The pseudo-index of horospherical Fano varieties

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September 24, 2009

Abstract

We prove a conjecture of L. Bonavero, C. Casagrande, O. Debarre and S. Druel, on the pseudo-index of smooth Fano varieties, in the special case of horospherical varieties.

Mathematics Subject Classification. 14J45 14L30 52B20

Keywords. Horospherical varieties, Picard number, Pseudo-index, Fano varieties.

Let X be a normal, complex, projective algebraic variety of dimension d. Assume that X is Fano, namely the anticanonical divisor $-K_X$ is Cartier (in other words, X is Gorenstein) and ample. The *pseudo-index* ι_X is the positive integer defined by

$$\iota_X := \min\{-K_X \cdot C \mid C \text{ rational curve in } X\}.$$

The aim of this paper is to prove the following result.

Theorem 1. *Let X be a $\mathbb{Q}$-factorial horospherical Fano variety of dimension d, Picard number ρ_X, and pseudo-index ι_X. Then*

$$(\iota_X - 1)\rho_X \leq d.$$

Moreover, equality holds if and only if X is isomorphic to $(\mathbb{P}^{\iota_X-1})^{\rho_X}$.

This inequality has been conjectured by L. Bonavero, C. Casagrande, O. Debarre and S. Druel for all smooth Fano varieties [1]. Moreover, they have proved Theorem 1 in the case of Fano varieties of dimension 3 and 4, all flag varieties, and also for some particular toric varieties. More generally, this result has been also proved by C. Casagrande for all $\mathbb{Q}$-factorial toric Fano varieties [4]. Theorem 1 generalizes these results to the family of horospherical varieties which contains in particular the families of toric varieties and flag varieties.

We will use arguments similar to the ones used by C. Casagrande, and we will use also a few results on horospherical varieties from [8]. So, let us summarize the theory of Fano horospherical varieties (see [8] or [7] for more details).

Let G be a connected algebraic group over $\mathbb{C}$. A homogeneous space is said to be *horospherical* if it is a torus bundle over a flag variety. Then a *horospherical variety* is a *G/H-embedding* (i.e., a normal G-variety

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with an open G-orbit isomorphic to G/H) of a horospherical homogeneous space G/H. The dimension of the torus is called the *rank* of the variety and is denoted by n. Toric varieties (where $G = (\mathbb{C}^*)^n$) and flag varieties (where $n = 0$) are the first examples of horospherical varieties.

Let G/H be a horospherical homogeneous space. Denote by P the normalizer of H in G. It is a parabolic subgroup of G. There exists a Borel subgroup B of G contained in P such that H contains the unipotent radical of B. We fix such a Borel subgroup. We denote by S the set of simple roots, by $(\omega_\alpha)_{\alpha\in S}$ the fundamental weights, and by $(P(\omega_\alpha))_{\alpha\in S}$ the associated maximal parabolic subgroups of G containing B. Then the application from the set of subsets of S to the set of parabolic subgroups of G containing B, defined by $J \subset S \longmapsto P_J := \cap_{\alpha\in S\backslash J} P(\omega_\alpha)$, is a bijection. We denote by I the subset of S such that $P = P_I$.

Let us define a lattice M as the set of characters of P whose restrictions to H are trivial. Let N be the dual lattice of M. Let $M_\mathbb{R} := M \otimes_\mathbb{Z} \mathbb{R}$ and $N_\mathbb{R} := N \otimes_\mathbb{Z} \mathbb{R}$.

For all $\alpha \in S\backslash I$, we denote by $\check{\alpha}_M$ the restriction of the coroot $\check{\alpha}$ to M. We also define $a_\alpha := \langle 2\rho^P, \check{\alpha}\rangle$ where $2\rho^P$ is the sum of all the positive roots not generated by the simple roots in I. (Remark that $a_\alpha \geq 2$.)

