LECTURE 1 Crash course on MMP

Varieties = normal projective $\mathbb{Q}$factorial varieties $/\mathbb{C}$
irreducible and reduced.

normality $\Rightarrow$ $\mathrm{Sing}(X) \subseteq X$ has codim ≥ 2

set $X^{REG} = X \setminus \mathrm{Sing}(X)$. On X^{REG} we have $T_{X^{REG}}$ and its dual $T^*_{X^{REG}}$. $\det T^*_{X^{REG}} \cong \mathcal{O}(K_{X^{REG}})$

line bdle on a smooth var

$K_{X^{REG}} = \sum_{i=1}^{k} d_i D_i$ divisor, we set $K_X = \sum d_i \overline{D_i}$ Zar closure.

$X \setminus X^{REG}$ does not contain any div $\Rightarrow$ fixed $K_{X^{REG}}$, K_X is unique.

projectivity $\exists X \hookrightarrow \mathbb{P}^N$ embedding (! not unique)

proj var have many subvar, e.g. H_i = zero locus of a hom poly of deg 1 in $\mathbb{P}^N$, $X \cap H_1 \cap \dots \cap H_d \subseteq X$.

$X \cap H$ is a very ample divisor on X

A div on X is ample if $\exists m$ $mA \sim A'$ very ample.

$\mathbb{Q}$factoriality: every Weil div D has a multiple mD which is Cartier.

In particular X $\mathbb{Q}$ factorial $\forall D$ Weil div, $\forall C$ curve we can define $D \cdot C = \frac{1}{m} \deg \mathcal{O}(mD)|_C$ mD Cartier

We set $D_1 \equiv D_2 \iff D_1 \cdot C = D_2 \cdot C$ $\forall C$ curve.

$N^1(X)_{\mathbb{R}} = \{\text{Weil div}\}/_{\equiv}$

$N_1(X)_{\mathbb{R}} = \{\sum q_i C_i \mid C_i \text{ curve}, q_i \in \mathbb{R}\}/_{\equiv}$

when we extend $\cdot$ by linearity and

$C_1 \equiv C_2 \iff \forall D$ $D \cdot C_1 = D \cdot C_2$.

The inters product is a perfect pairing.

$N^1(X)_{\mathbb{R}} \times N_1(X)_{\mathbb{R}} \longrightarrow \mathbb{R}$

we set $\rho(X) = \dim N^1(X)_{\mathbb{R}} = \dim N_1(X)_{\mathbb{R}}$ Picard number.

Ex: • $\rho(\mathbb{P}^n) = 1$ • $\rho(\mathbb{P}^n \times \mathbb{P}^m) = 2$

• $\tilde{X} = Bl_p X \Rightarrow \rho(\tilde{X}) = \rho(X) + 1$.

Def EFFECTIVE DIVISORS

Rk Ample divisors are sth $\forall C \subseteq X$ curve $A \cdot C > 0$.

Indeed replace A by $A' \sim mA$ very ample

$\cdot X \subseteq \mathbb{P}^N$ $A' = X \cap H$. Then

$A' \underset{X}{\cdot} C = H \underset{\mathbb{P}^n}{\cdot} C$ and we can deform H sth $H \not\supseteq C$, $H \cap C \neq \emptyset$

§ MMP

Birational geometry / the Minimal Model Program MMP is a program to classify varieties.

class $\to$ eq. relation. Our choice of eq rel is

Def X_1 and X_2 are bir equiv $X_1 \sim_{bir} X_2 \iff$

$\exists U_i \subseteq X_i$ Zar open sets $\neq \emptyset$ ($\Rightarrow$ dense) sth $U_1 \cong U_2$.

One of the main ideas of the MMP is that in any birat equivalence class we can find some special elements where the geometry is simpler. I want to explain what they are, starting w surfaces

NB $X_1 \cong X_2 \to$ "$X_1 = X_2$" everything is mod isom

Surfaces (All surf will be smooth)

Def A smooth surface S is minimal $\iff \forall \varphi: S \to Y$ birat morph to Y smooth φ is an isom.

EX

1) Y smooth surf, $\hat{Y} = Bl_p Y$ $p \in Y$, and

$\varepsilon: \hat{Y} \to Y$ bir morph. Then $\hat{Y}$ is not minimal.

$\varepsilon^{-1}(p) \mapsto p$

2) $S = \mathbb{P}^2$. Let $\mu: S \to Y$ be a birat morph. to Y proj.

A ample on Y. Then $\mu^* A$ is effective on $\mathbb{P}^2 \Rightarrow \mu^* A \sim kH$

H hyperplane (line), $k > 0$.

$C \subseteq Exc(\mu)$ curve contracted by μ.

Then $0 = \mu^* A \cdot C = kH \cdot C > 0$ ↯

These two examples suggest that a "non min" surf contains some special subvar. that can be contracted

Def $E \subseteq S$ irred curve is a (−1)-curve if the foll equiv conditions are satisfied

1) $E \cong \mathbb{P}^1$, $E^2 = -1$ 2) $K_S \cdot E = E^2 = 1$

1 $\iff$ 2 by the Adjunction formula

Thm (Castelnuovo) $E \subseteq S$ (−1) curve. Then there is a smooth surface $\bar{S}$ and $\varepsilon: S \to \bar{S}$ birat and $p \in \bar{S}$ sth

$\varepsilon|_{S \setminus E}: S \setminus E \longrightarrow \bar{S} \setminus \{p\}$ isom, $\varepsilon(E) = p$

S = bu of $\bar{S}$ along p. Moreover $\rho(S) = \rho(\bar{S}) + 1$.

Thm. Minimal models of surfaces exist

($\forall S$ smooth surf, there is S_{min} minimal $S \sim_{bir} S_{min}$).

proof

* S surface. If S contains a (−1)-curve E_1, by Castelnuovo thm. there is $S \to S_1$ contraction of E_1. $\rho(S_1) = \rho(S) - 1$.

If S_1 contains a (−1) curve E_2, we contract it again.

After a fin of steps we get a surface w no (−1) curves.

The process ends because $\rho(S_i) > 0$ $\forall i$

* If S does not contain any (−1) curve then it is minimal.

follows from factorisation thm for surf:

$\mu: S \to Y$ bir morph, then there are Y_i and $Y_i \to Y_{i+1}$ contr of (−1) curves sth

$\mu: S = Y_0 \to Y_1 \to \dots \to Y_{k-1} \to Y_k = Y$. □

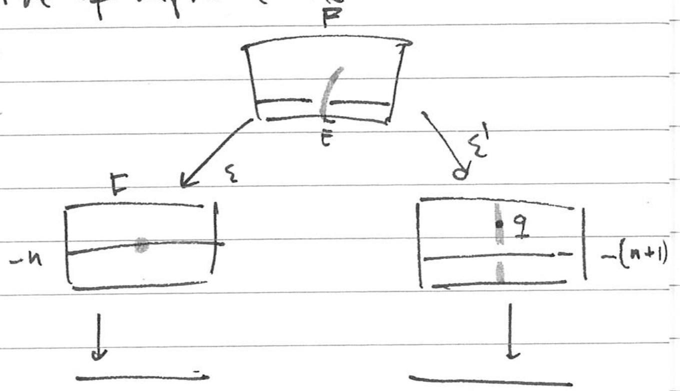
Thm (Uniqueness of minimal surfaces) S minimal surface.
There are two cases: either
1) K_S is nef ($K_S \cdot C \geq 0\ \forall C \subseteq S$), and the min model S is !, or
2) S is covered by rat curves and there is an infinity of min models $S' \sim_{bir} S$.

[Take a look closely at the bir eq elem of $\mathbb{P}^2$]

Ex. $\mathbb{P}^2 \sim_{bir} \mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ $n \in \mathbb{Z}$.
$\mathbb{F}_n \to \mathbb{P}^1$ fibration w fibr $\cong \mathbb{P}^1$. $\forall n \neq 1$ $\mathbb{F}_n$ minimal.
We look at $\mathbb{F}_n \dashrightarrow \mathbb{F}_{n+1}$.
Inside $\mathbb{F}_n$ there is a curve of selfint $(-n)$ which is a section of $\mathbb{F}_n \to \mathbb{P}^1$.

$$\begin{tikzcd} Bl_p \mathbb{F}_n \arrow[r, equal] \arrow[d, "\varepsilon"] & Bl_q \mathbb{F}_{n+1} \arrow[d, "\varepsilon'"] \\ \mathbb{F}_n \arrow[d] & \mathbb{F}_{n+1} \arrow[d] \\ \mathbb{P}^1 \arrow[r, equal] & \mathbb{P}^1 \end{tikzcd}$$

$p \in (-n)$ curve



$$0 = (E+\tilde F)^2 = \underset{-1}{E^2} + \tilde F^2 + \underbrace{2\, E\cdot\tilde F}_{2} \Rightarrow \tilde F^2 = -1,\ \tilde F \cong \mathbb{P}^1$$
$\Rightarrow$ can contr by Cast

