Vanishing theorem for the cohomology of line bundles on Bott-Samelson varieties

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Abstract

We use the Grossberg-Karshon's degeneration of Bott-Samelson varieties to toric varieties and the description of cohomolgy of line bundles on toric varieties to deduce vanishing results for the cohomology of lines bundles on Bott-Samelson varieties.

Introduction

Bott-Samelson varieties were originally defined as desingularizations of Schubert varieties and were used to describe the geometry of Schubert varieties. In particular, the cohomology of some line bundles on Bott-Samelson varieties were used to prove that Schubert varieties are normal, Cohen-Macaulay and with rational singularities (see for example [BK05]). In this paper, we will be interested in the cohomology of all line bundles of Bott-Samelson varieties.

We consider a Bott-Samelson variety $Z(\tilde{w})$ over an algebraically closed field $\Bbbk$ associated to an expression $\tilde{w} = s_{\beta_1} \dots s_{\beta_N}$ of an element w in the Weyl group of a Kac-Moody group G over $\Bbbk$ (see Definition 1.1 (i)).

In the case where G is semi-simple, N. Lauritzen and J.F. Thomsen proved, using Frobenius splitting, the vanishing of the cohomology in positive degree of line bundles on $Z(\tilde{w})$ of the form $\mathcal{L}(-D)$ where $\mathcal{L}$ is any globally generated line bundle on $Z(\tilde{w})$ and D a subdivisor of the boundary of $Z(\tilde{w})$ corresponding to a reduced expression of w [LT04, Th7.4]. The aim of this paper is to give the vanishing in some degrees of the cohomology of any line bundles on $Z(\tilde{w})$.

Let us define, for all $\epsilon = (\epsilon_k)_{k\in\{1,\dots,N\}} \in \{+,-\}^N$ and for all integers $1 \leq i < j \leq N$,

$$\alpha_{ij}^{\epsilon} := \langle \beta_i^{\vee}, (\prod_{i<k<j,\, \epsilon_k=-} s_{\beta_k})(\beta_j)\rangle.$$

These integers are natural geometric invariants of the Bott-Samelson variety, they also appear, for example, in [Wi06, Theorem 3.21] in product formula in the equivariant cohomology of complex Bott-Samelson varieties .

Since $Z(\tilde{w})$ is smooth, we can consider divisors instead of line bundles. Thus, let us denote by $Z_1, \dots, Z_N$ the natural basis of divisors of $Z(\tilde{w})$ (see Definition 1.1 (ii)). Let $D := \sum_{i=1}^{N} a_i Z_i$ be any divisor of $Z(\tilde{w})$.

Let $i \in \{1,\dots,N\}$. We say that D satisfies condition (C_i^+) if for all $\epsilon \in \{+,-\}^N$, we have

$$C_i^\epsilon := a_i + \sum_{j>i,\,\epsilon_j=+} \alpha_{ij}^\epsilon a_j \geq -1$$

and we say that D satisfies condition (C_i^-) if for all $\epsilon \in \{+,-\}^N$, we have

$$C_i^\epsilon := a_i + \sum_{j>i,\,\epsilon_j=+} \alpha_{ij}^\epsilon a_j \leq -1.$$

The main result of this paper is the following.

Theorem 0.1. *Let $X = Z(\tilde{w})$ be a Bott-Samelson variety and D a divisor of $Z(\tilde{w})$. Let $\eta \in \{+,-,0\}^N$. Define two integers $\eta^+ := \sharp\{1 \leq j \leq N \mid \eta_j = +\}$ and $\eta^- := \sharp\{1 \leq j \leq N \mid \eta_j = -\}$.*
Suppose that D satisfies conditions $(C_i^{\eta_i})$ for all $i \in \{1,\dots,N\}$ such that $\eta_i \neq 0$.
Then $H^i(X,D) = 0$, for all $i < \eta^-$ and for all $i > N - \eta^+$.

Let us remark that, Conditions (C_N^+) and (C_N^-) are respectively $a_N \geq -1$ and $a_N \leq -1$, so that η_N can always be chosen different from 0. Thus, for any divisor D of $Z(\tilde{w})$, Theorem 0.1 gives the vanishing of the cohomology of D in at least one degree.

Although Theorem 0.1 gives us a lot of cases of vanishing, it does not permit to recover all the result of N. Lauritzen and J.P. Thomsen. See Example 2.11 to illustrate this fact.

However, for lots of divisors, Theorem 0.1 gives the vanishing of their cohomology in all degrees except one. More precisely, we have the following.

Corollary 0.2. *Let $D = \sum_{i=1}^N a_i Z_i$ be a divisor of $X = Z(\tilde{w})$. Suppose that, for all $i \in \{1,\dots,N\}$, one of the following two conditions $\tilde{C}_i^+$ and $\tilde{C}_i^-$ is satisfied:*

$$\begin{array}{ll} \tilde{C}_i^+: & a_i \geq -1 + \max_{\epsilon\in\{+,-\}^N}(-\sum_{j>i,\,\epsilon_j=+} \alpha_{ij}^\epsilon a_j) \\ \tilde{C}_i^-: & a_i \leq -1 + \min_{\epsilon\in\{+,-\}^N}(-\sum_{j>i,\,\epsilon_j=+} \alpha_{ij}^\epsilon a_j) \end{array}.$$

Then, $H^i(X,D) = 0$ for all $i \neq \sharp\{1 \leq j \leq N \mid \tilde{C}_j^-$ is satisfied $\}$.

Let us remark that, for all $\eta \in \{+,-\}^N$, the set of points $(a_i) \in \mathbb{Z}^N$ satisfying $\tilde{C}_j^{\eta_j}$ for all $j \in \{1,\dots,N\}$ is a non empty cone. So that Corollary 0.2 can be applied to infinitely many divisors.

The strategy of the proof of Theorem 0.1 is the following. In Section 1, we define and describe a family of deformations with general fibers the Bott-Samelson variety and with special fiber a toric variety. The toric variety we obtain is a Bott tower, its fan has a simple and well understood structure (for example it has $2N$ cones of dimension 1 and 2^N cones of dimension N). In fact, we rewrite a part of the theory of M. Grossberg and Y. Karshon [GK94] on Bott-towers from an algebraic point of view. In Section 2, we describe how to compute the cohomology of divisors on the special fiber and we prove the same vanishings as in Theorem 0.1 but for divisors on this toric variety. Then Theorem 0.1 is a direct consequence of the semicontinuity Theorem [Ha77, III 12.8].

1 Toric degeneration of Bott-Samelson varieties

In this section we rewrite the theory of M. Grossberg and Y. Karshon [GK94] on Bott-towers, in the case of Bott-Samelson varieties and from an algebraic point of view over any algebraically closed field.

