On explicit representation and approximations of Dirichlet-to-Neumann semigroup

H. Emamirad & M. Sharifitabar

Contents

Online First
Immediately Online
www.springerlink.com

Springer

On explicit representation and approximations of Dirichlet-to-Neumann semigroup

H. Emamirad · M. Sharifitabar

Abstract In his book (Functional Analysis, Wiley, New York, 2002), P. Lax constructs an explicit representation of the Dirichlet-to-Neumann semigroup, when the matrix of electrical conductivity is the identity matrix and the domain of the problem in question is the unit ball in $\mathbb{R}^n$. We investigate some representations of Dirichlet-to-Neumann semigroup for a bounded domain. We show that such a nice explicit representation as in Lax book, is not possible for any domain except Euclidean balls. It is interesting that the treatment in dimension 2 is completely different than other dimensions. Finally, we present a natural and probably the simplest numerical scheme to calculate this semigroup in full generality by using Chernoff's theorem.

Keywords Dirichlet-to-Neumann operator · Lax's Dirichlet-to-Neumann semigroup · γ-harmonic lifting

H. Emamirad (✉) · M. Sharifitabar
School of Mathematics, Institute for Research in Fundamental Sciences (IPM),
P.O. Box 19395-5746, Tehran, Iran
e-mail: emamirad@ipm.ir

M. Sharifitabar
e-mail: sharifitabar@ipm.ir

H. Emamirad
Laboratoire de Mathématiques, Université de Poitiers, Teleport 2, BP 179, 86960 Chassneuil du
Poitou Cedex, France
e-mail: emamirad@math.univ-poitiers.fr

Springer

1 Introduction

We consider a bounded smooth domain $\Omega \subset \mathbb{R}^n$, and $\gamma(x)$ an $n \times n$ symmetric matrix with smooth (enough) real elements which has uniformly bounded positive eigenvalues, i.e., there exist $0 < c_1 < c_2$ such that for every $x \in \Omega$ and $\xi \in \mathbb{R}^n$,

$$c_1 \|\xi\|^2 \leq \xi^T \gamma(x)\xi \leq c_2 \|\xi\|^2.$$

This matrix is known as the electrical conductivity. Let $X := L^2(\Omega)$ or $C(\overline{\Omega})$ and let the corresponding boundary space be $\partial X := L^2(\partial\Omega)$ or $C(\partial\Omega)$. We solve the following Dirichlet problem

$$\begin{cases} \nabla \cdot (\gamma \nabla u) = 0, & \text{in } \Omega, \\ u = f, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

For any $f \in \partial X$, we write $u = L_\gamma f$. Such a function is called the γ-harmonic lifting of f and L_γ the γ-harmonic lifting operator. The function u represents the electrical potential where this potential on the surface of the substance is f and the substance is in electrical equilibrium. Now define the action of the generalized Dirichlet-to-Neumann operator on f as the normal outward derivative of u on the boundary, i.e.,

$$\Lambda_\gamma f := \nu \cdot \gamma \nabla u.$$

In other words, we define $\Lambda_\gamma := \nu \cdot \gamma \nabla L_\gamma$ where $\nu(y)$ is the unit outer normal vector at $y \in \partial\Omega$, and Λ_γ is called Dirichlet-to-Neumann operator. In the simplest case where γ is the identity matrix, we denote these operators by L_0, the harmonic lifting operator, and Λ_0, the corresponding Dirichlet-to-Neumann operator.

We consider Λ_γ as an unbounded operator on ∂X with the domain,

$$D(\Lambda_\gamma) = \{f \in \partial X : \Lambda_\gamma f \in \partial X\}.$$

See [1, 6, 9] for various properties of this domain. The operator $-\Lambda_\gamma$ generates an analytic compact semigroup of contractions on these spaces, $C(\partial\Omega)$ and $L^2(\partial\Omega)$. This semigroup can be identified as the trace of the solution to the following problem with dynamical boundary condition,

$$\begin{cases} \nabla \cdot (\gamma \nabla u(t, \cdot)) = 0, & \text{for every } t \in \mathbb{R}^+, \text{ in } \Omega, \\ \partial_t u + \nu \cdot \gamma \nabla u = 0, & \text{for every } t \in \mathbb{R}^+, \text{ on } \partial\Omega, \\ u(0, \cdot) = f, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

