SEMICLASSICAL LIMIT OF HUSIMI FUNCTION

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ABSTRACT. We will show that Liouville and quantum Liouville operators L and $L_\hbar$ generate two C_0-groups e^{tL} and $\mathrm{e}^{tL_\hbar}$ of isometries in $L^2(\mathbb{R}^{2n})$ and $\mathrm{e}^{tL_\hbar}$ converges ultraweakly to e^{tL}. As a consequence we show that the Gaussian mollifier of the Wigner function, called Husimi function, converges in $L^1(\mathbb{R}^{2n})$ to the solution of the Liouville equation.

1. INTRODUCTION.

In the Schrödinger picture $H_0 := -\frac{\hbar^2}{2}\Delta$ and $H := -\frac{\hbar^2}{2}\Delta + V$ are the free and perturbed hamiltonian operators in $L^2(\mathbb{R}^n)$, where $\hbar$ is the Planck's constant. If φ is the solution of the corresponding Schrödinger equation

$$\text{(Sch)} \qquad \begin{cases} \mathrm{i}\hbar \dfrac{\partial \varphi}{\partial t} = H\varphi \\ \varphi(x,0) = \varphi_0(x) \end{cases}$$

It is well-known that for some potential V the operator H is self-adjoint. For example, when V satisfies the Kato conditions: $V \in L^2_{loc}(\mathbb{R}^n)$, $V = V_1 + V_2$, $V_1 \in L^\infty(\mathbb{R}^n)$, $V_2 \in L^p(\mathbb{R}^n), p > \max(n/2, 2)$, then $-\frac{\mathrm{i}}{\hbar}H$ generates a unitary group $\mathrm{e}^{-\frac{\mathrm{i}t}{\hbar}H}$ and

$$\|\mathrm{e}^{-\frac{\mathrm{i}t}{\hbar}H}\varphi_0(x)\|_{L^2(\mathbb{R}^n)} = \|\varphi_0(x)\|_{L^2(\mathbb{R}^n)}, \tag{1.1}$$

for all $t \in \mathbb{R}$.

If we denote the Wigner transform of φ by

$$w := W_\varphi(x,\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}\xi\cdot y} \varphi\left(x + \frac{\hbar y}{2}\right) \overline{\varphi}\left(x - \frac{\hbar y}{2}\right) \mathrm{d}y \tag{1.2}$$

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and if the potential $V=0$, then w will satisfy the advection equation

(AE)
$$\begin{cases} \frac{\partial w}{\partial t} + \xi \cdot \nabla_x w = 0 \\ w(x,\xi,0) = w_0 \in L^2(\mathbb{R}^n_x \times \mathbb{R}^n_\xi). \end{cases}$$

and if $V \neq 0$, then w will satisfy the quantum Liouville equation

(QLE)
$$\begin{cases} \frac{\partial w}{\partial t} + \xi \cdot \nabla_x w - P_\hbar(x,\nabla_\xi) w = 0 = \frac{\partial w}{\partial t} - L_\hbar w \\ w(x,\xi,0) = w_0(x,\xi) = W_{\varphi_0} \in L^2(\mathbb{R}^n_x \times \mathbb{R}^n_\xi). \end{cases}$$

In this equation $P_\hbar$ is a pseudo-differential operator defined either in symbolic form
$$P_\hbar(x,\nabla_\xi) = \frac{\mathrm{i}}{\hbar}\left[V(x+\mathrm{i}\frac{\hbar}{2}\nabla_\xi) - V(x-\mathrm{i}\frac{\hbar}{2}\nabla_\xi)\right]$$
or by
$$P_\hbar(x,\nabla_\xi) w = k_\hbar \underset{\xi}{*} w = \int_{\mathbb{R}^n} k_\hbar(x,\xi-\eta) w(x,\eta)\mathrm{d}\eta$$
with
$$k_\hbar(x,\xi) = (2\pi)^{-n/2}\mathcal{F}_y\left[\frac{1}{\mathrm{i}\hbar}\left[V\left(x+\frac{\hbar}{2}y\right) - V\left(x-\frac{\hbar}{2}y\right)\right]\right](\xi).$$

