SCATTERING THEORY FOR THE WIGNER EQUATION

HASSAN EMAMIRAD AND PHILIPPE ROGEON

ABSTRACT. *We prove that the Wigner equation is well-posed in $L^p(\mathbb{R}^{2n})$ for some potential V. From the formalism established by Markovich, we show the completeness of wave operators for the Wigner equation in L^2. Using estimations proved by Castella and Perthame on the one hand, and the $L^p \to L^q$ estimations for the Schrödinger group on the other hand, we prove the existence of the wave operators in $L^{2,p}$ spaces.*

1. INTRODUCTION

Let us denote by $H_0 := -\frac{\hbar^2}{2}\Delta$ and $H := -\frac{\hbar^2}{2}\Delta + V$ the free and perturbed hamiltonian operators in $L^p(\mathbb{R}^n)$ $(1 \le p < \infty)$, where $\hbar$ is the Planck's constant. If φ is the solution of the correspondent Schrödinger equation

$$\text{(Sch)} \qquad \begin{cases} \mathrm{i}\hbar \dfrac{\partial \varphi}{\partial t} = H\varphi \\ \varphi(x,0) = \varphi_0(x) \end{cases}$$

then the Wigner transformation of φ, denoted

$$w := W_\varphi(x,\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}\xi\cdot y} \varphi\left(x + \frac{\hbar y}{2}\right) \overline{\varphi}\left(x - \frac{\hbar y}{2}\right) \,\mathrm{d}y$$

will satisfy the Wigner, or quantum Liouville equation

$$\text{(Wi1)} \qquad \begin{cases} \frac{\partial w}{\partial t} + \xi \cdot \nabla_x w - P_\hbar(x, \nabla_\xi) w = 0 \\ w(x,\xi,0) = w_0 = W_{\varphi_0}. \end{cases}$$

In this equation $P_\hbar$ is a pseudo-differential operator defined either in symbolic form

$$P_\hbar(x, \nabla_\xi) = \frac{\mathrm{i}}{\hbar} \left[V(x + \mathrm{i}\frac{\hbar}{2}\nabla_\xi) - V(x - \mathrm{i}\frac{\hbar}{2}\nabla_\xi) \right] \tag{1.1}$$

or by

$$P_\hbar(x, \nabla_\xi) w = k_\hbar \underset{\xi}{*} w$$

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with

$$k_\hbar(x,\xi) = (2\pi)^{-n/2}\mathcal{F}_y\left[\frac{1}{\mathrm{i}\hbar}\left[V\left(x+\frac{\hbar}{2}y\right) - V\left(x-\frac{\hbar}{2}y\right)\right]\right](\xi).$$

In (Wi1), the operator $L_\hbar := L_0 + P_\hbar(x,\nabla_\xi)$ is a bounded perturbation of $L_0 := -\xi\cdot\nabla_x$ whenever the perturbation $P_\hbar$ is bounded . It's in particular true if the potential $V \in L^\infty$ (see (2.3)). For more general consequence, see Theorem 4.3 below. It is well-known that L_0 generates a C_0-group of isometries in any $L^p(\mathbb{R}^{2n})$. The existence of the Wigner C_0-semigroup $\mathrm{e}^{tP_\hbar}$ on $L^2(\mathbb{R}^{2n})$ has been studied by Markovich and Ringhofer in [11], and on $L^1(\mathbb{R}^{2n})$ by Emamirad and Rogeon [6].

In a Banach space $(X, \|.\|)$ the scattering theory connects the asymptotic behaviour of the advection problem

$$(AP) \qquad \begin{cases} \dfrac{\partial u}{\partial t} & = L_0 u \\ u(0) & = u_0 \end{cases}$$

to those of (Wi1), which can be written in abstract form

$$(AW) \qquad \begin{cases} \dfrac{\partial w}{\partial t} & = L_\hbar w \\ w(0) & = w_0 \end{cases}$$

The study of the asymptotic behavior, in the past and the future for these two problems, leads to the following question in X: for an initial data, $w_0^\pm$ for (AW), is there an initial data u_0 for (AP) such that

$$\lim_{t\to\pm\infty}\left\|u(t) - w^\pm(t)\right\| = 0\ ?$$

Since e^{tL_0} is isometric on X, the question reduces to the existence of the *wave operators*, defined by

$$W_\pm(L_0, L_\hbar) = s - \lim_{t\to\pm\infty}\mathrm{e}^{-tL_0}\mathrm{e}^{tL_\hbar} \tag{1.2}$$

In fact if these operators exist, then for any $w_0^\pm \in X$ and $u_0^\pm = W_\pm(L_0, L_\hbar)w_0^\pm$, we would obtain :

$$\left\|\mathrm{e}^{tL_0}u_0^\pm - \mathrm{e}^{tL_\hbar}w_0^\pm\right\| = \left\|u_0^\pm - \mathrm{e}^{-tL_0}\mathrm{e}^{tL_\hbar}w_0^\pm\right\| \to 0$$

as $t \to \pm\infty$.

In [10], P. A. Markowich has treated the scattering problem in $L^2(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ and he showed the existence of of th wave operators $W_\pm(L_\hbar, L_0)$ by proving the equivalence between the Schrödinger and the Wigner, or quantum Liouville, equations. In this paper Markowich constructs a unitary transformation which leads back (Wi1) to a new Schrödinger type equation in $L^2(\mathbb{R}^{2n})$, then he deduced the existence of the wave operators via those of Schrödinger equation.

