EIGENVALUE PROBLEMS WITH FULLY DISCONTINUOUS OPERATORS AND CRITICAL EXPONENTS

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Abstract. We construct here more examples of $\overrightarrow{\rho}$-multivoque Leray-Lions operators that are strongly monotonic and we introduce a new abstract theorem on critical points for multivalued operators available for unbounded domains.

Keywords : Nonlinear eigenvalues, Multi-voque operators, Clarke subdifferential, Mountain pass Theorem, Banach function norms.

1. Introduction

The purpose of this paper is to give additional results on eigenvalue problems related to fully discontinuous operators. The study of this issue was initiated by the second author in [20] in which the notion of $\overrightarrow{\rho}$-multivoque Leray-Lions operators were introduced

$$-\mathrm{div}_{\overrightarrow{\rho}}\left(\partial\varphi_i\left(x,\frac{\partial u}{\partial x_i}(x)\right)\right),$$

associated to $\overrightarrow{\rho}=(\rho_0,\dots,\rho_N)$, ρ_i are Banach function norms, φ_i are Caratheodory functions on $\Omega\times\mathbb{R}$ such that for a.e. $t\to\varphi_i(x,t)$ is locally Lipschitz on $\mathbb{R}$, therefore is subdifferentiable in the sense of Clarke [7].

Here, we shall start to give an example associated to Orlicz-Sobolev spaces (see [17]). In this case, we shall consider as Banach function norm:

$$\rho(v)=\mathrm{Inf}\left\{\lambda>0:\int_\Omega A\left(\frac{|v(x)|}{\lambda}\right)\mathrm{d}x\leqslant 1\right\},$$

where A is a suitable N-function (see [1]). Or more generally, we shall consider

$$\rho(v)=\mathrm{Inf}\left\{\lambda>0:\int_\Omega A\left(x;\frac{|v(x)|}{\lambda}\right)\mathrm{d}x\leqslant 1\right\},$$

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and $L(\Omega,\rho)$ the associated Banach function space.

Moreover, we shall add an abstract result concerning the resolution of the eigenvalue problem

$$-\mathrm{div}_{\vec{\beta}}\left(\partial\varphi_i\left(x,\frac{\partial u}{\partial x_i}(x)\right)\right)=\lambda f\big(x,u(x)\big),\quad f\big(x,u(x)\big)\in\partial j_0\big(x,u(x)\big).$$

Namely, this result shall recover the following case, containing a critical exponent. There exists a function $u\geqslant 0,\ u\not\equiv 0,\ u\in W_0^{1,p_1,\dots,p_N}(\Omega)$ such that

$$-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right)-a(x)u^{p^*-1}\in\lambda\partial j_{N+1}(u)$$

with $\lambda^*>\lambda>0,\ w\in\partial j_{N+1}$

$$\lambda>0,\ w(x)=\begin{cases} q_1|u(x)|^{q_1-2}u(x) & \text{if } |u(x)|<1,\\ q_2|u(x)|^{q_2-2}u(x) & \text{if } |u(x)|>1,\\ \in[q_1,q_2]\,\mathrm{sign}\big(u(x)\big) & \text{if } |u(x)|=1,\end{cases}$$

$1<q_1<\min\limits_{1\leqslant i\leqslant N}p_i,\ p^*=\dfrac{N}{\sum_{i=1}^{N}\dfrac{1}{p_i}-1}$ provided $\sum_{i=1}^{N}\dfrac{1}{p_i}>1$, $1<q_2<p^*,\ q_1<q_2$, the function a is bounded and nonnegative.

2. Notations - Preliminary results.

For a Banach space $(V,\|\cdot\|)$, we shall denoted by V' its dual and duality bracket between $V'$and V shall be denoted by $<\cdot,\cdot>$. The set of all subsets of V' is denoted by $\mathcal{P}(V')=2^{V'}$. Sometimes, we shall denoted $\|\cdot\|_V$ or $\|\cdot\|_{V'}$ the norms in those spaces. The weak topology on V shall be denoted as usual by $\sigma(V,V')$.

Definition 1. (The first statement of definition is already known (see [3]), the second one was introduced in [20].)

- *A multivoque operator $A:V\to\mathcal{P}(V')$ is called a monotone operator if :*
  $\forall\,u_1\in V,\ \forall\,u_2\in V,\ \forall\,w_1\in Au_1,\ \forall\,w_2\in Au_2,$ *we have*
  $$<w_1-w_2,u_1-u_2>\geqslant 0.$$
- *A multivoque monotone operator A is strongly monotonic if it satisfies : for any sequence $(u_n)_n$ in V converging weakly to a function u and verifying:*
  $\forall\,w_n\in Au_n,\ \forall\,w\in Au$ *if* $\lim\limits_{n\to+\infty}<w_n-w,u_n-u>=0$ *then u_n converges to u strongly in V.*

Definition 2. (see [5, 8, 12].)

- *Let $j : V \to \mathbb{R}$ be a locally Lipschitz function if for all $u \in V$, there exist a ball $B(u,r), r > 0$ and a constant $K_r(u) = K(u)$ such that*

$$|j(v) - j(w)| \leqslant K(u)\|v - w\| \qquad \forall\, v \in B(u,r),\ \forall\, w \in B(u,r).$$

- *For each $v \in V$, the generalized-directional derivative of j (at a point u in the direction v), denoted $j^0(u;v)$, is defined as follows*

$$j^0(u;v) = \limsup_{\lambda \searrow 0,\ h \to 0} \frac{j(u+h+\lambda v) - j(u+h)}{\lambda}.$$

Property 1. of the generalized directional derivative (see [5, 8, 12].)
For a fixed $u \in V$

(1) *$v \to j^0(u;v)$ is subadditive, positively homogeneous, and convex.*
(2) *$|j^0(u;v)| \leqslant K(u)\|v\|$,*
(3) *the map $v \to j^0(u;v)$ is Lipschitz*

$$|j^0(u;v_1) - j^0(u;v_2)| \leqslant K(u)\|v_1 - v_2\|.$$

Definition 3. (see [5, 8, 12].) *The (Clarke) subdifferential of j at a point $u \in V$ is the subdifferential of convex map $v \to j^0(u;v)$ at zero.*
More precisely,

$$w \in \partial j(u) \subset V' \Longleftrightarrow < w, v > \leqslant j^0(u;v), \quad \forall\, v \in V$$

that is

$$\partial j(u) = \Big\{ w \in V' :< w, v > \leqslant j^0(u;v),\ \forall\, v \in V \Big\}.$$

For these reasons, the Clarke-subdifferential possesses the same properties as for the subdifferential in the sense of convex analysis.

Property 2. of the Clarke-subdifferential (see [5, 8, 12].)
For all $u \in V$,

(1) *$\partial j(u)$ is convex, weakly-* compact in V',*
(2) *for each $w \in \partial j(u)$, $\|w\|_{V'}$(dual norm) $\leqslant K(u)$, $K(u)$ is the Lipschitz constant at u.*
(3) *$\partial(j_1 + j_2)(u) \subset \partial j_1(u) + \partial j_2(u)$, whenever $j_1,\ j_2$ are locally Lipschitz functions,*
(4) *$\partial(\lambda j)(u) = \lambda \partial j(u)\ \forall\, \lambda \in \mathbb{R}$,*

(5) *the map* $u \in V \to \lambda(u) = \text{Min}\left\{\|w\|_{V'} : w \in \partial j(u)\right\}$ *is lower semi-continuous. In particular there exists*

$$w_u \in \partial j(u) : \|w_u\|_{V'} = \inf_{w \in \partial j(u)} \|w\|_{V'},$$

(6) *if* j *is convex, then the two notions of subdifferentials coincide.*

(7) $\forall\, v \in V\ \ j^0(u; v) = \text{Max}\left\{ < w, v >: w \in \partial j(u)\right\}$.

(8) *The set-valued mapping* $u \to \partial j(u)$ *is upper semi-continuous.*

Definition 4. (see [5, 8, 12].)
If $j : V \to \mathbb{R}$ *is locally Lipschitz we will say that* u *is a critical point if* $0 \in \partial j(u)$.

We shall use the following notions on Banach function norms and spaces (see [4] for more details, also [21]).
Let $L^0(\Omega) = \left\{v : \Omega \to \mathbb{R} \text{ Lebesgue measurable}\right\}$,
and $L^0_+(\Omega) = \left\{v \in L^0(\Omega),\ v \geqslant 0\right\}$.

Definition 5. (see [4].) *A mapping* $\rho : L^0_+(\Omega) \to \overline{\mathbb{R}}_+ = [0, +\infty]$ *is called a Banach function norm if for all* f, g, g_n *in* $L^0_+(\Omega)$, *for all measurable set* $E \subset \Omega$, *we have*

P1./ $\rho(f) = 0 \Longleftrightarrow f = 0$ *a.e.* $\rho(\lambda f) = \lambda\rho(f) \quad \forall\, \lambda > 0$, $\rho(f + g) \leqslant \rho(f) + \rho(g)$.

P2./ *If* $f \leqslant g$ *a.e. then* $\rho(f) \leqslant \rho(g)$.

P3./ *If* $f_n \leqslant f_{n+1} \nearrow f$ *then* $\rho(f_n) \to \rho(f)$.

P4./ *If* $|E| < +\infty$ *then* $\rho(\chi_E) < +\infty$. *(*$|E|$ *is the Lebesgue measure of* E *and* χ_E *is characteristic function of* E*).*

P5./ *If* $|E| < +\infty$ *then* $\displaystyle\int_E f(x)\mathrm{d}x \leqslant c_{E}\rho(f)$ *for some* c_{E} *such that* $0 < c_{E} < +\infty$ *depending only on* E *and* ρ.

Definition 6. (see [4].) *Let* ρ *be a Banach function norm. Then we define*

$$Y = L(\Omega, \rho) = \left\{f \in L^0(\Omega) : \rho(|f|) < +\infty\right\}$$

$L(\Omega, \rho)$ *is called a Banach function space and is endowed with the norm* $||f||_Y = \rho(|f|)$.

A Banach function space possesses many properties, we state some of them that we shall use

Definition 7. (see [4].)

- *If* ρ *is a Banach function norm, its associate norm* ρ' *is defined by*

$$\rho'(f) = \sup\left\{\int_\Omega fg\mathrm{d}x,\ g \in L^0_+(\Omega),\ \rho(g) \leqslant 1\right\} \text{ for } f \in L^0_+(\Omega).$$

- *The Banach function space associate to ρ' is called Banach function space associate to $L(\Omega,\rho)$. That is*

$$L(\Omega,\rho') = \Big\{ v \in L^0(\Omega) : \rho'(|v|) < +\infty \Big\}.$$

- *One has for $g \in L(\Omega,\rho') \doteq Z,\ Y = L(\Omega,\rho)$.*

$$||g||_Z = \sup \left\{ \int_\Omega |fg| \mathrm{d}x,\ f \in Y, ||f||_Y \leqslant 1 \right\}.$$

Definition 8. (see [4].)
A Banach function space Y is said to have absolutely continuous norm if

$$||f\chi_{E_n}||_Y \to 0 \text{ for every sequence } \Big\{ E_n \Big\}, \quad E_{n+1} \subset E_n, \quad |E_n| \to 0.$$

Theorem 1. (see [4].) *A Banach function space Y is reflexive if and only if both Y and its associate $Z = Y'$ have absolutely continuous norm.*

Remark 1. *For simplicity, we sometimes write $\rho(f) = \rho(|f|)$, and various constants depending on data shall be denoted by c or c_i.*

The following definitions and results were introduced in [20].
We shall consider in this section two reflexive Banach spaces $(V, \|\cdot\|)$ and $(X, |\cdot|)$ we assume that V is continuously embedded in X and let us consider two locally Lipschitz functions $J : V \to \mathbb{R}$ and $j : X \to \mathbb{R}$. Furthermore, we assume that

H1./ $\begin{array}{rl} A: & V \to \mathcal{P}(V') \\ & u \mapsto \partial J(u) \end{array}$ is strongly monotonic.

