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The chaotic heat equation

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Abstract. The one dimensional heat equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} + cu$$

for $x \in \mathbb{R}$, $t \geq 0$ is governed by a chaotic semigroup for certain values of the coefficients $(a,b,c) \in \mathbb{R}^3$ on certain weighted supremum norm spaces.

1 Introduction

In 1997, G. Herzog [1] proved the universality of the heat equation in an exponentially weighted space. Concerning the universal families K. Grosse-Erdmann published an expository paper [2] in which we can find the following definition.

Definition 1. Let X and Y be topological spaces and $T_\iota : X \to Y \ \ (\iota \in I)$ continuous mappings. Then an element $x \in X$ is called *universal* (for the family $\{T_\iota\}_{\iota\in I}$) if the set $\{T_\iota x \, : \, \iota \in I\}$ is dense in Y. The set of universal elements is denoted by $\mathcal{U} = \mathcal{U}(T_\iota)$. The family $\{T_\iota\}_{\iota\in I}$ is called universal if it has a universal element.

The set $\mathcal{U}$ of universal elements may be expected to be huge, in fact residual in the sense of Baire categories. In [1] G. Herzog defined for $\alpha > 0$, the separable Banach space

$$E_\alpha := \{\varphi \in C(\mathbb{R},\mathbb{R}) : \quad \lim_{x\to\pm\infty} e^{-\alpha|x|}\varphi(x) = 0\}$$

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equipped with the norm

$$\|\varphi\|_{E_\alpha} = \sup_{x\in\mathbb{R}} e^{-\alpha|x|}|\varphi(x)|$$

and by taking $T_t : E_\alpha \to C(\mathbb{R},\mathbb{R})$,

$$[T_t\varphi](x) = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} exp\left(-\frac{(x-s)^2}{4t}\right)\varphi(s)ds, \tag{1.1}$$

he showed that the set $\mathcal{U} := \{\varphi \in E_\alpha \ : \ \overline{\{T_n\varphi \ : \ n\in\mathbb{N}\}} = C(\mathbb{R},\mathbb{R})\}$ is a residual subset of E_α for some $\alpha > 0$.

In operator theory, when one studies the iterates $\{T^n\}_{n\in\mathbb{N}_0}$ of a (continuous linear) operator T, universal elements are usually called *hypercyclic*. More generally a (C_0) semigroup $T = \{T(t)\}_{t\geq 0}$ on a separable Banach space X is called *hypercyclic*, if there exists an element $x\in X$ such that its orbit $\{T(t)x \ : \ t\geq 0\}$ is dense in X.

The idea of universality is well known in topological dynamics under the name of *topological transitivity*. This definition has been translated for a (C_0) semigroup in the following manner.

Definition 2. Let $T = \{T(t)\}_{t\geq 0}$ be a (C_0) semigroup on a separable Banach space X. T is called *chaotic* if it is hypercyclic and there are vectors $y\in X$ such that y has a periodic orbit, i.e., there exists $\tau = \tau(y) > 0$ such that $T(\tau)y = y$; and the set of such y's is dense in X.

Recently F. Astengo and B. Di Blasio, in a nice paper [3], proved that if $A(x)$ defined in [3, p. 50] is the density of an absolutely continuous measure μ on the half line $(0,\infty)$, if $L^p = L^p((0,\infty),\mu)$ with $d\mu(x) = A(x)dx$, is the corresponding Lebesgue space with $2 < p < \infty$, and if $\Delta_p = d^2/dx^2$ is the Laplacian on L^p with maximal domain, then there is a constant $B(p)$ such that if $\beta > B(p)$, then $\Delta_p + \beta I$ generates a (C_0) chaotic semigroup on L^p. In this paper we prove a similar result in spaces of continuous functions.

