SEMCLASSICAL LIMIT OF HUSIMI FUNCTION

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Abstract. We will show that Liouville and quantum Liouville operators \( L \) and \( L_\hbar \) generate two \( C_0 \)-groups \( e^{tL} \) and \( e^{tL_\hbar} \) of isometries in \( L^2(\mathbb{R}^{2n}) \) and \( e^{tL_\hbar} \) converges ultraweakly to \( e^{tL} \). As a consequence we show that the Gaussian mollifier of the Wigner function, called Husimi function, converges in \( L^1(\mathbb{R}^{2n}) \) to the solution of the Liouville equation.

1. Introduction.

In the Schrödinger picture \( H_0 := -\frac{\hbar^2}{2} \Delta \) and \( H := -\frac{\hbar^2}{2} \Delta + V \) are the free and perturbed hamiltonian operators in \( L^2(\mathbb{R}^n) \), where \( \hbar \) is the Planck’s constant. If \( \varphi \) is the solution of the corresponding Schrödinger equation

\[
(Sch) \quad \begin{cases}
    i\hbar \frac{\partial \varphi}{\partial t} = H \varphi \\
    \varphi(x,0) = \varphi_0(x)
\end{cases}
\]

It is well-known that for some potential \( V \) the operator \( H \) is self-adjoint. For example, when \( V \) satisfies the Kato conditions: \( V \in L^2_{\text{loc}}(\mathbb{R}^n) \), \( V = V_1 + V_2 \), \( V_1 \in L^\infty(\mathbb{R}^n) \), \( V_2 \in L^p(\mathbb{R}^n), p > \max(n/2,2) \), then \( -\frac{i}{\hbar}H \) generates a unitary group \( e^{-\frac{i}{\hbar}H} \) and

\[
\|e^{-\frac{i}{\hbar}H} \varphi_0(x)\|_{L^2(\mathbb{R}^n)} = \|\varphi_0(x)\|_{L^2(\mathbb{R}^n)},
\]

for all \( t \in \mathbb{R} \).

If we denote the Wigner transform of \( \varphi \) by

\[
w := W_\varphi(x,\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-i\xi y} \varphi \left( x + \frac{\hbar y}{2} \right) \varphi^\ast \left( x - \frac{\hbar y}{2} \right) dy
\]

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and if the potential $V = 0$, then $w$ will satisfy the advection equation

\begin{align*}
\frac{\partial w}{\partial t} + \xi \cdot \nabla_x w &= 0 \\
w(x, \xi, 0) &= w_0 \in L^2(\mathbb{R}^n_x \times \mathbb{R}^n_\xi).
\end{align*}

and if $V \neq 0$, then $w$ will satisfy the quantum Liouville equation

\begin{align*}
\frac{\partial w}{\partial t} + \xi \cdot \nabla_x w - P_h(x, \nabla_x) w &= 0 = \frac{\partial w}{\partial t} - \mathcal{L}_h w \\
w(x, \xi, 0) &= w_0(x, \xi) = W_{\phi_0} \in L^2(\mathbb{R}^n_x \times \mathbb{R}^n_\xi).
\end{align*}

In this equation $P_h$ is a pseudo-differential operator defined either in symbolic form

$$
P_h(x, \nabla_x) = \frac{i}{\hbar} \left[ V(x + i \frac{\hbar}{2} \nabla_x) - V(x - i \frac{\hbar}{2} \nabla_x) \right]
$$

or by

$$
P_h(x, \nabla_x) w = k_h * w = \int_{\mathbb{R}^n} k_h(x, \xi - \eta) w(x, \eta) d\eta
$$

with

$$
k_h(x, \xi) = (2\pi)^{-n/2} \mathcal{F}_y \left[ \frac{1}{i\hbar} \left[ V \left( x + \frac{\hbar}{2} y \right) - V \left( x - \frac{\hbar}{2} y \right) \right] \right](\xi).
$$