Let $A = (a_{ij})_{1 \leq i,j \leq n}$ be a generalized Cartan matrix, *i.e.* such that (for all i, j) $a_{ii} = 2$, $a_{ij} \leq 0$ for $i \neq j$, and $a_{ij} = 0$ if $a_{ji} = 0$. Let G be the "maximal" Kac-Moody group over $\Bbbk$ associated to A constructed in [Ku02, Section 6.1] (see [Ti81a] and [Ti81b] in arbitrary characteristic). Note that, in the finite case, G is the simply-connected semisimple algebraic group over $\Bbbk$. Denote by B the standard Borel subgroup of G containing the standard maximal torus T. Let $\alpha_1, \ldots, \alpha_n$ be the simple roots of (G, B, T) and $s_{\alpha_1}, \ldots, s_{\alpha_n}$ the associated simple reflections generating the Weyl Group W. For all $i \in \{1, \ldots, n\}$, denote by $P_{\alpha_i} := B \cup B s_{\alpha_i} B$ the minimal parabolic subgroup containing B associated to α_i. Let $w \in W$, an expression of w is a sequence $(s_{\beta_1}, \ldots, s_{\beta_N})$ of simple reflections $s_{\beta_1}, \ldots, s_{\beta_N}$ such that $w = s_{\beta_1} \ldots s_{\beta_N}$. An expression of w is said to be reduced if the number N of simple refections is minimal. In that case N is called the lenght of w and equals the dimension of the Schubert variety $X(w)$. Let $w \in W$ and $\tilde{w} := s_{\beta_1} \ldots s_{\beta_N}$ be an expression (not neccessarily reduced) of w. For all i and j in $\{1, \ldots, N\}$, denote by β_{ij} the integer $\langle \beta_i^\vee, \beta_j \rangle$.

Definition 1.1. (i) The Bott-Samelson variety associated to $\tilde{w}$ is

$$Z(\tilde{w}) := P_{\beta_1} \times^B \cdots \times^B P_{\beta_N}/B$$

where the action of B^N on $P_{\beta_1} \times \cdots \times P_{\beta_N}$ is defined by

$$(p_1, \ldots, p_N).(b_1, \ldots, b_N) = (p_1 b_1, b_1^{-1} p_2 b_2, \ldots, b_{N-1}^{-1} p_N b_N),\ \forall p_i \in P_{\beta_i},\ \forall b_i \in B.$$

(ii) For all $i \in \{1, \ldots, N\}$, we denote by Z_i the divisor of $Z(\tilde{w})$ defined by $\{(p_1, \ldots, p_N) \in Z(\tilde{w}) \mid p_i \in B\}$. Thus $(Z_i)_{i \in \{1,\ldots,N\}}$ is a basis of the Picard group of $Z(\tilde{w})$, and if $\tilde{w}$ is reduced it is the basis of effective divisor [LT04, Section 3].

In order to define a toric degeneration of a Bott-Samelson variety, we need to introduce particular endomorphisms of G and B.

Since the simple roots are linearly independent elements in the character group of G, one can choose a positive integer q and an injective morphism $\lambda : \Bbbk^* \longrightarrow T$ such that for all $i \in \{1, \ldots, n\}$ and all $u \in \Bbbk^*$, $\alpha_i(\lambda(u)) = u^q$. And let us define, for all $u \in \Bbbk^*$,

$$\begin{array}{cccc} \tilde{\psi}_u : & G & \longrightarrow & G \\ & g & \longmapsto & \lambda(u) g \lambda(u)^{-1}. \end{array}$$

The morphism ψ from $\Bbbk^*$ to the set of endomorphisms of B defined by $\psi(u) = \tilde{\psi}_{u|B}$ can be continuously extended to 0. Indeed, the unipotent radical U of B lives in a group (denoted by $U^{(1)}$ in [Ti81b]) where the action of $t \in T$ by conjugation is, on some generators (except the identity), the multiplication by some positive powers of $\alpha_i(t)$ for some $i \in \{1, \ldots, n\}$. Then for all $x \in U$, $\psi(u)$ goes to the identity when u goes to zero.

We denote, for all $u \in \Bbbk$, by ψ_u the morphism $\psi(u)$. Remark that ψ_0 is the projection from B to T.

We are now able to give the following

Definition 1.2. (i) Let $\mathfrak{X} \longrightarrow \Bbbk$ be the variety defined by

$$\mathfrak{X} := \Bbbk \times P_{\beta_1} \times \cdots \times P_{\beta_N}/B^N$$

where the action of B^N on $\Bbbk \times P_{\beta_1} \times \cdots \times P_{\beta_N}$ is defined by $\forall u \in \Bbbk$, $\forall p_i \in P_{\beta_i}$, $\forall b_i \in B$,

$$(u, p_1, \ldots, p_N).(b_1, \ldots, b_N) = (u, p_1 b_1, \psi_u(b_1)^{-1} p_2 b_2, \ldots, \psi_u(b_{N-1})^{-1} p_N b_N).$$

(ii) For all $i \in \{1, \ldots, N\}$, we denote by $\mathcal{Z}_i$ the divisor of $\mathfrak{X}$ defined by

$$\{(u, p_1, \ldots, p_N) \in \mathfrak{X} \mid p_i \in B\}.$$

Since $\mathfrak{X}$ is integral and $\mathfrak{X} \longrightarrow \Bbbk$ is surjective, $\mathfrak{X}$ is a flat family over $\Bbbk$ [Ha77, Chap.III Prop. 9.7]. For all $u \in \Bbbk$, we denote by $\mathfrak{X}(u)$ the fiber of $\mathfrak{X} \longrightarrow \Bbbk$ over u.

Proposition 1.3. *(i) For all $u \in \Bbbk^*$, $\mathfrak{X}(u)$ is isomorphic to the Bott-samelson variety $Z(\tilde{w})$ such that, for all $i \in \{1, \ldots, N\}$, the divisor $\mathcal{Z}_i(u) := \mathfrak{X}(u) \cap \mathcal{Z}_i$ corresponds to the divisor Z_i of $Z(\tilde{w})$.*

(ii) There is a natural action of the torus $(\Bbbk^*)^N$ on $\mathfrak{X}(0)$ such that an open orbit is isomorphic to $(\Bbbk^*)^N$.

We will see later that $\mathfrak{X}(0)$ is a smooth toric variety with this action.

Proof. (i) Remark first that $\mathfrak{X}(1)$ is by definition the Bott-Samelson variety and, that for all $i \in \{1, \ldots, N\}$, $\mathcal{Z}_i(1) = Z_i$. Now let $u \in \Bbbk^*$ and check that

$$\begin{array}{cccc} \theta_u : & \mathfrak{X}(1) & \longrightarrow & \mathfrak{X}(u) \\ & (p_1, \ldots, p_N) & \longmapsto & (p_1, \tilde{\psi_u}(p_2), \tilde{\psi_u}^2(p_3), \ldots, \tilde{\psi_u}^{N-1}(p_N)). \end{array}$$

is well-defined and is an isomorphism. Moreover, for all $i \in \{1, \ldots, N\}$, p_i is in B if and only if $\tilde{\psi}_u(p_i)$ is in B, so that $\theta_u(Z_i) = \mathcal{Z}_i(u)$.

(ii) Let T_{β_i} be the maximal subtorus of T acting trivially on $P_{\beta_i}/B \simeq \mathbb{P}^1_{\Bbbk}$. Now, since $\psi_0(b)$ commutes with T for all $b \in B$, one can define an effective action of $\prod_{i=1}^N T/T_{\beta_i} \simeq (\Bbbk^*)^N$ on $\mathfrak{X}(0)$ as follows

$$\forall t_i \in T, \forall p_i \in P_{\beta_i}, (t_1, \ldots, t_N).(p_1, \ldots, p_N) = (t_1 p_1 t_1^{-1}, t_2 p_2 t_2^{-1}, \ldots, t_N p_N t_N^{-1}).$$

Moreover, since $T/T_{\beta_i} \simeq \Bbbk^*$ acts on $P_{\beta_i}/B \simeq \mathbb{P}^1_{\Bbbk}$ with an open orbit, $(\Bbbk^*)^N$ acts also with an open orbit in $\mathfrak{X}(0)$.