In [11, 10] it is proved that if $V \in H^1_{loc}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ then **(QLE)** admits a solution in $L^2(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ and the solution is unique if $V \in H^1_{loc}(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$. Furthermore the mild solution of **(QLE)** converges weakly in $L^2(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ to the weak solution of **(LE)**. In [4] the authors proved the well-posedness of **(QLE)** in $L^1(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ and it is proved that if $V \in H^s(\mathbb{R}^n)$ for $s > \max\{2, n/2\}$ then the operator $L_\hbar$ is a bounded perturbation of $L_0 := -\xi \cdot \nabla_x$ and generates a quasi-contractive C_0-group, which satisfies
$$\|\mathrm{e}^{tL_\hbar} f\|_{L^1(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)} \leq \mathrm{e}^{\delta_\hbar |t|}\|f\|_{L^1(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)}$$
where $\delta_\hbar = 2(2\pi)^{-n/2} C_\hbar \|V\|_{H^s}$.

In the sequel we suppose that the potential V is such that the C_0-group acts on $L^p(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$, for $p=1$ and $p=2$. With respect to such a potential we consider the Liouville equation

(LE)
$$\begin{cases} w_t = -\xi.\nabla_x w + \nabla_x V.\nabla_\xi w = Lw \\ w(x,\xi,0) = w_0(x,\xi) \in L^p(\mathbb{R}^n_x \times \mathbb{R}^n_\xi), \end{cases}$$

which generates also a C_0-group e^{tL} in $L^p(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ (see [1, Proposition 2.2]). In $L^2(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ this group is unitary, since the operator L is skew-adjoint operator (see [11]). This group has also an explicit representation via Koopman formalism which asserts that
$$\mathrm{e}^{tL} f(x_0,\xi_0) = f(x(-t),\xi(-t)), \tag{1.3}$$

where $(x(t), \xi(t))$ is the solution of the Hamiltonian system

$$\text{(HS)} \qquad \begin{cases} \dot{x} = \xi, & x(0) = x_0 \\ \dot{\xi} = -\nabla_x V(x), & \xi(0) = \xi_0. \end{cases}$$

In [11] it is also shown that $e^{tL_\hbar}$ converges weakly to e^{tL}. In other words if w is the solution of Liouville equation (LE), then $w_\hbar$ converges to w weakly in $L^2(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ as $\hbar \to 0$.

In this paper we will prove that this convergence is not only in weak sense but also in ultra-weakly in $L^2(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$. Our proof is based on the theory of the algebras of operators on the Hilbert spaces ([3]).

In the first section we develop in an abstract manner some results which come to make use in the second section.

It is well-known that the Wigner distribution function is not positive and therefore one cannot regard that as a density distribution in statistical mechanics. For alleviating this difficulty K. Husimi proposed in [6] to take a mollifier of Wigner function which is called *Husimi function* and defined by $H_\hbar(x, \xi) = [G_\hbar * W_\hbar](x, \xi)$, where the Gaussian $G_\hbar$ is

$$G_\hbar(x, \xi) = (\pi\hbar)^{-n} e^{-(|x|^2+|\xi|^2)/\hbar} \tag{1.4}$$

Let us denote by

$$C_\hbar : f \in L^1(\mathbb{R}^n_x \times \mathbb{R}^n_\xi) \mapsto G_\hbar * f \in \mathscr{S}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi), \tag{1.5}$$

then $H_\hbar(x, \xi) = C_\hbar W_\hbar$. The action of $C_\hbar$ on (QLE), gives a new perturbated system of **(LE)** called *Husimi equation*. The ill-posedness of the Husimi equation is already studied in [4]. In the section 3 we prove that the Husimi function $H_\hbar$ converges strongly to w solution of (LE), in $L^1(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$. This proof is based on the result of P. Markowich and C. Ringhofer [11, Lemma 8], who prove that if the potential $V \in H^1_{loc}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ then $w_\hbar$ the mild solution of **(QLE)** converges weakly in $L^2(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ to the weak solution of **(LE)**, the ultraweak convergence (see definition 2.1) of $e^{tL_\hbar}$ to e^{tL}, as $\hbar \to 0$, together with some results of the Gaussian upper bound.

