On some smooth projective two-orbit varieties with Picard number 1

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Abstract

We classify all smooth projective horospherical varieties with Picard number 1. We prove that the automorphism group of any such variety X acts with at most two orbits and that this group still acts with only two orbits on X blown up at the closed orbit. We characterize all smooth projective two-orbit varieties with Picard number 1 that satisfy this latter property.

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Introduction

Horospherical varieties are complex normal algebraic varieties where a connected complex reductive algebraic group acts with an open orbit isomorphic to a torus bundle over a flag variety. The dimension of the torus is called the rank of the variety. Toric varieties and flag varieties are the first examples of horospherical varieties (see [14] for more examples and background).

It is well known that the only smooth projective toric varieties with Picard number 1 are the projective spaces. This is not the case for horospherical varieties: for example any flag variety G/P with P a maximal parabolic subgroup of G is smooth, projective and horospherical with Picard number 1.

Moreover, smooth projective horospherical varieties with Picard number 1 are not necessarily homogeneous. For example, let ω be a skew-form of maximal rank on $\mathbb{C}^{2m+1}$. For

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$i \in \{1, \ldots, m\}$, define the odd symplectic grassmannian $\mathrm{Gr}_\omega(i, 2m+1)$ as the variety of i-dimensional ω-isotropic subspaces of $\mathbb{C}^{2m+1}$. Odd symplectic grassmannians are horospherical varieties (see Proposition 1.12) and, for $i \neq m$ they have two orbits under the action of their automorphism group which is a connected non-reductive linear algebraic group (see [13] for more details).

Our focus on smooth horospherical varieties with Picard number 1, that gives interesting examples of Fano varieties with Picard number 1, is also motivated by the main result of [14], where Fano horospherical varieties are classified in terms of rational polytopes. Indeed in [14, Th.0.1], the degree (*i.e.* the self-intersection number of the anticanonical bundle) of smooth Fano horospherical varieties is bounded. Two different bounds are obtained in the case of Picard number 1 and in the case of higher Picard number.

In Section 1, we classify all smooth projective horospherical varieties with Picard number 1. More precisely, we prove the following result.

Theorem 0.1. *Let G be a connected reductive algebraic group. Let X be a smooth projective horospherical G-variety with Picard number 1.*

Then we have the following alternative:

(i) X is homogeneous, or

(ii) X is horospherical of rank 1. Its automorphism group is a connected non-reductive linear algebraic group, acting with exactly two orbits.

Moreover in the second case, X is uniquely determined by its two closed G-orbits Y and Z, isomorphic to G/P_Y and G/P_Z respectively; and (G, P_Y, P_Z) is one of the triples of the following list.

1. $(B_m, P(\omega_{m-1}), P(\omega_m))$ *with* $m \geq 3$
2. $(B_3, P(\omega_1), P(\omega_3))$
3. $(C_m, P(\omega_i), P(\omega_{i+1}))$ *with* $m \geq 2$ *and* $i \in \{1, \ldots, m-1\}$
4. $(F_4, P(\omega_2), P(\omega_3))$
5. $(G_2, P(\omega_2), P(\omega_1))$

Here we denote by $P(\omega_i)$ the maximal parabolic subgroup of G corresponding to the dominant weight ω_i with the notation of Bourbaki [4].

Remark that Case 3 of Theorem 0.1 corresponds to odd symplectic grassmannians. It would be natural to investigate other complete smooth spherical varieties with Picard number 1 (a normal variety is spherical if it admits a dense orbit of a Borel subgroup, for example horospherical varieties and symmetric varieties are spherical). A classification has been recently given in the special case of projective symmetric varieties by A Ruzzi [16].

In the second part of this paper, we focus on another special feature of the non-homogeneous varieties classified by Theorem 0.1: the fact that they have two orbits even when they are blown up at their closed orbit. Two-orbit varieties (*i.e.* normal varieties where a linear algebraic group acts with two orbits) have already been studied by D. Akhiezer and S. Cupit-Foutou. In [1], D. Akhiezer classified those whose closed orbit is of codimension 1 and proved in particular that they are horospherical when the group is not semi-simple. In [8], S. Cupit-Foutou classified two-orbit varieties when the group is semi-simple, and she also proved that they are spherical. In section 2, we define two smooth projective two-orbit varieties $\mathbf{X_1}$ and $\mathbf{X_2}$ with Picard number one (see Definitions 2.11 and 2.12) and we prove the following:

Theorem 0.2. *Let X be a smooth projective variety with Picard number 1 and put $G := \mathrm{Aut}^0(X)$.*

Assume that X has two orbits under the action of G and denote by Z the closed orbit. Then the codimension of Z is at least 2.

Assume furthermore that the blow-up of Z in X still has two orbits under the action of G. Then, one of the following happens:

- *G is not semi-simple and X is one of the two-orbit varieties classified by Theorem 0.1;*
- *$G = F_4$ and $X = \mathbf{X_1}$;*
- *$G = G_2 \times \mathrm{PSL}(2)$ and $X = \mathbf{X_2}$.*

The varieties in Theorem 0.2 are spherical of rank one [6]. Remark also that odd symplectic grassmannians have been studied in detail by I.A. Mihai in [13]. In particular, he proved that an odd symplectic grassmannian is a linear section of a grassmannian [13, Prop.2.3.15]. It could be interesting to obtain a similar description also for the varieties of Theorem 0.2.

The paper is organized as follows.

In Section 1.1, we recall some results on horospherical homogeneous spaces and horospherical varieties, which we will use throughout Section 1. In particular we briefly summarize the Luna-Vust theory [12] in the case of horospherical homogeneous spaces.

In Section 1.2, we prove that any horospherical homogeneous space admits at most one smooth equivariant compactification with Picard number 1. Then we give the list of horospherical homogeneous spaces that admit a smooth compactification not isomorphic to a projective space and with Picard number 1. We obtain a list of 8 cases (Theorem 1.7).

In Section 1.3, we prove that in 3 of these cases, the smooth compactification is homogeneous (under the action of a larger group).

In Section 1.4, we study the 5 remaining cases (they are listed in Theorem 0.1). We compute the automorphism group of the corresponding smooth compactification with Picard number 1. We prove that this variety has two orbits under the action of its automorphism group and that the latter is connected and not reductive.

In Section 1.5, we prove Theorem 0.2 in the case where the automorphism group is not semi-simple. This gives another characterization of the varieties obtained in Section 1.4.

The aim of Section 2 is to prove Theorem 0.2 when G is semi-simple.

Definition 0.3. A projective G-variety X satisfies (*), if it is smooth with Picard number 1, has two orbits under the action of G such that its closed orbit Z has codimension at least 2, and the blowing-up of X along Z has also two orbits under the action of G.

In Section 2.1, we prove the first part of Theorem 0.2 and we reveal two general cases.

In Sections 2.2 and 2.3, we study these two cases respectively. First, we reformulate part of the classification of two-orbit varieties with closed orbit of codimension one due to D. N. Akhiezer, in order to give a complete and precise list of possible cases. Then we study separately all these possible cases. We prove that the two varieties $\mathbf{X_1}$ and $\mathbf{X_2}$ satisfy (*) and are non-homogeneous, and that in all other cases, the varieties satisfying (*) are homogeneous.

1 Smooth projective horospherical varieties with Picard number 1

1.1 Notation

Let G be a reductive and connected algebraic group over $\mathbb{C}$, let B be a Borel subgroup of G, let T be a maximal torus of B and let U be the unipotent radical of B. Denote by C the center of G and by G' the semi-simple part of G (so that $G = C.G'$). Denote by S the set of simple roots of (G, B, T), and by Λ (respectively Λ^+) the group of characters of B (respectively the set of dominant characters). Denote by W the Weyl group of (G, T) and, when $I \subset S$, denote by W_I the subgroup of W generated by the reflections associated to the simple roots of I. If α is a simple root, we denote by $\check{\alpha}$ its coroot, and by ω_α the fundamental weight corresponding to α (when the roots are $\alpha_1, \dots \alpha_n$, we will write ω_i instead of ω_{α_i}). Denote by $P(\omega_\alpha)$ the maximal parabolic subgroup containing B such that ω_α is a character of $P(\omega_\alpha)$. Let Γ be the Dynkin diagram of G. When $I \subset S$, we denote by Γ_I the full subgraph of Γ with vertices the elements of I. For $\lambda \in \Lambda^+$, we denote by $V(\lambda)$ the irreducible G-module of highest weight λ and by v_λ a highest weight vector of $V(\lambda)$. If

G is simple, we index the simple roots as in [4].