In a Hilbert space $L^2(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ we have to replace (1.2) by

$$W_\pm(L_0, L_\hbar) = s - \lim_{t\to\pm\infty}\mathrm{e}^{-tL_0}\mathrm{e}^{tL_\hbar}P_{ac}(L_\hbar) \tag{1.3}$$

where $P_{ac}(L_\hbar)$ is the orthogonal projection on $\mathcal{H}_{ac}(L_\hbar)$, the absolutely continuous space of $L_\hbar$. In fact it is known that L_0 is a skew-adjoint in $L^2(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ and it is shown in [11] that $P_\hbar$ is bounded skew-symmetric operator, since $D(L_\hbar) = D(L_0) = D(L_0^*) = D(L_\hbar^*)$, $L_\hbar$ is also skew-symmetric and its spectrum lies in i$\mathbb{R}$. This implies that in the Hilbert space $L^2(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ for the vectors $f_\hbar$ out of $\mathcal{H}_{ac}(L_\hbar)$, the operator acting on $f_\hbar$ have an oscillatory behavior and the limit (1.2) might not to exist.

The existence of two other wave operators $W_\pm(L_\hbar, L_0)$ depends on completeness of $W_\pm(L_0, L_\hbar)$. We say that the wave operators $W_\pm(L_0, L_\hbar)$ are *complete* if and only if

$$Im\left((W_\pm(L_0, L_\hbar)\right) = Im\left(P_{ac}(L_0)\right).$$

In [13, p. 19], it is shown that if the operators $W_\pm(L_0, L_\hbar)$ exist. Then they are complete if and only if $W_\pm(L_\hbar, L_0)$ exist. In the next section we complete [10] by proving the completeness of the wave operators.

The scattering theory for the Schrödinger problem in $L^2(\mathbb{R}^n)$ has been widely treated by many authors. We base ourself on the famous work of S. Agmon [1] and we use the assumptions of this work for ensuring the existence and completeness of the wave operators. Our proof is based on a result of Umeda [16] on the completeness of pseudo-differential wave operators.

The scattering theory for Schrödinger equation in $L^p(\mathbb{R}^n)$ is already undertook by many authors (see [18] and the references therein). In general these studies are motivated by nonlinear Shrödinger problems. In the last section we prove that one can obtain the wave operators $W_\pm(L_\hbar, L_0)$ on $L^{2,p}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ by weaken considerably the assumptions on the potential V for $L^2(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$. Our proof is based on the Weyl calculus formula

$$\mathrm{e}^{\mathrm{i}tH_0} V \mathrm{e}^{-\mathrm{i}tH_0} = V(x - \mathrm{i}t\nabla_x) \tag{Wey}$$

(see Lemma 2.2 for the precise definition of this expression) together with Strichartz type estimation due to Castella and Perthame. Some results of this paper are already annonced in a Note [7].

2. Well-posedness of the Wigner problem in $L^p(\mathbb{R}^{2n})$.

Before treating of scattering, we study the well-posedness of the Wigner problem in the $L^1(\mathbb{R}^{2n})$ and $L^2(\mathbb{R}^{2n})$ spaces. Then, by interpolation, we can write an estimation of $P_\hbar(x; \nabla_\xi)f$ in L^p, for $1 \le p \le 2$.

Let us denote by

$$\widehat{f}(\xi) = [\mathcal{F}f](\xi) := \int_{\mathbb{R}^n_x} \mathrm{e}^{-\mathrm{i}x\cdot\xi} f(x)\ \mathrm{d}x$$

the Fourier transform of f and by H^s the Sobolev space

$$H^s(\mathbb{R}^n) = \{u \in S'; \|u\|_{H^s} < \infty\},$$

with the norm

$$\|u\|_{H^s} = \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^s |\widehat{u}(\xi)|^2 \, \mathrm{d}\xi \right)^{1/2}.$$

We recall that, for $u \in H^s(\mathbb{R}^n)$, when $s > n/2$, we have $\widehat{u} \in L^1(\mathbb{R}^n)$, and

$$\|\hat{u}\|_1 \leq C\|u\|_{H^s},$$

with $C = \displaystyle\int_{\mathbb{R}^n} (1+|\xi|^2)^{-s} \, \mathrm{d}\xi$.

Lemma 2.1. *If the potentiel V is in an $H^s(\mathbb{R}^n)$ space, with $s > n/2$, then*

(a) *The operator $P_\hbar \equiv P_\hbar(x, \nabla_\xi)$ is bounded in $L^1(\mathbb{R}^{2n})$, and*

$$\|P_\hbar(x,\nabla_\xi)w\|_1 \leq \frac{\delta}{\hbar}\|w\|_1 \tag{2.1}$$

with $\delta = 2(2\pi)^{-n/2}C\|V\|_{H^s}$.

(b) *The Wigner operator $L_\hbar \equiv L_0 + P_\hbar$ generates a C_0-group $S_\hbar(t) = \mathrm{e}^{tL_\hbar}$ on $L^1(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ satisfying*

$$\|S_\hbar(t)f\|_1 \leq \mathrm{e}^{\delta|t|/\hbar}\|f\|_1 \qquad \forall t \in \mathbb{R}. \tag{2.2}$$

Proof.