2. Ultraweak convergence of the quantum Liouville equation as $\hbar \to 0$.

Let H be a complex separable Hilbert space with scalar product $(.,.)$ and norm $\|.\|$. In the theory of von-Neumann algebra $\mathcal{L}(H)$ designates the algebra of linear bounded operators equipped with the uniform norm $\|A\| := \sup_{\|x\|\le 1} \|Ax\|$ and $\mathscr{I}_1(H)$ its $^*-$ideal of the trace class operators with the norm

$$\|A\|_1 := \sum_{i=1}^{\infty} |\lambda_i|,$$

where $|\lambda_j|$ are the singular values of A, or eigenvalues of $|A| = \sqrt{AA^*}$. Let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis in H. It is clear that if $A \in \mathscr{I}_1(H)$ then

$$\mathrm{Tr}(A) := \sum_{i=1}^{\infty} \lambda_i < \infty.$$

Since $\sum_{i=1}^{\infty} |(Ae_i, e_i)|$ is independent of the choice of the orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$, so, if we replace $\{e_i\}_{i\in\mathbb{N}}$ by $\{\varphi_i\}_{i\in\mathbb{N}}$ the orthonormal basis constituted by the eigenfunctions of $A \in \mathscr{I}_1(H)$, we retrieve

$$\|A\|_1 := \sum_{i=1}^{\infty} |(Ae_i, e_i)|.$$

Definition and Theorem 2.1. *We say that the sequence of the bounded operators $\{A_\alpha\}$ converges ultraweakly to A and we write $A_\alpha \stackrel{uw}{\to} A$, if and only if*

$$\lim_{\alpha} \sum_{i=1}^{\infty} ((A_\alpha - A)x_i, y_i) = 0, \tag{2.1}$$

for any pair of sequences (x_i), (y_i) in H satisfying $\sum_{i=1}^{\infty} \|x_i\|^2 + \|y_i\|^2 < \infty$, which is equivalent to say that

$$\mathrm{Tr}(A_\alpha \rho) \to \mathrm{Tr}(A\rho), \tag{2.2}$$

for any $\rho \in \mathscr{I}_1(H)$.

Proof. Any ρ can be represented in his orthonormal eigenfunctions basis (ϕ_i) as $\rho\phi_i = \lambda_i\phi_i$ and $\mathrm{Tr}(A_\alpha\rho) = \sum_{i=1}^{\infty}(A_\alpha\lambda_i\phi_i, \phi_i)$. If $\lambda_i = |\lambda_i|\mathrm{e}^{\mathrm{i}\theta_i}$, by taking $x_i = \sqrt{|\lambda_i|}\mathrm{e}^{\mathrm{i}\theta_i/2}\phi_i$ and $y_i = \sqrt{|\lambda_i|}\mathrm{e}^{-\mathrm{i}\theta_i/2}\phi_i$, we get

$$\sum_{i=1}^{\infty} \|x_i\|^2 + \|y_i\|^2 \le 2\|\rho\|_1.$$

Since $\mathrm{Tr}(A_\alpha\rho) = \sum_{i=1}^{\infty}(A_\alpha\lambda_i\phi_i, \phi_i)$ and $\mathrm{Tr}(A\rho) = \sum_{i=1}^{\infty}(A\lambda_i\phi_i, \phi_i)$, so we have (2.2).

Conversely if we suppose that (2.2) is true, then given (x_i) and (y_i) satisfying $\sum_{i=1}^{\infty} \|x_i\|^2 + \|y_i\|^2 < \infty$, for ρ defined by $\rho x = \sum_{i=1}^{\infty}(x, y_i)x_i$, we have

$$\begin{aligned}\|\rho x\| &\le \sum_{i=1}^{\infty} |(x, y_i)|\|x_i\| \\ &\le \sum_{i=1}^{\infty} \|x\|\|y_i\|\|x_i\| \le C\|x\|\end{aligned}$$

where $C^2 = \left(\sum_{i=1}^{\infty} \|x_i\|^2\right)\left(\sum_{i=1}^{\infty} \|y_i\|^2\right)$, so we see that ρ is bounded. To show that ρ is trace class, let (ψ_j) be any orthonormal set in H, by using the Bessel's inequality we have

$$\begin{aligned}\sum_{j=1}^{n}|(\rho\psi_j,\psi_j)| &= \sum_{j=1}^{n}\left|\left(\sum_{i=1}^{\infty}(\psi_j,y_i)x_i,\psi_j\right)\right| \\ &\leq \sum_{i=1}^{\infty}\sum_{j=1}^{n}|(\psi_j,y_i)||(x_i,\psi_j)| \\ &\leq \sum_{i=1}^{\infty}\sum_{j=1}^{n}\frac{1}{2}\left(|(\psi_j,y_i)|^2+|(x_i,\psi_j)|^2\right) \\ &\leq \sum_{i=1}^{\infty}\frac{1}{2}\left(\|x_i\|^2+\|y_i\|^2\right)\end{aligned}$$

which is finite and independent of n, which implies that ρ is a trace class operator on H.