Higher dimensional varieties

[In higher dimension selfinters of curves/divisors does not work, so we switch from $E^2 = -1$ to $K_X \cdot E < 0$]

(-1) curves Replaced w C sth $K_X \cdot C < 0$

Def A variety X is called minimal if K_X is nef

[Replacement for Cast thm]

Def The Mori cone / cone of curves of X is
$NE(X) = \{$cone generated in $N_1(X)_{\mathbb{R}}$ by $[C]$ w C irred curve$\}$

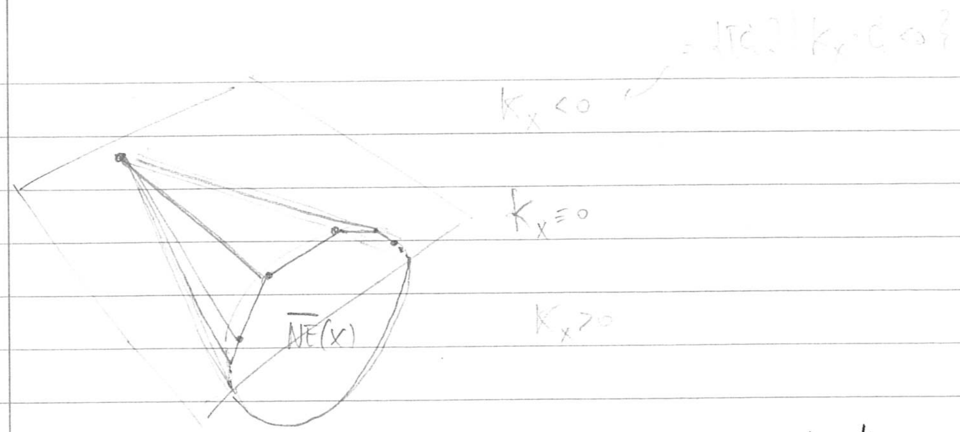
$\overline{NE}(X)$ = closure of $NE(X)$ in $N_1(X)_{\mathbb{R}}$ w euclidean topology.

EXAMPLES

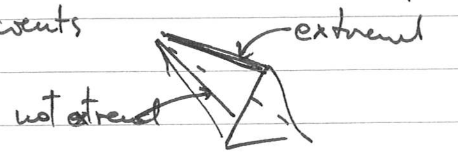
- $\mathbb{P}^n$ all the curves are num eq to $k\ell$ w $k \in \mathbb{Z}_{>0}$. ℓ line in $\mathbb{P}^n$. $\Rightarrow NE(\mathbb{P}^n) = \mathbb{R}_+[\ell]$
- $Bl_p \mathbb{P}^n =: X$. $\varepsilon : X \to \mathbb{P}^n$ $E \cong \mathbb{P}^{n-1} \cong \varepsilon^{-1}(p)$. $h \subseteq E$ line
  Let $\ell \subseteq \mathbb{P}^n$ line with $p \in \ell$. $\tilde\ell = \overline{\varepsilon^{-1}(\ell \setminus \{p\})}$
  EXERCISE $\overline{NE}(X) = \mathbb{R}_+[\tilde\ell] + \mathbb{R}_+[h]$
  hint: (1) $N_1(X)_{\mathbb{R}} = \mathbb{R}[\tilde\ell] + \mathbb{R}[h]$
  (2) prove that $\forall C$ irred $C \equiv a\tilde\ell + bh$ $a, b \geq 0$.
  $\varepsilon_* C = \deg(C \to \varepsilon(C))\, \varepsilon(C) \equiv k\ell$ $k \geq 0$ and $a = k$
  then intersect with $\tilde H$ to prove $b \geq 0$
  $\tilde H = \varepsilon^* H - E$.
- $Bl_{p,q} \mathbb{P}^n = X$ $h_p \subseteq E_p$ line $h_q \subseteq E_q$ line, ℓ thru p, q
  EX $\overline{NE}(X) = \mathbb{R}_+[h_p] + \mathbb{R}_+[h_q] + \mathbb{R}_+[\tilde\ell]$.
- $X = \mathbb{F}_n$ $\overline{NE}(X) = \mathbb{R}_+[\underset{\text{fibr}}{f}] + \mathbb{R}_+[\underset{(-n)\text{curve}}{s_n}]$

Cone and contraction thm. (Mori)

X smooth variety (terminal $\mathbb{Q}$ fact). Then $\overline{NE}(X) = \overline{NE}(X)_{K_X \geq 0} + \sum_{i \in I} R_i$
where I is countable and $\sum_{i \in I} R_i = \overline{NE}(X)_{K_X < 0}$.
Moreover
1) $\forall i\ \exists C_i$ rat curve sth $R_i = \mathbb{R}_+[C_i]$, $K_X \cdot C_i < 0$
2) $\forall A$ ample div the rays in $(K_X + A) < 0$ are finite
Therefore the R_i are discrete in $K < 0$ and accumulate only towards $K = 0$



[Def R is extremal if it is not lin comb of other rays w positive coefficients



3) If R_i is extremal then there is $c_{R_i} : X \to Y$ surjective w connected fibres sth $c_{R_i}(C) = p \iff [C] \in R_i$
c_{R_i} = contraction of R_i. Moreover $\rho(X) = \rho(Y)+1$.

Instead of the (-1) curves, we get the negative extr rays of the cone. So the idea is to contract all the extr rays

3 POSSIBILITIES FOR c_R

dim X > dim Y $X \to Y$ is called a Mf space. ← precise def will come later
Ex: $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$ $R = \mathbb{R}_+[\ell]$ $\ell \subseteq \mathbb{P}^n$ line.

dim X = dim Y and $Exc(c_R)$ has codim 1 in X
c_R is said divisorial.
EX $c_R : Bl_p Y \to Y$ $R = \mathbb{R}_+[\ell]$ $\ell \subseteq E \cong \mathbb{P}^{\dim X-1}$ line
BUT Y is not smooth anymore.
It is terminal and $\mathbb{Q}$ factorial ← comment on terminality
GOOD NEWS the cone and contr thm holds for X terminal, $\mathbb{Q}f$ ⊛

dim X = dim Y and $Exc(c_R)$ has codim ≥ 2 in X c_R is small

here Y is not $\mathbb{Q}f$, K_Y is not Cartier ← the K_Y-neg does not make sense, no C&C thm
EXAMPLES are trickier, one needs either X sing of dim 3, or dim ≥ 4 (but something simi in 4th lecture)

GOOD NEWS we can perform a flip, that is a diagram

$$\begin{tikzcd} X \arrow[dr, "c_R"'] & & X^+ \arrow[dl, "c^+"] \\ & Y & \end{tikzcd}$$

where c^+ is the contraction of a K_{X^+}-positive extr ray in $\overline{NE}(X^+)$, c^+ is small X^+ terminal $\mathbb{Q}f$ $c_R(Exc(c_R)) = c^+(Exc(c^+))$
In part $\rho(X) = \rho(X^+)$
Flips exists (Hacon McK) $X^+ = \mathrm{Proj}_Y\left(\bigoplus_m f_* \mathcal{O}_X(mK_X)\right)$

PROJECTION OF THE DIAGRAM

Notice that if we never go thru the flip part, the MMP terminates as $\rho(X) = \rho(Y)+1$ and $\rho(X) > 0$
~~LECTURE 2~~
Termination is not known in general but it is in the case that will interest us: $X \sim_{bir} \mathbb{P}^n$ ~~Notice that we made many~~
by BCHM it ends w a Mfs.

Def A Mori fibre space is the data of X terminal $\mathbb{Q}f$
$f : X \to S$ w conn fibres, surjective, sth
$\rho(X) = \rho(S)+1$, $-K_X|_{\text{general fibre}}$ is ample.

Uniqueness of the outcome of the MMP if it is a Mfs

Notice that we made many choices in the "algorithm"

Thm (Sarkisov program. Corti in dim 3, HMcK in dim n)
Let Z be a var and X/T, Y/S be two Mfs which are outcome of MMP on Z. Then the bir morph $X \dashrightarrow Y$ decomposes into a seq of diagrams called Sarkisov

* Sarkisov diagram

Links*, when

- $X_i' \to X_i$ is a div contr or an isomorph.
- $X_1' \dashrightarrow X_2'$ is an isom in codim 2
- $X_i \to T_i$ is a Mfs
- $T_i \to R$ are isom or $\rho(T_i/R)=1$
- $\rho(X_1') = \rho(R)+2$ ($\Rightarrow$ one of $X_i' \to X_i$ or $T_i \to R$ is isom)

$$\begin{tikzcd}
X_1' \arrow[rr, dashed] \arrow[d] & & X_2' \arrow[d] \\
X_1 \arrow[d] & & X_2 \arrow[d] \\
T_1 \arrow[dr] & & T_2 \arrow[dl] \\
& R &
\end{tikzcd}$$

Moreover, all the arrows involved are "bits" of MMP over R.