H2./ We have the following growth

(1) $\exists\, \beta > 0,\ \exists\, c_0 \geqslant 0:$

$$\beta j(u) \leqslant \inf_{v \in \partial j(u)} < v, u > + c_0 \beta \quad \forall\, u \in X.$$

(2) There are constants $c_1 > 0, c_2 \geqslant 0$

$$\frac{1}{\beta} \sup_{w^* \in \partial J(u)} < w^*, u > - c_2 + c_1 \|u\| \leqslant J(u), \qquad \forall\, u \in V.$$

Let us notice that

$$J^0(u;u) = \sup_{w^* \in \partial J(u)} < w^*, u >, \qquad -j^0(u;-u) = \inf_{v \in \partial j(u)} < v, u > .$$

Lemma 1. (see [20].) *Under the above conditions* **H1./**, **H2./** *if the injection of V into X is compact then the function*

$$\Phi(u) = J(u) - j(u),\ u \in V$$

satisfies the (P.S.) conditions.

H3./ $\Phi(0) = 0$ and there exist $r_0 > 0,\ R_0 > 0$ such that

$$\inf_{\|u\|=r} J(u) > 0 \text{ if } 0 < r < r_0 \text{ and } J(u) > 0 \text{ if } \|u\| > R_0.$$

H4./ $\lim_{\|u\|\to 0} \dfrac{j(u)}{J(u)} = 0$, and for some $u_0 \neq 0$, $\limsup_{t\to+\infty} \dfrac{j(tu_0)}{J(tu_0)} > 1$.

Definition 9. (see [20].) *Let Ω be an open set of $\mathbb{R}^N$ and $\rho_0, \dots, \rho_N$ be $(N+1)$ Banach function norms and $L(\Omega, \rho_i),\ i = 0, \dots, N$ corresponding Banach function space. We define the following Banach-Sobolev functions spaces*

$$\begin{aligned} \mathbf{W} &= W^{1,\rho_0,\dots,\rho_N}(\Omega) \\ &= \Big\{ v \in L(\Omega, \rho_0),\ \frac{\partial v}{\partial x_i} \in L(\Omega, \rho_i),\ i = 1, \dots, N \Big\}. \end{aligned}$$

The norm on $\mathbf{W}$ is then

$$||v||_{\mathbf{W}} = \sum_{i=1}^{N} \rho_i \left(\frac{\partial v}{\partial x_i} \right) + \rho_0(v).$$

Proposition 1. (see [20].) *Let $\rho_0, \dots, \rho_N$ be $(N+1)$ Banach function norms, ρ_i' is the associate norm of ρ_i. We assume that ρ_i and ρ_i' are absolutely continuous norms. Then, the dual space*

$$\begin{aligned} \mathbf{V}' &= \Big(W_0^{1,\rho_0,\dots,\rho_N}(\Omega) \Big)' \\ &= \Big\{ T : \text{there exist } f_0, \dots, f_N, f_i \in L(\Omega, \rho_i'), \\ &\quad < T, v > = \int_\Omega (v f_0)(x) \mathrm{d}x + \sum_{i=1}^{N} \int_\Omega f_i(x) \frac{\partial v}{\partial x_i}(x) \mathrm{d}x \ \forall\, v \in \mathbf{V} \Big\}. \end{aligned}$$

Lemma 2. Computation of a subdifferential (see [20].)
Let $\mathbf{V} = W_0^{1,\rho_0,\dots,\rho_N}(\Omega)$ where ρ_i and their associate ρ_i' are Banach function norms which are absolutely continuous. Let $\varphi_i (i = 0, \dots, N)$ be $(N+1)$ Caratheodory functions on $\Omega \times \mathbb{R}$ such that

(1) *For a.e. $x \in \Omega$, the mapping $\sigma \to \varphi_i(x, \sigma)$ is locally Lipschitz on $\mathbb{R}$.*
(2) *The mappings $v \in L(\Omega, \rho_i) \to \Phi_i(v) = \int_\Omega \varphi_i(x, v(x)) \mathrm{d}x$ satisfy for $i = 0, \dots, N$: for any bounded set $B \subset L(\Omega, \rho_i)$, there is a constant $k_B > 0$ such that $\forall\, u \in B,\ \forall\, v \in B$*

$$\begin{aligned} \int_\Omega \Big| \varphi_i(x, u(x)) - \varphi_i(x, v(x)) \Big| \mathrm{d}x &\leqslant k_B \rho_i(u - v), \\ \int_\Omega |\varphi_i(x, 0)| \mathrm{d}x &< +\infty. \end{aligned}$$

Then the functional defined by

$$J_0(v) = \int_\Omega \varphi_0(x, v(x))\mathrm{d}x + \sum_{i=1}^{N} \int_\Omega \varphi_i\left(x, \frac{\partial v}{\partial x_i}(x)\right)\mathrm{d}x$$

is locally Lipschitz on $\mathbf{V}$.

Moreover for all $u \in \mathbf{V}$*, all* $T \in \partial J_0(u)$ *there exist* $(N+1)$ *functions*

$$(w_0, \ldots, w_N) \in L(\Omega, \rho_0') \times \ldots \times L(\Omega, \rho_N')$$

such that

$$< T, v >= \int_\Omega w_0(x)v(x)\mathrm{d}x + \sum_{i=1}^{N} \int_\Omega w_i(x)\frac{\partial v}{\partial x_i}(x)\mathrm{d}x \qquad \forall\, v \in \mathbf{V},$$

for $i = 1, \ldots, N$,

$$w_i(x) \in \partial\varphi_i\left(x, \frac{\partial u}{\partial x_i}(x)\right) \text{ a.e. in } \Omega,$$

and

$$w_0(x) \in \partial\varphi_0(x, u(x)) \text{ a.e. in } \Omega.$$

Definition 10. (see [20].) *Under the notations of Lemma 2 we define the* $\overrightarrow{\rho}$*-multivoque Leray-Lions operator by setting*

$$-\mathrm{div}_{\overrightarrow{\rho}}\left(\partial\varphi_i\left(x, \frac{\partial u}{\partial x_i}(x)\right)\right) = -\sum_{i=1}^{N} \frac{\partial w_i}{\partial x_i}(x) + \omega_0 \text{ in } \mathcal{D}'(\Omega).$$

Here $\overrightarrow{\rho}$ *stands for* $(\rho_0, \ldots, \rho_N)$.

Lemma 3. (see [20].) *We assume the same conditions as for Lemma 2. Furthermore, we suppose*

(1) *for each* $j = 0, \ldots, N$*, for a.e.* $x \in \Omega$*, the mappings* $\sigma \to \partial\varphi_j(x, \sigma)$ *are monotone;*

(2) *for any sequence* $(u_n)_{n\geqslant 0}$ *converging weakly to a function* u *in* $\mathbf{V}$ *with*

$$\lim_{n\to+\infty} < T_n^* - T, u_n - u >= 0, \qquad \text{for some} \qquad T_n^* \in \partial J_0(u_n),\ T \in \partial J_0(u),$$

and

$$\lim_n \rho_0\left(|u_n - u|\right) = 0$$

satisfies:

There exists a subsequence still denoted (u_n) *such that :*

(a) $\frac{\partial u_n}{\partial x_i}(x) \to \frac{\partial u}{\partial x_i}(x)$ *a.e.* $x \in \Omega$*, for* $i = 1, \ldots, N$,

(b) *there exists a constant* $M > 1$ *such that*

$$\lim_n \rho_j \left(\left| \frac{\partial u_n}{\partial x_j}(x) - \frac{\partial u}{\partial x_j}(x) \right| \chi_{E_{nj}} \right) = 0 \text{ for } j = 1, \ldots, N,$$

where

$$E_{nj} = \left\{ x \in \Omega : \left| \frac{\partial u_n}{\partial x_j}(x) \right| > M, \left| \frac{\partial u}{\partial x_j}(x) \right| > M - 1 \right\}.$$

(c) *If*

$$\lim_n \int_{F_{nj}} w^*_{nj} \frac{\partial u_n}{\partial x_j}(x) \mathrm{d}x = 0 \text{ then } \lim_n \rho_j \left(\left| \frac{\partial u_n}{\partial x_j}(x) \right| \chi_{F_{nj}} \right) = 0,$$

with

$$F_{nj} = \left\{ x \in \Omega : \left| \frac{\partial u_n}{\partial x_j}(x) \right| > M, \quad \left| \frac{\partial u}{\partial x_j}(x) \right| \leqslant M - 1 \right\},$$

$w^*_{nj}(x) \in \partial \varphi_j \left(x, \frac{\partial u_n}{\partial x_j}(x) \right)$, $j = 1, \ldots, N$ *for a.e.* $x \in \Omega$. *Then*

(i) $u \to \partial J_0(u)$ *is monotone.*

(ii) $u_n \to u$ *strongly in* $\mathbf{V}$ *(for all the sequence).*

Theorem 2. (see [20].) *Under the assumptions of Lemma 1, and* **H3./** *to* **H4./**, *there exists a function* $u \in V, \; u \neq 0$ *a critical point of* Φ *i.e.*

$$0 \in \partial \Phi(u), \quad \Phi(u) = \gamma > 0.$$

3. Construction of a $\overrightarrow{\rho}$-multivoque Leray-Lions strongly monotonic operator

Let Ω be a bounded Lipschitz open set of $\mathbb{R}^N$ and consider N borelian functions $\phi_i : \Omega \times \mathbb{R} \to \mathbb{R}$, such that

A.1/ For a.e. $x \in \Omega$, the map $t \xrightarrow{\phi_i(x,\cdot)} \phi_i(x,t)$ is strictly increasing, odd, $\phi_i(x,0) = 0$. Moreover, $\phi_i(x,\cdot)$ is right continuous on $[0,+\infty[$ and $\lim\limits_{t\to+\infty} \operatorname*{essinf}\limits_{\Omega} \phi_i(x,t) = +\infty$.

A2./ There exist $2m$-numbers, $a_1 < \ldots < a_{2m}$ such that for a.e. $x \in \Omega, \; t \in \mathbb{R}\backslash\{a_1, \ldots, a_{2m}\} \to \phi_i(x,t)$ is continuous.

We define on $\Omega \times \mathbb{R}$ the following function

$$\varphi_i(x,t) = \int_0^{|t|} \phi_i(x,\sigma) \mathrm{d}\sigma.$$

Proposition 2. *For a.e.* $x \in \Omega$, *the function* $\varphi_i(x,\cdot)$ *is a* N*-function (see* [1] *for a definition) on* $[0,+\infty[$.

