Generalizations of the PRV conjecture, II

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Abstract

Let $G \subset \hat{G}$ be two complex connected reductive groups. We deal with the problem of finding sub-G-modules of a given irreducible $\hat{G}$-module. In the case where G is diagonally embedded in $\hat{G} = G \times G$, S. Kumar and O. Mathieu found some of them, proving the PRV conjecture. Recently, the authors generalized the PRV conjecture on the one hand to the case where $\hat{G}/G$ is spherical of minimal rank, and on the other hand giving more sub-G-modules in the classical case $G \subset G \times G$. In this paper, these two recent generalizations are combined in a more general result.

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1 Introduction

Let G be a complex connected reductive group. Let T be a maximal torus of G and let B be a Borel subgroup containing T. The Weyl group is denoted by W. The irreducible G-module of highest weight μ is denoted by $V_G(\mu)$. For any weight μ of T, there exists a unique dominant weight $\overline{\mu}$ in the W-orbit of μ. The representation $V_G(\overline{\mu})$ is called the *irreducible G-module with extremal weight* μ. Parthasarathy-Ranga Rao-Varadarajan conjectured in the sixties the following statement.

The PRV conjecture.

Let μ and ν be two dominant weights. Then, for any $w \in W$, the irreducible G-module $V_G(\overline{\mu + w\nu})$ with extremal weight $\mu + w\nu$, occurs with multiplicity at least one in $V_G(\mu) \otimes V_G(\nu)$.

This conjecture was proved independently by S. Kumar in [Kum88] and O. Mathieu in [Mat89]. This paper is a continuation of [MPR11] where the authors obtained a new proof and two generalizations of the PRV conjecture.

Assume that G is a subgroup of a bigger connected reductive group $\hat{G}$. The decomposition of the tensor product concerned by the PRV conjecture is an example (*i.e.* when $\hat{G} = G \times G$ and G is diagonally embedded in $\hat{G}$) of the following problem.

Find irreducible G-submodules in a given irreducible $\hat{G}$-module.

The homogeneous space $(G \times G)/G$ is spherical of minimal rank. From now on, we consider two reductive connected groups $G \subset \hat{G}$ such that $\hat{G}/G$ is spherical of minimal rank. Such homogeneous spaces are classified in [Res10b]. A list is given in the appendix.

Let $\hat{T}$ be the centralizer of T in $\hat{G}$; then, since G is a spherical subgroup of $\hat{G}$, it is well-known that $\hat{T}$ is a maximal torus of $\hat{G}$ (see Lemma 2 in the appendix). We can deduce that the Weyl group W of G is canonically a subgroup of the Weyl group $\hat{W}$ of $\hat{G}$ (see Corollary 2 in the appendix).

In [MPR11], we generalized the PRV conjecture: for any irreducible $\hat{G}$-module $\hat{V}$, we give an explicit family of irreducible G-submodules of $\hat{V}$ parametrized by $W \backslash \hat{W}$.

Let μ and ν be two dominant weights of G. In [MPR11], we also improved the PRV conjecture by constructing a family of irreducible submodules of $V_G(\mu) \otimes V_G(\nu)$, strictly containing the PRV components. The cardinality of this family is not bounded independently of the couple (λ, μ). More precisely, this family is the set of dominant weights belonging to some explicit line segments with (at least) one PRV component as end point.

The main result of this paper gives a similar family of irreducible sub-G-modules of any irreducible $\hat{G}$-module $\hat{V}$. In particular, this result is a common generalization of the two results of [MPR11] explained before.

The precise statement needs some preparation. If α is a root of (G, T), $\alpha^\vee$ denotes the corresponding coroot. There exists a unique Borel subgroup $\hat{B}$ containing $\hat{T}$ such that $\hat{B} \cap G = B$ (see for example [Res10b, Proposition 2.2]). Consider the restriction map $\rho : X(\hat{T}) \longrightarrow X(T)$ from the character group of $\hat{T}$ to this of T. Let $X(T)^+$ (resp. $X(\hat{T})^+$) denote the set of dominant weights of G (resp. of $\hat{G}$). Let $\hat{\nu} \in X(\hat{T})^+$; in [MPR11], we proved that for any $\hat{w} \in \hat{W}$, $V_G(\overline{\rho(\hat{w}\hat{\nu})})$ is a submodule of $V_{\hat{G}}(\hat{\nu})$. We call a submodule of $V_{\hat{G}}(\hat{\nu})$ a *PRV component*. Recall that W is canonically a subgroup of $\hat{W}$ and observe that, for any w in W, $\overline{\rho(\hat{w}\hat{\nu})} = \overline{\rho(w\hat{w}\hat{\nu})}$. In particular, the PRV components are parametrized by $W \backslash \hat{W}$.

Let Δ and $\hat{\Delta}$ denote the set of simple roots of G and $\hat{G}$. By [Res10b, Lemma 4.6], $\rho(\hat{\Delta}) = \Delta$. More precisely, for any $\alpha \in \Delta$, we have the following alternative:

(i) there exists a unique $\hat{\alpha}_0 \in \hat{\Delta}$ such that $\rho(\hat{\alpha}_0) = \alpha$; or

(ii) there exist exactly two simple roots $\hat{\alpha}_1$ and $\hat{\alpha}_2$ in $\hat{\Delta}$ such that $\rho(\hat{\alpha}_1) = \rho(\hat{\alpha}_2) = \alpha$. Moreover $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are orthogonal.

The set of simple roots satisfying the first condition is denoted by Δ_1 and the set of those satisfying the second one is denoted by Δ_2 (these sets are given for the spherical homogenous spaces of minimal rank listed in the appendix).

Theorem 1. *Recall that $G \subset \hat{G}$ are connected reductive groups and that $\hat{G}/G$ is assumed to be spherical of minimal rank. Let $\hat{\nu}$ be a dominant weight of $\hat{G}$ and let $\hat{w} \in \hat{W}$ such that $\rho(\hat{w}\hat{\nu})$ is a dominant weight of G. Let $\alpha \in \Delta_2$. Index the two simple roots $\hat{\alpha}_1$ and $\hat{\alpha}_2$ in $\hat{\Delta}$ belonging to the pullback of α by ρ in such a way that $\langle \hat{w}\hat{\nu}, \hat{\alpha}_1^\vee \rangle \le \langle \hat{w}\hat{\nu}, \hat{\alpha}_2^\vee \rangle$. Denote by $S_{\alpha,\hat{w},\hat{\nu}}$ the line segment in $X(T) \otimes \mathbb{Q}$ whose end points are $\rho(\hat{w}\hat{\nu})$ and $\rho(s_{\hat{\alpha}_1}\hat{w}\hat{\nu})$. And denote by $\Lambda \subset X(T)$ the root lattice of G.*

Then for any dominant weight $\nu \in S_{\alpha,\hat{w},\hat{\nu}} \cap (\rho(\hat{w}\hat{\nu}) + \Lambda)$ the irreducible G-module $V_G(\nu)$ occurs with multiplicity at least one in $V_{\hat{G}}(\hat{\nu})$.

