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Qualitative Properties of Eigenvectors Related to Multivalued Operators and some Existence Results

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Abstract In this work, we study a class of nonlinear eigenvalue problems related to fully discontinuous operators. In particular, we prove the existence of a critical point for two distinct problems. Connected with this problem, we also study a minimization problem with constraint and we investigate the existence of solutions for a resonant case near zero. Moreover, we give some estimates and qualitative properties of solutions by using the relative rearrangement theory.

Keywords Nonlinear eigenvalues · Multivalued operators · Clarke subdifferential · Relative rearrangement · Monotone rearrangement

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1 Introduction.

Eigenvalue problems involving nonhomogeneous elliptic operators have captured special attention in the last decade. Numerous papers have been devoted to the study of various phenomena, which occur in the spectrum of such operators (see [1–7] and references therein). The purpose of the present paper is to give additional results on eigenvalue problems related to fully discontinuous operators, which recover the results in [1, 2]; indeed, the operator given here,

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which is recently introduced by J.M. Rakotoson, becomes a natural extension of the univalued operator studied recently in [2].

2 Presentation of the Problem.

The study of this issue was initiated by J.M. Rakotoson in [8], in which the notion of $\overrightarrow{\rho}$-multivalued Leray-Lions operators was introduced by

$$-\mathrm{div}_{\overrightarrow{\rho}}\left(\partial\varphi_i\left(x,\frac{\partial u}{\partial x_i}(x)\right)\right), \tag{1}$$

associated to $\overrightarrow{\rho}=(\rho_0,\ldots,\rho_N)$, where ρ_i are Banach function norms and φ_i are Carathéodory functions on $\Omega\times\mathbb{R}$, such that, for a.e. $t\mapsto\varphi_i(x,t)$ is locally Lipschitz on $\mathbb{R}$; therefore the theoretical formulation of such concrete problems basically relies on the notion of Clarke's generalized gradient, which replaces the subdifferential in the sense of convex analysis (see [9]).
This operator can be used in the Euler-Lagrange functionals, that appear in the minimization problems of image restoration (see [10]).

In this paper, we shall consider the same Orlicz-Sobolev spaces used in [11] (see also [12]), that will enable us to study with sufficient accuracy problem (2). For this purpose, we consider as Banach function norm:

$$\rho(v):=\inf\left\{\lambda>0:\int_\Omega\varphi\left(x,\frac{|v(x)|}{\lambda}\right)\mathrm{d}x\leqslant 1\right\},$$

where φ is a suitable N-function (see [13]), and $L(\Omega,\rho)$ is the associated Banach function space.
Moreover, we shall give additional results concerning the resolution of the eigenvalue problem

$$-\mathrm{div}_{\overrightarrow{\rho}}\left(\partial\varphi_i\left(x,\frac{\partial u}{\partial x_i}(x)\right)\right)=\lambda f\big(x,u(x)\big),\quad f\big(x,u(x)\big)\in\partial j_0\big(x,u(x)\big), \tag{2}$$

under some assumptions on the function j_0, assumptions that place us in a different context from the one studied in [11]. Let us note that existence results are deeply influenced by the competition between the growth rates of the anisotropic coefficients.
The results presented here extend previous works of M. Mihăilescu and al. in [1, 2]. We define on $W_0^{1,p}(\Omega)$ the functionals J and j, such that, for all $v\in W_0^{1,p}(\Omega)$,

$$J(v):=\frac{1}{p}\sum_{i=1}^{N}\left[\int_{\left\{\left|\frac{\partial v}{\partial x_i}\right|\leqslant 1\right\}}\left|\frac{\partial v}{\partial x_i}\right|^q\mathrm{d}x+\int_{\left\{\left|\frac{\partial v}{\partial x_i}\right|>1\right\}}\left|\frac{\partial v}{\partial x_i}\right|^p\mathrm{d}x\right],$$

$$j_0(v):=\frac{1}{p_M}\left[\int_{\{|v|\leqslant 1\}}|v|^{q_M}\,\mathrm{d}x+\int_{\{|v|>1\}}|v|^{p_M}\mathrm{d}x\right].$$

Then, for all $\lambda \in \,]0, \lambda_*[$, there exists a function $u \geqslant 0,\ u \not\equiv 0,\ u \in W_0^{1,p}(\Omega)$ such that $0 \in \partial J(u) - \lambda \partial j_0(u)$, provided
$1 < q < p < +\infty$, $1 < q_M < p_M < p^* = \dfrac{Np}{N-p}$ and $q_M < q$.
Moreover, for all $\lambda > \lambda_{**}$, there exists a non-trivial solution $u \in W_0^{1,p}(\Omega)$ such that $0 \in \partial J(u) - \lambda \partial j_0(u)$, provided
$1 < q < p < +\infty$, $1 < q_M < p_M < p^*$ and $p_M < q$.

In order to go further, we will exploit the nonsmooth version of the Lagrange multiplier theorem (see [9]) to prove the existence of solutions for a more general problem. Moreover, this existence result will be used in the last section, in which we are able to characterize and prove the L^∞-regularity of the eigenvectors associated to $\overrightarrow{\rho}$-multivalued Leray-Lions operators by using the relative rearrangement theory (see [14]).
To our knowledge, this is the first paper dealing with the regularity of the eigenvectors related to $\overrightarrow{\rho}$-multivalued Leray-Lions operators.

We also obtain an existence result for an eigenvalue problem in a resonant case near zero in the sense that the potential has the same growth as the operator in a small neighborhood of the origin.
Our result shall recover the following problem. Consider J as above and

$$h(x,\sigma) := \begin{cases} \dfrac{c^*}{p}|\sigma|^p & \text{if } |\sigma| < 1, \\ \dfrac{1}{\beta}|\sigma|^\beta - |\sigma| + \dfrac{c^*}{p} + \dfrac{\beta-1}{\beta} & \text{otherwise,} \end{cases}$$

where $0 < c^* < \dfrac{1}{N^p \cdot S^p}$, the positive constant S is the Sobolev embedding constant of $W_0^{1,p}(\Omega)$ into $L^p(\Omega)$ and $p < \beta < p^*$. Then, there exists a non-trivial solution $u \in W_0^{1,p}(\Omega)$ such that $0 \in \partial J(u) - \partial j(u)$ and
$j(u) = \displaystyle\int_\Omega h(x, u(x))\mathrm{d}x$.

We start by recalling some basic facts about Clarke subdifferential, Banach function norms, $\overrightarrow{\rho}$-multivalued operators, Orlicz spaces and relative rearrangement. For more details we refer to [8, 9, 11, 13–18].

3 Notations - Preliminary Results.

For a Banach space $(V, \|\cdot\|)$, we shall denote by V' its dual, and the duality bracket between $V'$and V shall be denoted by $< \cdot, \cdot >$. The set of all subsets of V' is denoted by $\mathcal{P}(V') := 2^{V'}$. In order to avoid confusion, we shall also denote by $\|\cdot\|_V$ or $\|\cdot\|_{V'}$ the norms in those spaces.

Definition 3.1 The first statement of the definition is already known (see [19]), the second one was introduced recently in [8].

1. A multivalued operator $A : V \rightrightarrows \mathcal{P}(V')$ is called a monotone operator iff: $\forall\, u_1,\, u_2 \in V,\ \forall\, w_1 \in Au_1,\ \forall\, w_2 \in Au_2$, we have $< w_1 - w_2, u_1 - u_2 > \geqslant 0$.
2. A multivalued monotone operator A is strongly monotonic iff for any sequence $(u_n)_n$ in V converging weakly to a function u and verifying: $\forall\, w_n \in Au_n,\ \forall\, w \in Au$ if $\lim\limits_{n\to+\infty} < w_n - w, u_n - u >= 0$, then u_n converges to u strongly in V.

Definition 3.2 [20–22].

1. We call $j : V \to \mathrm{I\!R}$ to be a locally Lipschitz function iff for all $u \in V$, there exist a ball $B(u,r),\ r > 0$ and a constant $K_r(u) = K(u)$ such that
$$|j(v) - j(w)| \leqslant K(u)\|v - w\|, \qquad \forall\, v,\, w \in B(u,r).$$
2. For each $v \in V$, the generalized directional derivative of j (at a point u in the direction v), denoted by $j^0(u;v)$, is defined as follows:
$$j^0(u;v) := \limsup_{\lambda \searrow 0,\ h \to 0} \frac{j(u + h + \lambda v) - j(u + h)}{\lambda}.$$

On the basic properties of the generalized directional derivative we refer to [9, 20–22].

Definition 3.3 [20–22]. The (Clarke) subdifferential of j at a point $u \in V$ is the subdifferential of the convex map $v \mapsto j^0(u;v)$ at zero. More precisely,
$$\partial j(u) := \Big\{ w \in V' : < w, v > \leqslant j^0(u;v),\ \forall\, v \in V \Big\}.$$

For these reasons, the Clarke subdifferential possesses the same properties as the subdifferential in the sense of convex analysis. For more information on these properties we refer to [9, 20–22].

Definition 3.4 [20–22].
If $j : V \to \mathrm{I\!R}$ is a locally Lipschitz function, we say that u is a critical point of j iff $0 \in \partial j(u)$.

We shall use the following notions on Banach function norms and spaces (see [15] and also [14] for more details). Let
$L^0(\Omega) := \Big\{ v : \Omega \to \mathrm{I\!R}$ Lebesgue measurable$\Big\}$
and $L^0_+(\Omega) := \Big\{ v \in L^0(\Omega),\ v \geqslant 0 \Big\}$.

Definition 3.5 [15]. A mapping $\rho : L^0_+(\Omega) \to \overline{\mathrm{I\!R}}_+ := [0, +\infty]$ is called a Banach function norm iff, for all f, g and g_n in $L^0_+(\Omega)$, for all measurable set $E \subset \Omega$, we have

P1./ $\rho(f) = 0 \Longleftrightarrow f = 0$ a.e.,
$\rho(\lambda f) = \lambda \rho(f), \quad \forall\, \lambda > 0,$
$\rho(f + g) \leqslant \rho(f) + \rho(g)$.
P2./ If $f \leqslant g$ a.e. then $\rho(f) \leqslant \rho(g)$.

P3./ If $f_n \leqslant f_{n+1} \nearrow f$ then $\rho(f_n) \to \rho(f)$.
P4./ If $|E| < +\infty$ then $\rho(\chi_E) < +\infty$. ($|E|$ is the Lebesgue measure of E and χ_E is the characteristic function of E).
P5./ If $|E| < +\infty$ then $\int_E f(x)\mathrm{d}x \leqslant c_E \rho(f)$ for some finite, positive constant c_E depending only on E and ρ.

