THE NULL VOLATILITY LIMIT OF THE CHAOTIC BLACK-SCHOLES EQUATION

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Abstract. *In* [4, 5] *it is proved that the Black-Scholes semigroup is chaotic in various Banach spaces of continuous functions. Here we will see why in these spaces the null volatility has no sense, while if we consider the generalized Black-Scholes equation in which there are two interest rates r_1 and r_2, with $r_1 > r_2$ then the corresponding Black-Scholes semigroup converges strongly to a chaotic semigroup when the volatility $\sigma \to 0$. It is then shown that, keeping the volatility fixed and positive, the coefficients in the lower order terms in the generalized Black-Scholes equation can be replaced by any real constants, and one still obtains chaotic semigroups. Finally, the heat equation on the real line with arbitrary coefficients in the lower order terms is shown to be chaotic.*

1. Introduction and the main results.

In [4], it is proved that in the complex Banach spaces

$$Y^{s,\tau} := \{u \in C((0,\infty)) : \lim_{x\to\infty} \frac{u(x)}{1+x^s} = 0,\ \lim_{x\to 0} \frac{u(x)}{1+x^{-\tau}} = 0\}$$

with norm $\|u\|_{Y^{s,\tau}} = \sup_{x>0} \left| \frac{u(x)}{(1+x^s)(1+x^{-\tau})} \right| < \infty$, the semigroup of the Black-Scholes equation

$$(\mathbf{BS}) \qquad \begin{cases} \frac{\partial v}{\partial t} = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + rx\frac{\partial v}{\partial x} - rv & \text{in } \mathbb{R}^+ \times \mathbb{R}^+; \\ v(0,t) = 0 & \text{for } t \in \mathbb{R}^+; \\ v(x,0) = f(x) & \text{for } x \in \mathbb{R}^+. \end{cases}$$

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is chaotic for $s > 1$, $\tau \geq 0$ where $\sigma > 0$ is the volatility. In other words it is proved that the problem **(BS)** generates a chaotic semigroup in the following sense.

Definition 1. A strongly continuous semigroup (or (C_0) semigroup) $T = \{T(t) : t \geq 0\}$ of bounded linear operators on a Banach space Y is called *hypercyclic* if there exists a vector $f \in Y$ such that its orbit $\{T(t)f : t \geq 0\}$ is dense in Y, and T is called *chaotic* on Y if in addition the set of periodic points of T,

$$Y_{per} := \{f \in Y \ : \ \text{there exists } t_0 > 0 \text{ such that } T(t_0)f = f\},$$

is dense in Y. Finally, T is called chaotic if it is chaotic on some Banach space Y. (This is analogous to calling a problem wellposed if it is wellposed in some specific sense on some appropriate space.)

Our main aim in this note is to establish a version of the null volatility limit of chaotic Black-Scholes equation. Let us denote by $X = Y^{s,\tau}$ and by $Z = X_{\mathbb{R}}$ the space of real functions in X. First of all in the spaces X and Z this limit has no sense since the semigroup generated by **(BS)** is chaotic if $s > 1$ and $\tau \geq 0$ when $\sigma > 0$ and we cannot take $\sigma = 0$. For simplicity we take $\tau = 0$ and we consider this problem in

$$Y^s := \{f \in C([0,\infty)) \ : f(0) = 0 \ \text{ and } \ \lim_{x\to\infty} \frac{u(x)}{1+x^s} = 0\},$$

equipped with the norm

$$\|f\|_s := \sup_{x\in[0,\infty)} \frac{|f(x)|}{1+x^s}, \quad \text{for } \ s > 1,$$

in which the initial value that Black and Scholes used, $f(x) = (x-p)_+$ for some $p > 0$, of the Black-Scholes equation. For $\sigma = 0$, the null volatility Black-Scholes equation becomes

$$(\mathbf{NVBS}) \qquad \begin{cases} \frac{\partial v}{\partial t} = rx\frac{\partial v}{\partial x} - rv & \text{in } \ \mathbb{R}^+ \times \mathbb{R}^+; \\ v(0,t) = 0 & \text{for } \ t \in \mathbb{R}^+; \\ v(x,0) = f(x) & \text{for } \ x \in \mathbb{R}^+ \end{cases}$$