(a) We have $P_\hbar(x,\nabla_\xi)w = k_\hbar \underset{\xi}{*} w$ with

$$k_\hbar(x,\xi) = (2\pi)^{-n/2}\mathcal{F}_y\left[\frac{1}{\mathrm{i}\hbar}\left[V\left(x+\frac{\hbar}{2}y\right) - V\left(x-\frac{\hbar}{2}y\right)\right]\right](\xi).$$

But

$$\left[\mathcal{F}_y V\left(x+\frac{\hbar}{2}y\right)\right](\xi) = \frac{2^n}{\hbar^n}\left[\mathcal{F}_z V(x+z)\right]\left(\frac{2\xi}{\hbar}\right)$$

and then

$$\begin{aligned}\left\|\left[\mathcal{F}_y V\left(x+\frac{\hbar}{2}y\right)\right](\xi)\right\|_1 &= \frac{2^n}{\hbar^n}\left\|\left[\mathcal{F}_z V(x+z)\right]\left(\frac{2\xi}{\hbar}\right)\right\|_1 \\ &= \|\left[\mathcal{F}_z V(x+z)\right](\eta)\|_1.\end{aligned}$$

Now we write

$$\begin{aligned}\|P_\hbar(x,\nabla_\xi)w\|_1 &= \|k_\hbar \underset{\xi}{*} w\|_1 \\ &\leq \sup_x\left(\int_{\mathbb{R}^n}|k_\hbar(x,\xi)| \, \mathrm{d}\xi\right)\|w\|_1.\end{aligned}$$

From the fact that

$$\begin{aligned}\sup_x \|k_\hbar(x,\xi)\|_{L^1(\mathbb{R}^n_\xi)} &\leq \frac{2(2\pi)^{-n/2}}{\hbar}\|\mathcal{F}_z V\|_1 \\ &\leq \frac{2(2\pi)^{-n/2}}{\hbar}C\|V\|_{H^s}\end{aligned}$$

we get the estimation (2.1).

(b) The operator $L_\hbar$ is a perturbation of the advection operator L_0. When $V \equiv 0$, the advection equation is regented by the unitary C_0-group S_0, defined by $S_0(t)w(x,\xi) = w(x - \xi t, \xi)$. Since $P_\hbar$ is bounded, from the perturbation theorem by the bounded operators (see [9, 12]), $L_\hbar$ generates a C_0-group of type $(1, \delta/\hbar)$, satisfying (2.2). Consequentely, the problem (Wi1) is well-posed in $L^1(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$. (see [6, 15]).

□

There is a similar result in $L^2(\mathbb{R}^{2n})$

Lemma 2.2. *Suppose that $V \in L^\infty(\mathbb{R}^n)$ is a bounded potential. Then we have*

$$\|P_\hbar(x, \nabla_\xi) f\|_2 \leq \frac{2}{\hbar}\|V\|_\infty \|f\|_2. \tag{2.3}$$

Proof. We denote by V the function $V \in L^\infty(\mathbb{R}^n)$ and the multiplication operator

$$V: \quad \mathcal{S}'(\mathbb{R}^n) \ni f \mapsto V(x)f(x) \in \mathcal{S}'(\mathbb{R}^n).$$

Here we recall Weyl's formula:

$$\mathrm{e}^{\mathrm{i}tH_0} V \mathrm{e}^{-\mathrm{i}tH_0} = V(x - \mathrm{i}t\nabla_x) \tag{2.4}$$

where $H_0 = -1/2\Delta$ and $V(x - \mathrm{i}t\nabla_x)$ is a pseudo-differential operator defined by

$$[V(x - \mathrm{i}t\nabla_x)f](x) := (2\pi)^{-n} \int_{\mathbb{R}^n_\eta} V(x + t\eta)\mathrm{e}^{\mathrm{i}x\cdot\eta}\widehat{f}(\eta)d\eta.$$

For the seek of completness we give also an abridged proof of this formula. Take $V(x) = x_k, 1 \leq k \leq n$ and $f \in \mathcal{S}'$

$$\begin{aligned}
\mathcal{F}[\mathrm{e}^{\mathrm{i}tH_0} V \mathrm{e}^{-\mathrm{i}tH_0} f](\xi) &= \mathrm{e}^{\mathrm{i}t|\xi|^2/2}\mathcal{F}[x_k \mathrm{e}^{-\mathrm{i}tH_0} f](\xi) \\
&= \mathrm{e}^{\mathrm{i}t|\xi|^2/2}[\mathrm{i}\partial_k \mathrm{e}^{-\mathrm{i}t|\xi|^2/2}\widehat{f}(\xi)] \\
&= \mathrm{e}^{\mathrm{i}t|\xi|^2/2}[t\xi_k \mathrm{e}^{-\mathrm{i}t|\xi|^2/2}\widehat{f}(\xi) + \mathrm{i}\mathrm{e}^{-\mathrm{i}t|\xi|^2/2}(\widehat{-\mathrm{i}x_k f})(\xi)] \\
&= \mathcal{F}[(x_k - it\partial_k)f](\xi)
\end{aligned}$$

Hence for the vector x we retrieve

$$\mathrm{e}^{\mathrm{i}tH_0} x \mathrm{e}^{-\mathrm{i}tH_0} = x - \mathrm{i}t\nabla_x$$

Now by using the Weyl calculus formula (see [14, p. 294] for a generalization of this formula) we get (2.4) for any bounded function V.