A closed subgroup H of G is said to be *horospherical* if it contains the unipotent radical of a Borel subgroup of G. In that case we also say that the homogeneous space G/H is *horospherical*. Up to conjugation, one can assume that H contains U. Denote by P the normalizer $N_G(H)$ of H in G. Then P is a parabolic subgroup of G such that P/H is a torus. Thus G/H is a torus bundle over the flag variety G/P. The dimension of the torus is called the *rank* of G/H and denoted by n.

A normal variety X with an action of G is said to be a *horospherical variety* if G has an open orbit isomorphic to G/H for some horospherical subgroup H. In that case, X is also said to be a G/H-embedding. The classification of G/H-embeddings (due to D. Luna et Th. Vust [12] in the more general situation of spherical homogeneous spaces) is detailed in [14, Chap.1].

Let us summarize here the principal points of this theory. Let G/H be a fixed horospherical homogeneous space of rank n. This defines a set of simple roots

$$I := \{\alpha \in S \mid \omega_\alpha \text{ is not a character of } P\}$$

where P is the unique parabolic subgroup associated to H as above. We also introduce a lattice M of rank n as the sublattice of Λ consisting of all characters χ of P such that the restriction of χ to H is trivial. Denote by N the dual lattice to M.

In this paper, we call *colors* the elements of $S\backslash I$. For any color α, we denote by $\check{\alpha}_M$ the element of N defined as the restriction to M of the coroot $\check{\alpha} : \Lambda \longrightarrow \mathbb{Z}$. The point $\check{\alpha}_M$ is called the *image of the color* α. See [14, Chap.1] to understand the link between colors and the geometry of G/H.

Definition 1.1. A *colored cone* of $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ is an ordered pair $(\mathcal{C}, \mathcal{F})$ where $\mathcal{C}$ is a convex cone of $N_{\mathbb{R}}$ and $\mathcal{F}$ is a set of colors (called the set of colors of the colored cone), such that
(i) $\mathcal{C}$ is generated by finitely many elements of N and contains the image of the colors of $\mathcal{F}$,
(ii) $\mathcal{C}$ does not contain any line and the image of any color of $\mathcal{F}$ is not zero.

One defines a *colored fan* as a set of colored cones such that any two of them intersect in a common colored face (see [14, def.1.14] for the precise definition).

Then G/H-embeddings are classified in terms of colored fans. Define a *simple* G/H-embedding of X as one containing a unique closed G-orbit. Let X be a G/H-embedding and $\mathbb{F}$ its colored fan. Then X is covered by its simple subembeddings, and each of them corresponds to a colored cone of the colored fan of X. (See [7] or [10] for the general theory of spherical embeddings.)

1.2 Classification of smooth projective embeddings with Picard number 1

The Picard number ρ_X of a smooth projective G/H-embedding X satisfies

$$\rho_X = r_X + \sharp(S\backslash I) - \sharp(\mathcal{D}_X)$$

where $\mathcal{D}_X$ denotes the set of simple roots in $S\backslash I$ which correspond to colors of $\mathbb{F}$ and r_X is the number of rays of the colored fan of X minus the rank n [14, (4.5.1)]. Since X is projective, its colored fan is complete (*i.e.* it covers $N_{\mathbb{R}}$) and hence $r_X \geq 1$. Moreover $\mathcal{D}_X \subset S\backslash I$, so $\rho_X = 1$ if and only if $r_X = 1$ and $\mathcal{D}_X = S\backslash I$. In particular the colored fan of X has exactly $n+1$ rays.

Lemma 1.2. *Let G/H be a horospherical homogeneous space. Up to isomorphism of varieties, there exists at most one smooth projective G/H-embedding with Picard number 1.*

Proof. Let X and X' be two smooth projective G/H-embeddings with respective colored fans $\mathbb{F}$ and $\mathbb{F}'$ and both with Picard number 1. Denote by $e_1, \ldots, e_{n+1}$ the primitive elements of the $n+1$ rays of $\mathbb{F}$. By the smoothness criterion of [14, Chap.2], $(e_1, \ldots, e_n)$ is a basis of N, $e_{n+1} = -e_1 - \cdots - e_n$ and the images in N of the colors are distinct and contained in $\{e_1, \ldots, e_{n+1}\}$. The same happens for $\mathbb{F}'$. Then there exists an automorphism ϕ of the lattice N which stabilizes the image of each color and satisfies $\mathbb{F} = \phi(\mathbb{F}')$. Thus the varieties X and X' are isomorphic [14, Prop. 3.10]. □

If it exists, we denote by X^1 the unique smooth projective G/H-embedding with Picard number 1 and we say that G/H is "special".

Remark 1.3. By the preceding proof, we have $\sharp(\mathcal{D}_{X^1}) \leq n+1$.

1.2.1 Projective space

We first give a necessary condition for the embedding X^1 of a special homogeneous space not to be isomorphic to a projective space. In particular we must have $n = 1$, so that X^1 has three orbits under the action of G: two closed ones and G/H.

Theorem 1.4. *Let G/H be a "special" homogeneous space. Then X^1 is isomorphic to a projective space in the following cases:*

(i) $\sharp(\mathcal{D}_{X^1}) \leq n$,

(ii) $n \geq 2$,

(iii) $n = 1$, $\sharp(\mathcal{D}_{X^1}) = 2$ *and the two simple roots of* $\mathcal{D}_{X^1}$ *are not in the same connected component of the Dynkin diagram* Γ.

Proof. (i) In that case, there exists a maximal colored cone of the colored fan of X which contains all colors. Then the corresponding simple G/H-embedding of X^1, whose closed orbit is a point [14, Lem.2.8], is affine [10, th.3.1] and smooth. So it is necessarily a horospherical G-module V [14, Lem.2.10]. Thus $\mathbb{P}(\mathbb{C}\oplus V)$ is a smooth projective G/H-embedding with Picard number 1. Then by Lemma 1.2, X^1 is isomorphic to $\mathbb{P}(\mathbb{C}\oplus V)$.

(ii) We may assume that $\sharp(\mathcal{D}_{X^1}) = n+1$. Denote by $\alpha_1,\dots,\alpha_{n+1}$ the elements of $S\backslash I$ and by Γ_i the Dynkin diagram $\Gamma_{S\backslash\{\alpha_i\}}$. The smoothness criterion of horospherical varieties [14, Chap.2] applied to X^1 tells us two things.

Firstly, for all $i\in\{1,\dots,n+1\}$ and for all $j\neq i$, α_j is a simple end ("simple" means not adjacent to a double edge) of a connected component Γ_i^j of Γ_i of type A_m or C_m. Moreover the Γ_i^j are distinct, in other words, each connected component of Γ_i has at most one vertex among the $(\alpha_i)_{i\in\{1,\dots,n+1\}}$.

Secondly, $(\check{\alpha}_{iM})_{i\in\{1,\dots,n\}}$ is a basis of N and $\check{\alpha}_{(n+1)M} = -\check{\alpha}_{1M} - \cdots - \check{\alpha}_{nM}$. Thus a basis of M (dual of N) is of the form

$$(\omega_i - \omega_{n+1} + \chi_i)_{i\in\{1,\dots,n\}}$$

where χ_i is a character of the center C of G, for all $i\in\{1,\dots,n\}$.

Let us prove that a connected component of Γ contains at most one vertex among the $(\alpha_i)_{i\in\{1,\dots,n+1\}}$. Suppose the contrary: there exist $i,j\in\{1,\dots,n+1\}$, $i\neq j$ such that α_i and α_j are vertices of a connected component of Γ. One can choose i and j such that there is no vertex among the $(\alpha_k)_{k\in\{1,\dots,n+1\}}$ between α_i and α_j. Since $n\geq 2$, there exists an integer $k\in\{1,\dots,n+1\}$ different from i and j. Then we observe that Γ_k does not satisfy the condition that each of its connected component has at most one vertex among the $(\alpha_i)_{i\in\{1,\dots,n+1\}}$ (because $\Gamma_k^i = \Gamma_k^j$).

Thus we have proved that

$$\Gamma = \bigsqcup_{j=0}^{n+1} \Gamma^j \tag{1.4.1}$$

such that for all $j\in\{1,\dots,n+1\}$, Γ^j is a connected component of Γ of type A_m or C_m in which α_j is a simple end.

For all $\lambda\in\Lambda^+$, denote by by $V(\lambda)$ the simple G-module of weight λ. Then Equation 1.4.1 tells us that the projective space

$$\mathbb{P}(V(\omega_{n+1})\oplus V(\omega_1+\chi_1)\oplus\cdots\oplus V(\omega_n+\chi_n))$$

is a smooth projective G/H-embedding with Picard number 1. Thus X^1 is isomorphic to this projective space.

