X}_i, X_j] \subset \mathfrak{g}_{i-1}(f_{i-1}$. ∎

Proposition 3.4 — *There exists a good sequence of ideals of $\mathfrak{g}_{\mathbb{C}}$ such that $P(f,\mathcal{S})$ is a positive polarization at f.*

Proof — Let $\mathcal{S} = (\mathfrak{g}_i)_{i=1,\ldots,n}$ be a good sequence of ideals of $g_{\mathbb{C}}$, $f \in \mathfrak{g}^*$ and $P(f,\mathcal{S}) = \sum_{i=0}^{n} \mathfrak{g}_i(f_i)$. Let i_0 be the lowest index such that $P(f_{i_0}, \mathcal{S}_{i_0})$ is not positive. By the previous lemma, there exists an X_{i_0} in $\mathfrak{g}_{i_0}(f_{i_0})$ such that $H_f(X_{i_0}, X_{i_0}) < 0$ and $\mathfrak{g}_{i_0} \neq \overline{\mathfrak{g}}_{i_0}$. We replace the good sequence by the new good sequence where $\mathfrak{g}_{i_0}$ is replace by $\overline{\mathfrak{g}}_{i_0}$ and now if $\overline{X}_{i_0} \in \overline{\mathfrak{g}}_{i_0},\ H_f(\overline{X}_{i_0}, \overline{X}_{i_0}) = -H_f(X_{i_0}, X_{i_0})$ is positive. It is clear that $P(f_{i_0}, \mathcal{S}_{i_0}) = P(f_{i_0-1}, \mathcal{S}_{i_0-1}) + \mathbb{C}X_{i_0}$ is a positive polarization at f_{i_0}. We go on with the index $i_1 > i_0$ such that$P(f_{i_1}, \mathcal{S}_{i_1})$ is not positive and because n is finite, we see that it is possible to choose the good sequence such that $P(f,\mathcal{S})$ is positive. ∎

We remark that if $\mathfrak{a}$ is an ideal of $\mathfrak{g}$, we can choose the sequence $\mathcal{S}$ such that $\mathfrak{a}_{\mathbb{C}}$ is one of the $\mathfrak{g}_i$ (by choosing a good sequence of $\mathfrak{g}_{\mathbb{C}}/\mathfrak{a}_{\mathbb{C}}$ and a sequence of ideals of $\mathfrak{g}_{\mathbb{C}}$ included in $\mathfrak{a}$). So, $P(f,\mathcal{S})$ is admissible for $\mathfrak{a}$.

We have proved the following theorem.

Theorem 3.2 — *Let $\mathfrak{g}$ be a real solvable Lie algebra and $\mathfrak{a}$ an ideal of $\mathfrak{g}$, $f \in \mathfrak{g}^*, f' = f|_{\mathfrak{a}}$, $\boldsymbol{G}_f$ and $\boldsymbol{G}_{f'}$ the stabilizers of f and f' for the coadjoint representation of $\boldsymbol{G}$, the simply connected group with Lie algebra $\mathfrak{g}$. Then there exists a positive polarization $\mathfrak{h}$ at f which is admissible for $\mathfrak{a}$ and verify the Pukanszky condition. Furthermore, $\mathfrak{h}$ is invariant for the action of $\boldsymbol{G}_f$ and $\mathfrak{h} \cap \mathfrak{a}$ is invariant for the action of $\boldsymbol{G}_{f'}$.*

3.5.– REAL POLARIZATIONS.

If $\boldsymbol{G}$ is a nilpotent connected Lie group, its Lie algebra is of course completely solvable so, we may apply the results of 3.2 to $\mathfrak{g}$ with $\Bbbk = \mathbb{R}$. We obtain a subalgebra $\mathfrak{h}$ which is a maximal isotropic subspace of $\mathfrak{g}$: it is a real polarization at $f \in \mathfrak{g}^*$. The same is true if $\mathfrak{g}$ is completely solvable.

If $\boldsymbol{G}$ is solvable, real polarizations does not exist in general (Exercise 3.2), but real polarizations exist if $\boldsymbol{G}$ is an exponential group as we see below.

We say that $\mathcal{S} = (\mathfrak{g}_i)$ is a *good sequence of subalgebras* of $\mathfrak{g}$ if it verify the condition : if $\mathfrak{g}_i$ is not an ideal then $\mathfrak{g}_{i-1}$ and $\mathfrak{g}_{i+1}$ are ideals of $\mathfrak{g}$. If $\mathfrak{g}$ is solvable then there exists good sequences of ideals : to see this, get a (real) Jordan-Holder sequence of $\mathfrak{g}$. The quotients $\mathfrak{g}_{i+1}/\mathfrak{g}_i$ have the dimension 1 or 2. If $\dim \mathfrak{g}_{i+1}/\mathfrak{g}_i = 2$ we can take any subspace $\mathfrak{g}'$ of $\mathfrak{g}_i$ which contains $\mathfrak{g}_i$ and of codimension 1 in $\mathfrak{g}_{i+1}$. This subspace is clearly a subalgebra and the new sequence is a good sequence of ideals of $\mathfrak{g}$.

Now, we define a particular solvable Lie algebra : the *Diamond Lie algebra*, $\mathfrak{d}_4$. This algebra is a fondamental example. The non-zero brackets of $\mathfrak{d}_4$ are:

$$[e_1, e_2] = e_3 \qquad [e_1, e_3] = -e_2$$
$$[e_2, e_3] = e_4$$

Exercise 3.1 – Let $f = e_4^* \in \mathfrak{d}_4^*$. Show that there is no real subalgebra which is also a maximal isotropic subspace (cf. [3] Chap. IV).

Proposition 3.5 — *(Brezin-Dixmier criterion,* [3] *Chapter 4 p. 83) Let $\mathfrak{g}$ be a real solvable Lie algebra. If none of the subalgebra have a quotient isomorphic to $\mathfrak{d}_4$ then, for every $f \in \mathfrak{g}^*$ there exists a polarization at f and satisfying the Pukanszky condition.*

Corollary 3.3 — *If $\mathfrak{g}$ is an exponential Lie algebra and $f \in \mathfrak{g}^*$ then, there exists a real polarization $\mathfrak{h}$ at f and $\mathfrak{h}$ satisfy the Pukanszky condition.*

Exercise 3.2 – Consider the Diamond Lie algebra $\mathfrak{d}_4$ and $f = e_4^*$. We have $\mathfrak{g}(f) = \mathbb{R}e_4 \oplus \mathbb{R}e_1$. The dimension of polarizations at f is three.

There is no real subalgebra containing $\mathfrak{g}(f)$ and with dimension 3, but show that there are two and only two subalgebras of $\mathfrak{g}_{\mathbb{C}}$ which are polarizations at f:

$$\mathfrak{h} = \mathfrak{g}(f)_{\mathbb{C}} \oplus \mathbb{C}(e_2 + ie_3) \text{ and } \overline{\mathfrak{h}} = \mathfrak{g}(f)_{\mathbb{C}} \oplus \mathbb{C}(e_2 - ie_3)$$

Show that $f([e_2 + ie_3, e_2 - ie_3]) = -2i$ so, $\mathfrak{h}$ is positive and $\overline{\mathfrak{h}}$ is not positive.

If $\mathfrak{n} = \mathbb{R}e_2 \oplus \mathbb{R}e_3 \oplus \mathbb{R}e_4$, then $\mathfrak{n}$ is the Heisenberg Lie algebra and $\mathfrak{h} \cap \mathfrak{n}_{\mathbb{C}} = \mathbb{C} \oplus \mathbb{C}(e_2 + ie_3)$ is a polarization of $\mathfrak{n}_{\mathbb{C}}$ at $f|\mathfrak{n}$ which is not a real polarization. This proves that it is not always possible to find a polarization $\mathfrak{h}$ at f such that $\mathfrak{h} \cap \mathfrak{n}_{\mathbb{C}}$ is a real polarization at $f|_{\mathfrak{n}}$ for $\mathfrak{n}$ a nilpotent ideal of $\mathfrak{g}$.

Exercise 3.3 – There exists nilpotent Lie algebras such that polarizations are not all obtained by the M. Vergne construction. Let $\mathfrak{n} = \mathfrak{g}_5$ with the following nonzero brackets:

$$[e_1, e_2] = e_4,\ [e_1, e_4] = e_5,\ [e_2, e_3] = e_5$$

Let $f = e_5^*$. Show that $\mathfrak{n}(f) = \mathbb{R}e_5 = \mathfrak{z}$ and $\mathfrak{h} = \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_5$ cannot be obtained with a good sequence of $\mathfrak{n}$.

4. The dual space of a simply connected nilpotent Lie group

In this section we describe the irreducible unitary representations of a simply connected nilpotent Lie group. First we prove a classical useful lemma.

Lemma 4.1 — *(A.A. Kirillov) Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension greater than one and with a one dimensional center $\mathfrak{z}$. Then there exists linearly independant elements X, Y, Z of $\mathfrak{g}$ and a linear form λ on $\mathfrak{g}$ such that $\mathfrak{z} = \mathbb{R}Z$, $[X, Y] = Z$, $[T, Y] = \lambda(T)Z$, $\forall T \in \mathfrak{g}$. Furthermore, $\mathfrak{g}_0 = \ker\lambda$ is an ideal of $\mathfrak{g}$ containing Y and Z, $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathbb{R}X$, and $\mathfrak{a} = \mathbb{R}Y \oplus \mathbb{R}X$ is an abelian ideal central in $\mathfrak{g}_0$ such that $[\mathfrak{g}, \mathfrak{a}] = \mathfrak{z}$.*

Proof — We choose a non zero element $Z \in \mathfrak{z}$. The quotient $\mathfrak{g}/\mathfrak{z}$ is a nilpotent Lie algebra, so its center $\tilde{\mathfrak{z}}_1$ is non zero. The corresponding ideal $\mathfrak{z}_1$ in $\mathfrak{g}$ is such that $[\mathfrak{z}_1, \mathfrak{g}] \subset \mathfrak{z}$. We choose $Y \in \mathfrak{z}_1$ such that $Y \notin \mathfrak{z}$. Thus, for $T \in \mathfrak{g}$ we have $[T, Y] = \lambda(T)Z$ and λ is a non zero linear form. There is X such that $[X, Y] = Z$.

Let $T \in \mathfrak{g}$ and $U \in \ker\lambda$. By the Jacobi identity we have

$$[[T, U], Y] = -[[U, Y], T] - [[Y, T], U]$$

but $[U, Y] \in \mathfrak{z}$ and $[Y, T] \in \mathfrak{z}$ so, $\lambda([T, U]) = 0$. This shows that $\ker\lambda$ is an ideal in $\mathfrak{g}$. The relation $[T, Y] = \lambda(T)Z$ shows that $\mathfrak{a}$ is an ideal (abelian) and it is clear that $\mathfrak{g} = \ker\lambda \oplus \mathbb{R}X$. ∎

Theorem 4.1 — *Let $\boldsymbol{G}$ be a simply connected nilpotent Lie group and let ρ be an irreducible unitary representation of $\boldsymbol{G}$. Then there exists an analytic subgroup $\boldsymbol{H}$ of $\boldsymbol{G}$ and a character χ of $\boldsymbol{H}$ such that $\rho = \mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}}\chi$.*

Proof — We prove this result by induction on $\dim\boldsymbol{G}$. If $\dim\boldsymbol{G} = 1$ the result is clear. We suppose that the theorem is true for groups whose dimension is lower than $r > 1$.

Let $\boldsymbol{Z}$ be the center of $\boldsymbol{G}$ and $\mathfrak{z}$ its Lie algebra. For every $z \in \boldsymbol{Z}$ the operator $\rho(z) \in \mathrm{Hom}(\rho, \rho)$. By Schur lemma we have $\rho(z) = \omega(z)\,\mathrm{Id}$ where $\omega(z) \in \mathbb{C}$ and it is clear that ω is a unitary character of $\boldsymbol{Z}$. We consider the kernel $\boldsymbol{Z}'$ of ω in $\boldsymbol{Z}$. It is a closed subgroup of $\boldsymbol{Z}$ and its neutral component $\boldsymbol{Z}'_0$ is a simply connected subgroup of $\boldsymbol{G}$ (by corollary **2.2**) on which ρ is the identity operator.

Let p be the canonical map $p : \boldsymbol{G} \longrightarrow \boldsymbol{G}/\boldsymbol{Z}'_0$ There is a unique irreducible representation ρ' such that $\rho = \rho' \circ p$ (cf. **1.7** (e)). By the induction hypothesis, if $\dim\boldsymbol{Z}'_0 > 1$, we have a character χ' of an analytic subgroup $\boldsymbol{H}' \subset \boldsymbol{G}'$ such that $\rho' = \mathrm{Ind}_{\boldsymbol{H}'}^{\boldsymbol{G}'}\chi'$. Let $\chi = \chi' \circ p$ the character of the closed subgroup $p^{-1}(\boldsymbol{H}') = \boldsymbol{H}$. It is easy to see that

$$\rho = \rho' \circ p = \mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}}\chi' \circ p = \mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}}\chi$$

We are now concerned by the situation $\dim\boldsymbol{Z}'_0 = 0$ (or $\dim\boldsymbol{Z} = 1$). We can use the Kirillov's lemma. Let Z, X, Y be the three elements defined in this lemma. The subspace $\mathfrak{a} = \mathbb{R}Z \oplus \mathbb{R}Y$ is an abelian ideal of $\mathfrak{g}$. Let $\boldsymbol{A}$ be the (simply connected) analytic group whose Lie algebra is $\mathfrak{a}$. It is a vector group and the map $l \longrightarrow \chi_l$ defined by $\chi_l(\exp X) = e^{i<l,X>}$, $(\chi \in \boldsymbol{A})$ is an homomorphism from $\mathfrak{a}^*$ onto $\widehat{\boldsymbol{A}}$. For $x \in \boldsymbol{G}$ we have (remark that $l(Z) \neq 0$ otherwise $\boldsymbol{Z}_0' \neq 0$)

$$\begin{aligned} x.\chi_l(\exp X) &= \chi_l(x^{-1}\exp Xx) \\ &= \chi_l(\exp \mathrm{Ad}(x^{-1}).X) \\ &= e^{i<l,\mathrm{Ad}(x^{-1})X>} \\ &= e^{i<\mathrm{Ad}^*(x).l,X>} \\ &= \chi_{\mathrm{Ad}^*(x).l}(\exp X) \end{aligned}$$

so the action of $\boldsymbol{G}$ on $\widehat{\boldsymbol{A}}$ corresponds to the action of $\mathrm{Ad}^*(\boldsymbol{G})$ on $\mathfrak{a}^*$. We want to apply the Mackey's theorem so, we need to compute the $\boldsymbol{G}$-orbits in $\mathfrak{a}^*$ $(= \widehat{\boldsymbol{A}})$. According to the previous lemma we have $\mathfrak{g} = \mathbb{R}X \oplus \mathfrak{g}_0$ where $[\mathfrak{g}_0, \mathfrak{a}] = 0$ so, if $T \in \mathfrak{g}_0$, $\mathrm{Ad}^*(\exp T).l = l$ for every $l \in \mathfrak{a}^*$. This shows that the orbits of $\boldsymbol{G}$ in $\mathfrak{a}^*$ are only the orbits of the one parameter group $\exp(\mathbb{R}X)$ (note that $\{X\}$ is a Malcev basis for $\mathfrak{g}_0$).

Let $l = \alpha Y^* + \beta Z^* \in \mathfrak{a}^*$. We have to compute $\varphi = \mathrm{Ad}^*(\exp tX).l$ for $t \in \mathbb{R}$. Let $T \in \mathfrak{a}$.

$$\begin{aligned} \varphi(T) = \mathrm{Ad}^*(\exp tX).l(T) &= <l\,,\, \mathrm{Ad}\big(\exp(-tX).T\big)> \\ &= <l\,,\, \mathrm{Exp}\big(\mathrm{ad}(-tX).T\big)> \\ &= <l\,,\, T - t[X,T] + \frac{t^2}{2}[X,[X,T]] + \cdots> \end{aligned}$$

but $[X,T] \in \mathfrak{z}$ so $\varphi(T) = <l\,, T - t[X,T]>$ and we see that $\varphi = l - t\beta Y^*$. Thus, the orbit of αY^* is $\{\alpha Y^*\}$ and the orbit of $l \notin \mathbb{R}Y^*$ is $l + \mathbb{R}Y^*$. They are closed so, by Glimm's theorem, $\boldsymbol{A}$ is regularly embeded in $\boldsymbol{G}$. By the previous computation we also see that the stabilizer $\boldsymbol{G}_l$ of $l \in \widehat{\boldsymbol{A}}$ or $\chi_l \in \mathfrak{a}^*$ is $\boldsymbol{G}$ if $l \in \mathbb{R}Y^*$ and $\boldsymbol{G}_0 = \exp \mathfrak{g}_0$ otherwise. But we have seen that $l(Z) \neq 0$ so, by Mackey's theorem, there exists an irreducible representation ρ_0 of $\boldsymbol{G}_0$ such that $\rho = \mathrm{Ind}_{\boldsymbol{G}_0}^{\boldsymbol{G}} \rho_0$ and by the induction hypothesis $\rho_0 = \mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}_0} \chi_f$ for a connected subgroup $\boldsymbol{H}$ and an $f \in \mathfrak{g}_0^*$. Using induction by stage we see that $\rho = \mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}} \chi_f$. ∎

Proposition 4.1 — *Let ρ and ρ' be two irreducible representations of $\boldsymbol{G}$ which are not equal to identity on the center. Then $\rho \simeq \rho'$ if and only if they have the same restriction to the center of $\boldsymbol{G}$.*

Proof — We only sketch the proof which is almost the same that the previous one. If the two representations have the same restriction to the center, the crucial case is when $\dim \mathfrak{z} = 1$. Then, using the notations of the previous proposition, we remark that they are built by Mackey theory with two characters χ_l and $\chi_{l'}$ in the same $\boldsymbol{G}$-orbit in $\widehat{\boldsymbol{A}}$. So we may suppose they are identical. But $\boldsymbol{A}$ is the center of $\boldsymbol{G}_0$ and the two representations are induced from representations of $\boldsymbol{G}_0$ such that the restriction to $\boldsymbol{A}$ is χ_l. By induction hypothesis these representations of $\boldsymbol{G}_0$ are equivalent and by induction $\rho \simeq \rho'$. ∎

Proposition 4.2 — *Let $\mathfrak{h} \subset \mathfrak{g}$ be the Lie algebra of a subgroup $\boldsymbol{H}$ of $\boldsymbol{G}$ and χ a character of $\boldsymbol{H}$. There exists $f \in \mathfrak{h}^*$ such that for every $X \in \mathfrak{h}$ and $Y \in \mathfrak{h}$, $f([X,Y]) = 0$ and $\chi(\exp X) = e^{i<f,X>}$.*

Proof — The character χ is an homomorphism of Lie groups from $\boldsymbol{H}$ into $\mathbb{U} = \{z;\ z \in \mathbb{C},\ |z\bar{z}| = 1\}$ so, its differential $d\chi = if$ is a Lie algebra homomorphism from $\mathfrak{h}$ into the abelian Lie algebra $\mathbb{R}$. So, we have $f([X,Y]) = 0$ for $X, Y \in \mathfrak{h}$. The last expression of χ is clear because if is the differential of χ. ∎

Definition — *Let $f \in \mathfrak{g}^*$ and let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. Then we say that $\mathfrak{h}$ is isotropic at f if $f|_{[\mathfrak{h},\mathfrak{h}]} = 0$. Given $f \in \mathfrak{g}^*$ we denote by $\mathcal{S}(f)$ the set of subalgebras of $\mathfrak{g}$ which are isotropic at f.*

The previous theorem may be restated as follows.

Theorem 4.2 — *For each unitary irreducible representation ρ of $\boldsymbol{G}$, there exists $f \in \mathfrak{g}^*$ and $\mathfrak{h} \in \mathcal{S}(f)$ such that if $\boldsymbol{H}$ is the analytic subgroup of $\boldsymbol{G}$ with Lie algebra $\mathfrak{h}$ then $\rho = \mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}} \chi_f$ where χ_f is the character of $\boldsymbol{H}$ defined by f.*

For $f \in \mathfrak{g}^*$ we recall that we denote by $\boldsymbol{G}(f)$ or $\boldsymbol{G}_f$ the stabilizer of f with respect to the action of Ad^* in $\mathfrak{g}^*$ and by $\mathfrak{g}(f)$ or $\mathfrak{g}^f$ the kernel of the bilinear form B_f defined by $B_f(X,Y) = f([X,Y])$ for $X \in \mathfrak{g}$ and $Y \in \mathfrak{g}$.

Proposition 4.3 — *Let $\boldsymbol{G}$ be a simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$, and let $f \in \mathfrak{g}^*$. Then $\boldsymbol{G}_f$ is the analytic subgroup of $\boldsymbol{G}$ with Lie algebra $\mathfrak{g}^f$.*

Proof — By lemma **3.1** $\boldsymbol{G}_f$ is connected and it is closed. If $X, Y \in \mathfrak{g}^f$ then for $t \in \mathbb{R}$, $< f, \mathrm{Exp}(\mathrm{ad}\, tX).Y >= f(Y)$, so $\mathrm{Ad}^* \exp(-tX).f = f$ for all $t \in \mathbb{R}$. this shows that X belongs to the Lie algebra of $\boldsymbol{G}_f$. Conversely, by differentiating the relation $\mathrm{Ad}^* \exp(tX).f(Y) = f(Y)$ for $\exp(X) \in \boldsymbol{G}_f$, we see that $f([X,Y]) = 0$ so, $X \in \mathfrak{g}^f$. ∎

We have seen (cf. Theorem **2.4** and its proof) that every orbit $\Omega = \mathrm{Ad}^* \boldsymbol{G}.f$ is a closed manifold of $\mathfrak{g}^*$. It is isomorphic to the quotient space $\boldsymbol{G}/\boldsymbol{G}_f$ by the map $x \longrightarrow \mathrm{Ad}^*(x).f$. The tangent space at f to Ω is the image of $\mathfrak{g}$ under the differential $X \longrightarrow \mathrm{ad}^*(X).f$ whose kernel is $\mathfrak{g}^f$, so the dimension of Ω is equal to $\dim \mathfrak{g} - \dim \mathfrak{g}^f$. On the other hand, $\mathfrak{g}^f$ is the kernel of the bilinear form B_f on $\mathfrak{g}$ defined by $B_f(X,Y) = f([X,Y])$. We have seen that this form is skew symmetric and induces a non-degenerate form on $\mathfrak{g}/\mathfrak{g}^f$. Thus, the common dimension of $\mathfrak{g}/\mathfrak{g}^f$ and Ω is even.

Proposition 4.4 — *Let $\boldsymbol{G}$ be a simply connected nilpotent Lie group and $\mathfrak{g}$ its Lie algebra. Let $f \in \mathfrak{g}^*$. Then there exists a real polarization $\mathfrak{h}$ at f such that $\rho(f, \mathfrak{h}) = \mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}} \chi_f$ is irreducible.*

Proof — The proof is by induction on the dimension of $\boldsymbol{G}$. The result is clear if $\dim \boldsymbol{G} = 1$. Let $\mathfrak{z}$ be the center of $\mathfrak{g}$.