Now if we replace in (2.2) $\rho\psi_j$ by $\sum_{i=1}^{\infty}(\psi_j,y_i)x_i$, we get

$$\lim_{\alpha}\sum_{j=1}^{\infty}\left((A_\alpha-A)\sum_{i=1}^{\infty}(\psi_j,y_i)x_i,\psi_j\right)=0,$$

which implies that

$$\begin{aligned}\lim_{\alpha}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(\psi_j,y_i)((A_\alpha-A)x_i,\psi_j) &= \lim_{\alpha}\sum_{i=1}^{\infty}\left((A_\alpha-A)x_i,\sum_{j=1}^{\infty}(y_i,\psi_j)\psi_j\right) \\ &= \lim_{\alpha}\sum_{i=1}^{\infty}(A_\alpha-A)x_i,y_i)=0.\end{aligned}$$

□

Remark 2.1. In [2], (2.2) is taken as the definition of ultraweak convergence.

Lemma 2.2. *If the family $\{A_\alpha\}$ is uniformly bounded, then the ultraweak topology is equivalent to weak topology.*

Proof. It is clear that if choose the sequences $\{x_i\}$ and $\{y_i\}$ such that $x_1 = x, y_1 = y$, where x and y are arbitrary and $x_i = y_i = 0$ for any $i \geq 2$, then (2.1) can be written as $\lim_\alpha((A_\alpha - A)x, y) = 0$ which implies the weak convergence of $\{A_\alpha\}$. Conversely, if $L := \sup_\alpha \|A_\alpha\|$, for any $\varepsilon > 0$ we can find an integer $N \gg 1$ such that

$$\sum_{i\geq N}\|x_i\|^2+\|y_i\|^2<\frac{\varepsilon}{L+\|A\|}$$

and also saying $\lim_\alpha((A_\alpha - A)x, y) = 0$ is equivalent to say that for any finite set Λ, if $\alpha \notin \Lambda$, then $((A_\alpha - A)x_i, y_i) < \frac{\varepsilon}{2(N-1)}$ for all $i = 1, \cdot, N-1$. So we can write for any $\alpha \notin \Lambda$,

$$\begin{aligned}\sum_{i=1}^{\infty}((A_\alpha - A)x_i, y_i) &= \sum_{i\leq N-1}((A_\alpha - A)x_i, y_i) + \sum_{i\geq N}((A_\alpha - A)x_i, y_i)\\ &\leq \frac{\varepsilon}{2} + (L+\|A\|)\sum_{i\geq N}\|x_i\|\|y_i\|\\ &\leq \frac{\varepsilon}{2} + \frac{L+\|A\|}{2}\sum_{i\geq N}\|x_i\|^2 + \|y_i\|^2 < \varepsilon.\end{aligned}$$

□

Corollary 2.3. *The group $e^{tL_\hbar}$ is unitary and converges ultraweakly in $L^2(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ to e^{tL_0} as $\hbar \to 0$.*

Proof. Multiplication of (QLE) by $\overline{w}$, integration by parts and taking the real parts of the resulting equation gives

$$\frac{1}{2}\frac{d}{dt}\iint_{\mathbb{R}_x\times\mathbb{R}_\xi}|w|^2dxd\xi + \mathrm{Re}\iint_{\mathbb{R}_x\times\mathbb{R}_\xi}[P_\hbar(x,\nabla_\xi)w]\overline{w}dxd\xi = 0. \tag{2.3}$$

Applying the result of [11, Lemma 2] we get

$$\iint_{\mathbb{R}_x\times\mathbb{R}_\xi}[P_\hbar(x,\nabla_\xi)u]\overline{u}dxd\xi = \iint_{\mathbb{R}_x\times\mathbb{R}_\xi}[P_\hbar(x,\nabla_\xi)\overline{u}]udxd\xi$$

and since by definition of $P_\hbar(x,\nabla_\xi)$ we have $P_\hbar(x,\nabla_\xi)\overline{u} = \overline{P_\hbar(x,\nabla_\xi)u}$, it follows that $\mathrm{Re}\iint_{\mathbb{R}_x\times\mathbb{R}_\xi}[P_\hbar(x,\nabla_\xi)w]\overline{w}dxd\xi = 0$.

