APPROXIMATION IN THE SENSE OF KATO FOR TRANSPORT PROBLEM

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Abstract. *By using Chernoff's Theorem, we prove that an approximation of the family $\{S(t) : t \geq 0\}$ given by* (3.5) *converges in the sense of Kato to transport semigroup.*

1. Introduction.

Let us recall the Chernoff's Theorem as it is given in [Che].

Theorem 1.1. *Let X be a Banach space and $\{V(t)\}_{t\geq 0}$ be a family of contractions on X with $V(0) = I$. Suppose that the derivative $V'(0)f$ exists for all f in a set $\mathcal{D}$ and the closure Λ of $V'(0)\mid_{\mathcal{D}}$ generates a C_0–semigroup $S(t)$ of contractions. Then, for each $f \in X$,*

$$\lim_{n\to\infty} \|V(\frac{t}{n})^n f - S(t)f\| = 0, \tag{1.1}$$

uniformly for t in compact subsets of $\mathbb{R}_+$.

In the next section we will use the Chernoff's theorem to prove the following result.

Theorem 1.2. *Let A be the generator of a C_0-semigroup $S_0(t)$ such that $\|S_0(t)\| \leq \mathrm{e}^{-\omega t}$ $(\omega \geq 0)$, and B a bounded perturbation operator such that $\|B\| < \omega$, and $A + B$ defined in the $D(A)$ generates a C_0-semigroup $S(t)$ of contractions. Then, the conclusion of* (1.1) *holds for $V(t) := S_0(t) + \int_0^t S_0(s)Bds$.*

Proof. We remark that $V(0) = I$, $V'(0)f = (A + B)f$ for all $f \in D(A)$ and finally $V(t)$ is of contraction. In fact,

$$\begin{aligned}\|V(t)\| &\leq \|S_0(t)\| + \|\int_0^t S_0(s)Bds\| \\ &\leq \mathrm{e}^{-\omega t} + b\int_0^t \mathrm{e}^{-\omega s}ds = \left(1 - \frac{b}{\omega}\right)\mathrm{e}^{-\omega t} + \frac{b}{\omega} \leq 1,\end{aligned}$$

where $b = \|B\|$. Since all the assumptions of Theorem 1.1 are fulfilled, the conclusion infers from this Theorem. □

In the next section, we define the convergence in the sense of Kato. Finally in the last section we construct the approximation spaces convergence in the sense of Kato

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and we prove that an approximating family of operators constructed by mean of $V(t)$ in transport problem converges in the sense of Kato to the solution of this problem. This gives a new look to the transport processes given by J. Hejtmanek in [Hej]. In fact, Hejtmanek used this processes only to Euler approximation of transport equation, but we will show in our forthcoming paper that this processes can be applied not only to Euler schemes but also to Crank-Nicolson and Predictor-Corrector algorithms.

2. Convergence in the sense of Kato

In this paper we give an approximation processus for transport equation not only in time but also in space. For approximation in space we have to recall the convergence in the sense of Kato (see [Kat]). We say that a sequence of Banach spaces $\{(X_n, \|.\|_n) : n = 1, 2, \cdots\}$ *converges* to a Banach space $(X, \|.\|)$ *in the sense of Kato* and we write

$$X_n \xrightarrow{K} X$$

if for any n there is a linear operator $P_n \in \mathcal{L}(X, X_n)$ (called an *approximating* operator) satisfying the following two conditions:

(K1) $\lim_{n\to\infty} \|P_n f\|_n = \|f\|$ for any $f \in X$;

(K2) for any $f_n \in X_n$, there exists $f^{(n)} \in X$ such that $f_n = P_n f^{(n)}$ with $\|f^{(n)}\| \leq C\|f_n\|_n$ (C is independent of n).

Let $X_n \xrightarrow{K} X$, $B_n \in \mathcal{L}(X_n)$ and $B \in \mathcal{L}(X)$. We say that B_n *converges* to B *in the sense of Kato* and we write $B_n \xrightarrow{K} B$ if $\lim_{n\to\infty} \|B_n P_n f - P_n B f\|_n = 0$ for any $f \in X$. Let A_n and A be the generators of the C_0–semigroups $\{T_n(t)\}_{t\geq 0} \subseteq \mathcal{L}(X_n)$ and $\{T(t)\}_{t\geq 0} \subseteq \mathcal{L}(X)$, respectively. Consider the following three conditions:

(A) (**Consistency**). There is a complex number λ contained in the resolvent sets $\bigcap_{n\in\mathbb{N}} \rho(A_n)$ and $\rho(A)$, respectively, such that

$$(\lambda - A_n)^{-1} \xrightarrow{K} (\lambda - A)^{-1}.$$

(B) **(Stability)**. There exists a positive constant M and a real number ω such that

$$\|T_n(t)\| \leq Me^{\omega t}, \quad \text{for any } t \geq 0 \quad \text{and} \quad \text{for any } n \in \mathbb{N}.$$

(C) **(Convergence)**. For any finite $T > 0$

$$T_n(t) \xrightarrow{K} T(t)$$

uniformly on $[0, T]$, i.e.