First we suppose that $\mathfrak{z}' = \ker(f|_{\mathfrak{z}})$ is non zero. If $\boldsymbol{Z}' = \exp(\mathfrak{z}')$ we consider the simply connected Lie group $\boldsymbol{G}' = \boldsymbol{G}/\boldsymbol{Z}'$ whose Lie algebra is $\mathfrak{g}' = \mathfrak{g}/\mathfrak{z}'$. Let f' be the linear form induced by f on $\mathfrak{g}'$. By the induction hypothesis, there exists a polarization $\mathfrak{h}'$ at f' in $\mathfrak{g}'$ such that $\mathrm{Ind}_{\boldsymbol{H}'}^{\boldsymbol{G}'} \chi_{f'}$ is irreducible. We denote by p the canonical projections from $\mathfrak{g}$ onto $\mathfrak{g}'$ and from $\boldsymbol{G}$ onto $\boldsymbol{G}'$. Let be $\mathfrak{h} = p^{-1}(\mathfrak{h}^{'})$. It is a maximal isotropic subalgebra for f thus it is a real polarization at f. We have $\rho(f, \mathfrak{h}) = \rho(f', \mathfrak{h}') \circ p$. Thus $\rho(f, \mathfrak{h})$ is irreducible.

We suppose now that $\ker(f|_{\mathfrak{z}}) = 0$. This means that $\dim \mathfrak{z} = 1$. We choose three elements X_0, Y_0, Z_0 and λ and $\mathfrak{g}_0$ as in lemma **4.1**. Let $\boldsymbol{G}_0$ be the analytic subgroup with Lie algebra $\mathfrak{g}_0$. Then $\mathfrak{a} = \mathbb{R}Y_0 \oplus \mathfrak{z}$ is an ideal which is central in $\mathfrak{g}_0$. We denote by f_0 the restriction of f to $\mathfrak{g}_0$ and we consider a polarization $\mathfrak{h}_0$ at f_0 in $\mathfrak{g}_0$. As $\mathfrak{a}$ is central in $\mathfrak{g}_0$ we have $\mathfrak{a} \subset \mathfrak{h}_0$ thus $\mathfrak{h}_0^f \subset \mathfrak{a}^f$. But if $U \in \mathfrak{g}_0$ and $t \in \mathbb{R}$, we have $f([U + tX_0\,,\, Y_0]) = t\, f(Z_0)$, which implies $\mathfrak{a}^f = \mathfrak{g}_0$. So, $\mathfrak{h}_0^f \subset \mathfrak{g}_0$ and we have

$$\mathfrak{h}_0 = \mathfrak{h}_0^{f_0} = \mathfrak{h}_0^f \cap \mathfrak{g}_0 = \mathfrak{h}_0^f$$

Thus $\mathfrak{h}_0$ is a real polarization at f in $\mathfrak{g}$. By the induction hypothesis we can take $\mathfrak{h}_0$ such that $\mathrm{Ind}_{\boldsymbol{H}_0}^{\boldsymbol{G}_0} \chi_{f_0}$ is irreducible as a representation of $\boldsymbol{G}_0$.

Let $\boldsymbol{A} = \exp \mathfrak{a}$. In the proof of the Theorem **4.1** we have seen that $\boldsymbol{A}$ is regularly embedded in $\boldsymbol{G}$ and that $\boldsymbol{G}_{\chi_{f_0}} = \boldsymbol{G}_0$ because χ_f is non zero on $\mathfrak{z}$. Furthermore, $\boldsymbol{A}$ is contained in $\exp(\mathfrak{h}_0) = \boldsymbol{H}_0$ and is central so we have for $a \in \boldsymbol{A}$, $x \in \exp \mathfrak{g}_0$ and φ a fonction in the space of $\rho(f_0, \mathfrak{h}_0)$

$$\rho(f_0, \mathfrak{h}_0)(a)\varphi(x) = \varphi(a^{-1}x) = \varphi(xa^{-1}) = \chi_{f_0}(a)\varphi(x)$$

This shows that $\rho(f_0, \mathfrak{h}_0)|_{\boldsymbol{A}}$ is a multiple of χ_{f_0}. We can apply Mackey's theorem which shows that $\mathrm{Ind}_{\boldsymbol{G}_0}^{\boldsymbol{G}} \rho(f_0, \mathfrak{h}_0)$ is irreducible but by the theorem of induction by stages this representation is equivalent to $\rho(f, \mathfrak{h})$. ∎

Theorem 4.3 — *(A.A. Kirillov) Let $\boldsymbol{G}$ be a simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$ and let $f \in \mathfrak{g}^*$.*

1) If $\mathfrak{h}$ is isotropic at f, the following conditions are equivalent :

a) $\mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}} \chi_f$ is irreducible ;

b) $\mathfrak{h}$ is a real polarization at f ;

2) For two real polarizations $\mathfrak{h}_1$ and $\mathfrak{h}_2$ at f, the representations $\rho(f, \mathfrak{h}_1)$ and $\rho(f, \mathfrak{h}_2)$ are equivalent.

Proof — We first prove a lemma.

Lemma 4.2 — *If $\mathfrak{h}$ is an isotropic subalgebra at f such that $\mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}} \chi_f$ is irreducible then $\mathfrak{h}$ contains the center of $\mathfrak{g}$.*

Proof — (Of the lemma) Let $\mathfrak{z}$ be the center of $\mathfrak{g}$. Let $\mathfrak{h}^{'} = \mathfrak{z} + \mathfrak{h}$. We have $\boldsymbol{H}^{'} = \boldsymbol{H}\boldsymbol{Z}$ and by induction by stages $\rho \simeq \mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}} \chi_f \simeq \mathrm{Ind}_{\boldsymbol{H}'}^{\boldsymbol{G}} \mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{H}'} \chi_f$.

If ρ is irreducible, then $\pi = \mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{H}'} \chi_f$ is also irreducible. But $\boldsymbol{H}' = \boldsymbol{H}\boldsymbol{Z}$ and for a function φ in the space of π, $h \in \boldsymbol{H}$, $k \in \boldsymbol{H}$ and $z \in \boldsymbol{Z}$ we have $\pi(h)\varphi(zk) = \varphi(h^{-1}zk) = \chi_f(h^{-1})^{-1}\chi_f(k)^{-1}\varphi(z) = \chi_f(h)\varphi(zk)$, because z is in the center of $\boldsymbol{G}$. So π is a multiple of a character and is irreducible. This implies that the space of π is one dimensional and since π is an induced representation, this is possible only if $\boldsymbol{H} = \boldsymbol{H}'$ and thus $\mathfrak{z} \subset \mathfrak{h}$. ∎

We start with the proof of the theorem which is by induction on the dimension of $\boldsymbol{G}$. If $\dim \boldsymbol{G} = 1$, the result is clear. We suppose $\dim \boldsymbol{G} > 1$ and the result true for groups with dimension lower than $\dim \boldsymbol{G}$. We denote by $\rho(f, \mathfrak{h})$ or $\rho(f, \mathfrak{h}, \mathfrak{g})$ the representation $\mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}} \chi_f$.

1) We suppose $\mathfrak{z}' = \ker f \cap \mathfrak{z} \neq 0$ and let $\boldsymbol{G}' = \boldsymbol{G}/\boldsymbol{Z}'$, $p : \boldsymbol{G} \longrightarrow \boldsymbol{G}'$ the canonical projection (and by the same notation the coresponding projection for Lie algebras), and $p(f)$ the linear form on $\mathfrak{g}'*$ such that $f' \circ p = f$. If $\mathfrak{h}$ is a (real) polarization at f then $p(\mathfrak{h})$ is a polarization at $p(f)$ and conversely, if $\mathfrak{h}'$ is a polarization at $p(f) \in \mathfrak{g}'^*$, $p^{-1}(\mathfrak{h}')$ is a polarization at f because $\mathfrak{z}' \subset \mathfrak{g}(f)$. Thus, if $\rho(f, \mathfrak{h}) = \rho(f', p(\mathfrak{h})) \circ p$ is irreducible, $\rho(f', p(\mathfrak{h}))$ is irreducible and by induction hypothesis, $p(\mathfrak{h})$ is a polarization at $p(f)$ so $\mathfrak{h}$ is a polarization at f. Conversely, if $p(\mathfrak{h})$ is a polarization at $p(f)$, $\rho(f', p(\mathfrak{h}))$ is irreducible and $\rho(f, \mathfrak{h}) = \rho(f', p(\mathfrak{h})) \circ p$ is also irreducible.

For the second assertion of the theorem, we remark that if $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are two polarizations at f, $p(\mathfrak{h}_1)$ and $p(\mathfrak{h}_2)$ are two polarizations at $p(f)$ so, $= \rho(f', p(\mathfrak{h}_1)) \simeq \rho(f', p(\mathfrak{h}_2))$ by induction hypothesis, and

$$\rho(f, \mathfrak{h}_1) = \rho(f', p(\mathfrak{h}_1)) \circ \ p = \rho(f', p(\mathfrak{h}_2)) \circ \ p = \rho(f, \mathfrak{h}_2)$$

2) We may now suppose that $\mathfrak{z}' = 0$ or $\dim \mathfrak{z} = 1$ and $f|\mathfrak{z} \neq 0$. We use the lemma 6 and get three elements X, Y, Z in $\mathfrak{g}$ such that $[X, Y] = Z$. Let $\mathfrak{a} = \mathfrak{z} \oplus \mathbb{R}Y$ and $\mathfrak{g}_0$ the centralizer of $\mathfrak{a}$. The polarization $\mathfrak{h}$ contains $\mathfrak{z}$.

2.a) We first consider the case $\mathfrak{h} \subset \mathfrak{g}_0$. Of course if $\mathfrak{h}$ is a polarization at f, $\mathfrak{h}$ is a polarization at $f_0 = f|\mathfrak{g}_0$, then $\rho(f_0, \mathfrak{h}, \mathfrak{g}_0)$ is irreducible and as in the proof of theorem 9 $\rho(f, \mathfrak{h}, \mathfrak{g}) = \mathrm{Ind}_{\boldsymbol{G}_0}^{\boldsymbol{G}} \rho(f_0, \mathfrak{h}, \mathfrak{g}_0)$ is irreducible by the Mackey theorem. Conversely, if $\rho(f, \mathfrak{h}, \mathfrak{g}) = \mathrm{Ind}_{\boldsymbol{G}_0}^{\boldsymbol{G}} \rho(f_0, \mathfrak{h}, \mathfrak{g}_0)$ is irreducible, $\mathrm{Ind}_{\boldsymbol{G}_0}^{\boldsymbol{G}} \rho(f_0, \mathfrak{h}, \mathfrak{g}_0)$ is irreducible and $\mathfrak{h}$ is a polarization at f_0. But $\mathfrak{g}^f \subset \mathfrak{g}_0$ so $\mathfrak{g}^f \subset \mathfrak{g}_0^{f_0}$ by lemma 3, and $\mathfrak{g}^f \subset \mathfrak{h}$ thus, $\mathfrak{h}$ is a polarization at f in $\mathfrak{g}_0$.

2.b) It remains the case $\mathfrak{h} \not\subset \mathfrak{g}_0$ and the second assertion. We study this case by showing that there exists $\mathfrak{h}' \subset \mathfrak{g}_0$ such that $\rho(f, \mathfrak{h}) = \rho(f, \mathfrak{h}')$.

We set $\mathfrak{a} = \mathfrak{z} \oplus \mathbb{R}Y$. If $\mathfrak{a} \subset \mathfrak{h}$ we have $\mathfrak{h} = \mathfrak{h}^f \subset \mathfrak{a}^f = \mathfrak{g}_0$ and since $f|_{\mathfrak{z}} \neq 0$ we may suppose $f|_{\mathfrak{z}}(Z) = 1$. and $\lambda(Z) = 0$ (by lemma **4.1**), we see that $\lambda|_{\mathfrak{h}}$ and $f|\mathfrak{h}$ are linearly independent so, we can choose $X \in \mathfrak{h}$, $\mathfrak{h} = \mathfrak{h}_0 + \mathbb{R}X$ and $f(X) = 0$. We consider $\mathfrak{h}' = \mathfrak{h}_0 \oplus \mathbb{R}Y$. Since $f([\mathfrak{h}', \mathfrak{h}']) = 0$ and $\dim \mathfrak{h} = \dim \mathfrak{h}'$, $\mathfrak{h}'$ is a real polarization at f.

Lemma 4.3 — *We have* $\rho(f,\mathfrak{h}) \simeq \rho(f,\mathfrak{h}')$

Proof — Let $\mathfrak{k} = \mathfrak{h} + \mathfrak{a} \subset \mathfrak{g}$ and $\boldsymbol{K} = \exp \mathfrak{k}$. We have

$$\rho(f,\mathfrak{h}) \simeq \mathrm{Ind}_{\boldsymbol{K}}^{\boldsymbol{G}} \rho(f|_{\mathfrak{k}},\mathfrak{h},\mathfrak{k}) \text{ and } \rho(f,\mathfrak{h}') \simeq \mathrm{Ind}_{\boldsymbol{K}}^{\boldsymbol{G}} \rho(f|_{\mathfrak{k}},\mathfrak{h}',\mathfrak{k})$$

so, it is enough to prove $\rho(f|_{\mathfrak{k}},\mathfrak{h},\mathfrak{k}) \simeq \rho(f|_{\mathfrak{k}},\mathfrak{h}',\mathfrak{k})$. To prove this, we first consider the subspace $\mathfrak{k}_1 = \ker(f) \cap \mathfrak{h}_0$ and we show that $\mathfrak{k}_1$ is an ideal in $\mathfrak{k}$. We have $\mathfrak{k} = \mathfrak{h}_0 \oplus \mathbb{R}Y \oplus \mathbb{R}X$ and $\mathfrak{h} = \mathfrak{h}_0 + \mathbb{R}X$ (recall that $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{g}_0$). Since $f|_{\mathfrak{z}}$ is not zero and $\mathfrak{z} \subset \mathfrak{h}_0$ we have $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathbb{R}Y \oplus \mathbb{R}X \oplus \mathbb{R}Z$. Now, if we remark that $\mathfrak{h}_0$ is an ideal of codimension 1 in $\mathfrak{h}$ and $\mathfrak{h}'$ we see that

$$[X,\mathfrak{k}_1] \subset [\mathfrak{h},\mathfrak{h}] \subset \ker(f) \cap \mathfrak{h}_0$$
$$[Y,\mathfrak{k}_1] \subset [\mathfrak{h}',\mathfrak{h}'] \subset \ker(f) \cap \mathfrak{h}_0$$

and this proves our assertion.

We denote by $\boldsymbol{K}_1$ the analytic subgroup of $\boldsymbol{K}$ with Lie algebra $\mathfrak{k}_1$, $\overline{\boldsymbol{K}} = \boldsymbol{K}/\boldsymbol{K}_1$ which is a simply connected group, $p : \boldsymbol{K} \longrightarrow \overline{\boldsymbol{K}}$ the canonical projection, $\overline{X} = p(X)$, $\overline{Y} = p(Y)$, and $\overline{Z} = p(Z)$. It is clear that $\overline{\boldsymbol{K}}$ is isomorphic to the Heisenberg group. We look at the linear form g on $\overline{\mathfrak{k}}$ deduced from f on $\overline{\mathfrak{g}}$: we have $g(\overline{\boldsymbol{Z}}) = 1$, $g(\overline{X}) = g(\overline{Y}) = 0$. Thus g is $\overline{Z}^*$ and $p(\mathfrak{h}) = \mathbb{R}\overline{X} \oplus \mathbb{R}\overline{Z}$, $p(\mathfrak{h}') = \mathbb{R}\overline{Y} \oplus \mathbb{R}\overline{Z}$, so we have

$$\rho(g,\mathfrak{h}) \simeq \rho\big(\overline{Z}^*, p(\mathfrak{h})\big) \circ p_1 \quad \text{and} \quad \rho(g,\mathfrak{h}') \simeq \rho\big(\overline{Z}^*, p(\mathfrak{h}')\big) \circ p_1$$

and it is enough to prove $\rho\big(Z^*, \mathbb{R}X \oplus \mathbb{R}Z\big) \simeq \rho\big(Z^*, \mathbb{R}Y \oplus \mathbb{R}Z\big)$ for the Heisenberg group.

Computations on the Heisenberg group

We take a basis $\{X,Y,Z\}$ with $[X,Y] = Z$ and the polarizations $\mathfrak{h}_1 = \mathbb{R}X \oplus \mathbb{R}Z$, $\mathfrak{h}_2 = \mathbb{R}Y \oplus \mathbb{R}Z$. We want to prove $\rho(Z^*,\mathfrak{h}_1) \simeq \rho(Z^*,\mathfrak{h}_2)$. For this we can look at the computation of the first section, where Z^* correspond to the character with $y = 0,\ z = 1$, $\rho_1 = \rho(Z^*,\mathfrak{h}_1)$ acts on the space $\mathbf{L}^2(\mathbb{R})$ and

$$\rho_1(a,0,0)\varphi(\alpha) = e^{i\alpha a}\varphi(\alpha)$$
$$\rho_1(0,b,0)\varphi(\alpha) = \varphi(\alpha - b)$$
$$\rho_1(0,0,c)\varphi(\alpha) = e^{ic}\varphi(\alpha)$$

For $\rho_2 = \rho(Z^*,\mathfrak{h}_2)$ we have

$$\rho_2(a,0,0)\varphi(\alpha) = \varphi(\alpha - a)$$
$$\rho_2(0,b,0)\varphi(\alpha) = e^{-i\alpha b}\varphi(\alpha)$$
$$\rho_2(a,0,0)\varphi(\alpha) = e^{ic}\varphi(\alpha)$$

and an obvious computation shows that the Fourier transform defined by

$$\mathcal{F}\varphi(y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi(x) e^{ixy} dx$$

is an unitary operator realizing the equivalence.

The previous theorem shows that $\rho(f, \mathfrak{h})$ is independant of the polarization $\mathfrak{h}$ at f so, we have defined a map $f \longrightarrow \rho(f)$ from $\mathfrak{g}^*$ into $\widehat{\boldsymbol{G}}$. In fact, this map is constant on each orbit of the coadjoint representation of $\boldsymbol{G}$ as we show now.

Proposition 4.5 — *If $x \in \boldsymbol{G}$, $f \in \mathfrak{g}^*$ and $x.f = \mathrm{Ad}^* x.f$, then we have $\rho(x.f) \simeq \rho(f)$ and this defines a map from the quotient space $\mathfrak{g}^*/\boldsymbol{G}$ of orbits of the coadjoint representation of $\boldsymbol{G}$ into $\widehat{\boldsymbol{G}}$.*

Proof — Let $x \in \boldsymbol{G}$, $f \in \mathfrak{g}^*$ and $\mathfrak{h}$ a polarization at f. We show that $x.\mathfrak{h} = \mathrm{Ad}\, x(\mathfrak{h})$ is a polarization at $x.f$. First, we have for $h, h' \in \mathfrak{h}$, using the fact that $\mathrm{Ad}(x)$ is a Lie algebra isomorphism and $\mathfrak{h}$ is a polarization at f

$$\begin{aligned} x.f([x.h, x.h']) &= f\big(\mathrm{Ad}(x^{-1}).[\mathrm{Ad}\, x.h, \mathrm{Ad}\, x.h']\big) \\ &= f([h, h']) = 0 \end{aligned}$$

because Furthermore, $\mathfrak{g}^{x.f} = x.\mathfrak{g}^f$ (exercise), so $\dim \mathfrak{g}^{x.f} = \dim(x.\mathfrak{g}^f) = \dim \mathfrak{g}^f$. We deduce immediately by dimension argument, that $x.\mathfrak{h}$ is totally isotropic at $x.f$, so it is a polarization at $x.f$.

We can now compute $\rho(x.f)$

$$\begin{aligned} \rho(x.f, x.\mathfrak{h}) &\simeq \mathrm{Ind}_{x^{-1}\boldsymbol{H}x}^{\boldsymbol{G}}(\chi_{x.f}) \\ &\simeq \mathrm{Ind}_{x^{-1}\boldsymbol{H}x}^{\boldsymbol{G}}(x.\chi_f) \\ &\simeq \mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}} \chi_f \quad \text{(corollary } \mathbf{1.1} \text{ section } \mathbf{1.9}) \\ &\simeq \rho(f, \mathfrak{h}) \end{aligned}$$

This achieves the proof. ∎

We can state the main theorem of this section due to A.A. Kirillov.

Theorem 4.4 — *If $\boldsymbol{G}$ is a connected simply connected nilpotent Lie group, the map $f \longrightarrow \rho(f)$ induces a bijective map from $\mathfrak{g}^*/\boldsymbol{G}$ onto the dual space $\widehat{\boldsymbol{G}}$ of $\boldsymbol{G}$.*

Proof — According to the previous proposition we have only to prove the surjectivity but, by theorem **4.2**, if $\pi \in \widehat{\boldsymbol{G}}$, $\pi \simeq \mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}} \chi_f$ where χ_f is a character of $\boldsymbol{H}$. Thus, we can apply the first result of theorem **4.3** to see that $\boldsymbol{H} = \exp \mathfrak{h}$ with $\mathfrak{h}$ a polarization at f, so $\pi \simeq \rho(f, \mathfrak{h})$.

4.1.– THE CASE OF EXPONENTIAL GROUPS

Almost all the results of this section extend to groups of exponential type. The Kirillov bijection is a result of P. Bernat (cf. [3] chap. IV). The main new fact is that not every real polarization at $f \in \mathfrak{g}^*$ gives an irreducible representation of $\boldsymbol{G}$ but for each $f \in \mathfrak{g}^*$ there exists a real polarization $\mathfrak{h}$ at f such that $\mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}} \chi_f$ is irreducible. The necessary and sufficient condition for $\mathfrak{h}$ to give an irreducible representation is the Pukanszky condition discovered by L. Pukanszky (of course !) in [19]. If this condition is not verified, $\mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}} \chi_f$ is a finite sum of irreducible representations, the decomposition in irreducible ones is given by the orbits which intersect $f + \mathfrak{h}^{\perp}$ as an open set in $f + \mathfrak{h}^{\perp}$, each orbit Ω with this property corresponds to a subrepresentation of $\mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}} \chi_f$ with a multiplicity equal to the number of connected components of $\Omega \cap f + \mathfrak{h}^{\perp}$. This is due to M. Vergne (cf. [3] Chap. VII).

Exercise 4.1 – Compute the dual space of the "$ax + b$" group using the Kirillov-Bernat mapping. Let $\{X, Y\}$ be a basis of the Lie algebra such that $[X, Y] = Y$. Show that $\mathfrak{h} = \mathbb{R}Y$ and $\mathfrak{k} = \mathbb{R}X$ are polarizations at Y^* but $\mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}} \chi_{Y^*}$ is irreducible and $\mathrm{Ind}_{\boldsymbol{K}}^{\boldsymbol{G}} \chi_{Y^*}$ is not irreducible. Compute the decomposition of $\mathrm{Ind}_{\boldsymbol{K}}^{\boldsymbol{G}} \chi_{Y^*}$ into irreducible representations.

5. Holomorphical induction, irreducible representations of solvable Lie groups.

5.1.– DEFINITIONS AND GENERALITIES

Let $\boldsymbol{G}$ be a locally compact solvable group. To define holomorphical induced representations we have to consider some subrepresentations of an induced representation σ of a closed subgroup $\boldsymbol{H}$ of $\boldsymbol{G}$.

We denote by $\mathcal{E}'(\boldsymbol{G})$ the space of compact supported measures on $\boldsymbol{G}$. Let $\nu \in \mathcal{E}'(\boldsymbol{G})$ and let f be a locally integrable function on $\boldsymbol{G}$ (for the Haar measure μ_G). Then the function $x \longrightarrow \int f(xy)d\nu(y)$ is almost everywhere defined and locally μ_G-integrable. We denote by $\rho(\nu)f$ this function. Similarly we write $\lambda(\nu)f(x) = \int f(y^{-1}x)d\nu(y)$.