From each PRV component in $V_{\hat{G}}(\hat{\nu})$, Theorem 1 gives $|\Delta_2|$ line segments (that may be of length zero) of irreducible components of $V_{\hat{G}}(\hat{\nu})$. Note that the length of $S_{\alpha,\hat{w},\hat{\nu}}$ is $|\langle \hat{w}\hat{\nu}, \hat{\alpha}_1^\vee \rangle|$.

The proof of Theorem 1 above is the object of Section 2. In Section 2.3, Theorem 1 is illustrated with the example of $G = G_2$ in $\hat{G} = \mathrm{Spin}_7$ (see Figure 1).

In Section 3, we get a still larger family of irreducible G-submodules of $\hat{V}$. In Section 4, we get a still larger family of irreducible G-submodules of $V_G(\lambda) \otimes V_G(\mu)$. This last result is specific to the case of the tensor product decomposition. These two results are illustrated by examples. We do not detail here the results of these two sections, whose statements are quite technical, we refer the reader to Theorems 3, 4 and 6.

2 Proof of Theorem 1

2.1 A reduction

In this section we rewrite Theorem 1 as it was written in the tensor product case in [MPR11]. In fact, we prove that the following theorem implies Theorem 1.

Theorem 2. *Let ν and $\hat{\nu}$ be two dominant weights of T and $\hat{T}$. Assume that there exist a simple root $\alpha \in \Delta_2$, $\hat{w}$ in the Weyl group $\hat{W}$ and an integer k such that*

$$\nu = \rho(\hat{w}\hat{\nu}) - k\alpha. \tag{1}$$

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Denote by $\hat{\alpha}_1$ and $\hat{\alpha}_2$ the two roots in $\hat{\Delta}$ belonging to the pullback of α by ρ.

If

$$k \quad \geq \quad 0, \tag{2a}$$

$$k \quad \leq \quad \langle \hat{w}\hat{\nu}, \hat{\alpha}_1^\vee \rangle, \tag{2b}$$

$$k \quad \leq \quad \langle \hat{w}\hat{\nu}, \hat{\alpha}_2^\vee \rangle, \tag{2c}$$

then the irreducible G-module $V_G(\nu)$ occurs with multiplicity at least one in $V_{\hat{G}}(\hat{\nu})$.

Before proving that Theorem 2 implies Theorem 1, we state a basic remark.

Remark 1. *Let $\alpha \in \Delta_2$. If S is the neutral component of the kernel of α, [Res10b, Lemmas 4.3 and 4.4] shows that the inclusion of G^S in $\hat{G}^S$ is isomorphic to the diagonal inclusion of $(P)SL_2$ in $(P)SL_2 \times (P)SL_2$. It follows that the image of the one-parameter subgroup $\hat{\alpha}_1^\vee + \hat{\alpha}_2^\vee$ of $\hat{T}$ is contained in T, and that $\alpha^\vee = \hat{\alpha}_1^\vee + \hat{\alpha}_2^\vee$.*

Proof of: Theorem 2 implies Theorem 1. Let $\hat{\nu}$ be a dominant weight of $\hat{G}$, $\hat{w} \in \hat{W}$ such that $\rho(\hat{w}\hat{\nu})$ is a dominant weight of G, $\alpha \in \Delta_2$ and a dominant weight $\nu \in S_{\alpha,\hat{w},\hat{\nu}} \cap (\rho(\hat{w}\hat{\nu}) + \Lambda)$. Recall that $\langle \hat{w}\hat{\nu}, \hat{\alpha}_1^\vee \rangle \leq \langle \hat{w}\hat{\nu}, \hat{\alpha}_2^\vee \rangle$. Two different cases occur according to the sign of $\langle \hat{w}\hat{\nu}, \hat{\alpha}_1^\vee \rangle$.

(i) If $\langle \hat{w}\hat{\nu}, \hat{\alpha}_1^\vee \rangle$ is non-negative, then $\langle \hat{w}\hat{\nu}, \hat{\alpha}_2^\vee \rangle$ is also non-negative. Denoting by k the integer such that $\nu = \rho(\hat{w}\hat{\nu}) - k\alpha$, the definition of $S_{\alpha,\hat{w},\hat{\nu}}$ implies that k satisfies inequalities (2). Then, by Theorem 2, the irreducible module $V_G(\nu)$ occurs in $V_{\hat{G}}(\hat{\nu})$.

(ii) Suppose that $\langle \hat{w}\hat{\nu}, \hat{\alpha}_1^\vee \rangle$ is negative, so that $\langle s_{\hat{\alpha}_1}\hat{w}\hat{\nu}, \hat{\alpha}_1^\vee \rangle \geq 0$.

By Remark 1, $\langle \hat{w}\hat{\nu}, \hat{\alpha}_1^\vee + \hat{\alpha}_2^\vee \rangle \geq 0$. Then $\langle \hat{w}\hat{\nu}, \hat{\alpha}_2^\vee \rangle \geq 0$. Since $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are orthogonal, we deduce that $\langle s_{\hat{\alpha}_1}\hat{w}\hat{\nu}, \hat{\alpha}_2^\vee \rangle \geq 0$.

Denote by k the integer such that $\nu = \rho(s_{\hat{\alpha}_1}\hat{w}\hat{\nu}) - k\alpha$. Then k satisfies inequalities (2) with $s_{\hat{\alpha}_1}\hat{w}$ instead of $\hat{w}$. Hence the irreducible module $V_G(\nu)$ occurs in $V_{\hat{G}}(\hat{\nu})$.

$\square$

Remark 2. *In the first case, the two end points of the line segment $S_{\alpha,\hat{w},\hat{\nu}}$ are dominant and both correspond to a PRV component. In the second one, $\rho(s_{\hat{\alpha}_1}\hat{w}\hat{\nu})$ is not necessarily dominant.*

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2.2 Proof of Theorem 2

2.2.1 Preliminaries

The existence of k satisfying inequalities (2) implies that $\langle \hat{w}\hat{\nu}, \hat{\alpha}_1^\vee \rangle$ and $\langle \hat{w}\hat{\nu}, \hat{\alpha}_2^\vee \rangle$ are non-negative. If one of them is zero, ν is a PRV component, and in that case, Theorem 2 is a consequence of our previous work [MPR11, Theorem 1]. Hence, we assume from now that $\langle \hat{w}\hat{\nu}, \hat{\alpha}_1^\vee \rangle$ and $\langle \hat{w}\hat{\nu}, \hat{\alpha}_2^\vee \rangle$ are positive. In particular, by Lemma 1 in Appendix, we have $l(s_{\hat{\alpha}_1}\hat{w}) = l(s_{\hat{\alpha}_2}\hat{w}) = l(\hat{w}) + 1$.

2.2.2 An asymptotic result

In this section we apply the Borel-Weil theorem and an argument of Geometric Invariant Theory to get an asymptotic version of Theorem 2.

For any character ν of B, $\mathcal{L}_\nu$ denote the G-linearized line bundle on G/B such that B acts with weight $-\nu$ on the fiber over B/B. Similarly, we define the line bundle $\mathcal{L}_{\hat{\nu}}$ on $\hat{G}/\hat{B}$. The Borel-Weil theorem asserts that $H^0(G/B, \mathcal{L}_\nu)$ is the irreducible G-module $V_G(\nu)^*$.