In fact, by taking the trace of this solution (see [1] for uniqueness) and denoting it by $u(t, x)\big|_{\partial\Omega}$, we can also define the Dirichlet-to-Neumann semigroup as,

$$e^{-t\Lambda_\gamma} f := u(t, x)\big|_{\partial\Omega}, \quad \text{for every } f \in \partial X.$$

It is easy to see that the $L^2(\partial\Omega)$ version of Λ_γ is selfadjoint and nonpositive, and hence the generator of an analytic semigroup of maximal angle $\pi/2$. Further properties such as contraction, compactness, positivity, irreducibility and Markov character

of $e^{-t\Lambda_\gamma}$ can be found in [9] and [5]. In [7], J. Escher showed that the $C(\overline{\Omega})$ version of Λ_γ generates also an analytic semigroup of some positive angle θ.

In the sequel, we are specially interested in the case where γ is the identity matrix I_n, so u will satisfy the following system,

$$\begin{cases} \Delta u(t, \cdot) = 0, & \text{for every } t \in \mathbb{R}^+, \text{ in } \Omega, \\ \partial_t u + v \cdot \nabla u = 0, & \text{for every } t \in \mathbb{R}^+, \text{ on } \partial\Omega, \\ u(0, \cdot) = f, & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

In this case K.-J. Engel [6] showed that $\theta = \pi/2$ even for the $C(\overline{\Omega})$ version of Λ_γ. In his book [8, 36.2], P. Lax got the same result under the additional assumption that Ω is the unit ball in $\mathbb{R}^n$. But Lax also had a simple proof and a new explicit representation of this semigroup. The Lax semigroup is defined by,

$$\left(e^{-t\Lambda_0} f\right)(x) = (L_0 f)\left(e^{-t}x\right). \tag{1.4}$$

The main advantage of this representation is that, it is only necessary to solve the problem

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ u\big|_{\partial\Omega} = f, & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

in order to calculate the action of the semigroup in all times.

Here, two questions arise:

Question 1 Does Lax's representation hold for Ω not a ball?

Question 2 Is there an extension of Lax's ideas for γ not a multiple of the identity and for Ω not a ball that give an explicit representation of Dirichlet-to-Neumann semigroup?

We give a negative answer to the Question 1 in the two next sections. Concerning Question 2, this question remains open, however we will find a natural and good approximation to the Dirichlet-to-Neumann semigroup in his whole generality. In fact, by using the Chernoff's theorem we find an approximating family to this semigroup with some motivations in numerical calculations in the last section.

2 Optimality of Lax representation in $\mathbb{R}^n$, $n > 2$

In this section, we try to find a representation to this semigroup as is represented in [8]. Here we take $\gamma = I_n$. we begin by proving the following lemma.

Lemma 2.1 *Let $\Psi : \Omega \to \Omega$ be a function of class C^2. Then Ψ has the property that $u \circ \Psi$ is harmonic for all harmonic functions u, if and only if $\Psi(x) = Ax + b$, where the matrix A is a multiple of an orthogonal matrix and b is a constant vector, provided that $n > 2$.*

Proof Let $(\Psi_1, \ldots, \Psi_n)$ be the components of Ψ and suppose that $\Delta(u \circ \Psi) = 0$ whenever $\Delta u = 0$. Taking $u(x) = x_i$, $u(x) = x_i x_j$ and $u(x) = x_i^2 - x_j^2$ for $1 \leq i < j \leq n$ results in

$$\Delta \Psi_i = 0, \tag{2.1}$$

$$\nabla \Psi_i \cdot \nabla \Psi_j = 0, \tag{2.2}$$

$$|\nabla \Psi_i|^2 - |\nabla \Psi_j|^2 = 0. \tag{2.3}$$

The properties (2.2) and (2.3) are equivalent to say that the matrix $D\Psi$ is multiple of an orthogonal matrix, i.e. $D\Psi D\Psi^T = |\det D\Psi|^{2/n} I_n$. This is equivalent to say that the mapping Ψ is conformal (see [2]), i.e. the mapping Ψ preserves the angle between transversal curves. Note that this equation easily implies that $D\Psi$ is invertible or $D\Psi = 0$.