Let X be a complete G/H-embedding. Denote by $(X_i)_{i\in\{1,\dots,m\}}$ the irreducible G-stable divisor of X and by $(D_\alpha)_{\alpha\in S\backslash I}$ the irreducible B-stable but not G-stable divisors of X (in fact $D_\alpha := \overline{Bw_0s_\alpha P/H}$ where w_0 is the longest element of the Weyl group and s_α the simple reflection associated with α). Then $-K_X = \sum_{i=1}^m X_i + \sum_{\alpha\in S\backslash I} a_\alpha D_\alpha$. We denote by $l(-K_X)$ the piecewise linear function associated to $-K_X$ (see for example [2] for piecewise linear functions associated to divisors on spherical varieties). And we define $Q(X)$ as

$$Q(X) := \{u \in N_\mathbb{R} \mid l(-K_X)(u) \leq 1\}.$$

When X is Fano, $Q(X)$ is a G/H-*reflexive* polytope in the following sense.

Definition 2. Let G/H be a homogeneous horospherical space. A convex polytope Q in $N_\mathbb{R}$ is said to be G/H-*reflexive* if the following three conditions are satisfied:

(1) The vertices of Q are in $N \cup \{\frac{\check{\alpha}_M}{a_\alpha} \mid \alpha \in S\backslash I\}$, and the interior $\overset{o}{Q}$ of Q contains 0.

(2) $Q^* := \{v \in M_\mathbb{R} \mid \forall u \in Q,\ \langle v, u\rangle \geq -1\}$ is a lattice polytope (i.e., its vertices are in M).

(3) For all $\alpha \in S\backslash I$, $\frac{\check{\alpha}_M}{a_\alpha} \in Q$.

Moreover, we have the following.

Proposition 3. *Let G/H be a horospherical homogeneous space. Then the application $X \longmapsto Q(X)$ is a bijection between the set of isomorphism classes of Fano G/H-embeddings and the set of G/H-reflexive polytopes.*

There is also a classification of G/H-embeddings in terms of colored fans, due to D. Luna and T. Vust [6]. The *colored fan* of a Fano G/H-embedding X is the set of couples $(\mathcal{C}_F, \mathcal{D}_F)$, where $\mathcal{C}_F$ is the cone of $N_\mathbb{R}$ generated by a face F of $Q(X)$ and $\mathcal{D}_F := \{\alpha \in S\backslash S \mid \frac{\check{\alpha}_M}{a_\alpha} \in F\}$.

From now on, let us fix a horospherical homogeneous space G/H, a Fano G/H-embedding X, and the associated G/H-reflexive polytope Q. We define the *colors* of X to be the roots $\alpha \in S\backslash I$ such that $\frac{\check{\alpha}_M}{a_\alpha} \in Q\backslash \overset{o}{Q}$. Denote by $\mathcal{D}_X$ the set of colors of X.

Moreover, X is $\mathbb{Q}$-factorial if and only if Q is simplicial and the points $\frac{\check{\alpha}_M}{a_\alpha} \in Q\backslash \overset{o}{Q}$, $\alpha \in \mathcal{D}_X$ are disjoint vertices of Q. From now on, suppose that X is $\mathbb{Q}$-factorial. Then the Picard number of X is given by

$$\rho_X = r + \sharp(S\backslash I) - \sharp(\mathcal{D}_X) \tag{1}$$

where $n+r$ is the number of vertices of Q.[a] And for all vertices u of Q, we can now define an integer a_u to be a_α if there exists $\alpha \in S\backslash I$ such that $u = \frac{\check{\alpha}_M}{a_\alpha}$, and 1 otherwise.

Let us denote by $V(Q)$ and $V(Q^*)$ the set of vertices of Q and Q^* respectively. For each $v \in V(Q^*)$, the set $F_v := \{u \in Q \mid \langle v, u\rangle = -1\}$ is a facet of Q. Moreover the application $v \longmapsto F_v$ is a bijection between $V(Q^*)$ and the set of facets of Q (and likewise between $V(Q)$ and the set of facets of Q^*). By Luna-Vust theory, $V(Q^*)$ (i.e., the set of facets of Q) is in bijection with the set of closed G-orbits of X. Moreover, in our case, we can describe explicitly the closed G-orbits of X as follows. Let $v \in V(Q^*)$ and $J_v := \{\alpha \in S\backslash I \mid \frac{\check{\alpha}_M}{a_\alpha} \in F_v\}$. Then the G-orbit associated with v is isomorphic to $G/P_{I\cup J_v}$.

Let us now define

$$\epsilon_Q := \min\{a_u(1+\langle v,u\rangle) \mid u \in V(Q),\, v \in V(Q^*),\, u \notin F_v\}.$$

The pseudo-index of X is related to ϵ_Q as follows.