EX $\mathbb{F}_n \dashrightarrow \mathbb{F}_{n+1}$

§ Adding groups to the mix.

NOT $\mathrm{Aut}(X)$ = automorphism group of X

$\dim X = 1$ (bir equiv and isom are the same)

3 main classes

X	$\mathbb{P}^1$	elliptic curves	curves of genus ≥ 2
K_X	antiample	~ 0	ample
$\mathrm{Aut}(X)$	$PGL(2)$ linear affine group	finite extension of X (an elliptic curve is a group $\mathbb{C}^2/\Lambda$)	finite (of cardinality $\leq 84(g-1)$)

For an alg variety, the autom group has a topology and we set $\mathrm{Aut}^\circ(X)$ = the connected component containing id_X

$\mathrm{Aut}^\circ(X) \trianglelefteq \mathrm{Aut}(X)$ and is an algebraic group.

Def G is an alg group if G has a structure of alg variety sth the operations (mult & inverse) are morphisms.

Thm (Chevalley) G connected alg gp then there is a sequence

$$\{1\} \to H \to G \to A \to \{1\}$$

H affine linear, A abelian var. ← So the autom group is a mix of linear and abelian

Remarks

(1) If $\mathrm{Aut}^\circ(X)$ is a non-triv linear group then X is covered by rat curves ← because a lin group is

(2) If X is an abelian variety then $\mathrm{Aut}^\circ(X) = X$

(3) If X is of general type (i.e. $K_X = \text{Ample} + \text{Effective}$) then $\mathrm{Aut}(X)$ is finite by a result of Hacon-McKernan-Xu

The structure of the aut gp mirrors the geom of the variety

Remark 2 If $X_1 \sim_{bir} X_2$, then $\mathrm{Aut}^\circ(X_1)$ and $\mathrm{Aut}^\circ(X_2)$ are subgroups of $\mathrm{Bir}(X_1) = \mathrm{Bir}(X_2)$

Def $\mathrm{Bir}(X) = \left\{ \mu : X \dashrightarrow X \text{ rat map} \mid \exists\, U, V \subseteq X,\ \mu : U \xrightarrow{\cong} V \right\}$

IDEA:
- The varieties w bigger $\mathrm{Aut}^\circ(X)$ are more symmetric and nicer representatives in their bir class
- On the other hand, if we consider the Cremona group $\mathrm{Bir}(\mathbb{P}^n)$, studying its alg subgroups is a way of studying the group itself.

What we'll do now is assuming we fix

$G < \mathrm{Aut}^\circ(X)$ connected (and see how this will influence the setup we introduced yesterday)

1. $g(X^{REG}) = X^{REG}$ $\forall g \in G$.

$g^*(T^*_{X^{REG}}) \cong T^*_{X^{REG}} \Rightarrow g^* K_X \sim K_X$ $\forall g \in G$

$\to$ the canonical class is preserved.

2. $\overline{NE}(X)$. There is an action of G on $\overline{NE}(X)$ given by

$g \cdot [C] = [g(C)]$ $\forall g \in G$.

Moreover, if $K_X \cdot C < 0$, then $K_X \cdot C = g(g^{-1})^* K_X \cdot g_* C = K_X \cdot g(C)$

Then $\overline{NE}(X)_{K_X<0}$ $\left(\overset{def}{=} \overline{NE}(X) \cap \left\{ \sum \delta_i [C_i] \mid \sum \delta_i [C_i] \cdot K_X < 0 \right\} \right)$

is preserved by the action of G on $\overline{NE}$

By the cone thm $\overline{NE}(X)_{K_X<0} = \sum_{i \in I} R_i$ sum of countably many rays.

$\Rightarrow$ the action of G on $\overline{NE}(X)_{K<0}$ is trivial.

Indeed G permutes the extr rays, but they are discrete

$\Rightarrow$ perm is trivial.

$\Rightarrow$ $\forall [C] \in R_i$ $\quad [gC] \in R_i$.

- "the exceptional locus of c_R" $= \bigcup_{[C] \in R} C$ is G invariant.

3. MMP

Fundamental lemma for act of conn groups is

Blanchard's lemma Let $f : X \to Y$ be a proper surj morph w connected fibres. G conn gp acting on X.
Then there is an action of G on Y
s.th $\forall x \in X$ $g \cdot f(x) = f(g \cdot x)$ $\forall g \in G$
(making f G-equivariant)

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

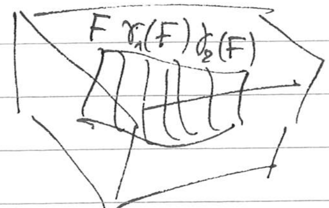
NB G is not a subgp of $\mathrm{Aut}^\circ(Y)$, we only have a hom $G \to \mathrm{Aut}^\circ(Y)$

IDEA of why this is true (not a sketch of pf)

F fibre of f. $F = f^{-1}(y)$. We want to set $g\cdot y = f(g\cdot F)$ ⇒ we need to that ∀F ∀g $g(F)$ is a fibre of f.

Fix $g\in G\setminus\{id\}$ G connected, let $\Gamma\subseteq G$ curve $id, g\in\Gamma$.

$$\Gamma\times F \xrightarrow{ev} X$$
$$(\gamma, x) \longmapsto \gamma(x)$$



The image of ev has dim = dim F + 1.

Replace X w $\overline{ev(\Gamma\times F)} = W$ (assume it is normal)

Then $F, g(F)$ are divisors in W, they are num eq because they sit in the same def family.

$\Rightarrow F|_{g(F)} \equiv 0$. They are ≠, and $F \geq 0 \Rightarrow F|_{g(F)} \sim 0$.

⇒ disjoint. Moreover A ample on Y ⇒ $f^*A|_{g(F)} \equiv 0 \Rightarrow \sim 0$

... $g(F)$ is contracted by f. □

In part if $c_R: X \to Y$ and $G < Aut^\circ(X)$
⇒ G acts on Y.

Lemma X, $G < Aut^\circ(X)$.
$$\begin{tikzcd} X \arrow[rr, dashed, "\varphi"] \arrow[dr, "c_R"'] & & X^+ \arrow[dl, "c^+"] \\ & Y & \end{tikzcd}$$
flop of R extr ray

then $G < Aut^\circ(X^+)$

(more precisely $\varphi G \varphi^{-1} < Aut^\circ(X^+)$)

proof. (sketch)

G acts on Y, $X^+ = Proj_Y\, f_*\mathcal{O}(mK_X)$

g ∀E ∼ mK_X ∀g∈G $g(E) \sim mK_X$ ⇒ g acts on X^+ □

SLOGAN: $G < Aut^\circ(X)$. Every MMP on X is a G-MMP.

$\boxed{\text{HALF}}$

From Aut to Bir

We switch point of view now from subgroups of Aut to subgps of Bir, the class of good subgps is the following

Def G is an alg group acting rationally on Z if G is an alg gp and there is
$$\begin{tikzcd} G\times Z \arrow[rr, dashed, "\varphi"] \arrow[dr, "p_1"'] & & G\times Z \arrow[dl, "p_2"] \\ & G & \end{tikzcd}$$

1. $\varphi(g,x) = (g, \varphi_2(g,x))$
2. isom on $U \subseteq G\times Z$ sth $p_1(U) = p_2(U) = G$.
3. $\rho: G \to Bir(Z)$
   $g \mapsto (x \mapsto \varphi_2(g,*))$ is a gp hom.

If $G < Aut^\circ(Z)$ the action is regular.

without this G we only have a bir action of G on Z. The "special" hyp is (2)

EXAMPLES:

1) $X = \mathbb{P}^n$ $G = (\mathbb{A}^1, +)$

$$\mathbb{A}^1\times\mathbb{A}^n \xleftrightarrow{\cong} \mathbb{A}^1\times\mathbb{A}^n$$
$$(t, (x_1 \ldots x_n)) \longmapsto (t, (x_1 \ldots x_{n-1}, x_n + t x_1))$$

2) $Aut^\circ(Z_1)$ for $Z \sim_{bir} Z_1$

Will not comment on the def further, but the reason why it is important is

Thm (Weil) G connected alg gp acting rat on Z

Then there is $W \sim_{bir} Z$ sth $G < Aut^\circ(W)$
$\varphi: W \dashrightarrow Z$ (more precisely $\varphi^{-1} G \varphi < Aut^\circ$)

Def $G < Bir(Z)$ is ~~max~~ connected alg gp acting rat on Z is maximal if ∀H conn alg gp acting rat on Z $G < H \Rightarrow G = H$.

From now on $Z = \mathbb{P}^n$. $\mathrm{Bir}(\mathbb{P}^n)$ = Cremona group

$n=1$ $\mathrm{Bir}(\mathbb{P}^1) = PGL(2) = \mathrm{Aut}(\mathbb{P}^1)$

$n=2$ $\mathrm{Bir}(\mathbb{P}^2) = \langle PGL(3), \sigma \rangle$

$\sigma[X_0:X_1:X_2] = [X_1X_2 : X_0X_2 : X_0X_1]$ (check $\sigma^2 = id$)

$n \geq 3$ no such description exists.
Many results in literature, eg recently Blanc Lamy Zimmermann.