Definition 3.6 [15]. Let ρ be a Banach function norm. Then, we define

$$Y := L(\Omega, \rho) = \Big\{ f \in L^0(\Omega) : \rho(|f|) < +\infty \Big\}.$$

Y is called a Banach function space and is endowed with the norm
$\|f\|_Y = \rho(|f|)$.

In what follows we state some useful properties of $L(\Omega, \rho)$:

Definition 3.7 [15].

1. If ρ is a Banach function norm, its associate norm ρ' is defined by

$$\rho'(f) := \sup \left\{ \int_\Omega fg\mathrm{d}x,\ g \in L^0_+(\Omega),\ \rho(g) \leqslant 1 \right\},\ \text{for } f \in L^0_+(\Omega).$$

2. The Banach function space determined by ρ' is called the associate Banach function space of $L(\Omega, \rho)$, that is $L(\Omega, \rho') := \Big\{ v \in L^0(\Omega) : \rho'(|v|) < +\infty \Big\}$.

Definition 3.8 [15].
A Banach function space Y is said to have absolutely continuous norm iff

$$\|f \chi_{E_n}\|_Y \to 0,\ \text{for every } \big\{E_n\big\} \text{ satisfying } E_{n+1} \subset E_n \text{ and } \quad |E_n| \to 0.$$

Theorem 3.1 [15]. *A Banach function space Y is reflexive if and only if both Y and its associate Y' have absolutely continuous norm.*

Remark 3.1 *For simplicity, we sometimes write $\rho(f) = \rho(|f|)$, and various constants depending on data shall be denoted by c or c_i.*

The following definitions and results were introduced in [8, 11] (see also [23]). Let $(V, \|\cdot\|)$ and $(X, |\cdot|)$ be two reflexive Banach spaces.

Lemma 3.1 [8]. *Consider two locally Lipschitz functions $J : V \to \mathbb{R}$ and $j : X \to \mathbb{R}$. We suppose that*

H1./ $\begin{array}{c} A : V \rightrightarrows \mathcal{P}(V') \\ u \mapsto \partial J(u) \end{array}$ *be strongly monotonic.*

H2./ *We have the following growth:*

1. $\exists\, \beta > 0,\ \exists\, c_0 \geqslant 0 : \quad \beta j(u) \leqslant \inf\limits_{v^* \in \partial j(u)} < v^*, u > + c_0 \beta, \quad \forall\, u \in X.$
2. *There are constants $c_1 > 0,\ c_2 \geqslant 0$:*

$$\frac{1}{\beta} \sup_{w^* \in \partial J(u)} < w^*, u > - c_2 + c_1 \|u\| \leqslant J(u), \qquad \forall\, u \in V.$$

Under these hypotheses and assuming that the injection of V into X be compact, the function $\Phi(u) := J(u) - j(u),\ u \in V$ satisfies the Palais-Smale conditions.

Definition 3.9 [8]. Let Ω be an open set of $\mathbb{R}^N$ and $\rho_0, \dots, \rho_N$ be $(N+1)$ Banach function norms and $L(\Omega, \rho_i),\ i = 0, \dots, N$ the corresponding Banach function space. We define the following Banach-Sobolev function space

$$\begin{aligned}\mathbf{W} &:= W^{1,\rho_0,\dots,\rho_N}(\Omega) \\ &:= \Big\{ v \in L(\Omega, \rho_0),\ \frac{\partial v}{\partial x_i} \in L(\Omega, \rho_i),\ i = 1, \dots, N \Big\}.\end{aligned}$$

The norm on $\mathbf{W}$ is then $\|v\|_{\mathbf{W}} = \displaystyle\sum_{i=1}^{N} \rho_i \left(\frac{\partial v}{\partial x_i} \right) + \rho_0(v)$.

Proposition 3.1 [8]. *Let $\rho_0, \dots, \rho_N$ be $(N+1)$ Banach function norms, ρ_i' is the associate norm of ρ_i. We assume that ρ_i and ρ_i' be absolutely continuous norms. Then, the dual space $\mathbf{V}' := \left(W_0^{1,\rho_0,\dots,\rho_N}(\Omega) \right)'$ is*

$$\begin{aligned}\mathbf{V}' = \Big\{ &T : \textit{there exist } f_0, \dots, f_N \textit{ such that } f_i \in L(\Omega, \rho_i') \textit{ and} \\ &< T, v > := \int_\Omega v(x) f_0(x) \mathrm{d}x + \sum_{i=1}^{N} \int_\Omega f_i(x) \frac{\partial v}{\partial x_i}(x) \mathrm{d}x,\ \forall\, v \in \mathbf{V} \Big\}.\end{aligned}$$

Definition 3.10 [8]. We define the $\overrightarrow{\rho}$-multivalued Leray-Lions operator as follows:

$$-\mathrm{div}_{\overrightarrow{\rho}} \left(\partial \varphi_i \left(x, \frac{\partial u}{\partial x_i}(x) \right) \right) := - \sum_{i=1}^{N} \frac{\partial w_i}{\partial x_i}(x) + \omega_0(x) \quad \text{in } \mathcal{D}'(\Omega).$$

Here, $\overrightarrow{\rho}$ stands for $(\rho_0, \dots, \rho_N)$ with $w_0(x) \in \partial \varphi_0(x, u(x))$ *a.e. in* Ω, and

$$w_i(x) \in \partial \varphi_i \left(x, \frac{\partial u}{\partial x_i}(x) \right) \ \textit{a.e. in } \Omega, \ \text{for} \ \ i = 1, \dots, N.$$

Lemma 3.2 (Construction of a $\overrightarrow{\rho}$-multivalued Leray-Lions Strongly Monotonic Operator) [11].

Let Ω be a bounded Lipschitz open set of $\mathbb{R}^N$ and consider N borelian functions $\phi_i : \Omega \times \mathbb{R} \to \mathbb{R}$, $i = 1, \dots, N$, such that

A.1/ For a.e. $x \in \Omega$, the map $t \xrightarrow{\phi_i(x,\cdot)} \phi_i(x,t)$ is strictly increasing, odd and $\phi_i(x, 0) = 0$. Moreover, $\phi_i(x, \cdot)$ is right continuous on $[0, +\infty[$ and $\displaystyle\lim_{t \to +\infty} \operatorname*{essinf}_{\Omega} \phi_i(x,t) = +\infty$.

A2./ There exist $2m$-numbers, $a_1 < \ldots < a_{2m}$ such that for a.e. $x \in \Omega,\ t \in \mathbb{R} \backslash \{a_1, \dots, a_{2m}\} \mapsto \phi_i(x,t)$ is continuous.

We define on $\Omega \times \mathbb{R}$ the following N-function (see [11,13])

$$\varphi_i(x,t) := \int_0^{|t|} \phi_i(x,\sigma)\mathrm{d}\sigma.$$

We need that $\varphi_i(x,\cdot)$ satisfies the so-called global Δ_2-condition (or near infinity). Following the necessary and sufficient condition given in [13,24], we shall assume

A3./ There exist two numbers $\underline{\alpha} > 1$, $\overline{\alpha} > 1$ such that, for a.e. x, $\forall\, t \in \mathbb{R}$

$$\underline{\alpha}\varphi_i(x,t) \leqslant t w_i(x,t) \leqslant \overline{\alpha}\varphi_i(x,t), \qquad \forall\, w_i(x,t) \in \partial\varphi_i(x,t).$$

As usual, we define the corresponding Orlicz space by considering

$$L^{\varphi_i}(\Omega) := \Big\{ v : \Omega \to \mathbb{R} \text{ measurable} : \int_\Omega \varphi_i\big(x, v(x)\big)\mathrm{d}x < +\infty \Big\},$$

we endowed this space with the Banach function norm

$$\rho_i(v) := \inf\left\{ \lambda > 0 : \int_\Omega \varphi_i\left(x, \frac{v(x)}{\lambda}\right)\mathrm{d}x \leqslant 1 \right\}. \tag{3}$$

We also assume that the conjugate function

$$\widetilde{\varphi}_i(x,\cdot) := \sup_{s \geqslant 0}\Big\{ ts - \varphi_i(x,s) \Big\}, \text{ for } t \geqslant 0,$$

satisfies the Δ_2-condition globally (or near infinity), say that there exists a constant $k > 0$, such that $\widetilde{\varphi}_i(x,2t) \leqslant k\widetilde{\varphi}_i(x,t)$ a.e., $\forall\, t \geqslant 0$ (or $t \geqslant t_0$).

Remark 3.2 [23,25]. *Assume that $\varphi_i(x,\cdot)$ satisfy the Δ_2-condition globally (or near infinity) and for a.e. the function $t \in [0,+\infty[\mapsto \varphi_i\big(x,\sqrt{t}\big)$ be convex. Then $\widetilde{\varphi}_i(x,\cdot)$ satisfies the Δ_2-condition globally (or near infinity).*

The Orlicz-Sobolev space associated to $\overrightarrow{\rho} = (\rho_1,\ldots,\rho_N)$ is

$$W_0^{1,\rho_1,\ldots,\rho_N}(\Omega) := \left\{ v \in L^1(\Omega) : \frac{\partial v}{\partial x_i}(x) \in L(\Omega,\rho_i),\ i = 1,\ldots,N, \gamma_0 v = 0 \right\}.$$

We define the function $J(v) := \displaystyle\sum_{i=1}^N \int_\Omega \varphi_i\left(x, \frac{\partial v}{\partial x_i}(x)\right)\mathrm{d}x.$

Theorem 3.2 [11]. *Assume* **A1./**, **A2./**, **A3./**. *Then, the $\overrightarrow{\rho}$-multivalued Leray-Lions operator defined by $Au := -\mathrm{div}_{\overrightarrow{\rho}}\left(\partial\varphi_i\left(x, \dfrac{\partial u}{\partial x_i}(x)\right)\right)$ is strongly monotonic on the Orlicz-Sobolev space $\mathbf{V}_0 := W_0^{1,\rho_1,\ldots,\rho_N}(\Omega)$.*

Remark 3.3 [11,23]. *The following properties hold:*

1. $W_0^{1,\rho_1,\ldots,\rho_N}(\Omega) \subset W_0^{1,1}(\Omega)$ is a continuous injection.

2. *One has for all $t > 0$, all $\sigma > 1$,*

$$\sigma^{\underline{\alpha}}\varphi_i(x,t) \leqslant \varphi_i(x,\sigma t) \leqslant \sigma^{\overline{\alpha}}\varphi_i(x,t), \text{ for a.e. } x \in \Omega.$$

3. *Moreover, we have $L^{\overline{\alpha}}(\Omega) \subset L^{\varphi_i}(\Omega) \subset L^{\underline{\alpha}}(\Omega)$ (see [16, 23]).*

The next definitions and results can be found in [14].
Let Ω be a bounded domain in $\mathbb{R}^N$ and $u : \Omega \to \mathbb{R}$ be a measurable function on Ω.