Let Σ be a set of such measures ν and let U be a representation of a closed subgroup $\boldsymbol{H}$ of $\boldsymbol{G}$. We consider the representation π of $\boldsymbol{G}$ induced by U. Let $\mathcal{H}_\pi$ be its space. We denote by $\mathcal{H}_\pi^\Sigma$ the subspace of functions $\varphi \in \mathcal{H}_\pi$ such that $\rho(\nu)\varphi = 0$ for every $\nu \in \Sigma$.

It can be shown that this space is closed and $\boldsymbol{G}$-invariant so, it defines a subrepresentation of π. To see the $\boldsymbol{G}$-invariance of $\mathcal{H}_\pi^\Sigma$ we can use the fact that

$$\pi_x f(y) = f(x^{-1}y) = \delta_{x^{-1}} f(y) \qquad \text{(definition)}$$

$$\rho(\nu)(\pi_x f)(y) = \rho(\nu)(\delta_{x^{-1}})f(y) = (\delta_{x^{-1}})\rho(\nu)f(y).$$

We have used the fact that $\rho(\nu_1)\lambda(\nu_2)f = \lambda(\nu_2)\rho(\nu_1)f$ (Exercise).

The assertion that $\mathcal{H}_\pi^\Sigma$ is closed, is a consequence of the fact that the map $f \longrightarrow \int_{\boldsymbol{G}} h(x)\rho(\nu)f(x)d\mu_{\boldsymbol{G}}(x)$ is continuous (see Duflo [3] p. 110).

So Σ defines a subrepresentation of $\mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}} U$, but of course, if Σ is badly choosen this subrepresentation may be 0.

We now apply this construction to Lie groups and polarizations. Let $\boldsymbol{G}$ be a Lie group, $\mathfrak{g}$ its Lie algebra and $f \in \mathfrak{g}^*$. We choose a polarization $\mathfrak{h}$ at f. We take for Σ a set of distributions with compact supports $\big(\Sigma \subset \mathcal{D}'_C(\boldsymbol{G})\big)$. We denote by $\mathcal{D}_n(\boldsymbol{G})$ the space of measures $h\mu_{\boldsymbol{G}}$ with $h \in \mathcal{D}(\boldsymbol{G})$. For $\alpha \in \mathcal{D}_n(\boldsymbol{G})$ and $\nu \in \mathcal{D}'_C(\boldsymbol{G})$ we have $\alpha * u \in \mathcal{D}_n(\boldsymbol{G})$ so we can consider $\rho(\alpha * \nu)$ for $\alpha \in \mathcal{D}_n(\boldsymbol{G})$ and $\nu \in \mathcal{E}'(\boldsymbol{G})$. If $\Sigma \subset \mathcal{D}'_C(\boldsymbol{G})$ we define

$$\mathcal{H}_\pi^\Sigma = \{f \in \mathcal{H}_\pi \mid \rho(\alpha * \nu)f = 0 \qquad \forall \alpha \in \mathcal{D}_n(\boldsymbol{G})\ \forall \nu \in \Sigma\}$$

For instance, if f is $\mathcal{C}^\infty$ on $\boldsymbol{G}$ we can define $\rho(\nu)f$ by $\rho(\nu)f(y) = \int_{\boldsymbol{G}} f(xy)d\nu(y)$, so if $X \in \mathfrak{g}$ is the distribution

$$\frac{d}{dt} f(\exp tX)|_{t=0}$$

we have

$$\rho(X)f(x) = \frac{d}{dt}\big(f(x\exp tX)\big)|_{t=0}$$

and we define $\rho(Z)$ by linearity for $Z = X + iY$ with $X \in \mathfrak{g}$, $Y \in \mathfrak{g}$

$$\rho(X+iY)f(x) = [\rho(X) + i\rho(Y)]f(x).$$

Now, let $\mathfrak{h}$ be a polarization at $f \in \mathfrak{g}^*$, $\mathfrak{d} = \mathfrak{h} \cap \mathfrak{g}$, $\mathfrak{e} = (\mathfrak{h} + \overline{\mathfrak{h}}) \cap \mathfrak{g}$. Suppose that $\mathfrak{h}$ is $\boldsymbol{G}_f$-invariant. We denote by $\boldsymbol{D}^0$, $\boldsymbol{E}^0$, $\boldsymbol{G}_f^0$ the subgroups with Lie algebras $\mathfrak{d}$, $\mathfrak{e}$, $\mathfrak{g}^f$. Since $\mathfrak{h}$ is $\boldsymbol{G}_f$-invariant, $\boldsymbol{G}_f$ normalizes $\boldsymbol{D}^0$ and $\boldsymbol{E}^0$ so $\boldsymbol{D} = \boldsymbol{G}_f\boldsymbol{D}^0$ and $\boldsymbol{E} = \boldsymbol{G}_f\boldsymbol{E}^0$ are subgroups of $\boldsymbol{G}$. If $\mathfrak{h}$ is a "good polarization" like those defined in the previous section, it can be shown that $\boldsymbol{E}$ and $\boldsymbol{D}$ are closed subgroups of $\boldsymbol{G}$ but $\boldsymbol{G}_f$ is not connected for all solvable groups so $\boldsymbol{E}$ and $\boldsymbol{D}$ are not always connected. The groups $\boldsymbol{D}$ and $\boldsymbol{D}^0$ have the same Lie algebra because $\boldsymbol{G}_f^0 \subset \boldsymbol{D}^0$ ($\mathfrak{g}_{\mathbb{C}}^f \subset \mathfrak{h} \cap \overline{\mathfrak{h}}$). The same argument can be used for $\boldsymbol{E}$ and $\boldsymbol{E}^0$.

We have $f([\mathfrak{d},\mathfrak{d}]) = 0$ so there exists a character $\chi_f \in \widehat{\boldsymbol{D}}^0$ such that $d\chi_f = if$. But it is not always true that this character extends to a character of $\boldsymbol{D}$.

Definition — *We say that $f \in \mathfrak{g}^*$ is an integral form if there exists a character η_f of $\boldsymbol{G}_f$ such that its differential is $if|_{\mathfrak{g}(f)}$.*

Until the end of the section we suppose that f is an integral form. Since η_f and χ_f are equal on $\boldsymbol{G}_f^0$ we have a character $\widetilde{\chi_f}$ on $\boldsymbol{D} = \boldsymbol{G}_f\boldsymbol{D}^0$ which extends χ_f and η_f. We note $\chi_f = \widetilde{\chi_f}$.

We denote by $\mathcal{H}(f, \chi_f, \mathfrak{h}, \mathfrak{g})$ the completion of the space of functions φ on $\boldsymbol{G}$ which are $\mathcal{C}^\infty$ and such that

$$\begin{cases} 1)\ \varphi(xd) = \Delta_{\boldsymbol{D},\boldsymbol{G}}(d)^{\frac{1}{2}}\chi_f(d)^{-1}\varphi(x) \qquad x \in \boldsymbol{G}, \quad d \in \boldsymbol{D} \\ 2)\ \oint_{\boldsymbol{G}/\boldsymbol{D}} |\ \varphi\ |^2\, d\mu_{\boldsymbol{G},\boldsymbol{D}} < \infty \\ 3)\ \rho(Y)\varphi(x) = [-if(Y) + \dfrac{1}{2}\operatorname{tr}\operatorname{ad}_{\mathfrak{g}/\mathfrak{e}}(Y)]\varphi(x) \quad Y \in \mathfrak{h} \quad x \in \boldsymbol{G} \end{cases}$$

and $\pi = \operatorname{ind}(f, \mathfrak{h}_f, \mathfrak{h}, \boldsymbol{G})$ acts by left translations on $\mathcal{H}(f, \chi_f, \mathfrak{h}, \mathfrak{g})$.

Remarks : 1) $\Delta_{\boldsymbol{D},\boldsymbol{E}} = 1$ because there is an invariant measure on $\boldsymbol{E}/\boldsymbol{D}$; to show this let $\widetilde{B}_f$ the bilinear form on $\mathfrak{e}/\mathfrak{d}$ deduced from B_f on $\mathfrak{g}$; it is non degenerate and $\omega = \widetilde{B}_f \wedge \widetilde{B}_f \cdots \wedge \widetilde{B}_f$ defines a $\boldsymbol{D}$-invariant differential form of maximal degree on $\boldsymbol{E}/\boldsymbol{D}$ so $\Delta_{\boldsymbol{D},\boldsymbol{E}} = 1$.

2) If $Y \in \boldsymbol{D} = \mathfrak{h} \cap \mathfrak{g}$ the condition 3) follows from 1) because $\Delta_{\boldsymbol{D},\boldsymbol{E}} = 1$ on $\boldsymbol{D}$ so $\Delta_{\boldsymbol{D},\boldsymbol{G}} = \Delta_{\boldsymbol{E},\boldsymbol{G}}$.

3) If $d \in \boldsymbol{D}_0$ the condition 1) follows from 3).

If $\boldsymbol{G} = \boldsymbol{E}$, it can be shown that the space $\mathcal{H}(f, \eta_f, \mathfrak{h}, \boldsymbol{E})$ is exactly the space of $\mathcal{C}^\infty$ functions on $\boldsymbol{E}$ such that

$$\begin{cases} \varphi(xd) = \chi_f(d)^{-1}\varphi(x) & d \in \boldsymbol{D} \quad x \in \boldsymbol{E} \\ \int_{\boldsymbol{E}/\boldsymbol{D}} \mid \varphi(x) \mid^2 d\mu_{\boldsymbol{E},\boldsymbol{D}}(x) < \infty \\ \rho(Y)\varphi = -i\langle f, Y\rangle\varphi & Y \in \mathfrak{h} \end{cases}$$

We see that the set Σ of the definition is equal to the set of distributions defined by $Y \in \mathfrak{h}$. For holomorphic induction there is also a theorem of induction by stages.

Theorem 5.1 — *([3] prop 4.2.1 p. 112) Let $\boldsymbol{K} \subset \boldsymbol{H} \subset \boldsymbol{G}$ and V a representation in $\mathcal{H}$. Let Σ be a set of measures with support in $\boldsymbol{H}$. Let $\nu \in \Sigma$ and ν' be the element of $\mathcal{E}'(\boldsymbol{G})$ defined by*

$$h \longrightarrow \int_{\boldsymbol{H}} \Delta_{\boldsymbol{H},\boldsymbol{G}}(u)^{-1/2} h(u) d\nu(u)$$

where h is a function on $\boldsymbol{H}$ and let $\Sigma' = \{\nu' \ ; \ \nu \in \Sigma\}$, thus we have

$$\mathrm{ind}(V, \mathfrak{k}, \mathfrak{g}, \Sigma') \simeq \mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}}\big(\mathrm{ind}(V, \mathfrak{k}, \mathfrak{h}, \Sigma)\big)$$

We have the following corollary.

Corollary 5.1 — *Let $\mathfrak{h}$ be a polarization at $f \in \mathfrak{g}^*$, $\mathfrak{e}, \mathfrak{d}, \boldsymbol{E}, \boldsymbol{D} \ldots$ defined as previously, thus if χ_f is a character of $\boldsymbol{D}$ whose differential is $if|_{\mathfrak{d}}$ we have*

$$\mathrm{Ind}_{\boldsymbol{E}}^{\boldsymbol{G}}\big(\mathrm{ind}(f, \chi_f, \mathfrak{h}, \boldsymbol{E})\big) \simeq \mathrm{ind}(f, \chi_f, \mathfrak{h}, \boldsymbol{G})$$

We will show that the space of an holomorphical induced representation may be 0 even if $\boldsymbol{G}$ is a simply connected nilpotent group. The theory of orbits for solvable Lie groups is to show that if $f \in \mathfrak{g}^*$ is an integral form, it is possible to build an irreducible representation by holomorphical induction and if $\boldsymbol{G}$ is type I every $\pi \in \widehat{\boldsymbol{G}}$ is obtained by this method.

5.2.– In these notes I only show how it is possible to build irreducible representations of solvable Lie groups and I give in the next section a survey of the other results with examples for typical cases.

We consider a simply connected real solvable Lie group $\boldsymbol{G}$ with Lie algebra $\mathfrak{g}$. We choose an ideal $\mathfrak{n} \subset \mathfrak{g}$ which is nilpotent and contains $[\mathfrak{g}, \mathfrak{g}]$. We denote by $\boldsymbol{N}$ the Lie subgroup $\exp \mathfrak{n}$ of $\boldsymbol{G}$. We fix $f \in \mathfrak{g}^*$ and we suppose that f is an integral form. We have to prove the following theorem :

Theorem 5.2 — *If f is an integral form, and if $\mathfrak{h}$ is a positive polarization at f, admissible for $\mathfrak{n}$, $\boldsymbol{G}_f$-invariant and verifies the Pukanszky condition, then if χ_f is a character of $\boldsymbol{D} = \boldsymbol{G}_f \boldsymbol{D}^0$ whose differential is $if|_{\mathfrak{d}}$, the representation $\mathrm{ind}(f, \chi_f, \mathfrak{h}, \mathfrak{g})$ is irreducible and does not depend of $\mathfrak{h}$ verifying these conditions.*

The idea of the proof is to show that the holomorphical induced representation is equivalent to a standard induced representation obtained by the Mackey machine from an extension of an irreducible representation σ of $\boldsymbol{N}$. Of course σ is equivalent to the representation $\rho(f|_{\mathfrak{n}})$ by the Kirillov theory but unfortunately the realization of σ is not obtained from a real polarization. So an important step of the proof is to show that for nilpotent groups the representation built by holomorphical induction from complex polarization is equivalent to the representation of the Kirillov theory.

An other serious problem is to show that it is possible to extend a multiple of $\rho(f|_{\mathfrak{h}})$ to a representation of the stabilizer $\boldsymbol{G}_{f|\mathfrak{h}}.\boldsymbol{N}$ of $\rho(f|\mathfrak{h})$. After this step, the Mackey machine gives an irreducible representation of $\boldsymbol{G}$ and the last step is to show that this representation is equivalent to $\mathrm{ind}(f, \chi_f, \mathfrak{h}, \boldsymbol{G})$.

5.3.– THE CASE OF NILPOTENT LIE GROUPS

We consider a nilpotent connected and simply connected Lie group $\boldsymbol{N}$ and $f \in \mathfrak{n}^*$. We choose a positive polarization $\mathfrak{h}$ at f. Recall that this means $if([X,\overline{X}]) \geq 0$ for all $X \in \mathfrak{h}$.

We have to consider the space $\mathcal{H}(f,\boldsymbol{N})$ of functions φ on $\boldsymbol{N}$ such that

$$\begin{cases} \dfrac{d}{dt}\big(\varphi(x\exp tY)\big)|_{t=0} = -if(Y)\varphi(x) \qquad x \in \boldsymbol{N},\ Y \in \mathfrak{h} \\ \displaystyle\int_{\boldsymbol{N}/\boldsymbol{D}} \mid \varphi(x) \mid^2 d\dot{x} < \infty \end{cases}$$

where $d\dot{x}$ is an invariant measure on $\boldsymbol{N}/\boldsymbol{D}$.

This space is exactly the space of the holomorphical induced representation $\mathrm{ind}(f, \chi_f, \mathfrak{h}, \boldsymbol{N})$ because $\mathrm{tr}\,\mathrm{ad}_{\mathfrak{n}|\mathfrak{d}} X = 0$ for a nilpotent Lie algebra $\mathfrak{n}$ and the first condition is equivalent to the global condition $\varphi(xd) = \chi_f(d)^{-1}\varphi(x)$ for $d \in \boldsymbol{D} = \exp\mathfrak{d}$ because $\boldsymbol{G}_g$ is connected, so $\boldsymbol{D}$ is connected.

Theorem 5.3 — *Let $\mathfrak{h}$ be a positive polarization at $f \in \mathfrak{h}^*$. Then the representation ind$(f, \chi_f, \mathfrak{h}, \boldsymbol{N})$ is irreductible and equivalent to the representation $\rho(f)$ constructed by the Kirillov orbits method.*

Proof — Let $\mathfrak{d} = \mathfrak{h} \cap \mathfrak{g}$, $\mathfrak{e} = \big((\mathfrak{h} + \overline{\mathfrak{h}}) \cap \mathfrak{g}\big)$, then $\boldsymbol{N}_f$ is connected so $\boldsymbol{E} = \boldsymbol{E}^0 = \exp\mathfrak{e}$ and $\boldsymbol{D} = \exp\mathfrak{d}$. If $f' = f|_{\mathfrak{e}}$ then we have by stage induction theorem

$$\mathrm{ind}(f, \chi_f, \mathfrak{h}, \boldsymbol{N}) \simeq \mathrm{Ind}_{\boldsymbol{E}}^{\boldsymbol{N}}\big(\mathrm{ind}(f, \chi_{f'}, \mathfrak{h}, \boldsymbol{E})\big)$$

thus we just have to prove that there exists a real polarization $\mathfrak{h}_0$ at $f' \in \mathfrak{e}^*$ such that

$$\mathrm{ind}(f', \mathfrak{h}, \boldsymbol{E}) \simeq \mathrm{ind}(f', \mathfrak{h}_0, \boldsymbol{E})$$

because for a real polarization $\mathfrak{h}_0$ we have

$$\mathrm{ind}(f', \mathfrak{h}_0, \boldsymbol{E}) \simeq \mathrm{Ind}_{\boldsymbol{D}_0'}^{\boldsymbol{E}}\chi_f \simeq \rho(f)$$

where $\rho(f)$ is the Kirillov representation. ∎

Let $\mathfrak{b} = \mathfrak{d} \cap \ker f$.

Lemma 5.1 — *The group $\boldsymbol{B} = \exp \mathfrak{b}$ is normal in $\boldsymbol{E}$ and $\boldsymbol{E}/\boldsymbol{B}$ is an Heisenberg group with center $\boldsymbol{D}/\boldsymbol{B}$.*

PROOF — Since $\boldsymbol{E}$ is nilpotent it is enough to show that $\mathfrak{e}/\mathfrak{b}$ is an Heisenberg Lie algebra of center $\mathfrak{d}/\mathfrak{b}$. First we show that $\ker f \neq \mathfrak{d}$. To this end we show that f is non zero on $\mathfrak{g}^f \subset \mathfrak{d}$. Let $(\mathfrak{g}_i)_{i=1,\cdots,n}$ be a Jordan-Holder sequence of ideals and j the first index such that $f|_{\mathfrak{g}_j} = 0$ and $f|_{\mathfrak{g}_{j-1}} \neq 0$. We have $[\mathfrak{g}, \mathfrak{g}_{j-1}] \subset \mathfrak{g}_j$ thus $\mathfrak{g}_{j-1} \subset \mathfrak{g}^f$. We have $\mathfrak{d}/\mathfrak{b}$ which is a one dimensional subspace. We prove now that $\mathfrak{d}$ is an ideal of $\mathfrak{e}$. We need some tedious computations concerning B_f on $\mathfrak{e}/\mathfrak{d}$. We have $(\mathfrak{e}/\mathfrak{d})_{\mathbb{C}} = \mathfrak{h}/\mathfrak{d}_{\mathbb{C}} \oplus \overline{\mathfrak{h}}/\mathfrak{d}_{\mathbb{C}}$. We define J as the operator on $\mathfrak{e}/\mathfrak{d}$ whose eigenspace with eigenvalue $-i$ is $\mathfrak{h}/\mathfrak{d}_{\mathbb{C}}$ and eigenspace with eigenvalue $+i$ is $\overline{\mathfrak{h}}/\mathfrak{d}_{\mathbb{C}}$. We have $J^2 = -\mathrm{Id}$ and this defines a complexe structure on $\mathfrak{e}/\mathfrak{d}$. ∎

We denote by S the bilinear form defined by $S(x,y) = B(x, Jy)$.

Lemma 5.2 — *S is a symetric bilinear form on $\mathfrak{e}/\mathfrak{d}$ non degenerated, positive if $\mathfrak{h}$ is positive.*

PROOF — We verify that B is J-invariant. If h_1, h_2, h_1', h_2' are in $\mathfrak{h}$ we have

$$\begin{aligned} B(h_1 + \overline{h}_2, h_1' + \overline{h}_2') &= B(h_1, \overline{h}_2') + B(\overline{h}_2, h_1') \\ &= B(-ih_1, i\overline{h}_2') + B(i\overline{h}_2, -ih_1') \\ &= B(Jh_1, Jh_2') + B(J\overline{h}_2, Jh_1') \\ &= B\Big(J(h_1 + \overline{h}_2), J(h_1' + \overline{h}_2')\Big) \end{aligned}$$

so, $S(x,y) = B(x,Jy) = B(Jx, J^2y) = B(Jx, -y) = B(y, Jx) = S(y,x)$
Furthermore, if $x \in \mathfrak{h}$, $iJx = x$ so,

$$\begin{aligned} if([x,\overline{y}]) &= \frac{i}{4} f([x + iJx, y - iJy]) \\ &= \frac{i}{4}\big[f([x,y]) + f([iJx,y]) + f([x,-iJy]) + f([iJx,-iJy])\big] \\ &= \frac{i}{4}\big(B(x,y) + if([Jx,y]) - if([x,Jy]) + f([Jx,Jy])\big) \\ &= \frac{i}{2}B(x,y) + \frac{i}{4}\big(iB(J^2x,Jy) - if([x,Jy])\big) \\ &= \frac{i}{2}B(x,y) + \frac{1}{2}S(x,y) \end{aligned}$$

so, $2if([x,\overline{x}]) = S(x,x) + iB(x,x)$ and

$$2if([x,\overline{x}]) = S(x,x)$$

Thus $\mathfrak{h}$ is a positive polarization if and only if $S \geq 0$.

Now let X be an element of $\mathfrak{d}$ and $\operatorname{ad} X = \pi(X)$ the operator in $\mathfrak{e}/\mathfrak{d}$. Let Y and Z in $\mathfrak{e}$. By Jacobi identity we have

$$f\Big([[X,Y],Z]\Big) + f\Big([[Y,Z],X]\Big) + f\Big([[Z,X],Y]\Big) = 0$$

and by the equality $\mathfrak{d}^f = \mathfrak{e}$, $f\Big([[Y,Z],X]\Big) = 0$, this means $B(\pi(X)Y,Z) + B(Y,\pi(X)Z) = 0$. But the spaces $\mathfrak{h}/\mathfrak{d}_{\mathbb{C}}$ and $\overline{\mathfrak{h}}/\mathfrak{d}_{\mathbb{C}}$ are invariant by $\operatorname{ad} X$, so $\pi(X)$ commutes with J and

$$B(\pi(X)Y, JZ) = -B(Y, \pi(X)JZ) = -B(Y, J\pi(X)Z)$$

or

$$S(\pi(X)Y, Z) = -S(Y, \pi(X)Z).$$

This proves that $\pi(X)$ is anti-orthogonal for S, so it is semi-simple but it is nilpotent, so it is zero. This means that $[X,\mathfrak{e}] \subset \mathfrak{d}$ and $\mathfrak{d}$ is an ideal in $\mathfrak{e}$. On the other hand we have $[\mathfrak{e},\mathfrak{d}] \subset \ker f$ so $[\mathfrak{e},\mathfrak{d}] \subset \mathfrak{b}$ and $\mathfrak{b}$ is an ideal of $\mathfrak{e}$, $\mathfrak{d}/\mathfrak{b}$ is central in $\mathfrak{e}/\mathfrak{b}$. We must show that $\mathfrak{d}/\mathfrak{b}$ is exactly the center of $\mathfrak{e}/\mathfrak{b}$ but if $X \in \mathfrak{d}/\mathfrak{b}$ there exists $Y \in \mathfrak{e}/\mathfrak{b}$ such that $f([X,Y]) \neq 0$ (because $\mathfrak{d}^f = \mathfrak{e}$) so, $[X,Y] \neq 0$ in $\mathfrak{e}/\mathfrak{b}$ and X is not in the center $\mathfrak{e}/\mathfrak{b}$.