Taking the gradient of (2.2) and (2.3) yields that,

$$D^2 \Psi_i \nabla \Psi_j + D^2 \Psi_j \nabla \Psi_i = 0, \tag{2.4}$$

$$D^2 \Psi_i \nabla \Psi_i - D^2 \Psi_j \nabla \Psi_j = 0, \tag{2.5}$$

and we mean the Hessian matrix by D^2. Now for i, j, k distinct, we multiply (2.4) by $\nabla \Psi_k$ and use the same equation for other indices to obtain

$$
\begin{aligned}
0 &= \nabla \Psi_k \cdot D^2 \Psi_i \nabla \Psi_j + \nabla \Psi_k \cdot D^2 \Psi_j \nabla \Psi_i \\
&= \nabla \Psi_j \cdot D^2 \Psi_i \nabla \Psi_k + \nabla \Psi_i \cdot D^2 \Psi_j \nabla \Psi_k \\
&= -\nabla \Psi_j \cdot D^2 \Psi_k \nabla \Psi_i - \nabla \Psi_i \cdot D^2 \Psi_k \nabla \Psi_j \\
&= -2\nabla \Psi_i \cdot D^2 \Psi_k \nabla \Psi_j.
\end{aligned}
$$

Therefore we have $\nabla \Psi_i \cdot D^2 \Psi_k \nabla \Psi_j = 0$. Again we multiply (2.4) by $\nabla \Psi_j$ and using (2.5) yields,

$$
\begin{aligned}
0 &= \nabla \Psi_j \cdot D^2 \Psi_i \nabla \Psi_j + \nabla \Psi_j \cdot D^2 \Psi_j \nabla \Psi_i \\
&= \nabla \Psi_j \cdot D^2 \Psi_i \nabla \Psi_j + \nabla \Psi_i \cdot D^2 \Psi_j \nabla \Psi_j \\
&= \nabla \Psi_j \cdot D^2 \Psi_i \nabla \Psi_j + \nabla \Psi_i \cdot D^2 \Psi_i \nabla \Psi_i.
\end{aligned}
$$

Hence if we look at the matrix $D\Psi D^2 \Psi_i D\Psi^T$, all of its entries are zero except on i-th row and column and also on its main diagonal and the trace of this matrix is $(2 - n)\nabla \Psi_i \cdot D^2 \Psi_i \nabla \Psi_i$. But since $\text{tr} D^2 \Psi_i = \Delta \Psi_i = 0$ and $D\Psi$ is an orthogonal matrix times a scalar, the trace of $D\Psi D^2 \Psi_i D\Psi^T$ is also zero. We conclude that $\nabla \Psi_i \cdot D^2 \Psi_i \nabla \Psi_i = 0$ which implies $D\Psi D^2 \Psi_i D\Psi^T = 0$. Hence $D^2 \Psi_i = 0$ and therefore we must have $\Psi_j(x) = a_j \cdot x + b_j$ which completes the proof.

The converse is almost obvious. Since

$$\Delta\big(u(Ax + b)\big) = \nabla \cdot \big(A^T \nabla u(Ax + b)\big) = \sum_{1 \leq i,j,k \leq n} a_{ji} a_{ki} \frac{\partial^2}{\partial x_j \partial x_k} u(Ax + b),$$

and A being a multiple of an orthogonal matrix, so $\sum_i a_{ji} a_{ki} = c^2 \delta_{jk}$. Furthermore $\sum_j \frac{\partial^2}{\partial x_j^2} u = 0$ and consequently we have $\Delta(u(Ax+b)) = 0$. $\qquad\square$

Remark 2.1 In two dimensions, the properties (2.2) and (2.3) automatically imply (2.1). In fact Ψ being conformal implies that the components of Ψ are harmonic functions. Hence in two dimensions we have Ψ or its conjugate is an analytic function.

Now let us hope that we may find $\Phi(t, \cdot) : \overline{\Omega} \to \overline{\Omega}$ such that $(L_0 f)(\Phi(t, x))$ is the solution to the system (1.3), where $\Phi(t, \cdot)$ is a smooth function for all $t \geq 0$ and also they are independent of the choice of initial condition f, i.e. the family $\Phi(t, \cdot)$ depends only on the geometry of Ω.

Theorem 2.2 *Let $\Phi(t, x)$ be as above and $u(x)$ be the solution of* (1.5). *In order that $u(\Phi(t, y))$, $y \in \partial\Omega$, to be the Dirichlet-to-Neumann semigroup, it is necessary* (*and also sufficient*) *that Ω is a ball in $\mathbb{R}^n$.*

Remark 2.2 Note that in this theorem we have no assumption on the dimension of space. The case $n > 2$ is proved here and we postpone the proof for $n = 2$ to the next section.