(iii) As in case (ii), one checks that X^1 is isomorphic to $\mathbb{P}(V(\omega_2)\oplus V(\omega_1+\chi_1))$ for some character χ_1 of C. □

1.2.2 When X^1 is not isomorphic to a projective space

According to Theorem 1.4 we have to consider the case where the rank of G/H is 1 and where there are two colors corresponding to simple roots α and β in the same connected component of Γ. As we have seen in the proof of Theorem 1.4, the lattice M (here of rank 1) is generated by $\omega_\alpha-\omega_\beta+\chi$ where χ is a character of the center C of G. Moreover, H is the kernel of the character $\omega_\alpha-\omega_\beta+\chi:\ P(\omega_\alpha)\cap P(\omega_\beta)\longrightarrow\mathbb{C}^*$.

We may further reduce to the case where G is semi-simple (recall that G' denotes the semi-simple part of G).

Proposition 1.5. *Let $H'=G'\cap H$. Then G/H is isomorphic to G'/H'.*

Proof. We are going to prove that G/H and G'/H' are both isomorphic to a horospherical homogeneous space under $(G'\times\mathbb{C}^*)$. In fact G/H is isomorphic to $(G'\times P/H)/\tilde{H}$ [14, Proof of Prop.3.10], where

$$\tilde{H}=\{(g,pH)\in G'\times P/H\mid gp\in H\}.$$

Similarly, G'/H' is isomorphic to $(G'\times P'/H')/\tilde{H}'$ where $P'=P\cap G'$ and $\tilde{H}'$ defined as the same way as $\tilde{H}$. Moreover the morphisms

$$\begin{array}{ccc} P/H & \longrightarrow & \mathbb{C}^* \\ pH & \longmapsto & (\omega_\alpha-\omega_\beta+\chi)(p) \end{array} \quad\text{and}\quad \begin{array}{ccc} P'/H' & \longrightarrow & \mathbb{C}^* \\ p'H' & \longmapsto & (\omega_\alpha-\omega_\beta)(p') \end{array}$$

are isomorphisms. Then

$$\begin{array}{rcl} \tilde{H} & = & \{(p',c)\in P'\times\mathbb{C}^*\mid(\omega_\alpha-\omega_\beta+\chi)(p')=c^{-1}\} \\ & = & \{(p',c)\in P'\times\mathbb{C}^*\mid(\omega_\alpha-\omega_\beta)(p')=c^{-1}\} \\ & = & \tilde{H}'. \end{array}$$

This completes the proof. □

Remark 1.6. In fact $P/H\simeq\mathbb{C}^*$ acts on G/H by right multiplication, so it acts on the $\mathbb{C}^*$-bundle $G/H\longrightarrow G/P$ by multiplication on fibers. Moreover, this action extends to X^1 (where $\mathbb{C}^*$ acts trivially on the two closed G-orbits).

So we may assume that G is semi-simple. Let $G_1,\dots,G_k$ the simple normal subgroups of G, so that G is the quotient of the product $G_1\times\cdots\times G_k$ by a central finite group C_0. We can suppose that C_0 is trivial, because $G/H\simeq\tilde{G}/\tilde{H}$ where $\tilde{G}=G_1\times\cdots\times G_k$ and $\tilde{H}$ is the preimage of H in $\tilde{G}$. If α and β are simple roots of the connected component corresponding to G_i, denote by H_i is the kernel of the character $\omega_\alpha-\omega_\beta$ of the parabolic subgroup $P(\omega_\alpha)\cap P(\omega_\beta)$ of G_i. Then

$$H=G_1\times\cdots\times G_{i-1}\times H_i\times G_{i+1}\times\cdots\times G_k$$

and $G/H=G_i/H_i$.

So from now on, without loss of generality, we suppose that G is simple.

Theorem 1.7. *With the assumptions above, G/H is "special" if and only if (Γ, α, β) appears in the following list (up to exchanging α and β).*

1. $(A_m, \alpha_1, \alpha_m)$, *with* $m \geq 2$*;*
2. $(A_m, \alpha_i, \alpha_{i+1})$, *with* $m \geq 3$ *and* $i \in \{1, \ldots, m-1\}$
3. $(B_m, \alpha_{m-1}, \alpha_m)$, *with* $m \geq 3$*;*
4. $(B_3, \alpha_1, \alpha_3)$
5. $(C_m, \alpha_{i+1}, \alpha_i)$ *with* $m \geq 2$ *and* $i \in \{1, \ldots, m-1\}$
6. $(D_m, \alpha_{m-1}, \alpha_m)$, *with* $m \geq 4$*;*
7. $(F_4, \alpha_2, \alpha_3)$
8. $(G_2, \alpha_2, \alpha_1)$

Proof. The Dynkin diagrams $\Gamma_{S\backslash\{\alpha\}}$ and $\Gamma_{S\backslash\{\beta\}}$ are respectively of type A_m or C_m by the smoothness criterion [14, Chap.2]. And for the same reason, α and β are simple ends of $\Gamma_{S\backslash\{\beta\}}$ and $\Gamma_{S\backslash\{\alpha\}}$ respectively.

Suppose Γ is of type A_m. If α equals α_1 then, looking at $\Gamma_{S\backslash\{\alpha\}}$, we remark that β must be α_2 or α_m. So we are in Case 1 or 2. If α equals α_m the argument is similar. Now if α is not an end of Γ, in other words if $\alpha = \alpha_i$ for some $i \in \{2, \ldots, m-1\}$ then, looking at $\Gamma_{S\backslash\{\alpha\}}$, we see that β can be α_1, α_{i-1}, α_{i+1} or α_m. The cases where β equals α_1 or α_m are already done and the case where β equals α_{i-1} or α_{i+1} is Case 2.

The study of the remaining cases is analogous and left to the reader. □

In the next two sections we are going to study the variety X^1 for each case of this theorem. In particular we will see that X^1 is never isomorphic to a projective space.

1.3 Homogeneous varieties

In this section, with the notation of Section 1.2.2, we are going to prove that X^1 is homogeneous in Cases 1, 2 and 6.

In all cases (1 to 8), there are exactly 4 projective G/H-embeddings and they are all smooth; they correspond to the 4 colored fans consisting of the two half-lines of $\mathbb{R}$, without color, with one of the two colors and with the two colors, respectively (see [14, Ex.1.19] for a similar example).

Let us realize X^1 in a projective space as follows. The homogeneous space G/H is isomorphic to the orbit of the point $[v_{\omega_\beta} + v_{\omega_\alpha}]$ in $\mathbb{P}(V(\omega_\beta) \oplus V(\omega_\alpha))$, where v_{ω_α} and v_{ω_β}

are highest weight vectors of $V(\omega_\alpha)$ and $V(\omega_\beta)$ respectively. Then X^1 is the closure of this orbit in $\mathbb{P}(V(\omega_\beta) \oplus V(\omega_\alpha))$, because both have the same colored cone (*i.e.* that with two colors)[1].

We will describe the other G/H-embeddings in the proof of Lemma 1.17.

Proposition 1.8. *In Case 1, X^1 is isomorphic to the quadric $Q^{2m} = \mathrm{SO}_{2m+2}/P(\omega_1)$.*

Proof. Here, the fundamental G-modules $V(\omega_\alpha)$ and $V(\omega_\beta)$ are the simple SL_{m+1}-modules $\mathbb{C}^{m+1}$ and its dual $(\mathbb{C}^{m+1})^*$, respectively. Let denote by Q the quadratic form on $\mathbb{C}^{m+1} \oplus (\mathbb{C}^{m+1})^*$ defined by $Q(u, u^*) = \langle u^*, u\rangle$. Then Q is invariant under the action of $\mathrm{SL}_{\mathrm{m+1}}$. Moreover $Q(v_{\omega_\alpha} + v_{\omega_\beta}) = 0$, so that X^1 is a subvariety of the quadric $(Q = 0)$ in $\mathbb{P}(\mathbb{C}^{m+1} \oplus (\mathbb{C}^{m+1})^*) = \mathbb{P}(\mathbb{C}^{2m+2})$.

We complete the proof by computing the dimension of X^1:

$$\dim X^1 = \dim G/H = 1 + \dim G/P = 1 + \sharp(R^+ \backslash R_I^+) \tag{1.8.1}$$

where R^+ is the set of positive roots of (G, B) and R_I^+ is the set of positive roots generated by simple roots of I. So $\dim X^1 = \dim Q^{2m} = 2m$ and $X^1 = Q^{2m}$. □

Proposition 1.9. *In Case 2, X^1 is isomorphic to the grassmannian $\mathrm{Gr}(i+1, m+2)$.*

Proof. The fundamental $\mathrm{SL}_{\mathrm{m+1}}$-modules are exactly the

$$V(\omega_i) = \bigwedge^i \mathbb{C}^{m+1}$$

and a highest weight vector of $V(\omega_i)$ is $e_1 \wedge \cdots \wedge e_i$ where $e_1, \ldots, e_{m+1}$ is a basis of $\mathbb{C}^{m+1}$.