In [11, 10] it is proved that if $V \in H^1_{\text{loc}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ then (QL) admits a solution in $L^2(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ and the solution is unique if $V \in H^1_{\text{loc}}(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$. Furthermore the mild solution of (QL) converges weakly in $L^2(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ to the weak solution of (LE). In [4] the authors proved the well-posedness of (QLE) in $L^1(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ and it is proved that if $V \in H^s(\mathbb{R}^n)$ for $s > \max\{2, n/2\}$ then the operator $L_h$ is a bounded perturbation of $L_0 := -\xi \cdot \nabla_x$ and generates a quasi-contractive $C_0$-group, which satisfies

$$
\|e^{tL_h}f\|_{L^1(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)} \leq e^{\delta_h t}\|f\|_{L^1(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)}
$$

where $\delta_h = 2(2\pi)^{-n/2} C_h \|V\|_{H^s}$.

In the sequel we suppose that the potential $V$ is such that the $C_0$-group acts on $L^p(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$, for $p = 1$ and $p = 2$. With respect to such a potential we consider the Liouville equation

\begin{align*}
\frac{d}{dt} w &= -\xi \cdot \nabla_x w + \nabla_x V \cdot \nabla_x w = L w \\
w(x, \xi, 0) &= w_0(x, \xi) \in L^p(\mathbb{R}^n_x \times \mathbb{R}^n_\xi),
\end{align*}

which generates also a $C_0$-group $e^{tL}$ in $L^p(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ (see [1, Proposition 2.2]). In $L^2(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ this group is unitary, since the operator $L$ is skew-adjoint operator (see [11]). This group has also an explicit representation via Koopman formalism which asserts that

$$
e^{tL}f(x_0, \xi_0) = f(x(-t), \xi(-t)),
$$

(1.3)
where \((x(t), \xi(t))\) is the solution of the Hamiltonian system
\[
\begin{cases}
    \dot{x} = \xi, & x(0) = x_0 \\
    \dot{\xi} = -\nabla_x V(x), & \xi(0) = \xi_0.
\end{cases}
\]

In [11] it is also shown that \(e^{tL_h}\) converges weakly to \(e^{tL}\). In other words if \(w\) is the solution of Liouville equation (LE), then \(w\) converges to \(w\) weakly in \(L^2(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)\) as \(\hbar \to 0\).

In this paper we will prove that this convergence is not only in weak sense but also in ultra-weakly in \(L^2(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)\). Our proof is based on the theory of the algebras of operators on the Hilbert spaces ([3]).

In the first section we develop in an abstract manner some results which come to make use in the second section.

It is well-known that the Wigner distribution function is not positive and therefore one cannot regard that as a density distribution in statistical mechanics. For alleviating this difficulty K. Husimi proposed in [6] to take a mollifier of Wigner function which is called Husimi function and defined by
\[
H(\hbar)(x, \xi) = \left[G(\hbar)*W(\hbar)\right](x, \xi),
\]
where the Gaussian \(G(\hbar)\) is
\[
G(\hbar)(x, \xi) = \frac{1}{\pi \hbar^n}e^{-\frac{|x|^2 + |\xi|^2}{\hbar}}.
\]

Let us denote by
\[
C(\hbar): f \in L^1(\mathbb{R}^n_x \times \mathbb{R}^n_\xi) \mapsto G(\hbar)*f \in \mathcal{S}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi),
\]
then \(H(\hbar)(x, \xi) = C(\hbar)W(\hbar)\). The action of \(C(\hbar)\) on (QLE), gives a new perturbated system of (LE) called Husimi equation. The ill-posedness of the Husimi equation is already studied in [4]. In the section 3 we prove that the Husimi function \(H(\hbar)\) converges strongly to \(w\) solution of (LE), in \(L^1(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)\). This proof is based on the result of P. Markowich and C. Ringhofer [11, Lemma 8], who prove that if the potential \(V \in H^1_{loc}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\) then \(w\) the mild solution of (QLE) converges weakly in \(L^2(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)\) to the weak solution of (LE), the ultraweak convergence (see definition 2.1) of \(e^{tL_h}\) to \(e^{tL}\), as \(\hbar \to 0\), together with some results of the Gaussian upper bound.