Now, as $\mathrm{e}^{tL_0}V(x) = V(x - t\xi)$, we can write

$$\begin{aligned}
\mathrm{e}^{-L_0}\mathrm{e}^{\mathrm{i}tH_{0,\xi}}\mathrm{e}^{L_0}V(x)\mathrm{e}^{-\mathrm{i}tH_{0,\xi}} &= \mathrm{e}^{-L_0}V\left(x - (\xi - \mathrm{i}t\nabla_\xi)\right) \\
&= V(x + \mathrm{i}t\nabla_\xi)
\end{aligned}$$

where $H_{0,\xi} = -1/2\Delta_\xi$. Taking $t = \pm\hbar/2$, we get

$$\begin{aligned}
P_\hbar(x, \nabla_\xi) = \frac{\mathrm{i}}{\hbar}[&\mathrm{e}^{-L_0}\mathrm{e}^{-\mathrm{i}\frac{\hbar}{2}H_{0,\xi}}\mathrm{e}^{L_0}V(x)\mathrm{e}^{\mathrm{i}\frac{\hbar}{2}H_{0,\xi}} \\
&- \mathrm{e}^{-L_0}\mathrm{e}^{\mathrm{i}\frac{\hbar}{2}H_{0,\xi}}\mathrm{e}^{L_0}V(x)\mathrm{e}^{-\mathrm{i}\frac{\hbar}{2}H_{0,\xi}}]
\end{aligned} \tag{2.5}$$

and we obtaint the result, since e^{tL_0} et $e^{itH_{0,\xi}}$ are unitary. □

Remark 2.3. Although that this Lemma implies that $L_\hbar$ generates a C_0-group $S_\hbar(t)$ with $\|S_\hbar(t)\|_2 \leq e^{(\frac{2}{\hbar}\|V\|_\infty)|t|}$, but this group is unitary in $L^2(\mathbb{R}^{2n})$. In fact as we have mentioned in Introduction by virtue of a result of [11], which uses the boundedness of V, $L_\hbar$ is skew-sysmmetric operator. We can even obtain this result as in [10] for unbounded potentials such as

$$V = V_1 + V_2, \quad V_1 \in L^\infty(\mathbb{R}^n), \quad V_2 \in L^p(\mathbb{R}^n), p > \max\{\frac{n}{2}, 2\},$$

or

$$V = V_1 + V_2, \quad V_1 \in L^\infty(\mathbb{R}^n), \quad V_2 \in L^2_{loc}(\mathbb{R}^n), V_2(x) \geq -\alpha|x|^2 - \beta.$$

with these assumptions the Hamilton operator H become essentially self-adjoint and by a technique which will be developped in the next chapter, Markowich proves that $L_\hbar$ is unitary equivalent of a skew-adjoint operator.

Now we can establish, by interpolation, an estimation of $P_\hbar$ on $L^p(\mathbb{R}^{2n})$ spaces, for $1 \leq p \leq 2$.

Proposition 2.4. *Let $V \in H^s \cap L^\infty(\mathbb{R}^n)$, with $s > n/2$. Then for all $p \in [1,2]$, and all $f \in L^p(\mathbb{R}^{2n})$, $P_\hbar f \in L^p(\mathbb{R}^{2n})$ and*

$$\|P_\hbar f\|_p \leq C_p \|f\|_p$$

with $C_p = \hbar^{-1}\delta^{\frac{2}{p}-1}2^{2-\frac{2}{p}}\|V\|_\infty^{2-\frac{2}{p}}$ and δ as in lemma 2.1.

Proof. From Lemmas 2.1 and 2.2, for $f \in L^1 \cap L^2(\mathbb{R}^{2n})$, we have

$$\|P_\hbar f\|_1 \leq M_1\|f\|_1 \qquad \text{and} \qquad \|P_\hbar f\|_2 \leq M_2\|f\|_2$$

with $M_1 = \frac{\delta}{\hbar}$ and $M_2 = \frac{2}{\hbar}\,\|V\|_\infty$. Then, by Riesz-Thorin theorem, for any $\alpha \in [0,1]$ we have $P_\hbar f \in L^p(\mathbb{R}^{2n})$, with

$$\|P_\hbar f\|_p \leq c_\alpha \|f\|_p$$

where $p^{-1} = \frac{\alpha}{2} + (1-\alpha) = 1 - \frac{\alpha}{2}$ and $c_\alpha = M_1^{1-\alpha}M_2^\alpha$.
Since $\alpha = 2 - \frac{2}{p}$, so we get

$$\begin{aligned} c_\alpha &= \left(\frac{\delta}{\hbar}\right)^{1-\alpha}\left(\frac{2}{\hbar}\|V\|_\infty\right)^\alpha \\ &= \frac{1}{\hbar}\delta^{\frac{2}{p}-1}[2\|V\|_\infty]^{2-\frac{2}{p}} = C_p. \end{aligned}$$

□

For the estimation of the generated C_0-group in $L^p(\mathbb{R}^{2n})$, we will use the unitary of $S_\hbar(t)$ in $L^2(\mathbb{R}^{2n})$.

Corollary 2.5. *If the potential $V \in H^s(\mathbb{R}^n)$ with $s > n/2$, then the C_0-group $S_\hbar(t)$ is also well defined in $L^p(\mathbb{R}^{2n})$, and we have*

$$\|S_\hbar(t)f\|_p \leq e^{c_p|t|}\|f\|_p \qquad \forall t \in \mathbb{R}.$$

where $c_p = \frac{(2-\frac{2}{p})\delta}{\hbar}$.