Thus, from (2.3) it follows that $\|w(.,.,t)\| = \|w_0(.,.)\|$, that is the group $e^{tL_\hbar}$ is unitary and for any $\hbar > 0$ we have $\|e^{tL_\hbar}\| = 1$ and the result infers from the above Lemma. □

3. Husimi transformation

If we denote the solution of the Schrödinger equation (Sch) by $\phi_\hbar$ and its Wigner transform (1.2) by $W_{\varphi_\hbar}$, the Wigner function $W_{\varphi_\hbar}(x,\xi)$ is not positive for all values (x,ξ) of the phase space, in spite of the fact that

$$\int_{\mathbb{R}_\xi} W_{\varphi_\hbar}(x,\xi)d\xi = \mathcal{F}^{-1}\big[\mathcal{F}_y[\varphi_\hbar(x+\frac{\hbar y}{2})\overline{\varphi}_\hbar(x-\frac{\hbar y}{2})]\big](0) = |\varphi_\hbar(x)|^2 > 0. \tag{3.1}$$

So, we cannot consider the Wigner function as a density function in the context of statistical mechanics. In [6], K. Husimi proposed the following procedure which ends to define a new function which is called *Husimi function*, which can be considered in some manner as a density function.

Define

$$H_\hbar(x,\xi,t) = [W_{\varphi_\hbar} * G_\hbar](x,\xi,t) = \iint_{\mathbb{R}^n_x\times\mathbb{R}^n_\xi} W_{\varphi_\hbar}(y,\eta,t)G_\hbar(x-y,\xi-\eta)dyd\eta,$$

where $G_\hbar(x,\xi)$ is given in (1.4).

By taking $\hbar = 4t$, it is well-known that $[T(t)f](x,\xi) = [G_{4t} * f](x,\xi)$ forms a Gaussian (or heat, or diffusion) semigroup and the strong continuity of this semigroup asserts that

$$\|G_{4t} * f - f\|_p \to 0 \qquad \text{for any} \quad f\in L^p(\mathbb{R}^n_x\times\mathbb{R}^n_\xi), \quad (1\le p<\infty) \tag{3.2}$$

as $t\to 0$.

Lemma 3.1. *For (x_0,ξ_0) in phase space, define the Gabor function*

$$\psi_{x_0,\xi_0}(x) = (\pi)^{-n/4}\mathrm{e}^{-\frac{|x-x_0|^2}{2\hbar}}\mathrm{e}^{\mathrm{i}\xi_0\cdot x/\hbar}$$

then the Wigner transform of Gabor function $W_{\psi_{x_0,\xi_0}}$ satisfies

$$W_{\psi_{x_0,\xi_0}}(x,\xi) = G_\hbar(x-x_0,\xi-\xi_0) = W_{\psi_{x,\xi}}(x_0,\xi_0) \tag{3.3}$$

Proof. By using the expression

$$W_{\psi_{x_0,\xi_0}}(x,\xi) = (2\pi\hbar)^{-n}\int_{\mathbb{R}^n}\mathrm{e}^{-\mathrm{i}(\xi/\hbar).y}\psi_{x_0,\xi_0}(x+\frac{y}{2})\overline{\psi}_{x_0,\xi_0}(x-\frac{y}{2})dy$$

we get by using the parallelogram identity

$$W_{\psi_{x_0,\xi_0}}(x,\xi) = (\pi\hbar)^{-n/2}(2\pi\hbar)^{-n}\int_{\mathbb{R}^n}\mathrm{e}^{-\mathrm{i}(\frac{\xi-\xi_0}{\hbar}.y)}\mathrm{e}^{-\left(\frac{|x-x_0|^2}{\hbar}+\frac{|y|^2}{4\hbar}\right)}dy$$
$$= (\pi\hbar)^{-n}\,\mathrm{e}^{-(|x-x_0|^2+|\xi-\xi_0|^2)/\hbar},$$

since it is well-known that $\int_{\mathbb{R}^n}\mathrm{e}^{-\mathrm{i}\xi.y}\mathrm{e}^{-k|y|^2}dy = \left(\frac{\pi}{k}\right)^{\frac{n}{2}}\mathrm{e}^{-\frac{|\xi|^2}{4k}}$. □