2. ULTRAWEAK CONVERGENCE OF THE QUANTUM LIOUVILLE EQUATION AS \(\hbar \to 0\).

Let \(H\) be a complex separable Hilbert space with scalar product \((.,.)\) and norm \(\|\|\|.\) In the theory of von-Neumann algebra \(\mathcal{L}(H)\) designates the algebra of linear bounded operators equipped with the uniform norm \(\|A\| := \sup_{\|x\|\leq 1} \|Ax\|\) and \(\mathcal{S}_1(H)\) its \(*\)-ideal of the trace class operators with the norm
\[
\|A\|_1 := \sum_{i=1}^\infty |\lambda_i|,
\]
where $|\lambda_j|$ are the singular values of $A$, or eigenvalues of $|A| = \sqrt{AA^*}$. Let $\{e_i\}_{i \in \mathbb{N}}$ be an orthonormal basis in $H$. It is clear that if $A \in \mathcal{S}_1(H)$ then

$$\text{Tr}(A) := \sum_{i=1}^{\infty} \lambda_i < \infty.$$  

Since $\sum_{i=1}^{\infty} |(Ae_i, e_i)|$ is independent of the choice of the orthonormal basis $\{e_i\}_{i \in \mathbb{N}}$, so, if we replace $\{e_i\}_{i \in \mathbb{N}}$ by $\{\phi_i\}_{i \in \mathbb{N}}$ the orthonormal basis constituted by the eigenfunctions of $A \in \mathcal{S}_1(H)$, we retrieve

$$\|A\|_1 := \sum_{i=1}^{\infty} |(Ae_i, e_i)|.$$  

**Definition and Theorem 2.1.** We say that the sequence of the bounded operators $\{A_\alpha\}$ converges ultraweakly to $A$ and we write $A_\alpha \xrightarrow{\text{uw}} A$, if and only if

$$\lim_{\alpha} \sum_{i=1}^{\infty} ((A_\alpha - A)x_i, y_i) = 0,$$  

(2.1)

for any pair of sequences $(x_i), (y_i)$ in $H$ satisfying $\sum_{i=1}^{\infty} \|x_i\|^2 + \|y_i\|^2 < \infty$, which is equivalent to say that

$$\text{Tr}(A_\alpha \rho) \to \text{Tr}(A \rho),$$  

(2.2)

for any $\rho \in \mathcal{S}_1(H)$.

**Proof.** Any $\rho$ can be represented in his orthonormal eigenfunctions basis $(\phi_i)$ as $\rho \phi_i = \lambda_i \phi_i$ and $\text{Tr}(A_\alpha \rho) = \sum_{i=1}^{\infty} (A_\alpha \lambda_i \phi_i, \phi_i)$. If $\lambda_i = |\lambda_i| e^{i \theta_i}$, by taking $x_i = \sqrt{|\lambda_i|} e^{i \theta_i/2} \phi_i$ and $y_i = \sqrt{|\lambda_i|} e^{-i \theta_i/2} \phi_i$, we get

$$\sum_{i=1}^{\infty} \|x_i\|^2 + \|y_i\|^2 \leq 2 \|\rho\|_1.$$  

Since $\text{Tr}(A_\alpha \rho) = \sum_{i=1}^{\infty} (A_\alpha \lambda_i \phi_i, \phi_i)$ and $\text{Tr}(A \rho) = \sum_{i=1}^{\infty} (A \lambda_i \phi_i, \phi_i)$, so we have (2.2).

Conversely if we suppose that (2.2) is true, then given $(x_i)$ and $(y_i)$ satisfying $\sum_{i=1}^{\infty} \|x_i\|^2 + \|y_i\|^2 < \infty$, for $\rho$ defined by $\rho x = \sum_{i=1}^{\infty} (x, y_i) x_i$, we have

$$\|\rho x\| \leq \sum_{i=1}^{\infty} |(x, y_i)| \|x_i\| \leq \sum_{i=1}^{\infty} \|x\| \|y_i\| \|x_i\| \leq C \|x\|$$