Set $X = G/B \times \hat{G}/\hat{B}$ and $\hat{v} = s_{\hat{\alpha}_1}.\hat{w}$. Let S be the neutral component of the Kernel of α. Let ρ_S denote the restriction map to S both for characters of T and $\hat{T}$. We denote by G^S (resp. by $\hat{G}^S$) the centralizer of S in G (resp. in $\hat{G}$).

Let w_0 denote the longest element in W and consider the following irreducible component of the S-fixed points set:

$$C = G^S w_0 B/B \times \hat{G}^S \hat{w}\hat{B}/\hat{B}.$$

Note that C is isomorphic to $\mathbb{P}^1 \times (\mathbb{P}^1)^2$ (see for example [Hum75, Chapter 9]). Finally consider the line bundle $\mathcal{M} = \mathcal{L}_{-w_0\nu} \boxtimes \mathcal{L}_{\hat{\nu}}$ on X. Then, for any non-negative integer n, $H^0(X, \mathcal{M}^{\otimes n}) = V_G(n\nu) \otimes V_{\hat{G}}(n\hat{\nu})^*$.

The torus S acts trivially on C and hence on the restriction $\mathcal{M}_{|C}$ of $\mathcal{M}$ on C by a character. This character equals $\rho_S(\nu) + \rho_S(-\hat{w}\hat{\nu})$. By assumption (1), this character is trivial. In particular, any point of C is semistable for the action of S.

The restriction of $\mathcal{M}$ to $C \simeq (\mathbb{P}^1)^3$ is isomorphic as a line bundle to $\mathcal{O}(a) \otimes \mathcal{O}(b) \otimes \mathcal{O}(c)$ for some integers a, b, and c. The torus T acts on the fiber over $w_0 B/B$ in the line bundle $\mathcal{L}_{-w_0\nu}$ by the weight ν. The SL_2-theory of $\mathbb{P}^1$ implies that $a = \langle \nu, \alpha^\vee \rangle$. Similarly we compute b and c and we get

$$a = \langle \nu, \alpha^\vee \rangle,$$
$$b = \langle \hat{w}\hat{\nu}, \hat{\alpha}_1^\vee \rangle,$$
$$c = \langle \hat{w}\hat{\nu}, \hat{\alpha}_2^\vee \rangle.$$

Then, using Remark 1, inequalities (2) are equivalent to

$$a + b \geq c,$$
$$a + c \geq b,$$
$$b + c \geq a.$$

They imply that $C^{ss}(\mathcal{M}, G^S/S)$ is not empty.

Since $C^{ss}(\mathcal{M}, G^S) \neq \varnothing$, Luna's theorem (see [Lun75, Corollary 2 and Remark 1] or see also [Res10a, Proposition 8]) implies that $X^{ss}(\mathcal{M}) \cap C \neq \varnothing$. Fix a positive integer n and a G-invariant section σ in $H^0(X, \mathcal{M}^{\otimes n})$ such that $\sigma_{|C}$ is not identically zero. Let $X_{\hat{v}}$ be the closure of the orbit $G \hat{v} \hat{B}/\hat{B}$ in $\hat{G}/\hat{B}$. By the remark of Section 2.2.1, the stabilizer of $\hat{v}$ in G^S equals T. Then $G^S . \hat{v}$ has dimension two and $\overline{G^S \hat{v}} = \hat{G}^S \hat{v}$. Hence $C \cap (\{w_0 B/B\} \times \hat{G}/\hat{B})$ is contained in $\{w_0 B/B\} \times X_{\hat{v}}$.

Consider now the restriction τ of σ to $Y = \{w_0 B/B\} \times \hat{G}/\hat{B}$. Identifying Y with the $\hat{G}/\hat{B}$ by the second projection, $\mathcal{M}_{|Y}$ is a B^--linearized line bundle on $\hat{G}/\hat{B}$. More precisely, it is the line bundle $\mathcal{L}_{\hat{\nu}}$ where the action of B^- is obtained by restricting the action of $\hat{G}$ and then by twisting by ν. But τ is a B^--invariant section of $\mathcal{M}_{|Y}^{\otimes n}$. Then τ identifies with a section $\bar{\tau}$ of $\mathcal{L}_{n\hat{\nu}}$ on $\hat{G}/\hat{B}$ which is a B^--eigenvector of weight $-n\nu$. Moreover, the relative position of C and $X_{\hat{v}}$ implies that $\bar{\tau}_{|X_{\hat{v}}}$ is not identically zero. Note that this implies that $V_G(n\nu)^*$ appears in $V_{\hat{G}}(n\hat{\nu})^*$.

2.2.3 Multiplicity one

We introduce a notation: let H be an algebraic group and let χ be a character of H; if V is a representation of H, we denote by $V^{(H)_\chi}$ the subspace of H-eigenvectors of weight χ.

Define $X_{\hat{v}}^\circ = G \hat{v} \hat{B}/\hat{B}$, and recall that $X_{\hat{v}} = \overline{X_{\hat{v}}^\circ}$. Now, we prove that the multiplicity of $V_G(\nu)^*$ in $H^0(X_{\hat{v}}^\circ, \mathcal{L}_{\hat{\nu}})$ is one. We have an injection

$$\iota \; : \; H^0(X_{\hat{v}}^\circ, \mathcal{L}_{\hat{\nu}}) \to \mathbb{C}[G]$$

and the image $\iota(H^0(X_{\hat{v}}^\circ, \mathcal{L}_{\hat{\nu}}))$ equals $\mathbb{C}[G]^{(\{1\} \times G_{\hat{v}})_{\rho(\hat{v}\hat{\nu})}}$. The isotropy subgroup $G_{\hat{v}}$ contains the torus T. Hence by using the Frobenius theorem, we can deduce the following:

$$\iota(H^0(X_{\hat{v}}^\circ, \mathcal{L}_{\hat{\nu}})) \subset \mathbb{C}[G]^{(\{1\} \times T)_{\rho(\hat{v}\hat{\nu})}} \simeq \bigoplus_{\chi \in X(T)^+} V_G(\chi)^* \otimes V_G(\chi)^{(T)_{\rho(\hat{v}\hat{\nu})}}.$$

In particular the multiplicity of $V_G(\nu)^*$ in $H^0(X_{\hat{v}}^\circ, \mathcal{L}_{\hat{\nu}})$ is at most the dimension of $V_G(\nu)^{(T)_{\rho(\hat{v}\hat{\nu})}}$. But, by assumption,

$$\rho(\hat{v}\hat{\nu}) = \rho(\hat{w}\hat{\nu} - \langle \hat{w}\hat{\nu}, \hat{\alpha}_1^\vee \rangle \hat{\alpha}_1^\vee) = \nu - l\alpha,$$

where $l = \langle \hat{w}\hat{\nu}, \hat{\alpha}_1^\vee \rangle - k$. Then the dimension of $V_G(\nu)^{(T)\rho(\hat{v}\hat{\nu})}$ is at most one and the multiplicity of $V_G(\nu)^*$ in $H^0(X_{\hat{v}}^\circ, \mathcal{L}_{\hat{\nu}})$ is at most one.