Now, let us consider the operator $C_\hbar$ given in (1.5) and denote by $Q_\hbar(x,\nabla_\xi) := C_\hbar P_\hbar(x,\nabla_\xi)$.

Theorem 3.2. *The Husimi function $H_\hbar$ is positive and belongs to $L^1(\mathbb{R}^n_x\times\mathbb{R}^n_\xi)$, further more $H_\hbar$ converges in this space to $w(x,\xi)$ the solution of Liouville equation* **(LE)**, *as $\hbar\to 0$.*

For the proof of this Theorem we will use the following Lemma.

Lemma 3.3. *Let us denote by $\chi_k(x,\xi,y,\eta)$ the characteristic function of $I_k = \{(x,\xi,y,\eta)\in\mathbb{R}^{4n}\ :\ (k-1)^2\hbar \le |x-y|^2+|\xi-\eta|^2 < k^2\hbar\}$ and*

$$g_k(y,\eta) = \int_{\mathbb{R}^n_x\times\mathbb{R}^n_\xi} kG_\hbar(x-y,\xi-\eta)\chi_k(x,\xi,y,\eta)dxd\xi$$

then

$$\sum_{k\geq 1} \|g_k\|^2_{L^2(\mathbb{R}^n_x\times\mathbb{R}^n_\xi)} < \infty. \tag{3.4}$$

Proof. For the proof of this Lemma we will use the Gaussian upper bound (see [9, Chapter 7]). We remark that $G_{4\hbar}(x-y,\xi-\eta) = p(\hbar,(x,\xi),(y,\eta))$ is the Gaussian kernel of Δ in $\mathbb{R}^n_x\times\mathbb{R}^n_\xi$ and satisfies

$$|p(\hbar,(x,\xi),(y,\eta))| \leq C(V((y,\eta),\sqrt{\hbar}))^{-1}\exp\left\{-c\left(\frac{|x-y|^2+|\xi-\eta|^2}{\hbar}\right)\right\} \tag{3.5}$$

where $V((y,\eta),r)$ is the volume of the ball $B(y,\eta),r)$ centered at (y,η) of radius r in $\mathbb{R}^n_x\times\mathbb{R}^n_\xi$ (see [9, formula (7.5)]). Using the facts that

$$\int_{\mathbb{R}^n_x\times\mathbb{R}^n_\xi} \chi_k(x,\xi,y,\eta)dxd\xi \leq V((y,\eta),k\sqrt{\hbar})$$

and

$$I_k \subset \bigcup_{|(y,\eta)|\leq k}\ \bigcup_{(x,\xi,y,\eta)\in I_k} B((y,\eta),k\sqrt{\hbar})$$

it follows from (3.5) that

$$\begin{aligned}\sum_{k\geq 1}\|g_k\|^2_{L^2(\mathbb{R}^n_x\times\mathbb{R}^n_\xi)} &= \sum_{k\geq 1}\int_{\mathbb{R}^n_x\times\mathbb{R}^n_\xi}\left|\int_{\mathbb{R}^n_x\times\mathbb{R}^n_\xi} kG_\hbar(x-y,\xi-\eta)\chi_k(x,\xi,y,\eta)dxd\xi\right|^2 dyd\eta \\ &\leq C^2\sum_{k\geq 1}\int_{|(y,\eta)|\leq k}\left(k\mathrm{e}^{-8c(k-1)^2}\frac{V((y,\eta),k\sqrt{\hbar}))}{V\left((y,\eta),\sqrt{\hbar}/2)\right)}\right)^2 dyd\eta,\end{aligned}$$

since in $\mathbb{R}^n_x\times\mathbb{R}^n_\xi$, $V((y,\eta),r)$ is proportional to r^{2n}, we get $\frac{V((y,\eta),k\sqrt{\hbar}))}{V\left((y,\eta),\sqrt{\hbar}/2)\right)} \leq C'(2k)^{2n}$. This shows that