where $C^2 = (\sum_{i=1}^{\infty} \|x_i\|^2)(\sum_{i=1}^{\infty} \|y_i\|^2)$, so we see that $\rho$ is bounded. To show that $\rho$ is trace class, let $(\psi_j)$ be any orthonormal set in $H$, by using the Bessel’s
inequality we have
\[
\sum_{j=1}^{n} |(\rho \psi_j, \psi_j)| = \sum_{j=1}^{n} \left| \left( \sum_{i=1}^{\infty} (\psi_j, y_i) x_i, \psi_j \right) \right|
\leq \sum_{i=1}^{\infty} \sum_{j=1}^{n} |(\psi_j, y_i)| |(x_i, \psi_j)|
\leq \sum_{i=1}^{\infty} \sum_{j=1}^{n} \frac{1}{2} \left( |(\psi_j, y_i)|^2 + |(x_i, \psi_j)|^2 \right)
\leq \sum_{i=1}^{\infty} \frac{1}{2} \left( \|x_i\|^2 + \|y_i\|^2 \right)
\]
which is finite and independent of \(n\), which implies that \(\rho\) is a trace class operator on \(H\).

Now if we replace in (2.2) \(\rho \psi_j\) by \(\sum_{i=1}^{\infty} (\psi_j, y_i) x_i\), we get
\[
\lim_{\alpha} \sum_{j=1}^{\infty} \left( (A_\alpha - A) \sum_{i=1}^{\infty} (\psi_j, y_i) x_i, \psi_j \right) = 0,
\]
which implies that
\[
\lim_{\alpha} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (\psi_j, y_i)((A_\alpha - A)x_i, \psi_j) = \lim_{\alpha} \sum_{i=1}^{\infty} \left( (A_\alpha - A)x_i, \sum_{j=1}^{\infty} (y_j, \psi_j) \psi_j \right)
\]
\[
= \lim_{\alpha} \sum_{i=1}^{\infty} (A_\alpha - A)x_i, y_i) = 0.
\]

\[\square\]

**Remark 2.1.** In [2], (2.2) is taken as the definition of ultraweak convergence.

**Lemma 2.2.** If the family \(\{A_\alpha\}\) is uniformly bounded, then the ultraweak topology is equivalent to weak topology.

**Proof.** It is clear that if choose the sequences \(\{x_i\}\) and \(\{y_i\}\) such that \(x_1 = x, y_1 = y\), where \(x\) and \(y\) are arbitrary and \(x_i = y_i = 0\) for any \(i \geq 2\), then (2.1) can be written as \(\lim_{\alpha}((A_\alpha - A)x, y) = 0\) which implies the weak convergence of \(\{A_\alpha\}\). Conversely, if \(L := \sup_\alpha \|A_\alpha\|\), for any \(\varepsilon > 0\) we can find an integer \(N \gg 1\) such that
\[
\sum_{i \geq N} \|x_i\|^2 + \|y_i\|^2 < \frac{\varepsilon}{L + \|A\|}
\]
and also saying \(\lim_{\alpha}((A_\alpha - A)x, y) = 0\) is equivalent to say that for any finite set \(\Lambda\), if \(\alpha \not\in \Lambda\), then \((A_\alpha - A)x_i, y_i) < \frac{\varepsilon}{2(\Lambda - 1)}\) for all \(i = 1, \ldots, N - 1\). So we can write
for any $\alpha \not\in \Lambda$,
\[
\sum_{i=1}^{\infty} ((A_{\alpha} - A)x_i, y_i) = \sum_{i \leq N-1} ((A_{\alpha} - A)x_i, y_i) + \sum_{i \geq N} ((A_{\alpha} - A)x_i, y_i) \\
\leq \varepsilon + (L + ||A||) \sum_{i \geq N} ||x_i|| ||y_i|| \\
\leq \varepsilon + \frac{L + ||A||}{2} \sum_{i \geq N} ||x_i||^2 + ||y_i||^2 < \varepsilon.
\]

\Box

**Corollary 2.3.** The group $e^{Lt}$ is unitary and converges ultraweakly in $L^2(\mathbb{R}^n \times \mathbb{R}_\xi^*)$ to $e^{L_0}$ as $h \to 0$.