In particular, $\varphi_i(x,\cdot)$ *is a convex function and*

a) $\partial\varphi_i(x,t) = \Big[\liminf_{\sigma\to t}\phi_i(x,\sigma), \limsup_{\sigma\to t}\phi_i(x,\sigma)\Big], \ \forall\, t\in\mathbb{R}$.

b) *Moreover, one has for all* $t\in\mathbb{R}$

$$\int_0^{|t|}\partial\varphi_i(x,\sigma)\mathrm{d}\sigma = \varphi_i(x,t).$$

(That is $\forall\, w(x,\sigma)\in\partial\varphi_i(x,\sigma), \quad \int_0^{|t|} w(x,\sigma)\mathrm{d}\sigma = \varphi_i(x,t)$.)

Proof :

- The condition A.1/ implies $\varphi_i(x,\cdot)$ is a N function according to the usual definition (see [1, 14]). The computation of the subdifferential is straightforward.
- We recall that for almost all $t\in\mathbb{R}$

$$\phi_i(x,t) = \liminf_{\sigma\to t}\phi_i(x,\sigma) = \limsup_{\sigma\to t}\phi_i(x,\sigma),$$

and since φ_i is locally Lipschitz, we deduce easily the statement b).

$\diamondsuit$

Proposition 3. *Let* $i\in\{1,\dots,N\}$ *and* (t,t_1) *be in* $\mathbb{R}$. *Assume that*

$$(t-t_1)(w-w_1) = 0 \quad \textit{for some } w\in\partial\varphi_i(x,t), \textit{ and } w_1\in\partial\varphi_i(x,t_1).$$

Then

$$t = t_1.$$

Proof :

If $t\neq t_1$ then $w = w_1$. We may assume that $t<t_1$, since $\phi_i(x,\cdot)$ is strictly increasing, we have

$$w \leqslant \limsup_{\sigma\to t}\phi_i(x,\sigma) < \liminf_{\sigma\to t_1}\phi_i(x,\sigma) \leqslant w_1,$$

which implies $w<w_1$. This is a contradiction so

$$t = t_1.$$

$\diamondsuit$

Remark 2. *Let us notice that*

$$\liminf_{\sigma\to t}\phi_i(x,\sigma) = \lim_{\varepsilon\to 0,\varepsilon>0}\phi_i(x,t-\varepsilon),$$

and

$$\limsup_{\sigma\to t}\phi_i(x,\sigma) = \lim_{\varepsilon\to 0,\varepsilon>0}\phi_i(x,t+\varepsilon).$$

We need that $\varphi_i(x,\cdot)$ satisfies the so-called global Δ_2-condition (or near infinity). Following the necessary and sufficient condition given in [1, 14], we shall assume

A3./ There are two numbers $\underline{\alpha} > 1$, $\overline{\alpha} > 1$ such that : for a.e. x, $\forall t \in \mathbb{R}$

$$\underline{\alpha}\varphi_i(x,t) \leqslant t w_i(x,t) \leqslant \overline{\alpha}\varphi_i(x,t), \text{ and } \forall\, w_i(x,t) \in \partial\varphi_i(x,t).$$

Remark 3. i) *According to the remark 2, this last condition is equivalent to*

$$\underline{\alpha}\varphi_i(x,t) \leqslant t \lim_{\varepsilon\to 0}\phi_i(x,t-\varepsilon) \leqslant t \lim_{\varepsilon\to 0}\phi_i(x,t+\varepsilon) \leqslant \overline{\alpha}\varphi_i(x,t).$$

ii) *Therefore there exists $K > 0$*

$$\varphi_i(x,2t) \leqslant K\varphi_i(x,t) \textit{ for a.e. } x \in \Omega,\ \forall t \in \mathbb{R}.$$

As usual, we define the associate Orlicz space by considering

$$L^{\varphi_i}(\Omega) = \Big\{ v : \Omega \to \mathbb{R} \text{ measurable } : \int_\Omega \varphi_i\big(x, v(x)\big)\mathrm{d}x < +\infty \Big\},$$

we endowed this space with Banach function norm

(1)
$$\rho_i(v) = \mathrm{Inf}\left\{ \lambda > 0 : \int_\Omega \varphi_i\left(x, \frac{v(x)}{\lambda}\right)\mathrm{d}x \leqslant 1 \right\}.$$

Thus

(2)
$$L^{\varphi_i}(\Omega) = L(\Omega, \rho_i).$$

Introducing the conjugate function of $\varphi_i(x,\cdot)$

$$\widetilde{\varphi}_i(x,t) = \sup_{s\geqslant 0}\Big\{ ts - \varphi_i(x,s) \Big\}, \text{ for } t \geqslant 0.$$

Then the associate norm of ρ_i is

(3)
$$\rho_i'(v) = \mathrm{Inf}\left\{ \lambda > 0 : \int_\Omega \widetilde{\varphi}_i\left(x, \frac{|v(x)|}{\lambda}\right)\mathrm{d}x \leqslant 1 \right\},$$

and we shall assume that $\widetilde{\varphi}_i(x,\cdot)$ satisfies the Δ_2-condition globally (or near infinity) say that there exists a constant $k > 0$

$$\widetilde{\varphi}_i(x,2t) \leqslant k\widetilde{\varphi}_i(x,t) \text{ for a.e. } x \text{ and } \forall t \geqslant 0,\ (\text{or } t \geqslant t_0),$$

then the dual and associate space of $L(\Omega,\rho_i)$ is $L(\Omega,\rho_i')$:

(4)
$$\Big(L^{\varphi_i}(\Omega)\Big)' = L(\Omega,\rho_i)' = L(\Omega,\rho_i') = L^{\widetilde{\varphi}_i}(\Omega).$$

◇

The Orlicz-Sobolev space associated to $\overrightarrow{\rho} = (\rho_1, \ldots, \rho_N)$ is

$$W_0^{1,\rho_1,\ldots,\rho_N}(\Omega) = \left\{ v \in L^1(\Omega) : \ \frac{\partial v}{\partial x_i}(x) \in L(\Omega, \rho_i), \ i = 1, \ldots, N, \gamma_0 v = 0 \right\}.$$

We define the function

$$J(v) = \sum_{i=1}^{N} \int_\Omega \varphi_i \left(x, \frac{\partial v}{\partial x_i}(x) \right) \mathrm{d}x.$$

The main result in this section is :

Theorem 3. *Assume* **A1./**, **A2./**, **A3./**. *Then the $\overrightarrow{\rho}$-multivoque Leray-Lions operator defined by*

$$Au = -\mathrm{div}_{\overrightarrow{\rho}} \left(\partial\varphi_i \left(x, \frac{\partial u}{\partial x_i}(x) \right) \right)$$

is strongly monotonic on the Orlicz-Sobolev space $W_0^{1,\rho_1,\ldots,\rho_N}(\Omega) \doteq \mathbf{V}_0$.

Remark 4. (1) *Since* $\operatorname{essinf}_{\Omega} \phi_i(x,t) \geqslant 1$ *for* $t \geqslant t_0 > 0$, *then, we have*

$$\int_{\left\{ \left| \frac{\partial v}{\partial x_i}(x) \right| \geqslant t_0 \right\}} \left| \frac{\partial v}{\partial x_i}(x) \right| \left| \phi_i \left(x, \frac{\partial v}{\partial x_i}(x) \right) \right| \mathrm{d}x \geqslant \int_{\left\{ \left| \frac{\partial v}{\partial x_i}(x) \right| \geqslant t_0 \right\}} \left| \frac{\partial v}{\partial x_i}(x) \right| \mathrm{d}x,$$

and we deduce from assumption **A3./** *that*

$$\int_\Omega \left| \frac{\partial v}{\partial x_i}(x) \right| \mathrm{d}x \leqslant c_1 \int_\Omega \varphi_i \left(x, \frac{\partial v}{\partial x_i}(x) \right) \mathrm{d}x + c_2,$$

(c_j denote various constants independent of v).

Thus

$$W_0^{1,\rho_1,\ldots,\rho_N}(\Omega) \subset W_0^{1,1}(\Omega).$$

(2) *One has for all $t > 0$, all $\sigma > 1$*

$$\sigma^{\underline{\alpha}} \varphi_i(x,t) \leqslant \varphi_i(x, \sigma t) \leqslant \sigma^{\overline{\alpha}} \varphi_i(x,t) \text{ for a.e. } x \in \Omega.$$

This relation is a consequence of the assumption **A3./**, *noticing that $t \to \varphi_i(x,t)$ is a locally Lipschitz function.*

Proof :

Let $(u_n)_{n\geqslant 0}$ be a bounded sequence of $\mathbf{V}_0$ converging weakly to a function $u \in \mathbf{V}_0$. Assume that $T_n \in \partial J(u_n)$ and $T \in \partial J(u)$ satisfies

$$\lim_{n\to+\infty} < T - T_n, u - u_n >= 0,$$

we have to show that $u_n \to u$ in $\mathbf{V}_0$. For this, we shall apply Lemma 5 of [20] (see Lemma 2 of the above section 2). We first need to show :

Lemma 4. *Up to a subsequence, one has*

$$\frac{\partial u_n}{\partial x_i}(x) \xrightarrow[n\to+\infty]{} \frac{\partial u}{\partial x_i}(x) \text{ a.e. } x \text{ and for all } i = 1, \dots, N.$$

Proof :

The fact that $\lim_{n\to\infty} < T - T_n, u - u_n >= 0$ implies that
$\forall w_{ni}^* \in \partial\varphi_i\left(x, \frac{\partial u_n}{\partial x_i}(x)\right)$, $\forall\, w_i^* \in \partial\varphi_i\left(x, \frac{\partial u}{\partial x_i}(x)\right)$

$$\lim_{n\to\infty} \sum_{i=1}^{N} \int_\Omega (w_{ni}^*(x) - w_i^*(x)) \left(\frac{\partial u_n}{\partial x_i}(x) - \frac{\partial u}{\partial x_i}(x)\right) \mathrm{d}x = 0.$$

Since $\varphi_i(x, \cdot)$ is convex, we deduce that

$$\Delta_{ni}(x) = (w_{ni}^*(x) - w_i^*(x)) \left(\frac{\partial u_n}{\partial x_i}(x) - \frac{\partial u}{\partial x_i}(x)\right) \geqslant 0 \text{ a.e. } x,$$

thus,

$$\lim_{n\to\infty} \int_\Omega (w_{ni}^*(x) - w_i^*(x)) \left(\frac{\partial u_n}{\partial x_i}(x) - \frac{\partial u}{\partial x_i}(x)\right) \mathrm{d}x = 0.$$

Then at least for a subsequence, we deduce

$$\lim_{n\to+\infty} (w_{ni}^*(x) - w_i^*(x)) \left(\frac{\partial u_n}{\partial x_i}(x) - \frac{\partial u}{\partial x_i}(x)\right) = 0 \text{ for a.e. } x. \tag{5}$$

From assumption **A3./** and the above remark we have

$$\underline{\alpha}\varphi_i\left(x, \frac{\partial u_n}{\partial x_i}(x)\right) \leqslant w_{ni}^*(x)\frac{\partial u_n}{\partial x_i}(x) \leqslant \overline{\alpha}\varphi_i\left(x, \frac{\partial u_n}{\partial x_i}(x)\right). \tag{6}$$