We have

$$\begin{array}{ccc} X & \hookrightarrow & \mathbb{P}(\bigwedge^i \mathbb{C}^{m+1} \oplus \bigwedge^{i+1} \mathbb{C}^{m+1}) \\ \uparrow & & \uparrow \\ G/H & = & G.[e_1 \wedge \cdots \wedge e_i + e_1 \wedge \cdots \wedge e_{i+1}] \end{array}$$

Complete $(e_1, \ldots, e_{m+1})$ to obtain a basis $(e_0, \ldots, e_{m+1})$ of $\mathbb{C}^{m+2}$, then the morphism

$$\begin{array}{ccc} \bigwedge^i \mathbb{C}^{m+1} \oplus \bigwedge^{i+1} \mathbb{C}^{m+1} & \longrightarrow & \bigwedge^{i+1} \mathbb{C}^{m+2} \\ x + y & \longmapsto & x \wedge e_0 + y \end{array}$$

is an isomorphism. Then X^1 is a subvariety of the grassmannian

$$\mathrm{Gr}(i+1, m+2) \simeq \mathrm{SL}_{m+2} .[e_1 \wedge \cdots \wedge e_i \wedge (e_0 + e_{i+1})] \subset \mathbb{P}(\bigwedge^{i+1} \mathbb{C}^{m+1}).$$

We conclude by proving that they have the same dimension using Formula 1.8.1. □

[1] See [14, Chap.1] for the construction of the colored fan of a G/H- embedding.

Proposition 1.10. *In Case 6, X^1 is isomorphic to the spinor variety* $\mathrm{Spin}(2m+1)/P(\omega_m)$.

Proof. The direct sum $V(\omega_\alpha)\oplus V(\omega_\beta)$ of the two half-spin $\mathrm{Spin}(2m)$-modules is isomorphic to the spin $\mathrm{Spin}(2m+1)$-module. Moreover $v_{\omega_\alpha}+v_{\omega_\beta}$ is in the orbit of a highest weight vector of the spin $\mathrm{Spin}(2m+1)$-module. Thus we deduce that X^1 is a subvariety of $\mathrm{Spin}(2m+1)/P(\omega_m)$. We conclude by proving that they have the same dimension using Formula 1.8.1. □

1.4 Non-homogeneous varieties

With the notation of Section 1.2.2 we prove in this section the following result.

Theorem 1.11. *In Cases 3, 4, 5, 7 and 8 (and only in these cases), X^1 is not homogeneous.*

Moreover the automorphism group of X^1 is $(\mathrm{SO}(2m+1)\times\mathbb{C}^*)\ltimes V(\omega_m)$, $(\mathrm{SO}(7)\times\mathbb{C}^*)\ltimes V(\omega_3)$, $((\mathrm{Sp}(2m)\times\mathbb{C}^*)/\{\pm1\})\ltimes V(\omega_1)$, $(\mathrm{F}_4\times\mathbb{C}^*)\ltimes V(\omega_4)$ *and* $(\mathrm{G}_2\times\mathbb{C}^*)\ltimes V(\omega_1)$ *respectively.*

Finally, X^1 has two orbits under its automorphism group.

One can remark that in Case 5, Theorem 1.11 follows from results of I. A. Mihai [13, Chap.3 and Prop.5.1] combined with the following result.

Proposition 1.12. *In Case 5, X^1 is isomorphic to the odd symplectic grassmannian* $\mathrm{Gr}_\omega(i+1,2m+1)$.

The proof is very similar to the proof of Proposition 1.9 and is left to the reader.

Now, let X be one of the varieties X^1 in Cases 3, 4, 5, 7 and 8.

Then X has three orbits under the action of G (the open orbit X_0 isomorphic to G/H and two closed orbits). Recall that X can be seen as a subvariety of $\mathbb{P}(V(\omega_\alpha)\oplus V(\omega_\beta))$. Let $P_Y:=P(\omega_\alpha)$, $P_Z:=P(\omega_\beta)$ and denote by Y and Z the closed orbits, isomorphic to G/P_Y and G/P_Z respectively. (In Case 8 where G is of type G_2, we have $\alpha=\alpha_2$ and $\beta=\alpha_1$.)

Let X_Y be the simple G/H-embedding of X with closed orbit Y, we have $X_Y=X_0\cup Y$. Then, by [14, Chap.2], X_Y is a homogeneous vector bundle over G/P_Y in the sense of the following.

Definition 1.13. Let P be a parabolic subgroup of G and V a P-module. Then the homogeneous vector bundle $G\times^P V$ over G/P is the quotient of the product $G\times V$ by the equivalence relation $\sim$ defined by

$$\forall g\in G,\ \forall p\in P, \forall v\in V,\quad (g,v)\sim(gp^{-1},p.v).$$

Specifically, $X_Y = G \times^{P_Y} V_Y$ where V_Y is a simple P_Y-module of highest weight $\omega_\beta - \omega_\alpha$, and similarly, $X_Z = G \times^{P_Z} V_Z$ where V_Z is a simple P_Z-module of highest weight $\omega_\alpha - \omega_\beta$.

Denote by $\mathrm{Aut}(X)$ the automorphism group of X and $\mathrm{Aut}^0(X)$ the connected component of $\mathrm{Aut}(X)$ containing the identity.

Remark 1.14. Observe that $\mathrm{Aut}(X)$ is a linear algebraic group. Indeed $\mathrm{Aut}(X)$ acts on the Picard group of X which equals $\mathbb{Z}$ (the Picard group of a projective spherical variety is free [5]). This action is necessarily trivial. Then $\mathrm{Aut}(X)$ acts on the projectivization of the space of global sections of a very ample bundle. This gives a faithful projective representation of $\mathrm{Aut}(X)$.

We now complete the proof of Theorem 1.11 by proving several lemmas.

Lemma 1.15. *The closed orbit Z of X is stable under* $\mathrm{Aut}^0(X)$.

Proof. We are going to prove that the normal sheaf N_Z of Z in X has no nonzero global section. This will imply that $\mathrm{Aut}^0(X)$ stabilizes Z, because the Lie algebra $\mathrm{Lie}(\mathrm{Aut}^0(X))$ is the space of global sections $H^0(X, T_X)$ of the tangent sheaf T_X [2, Chap.2.3] and we have the following exact sequence

$$0 \longrightarrow T_{X,Z} \longrightarrow T_X \longrightarrow N_Z \longrightarrow 0$$

where $T_{X,Z}$ is the subsheaf of T_X consisting of vector fields that vanish along Z. Moreover $H^0(X, T_{X,Z})$ is the Lie algebra of the subgroup of $\mathrm{Aut}^0(X)$ that stabilizes Z.

The total space of N_Z is the vector bundle X_Z. So using the Borel-Weil theorem [2, 4.3], $H^0(G/P_Z, N_Z) = 0$ if and only if the smallest weight of V_Z is not antidominant. The smallest weight of V_Z is $w_0^\beta(\omega_\alpha - \omega_\beta)$ where w_0^β is the longest element of $W_{S\backslash\{\beta\}}$. Let $\gamma \in S$, then

$$\langle w_0^\beta(\omega_\alpha - \omega_\beta), \check{\gamma} \rangle = \langle \omega_\alpha - \omega_\beta, w_0^\beta(\check{\gamma}) \rangle.$$

If γ is different from β then $w_0^\beta(\check{\gamma}) = -\check{\delta}$ for some $\delta \in S\backslash\{\beta\}$. So we only have to compute $w_0^\beta(\check{\beta})$.

In Case 3, $\beta = \alpha_m$, so w_0^β maps α_i to $-\alpha_{m-i}$ for all $i \in \{1, \ldots, m-1\}$. Here, $\omega_\beta = \frac{1}{2}(\alpha_1 + 2\alpha_2 + \cdots + m\alpha_m)$. Then, using the fact that $w_0^\beta(\omega_\beta) = \omega_\beta$, we have $w_0^\beta(\beta) = \alpha_1 + \cdots + \alpha_m$ so that $w_0^\beta(\check{\beta}) = 2(\check{\alpha}_1 + \cdots + \check{\alpha}_{m-1}) + \check{\alpha}_m$ and $\langle \omega_\alpha - \omega_\beta, w_0^\beta(\check{\gamma}) \rangle = 1$ (because $\alpha = \alpha_{m-1}$).