Proposition 4. *Let X and Q as above. Then,*
(i) $\iota_X \leq \epsilon_Q$.
(ii) $\forall \alpha \in (S\backslash I)\backslash \mathcal{D}_X$, $\iota_X \leq a_\alpha$.

To prove this result we need the following lemma.

Lemma 5. *Let $u \in V(Q)$ and $v \in V(Q^*)$ be such that $\epsilon_Q = a_u(1+\langle v,u\rangle)$. Then u is adjacent to F_v (i.e., F_v and some facet containing u intersect along an $(n-2)$-dimensional face).*

Proof. Let $v \in V(Q^*)$. Let $e_1,\dots,e_n$ be the vertices of the complex F_v. For all $j \in \{1,\dots,n\}$, denote by F_j the facet of Q, distinct from F_v, containing $e_1,\dots,e_{j-1},e_{j+1},\dots,e_n$ and denote by u^j the vertex of F_j distinct from $e_1,\dots,e_n$.

Let $(e_1^*,\dots,e_n^*)$ be the dual basis of $(e_1,\dots,e_n)$ in $M_\mathbb{R}$. Let $j \in \{1,\dots,n\}$. Then $\langle e_j^*, u^j\rangle \neq 0$ (otherwise, u^j is in the hyperplane generated by the $(e_i)_{i\neq j}$) and one can define

$$\gamma_j = \frac{-1-\langle v,u^j\rangle}{\langle e_j^*, u^j\rangle}.$$

Moreover, the vertex of Q^* associated to F_j is $v^j = v + \gamma_j e_j^*$. Then $\gamma_j > 0$ because $\langle v^j, e_j\rangle > -1$.

Let $v \in V(Q^*)$ and $u \in V(Q)$ be such that $\epsilon_Q = a_u(1+\langle v,u\rangle)$. Then

$$a_u(1+\langle v^j,u\rangle) = a_u(1+\langle v,u\rangle) + a_u\gamma_j\langle e_j^*,u\rangle,$$

so that

$$\langle e_j^*,u\rangle < 0 \Longleftrightarrow \langle v^j,u\rangle = -1 \Longleftrightarrow u \in F_j,$$

or, in other words:

$$u \notin F_j \Longleftrightarrow \langle e_j^*,u\rangle \geq 0.$$

Now, if $u \neq u^j$ for all $j \in \{1,\dots,n\}$, then $\langle e_j^*,u\rangle \geq 0$ for all $j \in \{1,\dots,n\}$ and then u is in the cone generated by $e_1,\dots,e_n$, which is a contradiction. So, u is one of the u^j, i.e., u is adjacent to F_v. □

We need also to define two different families of rational curves in X. The first family is very similar to the family of the curves stable under the action of the torus in the case of toric varieties. And the second one is a family of Schubert curves in the closed G-orbits of X.

[a]Note that, as in the toric case, the Picard number of a $\mathbb{Q}$-factorial horospherical Fano variety is bounded by twice its dimension [8, Theorem 1.2].

Denote by x_0 the point of X with isotropy group H. Recall that $P/H \simeq (\mathbb{C}^*)^n$. Then the closure X' of the P-orbit of x_0 in X is a toric variety associated to the fan consisting of the cones $(\mathcal{C}_{F_v})_{v\in Q^*}$ and their faces, i.e., the fan of X without colors.

Let μ be an $(n-2)$-dimensional face of Q (or, which is the same an $(n-1)$-dimensional colored cone of the colored fan of X, or also an $(n-1)$-dimensional cone of the fan of X'). Then there exists a unique P-stable curve of $X' \subset X$ containing two P-fixed points lying in the two closed G-orbits of X corresponding to the two facets of Q containing μ. This curve is rational; we denoted it by C_μ.

Let $v \in V(Q^*)$ and $\alpha \in (S\backslash I)\backslash\mathcal{D}_X$ such that $\check{\alpha}_M \in \mathcal{C}_{F_v}$. Recall that v corresponds to a closed G-orbit Y of X. Moreover, since α is not a color of X, $Y = G/P_Y$, where P_Y is a parabolic subgroup of G containing B such that ω_α is a character of P_Y. Then we can define the rational B-stable curve $C_{\alpha,v}$ in X to be the Schubert curve $\overline{Bs_\alpha P_Y/P_Y}$ where s_α is the simple reflection associated to the simple root α.