$\exists\ \mathrm{Bir}(\mathbb{P}^n) \twoheadrightarrow \bigoplus_I \mathbb{Z}/2\mathbb{Z}$ I uncountable.

Classification of max subgroups.

$n=2$ Enriques (1893) conn case
Blanc (2008) gen case

$n=3$ Umemura series of 4 papers '80s
results by Blanc Fanelli Terpereau using MMP

$n \geq 4$ We do not know if every alg subgp is included in a max one

→ also not true if $X \not\sim_{bir} \mathbb{P}^n$ Pascal Sokratis

MMP strategy for studying $G < \mathrm{Bir}(\mathbb{P}^n)$ maximal.

- $G < \mathrm{Bir}(\mathbb{P}^n)$ connected acting rat on $\mathbb{P}^n$.
- [Weil] there is $W \to \mathbb{P}^n$ bir $G < \mathrm{Aut}^\circ(W)$.
- We may assume W is smooth (G-equiv desing exist)
- Apply a (G)-MMP for W and get X/B MfS. $G < \mathrm{Aut}^\circ(X)$.
- Thm (G-Sarkisov program, F) Let W be w $G < \mathrm{Aut}^\circ(W)$ connected. X_1/B_1 X_2/B_2 two MfS outcome of two MMP on W. Then $X_1 \dashrightarrow X_2$ can be factored into G equiv Sarkisov links/diagrams, i.e. diagrams where all the arrows are G equiv

DIAGR?

The thm is a consequence of HMcK and the fact that every MMP is a G-MMP

Cor $G < \mathrm{Bir}(\mathbb{P}^n)$ conn, acting rat on $\mathbb{P}^n$.
Then G is maximal iff $G = \mathrm{Aut}^\circ(X)$ X/B MfS. and for every $X/B \dashrightarrow X_1/B_1 \dashrightarrow \dots \dashrightarrow X_k/B_k$ sequence of Sark links $G = \mathrm{Aut}^\circ(X_k)$.

proof (MAYBE NOT ENOUGH TIME 4 THIS)

$\Leftarrow$ $G \subseteq H$ W sth $H < \mathrm{Aut}^\circ(W)$ Y/T result of H-MMP on W. ~~Thus~~ This is also a G-MMP on W and there is $X/B \dashrightarrow X_1/B_1 \dots \dashrightarrow X_k/B_k = Y/T$ factoring $X \dashrightarrow Y$. Then $H < G$. □

EXAMPLES ON SURFACES.

__Construction__ $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus d} \oplus \mathcal{O}_{\mathbb{P}^1}(n)) = \mathrm{Proj}_{\mathbb{P}^1}(\mathcal{O}^{\oplus d} \oplus \mathcal{O}(n))$

$\mathbb{G}_m^2$ = mult group in 2 variables.

$\Rightarrow \mathbb{P}(\mathcal{O}^{\oplus d} \oplus \mathcal{O}(n))$ is the quot of $(\mathbb{A}^{d+1} \setminus \{0\}) \times (\mathbb{A}^2 \setminus \{0\})$ by the action of $\mathbb{G}_m^2$:

$\mathbb{G}_m^2 \times (\mathbb{A}^{d+1} \setminus \{0\}) \times (\mathbb{A}^2 \setminus \{0\}) \longrightarrow (\mathbb{A}^{d+1} \setminus \{0\}) \times (\mathbb{A}^2 \setminus \{0\})$

$((\lambda,\mu), (x_0 \dots x_d), (u_0 u_1)) \longmapsto ((\mu x_0 \dots \mu x_{d-1}, \lambda^{-n}\mu x_d), (\lambda u_0, \lambda u_1))$

notice that on the 2nd factor gives $\mathbb{P}^1$

Not: eq class of $((x_0 \dots x_d), (u_0 u_1))$ $[x_0 \dots x_d ; u_0 u_1]$

$\mathbb{F}_n = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(n))$

$\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ $\mathrm{Aut}^\circ(\mathbb{F}_0) = PGL(2)^2$ acts transitively on $\mathbb{P}^1 \times \mathbb{P}^1$.

$\Rightarrow$ any G-equiv bir transf is isom.

$\Rightarrow PGL(2)^2$ is max in $\mathrm{Bir}(\mathbb{P}^2)$

$\mathbb{F}_1$ $n \geq 1$

action of $\mathbb{C}^{n+1} \rtimes GL(2,\mathbb{C})$ on $\mathbb{A}^2 \setminus \{0\} \times \mathbb{A}^2 \setminus \{0\}$

$\mathbb{C}^{n+1} = \{a_0 Y^n + a_1 X Y^{n-1} + \dots + a_n X^n$ hom polynomials of deg $n\}$

$\left(a_0 Y^n + a_1 X Y^{n-1} + \dots + a_n X^n, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(x_0 x_1, u_0 u_1) =$

$= \left(\left(x_0 + x_1 \sum a_i u_0^i u_1^{n-i}, (c u_0 + d u_1)^n \cdot x_1\right), \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \right)$

EX: check $\forall (\lambda \mu) \in \mathbb{G}_m^2$ $\forall (P, A) \in \mathbb{C}^{n+1} \rtimes GL$

$(P, A) \cdot ((\lambda,\mu)(x_0 x_1)(u_0 u_1)) = (\lambda,\mu)(P,A)((x_0 x_1), (u_0 u_1))$

$\Rightarrow$ the action descends to $\mathbb{F}_n$.

Can prove that $\mathrm{Aut}^\circ(\mathbb{F}_n) = \mathbb{C}^{n+1} \rtimes GL(2\mathbb{C}) / \mu_n$ n-th roots of 1

$\boxed{n=1}$ The action preserve the only (-1) curve.

$\mathbb{F}_1 = Bl_p \mathbb{P}^2$. Then is

$$\begin{tikzcd} \mathbb{F}_1 \arrow[r, equal] \arrow[d] & Bl_p \mathbb{P}^2 \arrow[d] \\ \mathbb{P}^1 \arrow[dr] & \mathbb{P}^2 \arrow[d] \\ & \{p\} \end{tikzcd}$$

NB: this is a Sarkisov link

and $\mathrm{Aut}^\circ(\mathbb{F}_1) \subsetneq PGL(3)$

$\rightarrow$ not maximal

$\boxed{n \geq 2}$ Notice that the action on the base of $\mathbb{F}_n \longrightarrow \mathbb{P}^1$ is given by $\cancel{PG}$ $GL(2)$.

$\Rightarrow$ it is transitive.

Also, $\mathrm{Aut}^\circ(\mathbb{F}_n)$ preserves the only $(-n)$ curve $(x_1 = 0)$ and acts transitively on it

$\Rightarrow$ it is an orbit $(= G \cdot x \;\; x \in \mathbb{F}_n)$ $= \{g(x) \mid g \in G\}$

the other orbit is $\mathbb{F}_n \setminus (-n)$ curve.

$\Rightarrow$ no G-equiv bir maps.

$\Rightarrow$ no non-trivial Sarkisov program.

LECTURE 3 — Blanc Fanelli Terpereau

In this lecture we study rational varieties Q_g of dim 3 w MFS $Q_g \to \mathbb{P}^1$ sth $Aut^\circ(Q_g) \subseteq Bir(\mathbb{P}^3)$ is maximal. We prove it w the Cor from LECT 2.

We will have $Aut^\circ(Q_g) = PGL(2)$ and "as many Q_g as hyp curves".

Recall $G \subset Bir(\mathbb{P}^3)$ alg subgp. is maximal iff $G = Aut^\circ(X)$ X/B MFS and for every seq of Sarkisov links $X/B \dashrightarrow X_1/B_1 \dashrightarrow \dots \dashrightarrow X_k/B_k$ we have $G = Aut^\circ(X_k)$.

CONSTRUCTION

$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ is the quotient of $\mathbb{A}^4\setminus\{0\} \times \mathbb{A}^2\setminus\{0\}$ by $\mathbb{G}_m^2$ acting w

$(\lambda,\mu)\cdot((x_0\, x_1\, x_2\, x_3)(u_0\, u_1)) = ((\mu x_0, \mu x_1, \mu x_2, \lambda^{-n}\mu x_3)(\lambda u_0\, \lambda u_1))$

NOT: equivalence class $[x_0\, x_1\, x_2\, x_3; u_0\, u_1]$

Rk. action on $\mathbb{A}^2\setminus\{0\}$ gives $\mathbb{P}^1$
- fixed λ we get $\mathbb{P}^3$ as quot.