$$\sum_{k\geq 1}\|g_k\|^2_{L^2(\mathbb{R}^n_x\times\mathbb{R}^n_\xi)} \leq C''\sum_{k\geq 1}k^{6n+2}\mathrm{e}^{-8c(k-1)^2} < \infty,$$

where C'' is a constant independent of $\hbar$. □

Proof of Theorem 3.2. It follows from (3.1) and (3.3), that

$$H_\hbar(x,\xi) = (1/4\pi\hbar)^{-n}\left|\int_{\mathbb{R}^n}\varphi_\hbar(x)\overline{\psi_{x_0,\xi_0}(x)}dx\right|^2 > 0,$$

and the fact that $\|G_\hbar\|_1 = 1$, implies that $\|H_\hbar\|_1 < \infty$ (see [4, Equation (1.19)]).

Now, in order to prove the L^1-convergence of $H_\hbar$ toward w we will use Theorem 2.1, together with some aspects of wavelet theory. First we note that in $\big(L^1(\mathbb{R}^n_x \times \mathbb{R}^n_\xi), \|\cdot\|_1\big)$, we have

$$\|H_\hbar - w\|_1 \leq \|(W_\hbar - w) * G_\hbar\|_1 + \|G_\hbar * w - w\|_1. \tag{3.6}$$

As it is showed in [1], the Koopman formalism (1.3), implies that if $w_0 \in L^p(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$, then w belongs also to $L^p(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ and (3.2) holds for $p = 1$, which yields to the convergence of the last term of (3.6) to zero as $\hbar \to 0$. So, it remains to prove the convergence of $\|(W_\hbar - w) * G_\hbar\|_1$ to zero, as $\hbar \to 0$.

Let $I_k = \{(x,\xi) \in \mathbb{R}^n_x \times \mathbb{R}^n_\xi \;:\; k-1 \leq \sqrt{|x|^2+|\xi|^2} < k\}$ and χ_I the characteristic function of I. We remark that $\sum_{k\geq 1} \chi_{I_k} = 1$ for any $(x,\xi) \in \mathbb{R}^n_x \times \mathbb{R}^n_\xi$. Let $s(x,\xi) := \mathrm{e}^{-\mathrm{i}\theta(x,\xi)}$, where $\theta(x,\xi) = \arg(W_\hbar - w)(x,\xi)$, so $[(W_\hbar - w)s](x,\xi) \geq 0$ and $s(x,\xi)s^*(x,\xi) = 1$. Finally by writing $W_\hbar - w = (\mathrm{e}^{tL_\hbar} - \mathrm{e}^{tL})w_0$, we get

$$\|(W_\hbar - w) * G_\hbar\|_1 = \int_{\mathbb{R}^n_x \times \mathbb{R}^n_\xi} \left| \int_{\mathbb{R}^n_y \times \mathbb{R}^n_\eta} (\mathrm{e}^{tL_\hbar} - \mathrm{e}^{tL}) w_0(y,\eta) G_\hbar(x-y, \xi-\eta)\, dy d\eta \right| dx d\xi$$

$$\leq \int_{\mathbb{R}^n_x \times \mathbb{R}^n_\xi} \int_{\mathbb{R}^n_y \times \mathbb{R}^n_\eta} (\mathrm{e}^{tL_\hbar} - \mathrm{e}^{tL}) w_0(y,\eta) s(y,\eta) \left| s^*(y,\eta) G_\hbar(x-y, \xi-\eta) \right| dy d\eta dx d\xi$$

$$= \sum_{k\geq 1} \int_{\mathbb{R}^n_y \times \mathbb{R}^n_\eta} (\mathrm{e}^{tL_\hbar} - \mathrm{e}^{tL}) f_k(y,\eta) g_k(y,\eta)\, dy d\eta,$$

where $f_k(y,\eta) = \frac{1}{k} w_0(y,\eta)s(y,\eta)$ and g_k defined as in Lemma 3.3 .

Now, since $\mathrm{e}^{tL_\hbar}$ converges ultraweakly (in the sense of (2.1)) to e^{tL}, it follows that $H_\hbar$ converges in $L^1(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ toward w. □

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