which generates the group

$$S(t) := \mathrm{e}^{-rt} S_r(t), \tag{1.1}$$

where $(S_r(t)f)(x) = f(\mathrm{e}^{rt}x)$. According to [4, Theorem 2.4], the family S_r forms a (C_0) group on Y^s for $s \geq 1$ and furthermore

$$\|S_r(t)f\|_s \leq \mathrm{e}^{rs|t|}\|f\|_s. \tag{1.2}$$

In fact

$$\|S_r(t)f\|_s = \sup_{x>0} \frac{|f(e^{rt}x)|}{(1+x^s)} = \sup_{y>0} \left| \frac{f(y)}{(1+e^{-rst}y^s)} \right|$$
$$\leq \sup_{y>0} \left| \frac{1+y^s}{(1+e^{-rst}y)} \right| \|f\|_s.$$

and (1.2) follows from $\left| \frac{1+y^s}{(1+e^{-rst}y^s)} \right| \leq e^{rs|t|}$, for all $y \geq 0$. In this case $S(t)$ is not an appropriate candidate to be chaotic, since as $s \to 1$, then $S(t)$ converges to a contraction for each $t \geq 0$. Note that taking $\tau = 0$ incorporates explicitly the boundary condition $v(0,t) = 0$.

So, let us consider the following generalized Black-Scholes equation

$$\textbf{(GBS)} \qquad \begin{cases} \frac{\partial v}{\partial t} = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + r_1 x \frac{\partial v}{\partial x} - r_2 v & \text{in } \mathbb{R}^+ \times \mathbb{R}^+, \\ v(0,t) = 0 & \text{for } t \in \mathbb{R}^+, \\ v(x,0) = f(x), & \text{for } x \in \mathbb{R}^+, \end{cases}$$

where r_1 and r_2 are two different interest rates. Such a model arises in the pricing of Asian options (see [1]).

Remark 1. In the **(BS)**, **(NVBS)** and **(GBS)** systems, we have the boundary condition $v(0,t) = 0$. In $X = Y^{s,\tau}$ for fixed $s > 1$ and $\tau \geq 0$, the functions f in this space satisfy $f(0) = 0$ if and only if $\tau = 0$. Thus the $\tau = 0$ case is the best fit for these systems. What is interesting is that the theory holds in the appropriate spaces for all $\tau > 0$ as well. That means that the boundary condition at the origin is not needed for wellposedness, provided one works in the right spaces. In these cases, the space

$$W = \{f \in Y^{s,\tau} \ : \ f(0) = 0\}$$

is an invariant subspace for the semigroup. This means, mathematically, that the Dirichlet boundary condition at 0 is optional, and one can consider more general economic models.

The main theorem on chaos in [4, 5], which corresponds to $r_1 = r_2 = r$ below, extends to the more general case as follows. From now on, we let $\nu = \sqrt{\sigma}/2$.

Theorem 1. *Let $s > 1, \tau \geq 0$ and define the complex Banach space*

$$Y^{s,\tau} = \{u \in C(0,\infty) : \lim_{x\to 0} \left| \frac{u(x)}{1+x^{-\tau}} \right| = \lim_{x\to\infty} \left| \frac{u(x)}{1+x^s} \right| = 0\}$$

with norm

$$\|u\|_{s,\tau} = \sup_{x>0} \left| \frac{u(x)}{(1+x^{-\tau})(1+x^s)} \right|.$$

*The generalized Black-Scholes problem (**GBS**) for $r_1, r_2 > 0$ is governed by a (C_0) semigroup*

$$T_{r_1,r_2} = \{T_{r_1,r_2}(t) : t \geq 0\}$$

on $Y^{s,\tau}$ which is chaotic, provided that

$$r_2 < \nu s(\nu s - \nu + \frac{r_1}{2}). \tag{1.3}$$

If $Y^{s,\tau}_{\mathbb{R}}$ consists of the real functions in $Y^{s,\tau}$, then S_T, the restriction of T_{r_1,r_2} to $Y^{s,\tau}_{\mathbb{R}}$, is a chaotic (C_0) semigroup in this case.