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|T_n(t)P_n f - P_n T(t) f\|_n = 0 \quad \text{for any } f \in X. \tag{2.1}$$

In [Ush] one can retrieve the standard version of the Lax equivalence theorem which says that the conditions **(A)** and **(B)** hold if and only if **(C)** holds.

3. Approximation of transport equation

Here, we consider a matter of particles, constituted of neutrons, electrons, ions and photons. Each particle moves on a straight line with constant velocity until it collides with other particle of the supporting medium resulting in absorption, scattering or multiplication. The unknown of the transport equation is the particle density function $u(\mathbf{x}, \mathbf{v}, t)$. This is a function in the phase space $(\mathbf{x}, \mathbf{v}) \in \Omega \times V \subset \mathbb{R}^{2n}$ at the time $t \geq 0$, which belongs to its natural space $X = L^1(\Omega, V)$. Actually, $\int_{\Omega\times V} u(\mathbf{x}, \mathbf{v}, t)\mathrm{d}x\mathrm{d}v$ designates the total number of particles in the whole space $\Omega \times V$ at the time t. The general form of the transport problem is the following

$$(\mathbf{TP}) \qquad \begin{cases} \frac{\partial u}{\partial t} = -\mathbf{v} \cdot \nabla u & -\sigma(\mathbf{x}, \mathbf{v})u \\ & + \int_V p(\mathbf{x}, \mathbf{v}', \mathbf{v})u(\mathbf{x}, \mathbf{v}', t)d\mathbf{v}' \qquad \text{in } \ \Omega \times V, \\ u(\mathbf{x}, \mathbf{v}, t) = 0 & \text{if } \ \mathbf{x} \cdot \mathbf{v} < 0, \quad \text{for all } \ \mathbf{x} \in \partial\Omega \\ u(\mathbf{x}, \mathbf{v}, 0) \ \ = & f(\mathbf{x}, \mathbf{v}) \in X, \end{cases}$$

In this equation which is known as linear Boltzmann equation the first term of the right hand side $-\mathbf{v} \cdot \nabla u(\mathbf{x}, \mathbf{v}, t)$ illustrates the movement of the classical group of the particles in the absent of the absorption and production interactions. The second term represents the lost of the particles caused by the diffusion or absorption at the point $(\mathbf{x}, \mathbf{v})$ in the phase space. Finally the integral of the last term represents the production of the particles at the point $(\mathbf{x}, \mathbf{v})$ in the phase space. The kernel $p(\mathbf{x}, \mathbf{v}', \mathbf{v})$ in this integral generates the transition of the states of particles at the position $\mathbf{x}$ and having the velocity $\mathbf{v}'$ to the particles at the same position having the velocity $\mathbf{v}$. The velocity space V is in general a spherical shell in $\mathbb{R}^n$ as

$$V = \{\mathbf{v} \in \mathbb{R}^n \ \mid \ 0 \leq v_{min} \leq |\mathbf{v}| \leq v_{max} \leq +\infty\}.$$

In this paper we deal with a particular feature of the transport equation in which we replace Ω, with $(-a, a)$ and we take $V := [-1, 1]$. We assume that σ is a strictly positive continuous function with

$$0 < s_m \leq \sigma(x) \leq s_M \text{ for almost any } x \in (-a, a) \tag{3.1}$$

and the kernel $p(x, v, v')$ by $\frac{1}{2}p(x)$ which is a positive continuous function independent of (v, v'), such that

$$0 < \sup_{x\in[-a,a]} p(x) \leq k_M. \tag{3.2}$$

With these assumptions the transport problem **(TP)** can be replaced by the following particular one

$$(\mathbf{TP1}) \qquad \begin{cases} \frac{\partial u}{\partial t} = -v \cdot \nabla u - \sigma(x)u + \frac{1}{2}\int_{-1}^{1} p(x)u(x, v, t)dv \text{ in } \ (-a, a) \times [-1, 1]; \\ u(-a, v \geq 0, t) = 0 \text{ and } u(a, v \leq 0, t) = 0 \text{ for all } \ t > 0; \\ u(x, v, 0) = f(\mathbf{x}, v) \in L^1((-a, a) \times [-1, 1]). \end{cases}$$

Remark 3.1. If we denote the production term $Af = \frac{1}{2}\int_{-1}^{1} p(x)f(x,v)dv = p(x)Pf$, with

$$Pf = \frac{1}{2}\int_{-1}^{1} f(x,v)dv, \tag{3.3}$$

which is a projection on $L^1((-a,a)\times[-1,1])$. This space being generating we get $\|P\| = 1$, and $\|A\| = k_M$. Since $\|A\| \leq k_M$ and for the constant function $p(x) = k_M$ we get the equality.