The computation of $w_0^\beta(\check{\beta})$ in the other cases is similar and left to the reader. In all four cases,$\langle \omega_\alpha - \omega_\beta, w_0^\beta(\check{\beta}) \rangle > 0$ (this number equals 1 in Cases 3, 4, 5, 7 and 2 in Case 8). This proves that $w_0^\beta(\omega_\alpha - \omega_\beta)$ is not antidominant. □

Remark 1.16. Using the Borel-Weil theorem we can also compute $H^0(G/P_Y, N_Y)$ by the same method. We find that in Cases 3, 4, 5, 7 and 8, this G-module is isomorphic to the simple G-module $V(\omega_m)$, $V(\omega_3)$, $V(\omega_1)$, $V(\omega_4)$ and $V(\omega_1)$ respectively.

We now prove the following lemma.

Lemma 1.17. *In Cases 3, 4, 5, 7 and 8,* $\mathrm{Aut}^0(X)$ *is* $(\mathrm{SO}(2m+1)\times\mathbb{C}^*)\ltimes V(\omega_m)$, $(\mathrm{SO}(7)\times\mathbb{C}^*)\ltimes V(\omega_3)$, $(\mathrm{Sp}(2m)\times\mathbb{C}^*)\ltimes V(\omega_1)$, $(\mathrm{F}_4\times\mathbb{C}^*)\ltimes V(\omega_4)$ *and* $(\mathrm{G}_2\times\mathbb{C}^*)\ltimes V(\omega_1)$ *respectively.*

Remark 1.18. By Remark 1.6, we already know that the action of G on X extends to an action of $G\times\mathbb{C}^*(\simeq G\times P/H)$. Moreover $\tilde{C}:=\{(c,c^{-1}H)\in C\times P/H\}\subset G\times\mathbb{C}^*$ acts trivially on X.

Proof. Let $\pi:\tilde{X}\longrightarrow X$ be the blowing-up of Z in X. Since Z and X are smooth, $\tilde{X}$ is smooth; it is also a projective G/H-embedding. In fact $\tilde{X}$ is the projective bundle

$$\phi:\,G\times^{P_Y}\mathbb{P}(V_Y\oplus\mathbb{C})\longrightarrow G/P_Y$$

where P_Y acts trivially on $\mathbb{C}$ (it is the unique projective G/H-embeddings with unique color α). Moreover the exceptional divisor $\tilde{Z}$ of $\tilde{X}$ is G/P.

Let us remark that $\mathrm{Aut}^0(\tilde{X})$ is isomorphic to $\mathrm{Aut}^0(X)$. Indeed, it contains $\mathrm{Aut}^0(X)$ because Z is stable under the action of $\mathrm{Aut}^0(X)$. Conversely, we know, by a result of A. Blanchard, that $\mathrm{Aut}^0(\tilde{X})$ acts on X such that π is equivariant [2, Chap.2.4].

Now we are going to compute $\mathrm{Aut}^0(\tilde{X})$. Observe that $H^0(G/P_Y,N_Y)$ acts on $\tilde{X}$ by translations on the fibers of ϕ:

$$\forall s\in H^0(G/P_Y,N_Y),\ \forall(g_0,[v_0,\xi])\in G\times^{P_Y}\mathbb{P}(V_Y\oplus\mathbb{C}),\ s.(g_0,[v_0,\xi])=(g_0,[v_0+\xi v(g_0),\xi])$$

where $v(g_0)$ is the element of V_Y such that $s(g_0)=(g_0,v(g_0))$.

Then the group $((G\times\mathbb{C}^*)/\tilde{C})\ltimes H^0(G/P_Y,N_Y)$ acts effectively on $\tilde{X}$ (the semi-product is defined by $((g',c'),s').((g,c),s)=((g'g,c'c),c'g's+s'))$. In fact we are going to prove that

$$\mathrm{Aut}^0(\tilde{X})=((G\times\mathbb{C}^*)/\tilde{C})\ltimes H^0(G/P_Y,N_Y).$$

By [2, Chap.2.4] again, we know that $\mathrm{Aut}^0(\tilde{X})$ exchanges the fibers of ϕ and induces an automorphism of G/P_Y. Moreover we have $\mathrm{Aut}^0(G/P_Y)=G/C$ in our four cases [2, Chap.3.3]. So we have an exact sequence

$$0\longrightarrow A\longrightarrow\mathrm{Aut}^0(\tilde{X})\longrightarrow G/C\longrightarrow 0$$

where A is the set of automorphisms which stabilize each fiber of the projective bundle $\tilde{X}$. In fact A consists of affine transformations in fibers.

Then $H^0(G/P_Y,N_Y)$ is the subgroup of A consisting of translations. Let A_0 be the subgroup of A consisting of linear transformations in fibers. Then A_0 fixes Y so that A_0 acts on the blowing-up $\tilde{\tilde{X}}$ of Y in $\tilde{X}$. Moreover $\tilde{\tilde{X}}$ is a $\mathbb{P}^1$-bundle over G/P, it is in fact the unique G/H-embedding without colors [14, Ex.1.13 (2)]. As before we know that $\mathrm{Aut}^0(\tilde{\tilde{X}})$

exchanges the fibers of that $\mathbb{P}^1$-bundle and induces an automorphism of G/P. Moreover we have $\mathrm{Aut}^0(G/P) = G/C$ and then $\mathrm{Aut}^0(\tilde{\tilde{X}}) = (G \times \mathbb{C}^*)/\tilde{C}$. We deduce that $A_0 = \mathbb{C}^*$.

We complete the proof of Lemma 1.17 by Remark 1.16.

□

To complete the proof of Theorem 1.11 we prove the following lemma.

Lemma 1.19. *The automorphism group of X is connected.*

Proof. Let ϕ be an automorphism of X. We want to prove that ϕ is in $\mathrm{Aut}^0(X)$. But ϕ acts by conjugation on $\mathrm{Aut}^0(X)$. Let L be a Levi subgroup of $\mathrm{Aut}^0(X)$. Then $\phi^{-1}L\phi$ is again a Levi subgroup of $\mathrm{Aut}^0(X)$. But all Levi subgroups are conjugated in $\mathrm{Aut}^0(X)$. So we can suppose, without loss of generality, that ϕ stabilizes L.

Then ϕ induces an automorphism of the direct product of $\mathbb{C}^*$ with a simple group G of type B_m, C_m, F_4 or G_2 (Lemma 1.17). It also induces an automorphism of G which is necessarily an inner automorphism (because there is no non-trivial automorphism of the Dynkin diagram of G). So we can assume now that ϕ commutes with all elements of G.

Then ϕ stabilizes the open orbit G/H of X. Let $x_0 := H \in G/H \subset X$ and $x_1 := \phi(x_0) \in G/H$. Since ϕ commutes with the elements of G, the stabilizer of x_1 is also H. So ϕ acts on G/H as an element of $N_G(H)/H = P/H \simeq \mathbb{C}^*$ (where $N_G(H)$ is the normalizer of H in G). Then ϕ is an element of $\mathbb{C}^* \subset \mathrm{Aut}^0(X)$. □

Remark 1.20. In Case 5, we recover the result of I. Mihai: $\mathrm{Aut}(X) = ((\mathrm{Sp}(2m) \times \mathbb{C}^*)/\{\pm 1\}) \ltimes \mathbb{C}^{2m}$.

1.5 First step in the proof of Theorem 0.2

In this section, we prove the following result.

Theorem 1.21. *Let X be a smooth projective variety with Picard number 1 and put $\mathbf{G} := \mathrm{Aut}^0(X)$. Suppose that $\mathbf{G}$ is not semi-simple and that X has two orbits under the action of $\mathbf{G}$. Denote by Z the closed orbit.*

Then the codimension of Z is at least 2.

Suppose in addition that the blowing-up of X along Z has also two orbits under the action of $\mathbf{G}$. Then X is one of the varieties X^1 obtained in Cases 3, 4, 5, 7 and 8 of Theorem 1.7.

Remark 1.22. The converse implication holds by Theorem 1.11.

We will use a result of D. Akhiezer on two-orbit varieties with codimension one closed orbit.

Lemma 1.23 (Th.1 of [1])**.** *Let X be a smooth complete variety with an effective action of the connected linear non semi-simple group $\mathbf{G}$. Suppose that X has two orbits under the action of $\mathbf{G}$ and that the closed orbit is of codimension 1. Let G be a maximal semi-simple subgroup of $\mathbf{G}$.*

Then there exist a parabolic subgroup P of G and a P-module V such that:
(i) the action of P on $\mathbb{P}(V)$ is transitive;
((ii) there exists an irreducible G-module W and a surjective P-equivariant morphism $W \longrightarrow V$;)
(iii) $X = G \times^P \mathbb{P}(V \oplus \mathbb{C})$ as a G-variety.