Proof for $n > 2$ For a fixed $t \in \mathbb{R}^+$, the previous lemma states that $\Phi(t, x) = A(t)x + b(t)$. On the other hand, if $u = L_0 f$, then $u \circ \Phi$ must satisfy the second equation of the system (1.3). But,

$$\partial_t (u \circ \Phi) = \partial_t \Phi \cdot \nabla u \circ \Phi,$$

$$v \cdot \nabla(u \circ \Phi) = v \cdot \mathrm{D}\Phi^T \nabla u \circ \Phi = \mathrm{D}\Phi v \cdot \nabla u \circ \Phi.$$

Hence,

$$(\partial_t \Phi + \mathrm{D}\Phi v) \cdot \nabla u \circ \Phi = 0.$$

Now note that this equation is satisfied by every harmonic function u and taking $u(x) = x_i$ results in,

$$\partial_t \Phi + \mathrm{D}\Phi v = 0, \quad \text{on } \partial\Omega. \tag{2.6}$$

This implies $A'(t)y + b'(t) + A(t)v(y) = 0$. Since $\Phi(0, y) = y$, we have $A(0) = I_n$ and we conclude at least near $t = 0$ that $v(y) = -A(t)^{-1} A'(t) y - A(t)^{-1} b'(t)$. Since the left hand side is independent of the time, we have $v(y) = By + c$. Since $\|v(y)\| = 1$, this shows that $\partial\Omega$, at least locally, is a level set of $\|Bx + c\|^2$. Therefore v is in the direction of the gradient of $\|Bx + c\|^2$. But this gradient is calculated as,

$$\nabla\left((x^T B^T + c^T) \cdot (Bx + c)\right) = \nabla\left(x^T B^T Bx + 2c^T Bx\right)$$

$$= 2B^T Bx + 2B^T c$$

$$= 2B^T (Bx + c).$$

We conclude that for every $y \in \partial\Omega$, the vector $v(y)$ is an eigenvector of B^T and this may happen only when B^T is a multiple of the identity. Hence $v(y) = ky + c$ and again $\|v(y)\| = 1$ implies that the domain Ω is a ball of center $-k^{-1}c$ and radius k^{-1}. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The above argument shows that we may have an explicit representation of Dirichlet-to-Neumann semigroup as in [8], only when we are on a sphere for $n > 2$. We will see in the following section that this is also true for $n = 2$.

3 Optimality of Lax representation in $\mathbb{R}^2$

In the case $n = 2$, we consider the mappings as maps on the subdomains of complex plane, i.e if $h : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$, we consider it as $h(x + iy) = \begin{bmatrix} 1 \\ i \end{bmatrix} \cdot h(x, y)$. We saw in our analysis that $\Phi(t, \cdot)$ or its conjugate should be an analytic map and this is enough to satisfy the first equation of (1.2). Since $\Phi(0, z) = z$, the conjugate case is ruled out. Also we may write the condition (2.6) in complex sense as,

$$\partial_t \Phi + v \partial_z \Phi = 0, \quad \text{on } \partial\Omega.$$

Since $\partial_z \Phi(0, \cdot) \neq 0$, for small enough $t > 0$ we have $v = -\partial_t \Phi / \partial_z \Phi$ which means that the function $v(z)$ on $\partial\Omega$ is extendible to an analytic function inside Ω. Assuming that $g(z)$ is the analytic extension of v inside Ω, we have

$$\begin{cases} \partial_t \Phi + g(z)\partial_z \Phi = 0, & \text{in } \Omega, \\ \Phi(0, z) = z. \end{cases} \qquad (3.1)$$

So $\Phi(t, z) = z$ wherever $g(z) = 0$ and this can happen at finite number of points. Outside these finite possible points, we can locally find a primitive function for $1/g$, i.e. $\frac{d}{dz}G = 1/g$ and since $\frac{d}{dz}G \neq 0$, this function has local inverses and the solution to (3.1) may be written down as

$$\Phi(t, z) = G^{-1}\big(-t + G(z)\big).$$

It is worth mentioning here that the system (3.1) is equivalent to the following:

$$\begin{cases} \partial_t \Phi + g(\Phi) = 0, & \text{in } \Omega, \\ \Phi(0, z) = z. \end{cases}$$

So the problem is reduced to the problem of extending $v(z)$ and we are done by finding such extension.