Definition 3.11 We define the function $u_* : \Omega_* :=]0, |\Omega|[\to \mathbb{R}$ by setting

$$u_*(s) := \inf\left\{t \in \mathbb{R} : |u > t| \leqslant s\right\} \quad s \in \Omega_*,$$

and $u_*(0) := \operatorname*{esssup}_{\Omega} u$, $u_*(|\Omega|) := \operatorname*{essinf}_{\Omega} u$. The function u_* is called the decreasing rearrangement of u.

Property 3.1 [14, 26]. *The decreasing rearrangement of u satisfies:*

1. $\|u\|_{L^p(\Omega)} = \|u_*\|_{L^p(\Omega_*)}$, *for* $1 \leqslant p \leqslant +\infty$.
2. $\left(|u|^p\right)_* = \left(|u|_*\right)^p$ *a.e., for* $1 \leqslant p < +\infty$.

Theorem 3.3 [14]. *Let $u, v \in L^1(\Omega)$ and $w : \overline{\Omega_*} \longrightarrow \mathbb{R}$ defined by*

$$w(s) := \int_{\{u > u_*(s)\}} v(x)\mathrm{d}x + \int_0^{s - |u > u_*(s)|} \left(v|_{\{u = u_*(s)\}}\right)_*(\sigma) d\sigma,$$

where $v|_{\{u=u_*(s)\}}$ is the restriction of v to $\{u = u_*(s)\}$.
If $v \in L^p(\Omega)$, $1 \leqslant p \leqslant +\infty$, then

1. $w \in W^{1,p}(\Omega_*)$.
2. $\dfrac{(u + \lambda v)_* - u_*}{\lambda} \rightharpoonup_{\lambda \to 0} \dfrac{dw}{ds} \begin{cases} \text{in } L^p(\Omega_*)\text{-weak}, & \text{if } 1 \leqslant p < +\infty, \\ \text{in } L^\infty(\Omega_*)\text{-weak-}*, & \text{if } p = +\infty. \end{cases}$

Definition 3.12 The function $\frac{dw}{ds} \in L^p(\Omega_*)$ is called the relative rearrangement of v with respect to u and is denoted by $v_{*u} := \frac{dw}{ds}$.

Property 3.2 [14]. *Let $u \in L^1(\Omega)$ and $v \in L^p(\Omega)$, $1 \leqslant p \leqslant +\infty$, the relative rearrangement v_{*u} verifies:*

1. $\|v_{*u}\|_{L^p(\Omega_*)} \leqslant \|v\|_{L^p(\Omega)}$.
2. $\displaystyle\int_{\Omega_*} v_{*u}(s)ds = \int_\Omega v(x)\mathrm{d}x$.
3. *If $v_1 \leqslant v_2$ a.e., $v_i \in L^p(\Omega)$, then $v_{1*u} \leqslant v_{2*u}$ a.e.*
4. *If $\alpha > 0$, then $(\alpha v)_{*u} = \alpha v_{*u}$.*

Theorem 3.4 Poincaré-Sobolev Inequality for the Relative Rearrangement, see [14], Theorem 4.1.1. *Let $u \in W_0^{1,1}(\Omega)$, $u \geqslant 0$.*
Then, $u_* \in W_{loc}^{1,1}(\Omega_*)$ and

$$-u_*'(s) \leqslant \frac{s^{\frac{1}{N}-1}}{N\omega_N^{\frac{1}{N}}} |\nabla u|_{*u}(s), \quad \text{a.e. in } \Omega_*. \tag{4}$$

Lemma 3.3 [14]. *Let* $1 \leqslant p \leqslant +\infty$, $v \in L^p(\Omega)$ *such that* $v_* \in W^{1,1}_{loc}(\Omega_*)$, *then* $\forall g \in L^{p'}(\Omega)$, $G(s) = \int_\Omega g \cdot \big(v - v_*(s)\big)_+ \mathrm{d}x \in W^{1,1}_{loc}(\Omega_*)$, *and for a.e.* $s \in \Omega_*$, *we have:*

$$G'(s) = -v_*'(s) \int_{\{u>u_*(s)\}} g(x)\mathrm{d}x = -v_*'(s) \int_0^s g_{*v}(\sigma)d\sigma. \tag{5}$$

4 An Abstract Result for the Existence of Critical Value Near Zero.

Due to Ekeland variational principle (see [11]), we shall introduce here an existence result for the case where we have compact embedding from V into X. We also introduce the following hypothesis:

H3./ $\Phi(0) = 0$, and there exist $u_1 \in V$ and $\eta > \|u_1\|$ such that

$$\Phi(v) > 0,\ \|v\| = \eta,\ \Phi(u_1) < 0,\ \inf_{v \in \overline{B}(0,\eta)} \Phi(v) > -\infty.$$

Theorem 4.1 *Under the conditions* **H1./** *and* **H3./**, *if the injection of* V *into* X *is compact, then there exists a function* $u \not\equiv 0$, $u \in V$, *which is a critical point of* $\Phi := J - j$, *that is*

$$0 \in \partial J(u) - \partial j(u).$$

Proof. Let $B(0,\eta)$ be the open ball of radius η. Since we have a function $u_1 \not\equiv 0$, $u_1 \in V$ such that $\Phi(u_1) < 0$ (by **H3./**), then $u_1 \in B(0,\eta)$ and $c = \inf_{v \in \overline{B}(0,\eta)} \Phi(v)$ verifies

$$-\infty < c \leqslant \Phi(u_1) < 0.$$

By Ekeland variational principle (see [22, 27] and Appendix in [11]), we have a minimizing sequence $(u_n) \subset \overline{B}(0,\eta)$, and $u \in V$ such that

1. $\Phi(u_n) \xrightarrow[n\to+\infty]{} c$, $\Big(\text{thus } u_n \in \text{interior}\{\,\overline{B}(0,\eta)\} = B(0,\eta)\Big)$,
2. $\ell_n^* \in \partial\Phi(u_n)$, $\|\ell_n^*\|_{V'} \xrightarrow[n\to+\infty]{} 0$,
3. u_n converges weakly to u in V,
4. u_n converges strongly to u in X.

Thus, there exist $w_n \in \partial J(u_n)$, $v_n \in \partial j(u_n)$ such that

$$\ell_n^* = w_n - v_n,$$

and this implies

$$< \ell_n^*, u_n - u > = < w_n, u_n - u > - < v_n, u_n - u > . \tag{6}$$

Thus, for $w \in \partial J(u)$

$$< \ell_n^*, u_n - u > - < w, u_n - u > + < v_n, u_n - u > = < w_n - w, u_n - u > \quad (7)$$

Since we have the strong convergence of the sequence (u_n) in X, then $|u_n| \leqslant c_1$ and $|v_n|_{X'} \leqslant c_2$, for $v_n \in \partial j(u_n)$.
From (7), we then have

$$| < w_n - w, u_n - u > | \leqslant c_3 \|\ell_n^*\|_{V'} + | < w, u_n - u > | + c_2 |u_n - u|. \quad (8)$$

The above convergences, together with relation (8), yield:

$$\lim_{n \to +\infty} < w_n - w, u_n - u > = 0, \quad \text{for } w_n \in \partial J(u_n) \text{ and } w \in \partial J(u). \quad (9)$$

By the assumption **H1./**, we conclude that u_n converges strongly to u in V. Therefore, $\Phi(u_n) \xrightarrow[n \to +\infty]{} \Phi(u) = c < 0$. □

4.1 **Example of Applications of Theorem 4.1**

Let $\delta_1 > 0, \ldots, \delta_N > 0$ be N real numbers and $q_i, p_i,\ i = 1, \ldots, N$ be $2N$ bounded measurable functions on Ω satisfying:

$$1 < q_{i-} := \operatorname*{essinf}_{\Omega}\, q_i \leqslant q_i(x) < p_{i-} := \operatorname*{essinf}_{\Omega}\, p_i \leqslant p_{i+} := \operatorname*{esssup}_{\Omega}\, p_i < +\infty.$$

Consider

$$\varphi_i(x,t) := \begin{cases} |t|^{p_i(x)}, & \text{if } |t| > \delta_i, \\ \alpha_i(x)|t|^{q_i(x)}, & \text{if } |t| \leqslant \delta_i, \end{cases} \quad x \in \Omega,\ t \in \mathbb{R},\ \alpha_i(x) = (\delta_i)^{(p_i - q_i)(x)}.$$

We can choose $\phi_i(x,t)$ as follows:

$$\phi_i(x,t) := \begin{cases} p_i(x)|t|^{p_i(x)-2}t, & \text{if } |t| > \delta_i, \\ q_i(x)|t|^{q_i(x)-2}t, & \text{if } |t| < \delta_i, \\ p_i(x)\delta_i^{p_i(x)-1}, & \text{if } t = \delta_i, \\ -p_i(x)\delta_i^{p_i(x)-1}, & \text{if } t = -\delta_i. \end{cases}$$

One can check that all the assumptions **A1./** to **A3./** are fulfilled for $\underline{\alpha} = q_-^- := \min_{1 \leqslant i \leqslant N} \{q_{i-}\}$ and $\overline{\alpha} = p_+^+ := \max_{1 \leqslant i \leqslant N} \{p_{i+}\}$. We also have:

$$\partial\varphi_i(x,t) = \begin{cases} \phi_i(x,t), & \text{if } |t| > \delta_i \text{ or } |t| < \delta_i, \\ \delta_i^{p_i(x)-1} \Big[q_i(x), p_i(x)\Big] \operatorname{sign}(t), & \text{otherwise.} \end{cases}$$

We define the function $J : \mathbf{V}_0 \to \mathbb{R}$ as follows

$$J(v) := \sum_{i=1}^N \int_\Omega \varphi_i \left(x, \frac{\partial v}{\partial x_i}(x) \right) \frac{\mathrm{d}x}{p_i(x)}.$$

Theorem 4.2 *Let $\delta_M > 0$ be a real number and p_M, q_M be two bounded measurable functions on Ω satisfying:*

$$1 < q_{M-} := \operatorname*{essinf}_{\Omega} q_M(x) \leqslant q_M(x) < p_{M-} := \operatorname*{essinf}_{\Omega} p_M(x).$$

Assume that $\mathbf{V}_0$ be compactly embedded in $L^{\varphi_M}(\Omega)$ and

$$q_{M-} < q_-^-.$$

Then, there exists $\lambda_* > 0$, such that $\forall\ \lambda \in]0, \lambda_*[$, there exists a non-trivial function $u \in \mathbf{V}_0$ such that

$$\sum_{i=1}^N \int_\Omega w_i(x) \frac{\partial v}{\partial x_i}(x) \mathrm{d}x = \lambda \int_\Omega w_M(x) v(x) \mathrm{d}x, \quad \forall\ v \in \mathbf{V}_0,$$

with $p_i(x)w_i(x) \in \partial\varphi_i\left(x, \dfrac{\partial u}{\partial x_i}(x)\right)$ *and* $\quad p_M(x)w_M(x) \in \partial\varphi_M\big(x, u(x)\big)$ *a.e., where*

$$\varphi_M\big(x, u(x)\big) = \begin{cases} |u(x)|^{p_M(x)}, & \text{if } |u(x)| > \delta_M, \\ \alpha_M(x)|u(x)|^{q_M(x)}, & \text{if } |u(x)| \leqslant \delta_M, \end{cases}$$

where $\alpha_M(x) = (\delta_M)^{(p_M - q_M)(x)}$.