We have

$$\frac{\partial u_n}{\partial x_i}(x) w_{ni}^*(x) \leqslant w_i^*(x)\frac{\partial u_n}{\partial x_i}(x) - w_i^*(x)\frac{\partial u}{\partial x_i}(x) + w_{ni}^*(x)\frac{\partial u}{\partial x_i}(x) + \Delta_{ni}(x). \tag{7}$$

By the definition of subdifferential for φ_i convex

$$w_{ni}^*(x)\left(\frac{\partial u}{\partial x_i}(x) - \frac{\partial u_n}{\partial x_i}(x)\right) \leqslant \varphi_i\left(x, \frac{\partial u}{\partial x_i}(x)\right) - \varphi_i\left(x, \frac{\partial u_n}{\partial x_i}(x)\right). \tag{8}$$

Thus

$$w_{ni}^*(x)\frac{\partial u}{\partial x_i}(x) \leqslant w_{ni}^*(x)\frac{\partial u_n}{\partial x_i}(x) - \varphi_i\left(x, \frac{\partial u_n}{\partial x_i}(x)\right) + \varphi_i\left(x, \frac{\partial u}{\partial x_i}(x)\right). \tag{9}$$

Using relation(6)

$$w_{ni}^*(x)\frac{\partial u}{\partial x_i}(x) \leqslant \left(1 - \frac{1}{\overline{\alpha}}\right) w_{ni}^*(x)\frac{\partial u_n}{\partial x_i}(x) + \varphi_i\left(x, \frac{\partial u}{\partial x_i}(x)\right) \tag{10}$$

From (7) and (10), we deduce

$$\frac{1}{\overline{\alpha}} w_{ni}^*(x)\frac{\partial u_n}{\partial x_i}(x) \leqslant w_i^*(x)\frac{\partial u_n}{\partial x_i}(x) - w_i^*(x)\frac{\partial u}{\partial x_i}(x) + \varphi_i\left(x, \frac{\partial u}{\partial x_i}(x)\right) + \Delta_{ni}(x). \tag{11}$$

But according to Remark 4, statement 2.) one deduces from relation (6)

$$w_{ni}^*(x)\frac{\partial u_n}{\partial x_i}(x) \geqslant \underline{\alpha}\varphi_i(x,1)\left|\frac{\partial u_n}{\partial x_i}(x)\right|^{\underline{\alpha}} - \underline{\alpha}\varphi_i(x,1). \tag{12}$$

From relations (11), (5) and (12), we have for some number $K_3(x)$ independent of n

$$\frac{\underline{\alpha}}{\overline{\alpha}}\varphi_i(x,1)\left|\frac{\partial u_n}{\partial x_i}(x)\right|^{\underline{\alpha}} - \frac{\underline{\alpha}}{\overline{\alpha}}\varphi_i(x,1) \leqslant w_i^*(x)\frac{\partial u_n}{\partial x_i}(x) + K_3(x). \tag{13}$$

By Young's inequality, noticing that $\underline{\alpha} > 1$ and $\varphi_i(x,1) > 0$ we derive

$$\left|\frac{\partial u_n}{\partial x_i}(x)\right|^{\underline{\alpha}} \leqslant K_4(x) < +\infty. \tag{14}$$

We may assume that $\frac{\partial u_n}{\partial x_i}(x) \xrightarrow[n\to+\infty]{} \xi_i(x)$ (at least for a subsequence). Since relation (14) implies that $w_{ni}^*(x)$ remains in a bounded set of $\mathbb{R}$, we may also assume that

$$w_{ni}^*(x) \xrightarrow[n\to+\infty]{} \overline{w}_i(x).$$

Therefore, one has

$$\overline{w}_i(x) \in \partial\varphi_i\Big(x, \xi_i(x)\Big)$$

and from relation (5).

$$\Big(\overline{w}_i(x) - w_i^*(x)\Big)\left(\xi_i(x) - \frac{\partial u}{\partial x_i}(x)\right) = 0. \tag{15}$$

According to Proposition 3, we have

$$\xi_i(x) = \frac{\partial u}{\partial x_i}(x).$$

Arguing by contradiction, we deduce that all the sequence satisfying relation (5).

$$\frac{\partial u_n}{\partial x_i}(x) \to \frac{\partial u}{\partial x_i}(x), \text{ for a.e. } x.$$

$\diamondsuit$

Next, we have to show that

Lemma 5. *Let* $M \geqslant \sup\Big\{|a_j|,\ j = 1,\ldots,2m\Big\} + 2,$ *such that*

$$\text{measure}\left\{x \in \Omega : \left|\frac{\partial u}{\partial x_i}(x)\right| = M\right\} = 0,$$

$$E_{ni} = \left\{x \in \Omega : \left|\frac{\partial u_n}{\partial x_i}(x)\right| > M;\ \left|\frac{\partial u}{\partial x_i}(x)\right| > M - 1\right\}.$$ *Then*

$$\lim_{n\to+\infty}\int_{E_{ni}} \varphi_i\left(x, \frac{\partial u_n}{\partial x_i}(x) - \frac{\partial u}{\partial x_i}(x)\right)\mathrm{d}x = 0.$$

Proof :

For $\varepsilon > 0$, we set

$$\Omega_\varepsilon^n = \left\{ x \in \Omega : \left| \frac{\partial u_n}{\partial x_i}(x) - \frac{\partial u}{\partial x_i}(x) \right| > \varepsilon \right\}, \quad E_{ni}^\varepsilon = E_{ni} \cap (\Omega_\varepsilon^n)^c$$

$$(\Omega_\varepsilon^n)^c = \left\{ x \in \Omega : \left| \frac{\partial u_n}{\partial x_i}(x) - \frac{\partial u}{\partial x_i}(x) \right| \leqslant \varepsilon \right\} \subset \left\{ x : \left| \frac{\partial u_n}{\partial x_i}(x) \right| \leqslant \left| \frac{\partial u}{\partial x_i}(x) \right| + \varepsilon \right\}.$$

Then we observe that

$$\int_{E_{ni}} \varphi_i \left(x, \frac{\partial u_n}{\partial x_i}(x) - \frac{\partial u}{\partial x_i}(x) \right) dx = \tag{16}$$

$$= \int_{E_{ni} \setminus E_{ni}^\varepsilon} \varphi_i \left(x, \frac{\partial u_n}{\partial x_i}(x) - \frac{\partial u}{\partial x_i}(x) \right) \mathrm{d}x + \int_{E_{ni}^\varepsilon} \varphi_i \left(x, \frac{\partial u_n}{\partial x_i}(x) - \frac{\partial u}{\partial x_i}(x) \right) \mathrm{d}x.$$

By the Lebesgue dominate Theorem, we have

$$\lim_{n \to +\infty} \int_{E_{ni}^\varepsilon} \varphi_i \left(x, \frac{\partial u_n}{\partial x_i}(x) - \frac{\partial u}{\partial x_i}(x) \right) \mathrm{d}x = 0. \tag{17}$$

To show that

$$\lim_{n \to +\infty} \int_{E_{ni} \setminus E_{ni}^\varepsilon} \varphi_i \left(x, \frac{\partial u_n}{\partial x_i}(x) - \frac{\partial u}{\partial x_i}(x) \right) \mathrm{d}x = 0,$$

we note by the convexity property that we have

$$\varphi_i \left(x, \frac{\partial (u_n - u)}{\partial x_i}(x) \right) \leqslant \frac{1}{2} \varphi_i \left(x, 2 \frac{\partial u_n}{\partial x_i}(x) \right) + \frac{1}{2} \varphi_i \left(x, 2 \frac{\partial u}{\partial x_i}(x) \right), \tag{18}$$

and by the Δ_2-condition

$$\varphi_i(x, 2t) \leqslant 2^{\overline{\alpha}} \varphi_i(x, t). \tag{19}$$

Then one has

$$\begin{aligned} \limsup_{n \to +\infty} \int_{E_{ni} \setminus E_{ni}^\varepsilon} \varphi_i \left(x, \frac{\partial u_n}{\partial x_i}(x) - \frac{\partial u}{\partial x_i}(x) \right) \mathrm{d}x \quad &\leqslant \quad c \limsup_{n \to +\infty} \int_{\Omega_\varepsilon^n} \varphi_i \left(x, \frac{\partial u}{\partial x_i}(x) \right) \mathrm{d}x \\ &\quad + c \limsup_{n \to +\infty} \int_{E_{ni} \setminus E_{ni}^\varepsilon} \varphi_i \left(x, \frac{\partial u_n}{\partial x_i}(x) \right) \mathrm{d}x. \end{aligned} \tag{20}$$

But

$$\limsup_{n \to +\infty} \int_{\Omega_\varepsilon^n} \varphi_i \left(x, \frac{\partial u}{\partial x_i}(x) \right) \mathrm{d}x = 0, \tag{21}$$

since $\dfrac{\partial u_n}{\partial x_i}(x)$ converges to $\dfrac{\partial u}{\partial x_i}(x)$ for almost every x. Thus it suffices to show

$$\limsup_{n \to +\infty} \int_{E_{ni} \setminus E_{ni}^\varepsilon} \varphi_i \left(x, \frac{\partial u_n}{\partial x_i}(x) \right) \mathrm{d}x = 0. \tag{22}$$

Setting $\delta\phi_{ni}(x) = \left(\phi_i\left(x, \frac{\partial u_n}{\partial x_i}(x)\right) - \phi_i\left(x, \frac{\partial u}{\partial x_i}(x)\right)\right)\left(\frac{\partial u_n}{\partial x_i}(x) - \frac{\partial u}{\partial x_i}(x)\right)$ we have

$$\frac{\partial u_n}{\partial x_i}(x)\phi_i\left(x, \frac{\partial u_n}{\partial x_i}(x)\right) = \tag{23}$$

$$= \delta\phi_{ni}(x) + \phi_i\left(x, \frac{\partial u}{\partial x_i}(x)\right)\left(\frac{\partial(u_n - u)}{\partial x_i}(x)\right) + \frac{\partial u}{\partial x_i}(x)\phi_i\left(x, \frac{\partial u_n}{\partial x_i}(x)\right).$$

For each i and a.e. $x \in \Omega$ the map $t \to \phi_i(x,t)$ is continuous for $t > M-2$, we deduce that for $x \in E_{ni}$

$$\partial\varphi_i\left(x, \frac{\partial u_n}{\partial x_i}(x)\right) = \left\{\phi_i\left(x, \frac{\partial u_n}{\partial x_i}(x)\right)\right\},$$

and

$$\partial\varphi_i\left(x, \frac{\partial u}{\partial x_i}(x)\right) = \left\{\phi_i\left(x, \frac{\partial u}{\partial x_i}(x)\right)\right\}.$$

By the convexity of φ_i, one has from the definition of subdifferential

$$\frac{\partial u}{\partial x_i}(x)\phi_i\left(x, \frac{\partial u_n}{\partial x_i}(x)\right) \leqslant \varphi_i\left(x, \frac{\partial u}{\partial x_i}(x)\right) - \varphi_i\left(x, \frac{\partial u_n}{\partial x_i}(x)\right) + \frac{\partial u_n}{\partial x_i}(x)\phi_i\left(x, \frac{\partial u_n}{\partial x_i}(x)\right). \tag{24}$$

From relations (23) and (24), we have

$$\varphi_i\left(x, \frac{\partial u_n}{\partial x_i}(x)\right) \leqslant \delta\phi_{ni}(x) + \phi_i\left(x, \frac{\partial u}{\partial x_i}(x)\right)\frac{\partial(u_n - u)}{\partial x_i}(x) + \varphi_i\left(x, \frac{\partial u}{\partial x_i}(x)\right). \tag{25}$$

From the above relation, we deduce

$$\int_{E_{ni}\setminus E_{ni}^{\varepsilon}} \varphi_i\left(x, \frac{\partial u_n}{\partial x_i}(x)\right)\mathrm{d}x \leqslant \int_{E_{ni}} \delta\phi_{ni}(x)\mathrm{d}x + \int_{E_{ni}} \phi_i\left(x, \frac{\partial u}{\partial x_i}(x)\right)\frac{\partial(u_n - u)}{\partial x_i}(x)\mathrm{d}x$$
$$- \int_{E_{ni}^{\varepsilon}} \phi_i\left(x, \frac{\partial u}{\partial x_i}(x)\right)\frac{\partial(u_n - u)}{\partial x_i}(x)\mathrm{d}x + \int_{E_{ni}\setminus E_{ni}^{\varepsilon}} \varphi_i\left(x, \frac{\partial u}{\partial x_i}(x)\right)\mathrm{d}x. \tag{26}$$

Let us analyze each term of the right hand of the last relation (26).