Theorem 3.2. *In the Banach space $X = L^1((-a,a)\times[-1,1])$ let us define the operators $T_0 f := -v\partial f/\partial x$, $T_1 f := T_0 f - \sigma(x)f$, $\widetilde{T}f := T_0 f + Af$ and $Tf := T_1 f + Af$ (A being defined in Remark 3.1). Any of these operators defined on $D(T_0) := \{f \in X : v\partial f/\partial x \in X,\ f(-a, v \geq 0) = 0 \text{ and } f(a, v \leq 0) = 0\}$ generates a C_0-semigroup which is given respectively by:*

(0) *$U_0(t)$ which are of contractions;*

(1) *$U_1(t)$ with $\|U_1(t)\| \leq \mathrm{e}^{-s_m t}$;*

(2) *$V(t)$ with $\|V(t)\| \leq \mathrm{e}^{k_M t}$;*

(3) *$U(t)$ with $\|U(t)\| \leq \mathrm{e}^{(k_M - s_m)t}$.*

Proof. **(0).** For $t > 0$ such that, $|x - tv| < a$, the semigroup $U_0(t)f(x,v) = f(x - tv, v)$, satisfies $\|U_0(t)f\| = \|f\|$ and if $x - tv < -a$ or $x - tv > a$, then $U_0(t)f(x,v) = 0$.
(1). The C_0-semigroup generated by T_1 is

$$[U_1(t)f](\mathbf{x},\mathbf{v}) := \mathrm{e}^{-\int_0^t \sigma(\mathbf{x}-s\mathbf{v})ds} f(\mathbf{x} - t\mathbf{v}, \mathbf{v}) \tag{3.4}$$

and

$$\int_{-a}^{a}\int_{-1}^{1} |[U_1(t)f](\mathbf{x},\mathbf{v})|dxdv \leq \mathrm{e}^{-ts_m}\int_{-a}^{a}\int_{-1}^{1} |f(\mathbf{x} - t\mathbf{v}, \mathbf{v})|dxdv.$$

(2). For $V(t)$ we will use the Dyson-Phillips formula:

$$V_0(t) = U_0(t), \qquad V(t) := \sum_{n=0}^{\infty} V_n(t),$$

where

$$V_{n+1}(t) = \int_0^t V_0(t-s)AV_n(s)ds.$$

Suppose that $\|V_n(s)\| \leq (k_M s)^n/n!$, then by induction we get

$$\begin{aligned}\|V_{n+1}(t)f\| &\leq \int_0^t \|V_0(t-s)AV_n(s)f\|ds \\ &\leq \int_0^t \|AV_n(s)f\|ds \leq \int_0^t k_M \frac{(k_M s)^n}{n!}\|f\|ds \\ &= \frac{(k_M s)^{n+1}}{(n+1)!}\|f\|.\end{aligned}$$

in which we have used Remark 3.1. Consequently,

$$\|V(t)\| \leq \sum_{n=0}^{\infty} \|V_n(t)\| \leq \sum_{n=0}^{\infty} \frac{(k_M t)^n}{n!} = \mathrm{e}^{k_M t}.$$

(3). We argue as in **(2)**, but we replace the Dyson-Phillips formula by $U(t) := \sum_{n=1}^{\infty} U_n(t)$ and we deduce by induction for $\|U_{n+1}(t)\| \leq \mathrm{e}^{-ts_m}(k_M t)^n/n!$ that

$$\|U(t)\| \leq \sum_{n=1}^{\infty} \|U_n(t)\| \leq \sum_{n=1}^{\infty} \mathrm{e}^{-ts_m} \frac{(k_M t)^{n-1}}{(n-1)!} = \mathrm{e}^{(k_M - s_m)t}.$$