□

For the general theory of toric varieties, one can refer to [Fu93] or [Od88]. But let us recall briefly the principal points of this theory. A toric variety of dimension N is a normal variety with the action of $(\Bbbk^*)^N$ such that it has an open orbit isomorphic to $(\Bbbk^*)^N$. Toric varieties are classified in terms of fans: a fan $\mathbb{F}$ of $\mathbb{R}^N$ is a set of cones generated by finitely many lattice points (*i.e.* in $\mathbb{Z}^N$), containing no lines and such that all faces of a cone in $\mathbb{F}$ are also in $\mathbb{F}$, and such that the cones of $\mathbb{F}$ intersect along their faces. Let X be a toric variety of dimension N. The corresponding fan is constructed as follows. For all affine $(\Bbbk^*)^N$-stable subvarieties X' of X of dimension N (not necessarily closed), consider the set $C_{X'}$ of weights of $(\Bbbk^*)^N$ in the ring $\Bbbk[X']$. Then $\mathbb{R}_+C_{X'}$ is a cone of $\mathbb{R}^N$ generated by finitely many lattice points. Their duals are cones generated by finitely many lattice points, containing no lines (the dual of C is defined by the set $\{v \in \mathbb{R}^N \mid \forall u \in C \, \langle v, u\rangle \geq 0\}$). And they form a fan of $\mathbb{R}^N$, that is the fan associated to X. Remark that a fan is entirely defined by its maximal cones (which correspond to the closed affine $(\Bbbk^*)^N$-stable subvarieties X' of X of dimension N), so that we can only consider maximal cones. Moreover there is a natural one to one correspondence between $(\Bbbk^*)^N$-stable divisor of X and arrows of the fan of X, given via valuation of divisors.

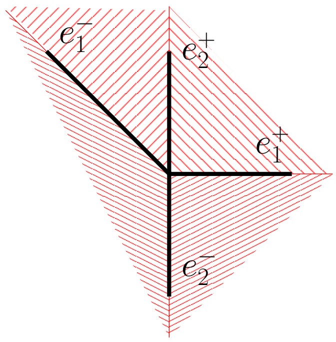
Proposition 1.4. *(i) $\mathfrak{X}(0)$ is a smooth toric variety.*

(ii) Let $(e_1^+, \dots, e_N^+)$ be a basis of $\mathbb{Z}^N$. Define, for all $i \in \{1, \dots, N\}$, the vector $e_i^- := -e_i^+ - \sum_{j>i} \beta_{ij} e_j^+$.

Then a fan $\mathbb{F}$ of $\mathfrak{X}(0)$ consists of cones generated by subsets of $\{e_1^+, \dots, e_N^+, e_1^-, \dots, e_N^-\}$ containing no subset of the form $\{e_i^+, e_i^-\}$. (In other words,the fan whose maximal cones are the cones generated by $e_1^{\epsilon_1}, \dots, e_N^{\epsilon_N}$ with $\epsilon \in \{+,-\}^N$.)

Moreover, for all $i \in \{1, \dots, N\}$, $\mathcal{Z}_i(0)$ is the irreducible $(\Bbbk^*)^N$-stable divisor of $\mathfrak{X}(0)$ corresponding to the one dimensional cone of $\mathbb{F}$ generated by e_i^+ and these divisors form a basis of the divisor classes of $\mathfrak{X}(0)$.

Example 1.5. If $G = \mathrm{SL}(3)$ and $\tilde{w} = s_{\alpha_1} s_{\alpha_1}$, we have the following fan.



In fact, one can prove that the Bott-Samelson variety of Example 1.5 is isomorphic the toric variety $\mathfrak{X}(0)$. But this is not the general case. For example, if $G = \mathrm{SL}(2)$ and $\tilde{w} = s_{\alpha_1} s_{\alpha_2}$, $Z(\tilde{w})$ is a toric variety (it is in fact $\mathbb{P}^1_\Bbbk \times \mathbb{P}^1_\Bbbk$) but it is not isomorphic to $\mathfrak{X}(0)$. And, if $G = \mathrm{SL}(4)$ and $\tilde{w} = s_{\alpha_2} s_{\alpha_1} s_{\alpha_3} s_{\alpha_2}$, then $Z(\tilde{w})$ is not a toric variety.

Proof. We will prove (i) and (ii) together. Let us first write a few technical results. For all simple roots α, there exists a unique closed subgroup $\mathcal{U}_\alpha$ of G and an isomorphism

$$u_\alpha : \mathbb{G}_a \longrightarrow \mathcal{U}_\alpha \text{ such that } \forall t \in T, \forall x \in \Bbbk, \, t u_\alpha(x) t^{-1} = u_\alpha(\alpha(t)x).$$

Moreover, the u_α can be chosen such that $n_\alpha := u_\alpha(1)u_{-\alpha}(-1)u_\alpha(1)$ is in the normalizer of T in G and has image s_α in W. And for all $x \in \Bbbk^*$ we have

$$u_{-\alpha}(x)u_\alpha(-x^{-1})u_{-\alpha}(x) = \alpha^\vee(x^{-1})n_{-\alpha}.$$

See [Sp98, Chapter8] for the finite case. And we can reduce from the general case to the finite case by construction of G.

Then for all $x \in \Bbbk^*$ we also have

$$n_{-\alpha}u_{-\alpha}(-x) = \alpha^\vee(x)u_{-\alpha}(x)u_\alpha(-x^{-1}) = u_{-\alpha}(x^{-1})\alpha^\vee(x)u_\alpha(-x^{-1}). \quad (1.5.1)$$

Remark also that, for all simple root α, the subgoup $\mathcal{U}_{-\alpha}$ is a subgroup of P_α and $n_{-\alpha} \in P_\alpha$.

Then, for all $\epsilon \in \{0,1\}^N$, we can define an embedding ϕ_ϵ of $\Bbbk^N$ in $\mathfrak{X}(0)$ by

$$\phi_\epsilon(x_1, \cdots, x_N) = ((n_{-\beta_1})^{\epsilon_1}u_{-\beta_1}((-1)^{\epsilon_1})x_1), \dots, (n_{-\beta_N})^{\epsilon_N}u_{-\beta_N}((-1)^{\epsilon_N})x_N).$$

Note that, the $\phi_\epsilon(\Bbbk^N)$ with $\epsilon \in \{0,1\}^N$, are the maximal affine $(\Bbbk^*)^N$-stable subvarieties of $\mathfrak{X}(0)$. In particular $\mathfrak{X}(0)$ is covered by affine spaces $\Bbbk^N$ so $\mathfrak{X}(0)$ is smooth and, by Proposition 1.3 (ii), $\mathfrak{X}(0)$ is a toric variety.