**Proof.** Multiplication of (QLE) by $\overline{w}$, integration by parts and taking the real parts of the resulting equation gives
\[
\frac{1}{2} \frac{d}{dt} \iint_{\mathbb{R}^n \times \mathbb{R}_\xi} |w|^2 dx d\xi + \text{Re} \iint_{\mathbb{R}^n \times \mathbb{R}_\xi} [P_h(x, \nabla_\xi)w] \overline{w} dx d\xi = 0. \tag{2.3}
\]
Applying the result of [11, Lemma 2] we get
\[
\iint_{\mathbb{R}^n \times \mathbb{R}_\xi} [P_h(x, \nabla_\xi)u] \overline{w} dx d\xi = \iint_{\mathbb{R}^n \times \mathbb{R}_\xi} [P_h(x, \nabla_\xi)\overline{w}] u dx d\xi
\]
and since by definition of $P_h(x, \nabla_\xi)$ we have $P_h(x, \nabla_\xi)\overline{w} = \overline{P_h(x, \nabla_\xi)u}$, it follows that $\text{Re} \iint_{\mathbb{R}^n \times \mathbb{R}_\xi} [P_h(x, \nabla_\xi)w] \overline{w} dx d\xi = 0$.

Thus, from (2.3) it follows that $||w(.,.,t)|| = ||w_0(.,.)||$, that is the group $e^{Lt}$ is unitary and for any $h > 0$ we have $||e^{L_0}|| = 1$ and the result infers from the above Lemma. \Box

3. **Husimi Transformation**

If we denote the solution of the Schrödinger equation (Sch) by $\phi_h$ and its Wigner transform (1.2) by $W_{\phi_h}$, the Wigner function $W_{\phi_h}(x, \xi)$ is not positive for all values $(x, \xi)$ of the phase space, in spite of the fact that
\[
\int_{\mathbb{R}_\xi} W_{\phi_h}(x, \xi) d\xi = \mathcal{F}^{-1} [\mathcal{F}_y [\mathcal{F}_x [\varphi_h(x + \frac{hy}{2}) \overline{\varphi_h(x - \frac{hy}{2})}]](0)] = |\varphi_h(x)|^2 > 0. \tag{3.1}
\]
So, we cannot consider the Wigner function as a density function in the context of statistical mechanics. In [6], K. Husimi proposed the following procedure which ends to define a new function which is called *Husimi function*, which can be considered in some manner as a density function.
Define
\[ H_h(x, \xi, t) = [W_{\varphi_h} \ast G_h](x, \xi, t) \]
\[ = \iint_{\mathbb{R}^2 \times \mathbb{R}^2} W_{\varphi_h}(y, \eta, t) G_h(x - y, \xi - \eta) dy d\eta, \]
where \( G_h(x, \xi) \) is given in (1.4).

By taking \( h = 4t \), it is well-known that \( [T(t)f](x, \xi) = [G_{4t} \ast f](x, \xi) \) forms a Gaussian (or heat, or diffusion) semigroup and the strong continuity of this semigroup asserts that
\[ \|G_{4t} \ast f - f\|_p \to 0 \quad \text{for any} \quad f \in L^p(\mathbb{R}^n_\epsilon \times \mathbb{R}^n_\epsilon), \quad (1 \leq p < \infty) \quad (3.2) \]
as \( t \to 0 \).

**Lemma 3.1.** For \((x_0, \xi_0)\) in phase space, define the Gabor function
\[ \psi_{x_0,\xi_0}(x) = (\pi)^{-n/4} e^{-\frac{|x-x_0|^2}{2\hbar}} e^{i\xi \cdot x/\hbar} \]
then the Wigner transform of Gabor function \( W_{\psi_{x_0,\xi_0}} \) satisfies
\[ W_{\psi_{x_0,\xi_0}}(x, \xi) = G_h(x - x_0, \xi - \xi_0) = W_{\psi_{x,\xi}}(x_0, \xi_0) \quad (3.3) \]