Proof. Introducing

$$\Phi_\lambda(u) = J(u) - \lambda j(u), \quad j(u) = \int_\Omega \varphi_M(x, u(x)) \frac{\mathrm{d}x}{p_M(x)},$$

one can check all assumptions of Theorem 4.1 (for more details, see [23]). □

4.2 **Other Example (with coerciveness)**

Theorem 4.3 *Under the same notations of Section 4.1, assume that $\mathbf{V}_0$ be compactly embedded in $L^{\varphi_M}(\Omega)$ and*

$$q_{M-} < p_{M+} < q_-^-.$$

Then, there exist $\lambda_* > 0$ and $\lambda_{**} > 0$, such that $\forall\ \lambda \in\ ]0, \lambda_*[\ \cup\ ]\lambda_{**}, +\infty[$, there exists a non-trivial function $u \in \mathbf{V}_0$ such that

$$\sum_{i=1}^N \int_\Omega w_i(x) \frac{\partial v}{\partial x_i}(x) \mathrm{d}x = \lambda \int_\Omega w_M(x) v(x) \mathrm{d}x, \quad \forall\ v \in \mathbf{V}_0,$$

with $p_i(x)w_i(x) \in \partial\varphi_i\left(x, \dfrac{\partial u}{\partial x_i}(x)\right)$ *and* $\quad p_M(x)w_M(x) \in \partial\varphi_M\big(x, u(x)\big)$ *a.e. in Ω.*

Proof. The existence of $\lambda_* > 0$ such that any $\lambda \in]0, \lambda_*[$ is an eigenvalue is an immediate consequence of Theorem 4.2. In order to prove the second part of Theorem 4.3, we will show that, for λ positive and large enough, the functional Φ_λ possesses a non-trivial global minimum point in $\mathbf{V}_0$.

First, the function Φ_λ is **coercive** on $\mathbf{V}_0$. Indeed, if $\max\limits_{1\leqslant i\leqslant N} \left\| \dfrac{\partial u}{\partial x_i} \right\|_{\Phi_i} \geqslant 1$, then

$$J(u) = \sum_{i=1}^{N} \int_\Omega \varphi_i\left(x, \frac{\partial u}{\partial x_i}(x)\right) \frac{\mathrm{d}x}{p_i(x)} \geqslant \frac{1}{p_+^+} \cdot \frac{1}{N^{q_-^-}} \cdot \|u\|_{\mathbf{V}_0}^{q_-^-}, \text{ (by Corollary 5.1).}$$

Moreover, we have $\forall\, u \in \mathbf{V}_0$,

$$\int_\Omega \varphi_M\big(x, u(x)\big)\mathrm{d}x \leqslant (S_0)^{q_{M-}} \|u\|_{\mathbf{V}_0}^{q_{M-}} + (S_0)^{p_{M+}} \|u\|_{\mathbf{V}_0}^{p_{M+}},$$

where $S_0 > 0$ is the Sobolev constant of the injection $\mathbf{V}_0$ into $L^{\varphi_M}(\Omega)$.
Then,

$$\Phi_\lambda(u) \geqslant \frac{1}{p_+^+ \cdot N^{q_-^-}} \cdot \|u\|_{\mathbf{V}_0}^{q_-^-} - \frac{\lambda (S_0)^{q_{M-}}}{p_{M-}} \|u\|_{\mathbf{V}_0}^{q_{M-}} + \frac{\lambda (S_0)^{p_{M+}}}{p_{M-}} \|u\|_{\mathbf{V}_0}^{p_{M+}},$$

for all $u \in \mathbf{V}_0$ such that $\max\limits_{1\leqslant i\leqslant N} \left\| \dfrac{\partial u}{\partial x_i} \right\|_{\varphi_i} \geqslant 1$.
Since we have $q_{M-} < p_{M+} < q_-^-$, then $\Phi_\lambda(u) \xrightarrow[\|u\|_{\mathbf{V}_0} \to +\infty]{} +\infty$.

On the other hand, the same arguments as in the proof of Lemma 3.4 of [28] can be used in order to show that Φ_λ is **weakly lower semi-continuous** on $\mathbf{V}_0$ (see also [23]). The functional Φ_λ is also coercive on $\mathbf{V}_0$. These two facts enable us to apply the first result of Section 2.1.1 in [22] in order to find that there exists $u_0 \in \mathbf{V}_0$, a global minimizer of Φ_λ.

Finally, we have to show that u_0 is a non-trivial solution for λ large enough. Indeed, letting $t_0 > \max\{1; \delta_M\}$ be a fixed real number and Ω_1 be an open subset of Ω with $\Omega_1 \subset\subset \Omega$ and $|\Omega_1| > 0$, we deduce that there exists $v_0 \in C_c^\infty(\Omega) \subset \mathbf{V}_0$ such that

$$\begin{cases} v_0(x) = t_0, & \text{if } x \in \overline{\Omega}_1, \\ 0 \leqslant v_0(x) \leqslant t_0, & \text{if } x \in \Omega \setminus \Omega_1. \end{cases}$$

We have

$$\begin{aligned}\Phi_\lambda(v_0) &= \sum_{i=1}^{N}\int_\Omega \varphi_i\left(x, \frac{\partial v_0}{\partial x_i}(x)\right)\frac{\mathrm{d}x}{p_i(x)} - \lambda\int_\Omega \varphi_M\big(x, v_0(x)\big)\frac{\mathrm{d}x}{p_M(x)}\\ &\leqslant \frac{1}{p_-^-}\sum_{i=1}^{N}\int_\Omega \varphi_i\left(x, \frac{\partial v_0}{\partial x_i}(x)\right)\mathrm{d}x - \frac{\lambda}{p_{M+}}\int_{\Omega_1}|v_0|^{p_M(x)}\mathrm{d}x\\ &\leqslant \frac{1}{p_-^-}\sum_{i=1}^{N}\int_\Omega \varphi_i\left(x, \frac{\partial v_0}{\partial x_i}(x)\right)\mathrm{d}x - \frac{\lambda}{p_{M+}}\int_{\Omega_1}(t_0)^{p_{M-}}\mathrm{d}x\\ &\leqslant C_1 - \frac{\lambda}{p_{M+}}(t_0)^{p_{M-}}|\Omega_1|,\end{aligned}$$

where $C_1 > 0$ is a positive constant. Thus, there exists $\lambda_{**} = \dfrac{C_1\ p_{M+}}{(t_0)^{p_{M-}}\ |\Omega_1|} > 0$ such that $\Phi_\lambda(v_0) < 0$, for any $\lambda > \lambda_{**}$. It follows that $\Phi_\lambda(u_0) < 0$, for any $\lambda > \lambda_{**}$ and thus u_0 is a non-trivial solution for λ large enough. □

Remark 4.1 *Let $f \in \mathbf{V}_0'$ be an element of the dual space of $\mathbf{V}_0$ and consider the following problem:*

$$-\mathrm{div}_{\vec{\beta}}\left(\partial\varphi_i\left(x, \frac{\partial u}{\partial x_i}(x)\right)\right) = f, \tag{10}$$

where all assumptions **A1./** *to* **A3./** *are satisfied. Then, there exists a global minimizer $u \in \mathbf{V}_0$ and u is a solution of* (10)*. Indeed, if we replace the function $j(v)$ of Theorem 4.3 by $< f, v >$ and we use the same arguments as in the proof of Theorem 4.3, we show that the functional Φ defined by*

$$\Phi(v) := J(v) - < f, v >, \qquad v \in \mathbf{V}_0,$$

is coercive and weakly lower semi-continuous.

5 Minimization Problem with Constraint.

In order to prove the existence for a minimization problem with constraint related to the fully discontinuous operator, we shall start by using the non-smooth version of the Lagrange Multiplier Theorem (see [9]). Due to the inhomogeneity, our operator is more complicated than the p-Laplacian.
Let Ω be a bounded Lipschitz open set of $\mathbb{R}^N$ and consider N borelian functions $\phi_i : \Omega \times \mathbb{R} \to \mathbb{R}$, such that ϕ_i satisfies **A1./** and **A2./**.
We define the N-function φ_i on $\Omega \times \mathbb{R}$:

$$\varphi_i(x, t) := \int_0^{|t|}\phi_i(x, \sigma)\mathrm{d}\sigma.$$

We also assume that φ_i satisfies **A3./**. Moreover, we consider the function J on $\mathbf{V}_0$, as follows

$$J(v) := \sum_{i=1}^{N} \int_{\Omega} \varphi_i \left(x, \frac{\partial v}{\partial x_i}(x) \right) \mathrm{d}x.$$

Theorem 5.1 *Under the above conditions, assume that $\mathbf{V}_0$ be continuously embedded in X. Let $j : X \to \mathbb{R}$ be an even and convex function such that $j(0) = 0$. If we suppose*

(i) *j be a continuous function and $\mathbf{V}_0$ be compactly embedded in X,*
or
(ii) *j be weakly-continuous function,*

then, there exist $u_1 \in \mathbf{V}_0$, $u_1 \geqslant 0$, $u_1 \not\equiv 0$ and $\lambda_1 > 0$ such that

$$0 \in \partial J(u_1) - \lambda_1 \partial j(u_1).$$

We start with two auxiliary results, and we recall the relation between the modular function and the norm in the Orlicz spaces.