Remark 1.24. It follows from (i) that V is an (irreducible) horospherical L-module of rank 1, where L is a Levi subgroup of G. This implies that X is a horospherical G-variety of rank 1.

(The G-module W is the set of global sections of the vector bundle $G \times^P V \longrightarrow G/P$.)

Proof of Theorem 1.21. If Z is of codimension 1, Lemma 1.23 tells us that X is a horospherical G-variety. Moreover the existence of the irreducible G-stable divisor Z tells us that one of the two rays of the colored fan of X has no color [14, Chap 1]. Then X satisfies the condition (i) of Theorem 1.4 and $X = \mathbb{P}(V \oplus \mathbb{C})$. Since X is not homogeneous we conclude that the codimension of Z is at least 2.

Denote by $\tilde{X}$ the blowing-up of X along Z. Then $\tilde{X}$ is a horospherical G-variety by Lemma 1.23. Moreover X and $\tilde{X}$ have the same open G-orbit, so that X is also a horospherical G-variety. We conclude by the description of smooth projective horospherical varieties with Picard number 1 obtained in the preceding sections . □

2 On some two-orbit varieties under the action of a semi-simple group

The aim of this section is to prove Theorem 0.2.

We keep the notation of the first paragraph of Section 1.1. In all this section, G is a semi-simple group and all varieties are spherical of rank one (but not horospherical).

We will often use the following result, that can be deduced from the local structure of spherical varieties [5, Chap.1].

Proposition 2.1. *Let X be a (spherical) two-orbit G-variety, with closed orbit isomorphic to G/Q (choose $Q \supset B \supset T$ such that X has an open B-orbit). Let $L(Q)$ be the Levi subgroup of Q containing T and Q^- the parabolic subgroup of G opposite to Q. Then there exists an affine $L(Q)$-subvariety V of X with a fixed point and an affine open subvariety X_0 of X, such that the natural morphism from $R_u(Q^-) \times V$ to X is an isomorphism into X_0. Moreover V is spherical of rank one.*

2.1 First results

Let us first prove the first part of Theorem 0.2.

Proposition 2.2. *Let X be a non-homogeneous projective two-orbit variety with Picard number 1. Then the codimension of the closed orbit Z is at least* 2.

Proof. Suppose that Z is of codimension 1. This implies that X is smooth.

Let G/H be a homogenous space isomorphic to the open orbit of X and P be a parabolic subgroup of G containing H and minimal for this property.

Then P satisfies one of the two following conditions [1, proof of Th.5]:

- $R(P) \subset H \subset P$,
- H contains a Levi subgroup of P.

Moreover, if $P = G$, then X is unique and homogeneous under its full automorphism group [1, Th.4]. So P is a proper subgroup of G.

We may assume, without loss of generality, that G is simply connected.

We have the following exact sequence.

$$\mathbb{Z}[Z] \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(G/H) \longrightarrow 0$$

so that the Picard group $\mathrm{Pic}(G/H)$ of G/H is finite. But, by [18, Th.2.2], $\mathrm{Pic}(G/H)$ is isomorphic to the group $\mathfrak{X}(H)$ of characters of H . This contradicts the fact that H contains the radical or a Levi subgroup of a proper parabolic subgroup of G. Indeed we have one of the two following commutative diagrams (respectively when $R(P) \subset H$ and $L(P) \subset H$),

$$\begin{array}{ccc} \mathfrak{X}(P) & \longrightarrow & \mathfrak{X}(H) \\ & \searrow & \downarrow \\ & & \mathfrak{X}(R(P)) \end{array} \qquad \begin{array}{ccc} \mathfrak{X}(P) & \longrightarrow & \mathfrak{X}(H) \\ & \searrow & \downarrow \\ & & \mathfrak{X}(L(P)) \end{array}$$

where the maps are restrictions and $L(P)$ is a Levi subgroup of P. Moreover the maps $\mathfrak{X}(P) \longrightarrow \mathfrak{X}(R(P))$ and $\mathfrak{X}(P) \longrightarrow \mathfrak{X}(L(P))$ are injective. Then $\mathfrak{X}(P) = 0$ so that $P = G$. □

From now on, let X be a two-orbit variety satisfying (*) (Definition 0.3). Let G/H be a homogenous space isomorphic to the open orbit of X and P be a parabolic subgroup of G containing H and minimal for this property. Then, we still have the two cases of Proposition 2.2: $R(P) \subset H \subset P$ or H contains a Levi subgroup of P.

Denote by $\pi : \tilde{X} \longrightarrow X$ the blow-up of X along Z. The variety $\tilde{X}$ is the unique G/H-embedding (*i.e.* a normal G-variety with an open orbit isomorphic to G/H) with closed orbit of codimension 1.

Note that $\tilde{X}$ cannot be homogeneous under its full automorphism group, because of a result of A. Blanchard [2, Chap.2.4].

Let us remark also that X is the unique G/H-embedding satisfying (*). Indeed, a spherical homogeneous space of rank one cannot have two different projective G/H-embeddings with Picard number one. In fact, if it exists, the projective embedding with Picard number one is the G/H-embedding associated to the unique complete colored fan with all possible colors (see, for example [10] for the classification of G/H-embeddings, and [5] for the description of the Picard group of spherical varieties).

To conclude this section, let us prove the following.

Lemma 2.3. *P is a maximal parabolic subgroup of G.*

Proof. Let us assume, without loss of generality, that G is simply connected. Since Z has codimension at least 2, the Picard groups of G/H and X are the same. The argument of the proof of Prop 2.2 implies that $\mathfrak{X}(P) = \mathbb{Z}$ so that P is maximal. □

2.2 When $R(P) \subset H$

We suppose in all this section that $R(P) \subset H$. Let us remark that P/H is isomorphic to $(P/R(P))/(H/R(P))$ and is still spherical of rank one. Then we deduce the following lemma from Theorem 4 of [1].

Lemma 2.4. *Suppose that there exists a projective G/H-embedding with Picard number 1 and that $R(P) \subset H \subset P \neq G$. Then P/H is isomorphic to one of the following homogeneous spaces G'/H' where G' is a quotient of a normal subgroup of $P/R(P)$ by a finite central subgroup:*

	G'	H'	X'	Q'
1a	SO(n), $n \geq 4$	SO(n-1)	Q^{n-1}	$P(\omega_{\alpha_1})$ *or* B *if* $n = 4$
1b	SO(n) $/C'$, $n \geq 4$	S(O(1) × O(n-1)) $/C'$	$\mathbb{P}^{n-1}$	$P(\omega_{\alpha_1})$ *or* B *if* $n = 4$
2	Sp(2n) $/C'$, $n \geq 2$	(Sp(2n-2) × Sp(2))$/C'$	$\mathrm{Gr}(2, 2n)$	$P(\omega_{\alpha_2})$
3a	Spin(7)	G_2	Q^7	$P(\omega_{\alpha_3})$
3b	SO(7)	G_2	$\mathbb{P}^7$	$P(\omega_{\alpha_3})$

Here we denote by C' the center of G'. The variety X' denotes the unique projective G'/H'-embedding. Its closed orbit is of codimension 1 and isomorphic to G'/Q' where Q' is the parabolic subgroup given in the last column. We denote by Q^{n-1} the quadric of dimension $n-1$ in $\mathbb{P}^n$ and by $\mathrm{Gr}(2, 2n)$ the grassmannian of planes in $\mathbb{C}^{2n}$.

Let us remark that, in case 1a and 1b when $n = 4$, the group G' is of type $A_1 \times A_1$. This is the only case where G' is not simple.

Moreover, the closed orbit Z of X is isomorphic to G/Q, where Q is a parabolic subgroup of G containing B such that $P/(P \cap Q) \simeq G'/Q'$.

Proof. We apply [1, Th.4] to the pair $(P/R(P), H/R(P))$ (note that $H/R(P)$ is reductive because P is a minimal parabolic subgroup containing H). So P/H is isomorphic to a homogeneous space listed in [1, Table 2]. Since P is a proper parabolic subgroup of G, the group G' cannot be G_2 or F_4.

Moreover we have already seen that the restriction morphism $\mathfrak{X}(H) \longrightarrow \mathfrak{X}(R(P))$ is injective, hence $\mathfrak{X}(H')$ is finite. So the cases ($G' = \mathrm{PSL(n+1)}$, $H' = \mathrm{GL(n)}$), ($G' = \mathrm{SO}(3)$, $H' = \mathrm{SO}(2)$) and ($G' = \mathrm{SO}(3)$, $H' = \mathrm{S}(\mathrm{O}(1) \times \mathrm{O}(2))$) cannot happen.