We now show that this multiplicity is one. Recall that $\alpha^\vee = \hat{\alpha}_1^\vee + \hat{\alpha}_2^\vee$ (see Remark 1). Hence inequalities (2) implies that $0 \leq l \leq \langle \nu, \alpha^\vee \rangle$. Therefore the dimension of $V_G(\nu)^{(T)\rho(\hat{v}\hat{\nu})}$ is one. By Section 2.2.2, we can choose a non-zero element f in the space $\mathbb{C}[G]^{(B_- \times T)(-n\nu, \rho(n\hat{v}\hat{\nu}))}$, for sufficiently large n. The subgroup $G_{\hat{v}}$ is a solvable subgroup of G containing T, so $G_{\hat{v}} = G_{\hat{v}}^u.T$, where $G_{\hat{v}}^u$ is the unipotent radical of $G_{\hat{v}}$. It is sufficient to prove that f is $G_{\hat{v}}^u$-invariant.

Consider the element $\iota(\bar{\tau})$: it is a non-zero element belonging to the one dimensional space: $\mathbb{C}[G]^{(B_- \times T_{\hat{v}})(-n\nu, \rho(n\hat{v}\hat{\nu}))}$. Hence, up to multiplying f by a non-zero scalar, we may assume that $f^n = \iota(\bar{\tau})$. We deduce that f^n is $G_{\hat{v}}^u$-invariant. Hence f is $G_{\hat{v}}^u$-invariant because $G_{\hat{v}}^u$ possesses no non-trivial character and the algebra $\mathbb{C}[G]$ is factorial.

2.2.4 Conclusion

We can now finish the proof of Theorem 2 by showing that $V_G(\nu)^* \subset V_{\hat{G}}(\hat{\nu})^*$. Consider the following morphisms:

$$
\begin{array}{c}
H^0(\hat{G}/\hat{B}, \mathcal{L}_{\hat{\nu}}) \\
\varphi \downarrow \\
H^0(X_{\hat{v}}, \mathcal{L}_{\hat{\nu}}) \\
\pi \uparrow\downarrow \\
H^0(X_{\hat{v}}^\circ, \mathcal{L}_{\hat{\nu}}) \\
\iota \uparrow\downarrow \\
\mathbb{C}[G]
\end{array}
$$

The morphisms π and ι are defined above and φ is the restriction morphism. By [Bri01, Corollary 8], the map φ is surjective. Since $H^0(\hat{G}/\hat{B}, \mathcal{L}_{\hat{\nu}})$ is isomorphic to $V_{\hat{G}}(\hat{\nu})^*$, it is sufficient to prove that $V_G(\nu)^* \subset H^0(X_{\hat{v}}, \mathcal{L}_{\hat{\nu}})$.

In Section 2.2.3, we showed that there exists a non-zero B^--equivariant section κ in $H^0(X_{\hat{v}}^\circ, \mathcal{L}_{\hat{\nu}})$ of weight $-\nu$. It remains to prove that κ extends to a regular section on $X_{\hat{v}}$ to conclude the proof. Since the space $H^0(X_{\hat{v}}^\circ, \mathcal{L}_{n\hat{\nu}})^{(B^-)n\nu}$ is one dimensional, we may assume that $\kappa^{\otimes n} = \bar{\tau}$, where $\bar{\tau}$ is the section obtained in 2.2.2, so $\kappa^{\otimes n}$ extends to a regular section on $X_{\hat{v}}$. The normality of $X_{\hat{v}}$ (see [Bri03, Theorem 1]) implies that κ extends to a regular section on $X_{\hat{v}}$.

Remark 3. *In general the multiplicity of $V_G(\nu)$ in $V_{\hat{G}}(\hat{\nu})$ does not equal one; but, as a consequence of the proof, the multiplicity of $V_G(\nu)^*$ in $H^0(X_{\hat{v}}, \mathcal{L}_{\hat{\nu}})$ is one.*

2.3 The example $G_2 \subset \mathrm{Spin}_7$

In this section $\hat{G} = \mathrm{Spin}_7$ and $G = G_2$. The first fundamental representation of G_2 has dimension 7 and induces an embedding of G_2 in SO_7. Since G_2 is simply connected this embedding can be raised to $\hat{G} = \mathrm{Spin}_7$. We use the numeration of Bourbaki [Bou02]. Denote by $\hat{s}_1$, $\hat{s}_2$ and $\hat{s}_3$ the simple reflections of $\hat{W}$. The fundamental weights of $\hat{G}$ and G are denoted by $\hat{\varpi}_i$ for $i = 1, 2, 3$ and by ϖ_i for $i = 1, 2$. Here ρ is characterized by: $\rho(\hat{\varpi}_1) = \varpi_1$, $\rho(\hat{\varpi}_2) = \varpi_2$ and $\rho(\hat{\varpi}_3) = \varpi_1$. We use the basis of fundamental weights for $\hat{G}$ and G to express the weights; in particular $\hat{\nu} = (\hat{\nu}_1, \hat{\nu}_2, \hat{\nu}_3)$ means $\hat{\nu} = \hat{\nu}_1 \varpi_1 + \hat{\nu}_2 \varpi_2 + \hat{\nu}_3 \varpi_3$.

We begin by giving the PRV components of any irreducible representation of $\hat{G}$.

Proposition 1. *Let $\hat{\nu} = (\hat{\nu}_1, \hat{\nu}_2, \hat{\nu}_3)$ be a dominant weight of $\hat{G}$ (i.e. $\hat{\nu}_i \geq 0$ for any i). Then the dominant weights of G, obtained as the restriction $\rho(\hat{w}\hat{\nu})$ of an element of the orbit $\hat{W}.\hat{\nu}$, are*

- $\nu^1 = (\hat{\nu}_1 + \hat{\nu}_3, \hat{\nu}_2)$;

- $\nu^2 = (-\hat{\nu}_1 + \hat{\nu}_3, \hat{\nu}_1 + \hat{\nu}_2)$ *if $\hat{\nu}_1 \leq \hat{\nu}_3$*
  or $\nu^2 = (\hat{\nu}_1 - \hat{\nu}_3, \hat{\nu}_2 + \hat{\nu}_3)$ if $\hat{\nu}_1 \geq \hat{\nu}_3$;

- $\nu^3 = (-\hat{\nu}_1 + \hat{\nu}_2 + \hat{\nu}_3, \hat{\nu}_1)$ *if $\hat{\nu}_1 \leq \hat{\nu}_2 + \hat{\nu}_3$*
  or $\nu^3 = (\hat{\nu}_1 - \hat{\nu}_2 - \hat{\nu}_3, \hat{\nu}_2 + \hat{\nu}_3)$ if $\hat{\nu}_1 \geq \hat{\nu}_2 + \hat{\nu}_3$;

- $\nu^4 = (-\hat{\nu}_1 + \hat{\nu}_2, \hat{\nu}_1)$ *if $\hat{\nu}_1 \leq \hat{\nu}_2$*
  or $\nu^4 = (\hat{\nu}_1 - \hat{\nu}_2, \hat{\nu}_2)$ if $\hat{\nu}_1 \geq \hat{\nu}_2$.