Moreover, if for all $i \in \{1, \dots, N\}$, $x_i \in \Bbbk^*$, we prove by induction, using Equation 1.5.1 and the defintion of $\mathfrak{X}(0)$, that

$$\begin{aligned} \phi_\epsilon(x_1, \cdots, x_N) &= (u_{-\beta_1}(x_1^{(-1)^{\epsilon_1}})\beta_1^\vee(x_1^{-1})^{\epsilon_1}, \dots, u_{-\beta_N}(x_N^{(-1)^{\epsilon_N}})\beta_N^\vee(x_N^{-1})^{\epsilon_N}) \\ &= (u_{-\beta_1}(x_1^{(-1)^{\epsilon_1}}), \dots, u_{-\beta_i}(x_i^{(-1)^{\epsilon_i}}\prod_{j<i} x_j^{-\epsilon_j\beta_{ji}}), \dots). \end{aligned} \quad (1.5.2)$$

Now, let us compute the weight of regular functions of all these affine subvarieties. We need first to fix a basis of characters of $(\Bbbk^*)^N$. Let us denote, for all $i \in \{1, \dots, N\}$, by f_i the function in $\Bbbk(\mathfrak{X}(0)) = \Bbbk(\phi_{(0,\dots,0)}((\mathbb{G}_a)^N))$ defined by $f_i((u_{-\beta_1}(x_1), \dots, u_{-\beta_N}(x_N)) = x_i$. Denote also by $(\chi_i)_{i\in\{1,\dots,N\}}$ the weights with $(\Bbbk^*)^N$ acts on $(f_i)_{i\in\{1,\dots,N\}}$, and by $(e_i^+)_{i\in\{1,\dots,N\}}$ the dual basis of $(\chi_i)_{i\in\{1,\dots,N\}}$. Then, if $\chi = \sum_{i=1}^N k_i\chi_i$, we can check, using Equation 1.5.2, that

$$\prod_{i=1}^N f_i^{k_i} \in \Bbbk[\phi_\epsilon((\mathbb{G}_a)^N))] \Longleftrightarrow \forall i \in \{1, \dots, N\}, \left\{ \begin{array}{ll} k_i = \langle \chi, e_i^+ \rangle \geq 0 & \text{if } \epsilon_i = 0 \\ -k_i - \sum_{j>i} \beta_{ij}k_j = \langle \chi, e_i^- \rangle \geq 0 & \text{if } \epsilon_i = 1 \end{array} \right. .$$

In other words, the cone associated to $\phi_\epsilon(\Bbbk^N)$ is generated by $e_1^{\epsilon_1'}, \dots, e_N^{\epsilon_N'}$, where $\epsilon_i' = +$ and $-$ if $\epsilon_i = 0$ and 1 respectively. It proves the first result of the proposition.

For the last statement, just remark that $\mathcal{Z}_i(0)$ is the divisor of $\mathfrak{X}(0)$ defined by the equation $f_i = 0$, and that f_i has weight χ_i so that the valuation of $\mathcal{Z}_i(0)$ corresponds to e_i^+. □

2 Cohomology of divisors on the toric variety $\mathfrak{X}(0)$

Let us first recall the result of M. Demazure [De70] on the cohomology of line bundles on smooth toric varieties. For this result, we can also refer to [Fu93, Chap.5.3] in the case of complex toric varieties.

Let X be a smooth complete toric variety of dimension N associated to a complete fan $\mathbb{F}$. Let $\Delta(1)$ be the set of primitive elements of one-dimensional cones of $\mathbb{F}$. For all $\rho \in \Delta(1)$, we denote by D_ρ the corresponding irreducible $(\mathbb{k}^*)^N$-stable divisor of X. The Picard group of X is generated by these divisors. Let $D := \sum_{\rho\in\Delta(1)} a_\rho D_\rho$. Let h_D be the piecewise linear function associated to D, *i.e.* if $\mathcal{C}$ is the cone generated by $\rho_1, \ldots, \rho_N$ then $h_{D|\mathcal{C}}$ is the linear function which takes values a_{ρ_i} at ρ_i.

Denote by $X((\mathbb{k}^*)^N)$ be the set of characters of $(\mathbb{k}^*)^N$. For all $m \in X((\mathbb{k}^*)^N)$, define the piecewise linear function $\phi_{D,m} : n \longmapsto \langle m, n\rangle + h_D(n)$. Let $\Delta(1)_{D,m} := \{\rho \in \Delta(1) \mid \phi_{D,m}(\rho) < 0\}$. And define the simplicial complex $\Sigma_{D,m}$ to be the set of all subset of $\Delta(1)_{D,m}$ generating a cone of $\mathbb{F}$ (we refer to [Go58, Chapter I.3] for cohomology of simplicial complexes).

The cohomology spaces $H^i(X, D)$ is a $(\mathbb{k}^*)^N$-module so that we have the following decomposition

$$H^i(X, D) = \bigoplus_{m\in X((\mathbb{k}^*)^N)} H^i(X, D)_m.$$

M. Demazure proved the following result.

Theorem 2.1 ([De70])**.** *With the notation above,*

(i) if $\Sigma_{D,m} = \emptyset$, then $H^0(X, D)_m = \mathbb{k}$ and $H^i(X, D)_m = 0$ for all $i > 0$;

(ii) if $\Sigma_{D,m} \neq \emptyset$, then $H^0(X, D)_m = 0$, $H^1(X, D) = H^0(\Sigma_{D,m}, \mathbb{k})/\mathbb{k}$ and $H^i(X, D)_m = H^{i-1}(\Sigma_{D,m}, \mathbb{k})$ for all $i > 1$.

Remark 2.2. I took above M. Demazure's notation and suppose that X is smooth. In his book ([Fu93]) W. Fulton consider the topological space

$$F_{D,m} := \{v \in \mathbb{R} \mid \langle m, v\rangle \geq -h_D(v)\}.$$

This space contains the origin and might be $\mathbb{R}^N$. Then for general toric varieties, we have

$$H^i(X, \mathcal{O}(D))_m \simeq H^i(\mathbb{R}^N, \mathbb{R}^N \backslash F_{D,m}, \mathbb{k}),$$

where the right side indicates relative singular cohomology with coefficients in $\mathbb{k}$. By the long exact sequence, this is isomorphic to the reduced singular cohomology $\tilde{H}^{i-1}(\mathbb{R}^N \backslash F_{D,m}, \mathbb{k})$ for $i \geq 1$. Now, for a simplicial fan, we can define a simplicial complex $\Sigma = \Sigma_{D,m}$ homotopic to $\mathbb{R}^N \backslash F_{D,m}$. So that we recover Demazure's Theorem above. Since M. Demazure only consider smooth varieties and W. Fulton gives the theory of complex toric varieties, I didn't know if all this is still true in positive characteristic for not smooth varieties!

Still now, in order to simplify the notation, we write Σ_m and ϕ_m instead of $\Sigma_{D,m}$ and $\phi_{D,m}$, respectively.

Applying Theorem 2.1 to $\mathfrak{X}(0)$, with the notation of the first section, we can deduce the following.