Lemma 5.1 [23,25]. *Let $u \in L^{\varphi_i}(\Omega)$. We have the following properties:*

(1) *If $\|u\|_{\varphi_i} \geqslant 1$, then $\|u\|_{\varphi_i}^{\underline{\alpha}} \leqslant \int_{\Omega} \varphi_i\left(x, u(x)\right) \mathrm{d}x \leqslant \|u\|_{\varphi_i}^{\overline{\alpha}}$.*

(2) *If $\|u\|_{\varphi_i} < 1$, then $\|u\|_{\varphi_i}^{\overline{\alpha}} \leqslant \int_{\Omega} \varphi_i\left(x, u(x)\right) \mathrm{d}x \leqslant \|u\|_{\varphi_i}^{\underline{\alpha}}$.*

As a consequence of Lemma 5.1, we have

Corollary 5.1 [23]. *Let $u \in \mathbf{V}_0$. Then:*

(1') *If $\max_{1 \leqslant i \leqslant N} \left\| \frac{\partial u}{\partial x_i} \right\|_{\varphi_i} \geqslant 1$, then $\frac{1}{N^{\underline{\alpha}}} \cdot \|u\|_{\mathbf{V}_0}^{\underline{\alpha}} \leqslant J(u) \leqslant N \cdot \|u\|_{\mathbf{V}_0}^{\overline{\alpha}}$.*

(2') *If $\max_{1 \leqslant i \leqslant N} \left\| \frac{\partial u}{\partial x_i} \right\|_{\varphi_i} < 1$, then $\frac{1}{N^{\overline{\alpha}}} \cdot \|u\|_{\mathbf{V}_0}^{\overline{\alpha}} \leqslant J(u) \leqslant N \cdot \|u\|_{\mathbf{V}_0}^{\underline{\alpha}}$.*

Proof of Theorem 5.1:
Let $\alpha > 0$ be fixed. We are in a position now to show that there exists $u_1 \in \mathbf{V}_0$, $u_1 \geqslant 0$ and $u_1 \not\equiv 0$ such that

$$J(u_1) = \inf \Big\{ J(v) : j(v) = \alpha \Big\} \geqslant 0.$$

For this, we define

$$m := \inf \Big\{ J(v) : j(v) = \alpha \Big\} \quad \text{and} \quad S := \Big\{ u \in \mathbf{V}_0 : j(u) = \alpha \Big\}.$$

We consider a minimizing sequence (v_n) in S such that

$$J(v_n) \xrightarrow[n \to +\infty]{} m. \tag{11}$$

By Corollary 5.1, (v_n) is bounded in $\mathbf{V}_0$. Then, there exists a subsequence of (v_n), still denoted by (v_n), that converges weakly in $\mathbf{V}_0$. The function J is weakly lower semi-continuous on $\mathbf{V}_0$ (see [23]), so we deduce that

$$J(u) \leqslant \liminf_{n\to+\infty} J(v_n) = m. \tag{12}$$

First, we suppose that **(i)** holds. Then, using the compact embedding and passing to a subsequence, if necessary, we deduce that (v_n) converges strongly in X. Moreover, by the continuity of j, S is closed. This implies that $u \in S$. Then,

$$J(u) = m. \tag{13}$$

We also suppose that **(ii)** holds. By the weak-continuity of j, we deduce that $j(v_n) = \alpha \xrightarrow[n\to+\infty]{} j(u) = \alpha$ and we also have (13).

On the other hand, exploiting Lagrange Multiplier Theorem (see [9]), we then get the existence of $(t, s) \neq (0, 0)$, such that $t \geqslant 0$ and

$$0 \in t\,\partial J(u_1) + s\,\partial j(u_1), \qquad u_1 = |u| \geqslant 0. \tag{14}$$

We observe that $t \neq 0$. Indeed, if we suppose that $t = 0$, then

$$0 \in s\,\partial j(u_1) \text{ and } s \neq 0.$$

We obtain

$$0 \in \partial j(u_1), \tag{15}$$

which is impossible since $\langle \partial j(u_1); u_1\rangle \geqslant \alpha > 0$. We then get $t > 0$ and

$$0 \in \partial J(u_1) - \lambda_1\,\partial j(u_1), \text{ where } \lambda_1 = -\frac{s}{t}. \tag{16}$$

Finally, we point out that there exist $w_J^* \in \partial J(u_1)$ and $w_j^* \in \partial j(u_1)$ such that

$$w_J^* = \lambda_1\, w_j^*. \tag{17}$$

Using the above equality, we find $\lambda_1 = \dfrac{\langle w_J^*; u_1\rangle}{\langle w_j^*; u_1\rangle}$. We note that $\lambda_1 > 0$. Indeed, we have $\langle w_j^*; u_1\rangle \geqslant \alpha > 0$. And there exists $w_i^*(x) \in \partial\varphi_i\left(x, \dfrac{\partial u_1}{\partial x_i}(x)\right)$ a.e. $x \in \Omega$, for $i = 1, \ldots, N$ (see Lemma 4.4 in [8]) such that

$$\begin{aligned}\langle w_J^*; u_1\rangle &= \sum_{i=1}^{N}\int_\Omega w_i^*(x)\frac{\partial u_1}{\partial x_i}(x)\mathrm{d}x\\ &\geqslant \underline{\alpha}\sum_{i=1}^{N}\int_\Omega \varphi_i\left(x, \frac{\partial u_1}{\partial x_i}(x)\right)\mathrm{d}x\\ &\geqslant \underline{\alpha}\, J(u_1) \geqslant 0.\end{aligned}$$

If $\lambda_1 = 0$, then $J(u_1) = 0$. By Corollary 5.1, we deduce that $\|u_1\|_{\mathbf{V}_0} = 0$, which is impossible since $u_1 \in S$. This implies $\lambda_1 > 0$. □

For more examples of applications of Theorem 5.1, see [23].

Remark 5.1 *In the case of p-Laplacian, the homogeneity of the operator allows to prove that λ_1 is the first eigenvalue.*

6 Resonant Case Near Zero.

In this section, we prove the existence of solutions for an eigenvalue problem, where the potential j has a partial interaction at zero with the best Sobolev constant (see **(j4)**).
Let Ω be a bounded Lipschitz open set of $\mathbb{R}^N$ and consider N borelian functions $\phi_i : \Omega \times \mathbb{R} \to \mathbb{R}$, for $i = 1, \dots, N$ and $\phi_M : \Omega \times \mathbb{R} \to \mathbb{R}$, such that ϕ_i and ϕ_M satisfy **A1./** and **A2./**. We define on $\Omega \times \mathbb{R}$ the N-functions

$$\varphi_i(x,t) := \int_0^{|t|} \phi_i(x,\sigma)\mathrm{d}\sigma, \qquad \text{and} \qquad \varphi_M(x,t) := \int_0^{|t|} \phi_M(x,\sigma)\mathrm{d}\sigma.$$

And we assume that φ_i and φ_M satisfy **A3./** i.e.
There exist $\underline{\alpha} > 1$, $\overline{\alpha} > 1$, $\underline{\beta} > 1$ and $\overline{\beta} > 1$ such that, for a.e. $x \in \Omega$, $\forall\, t \in \mathbb{R}$ and $\forall\, w_i^*(x,t) \in \partial\varphi_i(x,t)$, we have

$$\underline{\alpha}\, \varphi_i(x,t) \leqslant t\, w_i^*(x,t) \leqslant \overline{\alpha}\, \varphi_i(x,t), \quad \forall\, 1 \leqslant i \leqslant N.$$

For every $w_M^*(x,t) \in \partial\varphi_M(x,t)$, we have

$$\underline{\beta}\, \varphi_M(x,t) \leqslant t\, w_M^*(x,t) \leqslant \overline{\beta}\, \varphi_M(x,t).$$

We define the Banach-Sobolev function space $\mathbf{V}_0 := W_0^{1,\varphi_1,\dots,\varphi_N}(\Omega)$ as

$$\mathbf{V}_0 := \left\{ v \in W_0^{1,1}(\Omega) : \sum_{i=1}^{N} \left(\int_\Omega \varphi_i \left(x, \frac{\partial v}{\partial x_i}(x) \right) \mathrm{d}x \right) < +\infty \right\},$$

which is endowed with the norm $\|v\|_{\mathbf{V}_0} = \displaystyle\sum_{i=1}^{N} \left\| \frac{\partial v}{\partial x_i} \right\|_{\varphi_i}$.
Consider $J : \mathbf{V}_0 \to \mathbb{R}$ with $J(v) = \displaystyle\sum_{i=1}^{N} \int_\Omega \varphi_i \left(x, \frac{\partial v}{\partial x_i}(x) \right) \mathrm{d}x$.

Theorem 6.1 deals with the existence of solutions $u \in \mathbf{V}_0$ of (18) under the next hypotheses on the nonsmooth potential j.

Theorem 6.1 *Under the above conditions, let $j : \Omega \times \mathbb{R} \to \mathbb{R}$ be such that:*

(j1) *$x \mapsto j(x,\xi)$ is measurable for all $\xi \in \mathbb{R}$.*
$\xi \mapsto j(x,\xi)$ is locally Lipschitz for a.e. $x \in \Omega$.

(j2) *There exists a constant $a_1 > 0$, such that*

$$|j(x,\xi)| \leqslant a_1\Big(|\xi| + \varphi_M(x,\xi)\Big), \qquad a.e.\ x \in \Omega \text{ and for all } \xi \in \mathbb{R}.$$

(j3) *There exist some constants $M > 0$, $\beta > 0$ and $k_1 > 0$, such that*

$$k_1|\xi|^\beta \leqslant j(x,\xi), \qquad \forall\, |\xi| \geqslant M.$$

(j4) *There exists a constant* $0 < c^* < \dfrac{1}{N^{\overline{\alpha}} \cdot (S_2)^{\overline{\alpha}}}$, *such that*

$$j(x,\xi) \leqslant c^* |\xi|^{\overline{\alpha}}, \qquad \textit{in a neighborhood of } \xi = 0,$$

where $S_2 > 0$ *is the Sobolev constant of the embedding* $\mathbf{V}_0 \subset L^{\overline{\alpha}}(\Omega)$.

(j5) *We have the following growth properties:*

(1) $\exists\, c_0 \geqslant 0 \,:\, \beta g(u) \leqslant \inf\limits_{v^* \in \partial g(u)} \langle v^*; u\rangle + c_0 \beta, \quad \forall\, u \in L^{\varphi_M}(\Omega)$, *where* $g(u) = \displaystyle\int_\Omega j(x,u(x))\mathrm{d}x$.