It may happen of course that some of the four PRV components above are the same.

Proof. First remark that, since $|\hat{W}/W| = 4$ we have at most 4 PRV components. The weights ν^i, for $i = 1, \cdots, 4$ in the statement of the proposition are dominant. We have to prove that they equal $\rho(\hat{w}\hat{\nu})$, for some $\hat{w}$. This is checked by:

- $\nu^1 = \rho(\hat{\nu})$;

- $\nu^2 = \rho(\hat{s}_1 \hat{\nu}) = \rho(-\hat{\nu}_1, \hat{\nu}_1 + \hat{\nu}_2, \hat{\nu}_3)$
  or $\nu^2 = \rho(\hat{s}_3 \hat{\nu}) = \rho(\hat{\nu}_1, \hat{\nu}_2 + \hat{\nu}_3, -\hat{\nu}_3)$;

- $\nu^3 = \rho(\hat{s}_1 \hat{s}_2 \hat{\nu}) = \rho(-\hat{\nu}_1 - \hat{\nu}_2, \hat{\nu}_1, 2\hat{\nu}_2 + \hat{\nu}_3)$
  or $\nu^3 = \rho(\hat{s}_3 \hat{s}_2 \hat{\nu}) = \rho(\hat{\nu}_1 + \hat{\nu}_2, \hat{\nu}_2 + \hat{\nu}_3, -2\hat{\nu}_2 - \hat{\nu}_3)$;

- $\nu^4 = \rho(\hat{s}_1 \hat{s}_2 \hat{s}_3 \hat{\nu}) = \rho(-\hat{\nu}_1 - \hat{\nu}_2 - \hat{\nu}_3, \hat{\nu}_1, 2\hat{\nu}_2 + \hat{\nu}_3)$
  or $\nu^4 = \rho(\hat{s}_3 \hat{s}_2 \hat{s}_3 \hat{\nu}) = \rho(\hat{\nu}_1 + \hat{\nu}_2 + \hat{\nu}_3, \hat{\nu}_2, -2\hat{\nu}_2 - \hat{\nu}_3)$.

$\square$

The set Δ_2 is reduced to the short root α_1. Then Theorem 1 gives line segments parallel to α_1 and having (at least) one PRV component as end point.

Corollary 1. *The irreducible representation $V_{G_2}(\nu)$ occurs with multiplicity at least one in $V_{\mathrm{Spin}_7}(\hat\nu)$ for all dominant weights ν contained in one of the three following line segments (distinct in general):*

- *the line segment whose end points are ν^1 and ν^2, of length $\min(\hat\nu_1, \hat\nu_3)$;*

- *the line segment whose end points are ν^3 and $\nu^{3'} = (\hat\nu_1 + 3\hat\nu_2 + \hat\nu_3, -\hat\nu_2)$, of length $\min(\hat\nu_1 + \hat\nu_2, 2\hat\nu_2 + \hat\nu_3)$;*

- *the line segment whose end points are ν^4 and $\nu^{4'} = (\hat\nu_1 + 3\hat\nu_2 + 2\hat\nu_3, -\hat\nu_2 - \hat\nu_3)$, of length $\min(\hat\nu_1 + \hat\nu_2 + \hat\nu_3, 2\hat\nu_2 + \hat\nu_3)$.*
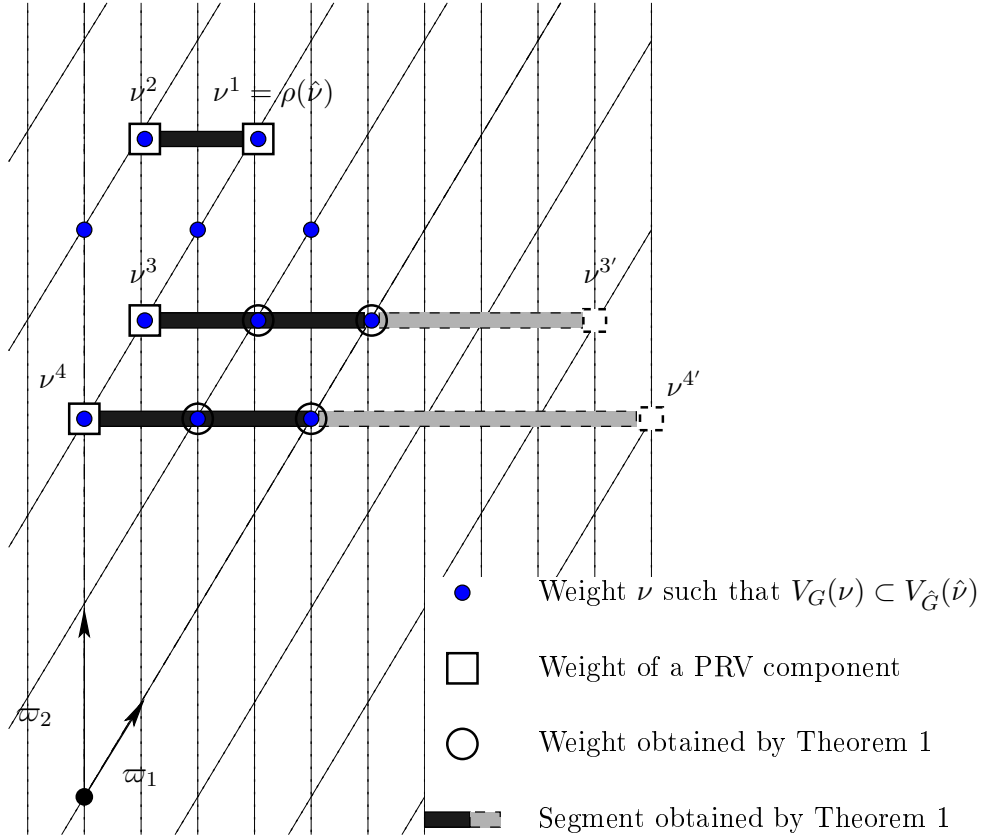

Figure 1: Illustration of Corollary 1

Remark that $\nu^{3'}$ and $\nu^{4'}$ are not dominant in general and are never strictly dominant. In particular the line segments with end points ν^3 and ν^4 intersect the wall of the dominant chamber corresponding to α_2.

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We illustrate this result in Figure 1 where $\hat{\nu} = (2, 2, 1)$.

Remark 4. *In that example, note that $\rho(\hat{\nu}) - \alpha_2$ is not a highest weight of a G-submodule of $V_{\hat{G}}(\hat{\nu})$. Nevertheless it is dominant and we have $\langle \hat{\nu}, \hat{\alpha}_2 \rangle = 2$. Hence we cannot expect to extend our result to the roots α in Δ_1.*

3   A generalization of Theorem 1

In this section, we come back to the general situation of connected reductive groups $G \subset \hat{G}$ such that $\hat{G}/G$ is spherical of minimal rank.

To state the result, we define the basic notion of hyperrectangle generated by a finite set of pairwise orthogonal line segments having one end point in common. For $1 \leq k \leq s$, let $I_k = [a, b_k]$ be k pairwise orthogonal line segments in an affine space. The hyperrectangle generated by the family $(I_k)_{1 \leq k \leq s}$ is

$$R = \{a + \sum_{k=1}^{s} \lambda_k \overrightarrow{ab_k} \mid \lambda_k \in [0, 1]\}.$$

We improve Theorem 1, by replacing the line segments $S_{\alpha, \hat{w}, \hat{\nu}}$ by hyperrectangles containing them.