Corollary 2.3. *Let $\mathcal{D} = \sum_{i=1}^N a_i \mathcal{Z}_i$ be a divisor of $\mathfrak{X}$ and $\mathcal{D}(0)$ be the corresponding divisor $\sum_{i=1}^N a_i \mathcal{Z}_i(0)$ of $\mathfrak{X}(0)$.*

(i) If there is an integer j such that $\phi_m(e_j^+) \geq 0$ and $\phi_m(e_j^-) < 0$, or, $\phi_m(e_j^+) < 0$ and $\phi_m(e_j^-) \geq 0$, then $H^i(\mathfrak{X}(0), \mathcal{D}(0))_m = 0$ for all $i \geq 0$.

(ii) If the condition above is not satisfied, let $j_m := \sharp\{i \in \{1, \dots, N\} \mid \phi_m(e_j^+) < 0\}$, then $H^i(\mathfrak{X}(0), \mathcal{D}(0))_m = 0$ for all $i \neq j_m$ and $H^{j_m}(\mathfrak{X}(0), \mathcal{D}(0))_m = \Bbbk$.

Remark 2.4. For all $i \in \{1, \dots, N\}$, the sets

$$\overline{\Pi_{D,i}} := \{m \in \mathbb{Z}^N \mid H^i(\mathfrak{X}(0), \mathcal{D}(0))_m = \Bbbk\}$$

are the lattice points of the union $\Pi_{D,i}$ of convex polytopes of $\mathbb{R}^N$. And the disjoint union $\Pi_D = \bigsqcup_{i \in \{1,\dots,N\}} \Pi_{D,i}$ is a twisted-cube (non necessarily convex) as defined in [GK94]. More precisely, Π_D is the set of $m \in \mathbb{R}^N$ satisfying for all $i \in \{1, \dots, N\}$,

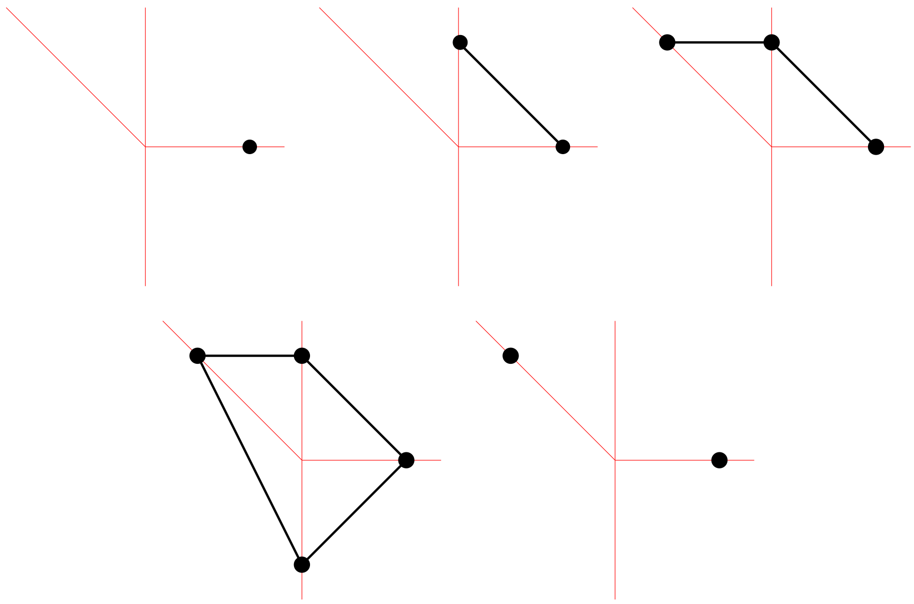
$$-a_i \leq m_i \leq -\sum_{j>i} \beta_{ij} m_j \text{ or } -\sum_{j>i} \beta_{ij} m_j < m_i < -a_i$$

and we denote its vertices $(x^\epsilon)_{\epsilon \in \{+,-\}^N}$, where $x^\epsilon \in \mathbb{Z}^N$ is given by induction as follows:

$$x_i^\epsilon = \begin{cases} -a_i & \text{if } \epsilon_i = + \\ -\sum_{j>i} \beta_{ij} x_j^\epsilon & \text{if } \epsilon_i = - \end{cases}$$

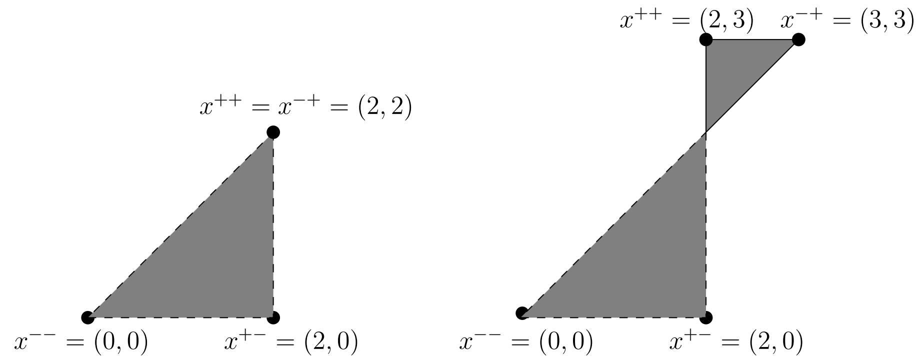
We will see this in Lemma 2.6.

Example 2.5. If $G = \mathrm{SL}(3)$ and $\tilde{w} = s_{\alpha_1} s_{\alpha_2}$, if the simplical complex Σ_m is not empty, it is one of the following modulo symmetries.



In the first three cases, $H^0(\Sigma_m, \Bbbk) = \Bbbk$ and the cohomology of Σ_m in positive degrees vanishes, and we are in the case (i) of Corollary 2.3. In fourth and fifth cases, we are in the case (ii) of Corollary 2.3. In fourth case, the non trivial cohomology are $H^0(\Sigma_m, \Bbbk) = \Bbbk^2$ and $H^1(\Sigma_m, \Bbbk) = \Bbbk$. In fifth case, the only non trivial cohomology is $H^0(\Sigma_m, \Bbbk) = \Bbbk^2$.

Now, in the same example, the twisted-cubes corresponding to $\mathcal{D}(0) = -2\mathcal{Z}_1(0) - 2\mathcal{Z}_2(0)$ and $\mathcal{D}(0) = -2\mathcal{Z}_1(0) - 3\mathcal{Z}_2(0)$ are respectively the following.



Sketch of proof of Corollary 2.3. (i) Suppose $\phi_m(e_j^+) \geq 0$ and $\phi_m(e_j^-) < 0$. Then, all maximal simplices of Σ_m contain e_j^-, so that Σ_m is contractible.

(ii) One can check that Σ_m is the set of faces of a j_m-dimensional convex polytope.

□

We will now prove two lemmas. In the first one, we give a necessary condition on $m \in X((\Bbbk^*)^N)$ to satisfy the condition of Corollary 2.3 (ii). The second lemma will be used to compute, in Case (ii), the possible values of j_m which depend on the Conditions $(C_i^{\pm})$.

Recall that we have already defined, in Remark 2.4, 2^N elements of $\mathbb{Z}^N$: the x^ϵ with $\epsilon \in \{+,-\}^N$.

Lemma 2.6. *Let $m \in \mathbb{Z}^N$ such that for all $i \in \{1, \dots, N\}$, we have either $\phi_m(e_i^+) \geq 0$ and $\phi_m(e_i^-) \geq 0$, or, $\phi_m(e_i^+) < 0$ and $\phi_m(e_i^-) < 0$.*

Then m is in the convex hull of the x^ϵ (in other words, the $(x_\epsilon)_{\epsilon \in \{+,-\}^N}$ are the vertices of Π_D).