Proof. The proof follows from the proof in [4] coupled with the remarks and Figure 1' in [5].

The point on the x axis labeled as ν in Figure 1' must be relabeled as a; and a must satisfy $0 < a < \nu s$. Here a must be the larger root of the quadratic function

$$x^2 + \left(\frac{r_1}{\nu} - \nu\right) x - r_2.$$

This root is given by

$$a = \frac{1}{2}\left\{\left(\nu - \frac{r_1}{\nu}\right) + \sqrt{\left(\nu - \frac{r_1}{\nu}\right)^2 + 4r_2}\right\}.$$

Clearly for $r_2 \leq r_1$ one has $0 < a \leq \nu < \nu s$. But in general in order to have $a < \nu s$ the range of values of r_2 for fixed r_1 is given by (1.3). □

By taking $\mathcal{B}_\nu = \nu^2 C_1 + C_2$, where $C_1 := x^2 \frac{d^2}{dx^2}$ and $C_2 := r_1 D_1 - r_2 I$, with $D_r = rx\frac{d}{dx}$, then **(GBS)** can be written as an abstract Cauchy problem

$$(\mathbf{AGBS}) \qquad \begin{cases} \frac{\partial v}{\partial t}(x,t) = \mathcal{B}_\nu v(x,t); \\ v(0,t) = 0; \\ v(x,0) = f(x), \qquad \text{for } x \in \mathbb{R}^+. \end{cases}$$

Then by applying Lemma 2.5 of [4] we can prove that $\{e^{t\mathcal{B}_\nu}\}_{t\geq 0}$ is an analytic semigroup in Y^s. Now we are in position to state our main theorem about the zero volatility limit.

If one allows r_1 and r_2 to be any real numbers, them the proof of Theorem 1 is easily modified to yield that the corresponding (C_0) semigroup is chaotic on $Y^{s,\tau}$ for any nonnegative τ and all sufficiently large s (with $\nu > 0$ fixed).

Theorem 2. *If $s > 1$ and $r_1 > r_2 > 0$ the semigroup $\{e^{t\mathcal{B}_\nu}\}_{t\geq 0}$ converges strongly to $\{e^{tC_2}\}_{t\geq 0}$, which is a chaotic semigroup in Y^s as $\nu \to 0$.*

The convergence of $\{e^{t\mathcal{B}_\nu}\}_{t\geq 0}$ to $\{e^{tC_2}\}_{t\geq 0}$ follows from the following abstract result.

Lemma 1. *Let A and B be the generators of two (C_0) semigroups $\{\mathrm{e}^{tA}\}_{t\geq 0}$ and $\{\mathrm{e}^{tB}\}_{t\geq 0}$, on a Banach space Y which commute on $D(A)\cap D(B)$, assumed to be a core for both A and B. Suppose that A_ε, the closure of $\varepsilon A + B$, is also the generator of a (C_0) semigroup $\{\mathrm{e}^{tA_\varepsilon}\}_{t\geq 0}$. Then $\mathrm{e}^{tA_\varepsilon}$ converges strongly to e^{tB}, as $\varepsilon \to 0$.*

Proof. For any $f \in Y$, we denote $g = \mathrm{e}^{tB} f$ and we use the strong continuity of $\{\mathrm{e}^{sA}\}_{s\geq 0}$ and we get $\|\mathrm{e}^{sA}g - g\| \to 0$ as $s \to 0$. For a fix $t > 0$, take $s = \varepsilon t$ as ε tends to zero. This proves that $\mathrm{e}^{t(\varepsilon A+B)}$ converges strongly to e^{tB}, as $\varepsilon \to 0$. For achieving the proof it is enough to prove that $\mathrm{e}^{\varepsilon tA}\mathrm{e}^{tB} = \mathrm{e}^{tA_\varepsilon}$. The fact that A and B commute on the (by assumption) common core $D(A) \cap D(B)$ shows that $\mathrm{e}^{\varepsilon tA}\mathrm{e}^{tB}$ is also a semigroup while for every $f \in D(A) \cap D(B)$,

$$\frac{d}{dt}\left(\mathrm{e}^{\varepsilon tA}\mathrm{e}^{tB}\right) f\Big|_{t=0} = \left(\varepsilon A\mathrm{e}^{\varepsilon tA}\mathrm{e}^{tB} + \mathrm{e}^{\varepsilon tA}B\mathrm{e}^{tB}\right) f\Big|_{t=0} = (\varepsilon A + B)f,$$

their generators coincide, which proves $\mathrm{e}^{\varepsilon tA}\mathrm{e}^{tB} = \mathrm{e}^{tA_\varepsilon}$. □