In [4, 5] we have proved that the Black-Scholes semigroup is chaotic in certain Banach spaces which will be define later. The Black-Scholes (or Black-Merton-Scholes) equation is

$$\frac{\partial u}{\partial t} = \frac{\sigma^2}{2}y^2\frac{\partial^2 u}{\partial y^2} + ry\frac{\partial u}{\partial y} - ru, \tag{1.2}$$

where $u = u(y,t), y > 0,\ t\geq 0$, and r,σ are positive constants representing the interest rate and the volatility respectively. In a series of papers in the period 1973-1976, F. Black, R. Merton and M. Scholes showed that the fair value of a stock option u satisfies this equation and developed a significant economic model around it. Here y is the current value of the asset. It was surprising to deduce a deterministic model for a study which started with an Ito stochastic differential equation. This equation had a profound effect on the development of financial mathematics and it led to Merton and Scholes receiving the 1998 Nobel prize in Economics; by that time Black had passed away.

The initial condition this model requires is

$$u(y,0) = max\{0, y-p\}$$

for a fixed given $p > 0$. Thus a rigorous study of the semigroup governing (1.2) must take place on a function space which includes functions which grows at least linearly at infinity. On this it is natrual of use supremum norm spaces with weights, namely, $X = \{f\in C(0,\infty) \ : \ f\rho\in C_0(0,\infty)\}$ where ρ is a continuous weight function on $(0,\infty)$ which grows faster than linearly at infinity. Here $C_0(0,\infty)$ consists of those continuous functions h on $(0,\infty)$ such that $\lim_{x\to L} h(x) = 0$ for $L\in\{0,\infty\}$.

The idea of *chaos* is most easily understood using the simple example of J. von Neumann in which the iterative process $x_{n+1} = 4x_n(1-x_n)$ generates a chaotic sequence. In his book [6], R. Devaney defines the notion of chaos for a dynamical system by basing itself on this example. Let

$$F : [0,1] \mapsto [0,1]$$

be defined by

$$F(x) = 4x(1-x).$$

The map $E : x \to \theta$, where $x = \sin^2\theta$, is a diffeomorphism from $[0,1]$ to $[0,\pi/2]$. Since $F(x) = 4\sin^2\theta(1-\sin^2\theta) = \sin^2(2\theta)$, we deduce that the n-fold composition F^n satisfies

$$F^n(x) = \sin^2(2^n\theta)$$

for all $n = 1,2,\cdots$. Then the orbits $\{F^n(x)\}_{n=0}^{\infty}$ are dense in $[0,1]$ whenever $\theta/2\pi$ is irrational, and they are periodic whenever $\theta/2\pi$ is rational. Thus F is a chaotic (nonlinear) map.

2 The chaotic Black-Scholes semigroup

Let $s,\tau \geq 0$ and define

$$Y_{s,\tau} := \{f \in C(0,\infty) \ : \ \frac{f(x)}{(1+x^s)(1+x^{-\tau})} \in C_0(0,\infty)\}$$

with the norm

$$\|u\|_{s,\tau} = \sup_{x\geq 0}\left|\frac{u(x)}{(1+x^s)(1+x^{-\tau})}\right| < \infty.$$

The above space is a complex separable Banach space.

Theorem 1. ([4, 5]) *Let $s \geq 1, \tau \geq 0$. Let σ, r be given positive constants. Let $S = \{S(t) \ : \ t \geq 0\}$ be the semigroup governing the Black-Scholes equation* (1.2) *on $Y_{s,\tau}$. Then S is a (C_0) semigroup. Moreover, S is chaotic if $s > 1$. The same conclusions hold for the restriction of S to*

$$Y_{s,\tau}^{\mathbb{R}} = \{f \in Y_{s,\tau} \ : \ f \text{ is a real function}\}.$$