It is not difficult to check that the curve C_μ is the one defined in [3, Prop.3.3]. Moreover, the curve $C_{\alpha,v}$ is the one defined in [3, Prop.3.6]. Indeed, $C_{\alpha,v}$ is contracted by the extremal contraction described in [3, Ch.3.4], which consists in "adding" the color α. In fact, the curves C_μ and $C_{\alpha,v}$ are exactly the irreducible B-stable curves in X and they generate the cone of effective 1-cycles.

Now, using the definition of the polytope Q and the intersection formulas given in [3, Ch.3.2], we prove the following lemma.

Lemma 6. *(i) Let μ an $(n-2)$-dimensional face of Q, and choose $v \in V(Q^*)$ and $u \in V(Q)$ such that F_v and some facet of Q containing u intersect along μ. Let χ_μ the primitive element of M such that $\langle\chi_\mu, x\rangle = 0$ for all $x \in \mu$ and $\langle\chi_\mu, u\rangle > 0$. Then we have*

$$-K_X \cdot C_\mu = \frac{a_u(1 + \langle v, u\rangle)}{\langle\chi_\mu, a_u u\rangle}.$$

(ii) Let $v \in V(Q^*)$ and $\alpha \in (S\backslash I)\backslash\mathcal{D}_X$ be such that $\check{\alpha}_M \in \mathcal{C}_{F_v}$. Then we have

$$-K_X \cdot C_{\alpha,v} = a_\alpha + \langle v, \check{\alpha}_M\rangle.$$

Proof. In both cases, let us consider and explain the formulas given in [3, Ch.3.2].

(i) The formula we will use is the following (with the notations of [3]):

$$\delta \cdot C_\mu = \frac{l(\delta)_{|\mu_+} - l(\delta)_{|\mu_-}}{\chi_\mu}.$$

In our case, let us choose μ_+ to be the cone generated by the facet of Q containing u and μ, and μ_- to be the cone generated by the facet F_v. The divisor δ is $-K_X$ so that, by the definition of the polytopes Q and Q^*, we have

$$l(\delta)_{|\mu_+} : x \longmapsto -\langle v', x\rangle \qquad \text{and} \qquad l(\delta)_{|\mu_-} : x \longmapsto -\langle v, x\rangle,$$

where v' is the vertex of Q^* such that μ_+ is the cone generated by $F_{v'}$.

Now we have

$$\frac{l(\delta)_{|\mu_+} - l(\delta)_{|\mu_-}}{\chi_\mu}(u) = \frac{1 + \langle v, u\rangle}{\langle\chi_\mu, u\rangle}.$$

(ii) Here we use the following formula of [3]:

$$\delta \cdot C_{D,Y} = n_D - v_D(l_Y).$$

In our case, $\delta = -K_X$, $D = D_\alpha$, $n_D = a_\alpha$, Y corresponds to v and $v_D(l_Y) = l(\delta)_{|F_v}(\check{\alpha}_M) = -\langle v, \check{\alpha}_M\rangle$, which induces the wanted formula. □

We are now able to prove Proposition 4.

Proof of Proposition 4. (i) Let $u \in V(Q)$ and $v \in V(Q^*)$ be such that $\epsilon_Q = a_u(1 + \langle v, u\rangle)$. By Lemma 5, u is adjacent to F_v. This means that u lies in a facet F of Q such that F and F_v intersect along an $(n-2)$-dimensional face μ. Then, by Lemma 6, we have

$$-K_X \cdot C_\mu = \frac{a_u(1 + \langle v, u\rangle)}{\langle \chi_\mu, a_u u\rangle},$$

where $\chi_\mu \in M$ and $\langle \chi_\mu, a_u u\rangle$ is a positive integer. Then we have

$$\iota_X \le -K_X \cdot C_\mu \le a_u(1 + \langle v, u\rangle) = \epsilon_Q.$$

(ii) Let $\alpha \in (S\backslash I)\backslash \mathcal{D}_X$. Let us choose $v \in V(Q^*)$ such that $\check{\alpha}_M \in \mathcal{C}_{F_v}$. Then, by Lemma 6,

$$\iota_X \le -K_X \cdot C_{\alpha,v} = a_\alpha + \langle v, \check{\alpha}_M\rangle \le a_\alpha$$

because $\langle v, \check{\alpha}_M\rangle$ is non-positive by the choice of v. □

Remark 7. In (ii), if $\check{\alpha}_M \neq 0$, we have in fact the strict inequality $\iota_X < a_\alpha$.