□

Let us define the approximating spaces X_n in this special case. We divide the phase space $(-a, a) \times [-1, 1]$ into a finite number of cells by chopping the x interval $(-a, a)$ into $2m_n$ equal parts and the v interval $[-1, 1]$ into $2\mu_n$ equal parts, h_n and k_n are the length of these parts, that is,

$$h_n = \frac{a}{m_n}, \qquad k_n = \frac{1}{\mu_n}.$$

Then each cell can be labeled by a pair of integers $(i, j) \in \mathcal{N}$,where

$$\mathcal{N} := \{(i, j) : \ i = -m_n, \cdots, -1, 0, 1, \cdots, m_n.\ j = -\mu_n, \cdots, -1, 0, 1, \cdots, \mu_n\}.$$

The number of the particles in cell $\gamma(i, j) = [ih_n, (i+1)h_n] \times [jk_n, (j+1)k_n]$ is written $\xi_{i,j}$.

We define the set of all real vectors $\xi_{i,j}$ as the Banach space X_n with the norm

$$\xi \in X_n, \qquad \|\xi\|_n = \sum_{i,j} |\xi_{i,j}|.$$

At this point let us prove that the approximating space X_n converges in the sense of Kato to X. In fact, from property of the positive cone X_+ of L^1 it follows that

Lemma 3.3. *For $P_n f = \{\xi_{i,j} \ : \ (i, j) \in \mathcal{N}\}$ where*

$$\xi_{i,j} = \int_{ih_n}^{(i+1)h_n} \int_{jk_n}^{(j+1)k_n} f(x, v)dxdv,$$

we have $\|P_n f\|_n = \|f\|$.

Proof. For every $f(x, v) \geq 0$, we get

$$\|P_n f\|_n = \sum_{i,j} \int_{ih_n}^{(i+1)h_n} \int_{jk_n}^{(j+1)k_n} f(x, v)dxdv = \|f\|.$$

Since $L^1((-a, a) \times [-1, 1])$ is generated by its positive cone X_+, that is $f = f_+ - f_-$ with $f_\pm \geq 0$. So we get also

$$\|P_n f\|_n = \|P_n(f_+ - f_-)\|_n = \|P_n f_+\|_n + \|P_n f_-\|_n = \|f_+\| + \|f_-\| = \|f\|.$$

□

The condition **(K1)** follows from 3.3 and for the condition **(K2)** we denote $\chi_{i,j}$ the characteristic function of the cell $\gamma(i,j)$ and for any $\{\xi_{i,j}\} \in X_n$ we define $f^{(n)} \in X$ as $f^{(n)}(x) = \sum_{i,j} \frac{\xi_{i,j}}{h_n k_n} \chi_{i,j}$ and we have

$$\int_{(-a,a)\times[-1,1]} |f^{(n)}(x)| dxdv \leq \sum_{i,j} \int_{\gamma(i,j)} \left| \frac{\xi_{i,j}}{h_n k_n} \chi_{i,j} \right| dxdv = \sum_{i,j} |\xi_{i,j}|,$$

since $\int_{\gamma(i,j)} \frac{\chi_{i,j}}{h_n k_n} dxdv = 1$.

In this section we consider the system **(TP1)**, with the notation of Remark 3.1, $Af = pPf$, where P is the projection defined in (3.3).

Here, we do not have at our disposition an explicit expression of the semigroup as $U_0(t)f(x,v) = f(x - tv, v)$ or $U_1(t)f(x,v) = \mathrm{e}^{-\int_0^t \sigma(x-sv)ds} f(x - tv, v)$, but we can introduce the operator

$$\begin{aligned}
[V(t)f](x,v) &:= \mathrm{e}^{-\int_0^t \sigma(x-sv)ds} f(x - tv, v) \\
&\quad + \frac{1}{2} \int_0^t \mathrm{e}^{-\int_0^s \sigma(x-rv)dr} p(x - sv) \int_{-1}^1 f(x - sv, v')dv'ds \qquad (3.5) \\
&= U_1(t)f + \int_0^t U_1(s)pPfds = U_1(t)f + \int_0^t U_1(s)Afds. \qquad (3.6)
\end{aligned}$$

The operator $V(t)$ is not himself a semigroup as $U_0(t)$ or $U_1(t)$, but it can act as the operator function $V(t)$ in Chernoff's theorem (Theorem 1.1).

We approximate this operator by

$$U_n(k\tau_n) := U_{1,n}(t)(I + \tau_n A_n)^k \qquad (3.7)$$

(see Remarks 3.5 **(a)**), where

$$[A_n \xi]_{i,j} := \frac{k_n p_i}{2} \sum_{l=-\mu_n}^{\mu_n - 1} \xi_{i,l} \qquad \forall j, \quad -\mu_n \leqslant j \leqslant \mu_n - 1. \qquad (3.8)$$

with $p_i = p(\theta),\ \theta \in [ih_n, (i+1)h_n)$.