As we have seen before the fact that

$$\lim_{n\to+\infty} < T_n - T, u_n - u >= 0,$$

implies

$$\lim_{n\to+\infty}\int_{\Omega}(w_{ni}^* - w_i^*)\left(\frac{\partial u_n}{\partial x_i}(x) - \frac{\partial u}{\partial x_i}(x)\right)\mathrm{d}x = 0,$$

and since

$$\int_{\Omega}(w_{ni}^* - w_i^*)\left(\frac{\partial u_n}{\partial x_i}(x) - \frac{\partial u}{\partial x_i}(x)\right)\mathrm{d}x \geqslant \int_{E_{ni}}(w_{ni}^* - w_i^*)\left(\frac{\partial u_n}{\partial x_i}(x) - \frac{\partial u}{\partial x_i}(x)\right)\mathrm{d}x,$$

we deduce

$$\lim_{n\to+\infty}\int_{E_{ni}}\left(\phi_i\left(x, \frac{\partial u_n}{\partial x_i}(x)\right) - \phi_i\left(x, \frac{\partial u}{\partial x_i}(x)\right)\right)\left(\frac{\partial(u_n - u)}{\partial x_i}(x)\right)\mathrm{d}x = \tag{27}$$

$$= \lim_{n\to+\infty} \int_{E_{ni}} \delta\phi_{ni}(x)\mathrm{d}x = 0.$$

For the second term of the right hand side of relation (26) one has

$$\phi_i\left(x, \frac{\partial u}{\partial x_i}(x)\right) \in \partial\varphi_i\left(x, \frac{\partial u}{\partial x_i}(x)\right) \text{ for a.e. } x \in E_{ni},$$

then,

$$\varphi_i\left(x, \frac{\partial u}{\partial x_i}(x)\right) + \widetilde{\varphi}_i\left(x, \phi_i\left(x, \frac{\partial u}{\partial x_i}(x)\right)\right) = \frac{\partial u}{\partial x_i}(x)\phi_i\left(x, \frac{\partial u}{\partial x_i}(x)\right), \tag{28}$$

where $\widetilde{\varphi}_i(x,t) = \sup\limits_{s\geqslant 0}\left\{st - \varphi_i(x,s)\right\}$ is the Legendre transform of $\varphi_i(x,\cdot)$.
Since $\varphi_i(x,0) = 0$ one has $\widetilde{\varphi}_i(x,0) = 0$ (remember $\varphi_i(x,s) \geqslant 0$).
Thus

$$\int_\Omega \widetilde{\varphi}_i\left(x, \chi_{E_{ni}}(x)\phi_i\left(x, \frac{\partial u}{\partial x_i}(x)\right)\right) \mathrm{d}x \leqslant c\int_\Omega \varphi_i\left(x, \frac{\partial u}{\partial x_i}(x)\right) \mathrm{d}x < +\infty, \tag{29}$$

for some constant c independent of u.

By the dominate convergence theorem, one deduces

$$\chi_{E_{ni}}\phi_i\left(x, \frac{\partial u}{\partial x_i}(x)\right) \xrightarrow[n\to+\infty]{} \chi_{E_{\infty,i}}\phi_i\left(x, \frac{\partial u}{\partial x_i}(x)\right), \tag{30}$$

(with $E_{\infty,i} = \left\{x \in \Omega : \left|\frac{\partial u}{\partial x_i}(x)\right| > M\right\}$), a.e. in Ω, and strongly in $L^{\widetilde{\varphi}_i}(\Omega)$. By the weak convergence of $\frac{\partial u_n}{\partial x_i}$ to $\frac{\partial u}{\partial x_i}$ in $L^{\varphi_i}(\Omega)$, we deduce that

$$\lim_{n\to+\infty} \int_{E_{ni}} \phi_i\left(x, \frac{\partial u}{\partial x_i}(x)\right)\left(\frac{\partial u_n}{\partial x_i}(x) - \frac{\partial u}{\partial x_i}(x)\right) \mathrm{d}x = 0. \tag{31}$$

By the Lebesgue dominate convergence's Theorem, on E^ε_{ni}

$$\lim_{n\to+\infty} \int_{E^\varepsilon_{ni}} \phi_i\left(x, \frac{\partial u}{\partial x_i}(x)\right)\left(\frac{\partial u_n}{\partial x_i}(x) - \frac{\partial u}{\partial x_i}(x)\right) \mathrm{d}x = 0. \tag{32}$$

We combine relations (32), (31) with (27) and we obtain from relation (26)

$$\limsup_{n\to+\infty} \int_{E_{ni}\setminus E^\varepsilon_{ni}} \varphi_i\left(x, \frac{\partial u_n}{\partial x_i}(x)\right) \mathrm{d}x \leqslant \limsup_{n\to+\infty} \int_{\Omega^n_\varepsilon} \varphi_i\left(x, \frac{\partial u}{\partial x_i}(x)\right) \mathrm{d}x = 0. \tag{33}$$

Relations (17) and (33) give the result.

◊

The last Lemma in order to conclude the proof of Theorem 3 is

Lemma 6. *Let $T_n \in \partial J(u_n)$, with T_n associated to $(w^*_{n1}, \dots, w^*_{nN})$, as before. If $\lim\limits_n \int_{F_{ni}} w^*_{ni}\frac{\partial u_n}{\partial x_i}(x) = 0$ then*

$$\lim_n \int_\Omega \varphi_i\left(x, \frac{\partial u_n}{\partial x_i}(x)\chi_{F_{ni}}\right) dx = 0,$$

whenever

$$F_{ni} = \left\{ x \in \Omega : \left| \frac{\partial u_n}{\partial x_i}(x) \right| > M, \ \left| \frac{\partial u}{\partial x_i}(x) \right| \leqslant M - 1 \right\},$$

M is as in the preceding Lemma.

Proof :

Since $w^*_{ni}(x)\dfrac{\partial u_n}{\partial x_i}(x) \geqslant \underline{\alpha}\varphi_i\left(x, \dfrac{\partial u_n}{\partial x_i}(x)\right)$ and $\varphi_i(x, 0) = 0$.

Then, if $\lim_n \displaystyle\int_{F_{ni}} w^*_{ni}(x)\frac{\partial u_n}{\partial x_i}(x)\mathrm{d}x = 0$ we deduce

$$\lim_{n\to\infty} \int_\Omega \varphi_i\left(x, \chi_{F_{ni}}\frac{\partial u_n}{\partial x_i}(x)\right)\mathrm{d}x = 0.$$

Thus

$$\lim_{n\to+\infty} \rho_i\left(\chi_{F_{ni}}\frac{\partial u_n}{\partial x_i}(x)\right) = 0.$$

♢

Applying Lemma 5 of [20] (see Lemma 3 of the above section 2), we conclude that $u_n \to u$ strongly in the Orlicz-Sobolev space $\mathbf{V}_0$. This ends the proof of Theorem 3. ♢

Remark 5. *The monotonicity comes from the convexity of φ_i.*

3.1. **Few examples of applications.**

Let $(q_i, p_i), i = 1, \ldots, N$ be $2N$-bounded measurable functions on Ω satisfying

$$1 < \operatorname*{essinf}_{\Omega} q_i(x) \leqslant q_i(x) < \operatorname*{essinf}_{\Omega} p_i(x).$$

Consider

$$\varphi_i(x,t) = \begin{cases} |t|^{p_i(x)} & if\ |t| > 1, \\ |t|^{q_i(x)} & if\ |t| \leqslant 1, \end{cases}, \text{ for } x \in \Omega,\ t \in \mathbb{R}.$$

We can choose $\phi_i(x,t)$ as

$$\phi_i(x,t) = \begin{cases} p_i(x)|t|^{p_i(x)-2}t & if\ |t| > 1, \\ q_i(x)|t|^{q_i(x)-2}t & if\ |t| < 1, \\ q_i(x) & if\ t = 1, \\ -q_i(x) & if\ t = -1. \end{cases}$$

One can check that all the assumption **A1./** to **A3./** are fulfilled. In this case, the Orlicz space $L^{\varphi_i}(\Omega)$ coincide with the variable exponent $L^{p_i(\cdot)}(\Omega)$. And

$$\partial\varphi_i(x,t) = \begin{cases} \phi_i(x,t) & if\ |t|>1\ or\ |t|<1, \\ \Big[q_i(x), p_i(x)\Big]\,\mathrm{sign}\,(t) & otherwise. \end{cases}$$

Applying Theorem 4 of [20] (see Theorem 2 of the above section 2), one has

Theorem 4. *Let* (p_{N+1}, q_{N+1}) *be two bounded measurable function satisfying*

$$1 < \operatorname*{essinf}_{\Omega}\ q_{N+1}(x) \leqslant q_{N+1}(x) < \operatorname*{essinf}_{\Omega}\ p_{N+1}(x).$$

Assume that $\mathbf{V}_0$ *is compactly embedded in* $L^{p_{N+1}}(\Omega)$ *and*

$$|q_{N+1}|_- = \operatorname*{essinf}_{\Omega}\ q_{N+1}(x) > p_\infty = \max_{1<j\leqslant N} |p_j|_\infty.$$

Then for all $\lambda > 0$*, there exists a non trivial function* $u \in \mathbf{V}_0$ *such that*

$$-\mathrm{div}_{\vec{\beta}}\left(\partial\varphi_i\left(x, \frac{\partial u}{\partial x_i}(x)\right)\right) = \lambda w_{N+1}\Big(x, u(x)\Big),$$

with $w_{N+1}\big(x,u(x)\big) \in \partial\varphi_{N+1}\big(x,u(x)\big)$.

$$\varphi_{N+1}\big(x,u(x)\big) = \begin{cases} |u(x)|^{p_{N+1}(x)} & if\ |u(x)|>1, \\ |u(x)|^{q_{N+1}(x)} & if\ |u(x)|\leqslant 1. \end{cases}$$

Proof :

Introduce

$$\Phi(u) = J(u) - j(u), \quad j(u) = \int_\Omega \varphi_{N+1}(x,u(x))\mathrm{d}x,$$

as in [20], one can check assumption H4./ ◊

3.2. **Other examples of function φ_i.**

The following example is related to Orlicz-Sobolev spaces with variable exponents

$$\begin{cases} \phi_i(x,t) = p_i(x)\,\mathrm{Log}\,\big(1+\alpha+|t|\big)|t|^{p_i(x)-2}t, & x\in\Omega,\ t\in \mathbb{R}, \\ \operatorname*{essinf}_{\Omega}\ p_i(x) > 1, & \alpha>0,\ p_i(x)\in L^\infty(\Omega), i=1,\dots,N, \end{cases}$$

$$\varphi_i(x,t) = \mathrm{Log}\,\big(1+\alpha+|t|\big)|t|^{p_i(x)} - \int_0^{|t|} \frac{s^{p_i(x)}}{1+\alpha+s}\mathrm{d}s,\ i=1,\dots,N.$$

4. An abstract result for the existence of critical value near zero

To complete the abstract Theorem 4 in [20] we shall introduce here a result where we don't need to impose a compact embedding of V in X.