Proof. This is an immediate consequence of (2.2) and the Riesz-Thorin theorem.□

3. Completeness of the wave operators in $L^2(\mathbb{R}^{2n})$

In [10], Markowich considered the following unitary transformation

$$\mathcal{C} : L^2(\mathbb{R}^n_x \times \mathbb{R}^n_\eta) \ni g \longmapsto \mathcal{C}g \in L^2(\mathbb{R}^n_r \times \mathbb{R}^n_s)$$

where $\mathcal{C}g(r,s) = g(x,\eta)$, with $r = x + \frac{\eta}{2}$ and $s = x - \frac{\eta}{2}$. He showed that the action of the Fourier transformation on (Wi1) leads back to a new equation of type Schrödinger. He established the

Lemma 3.1. (See [10].) *The problem* (Wi1) *is equivalent to*

$$\text{(Wi2)} \qquad \begin{cases} \frac{\partial}{\partial t} z(r,s,t) - \frac{i}{\hbar}\big[\big(-\hbar^2 \frac{\Delta_r}{2} + V(r)\big) \\ \qquad\qquad - \big(-\hbar^2 \frac{\Delta_s}{2} + V(s)\big)\big] z(r,s,t) = 0 \\ z(r,s,t=0) = z_I(r,s) \end{cases}$$

In order to rewrite this problem in an abstract form we denote by I_r and I_s the identities of $L^2(\mathbb{R}^n_r)$ and $L^2(\mathbb{R}^n_s)$ respectively. Now, if we use the operations $\otimes$ and $\oplus$, defined in the following way: For two linear operators R_r and R_s defined on $L^2(\mathbb{R}^n_r)$ and $L^2(\mathbb{R}^n_s)$

- $(R_r \otimes R_s)(f_1 \otimes f_2)(r,s) = R_r f_1(r) R_s f_2(s)$
- $R_r \oplus R_s = R_r \otimes I_s + I_r \otimes R_s$.

These operations unable us to define

$$Q = -H_r \oplus H_s \tag{3.1}$$

where H_r and H_s are the copies of H on $L^2(\mathbb{R}^n_r)$ and $L^2(\mathbb{R}^n_s)$, we can rewrites the problem (Wi1) under the form

$$\text{(Wi3)} \qquad \begin{cases} i\hbar \frac{\partial}{\partial t} z(r,s,t) = Qz(r,s,t) \\ z(r,s,t=0) = z_I(r,s). \end{cases}$$

The equivalence between the problems (Wi1) and (Wi3) comes from the following lemma.

Lemma 3.2. (See [10].) *If w is solution of* (Wi1)*, then $z = \mathcal{C}\mathcal{F}w$ is solution of* (Wi3)*. Conversely if z is the solution of* (Wi3)*, then $w = \mathcal{F}^{-1}\mathcal{C}^{-1}z$ is the solution of* (Wi1).

Thanks to the above lemma, the similarity between the quantum Liouville operator $L_\hbar$ and Q appears from the relation

$$L_\hbar = \frac{\mathrm{i}}{\hbar}\mathcal{F}^{-1}\mathcal{C}^{-1}Q\mathcal{C}\mathcal{F}. \tag{3.2}$$

We still denote by Q the closure of Q in $L^2(\mathbb{R}^n_r \times \mathbb{R}^n_s)$ and if we impose sufficient conditions on V in order H to be selfadjoint, then from (3.1), Q is also selfadjoint and by (3.2), $L_\hbar$ is skew-adjoint. Hence, the operators iH and iQ generate unitary C_0-groups $\{e^{itH}\}_{t\in\mathbb{R}}$ on $L^2(\mathbb{R}^n)$ and $\{e^{itQ}\}_{t\in\mathbb{R}}$ on $L^2(\mathbb{R}^n_r\times\mathbb{R}^n_s)$, linked by the relation

$$e^{itQ} = e^{-itH_r}\otimes e^{itH_s} \qquad (t\in\mathbb{R}).$$

Similarly, the group generated by the quantum evolution operator is given by

$$\exp(tL_\hbar) = \mathcal{F}^{-1}\mathcal{C}^{-1}\exp\left(\frac{i}{\hbar}tH_r\right)\otimes\exp\left(-\frac{i}{\hbar}tH_s\right)\mathcal{C}\mathcal{F}.$$

There are various conditions on the potential V which ensure the existence and completeness of the wave operators in the case of the Schrödinger problem. We have chosen a reference condition, so we worked with Agmon's potentials to illustrate our purpose. Our goal, in fact, is not to establish a supplementary condition on V, but to justify the transmission of the completeness from a problem to the other.

Theorem 3.3. *Suppose that for an $\varepsilon>0$ and a μ such that $0<\mu<4$, the potentiel V verifies*

$$\text{(Ag)}\qquad \sup_{x\in\mathbb{R}^n}\left[(1+|x|)^{2+2\varepsilon}\int_{|y-x|\le 1}|V(y)|^2|y-x|^{-n+\mu}\,dy\right]<\infty.$$

Then the wave operators

$$W_\pm(H,H_0) = s-\lim_{t\to\pm\infty} e^{itH}e^{-itH_0}$$

exist, and are complete. (see [1]).

For proving the completeness of the wave operators for Wigner equation, we need some preliminary lemmas.

Let A and B be the generators of two C_0-groups.

Lemma 3.4. (see [13, p. 19]) *Suppose that $W_\pm(A,B)$ exist. The following conditions are equivalent*

(a) $W_\pm(A,B)$ *are complete*

(b) $W_\pm(B,A)$ *exist.*

Lemma 3.5. (see [5, p. 476]). *Suppose that $W_\pm(A,B)$ exist and are complete. Then :*

$$\sigma(B)=\sigma_{ac}(B)=\sigma_{ac}(A).$$

Consequently, if the wave operators $W_\pm(A,B)$ exist and are complete, then $W_\pm(B,A)$ are too, and we have

$$\sigma(B)=\sigma_{ac}(B)=\sigma_{ac}(A)=\sigma(A).$$

Now, let us define Q_0 as in (3.1):

$$Q_0 = -(H_0)_r\oplus(H_0)_s$$

where $(H_0)_r$ and $(H_0)_s$ are the copies of H_0 on $L^2(\mathbb{R}^n_r)$ and $L^2(\mathbb{R}^n_s)$.