Concerning the fact that the semigroup $\{\mathrm{e}^{tC_2}\}_{t\geq 0}$ is chaotic, an ℓ_1 version of this is proved in [8] and a weighted L^p version in [2]. For proving this in our Banach space Y, we will use the following Lemma, which is proved in [3].

Lemma 2. (See [3, Theorem 2.3]) *Let $T = \{T(t)\}_{t\geq 0}$ be a (C_0) semigroup on the Banach space Y. Define $Y_0 := \{f \in Y : \lim_{t\to\infty} T(t)f = 0\}$ and $Y_\infty := \{f \in X :$ for any $\varepsilon > 0$, there exist $g \in Y$ and $t > 0$ such that $\|g\| < \varepsilon$ and $\|T(t)g - f\| < \varepsilon\}$. If Y_0 and Y_∞ are both dense in Y, then T is hypercyclic.*

Lemma 3. *If $r_1 > r_2 > 0$, then the (C_0) group $\{\mathrm{e}^{tC_2}\}_{t\in\mathbb{R}}$ is chaotic in Y^s.*

Proof. Let us denote by h_λ the function $x \mapsto x^\lambda$. For any $\lambda \in \Sigma := \{z \in \mathbb{C} : 0 < \mathrm{Re}\, z < 1\}, h_\lambda \in Y^s$ and the function $\lambda \mapsto h_\lambda$ is analytic on the open strip Σ. Let $D_1 f(x) = xf'(x)$. Furthermore $h_\lambda \in D(D_1) := \{f \in Y^s : x\frac{df}{dx} \in Y^s\}$, hence Σ belongs to the spectrum $\sigma(D_1)$ of D_1 and $D_1 h_\lambda = \lambda h_\lambda$ shows that for any $\lambda \in \Sigma, h_\lambda$ is its eigenfunction. Note that we have assumed that $0 < \frac{r_2}{r_1} < 1$.

(1) <u>Y_0 is dense in Y^s.</u> Indeed, for any $\mu \in [0, \frac{r_2}{r_1})$, one has $\mathrm{e}^{tC_2}h_\mu = \mathrm{e}^{t(r_1\mu - r_2)}h_\mu \to 0$ as $t \to \infty$. Hence h_μ and any finite linear combination of these vectors belongs to Y_0. Since any point of $(0, \frac{r_2}{r_1})$ is an accumulation point of this interval in Σ, $Span\{h_\mu : \mu \in (0, \frac{r_2}{r_1})\}$ is dense in Y^s. Indeed, if the span was not dense, there would exist a nonzero functional $\Phi \in (Y^s)^*$ such that $\langle \Phi, h_\mu\rangle = 0$ for any h_μ constructed as above. This would imply that the analytic function represented by $\mu \mapsto \langle \Phi, h_\mu\rangle$ is zero on a set of points μ with an accumulation point, which implies it is identically zero, in contradiction with assumption that $\Phi \neq 0$. Consequently, Y_0 is also dense in Y^s.

(2) Y_∞ is dense in Y^s. Since e^{tC_2} is a (C_0) group, for $1 > \mu > \frac{r_2}{r_1}$, one has $\mathrm{e}^{-tC_2}h_\mu = \mathrm{e}^{t(-r_1\mu+r_2)}h_\mu \to 0$ as $t \to \infty$. This shows any such $h_\mu \in Y_\infty$, in fact for any $\varepsilon > 0$ one can choose t large enough such that for $g_\mu = \mathrm{e}^{-tC_2}h_\mu$, $\|g_\mu\| < \varepsilon$ and $\|\mathrm{e}^{tC_2}g_\mu - h_\mu\| = 0$. The same argument as in the part (1) shows that Y_∞ is dense in Y^s.