We replace **H1./** by

H'1./We assume that the graph of $\partial\Phi$ satisfies the following property:

For any sequence $(u_n, w_n)_{n\geqslant 0}$, $w_n \in \partial\Phi(u_n)$ such that $u_n \rightharpoonup u$ weakly in V, $w_n \to 0$ strongly in V', one has

$$0 \in \partial\Phi(u).$$

We replace **H2./** by

H'2./ The function j can be split into two locally Lipschitz functions j_0, j_1 in V such that $j = j_0 + j_1$, $j_0(0) = j_1(0) = 0$, the map $V \to j_0(v)$ is weakly continuous. We have the following growth

(1) $\exists\,\beta > 0$:

$$\beta j_1(u) \leqslant \inf_{v\in\partial j(u)} < v, u > \qquad \forall\, u \in X.$$

(2) $\dfrac{1}{\beta} \sup_{w^*\in\partial J(u)} < w^*, u > \leqslant J(u), \qquad \forall\, u \in V.$

H'3./ $\Phi(0) = 0$ and there exist $u_1 \in V$ and $\eta > ||u_1||$ such that

$$\Phi(v) > 0,\ ||v|| = \eta,\ \Phi(u_1) < 0,\ \inf_{v\in \overline{B}(0,\eta)} \Phi(v) > -\infty.$$

Theorem 5. *Assume* **H'1./**, **H'2./**, *and* **H'3./**.

Then, there exists a function $u \not\equiv 0$, $u \in V$ a critical point of $\Phi = J - j$, that is

$$0 \in \partial J(u) - \partial j(u).$$

Proof :

Let $B(0, \eta)$ be the open ball of radius η. Since we have a function $u_1 \not\equiv 0$, $u_1 \in V$ such that $\Phi(u_1) < 0$ (by **H'3./**), then $u_1 \in B(0, \eta)$ and $c = \operatorname{Inf}_{v\in\overline{B}(0,\eta)} \Phi(v)$, verifies

$$-\infty < c \leqslant \Phi(u_1) < 0.$$

By Ekeland variational principle (see [12, 15] and Appendix A) we have a minimizing sequence $u_n \in B(0, \eta)$, and $u \in V$ such that

(1) $\Phi(u_n) \xrightarrow[n\to+\infty]{} c$, $\Big($thus $u_n \in \text{interior}\big(\overline{B}(0, \eta)\big) = B(0, \eta)\Big)$,

(2) $-\ell_n^* \in \partial\Phi(u_n)$, $||\ell_n^*||_{V'} \xrightarrow[n\to+\infty]{} 0$,

(3) u_n converges weakly to u in V.

But the assumption **H'1./**, we deduce

$$0 \in \partial\Phi(u).$$

It remains to show that $u \not\equiv 0$. There exist $w_n \in \partial J(u_n),\ v_n \in \partial j(u_n)$ such that

$$0 = w_n - v_n + \ell_n^*,$$

this implies

$$0 = \frac{1}{\beta} < w_n, u_n > -\frac{1}{\beta} < v_n, u_n > +\frac{1}{\beta} < \ell_n^*, u_n > . \tag{34}$$

We shall write

$$\frac{1}{\beta} < \ell_n^*, u_n > +\Phi(u_n) = J(u_n) - \frac{1}{\beta} < w_n, u_n > +\frac{1}{\beta} < v_n, u_n > -j_1(u_n) - j_0(u_n). \tag{35}$$

By assumption **H'2./**, we have

$$J(u_n) - \frac{1}{\beta} < w_n, u_n > \geqslant 0. \tag{36}$$

$$-j_1(u_n) + \frac{1}{\beta} < v_n, u_n > \geqslant 0. \tag{37}$$

From relation (35) to (37), one deduces

$$\frac{1}{\beta} < \ell_n^*, u_n > +\Phi(u_n) \geqslant -j_0(u_n),$$

we can let $n \to +\infty$ to derive

$$0 > c \geqslant -j_0(u) : j_0(u) \geqslant -c > 0 : u \not\equiv 0.$$

$\diamondsuit$

4.1. **Example of applications of Theorem 5.**

Theorem 6. *Let Ω be an open bounded Lipschitz set and $\varphi_i(x,t) = \dfrac{1}{p_i}|t|^{p_i}$, with $p_i > 1$ are N-real numbers such that*

$$\sum_{i=1}^{N} \frac{1}{p_i} > 1 \text{ and let } p^* = \frac{N}{\displaystyle\sum_{i=1}^{N} \frac{1}{p_i} - 1},\ p_+ = \max(p_1, \dots, p_N).$$

We assume $p_+ < p^*$, *and* $p_- = \min(p_1, \ldots, p_N)$. *Consider* $1 < q_1 < p_-$, $1 < q_2 < p^*$, $q_1 < q_2$ *two real numbers.*

Then there exist $\lambda^* > 0, \forall\, \lambda \in]0, \lambda^*[$, $u \in W_0^{1,p_1,\ldots,p_N}(\Omega)$, $u \geqslant 0$, $u \not\equiv 0$,

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}(x)\right|^{p_i-2}\frac{\partial u}{\partial x_i}(x)\right) - u^{p^*-1} \in \lambda \partial j_0(u),$$

with

$$j_0(u) = \int_\Omega g\big(u(x)\big)\mathrm{d}x, \quad g\big(u(x)\big) = \begin{cases} |u(x)|^{q_1} & if\ |u(x)| \leqslant 1, \\ |u(x)|^{q_2} & if\ |u(x)| > 1. \end{cases}$$

Proof :

We set

$$\mathbf{V} = \left\{ v \in L^{p_+}(\Omega), \int_\Omega \left|\frac{\partial v}{\partial x_i}(x)\right|^{p_i} \mathrm{d}x < +\infty, i = 1, \ldots, N,\ \gamma_0 v = 0 \right\}.$$

One has $\mathbf{V} \hookrightarrow L^{p^*}(\Omega)$, $p^* = \dfrac{N}{\sum_{i=1}^{N}\dfrac{1}{p_i} - 1}$ (see [13])

$$J(u) = \sum_{i=1}^{N}\frac{1}{p_i}\int_\Omega \left|\frac{\partial u}{\partial x_i}(x)\right|^{p_i} dx, \qquad j_1(u) = \frac{1}{p^*}\int_\Omega |u|^{p^*}\mathrm{d}x,$$

$$j_0(u) = \int_\Omega g(u)\mathrm{d}x,\ g(u) = \begin{cases} |u|^{q_1} & if\ |u| \leqslant 1, \\ |u|^{q_2} & if\ |u| > 1. \end{cases}$$

$$j_1'(u) = \left\{|u|^{p^*-2}u\right\},$$

j_0 is convex, vanishing at 0 and therefore $\partial j_0(u)$ is monotone, $< \partial j_0(u), u > \geqslant 0$

(38) $$< \partial j(u), u > \geqslant < j_1'(u), u > = \int_\Omega |u|^{p^*}\mathrm{d}x \geqslant p^* j_1(u),$$

$$J'(u) = \left\{ -\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}(x)\right|^{p_i-2}\frac{\partial u}{\partial x_i}(x)\right)\right\},$$

and then

$$< J'(u), u > = \sum_{i=1}^{N}\int_\Omega \left|\frac{\partial u}{\partial x_i}(x)\right|^{p_i}\mathrm{d}x \leqslant p_+ J(u).$$

Choosing β such that $p_+ < \beta < p^*$, we have **H'2./**.

Since the injection of $\mathbf{V}$ into $L^q(\Omega)$ is compact for any $q < p^*$, we conclude easily that the mappings

i) $u \to j_0(u)$ is weakly continuous,

ii) $u \in \mathbf{V} \to |u|^{p^*-2}u$ is continuous from $\mathbf{V}$ endowed the weak topology into $L^r(\Omega)$ endowed with the strong topology $r < \dfrac{p^*}{p^*-1}$.

Let S_0 be the Sobolev constant satisfying

$$|v|_{p^*} \leqslant S_0||v|| = S_0 \sum_{i=1}^{N} \left| \frac{\partial v}{\partial x_i} \right|_{p_i}.$$

Since $p^* > p_+$ there exists $0 < \eta < 1 : a = \dfrac{1}{N^{p_+}} \eta^{p_+ - q_2} - S_0^{p^*} \eta^{p^* - q_2} > 0$.
Then, we have, for $v \in \mathbf{V} : ||v|| = \eta$

$$J(v) - j_1(v) \geqslant \frac{1}{p^*} \eta^{q_2} a.$$

Let $\lambda^* > 0$ be a real number such that

$$\lambda^* \sup_{||v||=\eta} j_0(v) < a \frac{\eta^{q_2}}{p^*},$$

then $\forall\, v \in \mathbf{V}$ such that $||v|| = \eta$

$$\Phi(v) \doteq J(v) - j_1(v) - \lambda j_0(v) > 0, \quad \forall\, \lambda \in]0, \lambda^*[.$$

Let $u_0 \in C_c^1(\Omega)$, $\Big\{|u_0| \leqslant 1\Big\}$ and $\Big\{|u_0| > 1\Big\}$ are of positive measure, $0 < t < 1$

$$\begin{aligned} \Phi(tu_0) &\leqslant \sum_{i=1}^{N} \frac{t^{p_i}}{p_i} \int_\Omega \left| \frac{\partial u_0}{\partial x_i}(x) \right|^{p_i} \mathrm{d}x - t^{p^*} \int_\Omega \frac{|u_0|^{p^*}}{p^*} \mathrm{d}x - \lambda t^{q_1} \int_{\{|u_0| \leqslant 1\}} |u_0|^{q_1} \mathrm{d}x \\ &\leqslant t^{p_-} \Big(A_1 - A_2 t^{p^* - p_-} - A_3 t^{q_1 - p_-} \Big) \\ &< 0 \text{ for } t \text{ near zero since } q_1 < p_-,\ A_3 > 0,\ t||u_0|| < \eta. \end{aligned}$$

It is easy to check that

$$\inf \Big\{ \Phi(v), v \in \overline{B}(0, \eta) \Big\} > -\infty.$$

Since the operator $u \in \mathbf{V} \to \partial j(u)$ is bounded, we may use the compactness result used in [10] (see also [19]) to show that the graph of $u \in \mathbf{V} - \sigma(\mathbf{V}, \mathbf{V}') \to \partial\Phi(u)$ satisfies **H'1./**, with $\mathbf{V}$ is endowed with the weak topology. Indeed, let (u_n) be a sequence of $\mathbf{V}$ converging to u weakly and strongly in $L^r(\Omega)$ for $r < p^*$ and a.e. in Ω. Then $\partial j(u_n)$ remains in a bounded set of $(L^{p^*}(\Omega) + L^{q_2}(\Omega))'$. Let $k > 0$ and consider

$$T_k(\sigma) = \begin{cases} \sigma & \textit{if } |\sigma| < k, \\ k \operatorname{sign}(\sigma) & \textit{otherwise}. \end{cases}, \quad u^k = T_k(u).$$