Proposition 3.6. *The operators Q_0 and L_0 satisfy :*

(a) $\mathcal{H}_{ac}(Q_0) = L^2(\mathbb{R}^{2n})$
(b) $\mathcal{H}_{ac}(L_0) = L^2(\mathbb{R}^{2n})$

For proving this proposition we will use the following result which is proved by T. Umeda.

Lemma 3.7. (see [16].) *Let A be a (pseudo)-differential operator on $L^2(\mathbb{R}^{2n})$, of symbol p, and let*

$$\Xi = \{(r,s) \in \mathbb{R}^{2n}; \nabla p(r,s) = 0\}. \tag{3.3}$$

If $p \in C^1(\mathbb{R}^{2n})$ and if $\mathrm{Mes}(\Xi) = 0$, *then*

$$\mathcal{H}_{ac}(A) = L^2(\mathbb{R}^{2n}).$$

Proof of the Proposition 3.6.

(a) The symbol of $Q_0 = -H_{r,0} \oplus H_{s,0} = \frac{\hbar^2}{2}(\Delta_r - \Delta_s)$ being $\dfrac{\hbar^2}{2}(|r|^2 - |s|^2)$, so $\nabla p(r,s) = \hbar^2(r_1, \dots, r_n, -s_1, \dots, -s_n)$ and from (3.3), $\Xi = \{0\}$, so by applying the Lemma 3.7, we get $\mathcal{H}_{ac}(Q_0) = L^2(\mathbb{R}^{2n})$.
(b) By taking $V \equiv 0$, the relation (3.2) becomes $\hbar L_0 = -\mathcal{F}^{-1}\mathcal{C}^{-1} i Q_0 \mathcal{C}\mathcal{F}$, which implies that $\mathcal{H}_{ac}(L_0) = \mathcal{H}_{ac}(Q_0)$ and (a) yields (b).

□

Theorem 3.8. *Suppose that the potential V has been chosen in such a way that $W_\pm(H, H_0)$ exist and are complete as unitary operators on $L^2(\mathbb{R}^n)$. Then the wave operators $W_\pm(L_\hbar, L_0)$ also exist and are complete, as unitary operators on $L^2(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$, and are related by :*

$$W_\pm(L_\hbar, L_0) = \mathcal{F}^{-1}\mathcal{C}^{-1} W_\pm(H, H_0) \otimes W_\mp(H, H_0) \mathcal{C}\mathcal{F}.$$

Proof. By Proposition 3.6 (b) $P_{ac}(L_0) \equiv Id$. Hence

$$W_\pm(L_\hbar, L_0) = s - \lim_{t \to \pm\infty} \mathrm{e}^{tL_\hbar} \mathrm{e}^{-tL_0} P_{ac}(L_0) = s - \lim_{t \to \pm\infty} \mathrm{e}^{tL_\hbar} \mathrm{e}^{-tL_0}.$$

By hypothesis, we suppose that the operators $W_\pm(H, H_0)$ are complete. Then, by Lemma 3.4, $W_\pm(H_0, H)$ also exist and are complete. Moreover, by Lemma 3.5,

$$\sigma(H_0) = \sigma_{ac}(H_0) = \sigma_{ac}(H) = \sigma(H) = [0, +\infty).$$

This implies that $\mathcal{H}_{ac}(H) = L^2(\mathbb{R}^n)$. Thus we get

$$W_\pm(H_0, H) = s - \lim_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}tH_0} \mathrm{e}^{-\mathrm{i}tH} P_{ac}(H) = s - \lim_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}tH_0} \mathrm{e}^{-\mathrm{i}tH}.$$

Consequently

$$s - \lim_{t \to \pm\infty} \mathcal{F}^{-1}\mathcal{C}^{-1} W_\pm(H_0, H) \otimes W_\mp(H_0, H) \mathcal{C}\mathcal{F}$$

are defined on $L^2(\mathbb{R}^n) \otimes L^2(\mathbb{R}^n)$, and are exactly

$$s - \lim_{t \to \pm\infty} \mathrm{e}^{tL_0} \mathrm{e}^{-tL_\hbar} = W_\pm(L_0, L_\hbar).$$

As $W_\pm(L_0, L_\hbar)$ and $W_\pm(L_\hbar, L_0)$ exist on $L^2(\mathbb{R}^n) \otimes L^2(\mathbb{R}^n)$, these wave operators are complete. □

4. Existence of wave operators in $L^{2,p}(\mathbb{R}^{2n})$ spaces

In this section, we will fix $\hbar = 1$ to simplify the notations. We prove the existence of the wave operators for the Wigner problem in the $L^{2,p}(\mathbb{R}^{2n})$ space for a potential V suitably chosen. Let us keep the notation L_0 for the advection operator, and we use P and L for $P_\hbar$ and $L_\hbar$. We denote the norm on the $L^{r,p}(\mathbb{R}^{2n})$ spaces by

$$\|w\|_{r,p} = \left[\int_{\mathbb{R}^n_x} \left[\int_{\mathbb{R}^n_\xi} |w(x,\xi)|^p \, \mathrm{d}\xi \right]^{\frac{r}{p}} \mathrm{d}x \right]^{\frac{1}{r}}.$$

We first give two preliminary lemmas, in order to prove some estimation for the pseudo-differential operator P, and to establish the existence of the wave operators $W_\pm(L, L_0)$ in $L^{2,p}(\mathbb{R}^{2n})$.