Theorem 3. *Let $\hat{\nu}$ be a dominant weight of $\hat{G}$ and let $\hat{w} \in \hat{W}$ such that $\rho(\hat{w}\hat{\nu})$ is a dominant weight of G. Let $\alpha^1, \ldots, \alpha^s$ be s pairwise orthogonal simple roots of G in Δ_2 (with $s \geq 1$). Denote by $R_{\alpha^1, \ldots, \alpha^s, \hat{w}, \hat{\nu}}$ the hyperrectangle generated by the s pairwise orthogonal line segments $(S_{\alpha^i, \hat{w}, \hat{\nu}})_{i \in \{1, \ldots, s\}}$ (that have $\rho(\hat{w}\hat{\nu})$ as common end point).*

Then, for any dominant weight $\nu \in R_{\alpha^1, \ldots, \alpha^s, \hat{w}, \hat{\nu}} \cap (\rho(\hat{w}\hat{\nu}) + \Lambda)$, the irreducible G-module $V_G(\nu)$ occurs with multiplicity at least one in $V_{\hat{G}}(\hat{\nu})$.

Proof. The proof of this theorem is very similar to the proof of Theorem 1, so that we will only explain the main changes. First by a similar reasoning to that of Section 2.1, the proof reduce to the following: if $\nu = \rho(\hat{w}\hat{\nu}) + \sum_{i=1}^{s} k^i \alpha^i$, where the integers k^i satisfy three inequalities similar to inequalities (2), then $V_G(\nu)$ occurs in $V_{\hat{G}}(\hat{\nu})$.

For each $i \in \{1, \ldots, s\}$, there exist two simple roots $\hat{\alpha}_1^i$ and $\hat{\alpha}_2^i$ such that $\rho(\hat{\alpha}_1^i) = \rho(\hat{\alpha}_2^i) = \alpha^i$. Index them in such a way that, for any $i \in \{1, \ldots, s\}$,

$$\langle \hat{w}\hat{\nu}, (\hat{\alpha}_1^i)^{\vee} \rangle \leq \langle \hat{w}\hat{\nu}, (\hat{\alpha}_2^i)^{\vee} \rangle.$$

Arguing like in Section 2.2, we may assume that for $i \in \{1, \ldots, s\}$,

$$\langle \hat{w}\hat{\nu}, (\hat{\alpha}_1^i)^{\vee} \rangle > 0,$$

and hence that $l(s_{\hat{\alpha}_1^i} \hat{w}) = l(\hat{w}) + 1$.

Set $\hat{v} = s_{\hat{\alpha}_1^1} \ldots s_{\hat{\alpha}_1^s} \hat{w}$. Note that the simple reflections $s_{\hat{\alpha}_1^i}$ commute. Set $X = G/B \times \hat{G}/\hat{B}$, $S = \cap_{i=1}^{s} (\ker \alpha^i)^{\circ} \subset T$ and let C be the irreducible

component of X^S containing $(w_0 B/B, \hat{v} B/B)$. Observe that G^S/S is the product of s groups isomorphic to SL_2 or PSL_2. Note also that C is isomorphic to $((\mathbb{P}^1)^3)^s$, with the natural action of G^S/S. In particular, the inequalities satisfied by the integers k^i implies that $C^{ss}(\mathcal{M}_{|C}, G^S) \neq \varnothing$. Then $X^{ss}(\mathcal{M}) \cap C \neq \varnothing$. We deduce that there exists a positive integer n and a G-section $\bar{\tau}$ of $\mathcal{L}_{n\hat{\nu}}$ which is a eigenvector of weight $-n\nu$ for B^- and whose restriction to $X_{\hat{v}}$ is not identically zero.

Since the simple roots α_1^i are pairwise orthogonal, for any $0 \leq l^i \leq \langle \nu, \alpha^i \rangle$, the weight $\nu - \sum_i l^i \alpha^i$ of T occurs in $V_H(\nu)$ with multiplicity exactly one. We deduce (see Section 2.2.2) that $\dim H^\circ(X_{\hat{\nu}}^\circ, \mathcal{L}_{\hat{\nu}})^{(T)_{\nu - \Sigma_i l^i \alpha^i}} = 1$. We conclude as in Section 2.2.4. $\qquad \square$

We illustrate Theorem 3 by the following example.

Example 1. *Let $G = \mathrm{Sp}_6$ diagonally embedded in $\hat{G} = \mathrm{Sp}_6 \times \mathrm{Sp}_6$. In that case, the first and the third simple roots of Sp_6 are orthogonal. Consider the dominant weight $\hat{\nu} = ((2,0,4),(2,0,2))$ in the basis of fundamental weights. The irreducible components $V_G(\nu)$ of the tensor product $V_{\hat{G}}(\hat{\nu})$, obtained by Theorem 3, are represented in Figure 2. On this figure, the standard basis is the basis denoted by $(\epsilon_1, \epsilon_2, \epsilon_3)$ in Bourbaki.*

In this example, we observe that Theorem 3 gives two new components in addition of those obtained by Theorem 1. In fact, Theorem 3 gives here two rectangles, one of them being truncated by the dominant chamber (see Figure 2).

4 A specific result for tensor product decomposition

From now on, we specialize to the case where G is diagonally embedded in $\hat{G} = G \times G$. Fix two dominant weights of G. Then, Theorem 1 produces line segments in $X(T) \otimes \mathbb{Q}$ parallel to any simple roots of G. In this section, we use the symmetry induced by duality, in order to produce similar line segments whose direction is a root but not necessarily a simple one.

Let μ and ν be two dominant weights of G. We say that a weight λ, not necessarily dominant, is a *virtual PRV component* (with respect to μ and ν) if there exists v and w in W such that $\lambda = v\mu + w\nu$. Now we state the main result of this section.