Proof. Since, for all $i \in \{1, \dots, N\}$, $\phi_m(e_i^+) = m_i + a_i$ and $-\phi_m(e_i^-) = m_i + \sum_{j>i} \beta_{ij} m_j$ have opposite signs, there exists N real numbers $\lambda_1, \dots, \lambda_N$ in $[0, 1]$ such that, for all $i \in \{1, \dots, N\}$, $m_i = -\lambda_i a_i - (1 - \lambda_i) \sum_{j>i} \beta_{ij} m_j$. Denote, for all $\epsilon \in \{+,-\}^N$, by m^ϵ the product

$$\prod_{\substack{1 \leq i \leq N \\ \epsilon_i = +}} \lambda_i \times \prod_{\substack{1 \leq i \leq N \\ \epsilon_i = -}} (1 - \lambda_i).$$

Remark that $m^\epsilon \in [0, 1]$.

Let us prove by induction that for all $i \in \{1,\ldots,N\}$ we have $m_i = \sum_{\epsilon\in\{+,-\}^N} m^\epsilon x_i^\epsilon$, *i.e.* $m = \sum_{\epsilon\in\{+,-\}^N} m^\epsilon x^\epsilon$.

We will use the following easy fact: for all $i \in \{1,\ldots,N\}$,

$$\lambda_i = \sum_{\substack{\epsilon\in\{+,-\}^N \\ \epsilon_i=+}} m^\epsilon. \tag{2.6.1}$$

In particular, for $i = N$, we deduce, with the definition of x_N^ϵ, that $\sum_{\epsilon\in\{+,-\}^N} m^\epsilon x_N^\epsilon = -\lambda_N a_N = m_N$.

Now let $i < N$ such that, for all $j > i$, $m_j = \sum_{\epsilon\in\{+,-\}^N} m^\epsilon x_j^\epsilon$. Then

$$\begin{aligned} m_i &= -\lambda_i a_i - (1-\lambda_i)\sum_{j>i}\beta_{ij} m_j \\ &= -\lambda_i a_i - (1-\lambda_i)\sum_{j>i}\beta_{ij}\sum_{\epsilon\in\{+,-\}^N} m^\epsilon x_j^\epsilon \\ &= -\lambda_i a_i - (1-\lambda_i)\sum_{\epsilon\in\{+,-\}^N} m^\epsilon \sum_{j>i}\beta_{ij} x_j^\epsilon. \end{aligned}$$

Moreover, if for all $\epsilon \in \{+,-\}^N$, we define $\epsilon' \in \{+,-\}^N$ by $\epsilon'_j = \epsilon_j$ for all $j \neq i$ and $\epsilon'_i = -\epsilon_i$, we have

$$\sum_{j>i}\beta_{ij} x_j^\epsilon = \begin{cases} -x_i^\epsilon & \text{if } \epsilon_i = - \\ -x_i^{\epsilon'} & \text{if } \epsilon_i = + \end{cases}$$

Then

$$m_i = -\lambda_i a_i - (1-\lambda_i)\sum_{\substack{\epsilon\in\{+,-\}^N \\ \epsilon_i=-}} (m^\epsilon + m^{\epsilon'}) x_i^\epsilon.$$

We conclude by 2.6.1 and by checking that, for all $\epsilon \in \{+,-\}^N$ such that $\epsilon_i = -$, we have $(1-\lambda_i)(m^\epsilon + m^{\epsilon'}) = m^\epsilon$. ☐

Lemma 2.7. *For all $\epsilon \in \{+,-\}^N$ and all $i \in \{1,\ldots,N\}$, we have $\phi_{x^\epsilon}(e_i^+) = 0$ and $\phi_{x^\epsilon}(e_i^-) = a_i + \sum_{j>i,\,\epsilon_j=+} \alpha_{ij}^\epsilon a_j = C_i^\epsilon$ if $\epsilon_i = +$ (and the reverse if $\epsilon_i = -$).*

Proof. Fix $\epsilon \in \{+,-\}^N$. The lemma follows from the three following steps.

Step 1. Let us first prove by induction that, for all $i \in \{1,\ldots,N\}$,

$$x_i^\epsilon = \sum_{\substack{i+1\le h\le N \\ \epsilon_h=+}} \left(\sum_{k\ge 1} \sum_{\substack{i=i_0<i_1<\cdots<i_k=h \\ \forall x<k,\, \epsilon_{i_x}=-}} (-1)^{k+1} \prod_{x=0}^{k-1} \beta_{i_x i_{x+1}} \right) a_h + \begin{cases} -a_i & \text{if } \epsilon_i = + \\ 0 & \text{if } \epsilon_i = - \end{cases}$$

Let $i \in \{1, \ldots, N\}$. Remark that if $\epsilon_i = +$, this equality is clearly true because for all $k \geq 1$ there exists no $i = i_0 < i_1 < \cdots < i_k = h$ such that $\forall x < k$, $\epsilon_{i_x} = -$. Remark also, for similar reason, that the sum from $h = i+1$ to N can be replaced by the sum from $h = i$ to N (always with the condition $\epsilon_h = +$). Suppose now that $\epsilon_i = -$ and that for all $j > i$ the equality holds. Then

$$\begin{aligned}
x_i^\epsilon &= -\sum_{j>i} \beta_{ij} x_j^\epsilon \\
&= -\sum_{j>i} \sum_{\substack{j \leq h \leq N \\ \epsilon_h = +}} \left(\sum_{k \geq 1} \sum_{\substack{j=j_0<j_1<\cdots<j_k=h \\ \forall x<k,\, \epsilon_{j_x}=-}} (-1)^{k+1} \beta_{ij} \prod_{x=0}^{k-1} \beta_{j_x j_{x+1}} \right) a_h + \sum_{\substack{j>i \\ \epsilon_j=+}} \beta_{ij} a_j \\
&= \sum_{\substack{i+1 \leq h \leq N \\ \epsilon_h = +}} \left(\sum_{j=i+1}^{h} \sum_{k \geq 1} \sum_{\substack{j=j_0<j_1<\cdots<j_k=h \\ \forall x<k,\, \epsilon_{j_x}=-}} (-1)^{k+2} \beta_{ij} \prod_{x=0}^{k-1} \beta_{j_x j_{x+1}} \right) a_h + \sum_{\substack{j>i \\ \epsilon_j=+}} \beta_{ij} a_j \\
&= \sum_{\substack{i+1 \leq h \leq N \\ \epsilon_h = +}} \left(\sum_{k \geq 2} \sum_{\substack{i=i_0<i_1<\cdots<i_k=h \\ \forall x<k,\, \epsilon_{i_x}=-}} (-1)^{k+2} \prod_{x=0}^{k-1} \beta_{i_x i_{x+1}} \right) a_h + \sum_{\substack{h>i \\ \epsilon_h=+}} \beta_{ih} a_h.
\end{aligned}$$

But for all $h \in \{i+1, \ldots, h\}$, β_{ih} equals

$$\sum_{\substack{i=i_0<i_1<\cdots<i_k=h \\ \forall x<k,\, \epsilon_{i_x}=-}} (-1)^{k+2} \prod_{x=0}^{k-1} \beta_{i_x i_{x+1}}$$

when $k = 1$, so that we obtain the wanted equation.