Lemma 4.1. *The free transport operator L_0 generates :*
(a) *a C_0-group of isometies on $L^{r,p}(\mathbb{R}^{2n})$:*

$$\|\mathrm{e}^{tL_0} w\|_{r,p} = \|w\|_{r,p}.$$

(b) *a C_0-group on $L^{r,p}(\mathbb{R}^{2n})$ satisfiyng the following punctual estimation, for all p and r such that $1 \le p \le r \le +\infty$,*

$$\|\mathrm{e}^{tL_0} w\|_{r,p} \le |t|^{-n(\frac{1}{p} - \frac{1}{r})} \|w\|_{p,r}. \tag{4.1}$$

Proof.
(a) Since $\mathrm{e}^{tL_0} w = w(x - t\xi, \xi)$, this equality provides from the change of variable $y = x - t\xi$.
(b) This estimation is due to Castella and Perthame (see [4]), taking in account the results of Bardos and Degond in [2].

□

Lemma 4.2. *For any $p \ge 2$ and $t \ne 0$, the Schrödinger evolution group $\mathrm{e}^{-\mathrm{i}t\Delta}$ satisfies*

$$\|\mathrm{e}^{-\mathrm{i}t\Delta} f\|_{L^p(\mathbb{R}^n)} \le (4\pi|2t|)^{n(\frac{1}{p} - \frac{1}{2})} \|w\|_{L^{p'}(\mathbb{R}^n)},$$

where $p^{-1} + p'^{-1} = 1$. Hence if $w \in L^{r,p}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$

$$\|\mathrm{e}^{-\mathrm{i}tH_{0,\xi}} w\|_{r,p} \le (4\pi|2t|)^{n(\frac{1}{p} - \frac{1}{2})} \|w\|_{r,p'}. \tag{4.2}$$

(see [8])

From the expression (2.5) we can obtain the following results.

Theorem 4.3. (a) *If $V \in L^\infty(\mathbb{R}^n)$, then*

$$\|Pw\|_{2,p} \le 2\|V\|_\infty \|w\|_{2,p}. \tag{4.3}$$

(b) *Suppose* $2 \leq r \leq p$ *and* $s = (p-1)^2(p^2-2p)^{-1}$. *Then, if* $V \in L^s(\mathbb{R}^n_x)$,

$$\|Pw\|_{r,p} \leq 2(4\pi)^{n(\frac{1}{p}+\frac{1}{r}-1)}\|V\|_s\|w\|_{p,r'}. \tag{4.4}$$

Proof. (a) Since e^{tL_0} and $\mathrm{e}^{\mathrm{i}tH_{0,\xi}}$ are the isometric groups on $L^{2,p}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ and on $L^2(\mathbb{R}^n_\xi)$, the first term of the right hand side of (2.5) gives

$$\begin{aligned}\|\mathrm{e}^{-L_0}\mathrm{e}^{-\mathrm{i}/2H_{0,\xi}}\mathrm{e}^{L_0}V(x)\mathrm{e}^{\mathrm{i}/2H_{0,\xi}}w\|_{2,p} &= \|V(x)\mathrm{e}^{\mathrm{i}/2H_{0,\xi}}w\|_{2,p}\\ &\leq \|V\|_\infty\|\mathrm{e}^{\mathrm{i}/2H_{0,\xi}}w\|_{2,p}\\ &= \|V\|_\infty\|w\|_{2,p}.\end{aligned}$$

The same estimation can be obtained for the second term of the right hand side of (2.5), which implies (4.3).
(b) In order to prove (4.4), we will apply the results of lemma 4.1 and lemma 4.2 on the first term of the right hand side of (2.5). First, since e^{tL_0} are isometries on $L^{2,p}(\mathbb{R}^{2n})$, this gives

$$\|\mathrm{e}^{-L_0}\mathrm{e}^{-\mathrm{i}/2H_{0,\xi}}\mathrm{e}^{L_0}V(x)\mathrm{e}^{\mathrm{i}/2H_{0,\xi}}w\|_{r,p} \leq \|\mathrm{e}^{-\mathrm{i}/2H_{0,\xi}}\mathrm{e}^{L_0}V(x)\mathrm{e}^{\mathrm{i}/2H_{0,\xi}}w\|_{r,p}$$

By taking $t = \frac{1}{2}$, in (4.2), we obtain

$$\|\mathrm{e}^{-L_0}\mathrm{e}^{-\mathrm{i}/2H_{0,\xi}}\mathrm{e}^{L_0}V(x)\mathrm{e}^{\mathrm{i}/2H_{0,\xi}}w\|_{r,p} \leq (4\pi)^{n(\frac{1}{p}-\frac{1}{2})}\|\mathrm{e}^{L_0}V(x)\mathrm{e}^{\mathrm{i}/2H_{0,\xi}}w\|_{r,p'}$$

where $p^{-1} + p'^{-1} = 1$.
We now use the estimation (4.1) with $t = 1$ and we get

$$\|\mathrm{e}^{-L_0}\mathrm{e}^{-\mathrm{i}/2H_{0,\xi}}\mathrm{e}^{L_0}V(x)\mathrm{e}^{\mathrm{i}/2H_{0,\xi}}w\|_{r,p} \leq (4\pi)^{n(\frac{1}{p}-\frac{1}{2})}\|V(x)\mathrm{e}^{\mathrm{i}/2H_{0,\xi}}w\|_{p',r}.$$