**Proof.** By using the expression
\[ W_{\psi_{x_0,\xi_0}}(x, \xi) = (2\pi \hbar)^{-n} \int_{\mathbb{R}^n} e^{-i(\xi/\hbar) \cdot y} \psi_{x_0,\xi_0}(x + \frac{y}{2}) \psi_{x_0,\xi_0}(x - \frac{y}{2}) dy \]
we get by using the parallelogram identity
\[ W_{\psi_{x_0,\xi_0}}(x, \xi) = (\pi \hbar)^{-n/2} (2\pi \hbar)^{-n} \int_{\mathbb{R}^n} e^{-i(\xi/\hbar) \cdot y} e^{-\frac{(x-x_0)^2}{\hbar} + \frac{|y|^2}{4}} dy \]
\[ = (\pi \hbar)^{-n} e^{-\frac{(x-x_0)^2}{\hbar} + \frac{|\xi-\xi_0|^2}{\hbar}}, \quad \Box \]

Now, let us consider the operator \( C_h \) given in (1.5) and denote by \( Q_h(x, \nabla \xi) := C_h P_h(x, \nabla \xi) \).

**Theorem 3.2.** The Husimi function \( H_h \) is positive and belongs to \( L^1(\mathbb{R}^n_\epsilon \times \mathbb{R}^n_\epsilon) \), further more \( H_h \) converges in this space to \( w(x, \xi) \) the solution of Liouville equation (LE), as \( h \to 0 \).

For the proof of this Theorem we will use the following Lemma.

**Lemma 3.3.** Let us denote by \( \chi_k(x, \xi, y, \eta) \) the characteristic function of \( I_k = \{(x, \xi, y, \eta) \in \mathbb{R}^{4n} : (k - 1)^2 \hbar \leq |x - y|^2 + |\xi - \eta|^2 < k^2 \hbar \} \) and
\[ g_k(y, \eta) = \int_{\mathbb{R}^n_\epsilon \times \mathbb{R}^n_\epsilon} kG_h(x - y, \xi - \eta) \chi_k(x, \xi, y, \eta) dx d\xi \]
\[
\sum_{k \geq 1} \|g_k\|_{L^2(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)}^2 < \infty. \tag{3.4}
\]

**Proof.** For the proof of this Lemma we will use the Gaussian upper bound (see [9, Chapter 7]). We remark that \(G_\hbar(x - y, \xi - \eta) = p(\hbar, (x, \xi), (y, \eta))\) is the Gaussian kernel of \(\Delta\) in \(\mathbb{R}_x^n \times \mathbb{R}_\xi^n\) and satisfies

\[
|p(\hbar, (x, \xi), (y, \eta))| \leq C(V((y, \eta), \sqrt{\hbar}))^{-1} \exp \left\{ -c \left( \frac{|x - y|^2 + |\xi - \eta|^2}{\hbar} \right) \right\} \tag{3.5}
\]

where \(V((y, \eta), r)\) is the volume of the ball \(B(y, \eta), r\) centered at \((y, \eta)\) of radius \(r\) in \(\mathbb{R}_x^n \times \mathbb{R}_\xi^n\) (see [9, formula (7.5)])). Using the facts that

\[
\int_{\mathbb{R}_x^n \times \mathbb{R}_\xi^n} \chi_k(x, \xi, y, \eta) dxd\xi \leq V((y, \eta), k\sqrt{\hbar})
\]

and

\[
I_k \subset \bigcup_{|(y, \eta)| \leq k (x, \xi, y, \eta) \in I_k} B((y, \eta), k\sqrt{\hbar})
\]

it follows from (3.5) that

\[
\sum_{k \geq 1} \|g_k\|_{L^2(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)}^2 = \sum_{k \geq 1} \int_{\mathbb{R}_x^n \times \mathbb{R}_\xi^n} \left| \int_{\mathbb{R}_x^n \times \mathbb{R}_\xi^n} kG_\hbar(x - y, \xi - \eta) \chi_k(x, \xi, y, \eta) dxd\xi \right|^2 dyd\eta \\
\leq C^2 \sum_{k \geq 1} \int_{|(y, \eta)| \leq k} \left( k^6 e^{-8c(k-1)^2} V((y, \eta), k\sqrt{\hbar}) \right)^2 dyd\eta,
\]

since in \(\mathbb{R}_x^n \times \mathbb{R}_\xi^n\), \(V((y, \eta), r)\) is proportional to \(r^{2n}\), we get \(V((y, \eta), k\sqrt{\hbar}) \leq C'(2k)^{2n}\). This shows that

\[
\sum_{k \geq 1} \|g_k\|_{L^2(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)}^2 \leq C'' \sum_{k \geq 1} k^{6n+2} e^{-8c(k-1)^2} < \infty,
\]

where \(C''\) is a constant independent of \(\hbar\). \(\square\)