Theorem 4. *Let v and w be two elements of W, and let β be a root of G.*

The line through the virtual PRV component $\lambda_1 = v\mu + w\nu$ in the direction β contains the following four virtual PRV components: λ_1, $\lambda_2 = v\mu + s_\beta w\nu$, $\lambda_3 = s_\beta v\mu + w\nu$ and $\lambda_4 = s_\beta v\mu + s_\beta w\nu$. Choose i and j such that $\langle \lambda_i, \beta^\vee \rangle$ and $\langle \lambda_j, \beta^\vee \rangle$ are two largest among the family $(\langle \lambda_l, \beta^\vee \rangle)_{l=1,2,3,4}$. Denote by $S_{\beta, v, w, \mu, \nu}$ the line segment in $X(T) \otimes \mathbb{Q}$ whose end points are λ_i and λ_j.
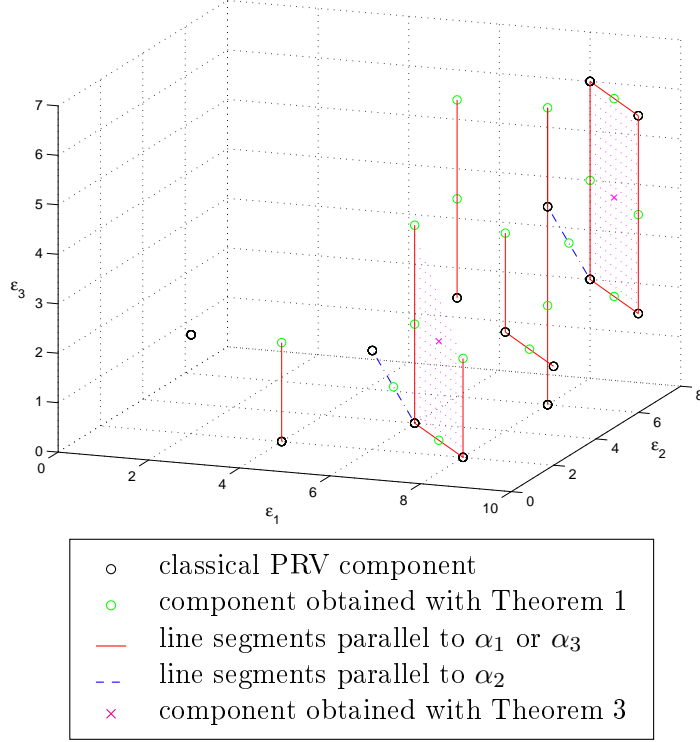
Figure 2: some weights of $V_{\mathrm{Sp}_6}(2,0,4) \otimes V_{\mathrm{Sp}_6}(2,0,2)$

Suppose that at least one of the following roots β, $v^{-1}\beta$ or $w^{-1}\beta$ is simple. Then, for any dominant weight $\lambda \in S_{\beta,v,w,\mu,\nu} \cap (v\mu + w\nu + \Lambda)$, the irreducible G module $V_G(\lambda)$ occurs with multiplicity at least one in $V_G(\mu) \otimes V_G(\nu)$.

Remark 5.   *(i) If $\langle \lambda_i, \beta^\vee \rangle = \langle \lambda_j, \beta^\vee \rangle$ then the line segment is reduced to a point.*

  (ii) Note that in the case of a simple direction, if $S_{\beta,v,w,\mu,\nu} \cap X^+(T)$ is not empty, the line segment $S_{\beta,v,w,\mu,\nu}$ defined as above always contains a PRV component (see [MPR11, Proposition 2]). But if the direction is not simple, there exist line segments intersecting $X^+(T)$ and that contain no PRV component (see the example at the end of the section).

As in the general case, we rewrite Theorem 4 as follows.

Theorem 5. *Let λ, μ, and ν be three dominant weights of G, let v, w be two elements of W, and let β be a root of G such that one of the following roots β, $v^{-1}\beta$ or $w^{-1}\beta$ is simple. Suppose that $\lambda = v\mu + w\nu - k\beta$ where k is*

an integer satisfying:

$$k \geq 0, \tag{3a}$$

$$k \leq \langle v\nu, \, \beta^\vee \rangle, \tag{3b}$$

$$k \leq \langle w\mu, \, \beta^\vee \rangle. \tag{3c}$$

Then, the irreducible G-module $V_G(\lambda)$ occurs with multiplicity at least one in $V_G(\mu) \otimes V_G(\nu)$.

Proof. The case where β is simple is not new: it is the second result of our preceding work [MPR11], or the simple translation of Theorem 2. The two other cases can be deduce from this one. Indeed, assume that $\alpha = v^{-1}\beta$ is simple. Rewrite the condition

$$\lambda = v\mu + w\nu - k\beta \tag{4}$$

as $\mu = v^{-1}\lambda - v^{-1}w\nu + k\alpha$, or equivalently as

$$\mu = v^{-1}\lambda + s_\alpha v^{-1}ww_0(-w_0\nu) + (k - \langle v^{-1}w\nu, \, \alpha^\vee \rangle)\alpha. \tag{5}$$

Setting $\nu' = -w_0\nu$, $v' = v^{-1}$, $w' = s_\alpha v^{-1}ww_0$ and $k' = \langle v^{-1}w\nu, \, \alpha^\vee \rangle - k$, equation (5) becomes:

$$\mu = v'\lambda + w'\nu' - k'\alpha \, . \tag{6}$$

One can check that conditions (3) are equivalent to

$$k' \geq 0,$$
$$k' \leq \langle v'\nu', \, \alpha^\vee \rangle,$$
$$k' \leq \langle w'\lambda', \, \alpha^\vee \rangle.$$

Hence, Theorem 2 or [MPR11, Theorem 2] implies that $V_G(\mu)$ appears in the tensor product $V_G(\lambda) \otimes V_G(\nu')$. But as $V_G(\nu') = V_G(-w_0\nu) = V_G(\nu)^*$, we deduce that $V_G(\lambda)$ appears in $V_G(\mu) \otimes V_G(\nu)$. □

Proof of: Theorem 5 implies Theorem 4. By eventually changing v by $s_\beta v$ and w by $s_\beta w$, we may assume that the two values $\langle v\nu, \, \beta^\vee \rangle$ and $\langle w\mu, \, \beta^\vee \rangle$ are non-negative. Assume also that $\langle v\nu, \, \beta^\vee \rangle \geq \langle w\mu, \, \beta^\vee \rangle$ so that two largest values among the family $(\langle \lambda_l, \, \beta^\vee \rangle)_{l=1,2,3,4}$ are $\langle \lambda_1, \, \beta^\vee \rangle$ and $\langle \lambda_2, \, \beta^\vee \rangle$. A weight λ belongs to the line segment $S_{\beta,v,w,\mu,\nu} = [\lambda_1, \lambda_2]$ if and only if $\lambda = v\mu + w\nu - k\beta$ with $0 \leq k \leq \langle v\nu, \, \beta^\vee \rangle$ and hence we can conclude with Theorem 5. □

The following theorem is obtained by combining Theorem 3 and Theorem 4.

Theorem 6. *Let v and w be two elements of the Weyl group, and let $(\beta^i)_{1 \leq i \leq s}$ be s pairwise orthogonal roots. Suppose that there exists $u \in \{1, v, w\}$ such that for any $i \in \{1, \ldots, s\}$, $u^{-1}\beta^i$ is simple.*

Denote by $R_{\beta^1, \ldots, \beta^s, v, w, \mu, \nu}$ the hyperrectangle generated by the s pairwise orthogonal line segments $(S_{\beta^i, v, w, \mu, \nu})_{i \in \{1, \ldots, s\}}$ defined in Theorem 4.

Then for any dominant weight $\lambda \in R_{\beta^1, \ldots, \beta^s, v, w, \mu, \nu} \cap (v\mu + w\nu + \Lambda)$, the irreducible G-module $V_G(\lambda)$ occurs with multiplicity at least one in $V_G(\mu) \otimes V_G(\nu)$.