By taking $\frac{1}{s} = \frac{1}{p'} - \frac{1}{p}$ (i.e. $s = (p-1)^2(p^2-2p)^{-1}$), the Hölder inequality implies that

$$\|\mathrm{e}^{-L_0}\mathrm{e}^{-\mathrm{i}/2H_{0,\xi}}\mathrm{e}^{L_0}V(x)\mathrm{e}^{\mathrm{i}/2H_{0,\xi}}w\|_{r,p} \leq (4\pi)^{n(\frac{1}{p}-\frac{1}{2})}\|V\|_s\|\mathrm{e}^{\mathrm{i}/2H_{0,\xi}}w\|_{p,r}.$$

Using once more (4.2) with $t = -\frac{1}{2}$, we finally get

$$\|\mathrm{e}^{-L_0}\mathrm{e}^{-\mathrm{i}/2H_{0,\xi}}\mathrm{e}^{L_0}V(x)\mathrm{e}^{\mathrm{i}/2H_{0,\xi}}w\|_{r,p} \leq (4\pi)^{n(\frac{1}{p}+\frac{1}{r}-1)}\|V\|_s\|w\|_{p,r'}.$$

The same estimation holds for the second term of the right hand side of (2.5) and therefore (4.4) is established. □

Corollary 4.4. (a) *Since* $\{\mathrm{e}^{tL_0}\}_{t\in\mathbb{R}}$ *is a group of isometries on* $L^{2,p}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$, *if* $V \in L^\infty(\mathbb{R}^n)$, $\{\mathrm{e}^{tL_0}\}_{t\in\mathbb{R}}$ *is also a strongly continuous group on* $L^{2,p}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$.
(b) *Suppose that* $p \geq 2$. *Then for* $\gamma = n(\frac{1}{2}-\frac{1}{p}), K = 2(4\pi)^{-\gamma}$, *we have*

$$\|P\mathrm{e}^{tL_0}w\|_{2,p} \leq K\|V\|_s|t|^{-\gamma}\|w\|_{2,p}. \tag{4.5}$$

Proof. It follows from (4.4) and (4.1) that

$$\begin{aligned}\|P\mathrm{e}^{tL_0}w\|_{r,p} &\leq 2(4\pi)^{n(\frac{1}{p}+\frac{1}{r}-1)}\|V\|_s\|\mathrm{e}^{tL_0}w\|_{p,r'}\\ &\leq 2(4\pi)^{n(\frac{1}{p}+\frac{1}{r}-1)}\|V\|_s|t|^{-n(\frac{1}{r'}-\frac{1}{p})}\|w\|_{r',p}\end{aligned}$$

In the particular case $r = 2$, we have also $r' = 2$ which gives the estimation (4.5). □

Theorem 4.5. *Let $p > 2$, such that $\gamma = n(1/2 - 1/p)$, $K = 2(4\pi)^{-\gamma}$ and $s = (p-1)^2(p^2-2p)^{-1}$. We suppose that the potential $V \in L^\infty \cap L^s(\mathbb{R}^n)$ is chosen such that $2\|V\|_\infty + K\|V\|_s(\gamma-1)^{-1} < 1$. Then the wave operators $W_\pm(L, L_0)$ exist in $L^{2,p}(\mathbb{R}^{2n})$.*

Proof. For the existence of the wave operator $W_+(L, L_0)$ (resp. $W_-(L, L_0)$)it suffices to prove that the assumptions of the Cook's Lemma are fulfilled. That is

(i) The semigroup e^{-tL} (resp. e^{tL}) is uniformly bounded on $[0, \infty)$.

(ii)

$$\int_0^\infty \|Pe^{tL_0}\|_{2,p}\ dt < \infty \qquad (resp.\ \int_{-\infty}^0 \|Pe^{tL_0}\|_{2,p}\ dt < \infty).$$

For proving that e^{tL} is uniformly bounded on $[0, \infty)$, the Duhamel's formula

$$e^{tL} = e^{tL_0} + \int_0^t e^{(t-s)L} Pe^{sL_0}\ ds$$

asserts that a sufficient condition is to have

$$\int_0^\infty \|Pe^{tL_0}\|_{2,p}\ dt < 1. \tag{4.6}$$

According to Theorem 4.3 (a) and Corollary 4.4 (b),

$$\begin{aligned}\int_0^\infty \|Pe^{tL_0}w\|_{2,p}\ dt &= \int_0^1 \|Pe^{tL_0}w\|_{2,p}\ dt + \int_1^\infty \|Pe^{tL_0}w\|_{2,p}\ dt \\ &\le 2\|V\|_\infty \int_0^1 \|e^{tL_0}w\|_{2,p}\ dt + K\|V\|_s \int_1^\infty t^{-\gamma}\|w\|_{2,p}\ dt \\ &\le \left(2\|V\|_\infty + \frac{K\|V\|_s}{\gamma-1}\right)\|w\|_{2,p},\end{aligned}$$

the condition (4.6) is satisfied. This gives simultaneously the assumption (ii) of the Cook's Lemma for the existence of $W_+(L, L_0)$. The boundedness of e^{-tL} on $[0, \infty)$ and the condition $\int_{-\infty}^0 \|Pe^{tL_0}\|_{2,p}\ dt < \infty$ can be justified in a similar manner. □

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Laboratoire de Mathématiques, Université de Poitiers. teleport 2, BP 179, 86960 Chassneuil du Poitou, Cedex, France
E-mail address: emamirad@math.univ-poitiers.fr

Centre de Recherches sur l'Intégration Economique et Financière, Université de Poitiers. 93 Avenue du Recteur Pineau, 86022 POITIERS Cedex, France
E-mail address: philippe.rogeon@univ-poitiers.fr