Proof of the inequality of Theorem 1. Since the origin lies in the interior of Q^*, there exists an equation

$$m_1 v_1 + \cdots + m_h v_h = 0 \tag{2}$$

where $h > 0$, $v_1, \ldots, v_h$ are vertices of Q^*, and $m_1, \ldots, m_h$ are positive integers. Let $I := \{1, \ldots, h\}$ and $M := \sum_{i\in I} m_i$. For any $u \in V(Q)$, we define

$$A(u) = \{i \in I \mid \langle v_i, u\rangle = -1\}.$$

Now, we have

$$\begin{aligned} 0 &= \sum_{i\in I} m_i\langle v_i, u\rangle \\ &= -\sum_{i\in A(u)} m_i + \sum_{i\notin A(u)} m_i\langle v_i, u\rangle \\ &\ge -\sum_{i\in A(u)} m_i + \sum_{i\notin A(u)} m_i\left(\frac{\epsilon_Q}{a_u} - 1\right) = \left(\frac{\epsilon_Q}{a_u} - 1\right)M - \frac{\epsilon_Q}{a_u}\sum_{i\in A(u)} m_i \end{aligned} \tag{3}$$

so that

$$\frac{\epsilon_Q - a_u}{\epsilon_Q} M \le \sum_{i\in A(u)} m_i.$$

Denote by r the positive integer such that the number of vertices of Q is $n+r$. Let us sum the preceding inequalities over all vertices u of Q. We obtain

$$(n+r)M - \frac{\sum_{u\in V(Q)} a_u}{\epsilon_Q} M \le \sum_{u\in V(Q)} \sum_{i\in A(u)} m_i = nM \qquad \text{and} \qquad \epsilon_Q r \le \sum_{u\in V(Q)} a_u.$$

(The last equality comes from the fact that X is $\mathbb{Q}$-factorial, so that each facet of Q has exactly n vertices.)

Then, by Proposition 4 (i), we have

$$\iota_X r \leq \epsilon_Q r \leq \sum_{u\in V(Q)} a_u. \tag{4}$$

But recall that $\rho_X = r + \sharp(S\backslash I) - \sharp(\mathcal{D}_X)$, hence

$$\sum_{u\in V(Q)} a_u = \sum_{\alpha\in\mathcal{D}_X} a_\alpha + \rho_X + n - \sharp(S\backslash I).$$

Thus, we have

$$\iota_X(\rho_X - \sharp(S\backslash I) + \sharp(\mathcal{D}_X)) \leq \sum_{\alpha\in\mathcal{D}_X} a_\alpha + \rho_X + n - \sharp(S\backslash I)$$

and, by Proposition 4, we obtain that

$$\begin{aligned}\rho_X(\iota_X - 1) &\leq \sum_{\alpha\in\mathcal{D}_X} a_\alpha + \sum_{\alpha\in(S\backslash I)\backslash\mathcal{D}_X} \iota_X - \sharp(S\backslash I) + n \\ &\leq \sum_{\alpha\in\mathcal{D}_X} a_\alpha + \sum_{\alpha\in(S\backslash I)\backslash\mathcal{D}_X} a_\alpha - \sharp(S\backslash I) + n = \sum_{\alpha\in S\backslash I}(a_\alpha - 1) + n. \end{aligned} \tag{5}$$

This yields the inequality of Theorem 1 by [8, Lemmma 4.8] which asserts that

$$\sum_{\alpha\in S\backslash I}(a_\alpha - 1) + n \leq d. \tag{6}$$

□

Let us study now the cases of equality. In view of the above proof, one can remark that $(\iota_X - 1)\rho_X = d$ if and only if equality holds in inequalities 3, 4, 5 and 6, i.e., all of the following conditions are satisfied:

(a) $\forall u \in V(Q)$, $\forall i \in \{1,\ldots,h\}$, $a_u(1 + \langle v_i, u\rangle)$ equals either 0 or ϵ_Q;

(b) $\iota_X = \epsilon_Q$;

(c) $\forall\alpha \in (S\backslash I)\backslash\mathcal{D}_X$, $\iota_X = a_\alpha$;

(d) $\sum_{\alpha\in S\backslash I}(a_\alpha - 1) + n = d$.