Def: g hom poly of deg $2n$
$$Q_g = \left\{ [x_0\, x_1\, x_2\, x_3; u_0\, u_1] \in \mathbb{P}(\mathcal{O}^{\oplus 3} \oplus \mathcal{O}(n)) \;\middle|\; x_0^2 - x_1x_2 - g(u_0,u_1)x_3^2 = 0 \right\}$$

Notice that the equation is homogeneous wrto the action of $\mathbb{G}_m^2$

We have $\pi: Q_g \to \mathbb{P}^1$ induced by $\pi: \mathbb{P}(\mathcal{O}^{\oplus 3} \oplus \mathcal{O}(n)) \to \mathbb{P}^1$

FIBRES

over $\bar u = [\bar u_0 : \bar u_1] \in \mathbb{P}^1$, $\gamma = g(\bar u_0, \bar u_1)$

$\gamma \neq 0 \Rightarrow \pi^{-1}(\bar u)$ is a smooth quadric $\mathbb{P}^1 \times \mathbb{P}^1$

$\gamma = 0$ sing quadric cone, sing pt $[0\,0\,0\,1]$

$\pi: Q_g \to \mathbb{P}^1$ is a MFS (terminal, $\rho(Q_g) = \rho(\mathbb{P}^1)+1 = 2$, $-K_{Q_g}$ rel ample)

1) Q_g has terminal singularities ← (given by the eqn of Q_g & its partial derivatives)

$Sing(Q_g) = \{x_0 = x_1 = x_2 = 0,\ g(u_0 u_1) = 0,\ \frac{\partial g}{\partial u_0} = 0,\ \frac{\partial g}{\partial u_1} = 0\}$

Thus Q_g is smooth $\iff$ g is square-free.

If there is a sing pt, it is $q = [0\,0\,0\,1; \bar u_0 \bar u_1]$ sth $[\bar u_0\, \bar u_1]$ multiple root of g.

Locally around q the eqn of Q_g becomes $x^2 - yz - t^m p(t)$, $m \geq 2$, $p(0) \neq 0$.

It is the eqn of a cA_1 sing $\Rightarrow$ terminal. (a sing sth the inters of X w a hyperplane is A_1 ← specific analytic equation)

It is $\mathbb{Q}$fact $\iff$ $x^2 - t^m p(t)$ irred $\iff$ $t^m p(t)$ not a sq.

($\Rightarrow$) $t^m p(t) = p_1(t)^2 \Rightarrow yz = (x - p_1(t))(x + p_1(t))$ ← not unique fact

($\Leftarrow$) exercise (eg w brute force)

From now on: g not a square.

2) $Pic(Q_g)$.

Set $H = (x_3 = 0)$ $\quad F = (u_1 = 0)$

$H_i = (x_i = 0)$ $(H = H_3)$ $\quad F_i = (u_i = 0)$ $\quad F = F_1$.

$H = \{x_0^2 - x_1x_2 - g(u)x_3^2 = 0,\ x_3 = 0\}$
$= \{x_0^2 - x_1x_2 = 0,\ x_3 = 0\} \cong \mathbb{P}^1 \times \mathbb{P}^1$

$f = F \cap H$
$h = \{x_0 = x_1 = x_2 = 0\}$
$H \neq F \to$ intersect w f

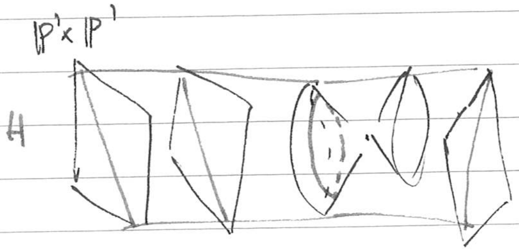
$H_0 = \{-x_1x_2 - g(u)x_3^2 = 0,\ x_0 = 0\} \sim H_3 + nF$

$\boxed{*}$ $\mathbb{P} = \mathbb{P}(\mathcal{O}(-n)^{\oplus 3} \oplus \mathcal{O})$

$H_1 \sim H_2 \sim H_3 + nF$

EX $H_2 = \alpha H_3 + \beta F$, intersect w h and f and use $H_2 \cdot h = 0$ ($H_2 \cap h = \emptyset$) and $H_3 \cdot h = \mathcal{O}(1)|_{h}^{\mathbb{P}*} = -1$

Q_g



$Q_g \setminus (H_2 \cup F) \cong \mathbb{A}^3$ by

$(x\ y\ z) \longmapsto [x : x^2 - g(1,z)y^2 : 1 : y : z : 1]$

Then there is a short exact sequence.

$$0 \to \mathbb{Z}H_2 + \mathbb{Z}F \to \mathrm{Pic}(Q_g) \to \underset{\{0\}}{\underset{\shortparallel}{\mathrm{Cl}(\mathbb{A}^3)}} \to 0$$

$\Rightarrow$ Pic is generated by H_2 and F $\Rightarrow$ gen by H_3 and F

Also $H \not\equiv F$ (intersect w curves in F, $F \cdot C = 0$, $H \cdot C \neq 0$)

so $\rho(Q_g) = 2$.

3) Canonical div of Q_g.

$K_{\mathbb{P}(\mathcal{O}^{\oplus 3} \oplus \mathcal{O}(n))} =: \mathbb{P}$

$\bar{H} = [\mathcal{O}(1)] = [(X_3 = 0)]$, P fibre of π

$K_{\mathbb{P}} \equiv -4\bar{H} - (n+2)P$

$Q_g \equiv 2\bar{H} + 2nP$ in $N^1 \mathbb{P}$ (bidegree)

Adjunction $K_{Q_g} = (K_{\mathbb{P}} + Q_g)|_{Q_g} \equiv (-2\bar{H} + (n-2)P)|_{Q_g}$

$= -2H + (n-2)F.$

Thus $-K_{Q_g}|_F = 2H|_F$ antiample.

$\boxed{\text{We proved that } \pi : Q_g \to \mathbb{P}^1 \text{ is a Mfs}}$

Automorphism gp of Q_g

* There is an action of $PGL(2)$ on Q_g.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot [X_0\, X_1\, X_2\, X_3 ; u_0\, u_1] = \left[M \cdot \begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{pmatrix} ; u_0\, u_1 \right]$$

$$M = \frac{1}{ad-bc} \begin{bmatrix} ad+bc & ac & bd & 0 \\ 2ab & a^2 & b^2 & 0 \\ 2cd & c^2 & d^2 & 0 \\ 0 & 0 & 0 & ad-bc \end{bmatrix}$$

NB: trivial action on the base | If g has ≥ 3 roots $\mathrm{Aut}^\circ(Q_g)$ acts trivially on $\mathbb{P}^1$ (Blanch) Can prove if g has ≥ 3 roots $\mathrm{Aut}^\circ(Q_g) = PGL(2)$

* There is a 2:1 covering $C \xrightarrow{t} \mathbb{P}^1$

in local coordinates on $\mathbb{A}^1 \times \mathbb{A}^1$ $C = \{(x, y) \mid y^2 = g(1,x)\}$

t is the 1st projection

We consider the basechange

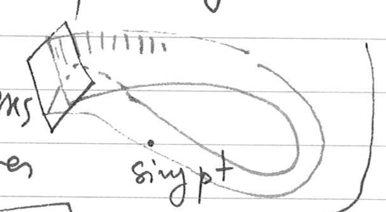
It is $PGL(2)$ invariant bec

$PGL(2)$ acts trivially on $\mathbb{P}^1$.

$$\begin{array}{ccc} Q_g \times_{\mathbb{P}^1} C & \longrightarrow & Q_g \\ \downarrow & & \downarrow \pi \\ C & \xrightarrow{\tau} & \mathbb{P}^1 \end{array}$$

NOTICE THAT $\rho(F) = 2$ $F \cong \mathbb{P}^1 \times \mathbb{P}^1$ & the two generators of $N_1(F)_{\mathbb{R}}$ are num eq in Q_g.