(2) *There exist constants* $c_1 > 0,\ c_2 \geqslant 0$:

$$\frac{1}{\beta} \sup_{w^* \in \partial J(u)} \langle w^*; u\rangle - c_2 + c_1 \|u\|_V \leqslant J(u), \quad \forall\, u \in \mathbf{V}_0.$$

If the injection $\mathbf{V}_0 \subset L^{\varphi_M}(\Omega)$ *is compact and if* $\overline{\alpha} < \beta < \underline{\beta}$, *then, there exists* $u \in \mathbf{V}_0$ *a non-trivial function, such that*

$$0 \in \partial J(u) - \partial g(u). \tag{18}$$

We start with two auxiliary results:

Lemma 6.1 *Assume that* **(j1)**, **(j2)** *and* **(j4)** *hold, then, there exist* $\eta > 0$ *and* $\alpha > 0$ *such that*

$$\Phi(u) := J(u) - g(u) \geqslant \alpha, \quad \textit{for all } u \in \mathbf{V}_0 \textit{ with } \|u\|_{\mathbf{V}_0} = \eta.$$

Proof. First, by **(j4)**, there exists $0 < \delta << 1$ such that

$$j(x,\xi) \leqslant c^* |\xi|^{\overline{\alpha}}, \qquad \text{for all } |\xi| \leqslant \delta. \tag{19}$$

By **(j2)**, we get that, for a.e. $x \in \Omega$, for all $\xi \in \mathbb{R}$ with $|\xi| > \delta$ and for all $w_M^*(x,\xi) \in \partial\varphi_M(x,\xi)$:

$$\begin{aligned}|j(x,\xi)| &\leqslant a_1\Big(|\xi| + \varphi_M(x,\xi)\Big)\\ &\leqslant a_1\Big(\frac{\overline{\beta}}{w_M^*(x,\delta)} + 1\Big)\varphi_M(x,\xi).\end{aligned}$$

It follows that there exists a constant $c(\delta) > 0$ independent of ξ, such that

$$j(x,\xi) \leqslant c^* |\xi|^{\overline{\alpha}} + c(\delta)\varphi_M(x,\xi), \qquad \forall\, \xi \in \mathbb{R}. \tag{20}$$

We recall that we have (see Remark 3.3)

$$\mathbf{V}_0 \subset L^{\varphi_M}(\Omega) \subset L^{\underline{\beta}}(\Omega) \subset L^{\overline{\alpha}}(\Omega). \tag{21}$$

Let $0 < \eta < \min\Big\{1; \dfrac{1}{S_0}\Big\}$ be fixed (small enough). Thus, by Corollary 5.1, we get:

$$J(u) \geqslant \frac{1}{N^{\overline{\alpha}}} \cdot \|u\|_{\mathbf{V}_0}^{\overline{\alpha}}, \qquad \text{for all } \|u\|_{\mathbf{V}_0} = \eta.$$

where $S_0 > 0$ is the Sobolev constant of the embedding $\mathbf{V}_0 \subset L^{\varphi_M}(\Omega)$.
On the other hand, there exists $\delta_1 > 0$ such that, for all u satisfying $\|u\|_{\mathbf{V}_0} = \eta$, we have:

$$\begin{aligned}\Phi(u) &\geqslant \frac{1}{N^{\overline{\alpha}}} \cdot \|u\|^{\overline{\alpha}}_{\mathbf{V}_0} - c^* \|u\|^{\overline{\alpha}}_{L^{\overline{\alpha}}} - \delta_1 \int_\Omega \varphi_M(x, u(x))\mathrm{d}x \\ &\geqslant \frac{1}{N^{\overline{\alpha}}} \cdot \|u\|^{\overline{\alpha}}_{\mathbf{V}_0} - c^*(S_2)^{\overline{\alpha}} \|u\|^{\overline{\alpha}}_{\mathbf{V}_0} - \delta_1 \|u\|^{\underline{\beta}}_{\varphi_M} \\ &\geqslant \left(\frac{1}{N^{\overline{\alpha}}} - c^*(S_2)^{\overline{\alpha}} \right) \|u\|^{\overline{\alpha}}_{\mathbf{V}_0} - \delta_1 (S_0)^{\underline{\beta}} \|u\|^{\underline{\beta}}_{\mathbf{V}_0},\end{aligned}$$

Finally, since $c^* < \dfrac{1}{N^{\overline{\alpha}} \cdot (S_2)^{\overline{\alpha}}}$ and $\overline{\alpha} < \underline{\beta}$, then, for $\eta > 0$ small enough, the lemma is proved. □

Lemma 6.2 *Assume that* **(j1)***,* **(j2)** *and* **(j3)** *hold, then, there exists* $u_0 \in \mathbf{V}_0$ *such that*

$$\Phi(u_0) < \alpha \quad \textit{and} \quad \|u_0\|_{\mathbf{V}_0} > \eta.$$

Proof. First, by **(j3)**, we have

$$k_1 |\xi|^\beta \leqslant j(x, \xi), \qquad \forall\ |\xi| \geqslant M.$$

Consider $u \in \mathbf{V}_0$, then there exists $k_2 > 0$, such that

$$\begin{aligned}k_1 \int_\Omega |u|^\beta \mathrm{d}x &= k_1 \int_{\{|u| \geqslant M\}} |u|^\beta \mathrm{d}x + k_1 \int_{\{|u| < M\}} |u|^\beta \mathrm{d}x \\ &\leqslant \int_{\{|u| \geqslant M\}} j(x, u(x))\mathrm{d}x + k_2.\end{aligned}$$

On the other hand, by **(j2)**, there exists $k_3 > 0$ such that

$$\begin{aligned}-g(u) &\leqslant - \int_{\{|u| \geqslant M\}} j(x, u(x))\mathrm{d}x + \int_{\{|u| < M\}} |j(x, u(x))|\mathrm{d}x \\ &\leqslant -k_1 \int_\Omega |u|^\beta \mathrm{d}x + k_2 + k_3 \\ &= -k_1 \|u\|^\beta_{L^\beta} + k_4.\end{aligned}$$

Taking $t > 1$ and using Remark 3.3, we find

$$\Phi(tu) \leqslant t^{\overline{\alpha}} J(u) - k_1 \cdot t^\beta \|u\|^\beta_{L^\beta} + k_4.$$

Finally, since $\overline{\alpha} < \beta$, we deduce that

$$\Phi(tu) \xrightarrow[t \to +\infty]{} -\infty.$$

Letting $t_0 > 1$ large enough and defining $u_0 := t_0 u \in \mathbf{V}_0$, we get

$$\Phi(u_0) < \alpha \quad \text{and} \quad \|u_0\|_{\mathbf{V}_0} > \eta.$$

The proof of Lemma 6.2 is complete. □

Proof of Theorem 6.1:
At this stage, using Lemma 6.1 and Lemma 6.2, we are able to apply SHI's result (see [27]) and we deduce the existence of a Palais-Smale sequence denoted by (u_n) in $\mathbf{V}_0$.
We end the proof of Theorem 6.1 by using **(j5)**, which allows us to apply Lemma 3.1. □

7 Qualitative Properties of Eigenvectors Related to $\overline{\rho}^{>}$-multivalued Leray-Lions Operators.

In this section, we are interested in showing the qualitative properties for the solutions of the eigenvalue problems related to $\overline{\rho}^{>}$-multivalued Leray-Lions operators.
Let $\varphi_i : \Omega \times \mathbb{R} \to \mathbb{R}$, $i = 1, \dots, N$ and $\varphi_M : \Omega \times \mathbb{R} \to \mathbb{R}$ be $N+1$ functions for which the assumptions **A.1/ - A.3/** are verified. We consider the Orlicz-Sobolev space $W_0^{1,\varphi_1,\dots,\varphi_N}(\Omega)$ and let $u \in W_0^{1,\varphi_1,\dots,\varphi_N}(\Omega) \subset L^{\varphi_M}(\Omega)$, $u \geqslant 0$ be a solution of the following problem:

$$-\text{div}\left(\partial\varphi_i\left(x, \frac{\partial u}{\partial x_i}\right)\right) = \lambda w_M^*, \tag{22}$$

where

$$w_M^*(x) \in \partial\varphi_M(x, u(x)) \text{ a.e. } x \in \Omega.$$

The subject of the next theorem is to study the L^∞-regularity of the solutions of such problems.

Theorem 7.1 *Let τ and α be two real numbers such that $1 < \tau < \alpha < +\infty$. Consider $\varphi_i : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by*

$$\varphi_i(x,t) := \begin{cases} |t|^\tau, & \text{if } |t| \leqslant 1, \\ |t|^\alpha, & \text{if } |t| > 1. \end{cases}$$

Then, there exist $\lambda_1 > 0$ and $u \in W_0^{1,\alpha}(\Omega)$, $u \geqslant 0$, solution of the following problem:

$$-\text{div}\left(\partial\varphi_i\left(x, \frac{\partial u}{\partial x_i}\right)\right) = \lambda_1 u^{\alpha-1}. \tag{23}$$

Furthermore, if

1. $1 \leqslant N \leqslant 5$,
or

2. $N \geqslant 6$ *and*
$$\begin{cases} 1 < \alpha < \alpha_N^-, & \text{where } \alpha_N^- = \dfrac{N+1-\sqrt{(N-3)^2-8}}{4}, \\ \text{or} & \\ \alpha_N^+ < \alpha < N, & \text{where } \alpha_N^+ = \dfrac{N+1+\sqrt{(N-3)^2-8}}{4}, \end{cases}$$

then $u \in L^\infty(\Omega)$.

For the reader's convenience, the proof of Theorem 7.1 is divided into several steps:

Lemma 7.1 *Under the assumptions of Theorem 7.1, there exist* $\lambda_1 > 0$ *and* $u \in W_0^{1,\alpha}(\Omega)$, $u \geqslant 0$, *solution of* (23).