Step 2. For all $i \in \{1, \ldots, N\}$ and $j \in \{i+1, \ldots, N\}$ we have

$$\alpha_{ij}^\epsilon = \sum_{k \geq 1} \sum_{\substack{i=i_0<i_1<\cdots<i_k=j \\ \forall x<k,\, \epsilon_{i_x}=-}} (-1)^{k+1} \prod_{x=0}^{k-1} \beta_{i_x i_{x+1}}.$$

The proof, by induction on j, of this formula is the same as in [Pe05, Lemma 3.5] and is left to the reader.

Step 3. Recall that $\phi_m(e_i^+) = m_i + a_i$ and that $\phi_m(e_i^-) = -m_i - \sum_{j>i} \beta_{ij} m_j$. Then, if $\epsilon_i = +$, we have $\phi_{x^\epsilon}(e_i^+) = 0$ and $\phi_{x^\epsilon}(e_i^-) = -x_i^\epsilon - \sum_{j>i} \beta_{ij} x_j^\epsilon$. And, if $\epsilon_i = -$, we have $\phi_{x^\epsilon}(e_i^-) = 0$ and $\phi_{x^\epsilon}(e_i^+) = a_i + x_i^\epsilon$. In fact, we only have to compute $\phi_{x^\epsilon}(e_i^+)$ in the case where $\epsilon_i = -$, *i.e.* $a_i + x_i^\epsilon$. Indeed, if $\epsilon_i = +$, define $\epsilon' \in \{+,-\}^N$ by $\epsilon'_j = \epsilon_j$ for all $j \neq i$ and $\epsilon'_i = -$. Then $\phi_{x^\epsilon}(e_i^-) = \phi_{x^{\epsilon'}}(e_i^+)$. □

We are now able to prove the vanishing theorem for divisors on the toric variety $\mathfrak{X}(0)$.

Theorem 2.8. *Let $\mathcal{D} = \sum_{i=1}^N a_i \mathcal{Z}_i$ be a divisor of $\mathfrak{X}$ and $\eta \in \{+,-,0\}^N$. Suppose that the coefficients $(a_i)_{i \in \{1,\dots,N\}}$ satisfy conditions $(C_i^{\eta_i})$ for all $i \in \{1,\dots,N\}$ such that $\eta_i \neq 0$. Then*

$H^i(\mathfrak{X}(0), \mathcal{D}(0)) = 0$, *for all* $i < \sharp\{1 \leq j \leq N \mid \eta_j = -\}$ *and for all* $i > N - \sharp\{1 \leq j \leq N \mid \eta_j = +\}$.

Proof. Let $m \in X((\Bbbk^*)^N)$ such that $H^i(\mathfrak{X}(0), \mathcal{D}(0))_m$ is not zero for some $i \in \{1,\dots,N\}$. Then, by Corollary 2.3 (i) and Lemma 2.6, there exist non negative real numbers m^ϵ with $\epsilon \in \{+,-\}^N$ such that $\sum_{\epsilon \in \{+,-\}^N} m^\epsilon = 1$ and $m = \sum_{\epsilon \in \{+,-\}^N} m^\epsilon x^\epsilon$.

Then, by Lemma 2.7,

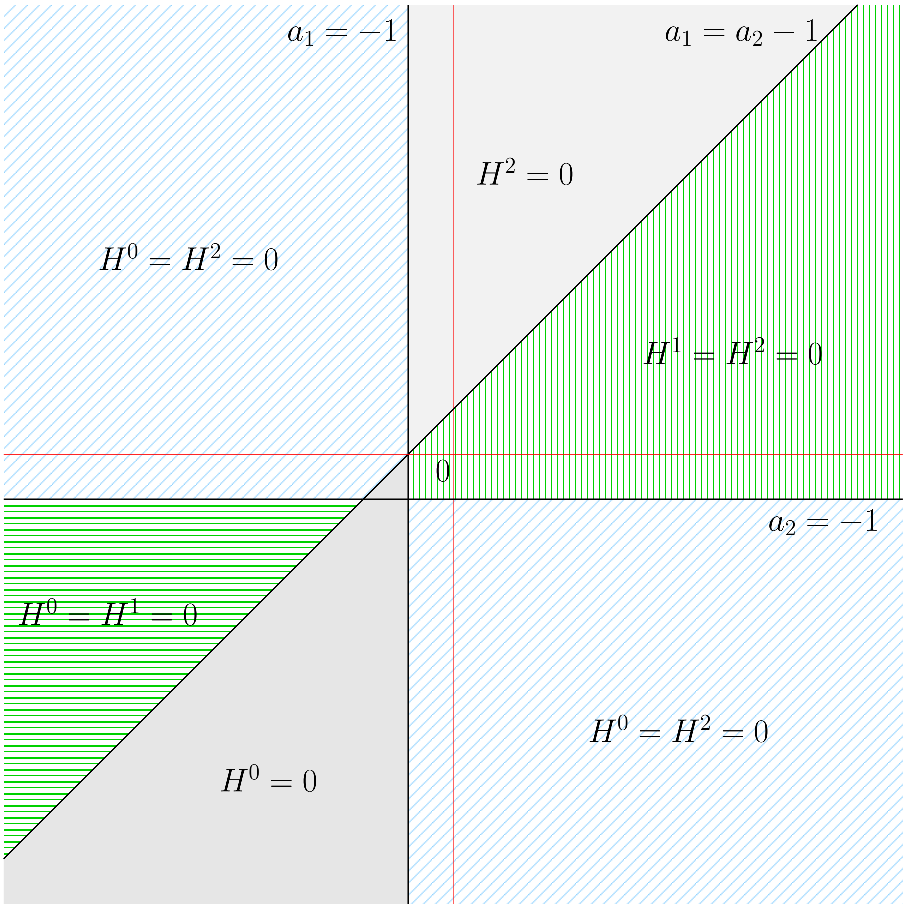
$$\phi_m(e_i^+) = \sum_{\substack{\epsilon \in \{+,-\}^N \\ \epsilon_i = -}} m^\epsilon C_i^\epsilon \qquad \text{and} \qquad \phi_m(e_i^-) = \sum_{\substack{\epsilon \in \{+,-\}^N \\ \epsilon_i = +}} m^\epsilon C_i^\epsilon.$$

Then, if Condition C_i^- is satisfied, we have $\phi_m(e_i^+)$ and $\phi_m(e_i^-)$ are both negative. And if Condition C_i^+ is satisfied and if the integers $\phi_m(e_i^+)$ and $\phi_m(e_i^-)$ are not both non-negative, then one of them equals -1 (say for example $\phi_m(e_i^+)$). It means that for all $\epsilon \in \{+,-\}^N$ such that $\epsilon_i = +$, we have $m^\epsilon = 0$. Then $\phi_m(e_i^-) = 0$ which is not possible by hypothesis on m and Corollary 2.3 (i).

We conclude the proof by Corollary 2.3 (ii). □

Remark 2.9. We cannot tell that Theorem 2.8 gives all possible vanishings. Indeed, in that case, the problem of non-vanishing is the same as the problem of existence of lattice points in parts of the twisted-cube Π_D defined in Remark 2.4.