{ general monodromy phenomenon — only on num eq classes, not on the actual curves



the morph t "unscrews" all these monodromy situations

and $Q_g \times_{\mathbb{P}^1} C \overset{\text{bir}}{\dashrightarrow} \mathbb{P}^1 \times \mathbb{P}^1 \times C$

$PGL(2)$-equiv

$[X_0\, X_1\, X_2\, X_3 ; u_0\, u_1\, u_2] \longmapsto ([X_0 + u_2 X_3 : X_2], [X_0 - u_2 X_3 : X_2]\, [u_0 : u_1 : u_2])$

The action of $PGL(2)$ on $\mathbb{P}^1\times\mathbb{P}^1\times C$ is diag on $\mathbb{P}^1\times\mathbb{P}^1$ $\quad g\in PGL(2) \quad g([x],[y]) = (g[x], g[y])$

Also this can be seen in the following way: over the g pt of $\mathbb{P}^1$
$Q_g^{\otimes \mathbb{C}(\mathbb{P}^1)}$ is a quadric w coefficients in $\mathbb{C}(\mathbb{P}^1)$

$$x_0^2 - x_1x_2 - g x_3^2 = 0 \qquad \text{coeff } 1, -1, -g. \quad g = y^2$$

in $\mathbb{C}(C)$, g is a square, thus over $\mathbb{C}(C)$ we can set $\bar{X}_3 = yX_3$ and

$$Q_g \otimes \mathbb{C}(C) = x_0^2 - x_1x_2 - \bar{X}_3^2 \cong \mathbb{P}^1(\mathbb{C}(C))\times\mathbb{P}^1(\mathbb{C}(C)).$$

Thus the behavg. is intrinsic to Q_g and

Aut

Matrix of the inters form

$N^1(Q_g)_{\mathbb{R}} = \mathbb{R}H \oplus \mathbb{R}F$ $\Longrightarrow$ no funny business

$\overline{NE}(Q_g)$ is a 2-dim cone

$f = H\cap F \qquad h = \{x_0 = x_1 = x_3 = 0\}$ section of π

$h \subseteq H \cong \mathbb{P}^1\times\mathbb{P}^1$ is one of the fibres and f the other

$\mathbb{R}_+[f]$ induces π $\Rightarrow$ it is an extremal ray

$$h\cdot H = h(H_2 - nF) = -n$$

$h\cdot H_2 = \emptyset$ (on a fibre $[0\,0\,1\,0] \notin (X_2 = 0)$)

$h\cdot F = 1$

$\Rightarrow$ $\mathbb{R}_+[h]$ is also extremal (EX)

SOL UTI ON

Indeed assume $h \equiv C_1 + C_2$, want to that $[C_1],[C_2] \in \mathbb{R}_+[h]$. $\quad \underbrace{H\cdot h}_{=-n} = H\cdot C_1 + H\cdot C_2$.

Can assume $H\cdot C_1 < 0$ and $H\cdot C_2 \geq 0$.

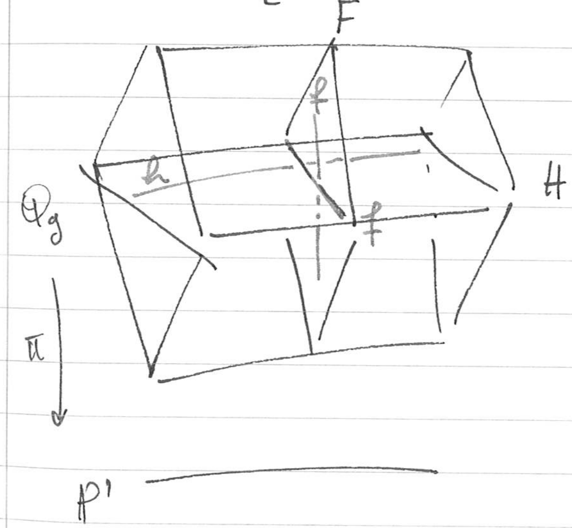
$\Rightarrow C_1 \subseteq H \cong \mathbb{P}^1\times\mathbb{P}^1 \quad C_1 \equiv_H ah + bf$ and $f\cdot H = 1$.

Then replace C_1 with ah, and C_2 w $C_2 + bf$.

$\Rightarrow$ get $h \equiv ah + C_2 \Rightarrow (1-a)h \equiv C_2$

$C_2\cdot H \geq 0 \Rightarrow 1-a \leq 0$ but eff $\equiv$ eff $\Rightarrow a = 1$, $C_2 \equiv 0$.

MATRIX $\begin{bmatrix} h\cdot H & f\cdot H \\ h\cdot F & f\cdot F \end{bmatrix} = \begin{bmatrix} -n & 1 \\ 1 & 0 \end{bmatrix}$



Orbits of the action

Assume g has ≥ 3 roots.

1) $p \in \mathbb{P}^1$ s.th. $g(p) \neq 0$.

Inside $\pi^{-1}(p) \cong (\mathbb{P}^1)^2$ we have $\begin{cases} \Gamma_p = \text{diag} = H \cap \pi^{-1}(p) \\ \pi^{-1}(p) \setminus \Gamma_p \end{cases}$

2) p s.th. $g(p) = 0$ $\quad \pi^{-1}(p)$ sing conic

Inside $\pi^{-1}(p)$ there is

$q = [0\,0\,0\,1\;\bar{u}_0\,\bar{u}_1]$ sing pt

$\Gamma_p = \pi^{-1}(p) \cap H \cong \mathbb{P}^1$

$\pi^{-1}(p) \setminus (\Gamma_p \cup \{q\})$

We are ready to study the Sarkisov program for Q_g. We start w constructions of links

g of deg $2n$ $\quad n \geq 2$, not a sq. $\quad$ w ≥ 3 roots.

h hom of deg 1

the bir map $\varphi : Q_g \dashrightarrow Q_g$

$[x_0 x_1 x_2 x_3; u_0 u_1] \longmapsto [hx_0 : hx_1 : hx_2 : x_3 ; u_0 u_1]$

fits into a Sarkisov link and

$\mathrm{Aut}^\circ(Q_g) = \mathrm{Aut}^\circ(Q_{h^2 g})$ $\quad \left(\varphi \, \mathrm{Aut}^\circ(Q_g) \varphi^{-1} = \mathrm{Aut}^\circ(Q_{h^2 g}) \right)$

how:

p s.th. $h(p) = 0$, Γ_p the 1-dim orbit in $\pi^{-1}(p)$

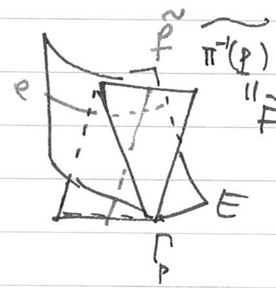
$W = \mathrm{Bl}_{\Gamma_p} Q_g$. Assume $\pi^{-1}(p)$ smooth for simplicity $\quad \varepsilon : W \longrightarrow Q_g$ bu.

$E \longrightarrow \Gamma_p$ $\quad \mathbb{P}^1$-bundle.

Then $\overline{NE}(W/\mathbb{P}^1) = \{ [C] \mid \pi(\varepsilon(C)) = pt \}$

$= \mathbb{R}_+[e] + \mathbb{R}_+[\tilde{f}]$

fibre of ε $\quad e \cong \mathbb{P}^1$ $\qquad$ one of the two rulings of $\mathbb{P}^1 \times \mathbb{P}^1$



$K_W \cdot e = -1 \qquad K_W = \varepsilon^* K_Q + E$

$K_W \cdot \tilde{f} = \underbrace{K_{Q_g} \cdot f}_{= -2} + \underbrace{E \cdot \tilde{f}}_{= 1} = -1$

$\Rightarrow$ we can contract $\mathbb{R}_+[\tilde{f}]$.

$$\begin{tikzcd} W \arrow[r, equal] \arrow[d] & W \arrow[d] \\ Q_g \arrow[d] & Q_{h^2 g} \arrow[d] \\ \mathbb{P}^1 \arrow[r, equal] & \mathbb{P}^1 \end{tikzcd}$$

NOTICE THAT THIS CREATES A SINGULARITY AS WE CONTRACT $\underset{\mathbb{P}^1 \times \mathbb{P}^1}{\tilde{F} \cong}$ TO A POINT.

$h^2 g$ has ≥ 3 distinct roots $\Rightarrow$ $\mathrm{Aut}^\circ(Q_{h^2 g}) = PGL(2)$

Rk: Notice that with φ^{-1} we can clear all the multiple factors of g. and get to Q_g smooth.

What other possible links from Q_g?

We have $\overline{NE}(Q_g) = \mathbb{R}_+[f] + \mathbb{R}_+[h]$.

The contraction of $\mathbb{R}_+[f]$ gives π.

Compute $K_{Q_g}[h] = (-2H + (n-2)F) \cdot h = -2(-n) + n - 2 = 3n - 2 > 0$

not a neg ray.

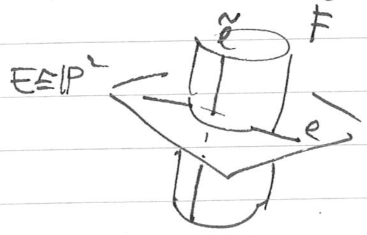
It could induce the inverse of a flip

but $\bigcup_{h'\in[h]} h' = H$ a divisor $\Rightarrow$ cannot be.

We cannot contract $\mathbb{R}_+[h]$.

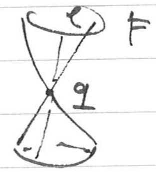
Then in the Sarkisov link: start w an "extraction" (the inverse of a contraction)

If we bu Γ_p, we get 7.



If we bu $q\in\pi^{-1}(p)$ singular

Assume for simplicity that Q_g is smooth. $W = Bl_q Q_g$. $\varepsilon: W\to Q_g$



$K_W = \varepsilon^* K_{Q_g} + 2E$

$e\subseteq E\cong\mathbb{P}^2$ line $\qquad \tilde{e}$ = str transf of a ruling.