Proof. The function $j : L^\alpha(\Omega) \to \mathbb{R}$ defined by $j(v) := \dfrac{1}{\alpha}\displaystyle\int_\Omega |v|^\alpha \mathrm{d}x$ is a convex, continuous, even function and $j(0) = 0$. $W_0^{1,\alpha}(\Omega) \subset L^\alpha(\Omega)$ is a compact embedding, so we can use Theorem 5.1 to achieve the proof. □

Lemma 7.2 *Let* u *be a solution of* (23), *then* u *verifies the following inequality:*

$$\big(|\nabla u|^\alpha\big)_{*u}(s) \leqslant \lambda_{1N}\left(-\frac{du_*}{ds}(s)\right)\int_0^s u_*^{\alpha-1}(t)dt + c_2, \tag{24}$$

where λ_{1N} *and* c_2 *are two positive constants.*

Proof. Since u satisfies (23), then, there exist

$$w_i^*(x) \in \partial\varphi_i\left(x, \frac{\partial u}{\partial x_i}(x)\right) \text{ a.e.,} \qquad \text{for } i = 1, \ldots, N, \text{ such that}$$

$$\sum_{i=1}^N \int_\Omega w_i^*(x)\frac{\partial \phi}{\partial x_i}(x)\mathrm{d}x = \lambda_1\int_\Omega u^{\alpha-1}(x)\phi(x)\mathrm{d}x, \quad \forall\, \phi \in W_0^{1,\alpha}(\Omega). \tag{25}$$

Consider $\phi(x) = \big(u(x) - u_*(s)\big)_+ \in W_0^{1,\alpha}(\Omega)$. From (25), we deduce:

$$\sum_{i=1}^N \int_{\{u>u_*(s)\}} w_i^*(x)\frac{\partial u}{\partial x_i}(x)\mathrm{d}x = \lambda_1\int_\Omega u^{\alpha-1}(x)\big(u(x) - u_*(s)\big)_+\mathrm{d}x. \tag{26}$$

We set

$$E_{\leqslant 1} := \{u > u_*(s)\} \cap \left\{\left|\frac{\partial u}{\partial x_i}\right| \leqslant 1\right\} \quad \text{and } E_{>1} := \{u > u_*(s)\} \cap \left\{\left|\frac{\partial u}{\partial x_i}\right| > 1\right\}.$$

We then have:

$$
\begin{aligned}
\Xi &:= \sum_{i=1}^{N} \int_{\{u>u_*(s)\}} w_i^*(x) \frac{\partial u}{\partial x_i}(x)\mathrm{d}x \\
&\geqslant \tau \sum_{i=1}^{N} \int_{\{u>u_*(s)\}} \varphi_i \left(x, \frac{\partial u}{\partial x_i}(x) \right) \mathrm{d}x \\
&= \tau \sum_{i=1}^{N} \int_{E_{\leqslant 1}} \left| \frac{\partial u}{\partial x_i} \right|^{\tau} (x)\mathrm{d}x + \tau \sum_{i=1}^{N} \int_{E_{>1}} \left| \frac{\partial u}{\partial x_i} \right|^{\alpha} (x)\mathrm{d}x \\
&\geqslant \tau \sum_{i=1}^{N} \int_{\{u>u_*(s)\}} \left| \frac{\partial u}{\partial x_i} \right|^{\alpha} (x)\mathrm{d}x - \tau \sum_{i=1}^{N} \int_{E_{\leqslant 1}} \left| \frac{\partial u}{\partial x_i} \right|^{\alpha} (x)\mathrm{d}x \\
&\geqslant \tau \sum_{i=1}^{N} \int_{\{u>u_*(s)\}} \left| \frac{\partial u}{\partial x_i} \right|^{\alpha} (x)\mathrm{d}x - c_1,
\end{aligned}
$$

where $c_1 = N \cdot \tau \cdot |\Omega|$ is a positive constant independent of u.
We know that there exists $c_2 = c_2(N, \alpha)$, a positive constant depending on N and α, such that:

$$
\sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|^{\alpha} \geqslant c_2 \left(\sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|^{2} \right)^{\frac{\alpha}{2}} = c_2 |\nabla u|^{\alpha}. \tag{27}
$$

Hence

$$
\sum_{i=1}^{N} \int_{\{u>u_*(s)\}} w_i^*(x) \frac{\partial u}{\partial x_i}(x)\mathrm{d}x \geqslant \tau \cdot c_2 \int_{\{u>u_*(s)\}} |\nabla u|^{\alpha} \mathrm{d}x - c_1. \tag{28}
$$

Therefore, (28) and (26) imply that

$$
\tau \cdot c_2 \int_{\{u>u_*(s)\}} |\nabla u|^{\alpha} \mathrm{d}x \leqslant \lambda_1 \int_{\Omega} u^{\alpha-1}(x) \big(u(x) - u_*(s)\big)_+ \mathrm{d}x + c_1. \tag{29}
$$

But $u \in W_0^{1,\alpha}(\Omega)$ and $u \geqslant 0$, thus $u_* \in W_{loc}^{1,1}(\Omega_*)$. By using Lemma 3.3, we deduce from (29) that:

$$
\begin{aligned}
\Big(|\nabla u|^{\alpha} \Big)_{*u}(s) &\leqslant \lambda_2 \Big(-u_*'(s) \Big) \int_0^s \big(u^{\alpha-1}\big)_{*u}(\sigma) d\sigma \\
&\leqslant \lambda_2 \Big(-u_*'(s) \Big) \int_0^s \big(u^{\alpha-1}\big)_{*}(\sigma) d\sigma \\
&\leqslant \lambda_2 \Big(-u_*'(s) \Big) \int_0^s \big(u_*\big)^{\alpha-1}(\sigma) d\sigma,
\end{aligned}
$$

where $\lambda_2 = \dfrac{\lambda_1}{\tau \cdot c_2}$. □

Proposition 7.1 *Let $u \in W_0^{1,\alpha}(\Omega)$, $u \geqslant 0$ verifying* (24)*, then there exist $c_6 > 0$ and $c_7 > 0$ two positive constants such that, for a.e. $s \in \Omega_* =]0, |\Omega|[$, one has*

$$|\nabla u|_{*u}(s) \leqslant c_6 s^{-\frac{N-1}{N(\alpha-1)}} \left(\int_0^s u_*^{\alpha-1}(t)dt\right)^{\frac{1}{\alpha-1}} + c_7. \tag{30}$$

Proof. Thanks to the Hölder inequality for the relative rearrangement, we have (see [14])

$$\Big[|\nabla u|_{*u}\Big]^{\alpha}(s) \leqslant \big(|\nabla u|^{\alpha}\big)_{*u}(s), \qquad \text{a.e. on } s. \tag{31}$$

Using Theorem 3.4, it follows

$$\begin{aligned}\Big[|\nabla u|_{*u}\Big]^{\alpha}(s) &\leqslant \lambda_{1N}\left(-\frac{du_*}{ds}(s)\right)\int_0^s u_*^{\alpha-1}(t)dt + c_2 \\ &\leqslant \frac{\lambda_{1N}}{N\omega_N^{\frac{1}{N}}}\, s^{\frac{1}{N}-1}\int_0^s u_*^{\alpha-1}(t)dt \cdot |\nabla u|_{*u}(s) + c_2.\end{aligned}$$

Using Young inequality, we also have:

$$\Big[|\nabla u|_{*u}\Big]^{\alpha}(s) \leqslant c_4 s^{\alpha'\left(\frac{1}{N}-1\right)}\left(\int_0^s u_*^{\alpha-1}(t)dt\right)^{\alpha'} + c_5, \tag{32}$$

where α' is such that $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$.
This implies that

$$|\nabla u|_{*u}(s) \leqslant c_6 s^{\frac{\alpha'}{\alpha}\left(\frac{1}{N}-1\right)}\left(\int_0^s u_*^{\alpha-1}(t)dt\right)^{\frac{\alpha'}{\alpha}} + c_7. \tag{33}$$

Thus, one has

$$|\nabla u|_{*u}(s) \leqslant c_6 s^{\frac{1}{\alpha-1}\left(\frac{1}{N}-1\right)}\left(\int_0^s u_*^{\alpha-1}(t)dt\right)^{\frac{1}{\alpha-1}} + c_7. \tag{34}$$

□

Proposition 7.2 *Under the same conditions as for Proposition 7.1, if $u \in L^r(\Omega)$ for $r > \dfrac{N}{\alpha'}$ and $\dfrac{1}{\alpha} + \dfrac{1}{\alpha'} = 1$, then $u \in L^\infty(\Omega)$.*

Proof. By Hölder inequality (for $\beta = \dfrac{r}{r-\alpha+1}$ and $\beta' = \dfrac{r}{\alpha-1}$), one has

$$\begin{aligned}\left(\int_0^s u_*^{\alpha-1}(t)dt\right)^{\frac{1}{\alpha-1}} &\leqslant \left[s^{\frac{1}{\beta}} \cdot \left(\int_0^{|\Omega|} u_*^{(\alpha-1)\beta'}(t)dt\right)^{\frac{1}{\beta'}}\right]^{\frac{1}{\alpha-1}} \\ &\leqslant s^{\frac{1}{\alpha-1}-\frac{1}{r}} \cdot \left(\int_0^{|\Omega|} u_*^{r}(t)dt\right)^{\frac{1}{r}}.\end{aligned}$$

Property 3.1 shows that

$$\left(\int_0^s u_*^{\alpha-1}(t)dt\right)^{\frac{1}{\alpha-1}} \leqslant s^{\frac{1}{\alpha-1}-\frac{1}{r}} \cdot \|u\|_{L^r(\Omega)}. \tag{35}$$

Using Theorem 3.4 and (34), one deduces

$$-u_*'(s) \leqslant \frac{s^{\frac{1}{N}-1}}{N\omega_N^{\frac{1}{N}}}\left[c_6 s^{\frac{1}{\alpha-1}\left(\frac{1}{N}-1\right)}\left(\int_0^s u_*^{\alpha-1}(t)dt\right)^{\frac{1}{\alpha-1}} + c_7\right]. \tag{36}$$

From relations (35) and (36), we obtain

$$-u_*'(s) \leqslant c_8 s^{\left(\frac{1}{N}-1\right)\left[1+\frac{1}{\alpha-1}\right]+\frac{1}{\alpha-1}-\frac{1}{r}} \cdot \|u\|_{L^r(\Omega)} + c_9 s^{\frac{1}{N}-1}.$$

The exponent $\left(\frac{1}{N}-1\right)\left[1+\frac{1}{\alpha-1}\right]+\frac{1}{\alpha-1}-\frac{1}{r}$, denoted γ, should verify:

$$\begin{aligned}\gamma+1>0 &\Longleftrightarrow \left(\frac{1}{N}-1\right)\left[1+\frac{1}{\alpha-1}\right]+\frac{1}{\alpha-1}-\frac{1}{r} > -1\\ &\Longleftrightarrow r > \frac{N}{\alpha'}.\end{aligned}$$

For all $\sigma, t \in \Omega_*$, one has

$$\begin{aligned}|u_*(\sigma)-u_*(t)| &\leqslant c_8 \cdot \|u\|_{L^r(\Omega)} \int_0^{|\Omega|} s^\gamma ds + c_9 \int_0^{|\Omega|} s^{\frac{1}{N}-1} ds\\ &\leqslant c_8 \frac{|\Omega|^{\gamma+1}}{\gamma+1} \cdot \|u\|_{L^r(\Omega)} + c_9 \frac{|\Omega|^{\frac{1}{N}}}{\frac{1}{N}}.\end{aligned}$$

Since $u \in W_0^{1,\alpha}(\Omega)$ and $u \geqslant 0$, then we have $u_*(|\Omega|)=0$. In particular, for all $\sigma \in \Omega_*$, one deduces

$$|u_*(\sigma)| \leqslant c_{10}|\Omega|^{\gamma+1} \cdot \|u\|_{L^r(\Omega)} + c_{11}|\Omega|^{\frac{1}{N}}. \tag{37}$$

Hence

$$\|u\|_{L^\infty(\Omega)} \leqslant c_{12} \cdot \|u\|_{L^r(\Omega)} + c_{13}. \tag{38}$$

□

We point out certain situations in which the condition $u \in L^r(\Omega)$, $r > \dfrac{N}{\alpha'}$ is satisfied. For this purpose, consider $1 < \alpha < N$. Thus, we have

$$\alpha^* = \frac{N\alpha}{N-\alpha} > \frac{N}{\alpha'} \Longleftrightarrow 2\alpha^2 - (N+1)\alpha + N > 0. \tag{39}$$

The discriminant of the last quadratic equation is given by

$$\Delta_N = (N+1)^2 - 8N = (N-3)^2 - 8.$$

If the discriminant is positive, then there are two distinct roots

$$\alpha_N^- = \frac{N+1-\sqrt{\Delta_N}}{4} \qquad \text{and} \qquad \alpha_N^+ = \frac{N+1+\sqrt{\Delta_N}}{4}. \tag{40}$$

Proposition 7.3 *If* $1 \leqslant N \leqslant 5$*, then* $\alpha^* > \dfrac{N}{\alpha'}$ *and* $u \in L^\infty(\Omega)$.