This procedure is used in [4] and we reuse it to recover the Lax semigroup. Since in the case of the unit ball we have $v(x, y) = (x, y)$, so $g(z) = z$ is the desired extension and therefore $G(z)$ can be chosen as any branch of $\log z$ and we have $G^{-1}(z) = \exp(z)$ and,

$$\Phi(t, z) = \exp(-t + \log z) = e^{-t}z.$$

Fig. 1 The ellipse $x^2 + b^2 y^2 = 1$
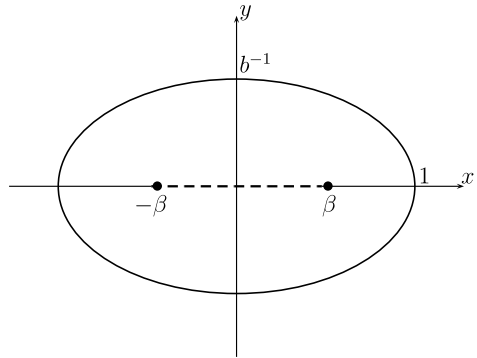
In order to understand what type of pathology appears if we choose $\partial\Omega$ different of a circle, let us consider Ω to be an ellipse in $\mathbb{R}^2$, where its boundary $\partial\Omega$ can be written as $x^2 + b^2 y^2 = 1$, $b > 1$ (see Fig. 1). Rewriting this equation in complex form, we obtain,

$$(1 - b^2)\bar{z}^2 + 2(1 + b^2)z\bar{z} + (1 - b^2)z^2 - 4 = 0. \tag{3.2}$$

Hence we may solve $\bar{z}$ as an analytic function of z around the boundary,

$$\bar{z} = \frac{(1 + b^2)z - 2b\sqrt{z^2 - \beta^2}}{b^2 - 1}, \quad \beta^2 = 1 - b^{-2}. \tag{3.3}$$

Note that by removing the interval $(-\beta, \beta)$ from the real line, we may choose an analytic branch for $\sqrt{z^2 - \beta^2}$. Now,

$$
\begin{aligned}
v^2 &= \left(\frac{x + ib^2 y}{\|x + ib^2 y\|} \right)^2 \\
&= \frac{((1 - b^2)\bar{z} + (1 + b^2)z)^2}{(z + \bar{z})^2 - b^4(z - \bar{z})^2} \\
&= \frac{4b^2(z^2 - \beta^2)}{(1 - b^4)(z^2 + \bar{z}^2) + 2(1 + b^4)z\bar{z}} \quad \text{(by using (3.3))} \\
&= \frac{4b^2(z^2 - \beta^2)}{(1 + b^2)(4 - 2(1 + b^2)z\bar{z}) + 2(1 + b^4)z\bar{z}} \quad \text{(by using (3.2))} \\
&= \frac{b^2(z^2 - \beta^2)}{1 + b^2 - b^2 z\bar{z}} \\
&= \frac{(1 - b^2)(z^2 - \beta^2)}{(1 - b^2)(b^{-2} + 1) + (1 + b^2)z^2 - 2bz\sqrt{z^2 - \beta^2}} \quad \text{(by using (3.3))} \\
&= \frac{(1 - b^2)(z^2 - \beta^2)}{(1 + b^2)(z^2 - \beta^2) - 2bz\sqrt{z^2 - \beta^2}}
\end{aligned}
$$

$$= \frac{1 - b^2}{1 + b^2 - 2bz/\sqrt{z^2 - \beta^2}}.$$

The denominator equals to zero exactly at $z = \pm(b + b^{-1})/(b - b^{-1})$ which are real numbers with absolute values greater than 1. Hence we may write,

$$\nu^{-1} = \sqrt{\frac{1 + b^2 - 2bz/\sqrt{z^2 - \beta^2}}{1 - b^2}}, \tag{3.4}$$

and we must choose the suitable square root such that ν is the outward normal and this definition may be extended inside the ellipse except on the real interval $(-\beta, \beta)$ where $\sqrt{z^2 - \beta^2}$ is undefined. In fact if we try to extend ν inside the domain, there exist two singularities at the points $z = \pm\beta$ (in the sense that $g(\pm\beta) = 0$) and we will also have a discontinuity along a curve which connects these two points.