Example 2.10. If $G = \mathrm{SL}(3)$ and $\tilde{w} = s_{\alpha_1} s_{\alpha_2}$, the vanishings of the cohomology of the divisor $\mathcal{D} = a_1 \mathcal{Z}_1 + a_2 \mathcal{Z}_2$ obtained by Theorem 2.8 is reprensented in the following picture.



Let us now discuss, with a more general example, what sort of vanishings Theorem 0.1 gives.

Example 2.11. Let $G = \mathrm{SL}(4)$ and $\tilde{w} = s_{\alpha_2} s_{\alpha_1} s_{\alpha_3} s_{\alpha_2}$ (with natural notation). Let $D = \sum_{i=1}^{4} a_i Z_i$ be a divisor of $Z(\tilde{w})$. Then, all the integers C_i^ϵ we obtain are the followings:

$i = 4$	a_4
$i = 3$	$a_3,\ a_3 - a_4$
$i = 2$	$a_2,\ a_2 - a_4$
$i = 1$	$a_1,\ a_1 - a_2,\ a_1 - a_3,\ a_1 - a_2 - a_3,\ a_1 - a_2 + a_4,\ a_1 - a_3 + a_4,\ a_1 - a_2 - a_3 + 2a_4.$

In particular, Conditions $(C_i^+)_{i \in \{1,2,3,4\}}$ are equivalent to $a_4 \geq -1$, $a_3 \geq a_4 - 1$, $a_2 \geq a_4 - 1$ and $a_1 \geq a_2 + a_3 - 1$. In that case, Theorem 0.1 tells us that the cohomology of D vanishes in non zero degree. But this fact can already be deduced by [LT04, Theorem 7.4]. Actually, the theorem of N. Lautitzen and J.F. Thomsen gives us the vanishing of the cohomology of D in non zero degree exactly for all D such that only if $a_4 \geq -1$, $a_3 \geq \max(a_4 - 1, -1)$, $a_2 \geq \max(a_4 - 1, -1)$ and $a_1 \geq \max(a_2 + a_3 - a_4 - 1, -1)$.

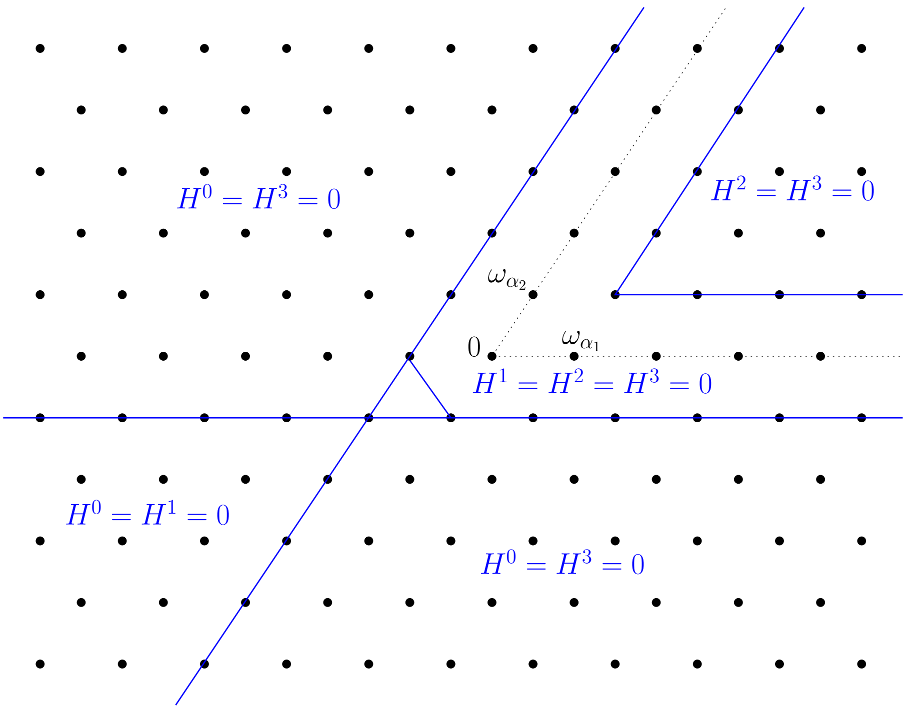
Let us consider $D = 2Z_1 + 2Z_2 + 2Z_3 + 2Z_4$, by the latter assertion the cohomology of D in non zero degree vanishes. But one can compute that the cohomology of the corresponding divisor on $\mathfrak{X}(0)$ is not trivial in degree 1 (indeed, we have for example $H^1(\mathfrak{X}(0), \mathcal{D}(0))_m = \Bbbk$ when $m = \frac{1}{2}(x^{(-,+,+,+)} + x^{(-,+,+,-)}) = (0, -2, -2, -3)$).

Theorem 0.1 is not as powerful as the results of N. Lautitzen and J.F. Thomsen for "positive" divisors (or also for "negative" divisors). But for all other divisors it gives many new vanishings results.

For example, if $a_4 \geq 0$, $a_3 \geq a_4$, $a_2 < 0$, Theorem 0.1 gives the vanishing of the cohomology of D in degree 0, 3 and 4.

Remark 2.12. Theorem 0.1 is easy to apply to a given divisor of a Bott-Sameslon variety. Indeed, we made a program that takes a triple $(A, \tilde{w}, Z)$ consisting of a Cartan matrix A, an expression $\tilde{w}$ and a divisor Z of $Z(\tilde{w})$, and that computes the vanishing results in the cohomology of Z given by Theorem 0.1 (contact the author for more detail).

We can also obtain vanishing results in the cohomology of line bundles on Schubert varieties. Indeed, if $\pi : Z(\tilde{w}) \longrightarrow X(w)$ is a Bott-Samelson resolution and D a divisor of the Schubert variety $X(w)$ then for all $i \geq 0$ we have $H^i(X(w), D) = H^i(Z(\tilde{w}), \pi^* D)$. These vanishings are case by case computable. For example, let us consider the simplest non trivial case: $G = \mathrm{SL}_3$ and $w = s_{\alpha_1} s_{\alpha_2} s_{\alpha_1} = s_{\alpha_2} s_{\alpha_1} s_{\alpha_2}$. Then, applying Theorem 0.1 to both reduced expression of w, we deduce vanishings of the cohomology of the lines bundles $\mathcal{L}(a_1\omega_{\alpha_1} + a_2\omega_{\alpha_2})$ on SL_3 / B, summarized in the following picture.



Remark that, to obtain this result, we needed to consider both reduced expression of w (each one gives different vanishings). We can also remark that we don't obtain the vanishing of the cohomology in degree 1 for all globally generated line bundles.

We can also compare these vanishings in few simple cases (for example when G is of rank 2) to the ones obtained in [Pa04] and [BSS04] in characteristic 0. We can't have the same results because our result is also true in positive characteristic (and even Borel-Weil-Bott Theorem is not true in positive characteristic). However, it is natural to think that, unfortunately, we are very far to obtain all possible vanishings. In fact we probably obtain all possible vanishings only when a Bott-Samelson resolution of the Schubert variety is toric (for example, it is the case for all proper Schubert varieties in SL_3/B).

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