**Proof of Theorem 3.2.** It follows from (3.1) and (3.3), that

\[
H_\hbar(x, \xi) = (1/4\pi\hbar)^{-n} \left| \int_{\mathbb{R}_n} \varphi_\hbar(x) \overline{\psi_{x_0, \xi_0}(x)} dx \right|^2 > 0,
\]

and the fact that \(\|G_\hbar\|_1 = 1\), implies that \(\|H_\hbar\|_1 < \infty\) (see [4, Equation (1.19)]).
Now, in order to prove the $L^1$-convergence of $H_h$ toward $w$ we will use Theorem 2.1, together with some aspects of wavelet theory. First we note that in $(L^1(\mathbb{R}^n_x \times \mathbb{R}^n_\xi), \| \cdot \|_1)$, we have

$$\|H_h - w\|_1 \leq \|(W_h - w) * G_h\|_1 + \|G_h * w - w\|_1. \quad (3.6)$$

As it is showed in [1], the Koopman formalism (1.3), implies that if $w_0 \in L^p(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$, then $w$ belongs also to $L^p(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ and (3.2) holds for $p = 1$, which yields to the convergence of the last term of (3.6) to zero as $h \to 0$. So, it remains to prove the convergence of $\|(W_h - w) * G_h\|_1$ to zero, as $h \to 0$.

Let $I_k = \{(x, \xi) \in \mathbb{R}^n_x \times \mathbb{R}^n_\xi : k - 1 \leq \sqrt{x^2 + \xi^2} < k\}$ and $\chi_I$ the characteristic function of $I$. We remark that $\sum_{k \geq 1} \chi_{I_k} = 1$ for any $(x, \xi) \in \mathbb{R}^n_x \times \mathbb{R}^n_\xi$. Let $s(x, \xi) := e^{-i\theta(x, \xi)}$, where $\theta(x, \xi) = \arg(W_h - w)(x, \xi)$, so $[W_h - w]s(x, \xi) \geq 0$ and $s(x, \xi)s^*(x, \xi) = 1$. Finally by writing $W_h - w = (e^{t_L} - e^{t_L})w_0$, we get

$$\|(W_h - w) * G_h\|_1 = \int_{\mathbb{R}^n_x \times \mathbb{R}^n_\xi} \left| \int_{\mathbb{R}^n_x \times \mathbb{R}^n_\xi} (e^{t_L} - e^{t_L})w_0(y, \eta)G_h(x - y, \xi - \eta) dy d\eta \right| dx d\xi$$

$$\leq \int_{\mathbb{R}^n_x \times \mathbb{R}^n_\xi} \int_{\mathbb{R}^n_x \times \mathbb{R}^n_\xi} (e^{t_L} - e^{t_L})w_0(y, \eta)s(y, \eta) |s^*(y, \eta)G_h(x - y, \xi - \eta)| dy d\eta dx d\xi$$

$$= \sum_{k \geq 1} \int_{\mathbb{R}^n_x \times \mathbb{R}^n_\xi} (e^{t_L} - e^{t_L})f_k(y, \eta)g_k(y, \eta) dy d\eta,$$

where $f_k(y, \eta) = \frac{1}{k}w_0(y, \eta)s(y, \eta)$ and $g_k$ defined as in Lemma 3.3.

Now, since $e^{t_L}$ converges ultraweakly (in the sense of (2.1) ) to $e^{t_L}$, it follows that $H_h$ converges in $L^1(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ toward $w$. \hfill $\Box$

References


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