For the last statement, remark that $\tilde{X} = G \times^P X'$ (see Definition 1.13), the closed orbit of $\tilde{X}$ is $G/(P \cap Q)$ and the closed orbit of X' is $G'/Q' \simeq P/(P \cap Q)$. Note that the blow-up $\pi : \tilde{X} \longrightarrow X$ is proper is G-equivariant and it sends the closed orbit of $\tilde{X}$ to the closed orbit Z of X, so that Z is isomorphic to G/Q, G/P or to a point. By Proposition 2.1, a two-orbit variety whose closed orbit is a point must be affine, so that Z cannot be point. Now if Z is isomorphic to G/P, again by Proposition 2.1, there exists an open subvariety of X isomorphic to the product of an open subvariety of G/P and a closed subvariety S of X spherical under the action of a Levi subgroup of P. Moreover S is necessarily isomorphic to a projective G'/H'-embedding with closed orbit a point, which gives us a contradiction. □

Let us now translate the fact that X is smooth.

Lemma 2.5. *Suppose that there exists a smooth G/H-embedding X with closed orbit G/Q.*

Denote by Q^- the parabolic subgroup of G opposite to Q and by $L(Q) := Q \cap Q^-$ the Levi subgroup of Q containing T.

Then, there is an affine open subvariety of X that is isomorphic to the product of $R_u(Q^-)$ and a $L(Q)$-module V included in X.

Moreover, V is either a fundamental module of a group of type A_n and of highest weight ω_1 or ω_n, or a fundamental module of a group of type C_n and of highest weight ω_1 (modulo the action of the center of Q).

Remark 2.6. For the variety X', one can directly compute that there is an affine open subvariety of X' that is isomorphic to the product of $R_u(Q'^-)$ and a line where $L(Q')$ acts respectively with weight:

$-\omega_1$ in Case 1a ($-\omega_1 - \omega_1'$ when $n = 4$);
$-2\omega_1$ in Case 1b ($-2\omega_1 - 2\omega_1'$ when $n = 4$);
$-\omega_2$ in Case 2;
$-\omega_3$ in Case 3a;
$-2\omega_3$ in Case 3b.

This remark will be used to prove that, in some cases, X cannot be smooth.

Proof. By Proposition 2.1, there exists a smooth affine $L(Q)$-subvariety V of X with a fixed point and an affine open subvariety of X that is isomorphic to $R_u(Q^-) \times V$. Moreover V is spherical of rank one.

But if L is a reductive connected algebraic group acting on a smooth affine variety V with a fixed point, then V is a $L(Q)$-module [15, Cor.6.7]. So V is a $L(Q)$-module.

Moreover V has two $L(Q)$-orbits, so that it is horospherical. Then, the last part of the lemma follows from [14, Lem.2.13]. □

Corollary 2.7. *Suppose that there exists a G/H-embedding X satisfying (*). Suppose also that Q is maximal (this holds in all cases except Case 1 of Lemma 2.4 with $n = 4$). Let i and j be the indices such that $P = P(\omega_i)$ and $Q = P(\omega_j)$.*

Let us define Γ_i^j (respectively Γ_j^i) to be the connected component of the subgraph $\Gamma_{S\backslash\{\alpha_i\}}$ (respectively $\Gamma_{S\backslash\{\alpha_j\}}$) of the Dynkin diagram of G containing α_j (respectively α_i).

Then Γ_i^j is of one of the following types:
B_n, $n \geq 2$, with first vertex α_j,
D_n, $n \geq 3$, with first vertex α_j,
C_n, $n \geq 2$, with second vertex α_j,
B_3, with third vertex α_j.

And Γ_j^i is of type A_n with first (or last) vertex α_i or of type C_n with first vertex α_i.

Proof. The first part of the corollary follows from Lemma 2.4.

We deduce the second part from Lemma 2.5. Recall that $G/H = G \times^P P/H$ and remark that Lemma 2.5 applied to the P/H-embedding X' says that there is an open affine subset of X' that is isomorphic to the product of $R_u((P \cap Q)^-)$ and a $L((P \cap Q)^-)$-module of dimension 1. Then the highest weight of the $L(Q)$-module V must be a character of $P \cap Q$. We conclude by the last statement of Lemma 2.5 applied to X. □

When G is simple, applying Corollary 2.7, we are able to give a first list of possible homogeneous spaces admitting an embedding that satisfies (*).

Lemma 2.8. *Suppose that there exists a G/H-embedding X satisfying (*). Suppose that G is simple and $R(P) \subset H$, then (G, P, Q) is one of the following:*

(a) $(A_4, P(\omega_1), P(\omega_3))$ *or, that is the same,* $(A_4, P(\omega_4), P(\omega_2))$;
(b) $(B_n, P(\omega_i), P(\omega_{i+1}))$, *with* $n \geq 3$ *and* $1 \leq i \leq n-2$,
or $(D_n, P(\omega_i), P(\omega_{i+1}))$, *with* $n \geq 4$ *and* $1 \leq i \leq n-3$,
or $(D_n, P(\omega_{n-2}), P(\omega_{n-1}) \cap P(\omega_n))$, *with* $n \geq 3$;
(c) $(B_4, P(\omega_4), P(\omega_2))$;
(d) $(B_4, P(\omega_1), P(\omega_4))$;
(e) $(B_3, P(\omega_2), P(\omega_1) \cap P(\omega_3))$;
(f) $(C_n, P(\omega_1), P(\omega_3))$ *with* $n \geq 3$;

(g) $(C_3, P(\omega_2), P(\omega_1) \cap P(\omega_3))$*;*
(h) $(F_4, P(\omega_1), P(\omega_3))$*;*
(i) $(F_4, P(\omega_4), P(\omega_1))$*;*
(j) $(F_4, P(\omega_4), P(\omega_3))$*.*

In the next Lemma, we consider the case where G is not simple.

Lemma 2.9. *Suppose that there exists a* G/H*-embedding* X *satisfying (*). Suppose that* G *is not simple, acts faithfully on* G/H *and* $R(P) \subset H$*. Then* (G, P, Q) *is one of the following:*

(a') $(A_n \times A_1, P(\omega_2) \times A_1, P(\omega_1) \times P(\omega_1))$ *or, that is the same,* $(A_n \times A_1, P(\omega_{n-1}) \times A_1, P(\omega_n) \times P(\omega_1))$*, with* $n \geq 2$*;*
(b') $(B_n \times A_1, P(\omega_{n-1}) \times A_1, P(\omega_n) \times P(\omega_1))$ *with* $n \geq 3$*;*
(c') $(C_n \times A_1, P(\omega_{n-1}) \times A_1, P(\omega_n) \times P(\omega_1))$ *with* $n \geq 2$*;*
(d') $(C_n \times A_1, P(\omega_2) \times A_1, P(\omega_1) \times P(\omega_1))$ *with* $n \geq 2$*;*
(e') $(G_2 \times A_1, P(\omega_1) \times A_1, P(\omega_2) \times P(\omega_1))$*;*
(f') $(G_2 \times A_1, P(\omega_2) \times A_1, P(\omega_1) \times P(\omega_1))$*;*

Proof. We may assume that $G = G_{i_1} \times \cdots \times G_{i_k}$ where $k \geq 2$ and $G_{i_1}, \ldots, G_{i_k}$ are simple groups. Since P is maximal, we may assume also that $P = P_{i_1} \times G_{i_2} \times \ldots G_{i_k}$ and then $P/R(P) = P_{i_1}/R(P_{i_1}) \times G_{i_2} \times \ldots G_{i_k}$.

Moreover P/H is isomorphic to one of the homogeneous space of Lemma 2.4. Since G is not simple and acts faithfully on G/H, H cannot contain the subgroup $G_{i_2} \times \cdots \times G_{i_k}$ so that P/H is isomorphic to $(\mathrm{SL}(2) \times \mathrm{SL}(2))/\mathrm{SL}(2)$ or $(\mathrm{PSL}(2) \times \mathrm{PSL}(2))/\mathrm{PSL}(2)$ (*i.e.* Case 1a and 1b of Lemma 2.4 with $n = 4$, respectively). One can deduce that $k = 2$, that G_{i_2} and a normal simple subgroup of $P_{i_1}/R(P_{i_1})$ are of type A_1. □

Remark 2.10. In each case of Lemmas 2.8 and 2.9 , there exist at most one homogeneous space G/H and one G/H-embedding satisfying (*). Indeed, suppose that there exist two varieties X_a and respectively X_b that satisfy the same case of Lemma 2.8 and Case 1a and respectively Case 1b (or Case 3a and respectively Case 3b) of Lemma 2.4. Then X_a is a double cover of X_b ramified along the closed orbit G/Q (because the quadric Q^n is a ramified double cover of $\mathbb{P}^n$). By the purity of the branch locus, both X_a and X_b cannot be smooth, which contradicts the hypothesis of the beginning.