To show that $\mathfrak{e}/\mathfrak{b}$ is Heisenberg it remains to prove that $W = \mathfrak{e}/\mathfrak{d}$ is abelian. Let $\mathfrak{a}$ be the center of W : then

Lemma 5.3 — *If $J(\mathfrak{a}) = \mathfrak{a}$ then W is abelian.*

Proof — First $\mathfrak{a}$ is non zero because $\mathfrak{e}$ is nilpotent. Moreover, $\mathfrak{a}$ is J-invariant so $J|_{\mathfrak{a}}$ is non degenerated, B is non degenerated on $\mathfrak{a}$ and we have $W = \mathfrak{a} \oplus \mathfrak{a}^B$. The center of $\mathfrak{a}^B$ is contained in the center of W so $W = \mathfrak{a}$. ∎

Let u be an element of the center of W and $M = \operatorname{ad} Ju$. We have to prove that $M = 0$.

1) $[M,J] = 0$. Let $u \in \mathfrak{a}$ and $v \in W$. We have $u + iJu \in \mathfrak{h}/\mathfrak{d}_{\mathbb{C}}$ and $v + iJv \in \mathfrak{h}/\mathfrak{d}_{\mathbb{C}}$ so

$$J(u+iJu) = Ju - iu = -i(u+iJu)$$
$$k = [u+iJu, v+iJv] = i[Ju,v] - [Ju,Jv] \in \mathfrak{h}/\mathfrak{d}_{\mathbb{C}}$$

so $Jk = -ik$ implies $iJ[Ju,v] - J[Ju,Jv] = [Ju,v] + i[Ju,Jv]$. This shows that $[Ju,v] = -J[Ju,Jv]$ or $J[Ju,v] = [Ju,Jv]$. This is exactly $J.M(v) = M(J(v))$. ∎

2) We suppose $M \neq 0$. Then we take $v \in W$ such that $Mv \neq 0$ and $M^2v = 0$. By Jacoby identity we have easily (exercice)

$$B(x,[x,[y,z]]) = B(\operatorname{ad}^2(x).y, z) + 2B(\operatorname{ad} x.y, \operatorname{ad} x.z) + B(y, \operatorname{ad}^2(x).z)$$

and $B(u,[y,z]) = 0$ for $u \in \mathfrak{a}$.

Now for $x = Ju$, $y = v$, $z = Jv$,

$$\begin{aligned} B(Ju,[Ju,[v,Jv]]) &= \underbrace{B(M^2y,z)}_{=0} + 2B([Ju,v],[Ju,Jv]) + \underbrace{B(Jv,M^2z)}_{=0} \\ &= 2B(Mv,MJv) \\ &= 2B(Mv,JMv) \\ &= 2S(Mv,Mv) \end{aligned}$$

but

$$\begin{aligned} B\big(Ju,\big[Ju,[v,Jv]\big]\big) &= B\Big(u,J\big(M([v,Jv])\big)\Big) \\ &= B\big(u,MJ([v,Jv])\big) \\ &= B\big(u,\big[Ju,J[v,Jv]\big]\big) = 0 \quad (u \in \mathfrak{a}) \end{aligned}$$

This is a contradiction so $M = 0$. ∎

We come back to our main proof. We have an Heisenberg group $\boldsymbol{N}_k = \boldsymbol{E}/\boldsymbol{B}$ and a polarization $\mathfrak{h}$ which is positive such that $\mathfrak{h} + \overline{\mathfrak{h}} = \mathfrak{n}_{k\mathbb{C}}$, $\mathfrak{h} \cap \overline{\mathfrak{h}} = \mathfrak{z}_{\mathbb{C}} = (\mathfrak{d}/\mathfrak{b})_{\mathbb{C}}$ and $f \neq 0$ on $\mathfrak{h} \cap \overline{\mathfrak{h}} = \mathfrak{g}^f_{\mathbb{C}}$. In this situation it is known that there exist an intertwining operator between this representation and the representation $\rho(f)$ which is an equivalence ([3] Chap. VII) but, since the two representations has the same restriction to the center, it is enough to show that the previous is irreducible. this can be proved by using a description of the space by holomorphical functions on $\mathfrak{n}/\mathfrak{z}$.

For most simplicity we give the proof only for $\boldsymbol{N} = \boldsymbol{N}_3$. In the course of the proof it is shown that for an Heisenberg group the space $\mathcal{H}(f,\mathfrak{h},\boldsymbol{G})$ is non zero if and only if $\mathfrak{h}$ is a positive polarization.

We take $\mathfrak{h} \in \mathrm{Pol}(f)$ (not a real polarization) such that $\mathfrak{h} + \overline{\mathfrak{h}} = \mathfrak{n}_{\mathbb{C}}$, $\dim_{\mathbb{C}} \mathfrak{h} = 2$. We denote $\mathfrak{h} = W \oplus \mathfrak{z}_{\mathbb{C}}$ where W is a one dimensional subspace. We have $[X,Y] = Z$ and $f = Z^*$. We can choose $\varepsilon = aX + bY \in W$, $\quad a = a_1 + ia_2$, $b = b_1 + ib_2 \in \mathbb{C}$

$$\varepsilon = (a_1X + b_1Y) + i(a_2X + b_2Y) = x + iy$$

and $[x,y] = (a_1b_2 - a_2b_1)Z$.

If $[x,y] = 0$, x and y are colinear : $\varepsilon = k.x$ or $k'y$ so $x \in \mathfrak{h}$, $y \in \mathfrak{h}$ and $\mathfrak{h}$ is not a complex polarization (it is real) so we can choose a_i, b_i, $i \in \{1,\ 2\}$ such that $[x,y] = Z$, $\mathfrak{h} = \mathbb{C}(x+iy) \oplus \mathbb{C}Z$ $\ \overline{\mathfrak{h}} = \mathbb{C}(x-iy) \oplus \mathbb{C}Z$.

We have

$$\begin{aligned} if([x+iy,\overline{x+iy}]) &= if([x,-iy]) + if([iy,x]) \\ &= f([x,y]) - f([y,x]) = 2 > 0 \end{aligned}$$

and $if([x-iy,\overline{x-iy}]) = if([x,iy]) + if([-iy,x]) = -2$.

This shows that $\mathfrak{h}$ is a positive polarization and $\overline{\mathfrak{h}}$ is not positive.

We now define the space of the holomorphic induced representation. It is the space of functions φ on $\boldsymbol{N}_3$ such that

$$\begin{cases} \varphi(\exp u \exp tZ) = e^{-it}\varphi(\exp u) \qquad u \in \mathfrak{n}, \quad t \in \mathbb{R} \\ \dfrac{d}{dt}\big(\varphi(\exp u \exp tx)\big)|_{t=0} + i\dfrac{d}{dt}\big(\varphi(\exp u \exp ty)\big)|_{t=0} = -if(x+iy)\varphi(\exp u) = 0 \\ \displaystyle\int_{\boldsymbol{N}/\boldsymbol{Z}} \|\varphi\|^2 dxdy \leq \infty \end{cases}$$

Let $u = ax + by + cZ$. We have by Hausdorff formula

$$\varphi(\exp u \exp tx) = \varphi(\exp(u + tx - tbZ))$$
$$\varphi(\exp u \exp ty) = \varphi(\exp(u + ty + taZ))$$

so the second formula becomes (we consider functions on $\mathfrak{g}$ by $\varphi(a,b,c) \longrightarrow \varphi(\exp u)$)

$$\Big(\frac{\partial \varphi}{\partial a} - b\frac{\partial \varphi}{\partial c}\Big)(\exp u) + i\Big(\frac{\partial \varphi}{\partial b} + a\frac{\partial \varphi}{\partial c}\Big)(\exp u) = 0$$

but by the first formula of the definition of the space of the representation : $\dfrac{\partial}{\partial c}\varphi(\exp u) = -i\varphi(\exp u)$ so $\varphi(\exp u) = e^{-ic}\psi(a,b)$ where ψ is any function and

$$\frac{\partial \varphi}{\partial a} + i\frac{\partial \varphi}{\partial b} + (-b + ia)(-i\varphi) = 0$$

$$\frac{\partial \varphi}{\partial a} + i\frac{\partial \varphi}{\partial b} + (a + ib)\varphi = 0$$

Now, if we look at the function ψ we have

$$\frac{\partial \psi}{\partial a} + i\frac{\partial \psi}{\partial b} = -(a + ib)\psi$$

It is clear that the function $\psi_0(a,b) = e^{-\frac{a^2+b^2}{2}}$ is a solution for this equation so, by taking $\psi = \psi_0 v$ we get $\dfrac{\partial v}{\partial a} + i\dfrac{\partial v}{\partial b} = 0$. This means that v is a holomorphic function so, we see that $\mathcal{H}(f, \mathfrak{h}, \boldsymbol{N})$ is the space of functions of the form

$$\varphi(\exp u) = e^{-ic}e^{-\frac{a^2+b^2}{2}}v(a+ib)$$

where v is holomorphic and such that

$$\int_{\mathbb{R}^2} |v(a+ib)|^2 e^{-(a^2+b^2)} da\ db < \infty$$

(The Lebesgue measure on $\mathbb{R}^2$ corresponds to the invariant measure for the action of $\boldsymbol{N}$ in $\boldsymbol{N}/\boldsymbol{Z}$).

REMARK. If $\mathfrak{h}$ is replaced by $\overline{\mathfrak{h}}$ then we obtain $\psi_0(a,b) = e^{\frac{a^2+b^2}{2}}$ and we see that $\mathcal{H}(f,\mathfrak{h},\boldsymbol{N})$ is equal to 0 so, the condition $\mathfrak{h}$ positive is an essential condition.

It remains to show that the representation is irreducible. This can be proved by using same method than for the irreducibility of the ordinary representation and we leave this computation to the reader. It is also possible to give an intertwining unitary operator between this representation and the Kirillov's one. We only sketch the proof (*see* [3] Chapter VII). We see that all the monomials z^n are in $\mathcal{H}(f,\mathfrak{h},\boldsymbol{N})$. To build an operator the idea is to associate to z^n a good multiple of the Hermite function on $\mathbf{L}^2(\mathbb{R})$. Precisely the intertwining operator is defined by

$$\begin{aligned} T\psi(t) &= \frac{1}{\sqrt{2}} \exp\Big[-\frac{a^2+b^2}{2}\Big] \int_{\mathbb{R}} \exp\big(-\frac{1}{4}(\alpha+i\beta)^2\big)\psi(\alpha+i\beta)d\beta \\ &= \frac{1}{\sqrt{2}} \int_{\mathbb{R}} e^{-\frac{\alpha^2+\beta^2}{4}} e^{2i\alpha\beta}\psi(\alpha+i\beta)d\beta \end{aligned}$$

We now go back to representation of the simply-connected Lie group $\boldsymbol{G}$. We denote by $\mathfrak{n}$ a nilpotent ideal of $\mathfrak{g}$ which contains $[\mathfrak{g},\mathfrak{g}]$. We fix $l \in \mathfrak{g}^*$ and denote by f the restriction of l to $\mathfrak{n}$ and by $\rho(f)$ the irreducible representation of $\boldsymbol{N}$ obtained by Kirillov's theory.

Lemma 5.4 — *The stabilizer $\boldsymbol{G}_\rho$ of $\rho(f)$ in $\boldsymbol{G}$ is $\boldsymbol{N}\boldsymbol{G}_f$ where $\boldsymbol{G}_f$ is the stabilizer of $f \in \mathfrak{n}^*$ under the action of $\boldsymbol{G}$ in $\mathfrak{n}^*$ by the coadjoint representation. Moreover $\boldsymbol{G}_\rho = \boldsymbol{N}\boldsymbol{G}_f$ is a closed subgroup of $\boldsymbol{G}$.*

PROOF — Let $x \in \boldsymbol{G}$ and $\mathrm{Ad}^*(x)\rho(f) \simeq \rho(f)$. We have $\mathrm{Ad}^*(x)\rho(f) \simeq \rho(Ad^*x.f) \simeq \rho(f)$. So $\mathrm{Ad}^*x.f$ and f are in the same $\boldsymbol{N}$-orbit in $\mathfrak{n}^*$. There exists $n \in \boldsymbol{N}$ such that $\mathrm{Ad}^*(x).f = \mathrm{Ad}^*(n).f$ so $n^{-1}x \in \boldsymbol{G}_f$ and $x \in \boldsymbol{N}\boldsymbol{G}_f$. This proves that $\boldsymbol{N}\boldsymbol{G}_f$ is a subgroup of $\boldsymbol{G}$ and $\boldsymbol{G}_f$ normalizes $\boldsymbol{N}$. To show that $\boldsymbol{G}_f\boldsymbol{N}$ is closed it is enough to prove that $\boldsymbol{G}_f\boldsymbol{N}$ is the set of $x \in \boldsymbol{G}$ such that $\mathrm{Ad}^*(x).f \subset \mathrm{Ad}^*(\boldsymbol{N}).f = \Omega_f$. The easy proof is left to the reader. ∎

Now let $\mathfrak{h}$ be a (complex) polarization at $f \in \mathfrak{n}^*$ and $\rho(f,\mathfrak{h},\chi_f,\boldsymbol{N})$ the holomorphical induced representation of $\boldsymbol{N}$. This representation is irreducible and equivalent to $\rho(f)$. We denote by $T(\mathfrak{h})$ a unitary intertwining operator between $\rho(f)$ and $\rho(f,\mathfrak{h},\chi_f,\boldsymbol{N})$; $T(\mathfrak{h})$ is an operator from $\mathcal{H}(f)$, space of $\rho(f)$ onto $\mathcal{H}(f,\mathfrak{h})$ space of $\rho(f,\mathfrak{h},\chi_f,\boldsymbol{N})$. For $\mathfrak{h} \subset \mathfrak{g}_{\mathbb{C}}$ we define the subalgebras $\mathfrak{e},\mathfrak{d}$ and the subgroups $\boldsymbol{E}$, $\boldsymbol{D}$ of $\boldsymbol{N}$.

For φ in the space of $\rho(f,\mathfrak{h},\chi_f,\boldsymbol{N})$ we define for all $x \in \boldsymbol{G}_f$, $n \in \boldsymbol{N}$

$$\sigma(x,\mathfrak{h})\varphi(n) = |\det_{\mathfrak{n}/\mathfrak{d}} \mathrm{Ad}(x)|^{-\frac{1}{2}}\varphi(x^{-1}nx)$$

In the following we denote by $r(x)$ the number $|\det_{\mathfrak{n}/\mathfrak{d}} \mathrm{Ad}(x)|^{-\frac{1}{2}}$.

It is clear that $\sigma(x,\mathfrak{h})$ is an unitary operator in $\mathcal{H}(f,\mathfrak{h})$ and we verify easily that for $n \in \boldsymbol{N}$

$$\sigma(x,\mathfrak{h})\rho(f,\mathfrak{h},\chi_f,\boldsymbol{N})(n)\sigma(x,\mathfrak{h})^{-1} = \rho(f,\mathfrak{h},\chi_f,\boldsymbol{N})(xnx^{-1}).$$

We define $\nu(x,\mathfrak{h}) = T(\mathfrak{h})^{-1}\sigma(x,\mathfrak{h})T(\mathfrak{h})$. It is clear by the previous formula that $\nu(x,\mathfrak{h})$ is an intertwining operator between $\rho(f)$ and $\rho(f)^{(x^{-1})}$, and $\nu(x,\mathfrak{h})$ does not depend of $T(\mathfrak{h})$.

Theorem 5.4 — *The operator $\nu(x,\mathfrak{h})$ does not depend of the choice of $\mathfrak{h} \in Pol(f)$.*

PROOF — See [3] p. 197.

The proof is by induction on the dimension of $\mathfrak{n}$. After several reductions, the crucial case is when $\mathfrak{e} = (\mathfrak{h} + \overline{\mathfrak{h}}) \cap \mathfrak{n}$ is an Heisenberg algebra of dimension 3 or 5. In this situation it is possible to write an explicit intertwining operator and to compute $\nu(x,\mathfrak{h})$. ∎

We choose a polarization $\mathfrak{h}$ at f which is invariant by every $x \in \boldsymbol{G}_f$ (recall that this is possible by the results of the section 3). Then we have clearly $\nu(xy,\mathfrak{h}) = \nu(x,\mathfrak{h}).\nu(y,\mathfrak{h})$ and ν defines a representation of $\boldsymbol{G}_f$ in $\mathcal{H}(\rho)$. We have the following result.

Proposition 5.1 — *Let $\widehat{\boldsymbol{K}}$ be the semi direct product of $\boldsymbol{G}_f$ and $\boldsymbol{N}$, $\widehat{\boldsymbol{K}} = \boldsymbol{G}_f \times_s \boldsymbol{N}$. Then there exists a canonical representation $\widehat{\mu}$ of $\widehat{\boldsymbol{K}}$ whose restriction to $\boldsymbol{N}$ is $\rho(f)$ and $\widehat{\mu}(x,n) = \nu(x).\rho(f)(n)$.*

To build a representation of $\boldsymbol{G}_f\boldsymbol{N}$ we need to find a representation of $\widehat{\boldsymbol{K}}$ which is identity on $\boldsymbol{G}_f \cap \boldsymbol{N}$ but it is not true for $\widehat{\mu}$.
We now compute a representation of $\boldsymbol{G}_f$ which is equal to $\widehat{\mu}$ on $\boldsymbol{G}_f \cap \boldsymbol{N}$.

Lemma 5.5 — $\boldsymbol{G}_f \cap \boldsymbol{N} = \boldsymbol{N}_f$.

PROOF — We leave it to the reader. ∎

Let $\mathfrak{h}$ be a polarization at $l \in \mathfrak{g}^*$ having all the properties defined in the section 3. We use the following notations and relations (Exercise) :

$$\mathfrak{h}_1 = \mathfrak{h} \cap \mathfrak{n}_{\mathbb{C}} \ ; \quad \mathfrak{h}_2 = \mathfrak{h} \cap \mathfrak{g}^f_{\mathbb{C}}, \qquad \text{we have } \mathfrak{h} = \mathfrak{h}_1 + \mathfrak{h}_2$$

$$\mathfrak{e} = \mathfrak{e}_1 + \mathfrak{e}_2 \ ; \quad \mathfrak{d} = \mathfrak{d}_1 + \mathfrak{d}_2 \ ; \quad [\mathfrak{e}_2,\mathfrak{e}_1] \subset \mathfrak{e}_1 \ ; \quad [\mathfrak{d}_2,\mathfrak{d}_1] \subset \mathfrak{d}_1.$$

We denote by m the restriction of l to $\mathfrak{g}^f = \mathfrak{n}^l$.

Lemma 5.6 — *The stabilizer of m in $\boldsymbol{G}_f$ is $\boldsymbol{G}_l\boldsymbol{N}_f$.*

PROOF — Let $x \in \boldsymbol{G}_f$ such that $x.l|_{\mathfrak{g}^f} = l|_{\mathfrak{g}^f}$. Then $x.l - l \in (\mathfrak{g}^f)^\perp$ and since $x \in \boldsymbol{G}_f$, $x.l - l \in \mathfrak{n}^\perp$ so $(x.l - l) \in (\mathfrak{g}^f + \mathfrak{n})^\perp$.
Let $X \in \mathfrak{g}^f$ and $Y \in \mathfrak{g}$, then $[X,Y] \in \mathfrak{n}$ then,

$$\exp X.l(Y) = l(Y + [X,Y]) = (l - X.l)(Y)$$

and $\boldsymbol{N}_f.l = l + \mathfrak{n}^f.l$ but $\mathfrak{n}^f.l = \left((\mathfrak{n}^f)^l\right)^\perp$ (exercise) and $\left((\mathfrak{n}^f)^l\right)^\perp \supset (\mathfrak{g}^f + \mathfrak{n})$ so $\boldsymbol{N}_f.l \supset l + (\mathfrak{g}^f + \mathfrak{n})^\perp$. This shows that $x.l = n.l$ where $n \in \boldsymbol{N}_f$ and $n^{-1}x \in \boldsymbol{G}_l$. ∎

We now choose l an integral form on $\mathfrak{g}$.

Let $\mathfrak{q} = \ker f|_{\mathfrak{n}\cap\mathfrak{g}^f}$ and $\boldsymbol{Q} = \exp \mathfrak{q}$. The following technical result is proved in [3] p. 207 and left to the reader.

Proposition 5.2 — *Let* $\boldsymbol{M} = \boldsymbol{G}_f^0 \boldsymbol{G}_l$ *;*

a) $[\boldsymbol{G}_f, \boldsymbol{G}_f] \subset \boldsymbol{N}_f$ *;* $[\boldsymbol{G}_f, \boldsymbol{N}_f] \subset \boldsymbol{Q}$ *;* $[\boldsymbol{G}_f^0, \boldsymbol{M}_m] \subset \boldsymbol{Q}$. $\boldsymbol{G}_f/\boldsymbol{Q}$ *is a nilpotent (non connected) group and* $\boldsymbol{G}_f^0/\boldsymbol{Q}$ *is a simply connected nilpotent group with center* $\boldsymbol{N}_f/\boldsymbol{Q}$.

b) if $l \neq 0$ *then denote by* χ_m *the character of* $\boldsymbol{M}_m$ *with differential im. Then* $\boldsymbol{M}/\ker\chi_m$ *is a connected Heisenberg group with center* $\boldsymbol{M}_m/\ker\chi_m$.

We denote $\boldsymbol{D}_2 = \boldsymbol{D}_2^0\boldsymbol{G}_l$. $\boldsymbol{E}_2 = \boldsymbol{E}_2^0\boldsymbol{G}_l$. Since $\boldsymbol{D}_2 \subset \boldsymbol{D}$ the character χ_l of $\boldsymbol{G}_l$ extends to a character χ_m of $\boldsymbol{D}_2$ and we consider now the representation $\rho(m, \chi_m, \mathfrak{h}_2, \boldsymbol{G}_f)$ of $\boldsymbol{G}_f$ and show the irreducibility.

Using stage induction theorem we only have to prove that $\rho(m, \chi_m, \mathfrak{h}_2, \boldsymbol{M})$ is irreducible. By the previous proposition we can consider $\boldsymbol{M}$ nilpotent (non connected) but $\boldsymbol{M} = \boldsymbol{M}^0\boldsymbol{G}_l$ and the representation $\rho_0 = \rho(m, \chi_m|_{\boldsymbol{D}_2^0}, \mathfrak{h}_2, \boldsymbol{M}^0)$ is irreducible and independant of $\mathfrak{h}_2$ ($\boldsymbol{M}^0$ is Heisenberg) then we only have to prove that $\rho(m, \chi_m, \mathfrak{h}_2, \boldsymbol{M})|_{\boldsymbol{M}^0} = \rho_0$.

Lemma 5.7 — $\rho(m, \chi_m, \mathfrak{h}_2, \boldsymbol{M})|_{\boldsymbol{M}_0} \simeq \rho_0$.

Proof — For $\varphi \in \mathcal{H}(m, \chi_n, \mathfrak{h}_2, \boldsymbol{M})$ we define $T_\varphi = \varphi|_{\boldsymbol{M}^0}$. By the definition of the spaces of the representations it is almost evident that T is a unitary operator which intertwines the two representations. ∎

Now we have an irreducible representation of $\boldsymbol{G}_f$. We consider the representation $\widehat{\rho}(m, \chi_m)$ of $\widehat{\boldsymbol{K}} = \boldsymbol{G}_f \times_s \boldsymbol{N}$ such that $\widehat{\rho}(m, \chi_m)(x, n) = \rho(m, \chi_m, \mathfrak{h}_2, \boldsymbol{G}_f)(x)$ and we cosider the representation $\widehat{\xi}(l, \eta_l) = \widehat{\mu}(f) \otimes \widehat{\rho}(m, \chi_f)$.