Example 2. *Consider the tensor product $V_{\mathrm{Sp}_6}(2,1,0) \otimes V_{\mathrm{Sp}_6}(0,3,1)$. The irreducible components of this tensor product obtained by Theorem 4 are represented in Figure 3.*
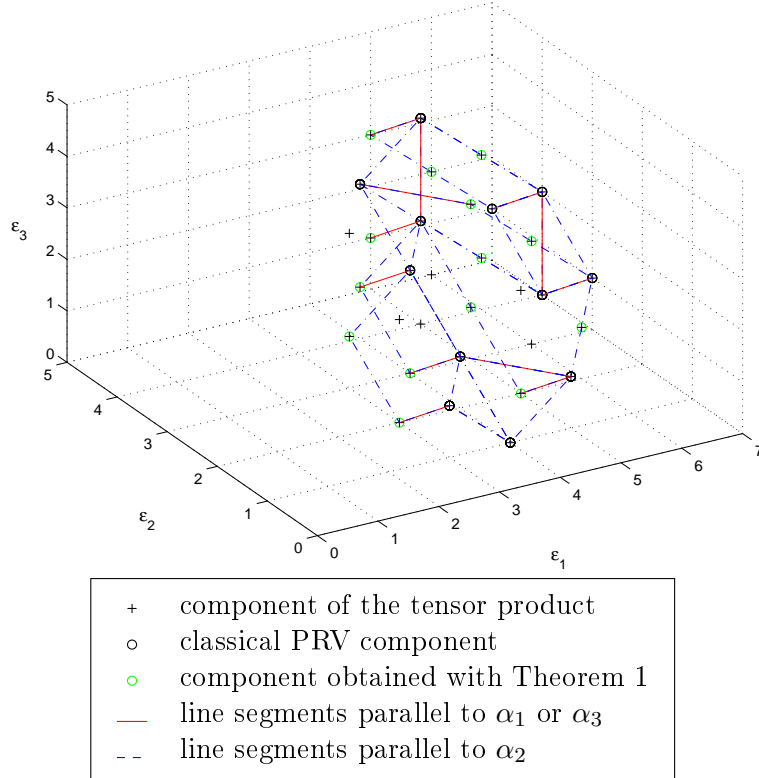

Figure 3: Weights of $V_{\mathrm{Sp}_6}(2,1,0) \otimes V_{\mathrm{Sp}_6}(0,3,1)$

There are 12 PRV components, 7 more weights given by Theorem 1, 7 more weighs given by Theorem 4, and no more with Theorem 6. To summarize, we obtain 26 weights (represented in Figure 3) among the 32 existing ones.

We observe a new phenomenon that cannot occur with Theorem 1: the two end points of a line segment can be both outside the dominant chamber

(for example, the one in the lower left in Figure 3).

Return to the example $V_{\mathrm{Sp}_6}(2,0,4) \otimes V_{\mathrm{Sp}_6}(2,0,2)$ seen in Example 1. In this case, there are 14 PRV components, plus 17 new dominant weights with Theorem 1, plus 2 new ones with Theorem 3, plus 16 new ones with Theorem 4, plus 9 new ones with Theorem 6. To summarize, the results of this paper give 58 dominant weights among the 83 existing ones.

5 Appendix

We recall the list of spherical homogeneous spaces $\hat{G}/G$ of minimal rank, such that $\hat{G}$ is semi-simple simply connected and G is simple (and $\hat{G} \neq G$). All spherical homogeneous spaces of minimal rank are obtained from these special homogeneous spaces and tori by products, finite covers and parabolic inductions.

$\hat{G}/G$	Δ_1	Δ_2
$(G \times G)/G$	$\varnothing$	Δ
$\mathrm{SL}_{2\mathrm{n}}/\mathrm{Sp}_{2\mathrm{n}}$, $n \geq 2$	$\{\alpha_n\}$	$\{\alpha_1, \ldots, \alpha_{n-1}\}$
$\mathrm{Spin}_{2\mathrm{n}}/\mathrm{Spin}_{2\mathrm{n}-1}$, $n \geq 4$	$\{\alpha_1, \ldots, \alpha_{n-2}\}$	$\{\alpha_{n-1}\}$
$\mathrm{Spin}_7/\mathrm{G}_2$	$\{\alpha_1\}$	$\{\alpha_2\}$
E_6/F_4	$\{\alpha_1, \alpha_2\}$	$\{\alpha_3, \alpha_4\}$

We recall that we use the numeration of Bourbaki [Bou02] for root systems.

And by lack of reference, we collect here technical results.

We begin with a result on the length of Weyl group elements.

Lemma 1. *Let G be a reductive group. Fix a Borel subgroup containing a maximal torus. Let W be the Weyl group of G. For any simple root α, denote by s_α the corresponding reflection in W.*

Let λ be a dominant weight and u be an element of W. Let β be a simple root such that $\langle u\lambda, \beta^\vee \rangle > 0$. Then $l(s_\beta u) = l(u) + 1$. In particular we also have that if $\langle u\lambda, \beta^\vee \rangle < 0$ then $l(u) = l(s_\beta u) + 1$.

Proof. The length of u equals the length of u^{-1}. It is the cardinality of

$$\{\gamma \text{ positive root } \mid u^{-1}\gamma \text{ is a negative root}\}.$$

Let γ be a positive root such that $u^{-1}\gamma$ is a negative root. We claim that γ cannot be the simple root β. Indeed, since $\langle \lambda, u^{-1}\beta^\vee \rangle = \langle u\lambda, \beta^\vee \rangle > 0$ and λ is dominant, $u^{-1}\beta$ is positive. Then the application

$$\{\gamma > 0 \mid u^{-1}\gamma < 0\} \longrightarrow \{\gamma > 0 \mid u^{-1}s_\beta\gamma < 0\}$$
$$\gamma \longmapsto s_\beta\gamma$$

is well-defined because $u^{-1}s_\beta s_\beta \gamma = u^{-1}\gamma$ and $s_\beta \gamma$ is a positive root for all positive roots γ different from β. This application is also clearly injective, so that the cardinality of the second set, which is the length of $u^{-1}s_\beta$ and also $s_\beta u$, is bigger than the length of u.

But we have either $l(s_\beta u) = l(u) + 1$ or $l(s_\beta u) = l(u) - 1$ because s_β is a simple reflection, so the proposition follows. $\qquad\square$

Now, we give a result on the maximal torus of a spherical reductive subgroup.

Lemma 2. *Suppose that G is a spherical reductive subgroup in a reductive group $\hat{G}$. Let T be a maximal torus of G. Then, the centralizer of T in $\hat{G}$ is a maximal torus of $\hat{G}$.*

Proof. By [Bri86], since G is spherical, it has finitely many orbits in $\hat{G}/\hat{B}$. Since G is reductive, T has finitely many fixed points in any G-orbit. Hence T has finitely many fixed points in $\hat{G}/\hat{B}$. It follows that T is regular in $\hat{G}$. $\qquad\square$

As a consequence of the preceding corollary, we have the following result.

Corollary 2. *In the situation of the preceding lemma, the Weyl group of G is canonically a subgroup of the Weyl group of $\hat{G}$.*

Proof. The lemma implies that the normalizer of T in G normalizes $\hat{T}$. In particular, it is contained in the normalizer of $\hat{T}$ in $\hat{G}$. The injection of the corollary is obtained by taking the quotient of this inclusion by $\hat{T}$. $\qquad\square$

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