Proof. Since $1 \leqslant N \leqslant 5$, then $\Delta_N < 0$. By (39), we deduce that $\alpha^* > \dfrac{N}{\alpha'}$.
This implies the existence of a constant $r > 0$ such that $\alpha^* > r > \dfrac{N}{\alpha'}$.
By Sobolev embedding, we deduce

$$W_0^{1,\alpha}(\Omega) \subset L^{\alpha^*}(\Omega) \subset L^r(\Omega).$$

Proposition 7.2 gives the result. □

Proposition 7.4 *If* $N \geqslant 6$*, then there are two distinct roots* α_N^- *and* α_N^+ *such that* $1 < \alpha_N^- < \alpha_N^+ < N$.

Proof. If $N \geqslant 6$, then $\Delta_N > 0$ and the roots are given by (40).
We also have

$$\begin{aligned} 1 < \alpha_N^- &\Longleftrightarrow 4 < N + 1 - \sqrt{(N-3)^2 - 8} \\ &\Longleftrightarrow \sqrt{(N-3)^2 - 8} < N - 3 \\ &\Longleftrightarrow -8 < 0. \end{aligned}$$

Moreover, one has

$$\begin{aligned} \alpha_N^+ < N &\Longleftrightarrow N + 1 + \sqrt{(N-3)^2 - 8} < 4N \\ &\Longleftrightarrow (N-3)^2 - 8 < (3N-1)^2 \\ &\Longleftrightarrow 0 < 8N^2. \end{aligned}$$

□

Proposition 7.5 *Assume that* $N \geqslant 6$*. If* $1 < \alpha < \alpha_N^-$ *or* $\alpha_N^+ < \alpha < N$*, then* $u \in L^\infty(\Omega)$.

Proof. By Proposition 7.4, we know that $1 < \alpha_N^- < \alpha_N^+ < N$. In addition, if $\alpha_N^+ < \alpha < N$, then $\alpha^* > \dfrac{N}{\alpha'}$, we conclude that $u \in L^\infty(\Omega)$ (see the proof of Proposition 7.3). If $1 < \alpha < \alpha_N^-$, the same argument holds.

Using now Lemmas 7.1, 7.2 and Propositions 7.1 - 7.5, the proof of Theorem 7.1 is complete. □

In the univalued case, we can refine the calculations and provide details of the constants. The method used for the p-Laplacian is based on the Poincaré-Sobolev and interpolation inequalities.

Theorem 7.2 *Let* λ_1, u_1 *be the first eigenvalue and eigenfunction of* Δ_p*:*

$$\begin{cases} -\Delta_p u_1 = \lambda_1 u_1^{p-1} > 0, & \text{for } 1 < p < N, \\ \displaystyle\int_\Omega u_1^p \mathrm{d}x = 1, & \text{and } u_1 \in W_0^{1,p}(\Omega). \end{cases} \tag{41}$$

Then,

$$\max_{\overline{\Omega}} u_1 \leqslant \left(\frac{B_{p^*}^{\frac{1}{p^*}}}{N\omega_N^{\frac{1}{N}}} \right)^{\frac{\theta}{\theta-1}} \left(\frac{\lambda_1}{p} \right)^{\frac{\theta}{p(\theta-1)}},$$

where $\theta = \dfrac{p^*}{p}$, $\lambda_1 = \inf \left\{ \int_\Omega |\nabla v|^p \mathrm{d}x : \int_\Omega v^p \mathrm{d}x = 1 \right\}$, $p^* = \dfrac{Np}{N-p}$, B_{p^*} *is the Bliss constant* [14] *and* ω_N *is the Lebesgue measure of the unit ball in* $\mathbb{R}^N$.

Proof. Let J be as follows

$$J = \int_\Omega |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla(u_1^{m+1}) \mathrm{d}x = (m+1) \int_\Omega |\nabla u_1|^{p-2} \nabla u_1 \cdot (\nabla u_1 \, u_1^m) \mathrm{d}x$$
$$= (m+1) \int_\Omega |\nabla u_1|^p \cdot u_1^m \mathrm{d}x = (m+1) \int_\Omega \left| \nabla u_1 \cdot u_1^{\frac{m}{p}} \right|^p \mathrm{d}x.$$

Since we have

$$\nabla u_1^{\frac{m+p}{p}} = \frac{m+p}{p} \, \nabla u_1 \cdot u_1^{\frac{m}{p}},$$

we conclude

$$J = \frac{p(m+1)}{m+p} \int_\Omega \left| \nabla u_1^{\frac{m+p}{p}} \right|^p \mathrm{d}x. \tag{42}$$

By (41), J can be written as

$$J = \lambda_1 \int_\Omega u_1^{p-1} \cdot u_1^{m+1} \mathrm{d}x = \lambda_1 \int_\Omega u_1^{m+p} \mathrm{d}x.$$

The Poincaré-Sobolev inequality (see [14]) implies:

$$\left(\int_\Omega u_1^{\frac{p^*(m+p)}{p}} \mathrm{d}x \right)^{\frac{1}{p^*}} \leqslant \frac{B_{p^*}^{\frac{1}{p^*}}}{N\omega_N^{\frac{1}{N}}} \left(\int_\Omega \left| \nabla u_1^{\frac{m+p}{p}} \right|^p \right)^{\frac{1}{p}}$$
$$= \frac{B_{p^*}^{\frac{1}{p^*}}}{N\omega_N^{\frac{1}{N}}} \left(\lambda_1 \frac{m+p}{p(m+1)} \right)^{\frac{1}{p}} \left(\int_\Omega u_1^{m+p} \mathrm{d}x \right)^{\frac{1}{p}},$$

from which we obtain

$$\|u_1\|_{L^{\theta(m+p)}(\Omega)}^{\frac{m+p}{p}} \leqslant \frac{B_{p^*}^{\frac{1}{p^*}}}{N\omega_N^{\frac{1}{N}}} \left(\lambda_1 \frac{m+p}{p(m+1)} \right)^{\frac{1}{p}} \|u_1\|_{L^{m+p}(\Omega)}^{\frac{m+p}{p}}. \tag{43}$$

Therefore,

$$\|u_1\|_{L^{\theta(m+p)}(\Omega)} \leqslant \left(\frac{B_{p^*}^{\frac{1}{p^*}}}{N\omega_N^{\frac{1}{N}}} \right)^{\frac{p}{m+p}} \left(\lambda_1 \frac{m+p}{p(m+1)} \right)^{\frac{1}{m+p}} \|u_1\|_{L^{m+p}(\Omega)}. \tag{44}$$

Using an interpolation inequality (see [29]) for $0 \leqslant \eta \leqslant 1$, one has

$$\|u_1\|_{L^{m+p}(\Omega)} \leqslant \|u_1\|_{L^p}^{1-\eta} \cdot \|u_1\|_{L^{\theta(m+p)}}^{\eta} \quad \text{and} \quad \frac{1}{m+p} = \frac{1-\eta}{p} + \frac{\eta}{\theta(m+p)}. \tag{45}$$

The above equality leads to

$$\eta = \frac{\theta m}{\theta(m+p)-p}, \quad \text{and} \quad 1-\eta = \frac{p(\theta-1)}{\theta(m+p)-p}. \tag{46}$$

Combining (44) and (45), we get

$$\|u_1\|_{L^{\theta(m+p)}} \leqslant \left(\frac{B_{p^*}^{\frac{1}{p^*}}}{N\omega_N^{\frac{1}{N}}} \right)^{\frac{p}{m+p}} \left(\lambda_1 \frac{m+p}{p(m+1)} \right)^{\frac{1}{m+p}} \|u_1\|_{L^p}^{1-\eta} \cdot \|u_1\|_{L^{\theta(m+p)}}^{\eta}. \tag{47}$$

Thus, we have:

$$\|u_1\|_{L^{\theta(m+p)}} \leqslant \left(\frac{B_{p^*}^{\frac{1}{p^*}}}{N\omega_N^{\frac{1}{N}}} \right)^{\frac{p}{m+p} \cdot \frac{1}{1-\eta}} \left(\lambda_1 \frac{m+p}{p(m+1)} \right)^{\frac{1}{m+p} \cdot \frac{1}{1-\eta}} \|u_1\|_{L^p}. \tag{48}$$

But

$$\frac{1}{m+p} \cdot \frac{1}{1-\eta} = \frac{1}{p(\theta-1)} \cdot \frac{\theta(m+p)-p}{m+p} \xrightarrow[m \to +\infty]{} \frac{\theta}{p(\theta-1)}.$$

This achieves the proof by letting $m \to +\infty$ in (48). □

Remark 7.1 *The L^∞-regularity of the eigenfunctions of p-Laplacian is already known (see, for example,* [30]*).*

8 Conclusion

In this paper, we treated eigenvalue problems related to $\overline{\rho}^{>}$-multivalued operators. We generalized the results of [1] and [2]. We studied the existence for a more general problem, by using the nonsmooth version of the Lagrange multiplier theorem. This existence result is then used to prove the L^∞-regularity of the eigenvectors associated with the $\overline{\rho}^{>}$-multivalued Leray-Lions operators by using the relative rearrangement theory. We also proved an existence result for an eigenvalue problem in a resonant case near zero.

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