Note that (1.2) determines a chaotic semigroup on any of the spaces

$$\{Y_{s,\tau} \ : \ \tau \geq 0, s > 1\}.$$

3 The chaotic heat equation

The exponential function $y = e^x$ is a Lie group diffeomorphism from $(\mathbb{R},+)$ to $((0,\infty),.)$. This induces a map

$$E : \ f \mapsto g,$$

mapping functions on $\mathbb{R}$ to function on $(0,\infty)$, defined by

$$(Ef)(y) = f(\log y) = g(y)$$

for $y \in (0,\infty)$ and $x = \log y \in \mathbb{R}$ so that $y = e^x$. Let w be a positive continuous function on $\mathbb{R}$. Let X_w be the weighted sup norm space

$$X_w = \{f \in C(\mathbb{R}) \ : \ \frac{f}{w} \in C_0(\mathbb{R})\}$$

with $\|f\|_{X_w} = \sup_{x\in\mathbb{R}} |f(x)|/w(x)$. Then E is an isometric isomorphism from X_w to $Y_{\widetilde{w}}$, where

$$Y_{\widetilde{w}} = \{g \in C(0,\infty) \ : \ \frac{g}{\widetilde{w}} \in C_0(0,\infty)\},$$

equipped with $\|g\|_{X_{\widetilde{w}}} = \sup_{y>0} |g(y)|/\widetilde{w}(y)$, and $\widetilde{w} = Ew$.

Let $Y_{s,\tau}$ be as in Section 2. Define

$$X_{s,\tau} = X_w,$$

where $w(x) = (1+e^{-\tau x})(1+e^{sx})$, for $x \in \mathbb{R}$. Then

$$\widetilde{w}(y) = (Ew)(y) = (1+y^{-\tau})(1+y^s),$$

and $Y_{s,\tau} = Y_{\widetilde{w}}$.

Now we work forward the goal of interpreting the result of [4] for functions on $(0,\infty)$ in the context of the functions on $\mathbb{R}$. The translation group T_a for functions on $\mathbb{R}$ with parameter a [resp., S_a for functions on $(0,\infty)$] is given by

$$[T_a(t)f](x) = f(x+at) \quad \text{for, } x \in \mathbb{R},\ t \in \mathbb{R}$$

[resp.

$$[S_a(t)g](y) = g(e^{at}y) \quad \text{for, } y > 0,\ t \in \mathbb{R}].$$

Here a is a fixed positive constant. The generator is, formally,

$$A_a = T_a'(0) = a\frac{d}{dx} \quad [\text{resp. } B_a = S_a'(0) = ay\frac{d}{dy}].$$

Then T_a [resp. S_a] is a (C_0) isometric group on $C_0(\mathbb{R})$ [resp. on $C_0(0,\infty)$. In [4] it was shown that S_a is a (C_0) group on $Y_{s,\tau}$ for $s > 1, \tau \geq 0$. Since E is an isometric isomorphism from $X_{s,\tau}$ to $Y_{s,\tau}$, T_a is a (C_0) group on $X_{s,\tau}$ for the same values of s and τ.

Now suppose G generates a (C_0) group $\{e^{tG} \ : \ t \geq 0\}$ on a Banach space Z. Then G^2 generates a (C_0) semigroup on Z, analytic in the right half plane, given by

$$e^{tG^2}h = \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-z^2/4t} e^{-zG} h dz,$$

for $h \in Z$ and $\mathrm{Re}(t) > 0$. This result is due to N. P. Romanoff (see [7] and [8, p. 120]). Consequently,

$$e^{tA_a^2} f(x) = \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-z^2/4t} f(x-az) dz, \tag{3.1}$$

$$e^{tB_a^2} g(y) = \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-z^2/4t} g(e^{-az}y) dz. \tag{3.2}$$

Since $A_a^2 = a^2\frac{d^2}{dx^2}$ and $B_a^2 = a^2y^2\frac{d^2}{dy^2} + a^2y\frac{d}{dy}$, the Black-Scholes equation (1.2) is

$$\frac{du}{dt} = B^2_{\sigma/\sqrt{2}}u + (r - \frac{\sigma^2}{2})B_1 u - rIu, \qquad (3.3)$$
$$= \sum_{j=1}^{3} G_j u.$$