Now we investigate the extension problem. Let $g(z)$ be a local analytic extension of $\nu(z)$ on the boundary, i.e. $g(z)$ is defined and analytic in a neighborhood U of $z_0 \in \partial\Omega$ and is identical to $\nu(z)$ along the boundary of Ω. Assume that $\zeta = \alpha + i\beta : I \subset \mathbb{R} \to \partial\Omega$ be a local parametrization of speed one, i.e. $\|\zeta'(t)\| = 1$. Now $g = w + iv$ being holomorphic says that,

$$w_x = v_y, \qquad w_y = -v_x, \quad \text{in } U. \tag{3.5}$$

Unitarity of the vector ν implies,

$$w(\zeta(t))^2 + v(\zeta(t))^2 = 1, \quad \text{for all } t \in I, \tag{3.6}$$

and the normality of the vector ν to the boundary says that,

$$w(\zeta(t))\alpha'(t) + v(\zeta(t))\beta'(t) = 0, \quad \text{for all } t \in I. \tag{3.7}$$

Differentiating (3.6) with respect to t gives,

$$\begin{aligned}
0 &= w(\zeta)(w_x(\zeta)\alpha' + w_y(\zeta)\beta') \\
&\quad + v(\zeta)(v_x(\zeta)\alpha' + v_y(\zeta)\beta') \\
&= w(\zeta)(w_x(\zeta)\alpha' + w_y(\zeta)\beta') \\
&\quad + v(\zeta)(-w_y(\zeta)\alpha' + w_x(\zeta)\beta') \quad \text{(by using (3.5))} \\
&= w_x(\zeta)(w(\zeta)\alpha' + v(\zeta)\beta') \\
&\quad + w_y(\zeta)(w(\zeta)\beta' - v(\zeta)\alpha') \\
&= w_y(\zeta)(w(\zeta)\beta' - v(\zeta)\alpha') \quad \text{(by using (3.7))}.
\end{aligned}$$

Equation (3.7) together with this result imply that $w_y(\zeta)\overline{g(\zeta)}\zeta' = 0$. Since both $g(\zeta)$ and ζ' has modulus 1, we conclude $w_y(\zeta) = 0$, i.e. $\partial\Omega$ is locally the 0-level set of the harmonic function w_y. Now we are in position to state the following theorem.

Theorem 3.1 *Let Ω be a bounded smooth subdomain of $\mathbb{R}^2$ and $v(z)$ is the normal outer unit vector on the boundary of Ω, considered as a complex valued map. One can extend v as an analytic function inside the domain Ω, if and only if Ω is a ball.*

Proof If we want to do the previously mentioned extension globally on whole Ω, then $w_y = 0$ on $\partial\Omega$. But w_y is harmonic, so w_y equals to zero in all of Ω which means that $g(z) = kz + c$, $k \in \mathbb{R}$, and this implies that $\partial\Omega = \{z \in \mathbb{C} : |g(z)| = |kz + c| = 1\}$ is the circle of center $-k^{-1}c$ and radius k^{-1}. Note that this also completes the proof of Theorem 2.2 for $n = 2$. $\qquad\qquad\square$

Remark 3.1 In [5, Theorem 5.4] it is proved that there exists a probability measure μ such that

$$\lim_{t \to \infty} e^{-t\Lambda_0} f = \int_{\partial\Omega} f(\sigma) d\mu(\sigma). \qquad (3.8)$$

So if such representation exists, then $\Phi(t, y)$ must converge to some point $\tilde{y}$ and $u(\tilde{y}) = \int_{\partial\Omega} f(\sigma) d\mu(\sigma)$. But such a point may not exist except when the domain is a ball and property (3.8) is the well-known formula

$$u(0) = \frac{1}{\text{meas}(S^{n-1})} \int_{S^{n-1}} u(\sigma) d\sigma.$$

4 An approximating family

Here we are going to approximate the Dirichlet-to-Neumann semigroup by means of Chernoff's Theorem. Let us recall this Theorem which is proved in [3].