Proposition 5.3 — *The representation* $\widehat{\xi}(l, \eta_l)$ *is irreducible and for* $a \in \boldsymbol{N}_f = \boldsymbol{G}_f \cap \boldsymbol{N}$, $\widehat{\xi}(l, \eta_l)(a, a^{-1}) = \mathrm{Id}$. *Thus* $\widehat{\xi}(l, \mathfrak{n}_l)$ *defines a representation* $\xi(l, \eta_l)$ *of* $\boldsymbol{G}_f\boldsymbol{N}$ *whose restriction to* $\boldsymbol{N}$ *is* $\rho(f)$.

Proof — The irreducibility of $\widehat{\mu}$ and $\widehat{\rho}$ implies easily that $\widehat{\xi}$ is irreducible.

It is clear that $\boldsymbol{N}_f = \boldsymbol{G}_f \cap \boldsymbol{N}$ and the kernel of the map $\widehat{\boldsymbol{K}} \xrightarrow{\theta} \boldsymbol{G}_f\boldsymbol{N}$ defined by $\theta(x, n) = xn$ is the set of (a, a^{-1}), $a \in \boldsymbol{N}_f$.

If $\varphi \in \mathcal{H}(f, \mathfrak{h}_1, \boldsymbol{N})$ (the space of $\widehat{\xi}(l, \eta_l)$) we compute easily that, using $|\det_{\mathfrak{n}/\mathfrak{d}_1} a| = 1$, $(a \in \boldsymbol{N})$

$$\widehat{\nu}(a, a^{-1})\varphi(n) = \eta_f(a)\varphi(n).$$

If $\varphi \in \mathcal{H}(m, \eta_m, \mathfrak{h}_2, \boldsymbol{G}_f)$

$$\rho\big(a^{-1}\varphi(x)\big) = \varphi(ax) = \varphi(x(x^{-1}ax)) = \eta_f(x^{-1}ax)\varphi(x)$$

(this is because $x^{-1}ax \in \boldsymbol{N}_f$, $[\boldsymbol{G}_f\boldsymbol{G}_f] \subset \boldsymbol{N}_f$) and now $x \in \boldsymbol{G}_f$ stabilizes f so $\eta_f(x^{-1}ax) = \eta_f(a)$ and the proposition is clear. ∎

By Mackey machine we have

Theorem 5.5 — *If l is an integral form on $\mathfrak{g}$ then* $\mathrm{Ind}_{\boldsymbol{G}_f\boldsymbol{N}}^{\boldsymbol{G}}\xi(l,\eta_l) \simeq \rho(l,\eta_l)$ *is an irreducible representation of* $\boldsymbol{G}$.

The last step is the following theorem

Theorem 5.6 — *We have* $\rho(l,\eta_l,\mathfrak{h},\boldsymbol{G}) \simeq \mathrm{Ind}_{\boldsymbol{G}_f\boldsymbol{N}}^{\boldsymbol{G}}(\xi(l,\eta_l))$.

The proof, long and tedious is, after some reduction, by computing an explicit intertwining operator between the two representations. (See the complete proof in [3] p. 212)

5.4.– THE MAIN RESULT OF AUSLANDER AND KOSTANT.

In this section we have built for every integral form l on $\mathfrak{g}$ an irreducible representation of $\boldsymbol{G}$. We remark that this representation depends of l but also of χ_l the character which extends $\chi_l|_{\boldsymbol{G}_l^0}$ ($\boldsymbol{G}_l^0$: the connected component of $\boldsymbol{G}_l$). We denote by $\mathcal{R}$ the set of (l,χ_l), $l \in \mathfrak{g}^*$, χ_l a character of $\boldsymbol{G}_l$ whose differential is $il|_{\mathfrak{g}(l)}$. The group $\boldsymbol{G}$ acts in $\mathcal{R}$ by $x.(l,\chi_l) = (\mathrm{Ad}^*\, x.l, \chi_l^x)$ because χ_l^x is a character of $\boldsymbol{G}_{\mathrm{Ad}^*\,x.l} = x.\boldsymbol{G}_l$.

It is not too difficult to see that if (l,χ_l) and $(l',\chi_{l'})$ are in the same $\boldsymbol{G}$-orbit of $\mathcal{R}$ then the representation $\rho(l,\chi_l)$ and $\rho(l',\chi_{l'})$ are equivalent.

We denote by $\mathcal{R}/\boldsymbol{G}$ the quotient space of $\mathcal{R}$ by $\mathrm{Ad}^*(\boldsymbol{G})$. A result of Auslander and Kostant is that the mapping from $\mathcal{R}/\boldsymbol{G}$ into $\widehat{\boldsymbol{G}}$ is bijective if and only if $\boldsymbol{G}$ is type I and we have the theorem .

Theorem 5.7 — *Let $\boldsymbol{G}$ be a simply connected solvable Lie group with Lie algebra $\mathfrak{g}$. Then $\boldsymbol{G}$ is type I if and only if*

1) The orbits of $\boldsymbol{G}$ in $\mathfrak{g}^*$ are locally closed;

2) All form $l \in \mathfrak{g}^*$ are integral.

The proof, which is difficult, is one of the most important of [2].

6. On the Plancherel formula and Kirillov character formula

The aim of this section is a generalization of the classical Fourier inversion formula on $\mathbb{R}^n$. This well known formula can be written as follows : if φ is a $\mathcal{C}^\infty$ function on $\mathbb{R}^n$ with compact support, then

$$\varphi(0) = \int_{\mathbb{R}^{n*}} \int_{\mathbb{R}^n} \varphi(x) e^{i<l,x>} \, dx \, dl \tag{$*$}$$

for a suitable choice of Lebesgue measures dx and dl on $\mathbb{R}^n$ and $\mathbb{R}^{n*}$.

For a non abelian Lie group $\boldsymbol{G}$ like a simply connected nilpotent group the character $x \longrightarrow e^{i<l,x>}$ may be replaced by an irreducible representation π, but the integral $\pi(\varphi) = \int \varphi(x)\pi(x)\, dx$ must be defined for $\pi \in \widehat{\boldsymbol{G}}$ and unfortunately it is not a number but an operator and to extend the formula $(*)$ we need to replace the value of the Fourier transform of φ at l by the trace of the operator $\pi(\varphi)$.

It will be shown in this section that the map $T_\pi : \varphi \longrightarrow \mathrm{tr}(\pi(\varphi))$ is a distribution on $\boldsymbol{G}$. It is the so called global character of π. By the exponential mapping, T_π becomes a distribution on $\mathfrak{g}$ and the famous Kirillov formula gives the value of the Fourier transform of $T_\pi \circ \exp$: it is exactly the canonical $\boldsymbol{G}$-invariant measure ν_π on the orbit Ω_π of π under the coadjoint representation of $\boldsymbol{G}$. The Plancherel formula is

$$\varphi(e) = \int_{\mathfrak{g}^*/\boldsymbol{G}} \int_{\Omega} (\varphi \circ \exp)\hat{}\,(l) d\nu_\Omega(l)\, d\mu(l) = \int_{\widehat{\boldsymbol{G}}} \mathrm{tr}(\pi(\varphi)) d\mu(\pi)$$

where μ is quotient measure of the Lebesgue measure on $\mathfrak{g}^*$ by the action of $\mathrm{Ad}^*(\boldsymbol{G})$.

In fact, in these notes I will give a more general Plancherel formula. I start from a character χ_f of a normal connected closed subgroup $\boldsymbol{H}$ of $\boldsymbol{G}$ (nilpotent simply connected) and a function $\varphi \in \mathcal{D}(\boldsymbol{G})$, then, there exists an (unbounded) operator U_π on the space $\mathcal{H}_\pi$ of $\pi \in \widehat{\boldsymbol{G}}$ such that $\pi(\varphi)U_\pi$ is a trace class operator and $\mathrm{tr}(\pi(\varphi)U_\pi)$ is the Fourier transform of a well-defined $\boldsymbol{H}^f$-invariant measure on $\Omega_\pi \cap (f + \mathfrak{h}^\perp)$ where $\mathfrak{h}^f = \{X \in \mathfrak{g},\ f([X, \mathfrak{h}]) = 0\}$ such that

$$\int_{\boldsymbol{H}} \varphi(h)\chi_f(h) dh = \int_{f+\mathfrak{h}^\perp/\boldsymbol{H}^f} \mathrm{tr}\big(\pi_\omega(\varphi)U_\omega\big)\, d\mu(\omega)$$

(ω is a $\boldsymbol{H}^f$-orbit in $f + \mathfrak{h}^\perp$) and μ is a measure on $(f + \mathfrak{h}^\perp/\boldsymbol{H}^f)$.
When $\boldsymbol{H} = \{e\}$ we have exactly the classical Plancherel formula and the character formula.

We organize the section as follows :

1) - Definition of $\pi(\varphi)$ for π a representation and $\varphi \in \mathcal{D}(\boldsymbol{G})$. Computation for an induced representation.

2) - The $\mathcal{C}^\infty$ vectors and the distribution vectors for $\pi \in \widehat{\boldsymbol{G}}$.

3) - The character formula.

4) - The Plancherel formula.

6.1.– Let $\boldsymbol{G}$ be a unimodular locally compact group, π a representation of $\boldsymbol{G}$ in $\mathcal{H}_\pi$ and $\varphi \in \mathbf{L}^1(\boldsymbol{G})$ for the Haar measure $\mu_{\boldsymbol{G}}$ on $\boldsymbol{G}$. We define $\pi(\varphi)$ by the formula

$$< \pi(\varphi)x, y >= \int_{\boldsymbol{G}} \varphi(u) < \pi(u)x, y > \, d\mu_{\boldsymbol{G}}(u) \qquad \forall x \in \mathcal{H}_\pi, \quad \forall y \in \mathcal{H}_\pi$$

and we have the notation

$$\pi(\varphi) = \int_{\boldsymbol{G}} \varphi(u)\pi(u)du$$

It is evident that $\|\pi(\varphi)\| \leq \|\varphi\|_1$ for $\varphi \in \mathbf{L}^1(\boldsymbol{G})$. Now, if φ, ψ are two functions on $\mathbf{L}^1(\boldsymbol{G})$ we see by using the Fubini theorem that for almost all $g \in \boldsymbol{G}$ the integral

$$\varphi * \psi(g) = \int_{\boldsymbol{G}} \varphi(h^{-1}g)\psi(h)d\mu_{\boldsymbol{G}}(h)$$

is convergent and $\varphi * \psi \in \mathbf{L}^1(\boldsymbol{G})$. Furthermore, $\|\varphi * \psi\|_1 \leq \|\varphi\|_1\|\psi\|_1$. This function $\varphi * \psi$ is the convolution product of φ and ψ. Moreover if $\varphi \in \mathbf{L}^1(\boldsymbol{G})$ we define the function φ^* by $\varphi^*(g) = \overline{\varphi(g^{-1})}$, $g \in \boldsymbol{G}$ and the map $\varphi \longrightarrow \varphi^*$ is an involution of $\mathbf{L}^1(\boldsymbol{G})$. The space $\mathbf{L}^1(\boldsymbol{G})$ endowed with convolution product and involution $*$ is an involutive Banach algebra.

If φ and ψ are in $\mathbf{L}^1(\boldsymbol{G})$ we easily prove (Exercise for the reader) that if π is a representation of $\boldsymbol{G}$ we have $\pi(\varphi * \psi) = \pi(\varphi)\pi(\psi)$ and $\pi(\varphi^*) = \pi(\varphi)^*$. This means that π becomes a representation of the involutive Banach algebra $\mathbf{L}^1(\boldsymbol{G})$.

6.2.– THE OPERATOR $\pi\big(\varphi\big)$ FOR AN INDUCED REPRESENTATION π.

We suppose that π is an unitary representation of $\boldsymbol{G}$ induced by a character χ_f, $f \in \mathfrak{g}^*$, of a closed connected subgroup $\boldsymbol{H}$ of $\boldsymbol{G}$. We have for $\pi = \mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}} \chi_f$:

Theorem 6.1 — *Let $\varphi \in \mathcal{D}(\boldsymbol{G})$, u and v two functions in $\mathcal{H}_\pi$ continuous with compact support modulus $\boldsymbol{H}$. Then,*

$$< \pi(\varphi)u, v >= \int_{\boldsymbol{G}/\boldsymbol{H}} \int_{\boldsymbol{G}/\boldsymbol{H}} K(x, y)u(y)\overline{v(x)} \, d\mu_{\boldsymbol{G},\boldsymbol{H}}(x) \, d\mu_{\boldsymbol{G},\boldsymbol{H}}(y)$$

where

$$K(x, y) = \int_{\boldsymbol{H}} \varphi(xhy^{-1})\chi_f(h)d\mu_{\boldsymbol{H}}(h).$$

In other words, $\pi(\varphi)$ is an operator defined by the continuous kernel $K(x, y)$.

Proof — Recall that we can choose Haar measures on $\boldsymbol{G}$ and $\boldsymbol{H}$ such that for $\varphi \in \mathcal{K}(\boldsymbol{G})$

$$\int_{\boldsymbol{G}} \varphi(x) d\mu_{\boldsymbol{G}}(x) = \int_{\boldsymbol{G}/\boldsymbol{H}} \int_{\boldsymbol{H}} \varphi(xh) d\mu_{\boldsymbol{H}}(h)\, d\mu_{\boldsymbol{G}/\boldsymbol{H}}(\dot{x}).$$

So we have for u and v continuous with compact support modulus $\boldsymbol{H}$,

$$\begin{aligned}
&< \pi(\varphi)u, v > = \\
&\quad = \int_{\boldsymbol{G}} \varphi(y) < \pi(y)u, v > d\mu_{\boldsymbol{G}}(y) \\
&\quad = \int_{\boldsymbol{G}} \varphi(y) \int_{\boldsymbol{G}/\boldsymbol{H}} u(y^{-1}x)\overline{v(x)} d\mu_{\boldsymbol{G},\boldsymbol{H}}(\dot{x}) d\mu_{\boldsymbol{G}}(y) \\
&\quad = \int_{\boldsymbol{G}/\boldsymbol{H}} \int_{\boldsymbol{G}} \varphi(y) u(y^{-1}x) d\mu_{\boldsymbol{G}}(y)\overline{v(x)} d\mu_{\boldsymbol{G},\boldsymbol{H}}(\dot{x}) \quad \text{(Fubini)} \\
&\quad = \int_{\boldsymbol{G}/\boldsymbol{H}} \int_{\boldsymbol{G}} \varphi(x, y) u(y^{-1}) d\mu_{\boldsymbol{G}}(y)\overline{v(x)} d\mu_{\boldsymbol{G},\boldsymbol{H}}(\dot{x}) \quad \text{(Invariance of } \mu) \\
&\quad = \int_{\boldsymbol{G}/\boldsymbol{H}} \int_{\boldsymbol{G}} \varphi(x, y^{-1}) u(y) d\mu_{\boldsymbol{G}}(y)\overline{v(x)} d\mu_{\boldsymbol{G},\boldsymbol{H}}(\dot{x}) \quad \text{(Invariance of } \mu) \\
&\quad = \int_{\boldsymbol{G}/\boldsymbol{H}} \int_{\boldsymbol{G}/\boldsymbol{H}} \int_{\boldsymbol{H}} \varphi\big(x(yh)^{-1}\big) u(yh) d\mu_{\boldsymbol{H}}(h) d\mu_{\boldsymbol{G},\boldsymbol{H}}(y)\overline{v(x)} d\mu_{\boldsymbol{G},\boldsymbol{H}}(\dot{x}) \\
&\quad = \int_{\boldsymbol{G}/\boldsymbol{H}} \int_{\boldsymbol{G}/\boldsymbol{H}} \int_{\boldsymbol{H}} \varphi\big(xhy^{-1}\big) u(yh^{-1}) d\mu_{\boldsymbol{H}}(h) d\mu_{\boldsymbol{G},\boldsymbol{H}}(\dot{y})\overline{v(x)} d\mu_{\boldsymbol{G},\boldsymbol{H}}(\dot{x}) \\
&\quad = \int_{\boldsymbol{G}/\boldsymbol{H}} \int_{\boldsymbol{G}/\boldsymbol{H}} \int_{\boldsymbol{H}} \varphi\big(xhy^{-1}\big) \chi_f(h) u(y) d\mu_{\boldsymbol{H}}(h) d\mu_{\boldsymbol{G},\boldsymbol{H}}(\dot{y})\overline{v(x)} d\mu_{\boldsymbol{G},\boldsymbol{H}}(\dot{x}) \\
&\quad = \int_{\boldsymbol{G}/\boldsymbol{H}} \int_{\boldsymbol{G}/\boldsymbol{H}} K(x, y) u(y)\overline{v(x)} d\mu_{\boldsymbol{G},\boldsymbol{H}}(\dot{y}) d\mu_{\boldsymbol{G},\boldsymbol{H}}(\dot{x}). \quad \blacksquare
\end{aligned}$$

Now let π be an irreducible representation obtained by the Kirillov method : we have $\boldsymbol{B} = \exp \mathfrak{h}$ where $\mathfrak{h}$ is a real polarization at f. Let $X_1, \ldots, X_k$ be a Malcev basis for $\boldsymbol{B}$. For $\varphi \in \mathcal{H}_\pi$ we denote by $\widetilde{\varphi}$ the function on $\mathbb{R}^k$ defined by

$$\widetilde{\varphi}(t_1, \ldots, t_k) = \varphi\big(\exp(t_1 X_1) \cdots \exp(t_k X_k)\big).$$

The covariance of the functions in $\mathcal{H}_\pi$ shows that the map $\varphi \longrightarrow \widetilde{\varphi}$ is bijective and the definition of invariant measures shows that it is in fact a norm preserving map between $\mathcal{H}_\pi$ and $\mathbf{L}^2(\mathbb{R}^k)$ where $2k$ is the dimension of the orbit of π.

6.3.– THE $\mathcal{C}^\infty$-VECTORS AND THE DISTRIBUTION VECTORS OF $\pi \in \widehat{\boldsymbol{G}}$.

Definition — *Let ρ be a representation of $\boldsymbol{G}$ in $\mathcal{H}_\rho$. We say that $v \in \mathcal{H}_\rho$ is a $\mathcal{C}^\infty$ vector if for any $\omega \in \mathcal{H}_\rho$, the coefficient $c_{v,w}$ defined by*

$$c_{v,w}(g) = < \rho(g)v, w >$$

is a $\mathcal{C}^\infty$ function on $\boldsymbol{G}$.

We denote by $\mathcal{H}_\rho^\infty$ the space of $\mathcal{C}^\infty$ vectors of ρ. It is clear that $\mathcal{H}_\rho^\infty$ is an invariant subspace of $\mathcal{H}_\rho$ but it is not closed for any ρ.

If $v \in \mathcal{H}_\rho^\infty$ the map $\omega \longrightarrow c_{v,w}$ is continuous from $\overline{\mathcal{H}_\rho}$ into $\mathcal{C}^\infty(\boldsymbol{G})$, thus, for $\lambda \in \mathcal{E}'(\boldsymbol{G})$ the set of compact supported distributions on $\boldsymbol{G}$, there exists a unique vector $\rho(\lambda)v$ such that $< \rho(\lambda)v, w >=< \lambda, c_{v,w} >$.

Lemma 6.1 — *For every $\lambda \in \mathcal{E}'(\boldsymbol{G})$, $\rho(\lambda)v \in \mathcal{H}_\rho^\infty$, $(v \in \mathcal{H}_\rho^\infty)$.*

PROOF —

$$
\begin{aligned}
< \rho(g)\rho(\lambda)v,\, w > &=< \rho(\lambda)v,\, \rho(g^{-1})w > \\
&= \int_{\boldsymbol{G}} < \rho(x)v,\, \rho(g^{-1})w > d\lambda(x) \\
&= \int_{\boldsymbol{G}} < \rho(gx)v,\, w > d\lambda(x) \\
&= c_{v,w} * \check{\lambda}(g)
\end{aligned}
$$

and $c_{v,w} * \check{\lambda}$ is a ∞ function if $\lambda \in \mathcal{E}'(\boldsymbol{G})$. ∎

Remarks. 1) If λ is a function in $\mathcal{D}(\boldsymbol{G})$, then this shows that $\rho(\lambda)v$ is a $\mathcal{C}^\infty$ vector ;

2) We easily check that if λ and μ are in $\mathcal{E}'(\boldsymbol{G})$ we have $\rho(\lambda * \mu) = \rho(\lambda)\rho(\mu)$, so, we have extended ρ to a representation of $\mathcal{E}'(\boldsymbol{G})$.

In the particular case of an irreducible representation π of a nilpotent Lie group $\boldsymbol{G}$ (simply connected), we have a useful structure theorem for $\mathcal{H}_\pi$ (cf. [5]).

Theorem 6.2 — *If $\pi \in \widehat{\boldsymbol{G}}$ is realized in the space $\mathbf{L}^2(\mathbb{R}^m)$ by using a Malcev basis, the space $\mathcal{H}_\pi^\infty$ is the Schwartz space $\mathcal{S}(\mathbb{R}^m)$ and if $\varphi \in \mathcal{D}(\boldsymbol{G})$, $\pi(\varphi)$ is a continuous operator from $\mathcal{S}'(\mathbb{R}^m)$ (the space of tempered distributions) into $\mathcal{S}(\mathbb{R}^m)$.*

We can now use the kernels theorem to state that $\pi(\varphi)$ is defined by a kernel $k_\pi \in \mathcal{S}(\mathbb{R}^m \times \mathbb{R}^m)$, but we have seen that this kernel is

$$
K(x,y) = \int_{\boldsymbol{H}} \varphi(xhy^{-1})d\mu_{\boldsymbol{H}}(h)
$$

Thus, in the realization of π in $\mathbf{L}^2(\mathbb{R}^m)$, this function is in $\mathcal{S}(\mathbb{R}^m \times \mathbb{R}^m)$. Furthermore, the kernels theorem also states that any continuous operator U from $\mathcal{S}(\mathbb{R}^m)$ into $\mathcal{S}'(\mathbb{R}^m)$ is defined by a kernel $k_U \in \mathcal{S}'(\mathbb{R}^m \times \mathbb{R}^m)$. So, for such an U, we have an other operator $\pi(\varphi) \circ U$ from $\mathcal{S}(\mathbb{R}^m)$ into $\mathcal{S}(\mathbb{R}^m)$.

The following proposition is the key of the Kirillov character formula.

Proposition 6.1 — *Let T be a nuclear operator from $\mathcal{S}'(\mathbb{R}^m)$ into $\mathcal{S}(\mathbb{R}^m)$ with kernel $K_T \in \mathcal{S}(\mathbb{R}^m \times \mathbb{R}^m)$ and S an operator from $\mathcal{S}(\mathbb{R}^m)$ into $\mathcal{S}'(\mathbb{R}^m)$ with kernel*

$K_S \in \mathcal{S}'(\mathbb{R}^m \times \mathbb{R}^m)$. Then, $T \circ S$ is a nuclear operator from $\mathcal{S}(\mathbb{R}^m)$ into $\mathcal{S}(\mathbb{R}^m)$ and the trace of $T \circ S$ is

$$\mathrm{tr}(T \circ S) =< K_T(x,y), \overline{K_S(y,x)} >$$

Proof — It is clear that $T \circ S$ is a continuous operator from $\mathcal{S}(\mathbb{R}^m)$ into $\mathcal{S}(\mathbb{R}^m)$ so it is nuclear. We have $K_T \in \mathcal{S}(\mathbb{R}^m \times \mathbb{R}^m) \simeq \mathcal{S}(\mathbb{R}^m)\widehat{\otimes}\mathcal{S}(\mathbb{R}^m)$ so there exists families of functions $\varphi_i \in \mathcal{S}(\mathbb{R}^m)$ and $\psi_i \in \mathcal{S}(\mathbb{R}^m)$ such that

$$K_T(x,y) = \sum_i t_i \varphi_i(x)\overline{\psi_i(y)}$$

where the series is convergent in $\mathcal{S}(\mathbb{R}^m \times \mathbb{R}^m)$.