Then $\forall\, w_n \in \partial j(u_n),\ \forall\, \varphi \in C_c^\infty(\Omega),\ \Big| < w_n, \varphi T_\varepsilon(u_n - u^k) > \Big| \leqslant |\varphi|_\infty K_0 \varepsilon$ for some constant K_0. If $w_n^* \in \partial\Phi(u_n),\ w_n^* \to 0$ strongly in $\mathbf{V}'$ then $w_n^* + w_n = J'(u_n), w_n \in \partial j(u_n)$, therefore we have

$$\sum_{i=1}^N \int_\Omega \left|\frac{\partial u_n}{\partial x_i}(x)\right|^{p_i-2} \frac{\partial u_n}{\partial x_i}(x) \frac{\partial}{\partial x_i}\Big(T_\varepsilon(u_n - u^k)\varphi\Big) \mathrm{d}x \leqslant |\varphi|_\infty K_0\varepsilon + K_1 ||w_n^*||_{\mathbf{V}'}.$$

We may apply Theorem 1 of [10] or Lemma 2 of [19] to conclude that

$$\nabla u_n(x) \xrightarrow[n \to +\infty]{} \nabla u(x) \text{ (up to subsequence).}$$

Therefore $\forall\, v \in \mathbf{V}$,

$$< J'(u_n), v > \to < J'(u), v >$$

and

$$\int_\Omega |u_n|^{p^*-2} u_n v \mathrm{d}x \to \int_\Omega |u|^{p^*-2} u v \mathrm{d}x.$$

Since $-w_n^* + J'(u_n) - |u_n|^{p^*-2} u_n \in \lambda \partial j_0(u_n)$, $u_n \to u$ in $L^r(\Omega)$, $r < p^*$. We deduce

$$J'(u) - |u|^{p^*-2} u \in \lambda \partial j_0(u) : 0 \in \partial\Phi(u). \tag{39}$$

Indeed, $u_n \to u$ in $L^{q_2}(\Omega)$ strongly, there exists h_n belonging to a bounded set of $L^{q_2'}(\Omega)$ such that $-w_n^* + J'(u_n) - |u_n|^{p^*-2} u_n = h_n \in \lambda \partial j_0(u_n)$.

We have a subsequence $h_{n'}$ and a function h such that $h_{n'} \rightharpoonup h$ weakly in $L^{q_2'}(\Omega)$. Thus $h \in \lambda \partial j_0(u)$. On the other hand, since

$$\forall\, v \in V, \lim_{n \to +\infty} < w_n^*, v >= 0, \tag{40}$$

($w_n^* \to 0$ strongly in $\mathbf{V}'$)

$$\lim_{n \to +\infty} < J'(u_n) - |u_n|^{p^*-2} u_n, v >=< J'(u) - |u|^{p^*-2} u, v > . \tag{41}$$

We deduce in the sense of the distribution that

$$J'(u) - |u|^{p^*-2} u = h \in \lambda \partial j_0(u).$$

This show that $0 \in \partial\Phi(u)$.

Since $\Phi(u) = \Phi(|u|)$, we may choose $u \geqslant 0$. This ends the proof. $\diamondsuit$

Remark 6. *The above Theorem 6 allows to recover many other situations, when λj_0 is small, in particular we can recover Theorem 1 in* [2]. *We may omit the term $|u|^{p^*-2}$, the result still holds true.*

Next, we want to study eigenvalue problem on unbounded domain eventually with a weight. To do so, we shall start with the following result due to V. Maz'ja.

5. Unbounded case with a weighted function

Theorem 7. *(see [16]). For all $u \in W^{1,1}(\mathbb{R}^N)$ with compact support, $\forall\, \alpha \in [0,1]$*

$$\left(\int_{\mathbb{R}^N} |u(x)|^{\frac{N-\alpha}{N-1}} \frac{\mathrm{d}x}{|x|^\alpha}\right)^{\frac{N-1}{N-\alpha}} \leqslant (N-\alpha)^{\frac{1-N}{N-\alpha}}\, \omega_N^{\frac{\alpha-1}{N-\alpha}} |\nabla u|_{L^1(\mathbb{R}^N)}.$$

Here ω_N is the measure of the unit ball of $\mathbb{R}^N$.

Corollary 1. *(***of Theorem 7)**
Let $p^*_\alpha = \dfrac{(N-\alpha)p}{(N-\alpha)-(1-\alpha)p}$, $\alpha \in [0,1]$, $p \geqslant 1$, $(N-\alpha) > (1-\alpha)p$. Then, $\forall\, u \in W^{1,p}(\mathbb{R}^N)$ with compact support, we have

$$\left(\int_{\mathbb{R}^N} |u|^{p^*_\alpha} \frac{\mathrm{d}x}{|x|^\alpha}\right)^{\frac{1}{p^*_\alpha}} \leqslant c_N \left(\int_{\mathbb{R}^N} |\nabla u|^p |x|^{\alpha(p-1)} \mathrm{d}x\right)^{\frac{1}{p}},$$

with $c_N = t(N-\alpha)^{\frac{1-N}{N-\alpha}} \omega_N^{\frac{\alpha-1}{N-\alpha}}$, $t = \frac{N-1}{N-\alpha} p^*_\alpha$.

Proof :
Let $u \in W^{1,p}(\mathbb{R}^N)$ with compact support. Then, $|u|^t \in W^{1,p}(\mathbb{R}^N)$ and has its support compact. Therefore, using Theorem 7, we have

$$\left(\int_{\mathbb{R}^N} |u(x)|^{t\frac{N-\alpha}{N-1}} \frac{\mathrm{d}x}{|x|^\alpha}\right)^{\frac{N-1}{N-\alpha}} \leqslant c_N \int_{\mathbb{R}^N} |u(x)|^{t-1} |\nabla u(x)| \mathrm{d}x. \tag{42}$$

Apply the Hölder's inequality to the right hand side of (42), we obtain (introducing the weight $|x|^{-\alpha}$

(43)
$$\left(\int_{\mathbb{R}^N} |u(x)|^{p^*_\alpha} \frac{\mathrm{d}x}{|x|^\alpha}\right)^{\frac{t}{p^*_\alpha}} \leqslant c_N \left(\int_{\mathbb{R}^N} |u(x)|^{(t-1)p'} \frac{\mathrm{d}x}{|x|^\alpha}\right)^{\frac{1}{p'}} \left(\int_{\mathbb{R}^N} |\nabla u(x)|^p |x|^{\alpha(p-1)} \mathrm{d}x\right)^{\frac{1}{p}}.$$

We have $(t-1)p' = p^*_\alpha$, then, from (43), we have

$$\left(\int_{\mathbb{R}^N} |u(x)|^{p^*_\alpha} \frac{\mathrm{d}x}{|x|^\alpha}\right)^{\frac{t}{p^*_\alpha} - \frac{1}{p'}} \leqslant c_N \left(\int_{\mathbb{R}^N} |\nabla u(x)|^p |x|^{\alpha(p-1)} \mathrm{d}x\right)^{\frac{1}{p}}. \tag{44}$$

And one can check that $\dfrac{t}{p^*_\alpha} - \dfrac{1}{p'} = \dfrac{1}{p^*_\alpha}$, this shows the result. ♢

Let Ω be an open set of $\mathbb{R}^N$, we denote for $v \in C^\infty_c(\Omega)$, $||v|| = \left(\int_\Omega |\nabla v(x)|^p |x|^{\alpha(p-1)} \mathrm{d}x\right)^{\frac{1}{p}}$. We shall denote by $\mathcal{D}^{1,p}_{0,\alpha}(\Omega)$ to closure of $C^\infty_c(\Omega)$ with respect to the above norm. As a consequence of Corollary 1 , we have

Lemma 7. *$\forall\, u \in \mathcal{D}^{1,p}_{0,\alpha}(\Omega)$, one has*

$$\left(\int_\Omega |u(x)|^{p^*_\alpha}\frac{\mathrm{d}x}{|x|^\alpha}\right)^{\frac{1}{p^*_\alpha}} \leqslant c_N \left(\int_\Omega |\nabla u(x)|^p |x|^{\alpha(p-1)}\mathrm{d}x\right)^{\frac{1}{p}}.$$

as in Corollary 1 of Theorem 7.

Theorem 8. *Let $1 < q < p$ and $a \in L^m_+\left(\Omega, |x|^{\frac{\alpha q}{p^*_\alpha - q}}\right)$, $m = \dfrac{p^*_\alpha}{p^*_\alpha - q}$. There exists a number λ^* such that $\forall\, \lambda \in]0, \lambda^*[$, there exists a function $u \geqslant 0$, $u \not\equiv 0$, $u \in \mathcal{D}^{1,p}_{0,\alpha}(\Omega)$ such that*

$$-\mathrm{div}\left(|x|^{\alpha(p-1)}|\nabla u|^{p-2}\nabla u\right) = \lambda a(x) u^{q-1} + \frac{u^{p^*_\alpha - 1}}{|x|^\alpha} \text{ in } \mathcal{D}'(\Omega).$$

Sketch of proof

The argument is similar to the proof of Theorem 6, using Theorem 5, we set

$$\begin{aligned} J(u) &= \frac{1}{p}\int_\Omega |\nabla u|^p |x|^{\alpha(p-1)}\mathrm{d}x,\\ j_1(u) &= \frac{1}{p^*_\alpha}\int_\Omega |u|^{p^*_\alpha}\frac{\mathrm{d}x}{|x|^\alpha},\\ j_0(u) &= \frac{1}{q}\int_\Omega a(x)|u(x)|^q\mathrm{d}x. \end{aligned}$$

One has

$$\begin{aligned} \partial J(u) &= \left\{-\,\mathrm{div}\left(|x|^{\alpha(p-1)}|\nabla u|^{p-2}\nabla u\right)\right\},\\ \partial j_1(u) &= \left\{\frac{|u|^{p^*_\alpha - 2}}{|x|^\alpha}u\right\},\\ \partial j_0(u) &= \left\{a(x)|u|^{q-2}u\right\},\\ j(u) &= \lambda j_0(u) + j_1(u) \text{ (see below for the choice of } \lambda),\\ <\partial j(u), u> &\geqslant \int_\Omega |u|^{p^*_\alpha}\frac{\mathrm{d}x}{|x|^\alpha} \geqslant p^*_\alpha j_1(u),\\ <J'(u), u> &= pJ(u). \end{aligned}$$

Thus, we can take $p < \beta < p^*_\alpha$ in conditions (1) and (2) of **H'2./**. Due to the integrability of the function a , we deduce that the mapping $u \in V \to j_0(u)$ is continuous from V-weak into $\mathbb{R}$. Indeed if $(u_n)_n$ is a bounded sequence in $\mathcal{D}^{1,p}_{0,\alpha}(\Omega)$ then there exist a function $u \in \mathcal{D}^{1,p}_{0,\alpha}(\Omega)$ and a subsequence still denoted u_n such that $u_n(x) \underset{n}{\to} u(x)$ a.e. and $u_n \rightharpoonup u$ weakly in $\mathcal{D}^{1,p}_{0,\alpha}(\Omega)$. Then $\lim_n \int_\Omega a(x)|u_n - u|^q\mathrm{d}x = 0$ using Vitali's Theorem. Thus **H'2./** is satisfied. While for **H'3./**, we have $\Phi(0) = 0 = J(0) = j(0)$. There exists $\eta > 0$ such that $\eta_1 = \eta^{p-q} - c_N^{p^*_\alpha}\eta^{p^*_\alpha - q} > 0$, c_N is the Sobolev constant given in the above Lemma.