(3) Y_{per} is dense in Y^s. Indeed, since for $0 < \operatorname{Re}\mu = \frac{r_2}{r_1} < 1$, the spectrum of the operator C_2 contains a nonempty segment of the imaginary axis. On that segment, the complex numbers $\kappa = \mathrm{i}\frac{m}{n}, (m,n) \in \mathbb{Z}^2$ form a dense set. Each such κ is an eigenvalue of C_2 with eigenvector $h_{(\kappa+r_1)/r_2}$ which belongs to Y_{per} and once more by the same argument linear combinations of such vectors are dense in Y^s.

□

Consequently for any $f \in Y^s$, writing $\mathrm{e}^{t\mathcal{B}_\nu}f$ as $\left(\mathrm{e}^{t\nu^2C_1}\right)\mathrm{e}^{tC_2}f$ and using Lemma 1 by taking $\nu^2 = \varepsilon$, $C_1 = A$ and $C_2 = B$, this implies that $\mathrm{e}^{t\mathcal{B}_\nu}$ converges to e^{tC_2} which is chaotic. This completes the proof of our main Theorem 2.

2. Chaos for generalized heat and Black-Scholes equations.

In this section, we obtain a generalization of Theorem 1 and we prove its analogue in the heat equation context. This gives an extension of an earlier work [6].

Consider on a core the most basic one dimensional Lie groups : G_1, the real numbers under addition, and G_2 the positive real numbers under multiplication. The exponential function

$$\exp\ :\ G_1 \to G_2$$

is a Lie group isomorphism. For complex valued functions f_j on G_j, the translation groups T_j are defined by

$$\begin{aligned} T_1(t)f_1(y) &= f_1(y+t) \qquad t \in \mathbb{R},\ y \in \mathbb{R} = G_1,\\ T_2(t)f_2(x) &= f_2(\mathrm{e}^t x) \qquad t \in \mathbb{R},\ x \in G_2. \end{aligned}$$

Soon we will specialize the functions f_j so that

$$T_j = \{T_j(t)\ :\ t \in \mathbb{R}^+\}$$

becomes a (C_0) group on a suitable Banach space Z_j. If A_j denotes the generator of T_j, then, formally, since

$$A_j f_j(z) = \frac{d}{dt}T_j(t)f(z)|_{t=0},$$

we deduce

$$A_1 f_1(y) = \frac{d}{dy} f_1(y),$$
$$A_2 f_2(x) = x\frac{d}{dx} f_2(x).$$

We also have

$$A_1^2 = D_y^2, A_2^2 = x^2 D_x^2 + x D_x.$$

If f_1 is locally integrable, and grows no faster than $\exp(c|y|^{2-\varepsilon})$ as $|y| \to \infty$, then the unique solution of

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} \qquad t > 0,\ y \in \mathbb{R}$$
$$u(y,0) = f_1(y) \qquad y \in \mathbb{R}$$

(and $u(y,t)$ satisfies the integrability and growth conditions imposed on f_1) is given by

$$u(y,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} \exp(-z^2/4t) f_1(y-z) dz.$$

Regard $f_1(y-z)$ as $T_1(-y)f_1(z)$. In 1947, Romanoff (see [7, 9]) proved that this works in the general (C_0) group context. Namely, if A generates a (C_0) group T on a Banach space X, then

$$\frac{du}{dt} = A^2 u, \quad u(0) = f$$

is uniquely solved by

$$u(t) = S(t)f = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} \exp(-z^2/4t) T(-z) f dz$$

which defines a (C_0) semigroup S on X, which is analytic in the right half plane $\{t \in \mathbb{C} \ : \ \mathrm{Re}(t) > 0\}$. Thus the solution of

$$\frac{du}{dt} = A_2^2 u, \quad u(0) = f_2 \tag{2.1}$$

is given by

$$u(y,t) = S_2(t)f = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} \exp(-z^2/4t) T_2(-z) f(y) dz$$
$$= \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} \exp(-z^2/4t) f(\mathrm{e}^{-z} y) dz.$$