We denote by $\varphi \otimes \overline{\psi}$ the rank one operator defined by $\varphi \otimes \overline{\psi}(v) = (v, \psi)\varphi$. It is a trace class operator and $\mathrm{tr}(\varphi \otimes \overline{\psi}) =< \varphi, \psi >$.

Now, if v and w are two functions in $\mathcal{S}(\mathbb{R}^m)$ we have

$$\begin{aligned} < \varphi_i \otimes \overline{\psi_i}.v, w > &=<< v, \psi_i > \varphi_i, w > \\ &=< v, \psi_i >< \phi_i, w > \\ &= \iint v(y)\overline{w(x)\psi_i(y)}\varphi_i(x)\, dydx \end{aligned}$$

This shows that $\varphi_i \otimes \overline{\psi_i}$ is defined by the kernel $\varphi_i(x)\overline{\psi_i(y)}$ and clearly

$$< Tv, w >= \sum_{i=0}^{\infty} t_i < \varphi_i \otimes \overline{\psi_i} v, w >$$

so,

$$\begin{aligned} T \circ S(v) &= \sum_{i=0}^{\infty} t_i \varphi_i \otimes \overline{\psi_i}(Sv) \\ &= \sum_{i=0}^{\infty} t_i < Sv, \psi_i > \varphi_i \\ &= \sum_{i=0}^{\infty} t_i < v, S^*\psi_i > \varphi_i \\ &= \sum_{i=0}^{\infty} t_i \varphi_i \otimes \overline{S^*\psi_i}(v) \end{aligned}$$

an we can compute the trace of $T \circ S$

$$\begin{aligned} \mathrm{tr}(T \circ S) &= \sum_{i=0}^{\infty} t_i \,\mathrm{tr}(\varphi_i \otimes \overline{S^*\psi_i}) \\ &= \sum_{i=0}^{\infty} t_i < \varphi_i, S^*\psi_i > \end{aligned}$$

but S^* is defined by the kernel $K_S^*(x,y) = \overline{K_S(y,x)}$ so

$$\begin{aligned}\operatorname{tr}(T\circ S) &= \sum_i t_i < \varphi_i(x)\overline{\psi_i(y)},\, \overline{K_S(y,x)} > \\ &= \Big\langle \sum_i t_i\varphi_i(x)\overline{\psi_i(y)},\, \overline{K_S(y,x)} \Big\rangle \\ &= < K_T(x,y)\,,\, \overline{K_S(y,x)} > \end{aligned}$$

and the proof is complete. ∎

We want to apply this result to a special operator $U \in \mathcal{L}\big(\mathcal{S}(\mathbb{R}^m), \mathcal{S}'(\mathbb{R}^m)\big)$. We recall that $\mathfrak{h}$ is such that $f([\mathfrak{h},\mathfrak{h}]) = 0$ and $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. For $g \in \mathfrak{g}$ we define $\mathfrak{h}^g = \{X \in \mathfrak{g}\ , g([X,\mathfrak{h}]) = 0\}$.

Lemma 6.2 — *1) For all $g \in f + \mathfrak{h}^\perp$, $\mathfrak{h}^f = \mathfrak{h}^g$;*

2) for all $g \in f + \mathfrak{h}^\perp$, there exists a real polarization $\mathfrak{b}_g$ at g such that $\mathfrak{h} \subset \mathfrak{b}_g \subset \mathfrak{h}^f$;

3) for all $g \in f + \mathfrak{h}^\perp$, $\mathrm{Ad}^*(\boldsymbol{G}).g \cap (f + \mathfrak{h}^\perp) = \mathrm{Ad}^*(\boldsymbol{H}^f).g$, where $\boldsymbol{H}^f = \exp \mathfrak{h}^f$.

Proof — 1) If $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$ then for $g \in f + \mathfrak{h}^\perp$, $g([X,Y]) = f([X,Y])$;

2) Let $0 = \mathfrak{a}_0 \subset \cdots \subset \mathfrak{a}_n = \mathfrak{g}$ be a Jordan-Holder sequence of $\mathfrak{g}$ such that $\mathfrak{h}$ is one of the $\mathfrak{a}_i$. Then if

$$\mathfrak{a}_i(g_i) = \{X \in \mathfrak{a}_i,\ g|_{\mathfrak{a}_i}([X,\mathfrak{a}_i]) = 0\}$$

the polarization $\mathfrak{b}_g = \sum_{i=1}^n \mathfrak{a}_i(g_i)$ at g is an answer to the problem ;

3) It is clear that $\mathrm{Ad}^*(\boldsymbol{H}^g).g \subset \Omega_g \cap (f + \mathfrak{h}^\perp)$ where Ω_g is the orbit of $g \in \mathfrak{g}^*$. Conversely, if $h \in \Omega_g \cap (f + \mathfrak{h}^\perp)$, there exists $x \in \boldsymbol{G}$ such that $h = \mathrm{Ad}^*\, x.g$. But $f|_\mathfrak{h} = h|_\mathfrak{h} = (\mathrm{Ad}^*\, x.g)|_\mathfrak{h} = \mathrm{Ad}^*(x).(g|_\mathfrak{h}) = \mathrm{Ad}^*(x).(f|_\mathfrak{h})$. So, $x \in \boldsymbol{H}^f = \boldsymbol{H}^g$. ∎

We denote by $\boldsymbol{B}_g$ the subgroup $\exp \mathfrak{b}_g$.

Proposition 6.2 — *Let φ and ψ two C^∞ vectors for π_g, the Kirillov representation defined by $g \in f + \mathfrak{h}^\perp$. Then, $\varphi\overline{\psi}$ is $\boldsymbol{B}_g$-invariant and the formula*

$$< U_g\varphi, \psi >= \int_{\boldsymbol{H}^g/\boldsymbol{B}_g} \varphi(x)\overline{\psi(x)}d\dot{x}$$

where dx is the $\boldsymbol{H}^g$-invariant measure on $\boldsymbol{H}^g/\boldsymbol{B}_g$, defines in the realization of π_g in $\mathbf{L}^2(\mathbb{R}^m)$ a nuclear operator U_g from $\mathcal{S}(\mathbb{R}^m)$ into $\mathcal{S}'(\mathbb{R}^m)$.

Proof — We choose a Malcev basis of $\mathfrak{b}_g$ such that the vectors $X_1, \dots X_k$ are in $\mathfrak{h}^f$ and the others form a malcev basis of $\mathfrak{h}^f$ in $\mathfrak{g}$. The operator U_g becomes $\tilde{U}_g$

$$< \tilde{U}_{\pi_g}\tilde{\varphi}, \tilde{\psi} >= \int_{\mathbb{R}^k} \tilde{\varphi}(t_1, \dots, t_k, 0, \dots, 0)\overline{\tilde{\psi}(t_1, \dots, t_k, 0, \dots, 0)}\, dt_1 \cdots dt_k$$

and the result is clear. ∎

Now we have to choose carefully a basis of $\mathfrak{g}$ (which depends on $g \in f + \mathfrak{h}^\perp$). We denote by B_g the bilinear form $B_g(X,Y) = g([X,Y])$. We deduce a nondegenarate form on $\mathfrak{h}^f/(\mathfrak{g}(g)+\mathfrak{h})$ and on $\mathfrak{g}/\mathfrak{g}(g)$. Since $\mathfrak{b}_g$ is a maximal isotropic subspace for B_g, we have a duality between $\mathfrak{g}/\mathfrak{b}_g$ and $\mathfrak{b}_g/\mathfrak{g}(g)$. We denote by p the dimension of $\mathfrak{b}_g/\mathfrak{h}$ and by k the common dimension of $\mathfrak{g}/\mathfrak{h}^f$ and $(\mathfrak{g}(g)+\mathfrak{h})/\mathfrak{g}(g)$. We choose a basis of $\mathfrak{g}$

$$X_1, \dots, X_l, h_1, \dots, h_k, e_1, \dots, e_p, e_{p+1}, \dots, e_{2p}, h_{k+1}, \dots, h_{2k}$$

such that

— $X_1, \dots, X_l$ is a basis of $\mathfrak{g}(g)$;
— $X_1, \dots, X_l$, $h_1, \dots, h_k$ is a basis of $\mathfrak{g}(g)+\mathfrak{h}$;
— $X_1, \dots, X_l$, $h_1, \dots, h_k$, $e_1, \dots, e_p$ is a basis of $\mathfrak{b}_g$;
— $X_1, \dots, X_l$, $h_1, \dots, h_k$, $e_1, \dots, e_p$, $e_{p+1}, \dots, e_{2p}$ is a basis of $\mathfrak{h}^f$

and such that if $i \le k$, $j \le k$, $B_g(h_i, h_{k+j}) = \delta_{i,j}$ and if $i \le p$, $j \le p$, $B_g(e_i, e_{p+j}) = \delta_{i,j}$.

Now we have measures $dh = dX_1 \cdots dX_l\, dh_1 \cdots dh_k$ on $\mathfrak{g}(g)+\mathfrak{h}$, $db = dh\, de_1 \cdots de_p$ on $\mathfrak{b}_g$, $dy = db\, de_{p+1} \cdots de_{2p}$ on $\mathfrak{h}^f$ and $dx = dy\, dh_{k+1} \cdots dh_{2k}$ on $\mathfrak{g}$, and measures on the quotient spaces $\mathfrak{g}/\mathfrak{h}^f$, $\mathfrak{h}^f/\mathfrak{b}_g$, $\mathfrak{b}_g/(\mathfrak{h}+\mathfrak{g}(g))$ and $(\mathfrak{h}+\mathfrak{g}(g))/\mathfrak{g}(g)$. This gives corresponding measures on groups and quotients on $\boldsymbol{G}$ by the exponential mapping. If E is a vector subspace of $\mathfrak{g}$ with a Lebesgue measure, we fix on $E^\perp$ the Lebesgue measure which is the Fourier transform of the Lebesgue measure on E.

We denote by ω_g the $\boldsymbol{H}^f$-orbit of the restriction $q(g)$ of g to $\mathfrak{h}^f$. We define the canonical measure $d\omega_g$ on ω_g by the form

$$\hat{\nu}_g = \frac{1}{(2\pi)^p p!}\, B_g \wedge \cdots \wedge B_g = \frac{1}{(2\pi)^p}\, e_1^* \wedge \cdots \wedge e_{2p}^*$$

At the end, we choose on $\Omega_g \cap (f+\mathfrak{h}^\perp)$ the measure $d\Omega_g$ product of the measure $d\omega_g$ and of the measure $dh^\perp$ on $(\mathfrak{h}^f)^\perp$. Thus, $d\Omega_g$ is defined by

$$\nu_g = \frac{1}{(2\pi)^{p+k}}\, e_1^* \wedge \cdots \wedge e_{2p}^* \wedge h_{k+1}^* \wedge \cdots \wedge h_{2k}^*$$

It is now possible to formulate the character formula.

Theorem 6.3 — *Let $g \in f + \mathfrak{h}^\perp$ and $\theta \in \mathcal{D}(\boldsymbol{G})$. Then*

$$\operatorname{tr}\left(\pi_g(\theta) U_{\pi_g}\right) = \int_{\Omega_g \cap (f+\mathfrak{h}^\perp)} (\theta \circ \exp)^\wedge(l)\, d\Omega_g(l)$$

where $(\theta \circ \exp)^\wedge$ is the Fourier transform of $\theta \circ \exp$ and $d\Omega_g$ the canonical measure defined by ν_g.

Proof — Let $\pi = \pi_g$ and $U = U_g$ until the end of this proof. By the previous computations of the trace of $\pi(\theta)U$ we have, because $U = U^*$

$$\begin{aligned}\operatorname{tr}\big(\pi(\theta)U\big) &= < K_\theta(x,y),\, U > \\ &= \int_{\boldsymbol{H}^f/\boldsymbol{B}_g} K_\theta(x,x)\, d\dot{x}\end{aligned}$$

where $K_\theta(x,y) = \int_{\boldsymbol{B}_g} \theta(xby^{-1})\chi_g(b)\, db \qquad (x,y) \in \boldsymbol{G}\times\boldsymbol{G}$ by a previous result (note that $\boldsymbol{H}_f = \boldsymbol{H}_g$). Thus,

$$\begin{aligned}\operatorname{tr}\big(\pi(\theta)U\big) =& \int_{\boldsymbol{H}^f/\boldsymbol{B}_g}\int_{\boldsymbol{B}_g} \theta(xbx^{-1})\chi_g(b)\, dbd\dot{x} \\ =& \int_{\boldsymbol{H}^f/\boldsymbol{B}_g}\int_{\mathfrak{b}_g} \theta\big(\exp(\operatorname{Ad}x.Y)\big)\,\chi_g(\exp Y)\, dY d\dot{x} \\ =& \int_{\boldsymbol{H}^f/\boldsymbol{B}_g}\int_{\mathfrak{b}_g} \big(\theta\circ\exp\circ\operatorname{Ad}x\big)(Y)\, e^{i<g,\,Y>}\, dY d\dot{x} \\ =& \int_{\boldsymbol{H}^f/\boldsymbol{B}_g}\int_{\mathfrak{b}_g^\perp} \big(\theta\circ\exp\circ\operatorname{Ad}x\big)^\wedge(g+l)dld\dot{x} \\ =& \int_{\boldsymbol{H}^f/\boldsymbol{B}_g} |\det(\operatorname{Ad}x)^{-1}| \int_{\mathfrak{b}_g^\perp} (\theta\circ\exp)^\wedge\big(\operatorname{Ad}^* x(g+l)\big)dld\dot{x} \\ =& \int_{\boldsymbol{H}^f/\boldsymbol{B}_g}\int_{\mathfrak{b}_g^\perp} (\theta\circ\exp)^\wedge\big(\operatorname{Ad}^* x(g+l)\big)dld\dot{x}\end{aligned}$$

(because $\det(\operatorname{Ad}x) = 1$).

But $\mathfrak{b}_g$ verifies the Pukanszky condition so $g + \mathfrak{b}_g^\perp = \operatorname{Ad}^*\boldsymbol{B}_g.g$. The mapping Ad^* from $\boldsymbol{B}_g/\boldsymbol{G}(g)$ on $g+\mathfrak{b}_g^\perp$ is bijective and transforms the measure $dh_1\cdots dh_k de_1\cdots de_p$ in the measure $dh^*_{k+1}\cdots dh^*_{2k}de^*_{p+1}\cdots de^*_{2p}$ so,

$$\int_{\mathfrak{b}_g^\perp} (\theta\circ\exp)^\wedge(\operatorname{Ad}^* x.(g+l))\, dl = \int_{\boldsymbol{B}_g/\boldsymbol{G}(g)} (\theta\circ\exp)^\wedge\big(\operatorname{Ad}^*(xb).g\big)\frac{1}{(2\pi)^{p+k}}\, d\dot{b}$$

Now we have

$$\begin{aligned}\operatorname{tr}\big(\pi(\theta)U\big) &= \int_{\boldsymbol{H}^f/\boldsymbol{B}_g}\int_{\boldsymbol{B}_g/\boldsymbol{G}(g)} (\theta\circ\exp)^\wedge\big(\operatorname{Ad}^*(xb).g\big)\frac{1}{(2\pi)^{p+k}}\, d\dot{b}d\dot{x} \\ &= \int_{\boldsymbol{H}^f/\boldsymbol{G}(g)} (\theta\circ\exp)^\wedge\big(\operatorname{Ad}^*(x).g\big)\frac{1}{(2\pi)^{p+k}}\, d\dot{x}\end{aligned}$$

but $\Omega_g \cap (\ f+\mathfrak{h}^\perp\ ) = \operatorname{Ad}^*(\boldsymbol{H}^f).g$ and Ad^* carries the $\boldsymbol{H}^f-$invariant measure on $\boldsymbol{H}^f/\boldsymbol{G}(g)$ in $de^*_1\cdots de^*_{2p}dh^*_1\cdots dh^*_k$ so, we have

$$\operatorname{tr}\big(\pi(\theta)U\big) = \int_{\Omega_g\cap(f+\mathfrak{h}^\perp)} (\theta\circ\exp)^\wedge(l)\, d\Omega_g(l)$$

and this is our character formula. ∎

Corollary 6.1 — *For every $\theta \in \mathcal{D}(\boldsymbol{G})$ we have*

$$\operatorname{tr}\big(\pi(\theta)\big) = \int_{\Omega_g} (\theta \circ \exp)^\wedge(l)\, d\Omega_g(l)$$

where $d\Omega_g$ is the invariant measure on Ω_g defined by $\frac{1}{(2\pi)^p p!} B_g \wedge \cdots \wedge B_g$, the exterior product of p forms B_g with $2p = \dim \Omega_g$.

Proof — We have only to apply the previous theorem with $\mathfrak{h} = \{0\}$. ∎

Corollary 6.2 — *The previous corollary gives a new proof of the injectivity of the Kirillov mapping*

Proof — If $\rho(f) = \rho(g)$ we have $\operatorname{tr}(\pi_f(\theta)) = \operatorname{tr}(\pi_g(\theta))$ for all $\theta \in \mathcal{D}(\boldsymbol{G})$ so, by the corollary above this distribution has a Fourier transform supported by Ω_g and also Ω_f so, $\Omega_f = \Omega_g$. ∎

6.4.– THE PLANCHEREL FORMULA

We denote by r the maximum of the dimension of $\mathfrak{g}(f)$ for $f \in \mathfrak{g}^*$. The number $n - r$ is the maximum of the dimension of the $\boldsymbol{G}$-orbits in $\mathfrak{g}^*$ and the maximum of the rank of the map $X \longrightarrow \operatorname{ad}^* X.g$ for $g \in \mathfrak{g}^*$.

By fixing a basis on $\mathfrak{g}$ and a basis on $\mathfrak{g}^*$ this map is a $n \times n$ matrix the coefficients of which are linear polynomials in the coordinates of g. So, if we consider a minor $m(g)$ with rank $n - r$, nonzero on $\mathfrak{g}^*$, we see that the set $\mathcal{O}$ of $g \in \mathfrak{g}^*$ such that $\det\big(m(g)\big) \neq 0$ is a Zariski dense open set of $\mathfrak{g}^*$ on which the dimension of $\mathfrak{g}(f)$ (and the dimension of orbits) is constant and maximum. By the same proof there exists a set $\mathcal{O}_f \subset f + \mathfrak{h}^\perp$ which is Zariski dense in $f + \mathfrak{h}^\perp$ and on which the dimension of $\mathfrak{g}(f)$ is constant. Moreover $\mathfrak{g}(g) \cap \mathfrak{h}$ is independant of $g \in f + \mathfrak{h}^\perp$ (Exercise).

Now we have a to use a lemma from Duflo-Raïs in [9].

Lemma 6.3 — *Let dl be the Lebesgue measure on $\mathfrak{g}^*$ and $d\Omega_g$ the canonical $\boldsymbol{G}$-invariant measure on the orbit $\Omega_g = \operatorname{Ad}^*(\boldsymbol{G}).g$. Then, there exists a measure μ on the quotient space $\mathcal{O}/\operatorname{Ad}^*(\boldsymbol{G})$ such that for each integrable or positive Borel function φ*

$$\int_{\mathfrak{g}^*} \varphi(l) dl = \int_{\mathcal{O}/\boldsymbol{G}} \int_{\Omega_g} \varphi(l) d\Omega_g(l) d\mu(\dot{g})$$

From this lemma we deduce an other result.

Lemma 6.4 — *There exists a positive Borel measure $d\mu$ on the space $(f + \mathfrak{h}^\perp)/\boldsymbol{H}^f$ of orbits of the group $\boldsymbol{H}^f$ in $f + \mathfrak{h}^\perp$ such that for each integrable or positive Borel function on $f + \mathfrak{h}^\perp$*

$$\int_{f + \mathfrak{h}^\perp} \varphi(l)\, dl = \int_{\mathcal{O}_f/\boldsymbol{H}^f} \int_{\Omega_g \cap (f + \mathfrak{h}^\perp)} \varphi(l)\, d\Omega_g(l)\, d\mu(l)$$

Proof — We have to consider the group $\boldsymbol{H}^f$ with Lie algebra $\mathfrak{h}^f$. Let $\mathfrak{a} = \mathfrak{h} \cap \ker(f|_{\mathfrak{h}})$. It is an ideal of $\mathfrak{h}^f$ which is contained in $\ker g$ for all $g \in f + \mathfrak{h}^\perp$ (Exercise), so, $\mathfrak{h}/\mathfrak{a}$ is central in $\mathfrak{h}^f/\mathfrak{a}$. We denote by $(\mathfrak{h}^f_{\mathfrak{a}})^*$ the set of linear forms on $\mathfrak{h}^f/\mathfrak{a}$ the restriction of which to $\mathfrak{h}/\mathfrak{a}$ is equal to f. So by the previous lemma we have a measure $\tilde{\mu}$ on $(\mathfrak{h}^f_{\mathfrak{a}})^*/\boldsymbol{H}^f$ such that

$$\int_{\mathfrak{h}^f_{\mathfrak{a}}} \varphi(l)\, d\tilde{l} = \int_{(\mathfrak{h}^f_{\mathfrak{a}})^*/\boldsymbol{H}^f} \int_{\boldsymbol{H}^f.(g|\mathfrak{h}^f_{\mathfrak{a}})} \varphi(l)\, d\omega_g(\tilde{l})$$

where $d\omega_g$ is the the measure on the orbit $\mathrm{Ad}^*(\boldsymbol{H}^f).(g|_{\mathfrak{h}^f}) \subset (\mathfrak{h}^f_{\mathfrak{a}})^*$. (here is an identification of $\mathrm{Ad}^*(\boldsymbol{H}^f)(g|_{\mathfrak{h}^f})$ and its image in $(\mathfrak{h}^f_{\mathfrak{a}})^*$ because $\boldsymbol{H}$ stabilizes $(g|_{\mathfrak{h}^f})$). But we have $d\Omega_g = d\omega_g\, dh^\perp$ where $dh^\perp$ is the Fourier transform of dh on $\mathfrak{h}^f$. It is a measure (Lebesgue) on $\mathfrak{h}^f$. So, $d\tilde{l}dh^\perp$ is a Lebesgue measure on $(\mathfrak{h}^f_{\mathfrak{a}})^* \times (\mathfrak{h}^f)^\perp \simeq \big((\mathfrak{g}/\mathfrak{a})|_{(\mathfrak{h}/\mathfrak{a})}\big)^*$. Now it is an easy computation to state the result by using the map $\mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{a}$. ∎

Theorem 6.4 — *(Plancherel formula). Let $\mathcal{O}_f$ the open set of $f + \mathfrak{h}^\perp$ defined previously. Let $d\mu$ the measure defined in the previous lemma and U_{w_l} the operator associated to $\pi_l \in \widehat{\boldsymbol{G}}$ then, for $\theta \in \mathcal{D}(\boldsymbol{G})$ we have*

$$\int_{\boldsymbol{H}} \theta(h)\chi_f(h)\, dh = \int_{\mathcal{O}_f/\boldsymbol{H}^f} \mathrm{tr}(\pi_{\omega_{\tilde{l}}}(\theta)U_{\omega_{\tilde{l}}})d\mu(\tilde{l})$$

Proof — Let $I(\theta) = \int_{\boldsymbol{H}} \theta(h)\chi_f(h)\, dh$. By using the classical Fourier transform we have

$$\begin{aligned} I(\theta) &= \int_{\mathfrak{h}} \theta(\exp Y)\, e^{if(Y)}\, dY \\ &= \int_{\mathfrak{h}^\perp} (\theta \circ \exp)^\wedge(l+f)\, dl \end{aligned}$$

and by the previous theorem :

$$I(\theta) = \int_{\mathcal{O}_f/\boldsymbol{H}^f} \int_{\Omega_g \cap (f+\mathfrak{h}^\perp)} (\theta \circ \exp)^\wedge(l)\, d\Omega_g(l)\, d\mu(\tilde{l})$$

and now by the character formula :

$$I(\theta) = \int_{\mathcal{O}_f/\boldsymbol{H}^f} \mathrm{tr}(\pi_{\omega_{\tilde{l}}}(\theta)U_{\omega_{\tilde{l}}})d\mu(\tilde{l})$$

which completes the proof. ∎

Exercise 6.1 – Compute the Plancherel Formula for the Heisenberg group with dimension $2n+1$.