$\overline{NE}(W/\mathbb{P}^1) = \mathbb{R}_+[e] + \mathbb{R}_+[\tilde{e}]$

contr given ε

$K_W\cdot\tilde{e} = \underbrace{K_{Q_g}\cdot\tilde{e}}_{-2} + 2 = 0.$ $\Rightarrow$ the ray is not K-neg & swipes a divisor

detail

We cannot contract it

$\Rightarrow$ the only nontriv Sarkisov links are the 7.

If g has no mult factors then every Sarkisov program leads to $Q_{g,e}$ and $\mathrm{Aut}^0(Q_{g,e}) = PGL(2)$

Conclusion $PGL(2)$ max in $\mathrm{Bir}(\mathbb{P}^3)$

LECTURE 4

This lecture will be a sneak peek of the higher dim^l case. We want to study Mfs $X\to\mathbb{P}^1$ w dim $X\geq 4$ when we know well the geom of the fibres.

AIM under certain hyp, we can perform "in families" the bir transf that we know on fibre.

$X\to\mathbb{P}^1$ dim $X\geq 4$, F general fibre

info on F $\rightsquigarrow$ info on the bir geom of X

Our main tool is the following result

Thm (Blanc, F) Let $\pi: X\to\mathbb{P}^1$ be a Mfs sth the general fibre F has $\rho(F)\geq 2$.
Then there is $Z\subsetneq X$ closed, $\mathrm{Aut}^0(X)$-invariant $\pi(Z) = \mathbb{P}^1$ ("horizontal")

This Z will be the center of the 1st blow up to start the Sarkisov program
$$W \to X \to \mathbb{P}^1$$

How to prove the result

* [Graber-Harris-Starr] $\pi: X\to\mathbb{P}^1$ X RC has a section $\varphi:\mathbb{P}^1\to X$. sth $\pi\circ\varphi = \mathrm{Id}$. We set $s = \varphi(\mathbb{P}^1)$.

* We consider $\mathcal{K} = \{-K_X\cdot s \mid s \text{ section}\}$.
(1) $\mathcal{K}\neq\emptyset$ (2) $\mathcal{K}$ discrete and (3) bounded from below.

proof
(1) ok $\exists s$ section by [GHS]
(2) r Cartier index of K ($r^\#$ sth rK Cartier) then $-K_X\cdot s\in\frac{1}{r}\mathbb{Z}$.
(3) A ample Cartier sth $-K_X + \pi^* A$ ample.

$\Rightarrow r(-K_X + \pi^*A)\cdot s \geq 1.$

$\Rightarrow -K_X \cdot s \geq \frac{1}{r} - \pi^*A\cdot s = \frac{1}{r} - \deg A$

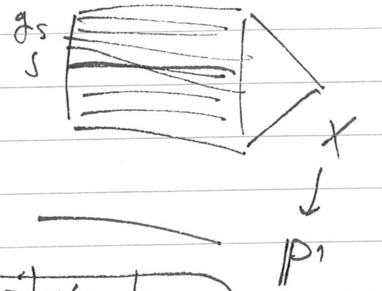
s section

* Thus $\exists s$ sth $-K_X\cdot s$ minimal.

Our Z will be of the form $\mathrm{Aut}^0(X)\cdot s$ w $-K_X\cdot s$ minimal.

What is $\mathrm{Aut}^0(X)\cdot s$

Explain



The key lemma is a bkb-type lemma

bkb : def w 2 fixed pts, for us 1 fixed pt and a fibration

Prop. $\pi : X \to \mathbb{P}^1$ fibration w $-K_X$ rel ample

s section, $x\in s$

Suppose there is $\Gamma \subseteq \mathrm{Aut}^0(X)$ irred sth

$g(x) = x \quad \forall g\in\Gamma$

$g(s)\neq s$ for $g\in\Gamma$ general.

Then $s \equiv s' + \gamma$ w s', γ int curves

s' section

$\gamma \subseteq$ fibres of π.

In particular, s is not minimal in the above sense.

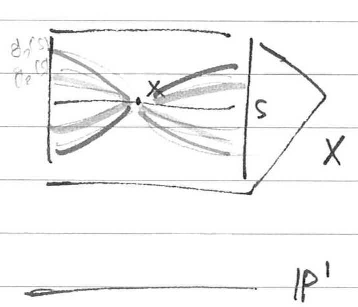
proof

$\varphi : \mathbb{P}^1 \to s \subseteq X$. Assume Γ normal (for simplicity, no big deal)

Let $F : \mathbb{P}^1\times\Gamma \longrightarrow X\times\Gamma$

$(p, g) \longmapsto (g\cdot\varphi(p), g)$

on the 1st coord



F is finite, indeed $\forall g\in\Gamma$

$F(\mathbb{P}^1\times\{g\})$ projects onto $\mathbb{P}^1$ by $\pi\circ p_X$.

Thus $F(\mathbb{P}^1\times\Gamma)$ has dim 2.

Let $\bar\Gamma$ be a smooth cpt of Γ, S the normalisation of $\overline{F(\mathbb{P}^1\times\Gamma)}$ inside $X\times\bar\Gamma$, $\bar F : S \longrightarrow X\times\bar\Gamma$

$$\begin{tikzcd} \mathbb{P}^1\times\Gamma \arrow[r, hook] \arrow[dd] & S \arrow[r, "e"] \arrow[d, "\bar F"] \arrow[dd, bend right, "k"'] & X \\ & X\times\bar\Gamma \arrow[ur, "p_1"] \arrow[d, "p_2"] & \\ \Gamma \arrow[r, hook] & \bar\Gamma & \end{tikzcd}$$

$\bar F$ finite (normal) $\Rightarrow$ no component of a f of k is contracted by e.

Our goal is to p that there is a non irred fibre of k.

We set $p_0 = \varphi(x)$.

$\{p_0\}\times\Gamma$ is contracted to x by $p_1\circ F$.

$\Gamma_0 = \overline{\{p_0\}\times\Gamma}$ Zar closure in S. Then $e(\Gamma_0) = \{x\}$.

We have $\tau : (k, \pi\circ e) : S \longrightarrow \bar\Gamma\times\mathbb{P}^1$

τ is birational

If it is isom, then $(\Gamma_0)^2_S = 0$, contradiction because Γ_0 is contracted by e and $\mathrm{Im}\, e$ has dim 2

Thus there is $y_0 \in \overline{\Gamma} \times \mathbb{P}^1$ w $\tau^{-1}(y_0)$ a curve.
$z_0 = p_1(y_0) \in \overline{\Gamma} \setminus \Gamma$.

$\kappa^*(z_0)$ is not ired and $e_*(\kappa^*(z_0))$ is not either

Moreover $\underbrace{e_* \kappa^* z_0}_{= C_1 + C_2} \equiv s$.

We have $1 = s \cdot \underbrace{F}_{\text{fibr of } \pi} = C_1 \cdot F + C_2 \cdot F$

Wlog C_1 has an ired comp s' sth $s' \cdot F = 1$
and $\underbrace{\text{everything else}}_{= \gamma}$ is sth $C \cdot F = 0$.

$\gamma \cdot F = 0 \Rightarrow \gamma \subseteq$ fibres of π.

Thus $s(-K_X) = s'(-K_X) + \underbrace{\gamma(-K_X)}_{> 0 \; -K \text{ rel ample}} > s'(-K_X)$

and s not minimal $\square$

Towards the thm.

$\pi : X \to \mathbb{P}^1$ w conn fibres

By Blanchard's lemma there is $H < PGL(2)$ and

$\{1\} \to \operatorname{Aut}^\circ(X)_{\mathbb{P}^1} \to \operatorname{Aut}^\circ(X) \xrightarrow{\alpha} H \to \{1\}$

NOT $\operatorname{Aut}^\circ(X)_{\mathbb{P}^1} = \ker(\alpha)$ ← the aut of X that act fibrewise

Lemma $\pi : X \to \mathbb{P}^1$ sth $-K_X$ rel ample, F general fibre
Assume $\operatorname{Aut}^\circ(X)_{\mathbb{P}^1}$ acts transitively on a gen fibre. Thus $\exists\ \theta : U \times F \to \pi^{-1} U$ birat.

For π Mfs, $\rho(F) \geq 2$. The action of $\operatorname{Aut}^\circ_{\mathbb{P}^1}$ on the gen fibre is not transitive. ← explain why this is almost the thm

proof of lemma.

s section w min $-K_X \cdot s$.

$H = \{ g \in \operatorname{Aut}^\circ(X)_{\mathbb{P}^1} \mid gs = s \}$. $V = \operatorname{Aut}^\circ(X)_{\mathbb{P}^1} / H$. $\{x\} = F \cap s$.

$\Phi : V \to F$
$[g] \mapsto g(x)$.

If the act is transitive, Φ is surjective.
$\Rightarrow \dim V \geq \dim F$.