Now, let $U(t)$ be the transport semigroup defined in Theorem 3.2, then we have

Theorem 3.4. *Under the assumption $2k_M < s_m$, we have the convergence of $U_n(t)$ to $U(t)$ in the sense of Kato.*

Proof. We have to prove that

$$\|U_n(t)P_n f - P_n U(t)f\|_n \to 0, \qquad (3.9)$$

as $n \to \infty$.

First we prove that

$$U_n(k\tau_n)P_n f = P_n V(\tau_n)^k f. \qquad (3.10)$$

In fact,

$$\begin{aligned}
P_nV(\tau_n)f &= P_n\Big[\mathrm{e}^{-\int_0^{\tau_n}\sigma(x-sv)ds}f(x-\tau_n v,v)\\
&\qquad+\frac{1}{2}\int_0^{\tau_n}\mathrm{e}^{-\int_0^{s}\sigma(x-rv)dr}p(x-sv)\int_{-1}^{1}f(x-sv,v')dv'ds\Big]\\
&=\exp(-\tau_n\sigma_{i-j})\xi_{i-j,j}+\frac{k_n\tau_n}{2}p_{i-j}\mathrm{e}^{-\tau_n\sigma_{i-j}}\sum_{l=-\mu_n}^{\mu_n-1}\xi_{i-j,l}\\
&=\left[U_{1,n}(\tau_n)(I+\tau_nA_n)\xi\right]_{i,j}\\
&=U_{1,n}(\tau_n)(I+\tau_nA_n)P_nf=U_n(\tau_n)P_nf.
\end{aligned}$$

Hence, by taking $g=V(\tau_n)f$, we obtain

$$P_nV(\tau_n)^2f=P_nV(\tau_n)g=U_n(\tau_n)P_ng=U_n(\tau_n)^2P_nf,$$

and by induction we retrieve (3.10). Once the identity (3.10) is proven, we replace $U_n(t)P_nf$ by $P_nV(\tau_n)^nf$ in (3.9) and we use the isometric character of P_n (see Lemma 3.3), then we get

$$\|U_n(t)P_nf-P_nU(t)f\|_n=\|V(t/n)^nf-U(t)f\|.$$

Now, if $\omega=s_m-k_M$, thanks to Theorem 3.2 **(3)**, $U(t)$ satisfies $\|U(t)\|\le \mathrm{e}^{-\omega t}$, and since $2k_M<s_m$, we get $k_M<\omega$. So we can replace in Theorem 1.2, $S_0(t)$ by $U_1(t)$ and B by the production operator A, the formula (3.6) show that we can use this Theorem to prove that (3.9) holds. □

Remark 3.5. (a) *After replacing the integral $\int_0^t\sigma(ih_n-sjk_n)ds$ by $\boldsymbol{\sigma}^{(n)}_{i,j}$, where*

$$\boldsymbol{\sigma}^{(l)}_{i,j}:=\tau_n\sum_{k=1}^{l}\sigma(ih_n-jk\tau_nk_n). \tag{3.11}$$

The approximation of U_1 given by (3.4) *would be*

$$U_{1,n}(t)=\exp\left(-\boldsymbol{\sigma}^{(n)}_{i,j}\right)f(ih_n-nj\tau_nk_n,jk_n),$$

where $\sigma_{i-kj}=\sigma(h_n(i-kj))$. Replacing $f(ih_n-jn\tau_nk_n,jk_n)$ by $\xi_{i-nj,j}$ as before we get

$$[U_{1,n}(t)\xi]_{i,j}=\exp\left(-\boldsymbol{\sigma}^{(n)}_{i,j}\right)\xi_{i-nj,j}. \tag{3.12}$$

So

$$\left[U_{1,n}(\tau_n)\xi\right]_{i,j}=\mathrm{e}^{-\tau_n\sigma_{i-j}}\xi_{i-j,j}.$$

(b) We note that by taking $k=n$, $U_n(t)$ given in (3.7), can be written as

$$U_n(t)=U_{1,n}(t)\left(\sum_{k=0}^{n}C_n^k(\tau_nA_n)^k\right).$$

Hence

$$[U_n(t)\xi]_{i,j} = [U_{1,n}(t)\xi]_{i,j} + U_{1,n}(t)\left(\sum_{k=1}^{n} C_n^k(\tau_n p_i)^k\right)\frac{k_n}{2}\sum_{l=-\mu_n}^{\mu_n-1}\xi_{i,l}.$$

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