Prove that for these Heisenberg groups the results in this section are true without the hypothesis of $\boldsymbol{H}$ normal in $\boldsymbol{G}$.

7. A survey on representation theory for non type I solvable Lie groups.

Let $\boldsymbol{G}$ be a connected, simply connected solvable Lie group which is not type I. There are two obstructions to build irreducible unitary representations by the method described in the section **5**.

1) A linear form $g \in \mathfrak{g}$ is not always integral ;

2) The $\boldsymbol{G}$-orbits in $\mathfrak{g}^*$ are not always locally closed.

We will consider in this section two clasical examples of such groups and we describe for these groups the most important features due to L. Pukanszky ([20],[21] and [22]).

7.1.– TWO EXAMPLES

1– *The Mautner group*

The Mautner group is the five dimensional connected simply connected Lie group $\boldsymbol{G}$ whose Lie algebra is given by a basis $e_1, \ldots, e_5$ and the nonzero brackets :

$$\begin{aligned} [e_1, e_2] &= e_3 \quad ; \quad [e_1, e_3] = -e_2 \\ [e_1, e_4] &= \theta e_5 \quad ; \quad [e_1, e_5] = -\theta e_4 \end{aligned}$$

where θ is an irrational number.

This Lie algebra is the semi direct product of $\mathbb{R}$ and the abelian Lie algebra $\mathbb{R}^4$. Of course, we have $\boldsymbol{G} = \exp(\mathbb{R}e_1)\exp(\oplus_{i=1}^4 \mathbb{R}e_i)$. There is another realization of $\boldsymbol{G}$ as a semi direct product $\mathbb{R} \times_s \mathbb{C}^2$. The action of $\mathbb{R}$ on $\mathbb{C}^2$ is given by

$$t.(z_1, z_2) = (e^{it}z_1,\ e^{i\theta t}z_2) \quad t \in \mathbb{R},\ z_1 \in \mathbb{C},\ z_2 \in \mathbb{C}$$

This group is not a regular semi direct product (exercise **7.1**), so we cannot apply Mackey theory to compute the dual space of $\boldsymbol{G}$. We will prove that this group is non type I by using the Auslander-Kostant's caracterization of type I groups described at the end of section **5**.

a) *Computation of orbits in* $\mathfrak{g}^*$.

Set $X = \sum_{i=1}^{5} \alpha_i e_i \in \mathfrak{g}$. We realize $\operatorname{ad} X$ by a matrix

$$\operatorname{ad} X = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \alpha_3 & 0 & -\alpha_1 & 0 & 0 \\ -\alpha_2 & \alpha_1 & 0 & 0 & 0 \\ \theta\alpha_5 & 0 & 0 & 0 & -\theta\alpha_1 \\ -\theta\alpha_4 & 0 & 0 & \theta\alpha_1 & 0 \end{pmatrix}$$

and we obtain $\operatorname{Exp}\operatorname{ad} X = \operatorname{Ad}\exp X$

$$\operatorname{Ad}\exp X = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{\alpha_2-\alpha_2\cos\alpha_1+\alpha_3\sin\alpha_1}{\alpha_1} & \cos\alpha_1 & -\sin\alpha_1 & 0 & 0 \\ \frac{\alpha_3-\alpha_2\sin\alpha_1-\alpha_3\cos\alpha_1}{\alpha_1} & \sin\alpha_1 & \cos\alpha_1 & 0 & 0 \\ \frac{\alpha_4-\alpha_4\cos\theta\alpha_1+\alpha_5\sin\theta\alpha_1}{\alpha_1} & 0 & 0 & \cos\theta\alpha_1 & -\sin\theta\alpha_1 \\ \frac{\alpha_5-\alpha_4\sin\theta\alpha_1-\alpha_5\sin\theta\alpha_1}{\alpha_1} & 0 & 0 & \sin\theta\alpha_1 & \cos\theta\alpha_1 \end{pmatrix}$$

It will be useful to write $x = \alpha_1 e_1$, and $y = \alpha_2 e_2 + \cdots + \alpha_5 e_5$. We have

$$\operatorname{Ad}^*\exp x = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cos\alpha_1 & \sin\alpha_1 & 0 & 0 \\ 0 & -\sin\alpha_1 & \cos\alpha_1 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta\alpha_1 & \sin\theta\alpha_1 \\ 0 & 0 & 0 & -\sin\theta\alpha_1 & \cos\theta\alpha_1 \end{pmatrix}$$

and by using $(\operatorname{ad} y)^2 = 0$ we have

$$\operatorname{Ad}^*\exp y = \begin{pmatrix} 1 & -\alpha_3 & \alpha_2 & -\theta\alpha_5 & \theta\alpha_4 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

These matrix are written in the dual basis $e_1^*, \ldots, e_5^*$

Proposition 7.1 — *The orbit of $g = e_3^* + e_5^*$ is not locally closed.*

PROOF — With the previous notations we have

$$\exp x \exp y.g = (\alpha_2 + \theta\alpha_4)e_1^* + \sin\alpha_1 e_2^* + \cos\alpha_1 e_3^* + \sin(\theta\alpha_1)e_4^* + \cos(\theta\alpha_1)e_5^*$$

This shows that the trace of the orbit in the plan $P = \mathbb{R}e_4^* \oplus \mathbb{R}e_5^*$ is obtained for $\alpha_2 + \theta\alpha_4 = 0$ and $\alpha_1 = 2k\pi$, $k \in \mathbb{Z}$. So, the trace of the orbit is the set

$$S = \{\sin(2k\theta\pi)e_4^* + \cos(2k\theta\pi)e_5^* \ ; \ k \in \mathbb{Z}\}$$

and this set S is not locally closed because $\theta \notin \mathbb{Q}$ so its closure is a circle C and S is not open in C. ∎

This result and theorem **5.7** show that $\boldsymbol{G}$ is not type I. We now look at the integral forms. We have $\mathfrak{g}^g = \mathbb{R}e_3 \oplus \mathbb{R}e_5 \oplus \mathbb{R}(\theta e_2 - e_4)$ and $(\boldsymbol{G})_g^0 = \exp\mathfrak{g}^g$. To compute the stabilizer $\boldsymbol{G}_g$ of g, we set $u = \exp x \exp y$ and we see that $u \in \boldsymbol{G}_g$ if and only if $\alpha_2 + \theta\alpha_4 = 0$, $\alpha_1 = 2k\pi$ and $\theta\alpha_1 = 2k'\pi$ where k and k' are integers. Since θ is not a rational number, we have $\alpha_1 = 0$ so $\boldsymbol{G}_g$ is connected. It follows that the form g is integral and there is only one class of irreducible representations obtained by Auslander-Kostant method described in section **5** : it is $\pi_g = \mathrm{Ind}_{\boldsymbol{K}}^{\boldsymbol{G}} \chi_g$ where $\boldsymbol{K} = \exp(\oplus_{i=2}^{5} \mathbb{R} e_i)$ and χ_g is the character of $\boldsymbol{K}$ corresponding to $g|_{\mathfrak{k}}$.

Exercise 7.1 – Show that the Mautner group is not a regular semi direct product of $\exp(\mathbb{R}e_1)$ and $\exp(\oplus_{i=1}^{4} \mathbb{R} e_i)$. Use the realization in $\mathbb{R} \times_s \mathbb{C}^2$ with the action of $\mathbb{R}$ on $\mathbb{C}^2$ given by $t.(z_1, z_2) = (e^{it} z_1,\ e^{i\theta t} z_2) \quad t \in \mathbb{R},\ z_1 \in \mathbb{C},\ z_2 \in \mathbb{C}$.

Exercise 7.2 – Verify that every $g \in \mathfrak{g}^*$ is integral but $\boldsymbol{G}_g$ is not always connected. For instance, prove that $\boldsymbol{G}_{e_2^*} = \exp(2\pi \mathbb{Z} e_1) \exp(\mathbb{R}e_2 \oplus \mathbb{R}e_4 \oplus \mathbb{R}e_5)$.

2– *The Dixmier group*

This group is given by its Lie algebra $\mathfrak{g}$ which is seven dimensional. Let $e_1, \ldots, e_7$ be a basis of $\mathbb{R}^7$. The nonzero brackets are :

$$[e_1, e_2] = e_3 \quad ; \quad [e_1, e_4] = e_5 \quad ; \quad [e_1, e_5] = -e_4$$
$$[e_2, e_6] = e_7 \quad ; \quad [e_2, e_7] = -e_6$$

This algebra is the semi direct product of the Heisenberg algebra $\mathfrak{k} = \oplus_{i=1}^{3} \mathbb{R} e_i$ and the abelian Lie algebra $\mathfrak{b} = \oplus_{i=4}^{7} \mathbb{R} e_i$ so, $\boldsymbol{G} = \exp \mathfrak{k} \exp \mathfrak{h} = \exp \mathfrak{h} \exp \mathfrak{k}$ is the simply connected Lie group with Lie algebra $\mathfrak{g}$. The center of $\mathfrak{g}$ is $\mathbb{R}e_3$ and its greatest nilpotent ideal is $\oplus_3^7 \mathbb{R} e_i = [\mathfrak{g}, \mathfrak{g}]$ and is abelian. It can be shown that $\mathfrak{g}$ is a regular semi direct product of $\exp \mathfrak{k}$ and $\exp \mathfrak{h}$ ([8]).

a) *Computation of orbits in* $\mathfrak{g}^*$.

Let $X = \sum_{i=1}^{7} \theta_i e_i \in \mathfrak{g}$. The matrix of $\mathrm{ad}\, X$ is :

$$\mathrm{ad}\, X = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\theta_2 & \theta_1 & 0 & 0 & 0 & 0 & 0 \\ \theta_5 & 0 & 0 & 0 & -\theta_1 & 0 & 0 \\ -\theta_4 & 0 & 0 & \theta_1 & 0 & 0 & 0 \\ 0 & \theta_7 & 0 & 0 & 0 & 0 & -\theta_2 \\ 0 & -\theta_6 & 0 & 0 & 0 & \theta_2 & 0 \end{pmatrix}$$

In the following, we set $x = \theta_1 e_1 + \theta_2 e_2 + \theta_3 e_3$ and $y = \theta_4 e_4 + \theta_5 e_5 + \theta_6 e_6 + \theta_7 e_7$. We can then compute $\mathrm{Ad} \exp = \mathrm{Exp}\, \mathrm{ad}\, x$:

$\mathrm{Exp}\, \mathrm{ad}\, x.e_1 = e_1 - \theta_2 e_2$ $\qquad$ $\mathrm{Exp}\, \mathrm{ad}\, x.(e_4 + ie_5) = e^{-i\theta_1}(e_4 + ie_5)$

$\mathrm{Exp}\, \mathrm{ad}\, x.e_2 = e_2 - \theta_1 e_3$ $\qquad$ $\mathrm{Exp}\, \mathrm{ad}\, x.(e_6 + ie_7) = e^{-i\theta_2}(e_6 + ie_7)$

$\mathrm{Exp}\, \mathrm{ad}\, x.e_3 = e_3$

We can write the matrix of $\mathrm{Ad}^* \exp x$:

$$\mathrm{Ad}^* \exp x = \begin{pmatrix} 1 & 0 & \theta_2 & 0 & 0 & 0 & 0 \\ 0 & 1 & -\theta_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ 0 & 0 & 0 & \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cos\theta_2 & -\sin\theta_2 \\ 0 & 0 & 0 & 0 & 0 & \sin\theta_2 & \cos\theta_2 \end{pmatrix}$$

and by a similar computation :

$$\mathrm{Ad}^* \exp y = \begin{pmatrix} 1 & 0 & 0 & -\theta_5 & \theta_4 & 0 & 0 \\ & 1 & 0 & 0 & 0 & -\theta_7 & \theta_6 \\ & & 1 & 0 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 & 0 \\ & \mathbf{(0)} & & & 1 & 0 & 0 \\ & & & & & 1 & 0 \\ & & & & & & 1 \end{pmatrix}$$

We denote as usual by $\{e_1^*, \ldots, e_7^*\}$ the dual basis of $\{e_1, \ldots, e_7\}$ in $\mathfrak{g}^*$. It is easy to show that for any $f \in \mathfrak{g}^*$, the coadjoint orbit $\boldsymbol{G}.f$ of f is locally closed, so the first condition of theorem **5.7** is verified. We now prove that some orbits are not integral.

Proposition 7.2 — *The linear form $g = e_3^* + e_4^* + e_6^*$ is not integral.*

Proof — We first compute stabilizers. We have :

$$(\exp x \exp y).g = (\theta_2 - \theta_5)e_1^* - (\theta_1 + \theta_7)e_2^* + e_3^* + \cos\theta_1 e_4^* + \sin\theta_1 e_5^* + \cos\theta_2 e_6^* + \sin\theta_2 e_7^* \tag{22}$$

Thus, it is clear that the orbit is homeomorphic to $\mathbb{R}^2 \times \mathbb{T}^2$. On the other hand, we compute easily

$$\mathfrak{g}^g = \mathbb{R}e_3 \oplus \mathbb{R}e_4 \oplus \mathbb{R}e_6 \quad ; \quad \boldsymbol{G}_g = \exp(\mathbb{R}e_3 \oplus \mathbb{R}e_4 \oplus \mathbb{R}e_6)$$

From (22) we deduce that $\exp x \exp y \in \boldsymbol{G}_g$ if and only if

$$\begin{cases} \theta_1 \in 2\pi\mathbb{Z} & ; \quad \theta_2 - \theta_5 = 0 \\ \theta_2 \in 2\pi\mathbb{Z} & ; \quad \theta_1 + \theta_7 = 0 \end{cases}$$

so $Z \in \boldsymbol{G}_g$ if and only if it can be written

$$Z = \exp(\theta_3 e_3)\exp(2k\pi e_1 + 2k'\pi e_2)\exp(2k'\pi e_5 - 2k\pi e_7)\exp(\theta_4 e_4 + \theta_6 e_6)$$
$$\text{where } k,\ k' \in \mathbb{Z} \quad ; \quad \theta_3,\ \theta_4, \theta_6 \in \mathbb{R}$$

But $\exp(2k\pi e_1 + 2k'\pi e_2) = \exp(2k'\pi e_2)\exp(2k\pi e_1)\exp(\theta_3' e_3)$ where $\theta_3' \in \mathbb{R}$. Furthermore, the two elements $\exp(2k\pi e_1)$ and $\exp(2k'\pi e_5)$ are commuting so $Z \in \boldsymbol{G}_g$ if and only if :

$$\begin{aligned} Z &= \exp(\theta_3'' e_3)\exp(2k'\pi e_2)\exp(2k'\pi e_5)\exp(2k\pi e_1)\exp(-2k\pi e_7)\exp(\theta_4 e_4 + \theta_6 e_6) \\ &= \exp(\theta_3'' e_3)\exp\big(2k'\pi(e_2 + e_5)\big)\exp\big(2k\pi(e_1 - e_7)\big)\exp(\theta_4 e_4 + \theta_6 e_6) \end{aligned}$$

It is now clear that

$$\boldsymbol{G}_g = \exp\big(2\pi\mathbb{Z}(e_2 + e_5)\big)\exp\big(2\pi\mathbb{Z}(e_1 - e_7)\big)\exp(\mathbb{R}e_3 \oplus \mathbb{R}e_4 \oplus \mathbb{R}e_6)$$

Now let $\mathring{\boldsymbol{G}}_g$ be the kernel of the restriction of χ_g to $(\boldsymbol{G}_g^0)$ (connected component of the neutral element of $\boldsymbol{G}_g$). This group is $\mathring{\boldsymbol{G}}_g = \exp\big(\mathbb{R}(e_3 - e_4) \oplus \mathbb{R}(e_6 - e_4)\big)$. At this step, we have to use a first result of L. Pukanszky in [20].

Proposition 7.3 — *Let g be a linear form on a solvable Lie algebra $\mathfrak{g}$. With the previous notations, χ_g extends to a character of $\boldsymbol{G}_g$ (or equivalently g is an integral form) if and only if the quotient group $\boldsymbol{G}_g/\mathring{\boldsymbol{G}}_g$ is abelian.*

To use this proposition we consider another group $\overline{\boldsymbol{G}}_g$: it is the pullback image of the center of $\boldsymbol{G}_g/\mathring{\boldsymbol{G}}_g$ in $\boldsymbol{G}_g$ by the canonical projection.

Exercise 7.3 – Prove that

$$\overline{\boldsymbol{G}}_g = \{x \in \boldsymbol{G}_g \ ; \ xax^{-1}a^{-1} \in \mathring{\boldsymbol{G}} \quad \forall a \in \mathring{\boldsymbol{G}}_g\}$$

and $\mathring{\boldsymbol{G}}_g \subset \overline{\boldsymbol{G}}_g$.

We show that $\overline{\boldsymbol{G}}_g \neq \boldsymbol{G}_g$. Let z and z' be two elements of $\boldsymbol{G}_g$

$$\begin{aligned} z &= \exp(\theta_3 e_3)\exp(2k\pi e_1 + 2k'\pi e_2)\exp(2k'\pi e_5 - 2k\pi e_7)\exp(\theta_4 e_4 + \theta_6 e_6) \\ z' &= \exp(\theta'_3 e_3)\exp(2k_1\pi e_1 + 2k'_1\pi e_2)\exp(2k'_1\pi e_5 - 2k_1\pi e_7)\exp(\theta'_4 e_4 + \theta'_6 e_6) \end{aligned}$$

By using the formula $\exp x \exp y = \exp(\operatorname{Exp} \operatorname{ad} x.y)\exp x$, we see that elements of $\exp(2\pi\mathbb{Z}e_1 + 2\pi\mathbb{Z}e_2)$ and $\exp\mathfrak{h}$ are commuting, so

$$\begin{aligned} zz' = {} & \exp(\theta'_3 e_3)\exp(2k\pi e_1 + 2k'\pi e_2)\exp(2k_1\pi e_1 + 2k'\pi e_2)\exp(\theta_3 e_3) \\ & \times \exp(2k'\pi e_5 - 2k\pi e_5)\exp(2k'_1\pi - 2k_1\pi e_7)\exp(\theta_4 e_4 + \theta_6 e_6)\exp(\theta'_4 e_4 + \theta'_6 e_6) \end{aligned}$$

Furthermore,

$$\begin{aligned} & \exp(2k\pi e_1 + 2k'\pi e_2)\exp(2k_1\pi e_1 - 2k'\pi e_2) = \\ & \qquad \exp\big(2k_1\pi e_1 + 2k'\pi e_2 + 4\pi^2(kk'_1 - k'k_1)e_3\big)\exp(2k\pi e_1 + 2k'\pi e_2) \end{aligned}$$

and at the end

$$\begin{aligned} zz' = {} & \exp(\theta'_3 e_3)\exp\big(2k_1\pi e_1 + 2k'_1\pi e_2 + 4\pi^2(kk'_1 - k'k_1)e_3\big) \\ & \times \exp(2k'_1\pi e_5 - 2k_1\pi e_7)\exp(\theta'_4 e_4 + \theta'_6 e_6).z \end{aligned}$$

It results from this that $z' \in \overline{\boldsymbol{G}}_g$ if and only if for all $k, k' \in \mathbb{Z}$

$$Z = \exp(2k_1\pi e_1 + 2k'_1\pi e_2 + 4\pi^2(kk'_1 - k'k_1)e_3)\exp(-2k_1\pi e_1 - 2k'_1 e_2) \in \mathring{\boldsymbol{G}}_g$$

But, using the formula giving the product in $\exp\mathfrak{k}$ (Heisenberg group), we have

$$Z = \exp\big(4\pi^2(kk'_1 - k'k_1)e_3\big)$$

Since $\exp e_3 \notin \mathring{\boldsymbol{G}}_g$, $z' \in \mathring{\boldsymbol{G}}_g$ if and only if $kk'_1 - k'k_1 = 0$ for all k and k' in $\mathbb{Z}$, thus, $k_1 = 0$, $k'_1 = 0$ and $\overline{\boldsymbol{G}}_g = \boldsymbol{G}_g^0$ as we expected. The form g is not integral. ∎

If we look carefully at the construction of the section **5**, we see that to build a factorial or irreducible representation of a solvable Lie group $\boldsymbol{G}$, it is enough to get a representation (factorial or irreducible) of $\boldsymbol{G}_g$ whose restriction to $\overset{\circ}{\boldsymbol{G}}_g$ is a multiple of χ_g. L. Pukanszky has shown that there exists factorial representations whose restriction to $\boldsymbol{G}_g^0$ are multiple of χ_g. The complete result is the following :

Proposition 7.4 — *Let $\boldsymbol{G}$ be any connected, simply connected solvable Lie group and $g \in \mathfrak{g}^*$. Then the character χ_g extends to a character $\overline{\chi}_g$ of $\overline{\boldsymbol{G}}_g$. The representation $\sigma_g = \operatorname{Ind}_{\overline{\boldsymbol{G}}_g}^{\boldsymbol{G}_g} \overline{\chi}_g$ is a type I or type II factorial representation of $\boldsymbol{G}_g$. Furthermore it is type I if and only if $\boldsymbol{G}_g/\overline{\boldsymbol{G}}_g$ is a finite group.*

This proposition gives rise to two descriptions of a representation of $\boldsymbol{G}$:

1) The first one by holomorphical induction as in section **5**. We choose a good polarization $\mathfrak{b}$ at g and let $\overline{\boldsymbol{D}} = \boldsymbol{D}^0\overline{\boldsymbol{G}}_g$. The character $\overline{\chi}_g$ extends to a character of $\overline{\boldsymbol{D}}$ denoted by χ_g yet. It is clear that we can form the holomorphical representation $\rho_g = \operatorname{ind}(g, \mathfrak{h}, \overline{\chi}_g)$.

2) To state the second form, let $\boldsymbol{N} = [\boldsymbol{G}, \boldsymbol{G}]$. The Lie algebra of $\boldsymbol{N}$ is $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}]$. This group is a simply connected nilpotent Lie group (see section **3**), so we can form the Kirillov representation π corresponding to the orbit of $g|_{\mathfrak{n}}$ in $\mathfrak{n}^*$. A result of L. Pukanszky says that $\boldsymbol{K}_\pi = \boldsymbol{N}\overline{\boldsymbol{G}}_g$ is the greatest closed subgroup of $\boldsymbol{G}$ such that π extends to a representation $\overline{\pi}$ of $\boldsymbol{K}_\pi$. We can choose the represntation $\overline{\pi}$ such that its restriction to $\overline{\boldsymbol{G}}_g$ is a multiple of $\overline{\chi}_g \in \widehat{\overline{\boldsymbol{G}}}_g$. We can now form the representation $\rho'_g = \operatorname{Ind}_{\boldsymbol{K}_\pi}^{\boldsymbol{G}} \overline{\pi}$.