Let us now interpret these conditions.

Lemma 8. *(i) Condition (a) is equivalent to*

(a') $\forall u \in V(Q)$, $\forall v \in V(Q^*)$, $a_u(1 + \langle v, u\rangle)$ equals either 0 or ϵ_Q.

(ii) Conditions (a') and (b) imply that X is locally factorial.

(iii) Condition (c) implies that every simple root $\alpha \in S\backslash I$ such that $\check{\alpha}_M \neq 0$ is a color of X.

(iv) Condition (d) is equivalent to

(d') The decomposition $\Gamma = \sqcup_{i=1}^{\gamma}\Gamma_i$ into connected components of the Dynkin diagram Γ of G satisfies: $\forall i \in \{1,\ldots,\gamma\}$, either no vertex of Γ_i corresponds to a simple root of $S\backslash I$ or only one vertex of Γ_i corresponds to a simple root α_i of $S\backslash I$. In the latter case, Γ_i is of type A_m or C_m, and α_i corresponds to a simple end of Γ_i ("simple" means not adjacent to a double edge).

(v) Conditions (a'), (b) and (d') imply that X is smooth.

Proof. (i) Just choose Equation 2 such that it involves all vertices of Q^*.

(ii) Let us suppose that X is not locally factorial (but $\mathbb{Q}$-factorial). Then there exists a facet of Q with vertices $u_1,\dots,u_n$ such that $(a_{u_1}u_1,\dots,a_{u_n}u_n)$ is a basis of $N_{\mathbb{R}}$ but not of N (just compare locally factoriality and $\mathbb{Q}$-factoriality criteria given respectively in [8, Propositions 4.4 and 4.7]). This means that there exist a non-zero element $u_0=\sum_{i=1}^n\lambda_i a_{u_i}u_i$ of N such that $\forall i\in\{1,\dots,n\}$, $\lambda_i\in\mathbb{Q}$, $0\leq\lambda_i\leq 1$ and some j satisfies $0<\lambda_j<1$. Let μ be the wall generated by $u_1,\dots,u_{j-1},u_{j+1},\dots,u_n$ and let χ_μ be the associated element of M as in the proof of Proposition 4 (i) (with $\chi_\mu(u_j)>0$). Then $\chi_\mu(a_{u_j}u_j)=\frac{1}{\lambda_j}\chi_\mu(u_0)>1$ because $\chi_\mu(u_0)\in\mathbb{Z}_{>0}$. This implies, by the proof of Proposition 4 (i) together with Condition (a'), that $\iota_X<\epsilon_Q$.

(iii) It is Remark 7.

(iv) Without loss of generality, we assume that Γ is connected and we also suppose that $S\backslash I\neq\emptyset$.

Let us consider [8, Lemma 4.8], where the inequality $\sum_{\alpha\in S\backslash I}(a_\alpha-1)+n\leq d$ is proved in 4 steps. Then we have to look at equality cases in Steps 2, 3 and 4.

In Step 2, we reduced to the case where the subdiagram Γ_I of Γ, with edges the simple roots in I and all possible arrows, is connected. One can check that equality occurs in that step if and only if Γ_I is already connected.

In Step 3, we reduced to the case where $S\backslash I$ consists of one point. Here also, it is not difficult to check that equality occurs if and only if $S\backslash I$ already consists of a point.

In Step 4, we computed explicitly both terms of the desired inequality in all possible cases. Then, there are only two cases where equality holds:
Γ is of type A_{m+1} and Γ_I is of type A_m with $m\geq 0$;
Γ is of type C_{m+1} and Γ_I is of type C_m with $m\geq 1$.

(v) By (ii) X is locally factorial, and the smoothness criterion of horospherical varieties [7, Theorem 2.6] then tells us that Condition (d') implies that X is smooth. □

We may now complete the proof of Theorem 1.

