A geometrical realization of the generic representations of $\mathrm{PGL}(2, \mathbb{F}_q)$

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Abstract

We construct a simplicial complex, equipped with an action of $\mathrm{PGL}(2, \mathbb{F}_q)$, whose degree 1 cohomology affords the generic representations with multiplicity one, except the Steinberg representation.

1 notation

Here $q > 1$ is an integer of the form p^n, $n \in \mathbb{N}$, where p is a prime. We denote by $\mathbb{F}_q$ the field with q elements and set $G = \mathrm{PGL}(2, \mathbb{F}_q)$. We shall only consider *complex* representations of G. Any element (resp. subset) of G will be denoted by a representative (resp. a set of representatives) in $\mathrm{GL}(2, \mathbb{F}_q)$. We denote by T the diagonal torus of G. A character of T is given by

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \mapsto \chi(x)\chi^{-1}(y) \ ,$$

where χ is a character of $\mathbb{F}_q^\times$; we denote this character by $\chi \otimes \chi^{-1}$.

By fixing an isomorphism of $\mathbb{F}_q$-vector spaces $\mathbb{F}_q^2 \simeq \mathbb{F}_{q^2}$, we may identify the group $\mathbb{F}_{q^2}^\times / \mathbb{F}_q^\times$ with a subtorus Θ of G. The Frobenius automorphism σ of $\mathrm{Gal}(\mathbb{F}_{q^2} / \mathbb{F}_q)$ acts on Θ. A character θ of Θ is said to be regular if $\theta^\sigma \not\simeq \theta$.

We denote by Det the determinant $G \longrightarrow \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2$. When q is odd, let χ_o be the unique non trivial character of $\mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2$

2 The irreducible representations of $\mathrm{PGL}(2,\mathbb{F}_q)$

Let us recall the list of (isomorphy classes of) irreducible representations of G. They splits into different series.

a) *Induced representations in the principal series.* We denote by $\pi(\chi,\chi^{-1})$, the representation parabolically induced from $\chi\otimes\chi^{-1}$, parametrized by the pairs $\{\chi,\chi^{-1}\}$, where χ runs over the characters of $\mathbb{F}_q^\times$ such that $\chi^2 \neq 1$.
b) *Steinberg representations.* The Steinberg representation $\mathbf{St}$ and (when q is odd) $\chi_o \circ \mathrm{Det} \otimes \mathbf{St}$ (that we abbreviate $\chi_o \otimes \mathbf{St}$).
c) *Cuspidal representations.* We have one representation $\pi(\theta)$ for each pair $\{\theta,\theta^\sigma\}$ of regular characters of Θ.
d) *Characters.* The trivial character $\mathbf{1}_G$ and (when q is odd) $\chi_o \circ \mathrm{Det}$.

An irreducible representation is said to be *generic* if it is not a character.

3 A simplicial complex

Recall that G acts on the projective line $\mathbb{P}^1(\mathbb{F}_q)$. We first construct a (non-oriented) graph Y as follows. Its vertices are the elements of $\mathbb{P}^1(\mathbb{F}_q)$ plus an extra point o; its edges are the $\{o,x\}$, where x runs over $\mathbb{P}^1(\mathbb{F}_q)$. Let X be the simplicial complex whose vertices are the oriented edges of Y and where two vertices form an edge when their union is an oriented 2-path in Y. So :

the vertices of X are the $[x,o]$, $[o,y]$, $x,y \in \mathbb{P}^1(\mathbb{F}_q)$;
the edges of X are the $[x,o,y]$, $x,y \in \mathbb{P}^1(\mathbb{F}_q)$ and $x \neq y$.

We have a natural action of G on X via simplicial automorphisms.

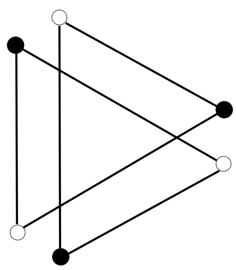

fig.1 The complex X for $q = 2$.

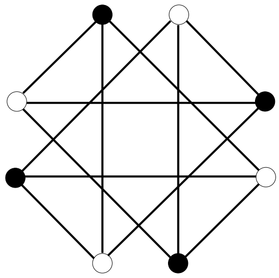

fig.2 The complex X for $q = 3$.

Lemma 3.1: The complex X is connected.

Proof. Indeed let s and t be two distinct vertices in X. If $s = [x, o]$, $t = [o, y]$, then s and t are linked by the edge $[x, o, y]$. If $s = [x, o]$, $t = [y, o]$, let z be a point of $\mathbb{P}^1(\mathbb{F}_q)$ distinct from x and y (recall that $\mathrm{Card}(\mathbb{P}^1(\mathbb{F}_q)) \geqslant 3$). Then $([x, o], [o, z], [y, o])$ is a path linking s and t.

Proposition 3.2: As a G-module the cohomology space $H^1(X, \mathbb{C})$ consists of the generic representations, with multiplicity 1, except the Steinberg representation.

The proof will be given in section 4. We start here with some preliminary reductions.