The three generators G_1, G_2, G_3 all commute, thus $e^{t\sum_{j=1}^{3} G_j} = \prod_{j=1}^{3} e^{tG_j}$, and the three factors in the product can be written in any order. Specifically, we have

$$G_1 = B^2_{\sigma/\sqrt{2}}, \quad G_2 = (r - \frac{\sigma^2}{2})B_1, \quad G_3 = -rI.$$

Consequently the unique solution of (3.3) with initial condition $u(0) = g$ is given by

$$\begin{aligned} u(t,y) &= e^{tG_3} e^{tG_1} e^{tG_2} g \\ &= \frac{e^{-rt}}{\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-z^2/4t} \left[e^{tG_2} g\right] \left(e^{-\frac{\sigma}{\sqrt{2}}z} y\right) dz \\ &= \frac{e^{-rt}}{\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-z^2/4t} g\left(e^{-\frac{\sigma}{\sqrt{2}}z} e^{(r-\frac{\sigma^2}{2})t} y\right) dz. \end{aligned}$$

This formula was derived in a slightly more complicated but equivalent form in [9] and [10, p. 162].

The heat equation corresponding to (3.3) is

$$\frac{du}{dt} = A^2_{\sigma/\sqrt{2}}u + (r - \frac{\sigma^2}{2})A_1 u - rIu, \qquad (3.4)$$
$$= \sum_{j=1}^{3} H_j u.$$

Using this and (3.1), we conclude that the unique solution (in $X_{s,\tau}$) of (3.4) is given by

$$\begin{aligned} u(t,y) &= \frac{e^{-rt}}{\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-z^2/4t} \left[e^{t(r-\frac{\sigma^2}{2})} f\right](x-z) dz \\ &= \frac{e^{-rt}}{\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-z^2/4t} f(x - z - (r - \frac{\sigma^2}{2})t) dz, \end{aligned}$$

provided $u(0,x) = f(x),\ f \in X_{s,\tau}$. We rewrite the initial value problem as

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + (r - \frac{\sigma^2}{2})\frac{\partial u}{\partial x} - ru, & x \in \mathbb{R},\ t > 0, \\ u(0,x) = f(x), & x \in \mathbb{R}. \end{cases} \qquad (3.5)$$

The following analogue of Theorem 1 now follows.

Theorem 2. *Let $s > 1, \tau \geq 0$. Let σ, r be given positive constants. Let $T = \{T(t) : t \geq 0\}$ be the semigroup governing (3.5) on $X_{s,\tau}$. Then T is a chaotic (C_0) semigroup. The same conclusions hold for the restriction of T to*

$$X^{\mathbb{R}}_{s,\tau} = \{f \in X_{s,\tau} : f \text{ is a real function}\}.$$

The following Example concerning a convection-diffusion type equation can be found in [11, Example 4.12].

Example 1. The operator A defined in the following linear evolution problem

$$\begin{cases} \frac{\partial u}{\partial t} = Au := a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} + cu, & x \in [0,\infty),\ t \geq 0, \\ u(0,t) = 0, \quad \text{for } t \geq 0, \\ u(x,0) = f(x) \quad \text{for } x > 0, \end{cases}$$

in the Hilbert space $L^2(\mathbb{R}_+, \mathbb{C})$ with domain $D(A) = \{f \in W^{2,2}(\mathbb{R}_+) \,:\, f(0) = 0\}$ generates an analytic chaotic semigroup, provided that $(a,b,c) \in (0,\infty)^3$ and $0 < c < b^2/2a < 1$.

As a comparison our result treats also the cases $\{\ b = 0$ and $c < 0\ \}$ and $\{\ c < 0 < a$ and $a + b + c = 0\ \}$.

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