The parabolic problem (2.1) reduces to the concrete form

$$\frac{\partial u}{\partial t} = y^2 \frac{\partial^2 u}{\partial y^2} + y \frac{\partial u}{\partial y}.$$

The following generalized version of the Black-Scholes equation

$$\frac{\partial u}{\partial t} = \frac{\sigma^2}{2} y^2 \frac{\partial^2 u}{\partial y^2} + ay\frac{\partial u}{\partial y} + bu$$

can be written in the form

$$\frac{du}{dt} = Bu := \left(\frac{\sigma^2}{2} A_2^2 + (a - \frac{\sigma^2}{2}) A_2 + bA_2 \right) u, \tag{2.2}$$

where B is a quadratic polynomial in A_2. Here a, $b \in \mathbb{R}$ are arbitrary. Since B is a positive linear combination of commuting (C_0) semigroup generators, the unique solution of (2.2) with initial condition $u(0) = f$ is given by

$$u(.,t) = \mathrm{e}^{tB} f = \left(\exp((t\sigma^2/2)A_2^2) \mathrm{e}^{t\alpha A_2} \mathrm{e}^{tb} \right) f$$

for $t > 0$ and $\alpha = a - \dfrac{\sigma^2}{2}$.

By the proof of Theorem 1, the (C_0) semigroup S_B generated by B is chaotic on (uncountably) infinitely many of the spaces $Y^{s,\tau}$ for $s > 1$ and $\tau \geq 0$. One simply needs to fix $\sigma > 0$, a, $b \in \mathbb{R}$ and then take s large enough. (For many choices of σ, a, b, S_B is chaotic on all the $Y^{s,\tau}$ for $s > 1$ and $\tau \geq 0$, but that is not our concern here.)

The space

$$X_{s,\tau} := \left\{ f \in C(\mathbb{R}) \ : \ \frac{f(y)}{(1 + \mathrm{e}^{ys})(1 + \mathrm{e}^{-y\tau})} \in C_0(\mathbb{R}) \right\}$$

with norm

$$\|f\|_{X_{s,\tau}} = \left\| \frac{f(y)}{(1 + \mathrm{e}^{ys})(1 + \mathrm{e}^{-y\tau})} \right\|_\infty$$

is a Banach space which is isometrically isomorphic to $Y^{s,\tau}$ via the map

$$U_{s,\tau} \ : \ Y^{s,\tau} \to X_{s,\tau},$$

induced from $y = \mathrm{e}^x : G_1 \to G_2$. Thus

$$g(x) = U_{s,\tau} f(x) = f(\mathrm{e}^x) = f(y).$$

This formula shows that $U_{s,\tau}$ maps

$$\frac{1}{(1 + x^s)(1 + x^{-\tau})} \text{ to } \frac{1}{(1 + \mathrm{e}^{ys})(1 + \mathrm{e}^{-y\tau})}$$

and the asserted properties of $U_{s,\tau}$ follow.

Summarizing, this discussion yields the following result.

Theorem 3. *Define*

$$C_{\alpha\beta\gamma} = \alpha A_1^2 + \beta A_1 + \gamma I$$

with $\alpha > 0$ *and* β, $\gamma \in \mathbb{R}$. *Then* $C_{\alpha\beta\gamma}$ *generates a* (C_0) *semigroup which is chaotic on infinitely many of the spaces* $X_{s,\tau}$ *and* $X^{\mathbb{R}}_{s,\tau}$ *for* $s > 1$ *and* $\tau \geq 0$.

The superscript $\mathbb{R}$ means the real functions in the space.

Thus *all* of the heat equations

$$u_t = \alpha u_{xx} + \beta u_x + \gamma u$$

for $t \geq 0$, $x \in \mathbb{R}$ and $\alpha > 0$, β, $\gamma \in \mathbb{R}$ are chaotic. The same is true for *all* the generalized Black-Scholes equations

$$u_t = \alpha x^2 u_{xx} + \beta x u_x + \gamma u$$

for $t \geq 0$, $x \in (0, \infty)$ and $\alpha > 0$, β, $\gamma \in \mathbb{R}$.

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