Let $C^i_{or}(X)$, $i = 0, 1$, be the space of complex oriented i-cochains of X. So $C^0_{or}(X)$ is the space of linear forms on $\mathbb{C}[X_o]$, the $\mathbb{C}$-vector space with basis the vertices of X. Write $\mathbb{C}[X_1]^{or}$ for the $\mathbb{C}$-vector space with basis the set of oriented edges in X; then $C^0_{or}(X)$ is the set of linear forms ω on $\mathbb{C}[X_1]$ satisfying $\omega(\bar{e}) = -\omega(e)$, for any oriented edge e, and where $\bar{e}$ denotes the opposed edge. We have a cobord operator $d : C^0_{or}(X) \longrightarrow C^1_{or}(X)$ defined by $df([s, t]) = f(t) - f(s)$, for any oriented edge $[s, t]$. The following sequence of $\mathbb{C}$-vector spaces

$$0 \longrightarrow C^0_{or}(X) \xrightarrow{\ d\ } C^1_{or}(X) \longrightarrow 0$$

is a cochain complex with an obvious action of G, and it cohomology computes $H^i(X, \mathbb{C})$, $i = 1, 2$, as G-modules.

The space $C^1_{or}(X)$ is isomorphic, as a G-module, to the space of linear forms on $\mathbb{C}[X_1]$, the space with basis the (non-oriented) edges of X. This is due to the fact that X has a G-equivariant labelling. The set of edges in X is isomorphic as a G-set to the set of pairs (x, y) of distinct points in $\mathbb{P}^1(\mathbb{F}_q)$, so to G/T. So we have an isomorphism of G-sets :

$$C^1_{or}(X) \simeq [\mathrm{Ind}_T^G \mathbf{1}_T]^*,$$

where the symbol $*$ denotes the contragredient representation.

On the other hand, the image of the cobord d is isomorphic to $C^0_{or}(X)/\mathrm{Ker}d$. One may identify $C^o_{or}(X)$ with the space of functions on the disjoint union $\mathbb{P}^1(\mathbb{F}_q)\coprod\mathbb{P}^1(\mathbb{F}_q)$; here an edge of the form $[x,o]$ (resp. $[o,x]$) is identified with x in the first (resp. second) set. A function on this set is a pair of functions (f,g) on $\mathbb{P}^1(\mathbb{F}_q)$; it lies in Ker$d$ when $d(f,g)[x,y]=0$, for all $x\neq y$ in $\mathbb{P}^1(\mathbb{F}_q)$, that is when $f=g$ is a constant function.

It is classical that, as a G-module, the space of functions on $\mathbb{P}^1(\mathbb{F}_q)$ is isomorphic to $\mathbf{1}_G\oplus\mathbf{S}t$. So we have

$$C^0_{or}(X)\simeq\mathbf{1}_G\oplus\mathbf{S}t\oplus\mathbf{1}_G\oplus\mathbf{S}t\ ,$$

and

$$\mathrm{Imd}\simeq C^0_{or}(X)/\mathrm{Ker}d\simeq\mathbf{1}_G\oplus\mathbf{S}t\oplus\mathbf{S}t.$$

4 The determination of $[\mathrm{Ind}_T^G \mathbf{1}_T]^*$

To prove proposition (3.2), we are reduce to determining this induced representation. We have the following equality (due to Imin Chen, see [1] Théorème (2.1)) :

$$\mathrm{Ind}_T^G\mathbf{1}_T = Ind_\Theta^G\mathbf{1}_G\oplus\mathbf{S}t\oplus\mathbf{S}t.$$

The multiplicity of an irreducible representation π of G in $Ind_\Theta^G\mathbf{1}_G$ is given in [2] (section **3.** *Tableau* 1) :

a) For $\pi=\pi(\chi,\chi^{-1})$, $m(\pi)=1$;
b) for $\pi=\mathbf{S}t$, $m(\pi)=0$, and for $\pi=\chi_o\otimes\mathbf{S}t$, $m(\pi)=1$;
c) for $\pi=\pi(\theta)$, $m(\pi)=1$;
d) for $\pi=\mathbf{1}_G$, $m(\pi)=1$, and for $\pi=\chi_o\circ\mathrm{Det}$, $m(\pi)=0$.

As a consequence, $Ind_\Theta^G\mathbf{1}_G$ is the direct sum of the generic representation of G, with multiplicity 1, plus $\mathbf{1}_G-\mathbf{S}t$ (in the Grothedieck group). To sum up, $\mathrm{Ind}_T^G\mathbf{1}_T$ is the sum of the generic representations, with multiplicity 1, plus $\mathbf{S}t\oplus\mathbf{S}t\oplus\mathbf{1}_G-\mathbf{S}t$ $=\mathbf{S}t\oplus\mathbf{1}_G$. This set of representations is stable under the operation of taking the contragredient, so $[\mathrm{Ind}_T^G\mathbf{1}_T]^*\simeq\mathrm{Ind}_T^G\mathbf{1}_T$, and $H^1(X)\simeq[\mathrm{Ind}_T^G\mathbf{1}_T]^*/\mathrm{Im}d$ consists of the generic representations, with multiplicity 1, plus :

$$[\mathbf{S}t\oplus\mathbf{1}_G]-[\mathbf{1}_G\oplus\mathbf{S}t\oplus\mathbf{S}t]=-\mathbf{S}t\ ,$$

and our result follows.

1

References

[1] F. Sauvageot, *Représentations de Steinberg et identités de projecteurs*, to appear.

[2] J. Soto-Andrade et J. Vargas, *Analyse harmonique sur le demi-plan de Poincaré tordu*, CRAS, Paris, t. 328, Série I, p. 375-380, 1999.