Theorem 4.1 (Chernoff's product formula) *Let X be a Banach space and $\{V(t)\}_{t \geq 0}$ be a family of contractions on X with $V(0) = I$. Suppose that the derivative $V'(0)f$ exists for all f in a set $\mathcal{D}$ and the closure Λ of $V'(0)|_{\mathcal{D}}$ generates a C_0-semigroup $S(t)$ of contractions. Then, for each $f \in X$,*

$$\lim_{n \to \infty} V\left(\frac{t}{n}\right)^n f = S(t)f,$$

uniformly for t in compact subsets of $\mathbb{R}^+$.

This procedure was done in [5] by choosing the approximating family as $V(t)f = u(e^{t\gamma}x)$, where $\|x\| = 1$ and $\Omega = \{x \in \mathbb{R}^n : \|x\| < 1\}$. In this section we find a natural approximation to the generalized Dirichlet-to-Neumann semigroup.

Now let $0 < \alpha \leq 1$ be a parameter. We may construct an approximating family as follows. Since $v \cdot \gamma v \geq c_1$ everywhere on $\partial\Omega$, there exist $T > 0$ such that for all $t \leq T$ and every $x \in \partial\Omega$, we have $x - t\gamma(x)v(x) \in \overline{\Omega}$. Now define,

$$V(t)f(x) = \begin{cases} (1-\alpha)u(x) + \alpha u(x - \alpha^{-1}t\gamma(x)v(x)), & 0 \leq t \leq \alpha T, \\ V(\alpha T)f(x), & t > \alpha T, \end{cases} \qquad (4.1)$$

where $u = L_\gamma f$ and $x \in \partial\Omega$.

Obviously $V(0)f = f$ and maximum principle yields that $V(t)$ is a contraction of $C(\partial\Omega)$. Also we have,

$$V'(0)f = -\nu \cdot \gamma \nabla u = -\Lambda_\gamma f.$$

Now we apply Chernoff theorem to conclude,

$$e^{-t\Lambda_\gamma} f = \lim_{n\to\infty} V\left(\frac{t}{n}\right)^n f.$$

This result has a motivation in numerical point of view. Let us fix two parameters Δt and Δx and try to discretize the semigroup equation in a finite difference explicit scheme. The result would be the following,

$$\frac{u^{(j+1)}(x) - u^{(j)}(x)}{\Delta t} + \frac{u^{(j)}(x) - u^{(j)}(x - \Delta x \gamma(x)\nu(x))}{\Delta x} = 0.$$

Equivalently,

$$u^{(j+1)}(x) = \left(1 - \frac{\Delta t}{\Delta x}\right)u^{(j)}(x) + \frac{\Delta t}{\Delta x}u^{(j)}\left(x - \frac{\Delta x}{\Delta t}\Delta t \gamma(x)\nu(x)\right).$$

So in fact $u^{(j+1)} = V(\Delta t)u^{(j)}$ with the choice of parameter $\alpha = \Delta t/\Delta x$. We conclude that the finite difference explicit scheme is convergent provided that $\Delta t \leq \Delta x$. In the simplest case $\Delta t = \Delta x$, the recursion formula is reduced to $u^{(j+1)}(x) = u^{(j)}(x - \Delta t \gamma(x)\nu(x))$. This result has a significant effect on numerical calculations because one only need to solve the lifting problem recursively and there is no need to deal directly with the operator Λ_γ and specially its resolvent.

References

1. Arendt, W., Ter Elst, A.F.M.: The Dirichlet-to-Neumann operator on rough domains. arXiv:1010.1703 (2010)
2. Blair, D.E.: Inversion Theory and Conformal Mapping. American Mathematical Society, Providence (2000)
3. Chernoff, P.R.: Note on product formulas for operator semigroups. J. Funct. Anal. **2**, 238–242 (1968)
4. deLaubenfels, R.: Well-behaved derivations on $C[0\,1]$. Pac. J. Math. **115**, 73–80 (1984)
5. Emamirad, H., Laadnani, I.: An approximating family for the Dirichlet-to-Neumann semigroup. Adv. Differ. Equ. **11**, 241–257 (2006)
6. Engel, K.-J.: The Laplacian on $C(\overline{\Omega})$ with generalized Wenzell boundary conditions. Arch. Math. **81**, 548–558 (2003)
7. Escher, J.: The Dirichlet-Neumann operator on continuous functions. Ann. Sc. Norm. Super. Pisa **21**, 235–266 (1994)
8. Lax, P.D.: Functional Analysis. Wiley, New York (2002)
9. Vrabie, I.I.: C_0-Semigroups and Applications. North-Holland, Amsterdam (2003)