Proposition 7.5 — *We have $\rho_g \simeq \rho'_g$ and this representation is a factorial representation which have the type of $\sigma_g = \operatorname{Ind}_{\overline{\boldsymbol{G}}_g}^{\boldsymbol{G}_g} \overline{\chi}_g$ so, by the previous theorem, it is type I if and only if $\boldsymbol{G}_g/\overline{\boldsymbol{G}}_g$ is a finite group.*

If we apply this result to the Dixmier group and the linear form $g = e_3^* + e_4^* + e_6^*$ discussed above, we see that $\overline{\boldsymbol{G}}_g = \boldsymbol{G}_g^0 \subset \boldsymbol{N}$, $\boldsymbol{K}_\pi = \boldsymbol{N}$ and since $\boldsymbol{G}_g/\overline{\boldsymbol{G}}_g$ is not a finite group, the representation $\rho = \operatorname{Ind}_{\boldsymbol{N}}^{\boldsymbol{G}}(\pi_{g|_{\mathfrak{n}}})$ is a type II factorial representation.

7.2.– THE GENERALIZED ORBITS

For general simply connected solvable Lie groups, the factorial representation ρ_g is not the best one. For instance, it has not good properties for Plancherel formula. The "good" representation is somewhat more subtle: we need to pack the previous factorial representations associated to orbits for all the "bad" orbits (not locally closed), closed to one of them. We now describe this "package".

Let $\boldsymbol{K}$ be a locally compact separable group acting on a metric space X. We recall that the action is said to be regular if every orbit is locally closed (see the end of section **1**). We have seen that if the orbits in $\mathfrak{g}^*$ of a connected simply connected solvable Lie group $\boldsymbol{G}$ are not all locally closed, this group is not type I.

Definition — *Let $\boldsymbol{G}$ be a locally compact group, X a topological space and $(k, x) \longrightarrow k.x$ a continuous action of $\boldsymbol{K}$ on X. We define the relation $\mathcal{R}$ on X by*

$$x\mathcal{R}y \Longleftrightarrow \overline{\boldsymbol{K}.x} = \overline{\boldsymbol{K}.y}$$

where $\overline{\boldsymbol{K}.x}$ and $\overline{\boldsymbol{K}.y}$ are the closures of the $\boldsymbol{K}$-orbits of x and y.

Exercise 7.4 – Prove the following lemma.

a) If $Y \subset X$ such that Y is locally closed and $\boldsymbol{K}$-invariant, the $\mathcal{R}$-class of any $x \in Y$ is contained in Y.

b) If the orbit $\boldsymbol{K}.x$ is locally closed, then the $\mathcal{R}$-class of x is the $\boldsymbol{K}$-orbit $\boldsymbol{K}.x$.

[b) is implied by a)].

L. Pukanszky has shown that if $\boldsymbol{K}$ is a solvable group (connected and simply connected), then the $\mathcal{R}$ classes are locally closed. More generally, we have the following theorem.

Theorem 7.1 — *Suppose there exists a Lie group $\widetilde{\boldsymbol{K}}$ containing $\boldsymbol{K}$ such that $[\widetilde{\boldsymbol{K}}, \widetilde{\boldsymbol{K}}] \subset \boldsymbol{K}$ and whose action on X is regular. Let x be an element of X, Ω its $\mathcal{R}$ class, $\widetilde{\boldsymbol{K}}.x$ the stabilizer of x in $\widetilde{\boldsymbol{K}}$. Then,*

1) $\widetilde{\boldsymbol{K}}.y$ does not depend on $y \in \widetilde{\boldsymbol{K}}.x$ and we have $\Omega = \overline{(\boldsymbol{K}\widetilde{\boldsymbol{K}}_x)}.x$.

2) Ω is locally closed and is homeomorphic to the homogeneous space $\boldsymbol{K}(\widetilde{\Omega})/\boldsymbol{K}(\widetilde{\Omega})_x$.

If $\boldsymbol{G}$ is a solvable group, we can choose for $\boldsymbol{K}$ the connected simply connected Lie group $\widetilde{\boldsymbol{G}}$ whose Lie algebra is an algebraic Lie algebra $\tilde{\mathfrak{g}}$ containing $\mathfrak{g}$ and such that $[\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}] = [\mathfrak{g}, \mathfrak{g}] = \mathfrak{n}$. Until the end of this section we suppose that $\widetilde{\boldsymbol{K}} = \widetilde{\boldsymbol{G}}$.

We denote by $\mathcal{R}(\mathfrak{g}^*)$ the set of $\{(g, \chi)\ ;\ g \in \mathfrak{g}^*,\ \chi \in \overline{\boldsymbol{G}}_g\}$. The action of $\boldsymbol{G}$ on this space is the natural action on g and χ. The topology on $\mathcal{R}(\mathfrak{g}^*)$ is given by convergent sequences : $(g_n, \chi_n) \longrightarrow (g, \chi)$ if and only if $g_n \longrightarrow g$ and for all $c_n \in \overline{\boldsymbol{G}}_{g_n}$ such that $c_n \longrightarrow c \in \overline{\boldsymbol{G}}_g$, $\chi_n(c_n) \longrightarrow \chi(c)$. The $\boldsymbol{G}$-orbits in $\mathcal{R}(\mathfrak{g}^*)$ are *the generalized orbits*. As we said before, the generalized orbits are locally closed.

It is not difficult to see that the groups $\widetilde{\boldsymbol{H}} = \widetilde{\boldsymbol{G}}_g\boldsymbol{N}$, $\boldsymbol{H} = \boldsymbol{G}_g\boldsymbol{N}$, $\overline{\boldsymbol{H}} = \overline{\boldsymbol{G}}_g\boldsymbol{N}$ and $\mathring{\boldsymbol{H}} = \boldsymbol{G}_g^0\boldsymbol{N}$ are invariant for g in $\mathcal{O}$. They are closed subgroups of $\widetilde{\boldsymbol{G}}$ with connected component equal to $\boldsymbol{H}^0 = \boldsymbol{G}_g^0\boldsymbol{N}$. We denote by $\mathring{\overline{\boldsymbol{H}}}$ the set of characters of $\overline{\boldsymbol{H}}$ equal to 1 on $\boldsymbol{H}^0$.

Exercise 7.5 – Prove that $\boldsymbol{G}_g/\boldsymbol{G}_g^0 \simeq \boldsymbol{H}/\boldsymbol{H}^0$ and $(\overline{\boldsymbol{G}}_g/\boldsymbol{G}_g^0)\widehat{\ } \simeq \mathring{\overline{\boldsymbol{H}}}$.

Theorem 7.2 — *Let $\mathcal{O}$ be a generalized orbit of $\mathcal{R}(\mathfrak{g}^*)$. There exists a non zero positive $\boldsymbol{G}$-invariant Borel measure on $\mathcal{O}$, unique up to a multiplicative scalar.*

Proof — Let $p = (g, \chi) \in \mathcal{O}$. We use the previous notations, and we denote by J the closure of $\big((\boldsymbol{G} \times \{1\})(\widetilde{\boldsymbol{G}} \times \mathring{\overline{\boldsymbol{H}}}\boldsymbol{H})_\rho\big)^\circ$ in $\widetilde{\boldsymbol{G}} \times \mathring{\overline{\boldsymbol{H}}}\boldsymbol{H})$. By what preceeds we know that $\mathcal{O}$ is homeomorphic to J/J_p where the $\boldsymbol{G}$ action on $\mathcal{O}$ is the action of $\boldsymbol{G} \times \{1\}$ on J/J_p. Since J_p contains $(\boldsymbol{G} \times \mathring{\overline{\boldsymbol{H}}}\boldsymbol{H})_p$, the closure of $(\boldsymbol{G} \times \{1\}) \times J_p$ is also J-invariant so, if there exists a $\boldsymbol{G}$-invariant measure on $\mathcal{O}$, it is unique up to a multiplicative scalar.

We now prove that such a measure exists. It is enough to show that the restriction of the modular function of J to J_p is equal to the modular function of J_p, that is, for all $a \in J_p$: $\big|\det(\mathrm{Ad}\, a|_{\mathfrak{j}})\big| = \big|\det(\mathrm{Ad}\, a|_{\mathfrak{j}_p})\big|$ where $\mathfrak{j}$ is the Lie algebra of J. But we have $[\mathfrak{j},\mathfrak{j}] = [\mathfrak{g},\mathfrak{g}] = \mathfrak{n}$ so, $\big|\det(\mathrm{Ad}\, a|_{\mathfrak{j}})\big| = \big|\det(\mathrm{Ad}\, a|_{\mathfrak{n}})\big|$ and $\big|\det(\mathrm{Ad}\, a|_{\mathfrak{j}_p})\big| = \big|\det(\mathrm{Ad}\, a|_{\mathfrak{j}_p\cap\mathfrak{n}})\big|$ and since $[\mathfrak{g}(g),\mathfrak{g}(g)] \subset \mathfrak{n}(g)$ we have $\big|\det(\mathrm{Ad}\, a|_{\mathfrak{j}_p\cap\mathfrak{n}})\big| = \big|\det\big(\mathrm{Ad}\, a|_{\mathfrak{g}(g)}\big)\big|$. It is enough to verify that $\big|\det(\mathrm{Ad}\, a|_{\mathfrak{g}})\big| = \big|\det\big(\mathrm{Ad}\, a|_{\mathfrak{g}(g)}\big)\big|$ or equivalently $\big|\det\big(\mathrm{Ad}\, a|_{\mathfrak{g}(g)}\big)\big| = 1$. We know that a stabilizes g so also stabilizes the non degenerated skew bilinear form on $\mathfrak{g}/\mathfrak{g}(g)$ defined by $B(u,v) = g([u,v])$ and this completes our proof. ∎

7.3.– THE CENTRAL REPRESENTATIONS OF PUKANSZKY

By a previous proposition, for each $p = (g,\chi) \in \mathcal{R} = \mathcal{R}(\mathfrak{g}^*)$ we can build a factorial representation $\pi(p)$ of $\boldsymbol{G}$ which is type I or type II. Just before, we have defined a positive $\boldsymbol{G}$-invariant measure $\mu_{\mathcal{O}}$ on each generalized orbit $\mathcal{O} \subset \mathcal{R}$. We shall prove now that the set of representations $\pi(p)$ is a $\mu_{\mathcal{O}}$-measurable field of representations and that the representation $\mathcal{F}(\mathcal{O}) = \int^{\oplus} \pi(g,\chi) d\mu_{\mathcal{O}}(g,\chi)$ is factorial and we shall give conditions such that this representation is type I.

Let $p_0 = (g_0,\chi)$ a fixed element of $\mathcal{O}$. Then, with the above notations $\mathcal{O} = J.p_0$. Let $\mathfrak{h}_0$ be a polarization at g_0 which verifies all the “good” conditions and invariant under the action of J_{p_0}. For $p = a.p_0$ $(a \in J)$, let be $\mathfrak{h}_p = a.\mathfrak{h}_0$ which is a polarization at g because $\mathfrak{h}_0$ is invariant under J_p. Thus, we have defined a field of polarizations and with these polarizations, a field of concrete representations on $\mathcal{O}$:

$$T(p) = \rho(g,\mathfrak{h}_p,\chi) \qquad p = (g,\chi) \in \mathcal{O}$$

We now consider a borel section s from $\mathcal{O}$ into J, that is a borel map such that $p = s(p).p_0$. This is possible because J/J_p is homeomorphic to $\mathcal{O}$. we put $T'(p) = s(p).T(p_0)$. It is clear that $T'(p)$ is equivalent to $T(p)$. To prove that the field $\{T(p)\}_{p\in\mathcal{O}}$ is $\mu_{\mathcal{O}}$-integrable, it is enough to prove that the field $\{T'(p)\}_{p\in\mathcal{O}}$ is $\mu_{\mathcal{O}}$-integrable. Since the space $\mathcal{H}'_p$ of $T'(p)$ is equal to $\mathcal{H}'_{p_0}$, the space of $T(p_0)$ for all $p \in \mathcal{O}$, so it is $\mu_{\mathcal{O}}-$measurable. Now it is enough to verify that the field of representations $\{T'(p)\}_{p\in\mathcal{O}}$ is $\mu_{\mathcal{O}}-$measurable, that is for all u and v in $\mathcal{H}_{p_0}$ and $x \in \boldsymbol{G}$ the function : $p \longrightarrow (T'(p)_x(u)\,,\, v)$ is measurable. But as we have $T'(p)_x = T(p_0)\big(s(p)^{-1}xs(p)\big)$ we see that this is a borel function hence a measurable function.

So the field $\{T'(p)\}_{p\in\mathcal{O}}$ is $\mu_{\mathcal{O}}-$integrable and as $\int^{\oplus} T'(p)d\mu_{\mathcal{O}}(p)$ depends only on the class of the $T'(p)$ we can write

$$\mathcal{F}(\mathcal{O}) = \int^{\oplus} \pi(g,\chi)d\mu_{\mathcal{O}}(g,\chi)$$

Theorem 7.3 — *The representations $\mathcal{F}(\mathcal{O})$ are factorial representations.*

Proof — We write $T = \int^{\oplus} T(p)d\mu_{\mathcal{O}}(p)$, W for the space of diagonal operators, W' the commuting algebra of W, $R(T)$ the Von Neumann algebra generated by $T(\boldsymbol{G})$ and $\boldsymbol{Z}(T)$ its center. Since for each $x \in \boldsymbol{G}$, $T(x) \in W'$, we have $R(T) \subset W'$.

Thus, if $A \in \boldsymbol{Z}(T)$ we have $A \in W'$ and A is a decomposable operator.

$$A = \int^{\oplus} A(p) d\mu_{\mathcal{O}}(p)$$

We have to verify that A is a scalar multiple of identity operator. If $x \in \boldsymbol{G}$ then, $T(x)A = AT(x)$ so $T(p)_x A(p) = A(p)T(p)_x$ for $\mu_{\mathcal{O}}$-almost all x. Using a counting dense subset of $\boldsymbol{G}$ and changing if it needs $A(p)$ on a $\mu_{\mathcal{O}}$-null set, we have $T(p)_x A(p) = A(p)T(p)_x$ for all $p \in \mathcal{O}$ and $x \in \boldsymbol{G}$. Thus, for all $p \in \mathcal{O},\ A(p) \in R\big(T(p)\big)'$. Since $A \in R(T)$ we can find a sequence (A_n) of linear combinations of operators $T(x),\ x \in \boldsymbol{G}$ such that $A_n \longrightarrow A$ according to the strong topology. We now replace A_n by a subsequence such that $A_n(p) \longrightarrow A(p)$ strongly for almost all p or all p if we change $A_n(p)$ on a $\mu_{\mathcal{O}}$-null set. We have $A_n(p) \in R\big(T(p)\big)$ so $A(p) \in R\big(T(p)\big)$. At last, $A(p) \in \boldsymbol{Z}\big(T(p)\big)$ and since $T(p)$ is a factorial representation, we have $A(p) = \varphi(p) I_p$ where I_p is the identity operator in the space of $T(p)$.

We have now to show that φ is constant almost everywhere. Let x a fixed element in $\boldsymbol{G}$. The representation $T(x.p)$ is equivalent to $T(p)$ and there exists an unitary operator U from $\mathcal{H}_p$ on $\mathcal{H}_{x.p}$ such that $T(x.p)_y = UT(p)_y U^{-1}$ for all $y \in \boldsymbol{G}$ so, for all $n \in \mathbb{N},\ A_n(x.p) = UA_n(p)U^{-1}$. By considering the limit when $n \longrightarrow \infty$ we see that $A(x.p) = UA(p)U^{-1}$, that is $\varphi(x.p) = \varphi(p)$ for all $x \in \boldsymbol{G}$. This shows that φ, which is a borel function, is also $\boldsymbol{G}$-invariant on $\mathcal{O}$, but the action of $\boldsymbol{G}$ into $\mathcal{O}$ is ergodic (because the action of J/J_p is ergodic since $(\boldsymbol{G} \times \{1\})J_p$ is dense in J) so φ is a constant function. ∎

Proposition 7.6 — *The representation $\mathcal{F}(\mathcal{O})$ is type I if and only if*

1) $\mathcal{O}$ is a $\boldsymbol{G}$-orbit;

2) For one $p = (g, \chi) \in \mathcal{O}$, thus for all p, $\boldsymbol{G}_g/\overline{\boldsymbol{G}}_g$ is a finite group.

Proof — If the conditions are satisfied, $\mathcal{F}(\mathcal{O}) = \int^{\oplus} \pi(g,\chi) d\mu_{\mathcal{O}}(g,\chi)$ where $\mathcal{O}$ is a $\boldsymbol{G}$-orbit, so all the representations $\pi(g,\chi)$ are equivalent and type I (by theorem **7.5**). A wellknown result on Hilbert integrals (cf. [16]) say that $\mathcal{F}(\mathcal{O})$ is also a multiple of the same representation $\pi(g,\chi)$.

Conversely, if $\mathcal{F}(\mathcal{O})$ is type I it is a multiple of an irreducible representation σ and another wellknown fact on Hilbert integrals (cf. [16] Th. 2.7 p. 201) say that $\pi(g,\chi)$ is type I for almost all $(g,\chi) \in \mathcal{O}$ and factorial representations multiple of σ. We deduce from this that the measure $\mu_{\mathcal{O}}$ is concentrated on a single $\boldsymbol{G}$-orbit Ω and the second condition is verified. Now, if $\mathcal{O}$ contains an orbit $\Omega' \neq \Omega$, there exists $a \in J$ such that $a.\Omega = \Omega'$ and $a.\mu_{\mathcal{O}}$ would be a $\boldsymbol{G}$-invariant measure on $\mathcal{O}$ concentrated on Ω' hence not a multiple of $\mu_{\mathcal{O}}$. By theorem **7.2** this is not possible and this shows that $\mathcal{O}$ is a single orbit. ∎

We now state without proof some others important results of L. Pukanszky about these representations $\mathcal{F}(\mathcal{O})$ for $\mathcal{O} \in \mathcal{R}(\mathfrak{g}^*)$.

Theorem 7.4 — *Let $\boldsymbol{G}$ be a simply connected solvable Lie group.*

1) There exists a positive Borel measure μ on a Borel space X (the space of classes of factorial representations of $\boldsymbol{G}$) such that

$$\lambda \simeq \int_X^{\oplus} \mathcal{F}(\mathcal{O}_x) d\mu(x)$$

where λ is the left regular representation of $\boldsymbol{G}$ and except on a μ-null set the representations $\mathcal{F}(\mathcal{O}_x)$ and $\mathcal{F}(\mathcal{O}_y)$ are not equivalent for $x \neq y$. This is a weak form of the Plancherel formula (cf. [20]).

2) If we denote by $\mathcal{F}(\mathcal{O})^*$ the representation of the C^*-algebra of $\boldsymbol{G}$, corresponding to $\mathcal{F}(\mathcal{O})$ and by $Ker(\mathcal{F}(\mathcal{O})^*)$ its kernel, the map $\mathcal{O} \longrightarrow Ker(\mathcal{F}(\mathcal{O})^*)$ is one to one from the set of generalized orbits onto the space of primitive ideals or kernels of irreducible representations of the C^*-algebra of $\boldsymbol{G}$ (cf. [22]).

3) The representations $\mathcal{F}(\mathcal{O})$ are semi-finite, this means that they are type I or type II (cf. [21]).

Exercise 7.6 – Let $\boldsymbol{G}$ be the Mautner group with the notations of the begining of this section. Let $g = e_3^* + e_5^*$, let $\mathcal{O}$ be its generalized orbit and let $\boldsymbol{H}$ be the connected group with Lie algebra $\mathfrak{h} = [\mathfrak{g}, \mathfrak{g}] = \oplus_{i=2}^5 \mathbb{R} e_i$. Show that

$$\mathcal{F}(\mathcal{O}) = \int_{\mathbb{R}} \int_0^{2\pi} \int_0^{2\pi} {}^{\oplus} \operatorname{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}} (\chi_{g_{\beta,\alpha_0,\alpha_1}}) d\beta d\alpha_0 d\alpha_1$$

where $g_{\beta,\alpha_0,\alpha_1} = \beta e_1^* + \sin \alpha_1 e_2^* + \cos \alpha_1 e_3^* + \sin \alpha_0 e_4^* + \cos \alpha_0 e_5^*$.

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References

[1] L. Auslander, C.C. Moore, Mem. of the A.M.S. **62**.

[2] L. Auslander, B. Kostant, *Polarizations and representations of solvable Lie groups*, Invent. Math. **14** (1971).

[3] P. Bernat & Al., Représentations des groupes de Lie Résolubles, Dunod, Paris **4** (1972).

[4] N. Bourbaki, *Act. Sc. et Ind. Hermann ; Paris*, Integration, Chap. VII §**2** (1963).

[5] L. Corwin, F. Greenleaf, Representations of nilpotent Lie groups and applications. Part 1 ; Cambridge studies in advanced mathematics **18** (1991).

[6] J. Dixmier, *Groupes de Lie Nilpotents III*, Canadian Journal of Mathematics **10** (1958).

[7] J. Dixmier, C*−Algèbres, Gauthier-Villars, Paris (1964).

[8] J. Dixmier, *Sur le revêtement universel d'un groupe de Lie de type I*, C.R. Acad. Sc. Paris **252** (1961), 2805-2806.

[9] M. Duflo, M. Raïs, *Sur l'analyse harmonique sur les groupes de Lie résolubles*, Annales Scient. de l'E.N.S 4ème série, **t. 9** (1976), 107-144.

[10] E.G. Effros, *Transformation Groups and* $\mathbb{C}^*-$*algebras*, Ann. of Math. **81** (1965), 38-55.

[11] J. Glimm, *Locally compact transformation groups*, Trans. Amer. Math. Soc. **101** (1961), 124-138.

[12] S. Helgason, Differential Geometry and Symetric Spaces. Academic Press. (1962).

[13] G. Hochschild, The structure of Lie groups. Holden-Day, San Franscisco, London Amsterdam. (1965).

[14] A.A. Kirillov, Éléments de la théorie des représentations. M.I.R. Moscou. (1972).

[15] G.W. Mackey, *Unitary representations of group extensions*, Acta Math. **99** (1958), 265-311.

[16] G.W. Mackey, *The theory of unitary group representations*, Chicago Lectures in Math. (1976).

[17] C.C. Moore, *Representations of solvable and nilpotent Lie Groups*, Proc. of the Williamstown conference on harmonic analysis (1972).

[18] N.V Pedersen, *Geometric quantization and nilpotent Lie groups ; a collection of examples*, Preprint, University of Copenhagen, Denmark (1988).

[19] L. Pukanszky, *On unitary representations of exponential groups*, Functional Analysis **3** (1969), 73-113.

[20] L. Pukanszky, *Unitary representations of solvable Lie groups*, Annales Scient. de l'E.N.S **4** (1971), 464-608.

[21] L. Pukanszky, *Characters of connected Lie groups*, Acta Math. (Upsala) **133** (1974), 81-137.

[22] L. Pukanszky, *The primitive ideal space of solvable Lie groups*, Invent. Math. **22** (1973), 75-118.

[23] M. Raïs, *Représentations des groupes de Lie nilpotents et méthode des orbites, Cours du CIMPA (Chap. 5)* (1983).

[24] P. Torasso, *Representations of nilpotent Lie groups*, Cours de DEA, Prépublication de l'université de Poitiers (1991).