Proof of the case of equality of Theorem 1. Step 1: we can suppose that G is the direct product of a semi-simple group G' and a torus (for example, one can replace G by the product of the semi-simple part of G by the torus P/H, see [8, proof of Proposition 3.5]). Then one can also suppose that G' is a direct product of simple groups and a torus. Then, because of Condition (d'), we have

$$X=X'\times\prod_{\alpha\in S\backslash I,\check{\alpha}_M=0}G/P(\omega_\alpha),$$

where X' is a horospherical variety. Moreover, for all $\alpha\in S\backslash I$ such that $\check{\alpha}_M=0$, the variety $G/P(\omega_\alpha)$ is isomorphic to $\mathbb{P}^{a_\alpha-1}$ by condition (d'), and hence to $\mathbb{P}^{\iota_X-1}$ by Condition (c). Also, X' has the same combinatorial data as X, except for the colors with zero image.

Then we have to prove the result in the case where there is no $\alpha\in S\backslash I$ such that $\check{\alpha}_M=0$. In that case, we have $\mathcal{D}_X=S\backslash I$.

Step 2: Let v be a vertex of Q^* and let $e_1,\dots,e_n$ be the vertices of F_v (which satisfy $\langle v,e_i\rangle=-1$). Let us denote by $f_1,\dots,f_r$ the remaining vertices of Q. Let $K:=\{1,\dots,n\}$ and $J:=\{1,\dots,r\}$. For any

$k \in K$, the face of Q with vertices $e_1, \dots, e_{k-1}, e_{k+1}, \dots, e_n$ lies on exactly two facets, which are F_v and another one we denote by F_k. Thus, there exists a unique $\phi(k) \in J$ such that $f_{\phi(k)} \in F_k$. This defines a function $\phi : K \longrightarrow J$.

Let $(e_1^*, \dots, e_n^*)$ the dual basis of $(e_1, \dots, e_n)$ in $M_{\mathbb{R}}$. We have $v = -e_1^* - \cdots - e_n^*$. Fix $k \in K$ and let v_k be the vertex of Q^* corresponding to the facet F_k of Q. We have, by Condition (a'),

$$\forall i \in K\backslash\{k\},\ a_{e_i}(1 + \langle v_k, e_i\rangle) = 0 \qquad \text{and} \qquad a_{e_k}(1 + \langle v_k, e_k\rangle) = \epsilon_Q,$$

so that $v_k = v + \frac{\epsilon_Q}{a_{e_k}} e_k^*$.

Now for any $j \in J$, using condition (a') for $u = f_j$, we have

$$\langle e_k^*, f_j\rangle = \frac{a_{e_k}}{\epsilon_Q}\langle v_k - v, f_j\rangle = \begin{cases} -\frac{a_{e_k}}{a_{f_j}} & \text{if } \phi(k) = j, \\ 0 & \text{otherwise.} \end{cases}$$

And then, for any $j \in J$,

$$a_{f_j} f_j + \sum_{k \in \phi^{-1}(j)} a_{e_k} e_k = 0. \tag{7}$$

Let us define M_j to be the intersection of M with the supspace of $M_{\mathbb{R}}$ generated by $\{e_k^* \mid k \in \phi^{-1}(\{j\})\}$. Then $M = \bigoplus_{j \in J} M_j$ (remark that $M_j \neq \{0\}$ because of Condition (a') and Lemma 5). By Step 1 and Condition (d'), we can assume that $G = G_1 \times \cdots \times G_r$ and $M_j = M \cap \Lambda_{G_j}$ where Λ_{G_j} is the character group of a maximal torus of G_j. Then G/H is isomorphic to $G_1/H_1 \times \cdots \times G_r/H_r$ for some horospherical subgroups H_j of G_j. And, by Equation 7, the colored fan of X is the same as the colored fan of the product $X^1 \times \cdots \times X^r$ where X^j is the G_j/H_j-embedding associated to the reflexive polytope defined as the convex hull of f_j and all e_k such that $\phi(k) = j$.

Moreover, for all $i \in \{1, \dots, r\}$, with formula 1, we compute that the Picard number of X^i is 1. But a smooth projective horospherical variety X with Picard number one such that two simple roots in $\mathcal{D}_X \subset S\backslash I$ are not in the same connected component of the Dynkin diagram of G, is a projective space [9, Theorem 1.4]. This completes the proof of Theorem 1. □

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