For all $v \in \mathcal{D}^{1,p}_{0,\alpha}(\Omega)$ with $||v|| = \eta$

$$J(v) - j_1(v) \geqslant \frac{1}{p^*_\alpha}\eta^q \eta_1.$$

Let λ^* such that

$$\lambda^* \sup_{||v||=\eta} j_0(v) < \eta_1 \frac{\eta^q}{p^*_\alpha}.$$

Then $\forall\, v \in V$ with $||v|| = \eta$, $\forall\, \lambda \in]0, \lambda^*[$

$$\Phi(v) \doteq J(v) - j_1(v) - \lambda j_0(v) > 0.$$

Let $u_0 \in C^\infty_c(\Omega)$ such that support $(a) \cap \Big\{|u_0| \geqslant 1\Big\}$ or support $(a) \cap \Big\{|u_0| \leqslant 1\Big\}$ is of positive measure, $0 < t < 1$

$$\begin{aligned}\Phi(tu_0) &\leqslant \frac{t^p}{p}\int_\Omega |\nabla u_0|^p |x|^{\alpha(p-1)}\mathrm{d}x - \frac{t^{p^*_\alpha}}{p^*_\alpha}\int_\Omega |u_0|^{p^*_\alpha}\frac{\mathrm{d}x}{|x|^\alpha} - \frac{\lambda t^q}{q}\int_\Omega a(x)|u_0|^q \mathrm{d}x \\ &\leqslant t^p\Big(A_1 - t^{p^*_\alpha - p}A_2 - A_3 t^{q-p}\Big)\end{aligned}$$

since $q < p$, $A_3 = \dfrac{\lambda}{q}\displaystyle\int_\Omega a(x)|u_0|^q \mathrm{d}x > 0$, the $\Phi(tu_0) < 0$ for t small.
It is easy to check that

$$\mathrm{Inf}\ \Big\{\Phi(v),\ ||v|| \leqslant \eta\Big\} > -\infty$$

since

$$j_1(v) \leqslant c||v||^{p^*_\alpha}, \quad j_0(v) \leqslant c||v||^q.$$

Thus all the conditions in **H'3./** are satisfied.
The graph of the mapping $v \in \mathcal{D}^{1,p}_{0,\alpha}(\Omega) \to J'(v) - j'(v)$ satisfies **H'1./** , when we endow $\mathcal{D}^{1,p}_{0,\alpha}(\Omega)$ with the weak topology. Indeed, its graph is

$$\Big\{(w, v) \in V' \times V : w = J'(v) - j'(v)\Big\} \doteq Gr(J', j').$$

If $(w_n, v_n) \in Gr(J', j')$ is a sequence such that $v_n \rightharpoonup v$ weakly in $\mathcal{D}^{1,p}_{0,\alpha}(\Omega)$ and $w_n \to 0$ in $\Big(\mathcal{D}^{1,p}_{0,\alpha}(\Omega)\Big)'$ strongly, then we have to show that

$$0 = J'(v) - j'(v).$$

Indeed, considering, $T_k(\sigma) = \begin{cases}\sigma & \text{if } |\sigma| < k \\ k\,\mathrm{sign}\,(\sigma) & \text{otherwise,}\end{cases}$
we have $\forall\, 0 < \varepsilon < 1$, $\forall\, \varphi \in C^\infty_c(\Omega)$.

$$\int_\Omega |x|^{\alpha(p-1)}|\nabla v_n|^{p-2}\nabla v_n \nabla\Big(\varphi T_\varepsilon\big(v_n - T_k(v)\big)\Big)\mathrm{d}x \leqslant K_0\Big(\varepsilon + ||w_n||_*\Big),$$

for some $K_0 > 0$, thus

$$\limsup_{n\to+\infty} \int_\Omega |x|^{\alpha(p-1)}|\nabla v_n|^{p-2}\nabla v_n \nabla\Big(\varphi T_\varepsilon\big(v_n - T_k(v)\big)\Big) \leqslant K_0\varepsilon.$$

We may apply Theorem 1 of [10] to conclude that up to a constant $\nabla v_n(x) \to \nabla v(x)$ a.e.. Using the growth of the gradient, we deduce easily: $\forall \varphi \in \mathcal{D}^{1,p}_{0,\alpha}(\Omega)$

$$\begin{array}{rcl} < J'(v_n), \varphi > & \to & < J'(v), \varphi >, \\ \displaystyle\int_\Omega |v_n|^{p^*_\alpha-2} v_n \varphi \mathrm{d}x & \to & \displaystyle\int_\Omega |v|^{p^*_\alpha-2} v\varphi \mathrm{d}x. \end{array}$$

Thus

$$0 = J'(v) - j'(v) \text{ in}\Big(\mathcal{D}^{1,p}_{0,\alpha}(\Omega)\Big)'.$$

This shows **H'1./**.

We may appeal Theorem 5 to conclude that, there exits $u \geqslant 0,\ u \not\equiv 0$ such that

$$0 = J'(u) - j'(u).$$

♢

Combining the arguments of Theorem 6 and Theorem 8, we have also

Theorem 9. *Let* (q_1, q_2) *be two real numbers such that* $1 < q_1 < q_2 < p$ *and* $a \in L^m_+\Big(\Omega, |x|^{\frac{\alpha q_2}{p^*_\alpha - q_2}}\Big)$, $m = \dfrac{p^*_\alpha}{p^*_\alpha - q_2}$. *There exists a number* λ^* *such that* $\forall\, \lambda \in]0, \lambda^*[$, *there exists a function* $u \geqslant 0,\ u \not\equiv 0,\ u \in \mathcal{D}^{1,p}_{0,\alpha}(\Omega)$ *such that*

$$-\mathrm{div}\Big(|x|^{\alpha(p-1)}|\nabla u|^{p-2}\nabla u\Big) - \frac{u^{p^*_\alpha - 1}}{|x|^\alpha} \in \lambda \partial j_a(u)$$

with $j_a(u) = \dfrac{1}{q_1}\displaystyle\int_{\{|u|\leqslant 1\}} |u(x)|^{q_1} a(x)\mathrm{d}x + \dfrac{1}{q_2}\displaystyle\int_{\{|u|>1\}} |u(x)|^{q_2} a(x)\mathrm{d}x.$

Remark for the proof of Theorem 9

Let us notice that setting $j(u) = j_1(u) + \lambda j_a(u)$ then $\lambda < \partial j_a(u), u > \geqslant 0$ so that

- $< \partial j(u), u > \geqslant p^*_\alpha j_1(u)$
- the map $u \in V \to j_a(u)$ is continuous from V-weak into $\mathbb{R}$.
- λ^* is defined as before

$$\lambda^* \sup_{||v||=\eta} j_a(v) < \eta_1 \frac{\eta^{q_2}}{p^*_\alpha}.$$

♢

For the case $p < q < p^*_\alpha$, we may adapt the proof given in [11] to obtain

Theorem 10. *Let $a \in L^\infty(\Omega)$ with compact support, $a \not\equiv 0$, $p < q < p_\alpha^*$. Then there exists λ^*, such that $\forall \lambda > \lambda^*$ the problem*

$$\begin{cases} u \in \mathcal{D}^{1,p}_{0,\alpha}(\Omega), \ u \geqslant 0, \ u \not\equiv 0, \\ -\text{div}\left(|x|^{\alpha(p-1)}|\nabla u|^{p-2}\nabla u\right) = \lambda a(x) u^{q-1} + \dfrac{u^{p_\alpha^*-1}}{|x|^\alpha} \end{cases}$$

possesses at least one solution.

For details, we refer [11, 6].

Appendix A

We recall the following lemma due to Ekeland [9, 8]

Lemma 8. *Let $B(0,\rho)$ an open ball of radius $\rho > 0$ of a Banach $(V, ||\cdot||)$ and $\Phi : \overline{B}(0,\rho) \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous function on $\overline{B} = \overline{B}(0,\rho)$ with $\inf\limits_{\overline{B}} \Phi > -\infty$.*
Let $\varepsilon > 0$, $z \in \overline{B} : \Phi(z) \leqslant \inf\limits_{\overline{B}} \Phi + \varepsilon$.
Then

$$\exists y \in \overline{B} \quad \begin{array}{rcl} \Phi(y) & \leqslant & \Phi(z), \\ d(z,y) & \leqslant & \sqrt{\varepsilon}, \\ \forall x \in \overline{B}: \quad \Phi(x) & \geqslant & \Phi(y) - \sqrt{\varepsilon} d(x,y). \end{array}$$

Corollary 2. of Lemma 8 *Under the same assumptions as the above Lemma there exist a sequence $(x_n)_n$, $x_n \in \overline{B}$:*

- $\Phi(x_n) \to \operatorname{Inf}\limits_{\overline{B}} \Phi$,
- $\Phi(x) \geqslant \Phi(x_n) - \dfrac{1}{n}||x - x_n||$, $\forall x \in \overline{B}$,
- *assume that $x_n \in B(0,\rho) = interior(\overline{B})$ then*

$$m_\Phi(x_n) = \inf \left\{ ||w_n||_{V'}, \ w_n \in \partial\Phi(x_n) \right\} \xrightarrow[n \to +\infty]{} 0.$$

Proof :
Consider $\varepsilon = \dfrac{1}{n^2}$, $y = x_n$ in Lemma 8 then

$$\begin{array}{rcl} \Phi(x_n) & \leqslant & \inf\limits_{\overline{B}} \Phi + \dfrac{1}{n^2}, \\ \Phi(x) & \geqslant & \Phi(x_n) - \dfrac{1}{n}||x - x_n|| \quad \forall x \in \overline{B}. \end{array}$$

Thus, if we set $F(x) = \Phi(x) - \dfrac{1}{n}||x - x_n||$, then $F(x_n) \leqslant F(x) \quad \forall x \in \overline{B}$. This implies in the case that $x_n \in B(0,\rho) = interior(\overline{B})$ that

$$0 \in \partial F(x_n) \subset \partial \Phi(x_n) - \frac{1}{n}\partial f_n(x_n) \qquad with\ f_n(x) = ||x - x_n||. \tag{45}$$

Using the definition of subdifferential we see that

$$\partial f_n(x_n) \subset \Big\{x^* \in V' : ||x^*||_{V'} \leqslant 1\Big\}.$$

Thus, relation (45) implies that

$$0 = w_n^* - \frac{1}{n}x_n^*, \qquad w_n^* \in \partial\Phi(x_n), \qquad ||x_n^*|| \leqslant 1.$$

Therefore,

$$m_\Phi(x_n) \leqslant \frac{1}{n} \xrightarrow[n\to+\infty]{} 0.$$

◊

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