~~If~~ $\dim V > \dim F \iff \exists\ \Gamma \subseteq \operatorname{Aut}^\circ(X)_{\mathbb{P}^1}$ sth $gx = x$
$\forall g \in \Gamma$ $gs \neq s$ for g general. ↯ w 6&6 lemma by min of s.

Thus Φ gen finite, but F and V are homogeneous varieties $\Rightarrow$ no exc locus and no ramification (they should be invariant by the act)

$\Phi : V \to F$ étale bw Fano mfds.

Thus $\underset{1}{\chi(V)} = \deg \Phi \, \underset{1}{\chi(F)} \Rightarrow \deg \Phi = 1$.

Kawamata Viehweg vanishing.

Φ isom and $\exists U \subseteq \mathbb{P}^1$. $\square$

Applications in dim 4

$F = (\mathbb{P}^1)^3 = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

$N_1(F)_{\mathbb{R}} = \oplus \mathbb{R}[f_i]$, $\overline{NE}(F) \overset{Ex}{=} \mathbb{R}_+[f_1] + \mathbb{R}_+[f_2] + \mathbb{R}_+[f_3]$

$\pi : X \to \mathbb{P}^1$ w gen fbr F.

$Z \subsetneq X$ $\mathrm{Aut}^\circ(X)$ invariant.

FACT $[Z \cap F]$ (num eq class) is symmetric in N_1 or N^1.

$Z \cap F$ is a closed orbit of a subgp of $\mathrm{Aut}^\circ((\mathbb{P}^1)^3)$. $\subsetneq F$

$\dim Z \cap F \in \{0,1,2\}$.

▷ $\dim Z \cap F = 2$ never happens (adj formula & comp)

▷ $\dim Z \cap F = 1$ $\Sigma = Z \cap F$ ⊛ the projections are equivariant $p_i : \Sigma \to \mathbb{P}^1$ and non const $(a>0)$

⊛ $[\Sigma]$ in $\overline{NE}(F)$ is $a([f_1]+[f_2]+[f_3])$ $a \in \mathbb{Z}_{>0}$.

~~Thus~~ $\Sigma \cong \mathbb{P}^1$ because it is an orbit of a lin gp.

If $a \geq 2$, $p_i|_\Sigma$ has deg $a \geq 2 \Rightarrow$ ramifies somewhere

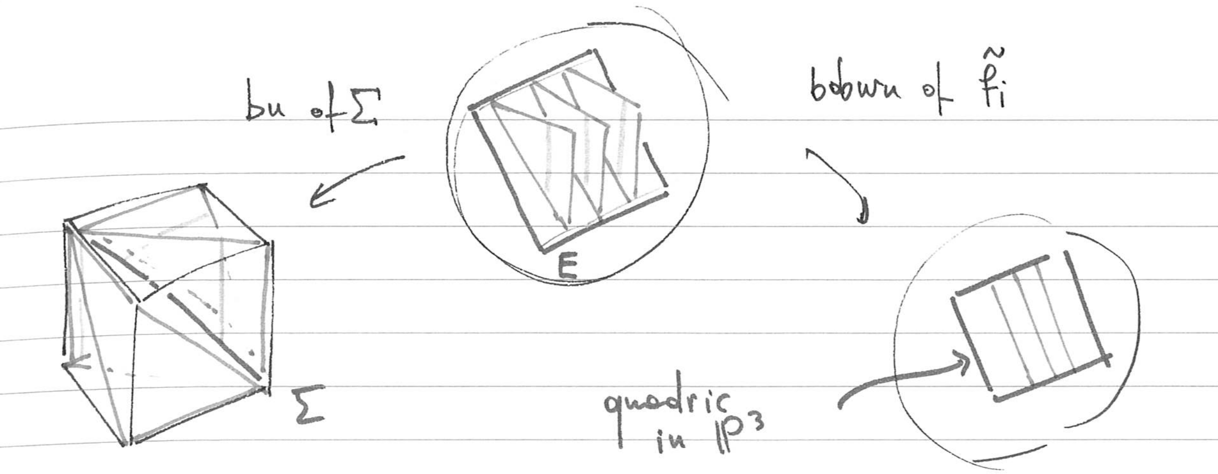
$\Rightarrow$ not equiv.

$\Rightarrow$ $a = 1$ and Σ is the diag

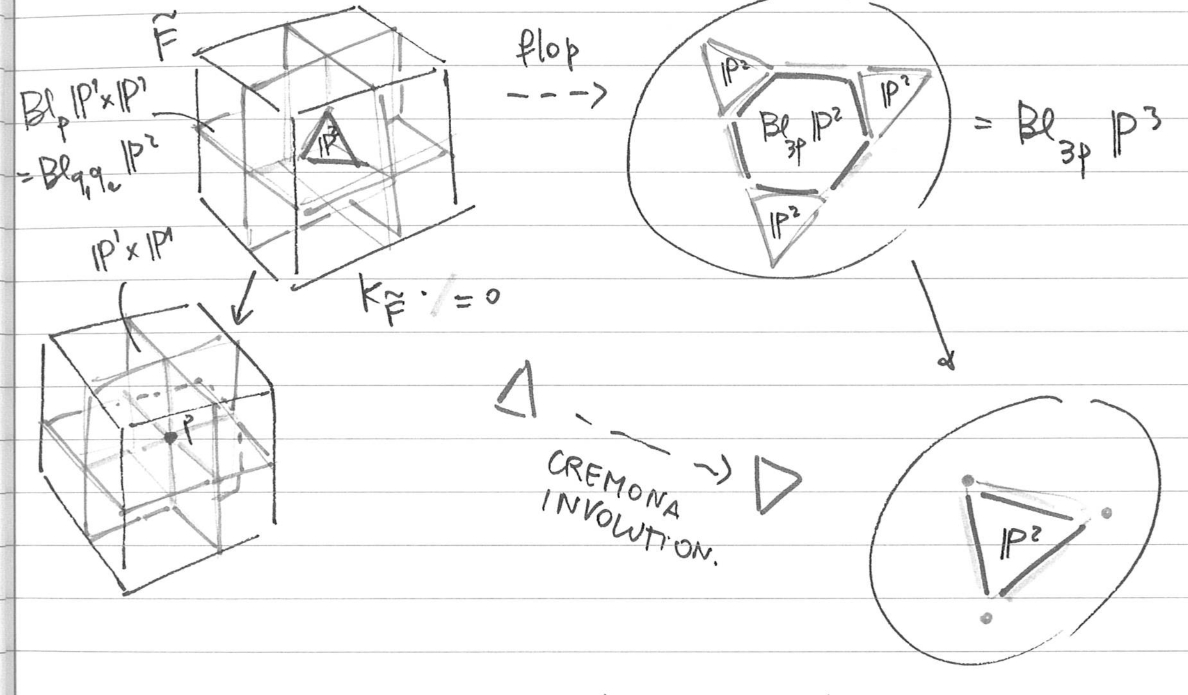
$f_i \subseteq$ green dir of 3 deg $(1,0,0)$ or $(0\,1\,0)$ or $(0\,0\,1)$

$\tilde{f}_i$ str transf. $K_{\tilde F} \cdot \tilde{f}_i = \varepsilon^* K_F \cdot \tilde{f}_i + E = -2 + 1 = -1$

Can also prove that generates an extremal ray of NE



▷ $\dim Z \cap F = 0$ + $Z = \mathrm{Aut}^\circ(X) \cdot s \Rightarrow Z = s$. $Z \cap F$ is 1 pt.



One has to prove then that those bir transf can be performed in family Aut° equiv. (Shokurov semiampleness)

If $X \to \mathbb{P}^1$ has g fbr $(\mathbb{P}^1)^3$ then there is a diagram w Aut° equiv rows

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[d, "\pi"'] & Y \arrow[d] \\ \mathbb{P}^1 \arrow[r, equal] & \mathbb{P}^1 \end{tikzcd}$$

gen fbr $\mathbb{P}^3$

{ Thus for max subgroups we do not need to study g fbr $(\mathbb{P}^1)^3$, only g fbr $\mathbb{P}^3$, $\mathbb{P}^3$ bdles }

BIBLIOGRAPHY / REFERENCES

Cone and contraction theorem

→ J. Kollár - S. Mori Birational Geometry of Algebraic Varieties 1.3

→ Y. Kawamata - K. Matsuda - K Matsuki Introduction to the MMP, Chapter 3

Blanchard's lemma

→ M. Brion - P. Samuel - V. Uma Lectures on the structure of algebraic groups and geometric applications (online notes) Prop 4.2.1.

Every MMP is a G-MMP

→ E Floris A note on the G-Sarkisov program.

Weil's regularisation theorem.

→ A. Weil On algebraic groups of transformations American J. of Maths.

→ H. Kraft Regularisation of rational group actions

Lecture 3

→ J. Blanc, A. Fanelli, R. Terpereau Connected algebraic groups acting on 3-dim Mori fibrations SECTION 4.4, 6.4

BOUTOT

FONCTEUR DE PICARD LOCAL