In the next two subsections we study all the cases enumerated in Lemmas 2.8 and 2.9.

2.2.1 Two non-homogeneous varieties

Let us first define two varieties as follows:

Definition 2.11. Let $G = F_4$, $P = P(\omega_1)$ and $Q = P(\omega_3)$.

Denote by $L(P)$ the Levi subgroup of P containing T and by ω_i' the fundamental weights of $(P/R(P), B/R(P))$ (here $P/R(P)$ is of type C_3). Let V be the G-module $V(\omega_1) \oplus V(\omega_3)$, V' be the sub-$L(P)$-module of V generated by v_{ω_3} and $\mathbb{C}.v_{\omega_1}$ be the line of V generated by v_{ω_1}. Remark that V' is the fundamental $L(P)$-module of weight $\omega_3 = \omega_2' + \omega_1$.

Let X' be the two-orbit $P/R(P)$-variety of Case 2 of Lemma 2.4, included in $\mathbb{P}(\mathbb{C}.v_{\omega_1} \oplus V')$ (X' is isomorphic to the grassmannian $\mathrm{Gr}(2,6)$).

One can now define $\mathbf{X_1} := \overline{G.X'} \subset \mathbb{P}(V)$.

Definition 2.12. Let $G = G_2 \times \mathrm{PSL}(2)$ acting on $\mathbb{P}(\mathrm{Im}(\mathbb{O}) \otimes \mathbb{C}^2)$ where $\mathrm{Im}(\mathbb{O})$ is the non-associative algebra consisting of imaginary octonions and G_2 is the group of automorphism of $\mathrm{Im}(\mathbb{O})$. (See proof of Proposition 2.22 for more details about octonions).

Let (e_1, e_2) be a basis of $\mathbb{C}^2$ and let z_1, z_2 be two elements of $\mathrm{Im}(\mathbb{O})$ such that $z_1^2 = z_2^2 = z_1 z_2 = 0$. Define

$$x := [z_1 \otimes e_1 + z_2 \otimes e_2] \in \mathbb{P}(\mathrm{Im}(\mathbb{O}) \otimes \mathbb{C}^2)$$

and $\mathbf{X_2}$ as the closure of $G.x$ in $\mathbb{P}(\mathrm{Im}(\mathbb{O}) \otimes \mathbb{C}^2)$.

Proposition 2.13. *The varieties $\mathbf{X_1}$ and $\mathbf{X_2}$ satisfy (*) respectively in Case (h) and Case (f').*

Proof. By construction, the open orbit G/H of $\mathbf{X_1}$ satisfies $R(P) \subset H \subset P$ such that P/H is as Case 2 of Lemma 2.4. Moreover, its closed orbit is $G.[v_{\omega_3}]$ and isomorphic to G/Q.

Let $x \in \mathbf{X_2}$ as in Definition 2.12 and $H := \mathrm{Stab}_G\, x$. Then H is included in the maximal parabolic subgroup P of G that stabilizes the plane generated by z_1 and z_2. Moreover $R(P) \subset H$ and P/H is isomorphic to $(\mathrm{PSL}(2) \times \mathrm{PSL}(2))/\mathrm{PSL}(2)$ where $\mathrm{PSL}(2)$ is included in $\mathrm{PSL}(2) \times \mathrm{PSL}(2)$ as follows:

$$\begin{array}{ccc} \mathrm{PSL}(2) & \longrightarrow & \mathrm{PSL}(2) \times \mathrm{PSL}(2) \\ A & \longmapsto & (A, {}^t A^{-1}). \end{array}$$

Now let us remark that the closed orbit of the closure X of $G.x$ in $\mathbb{P}(\mathrm{Im}(\mathbb{O}) \otimes \mathbb{C}^2)$ is $G.[z_1 \otimes e_1]$ that is isomorphic to G/Q where $Q = P(\omega_1) \times P(\omega_0)$ (to avoid confusion, we denote here by ω_0 the fundamental weight of $\mathrm{PSL}(2)$).

Now let us check that $\mathbf{X_1}$ and $\mathbf{X_2}$ are smooth. By Remark 2.6, there exists an affine open subset of X' that is a product of $R_u((P \cap Q)^-)$ and a line where $P \cap Q$ acts with weight $\omega_1 - \omega_3$ in the case of $\mathbf{X_1}$ and $\omega_2 - 2\omega_1 - 2\omega_0$ in the case of $\mathbf{X_2}$. Then there exists an open subset of $\mathbf{X_1}$ (respectively $\mathbf{X_2}$) that is a product of $R_u(Q^-)$ and the irreducible horospherical $L(Q)$-module of highest weight $\omega_1 - \omega_3$ (respectively $\omega_2 - 2\omega_1 - 2\omega_0$). $\square$

Lemma 2.14. *The two varieties $\mathbf{X_1}$ and $\mathbf{X_2}$ are not homogeneous.*

In fact, this lemma is a corollary of the following lemma, using the same arguments as in the proof of Lemma 1.15.

Lemma 2.15. *The spaces of global sections of the normal sheaves of the closed orbits Z_1 and Z_2 in $\mathbf{X_1}$ and $\mathbf{X_2}$ respectively are both trivial.*

Proof. By the local structure of $\mathbf{X_1}$ and $\mathbf{X_2}$ given in the latter two proofs, one can remark that the total spaces of the normal sheaves are the vector bundles $G\times^Q V(\omega_1-\omega_3)$ and $G\times^Q V(\omega_2-2\omega_1-2\omega_0)$ respectively (and with the respective notations). We conclude by the Borel-Weil Theorem using the same method as in the proof of Lemma 1.15. □

Now, to prove Theorem 0.2, we need to prove that, in all other cases of Lemmas 2.8 and 2.9, the variety satisfying (*) does not exist or is homogeneous. The study of all cases in detail is very tedious and often made of simple arguments. For this reason, we will not give the complete proofs in the following section.

2.2.2 Homogeneous cases

Proposition 2.16. *In Cases (e), (g), (j) and (e'), there exists no variety satisfying (*).*

Proof. We only give the proof in Case (e), the sketch of the proof is the same (even easier) for the other cases.

Then $G=\mathrm{Spin}(7)$ (or $\mathrm{SO}(7)$), and $P=P(\omega_2)$. Then $P/R(P)$ is isomorphic to $\mathrm{PSL}(2)\times\mathrm{SL}(2)$ (or $\mathrm{PSL}(2)\times\mathrm{SL}(2)$).

Suppose there exists a variety X satisfying (*) in Case (e) of 2.8. Then, by the preceding paragraph, the associated variety X' must be the one of Case 1b of Lemma 2.4 with $n=4$.

Let us now check that X is not smooth, in order to obtain a contradiction. By Remark 2.6, there exists an affine open subset of X' that is a product of $R_u((P\cap Q)^-)$ and a line where $P\cap Q$ acts with weight $2\omega_2-2\omega_1-2\omega_3$ (it is Lemma 2.5 in this particular case). Then there exists an open subset of X that is a product of $R_u(Q^-)$ and a cone on the flag variety in the irreducible $L(Q)$-module of highest weight $2\omega_2-2\omega_1-2\omega_3$. This implies that X is not smooth. □

Proposition 2.17. *In Case (a), $G=\mathrm{PSL}(5)$ and $X=\mathbb{P}(\bigwedge^2\mathbb{C}^5)$.*
In Case (b), $G=\mathrm{SO(n)}$ and X is the orthogonal grassmannian $\mathrm{Gr}_q(i+1,n+1)$.
In Case (d), X is a quadric of dimension 14 in the projectivization of the spinorial $\mathrm{Spin}(9)$*-module.*
In Case (f), $G=\mathrm{SP(2n)}$ and X is the grassmannian $\mathrm{Gr}(3,2n)$.
In Case (a'), $G=\mathrm{PSL(n+1)}\times\mathrm{PSL}(2)$ and $X=\mathbb{P}(\mathbb{C}^{n+1}\otimes\mathbb{C}^2)$.
In Case (b'), $G=\mathrm{SO(2n+1)}\times\mathrm{SO}(3)$ and X is the spinorial variety $\mathrm{Gr}_q^+(n+2,2n+4)$.
In Case (c'), $G=\mathrm{Sp(2n)}\times\mathrm{SL}(2)$ and X is the symplectic grassmannian $\mathrm{Gr}_\omega(n+1,2n+2)$.
In Case (d'), $G=\mathrm{Sp(2n)}/\{\pm1\}\times\mathrm{PSL}(2)$ and X is a quadric in $\mathbb{P}(\mathbb{C}^{2n}\otimes\mathbb{C}^2)$.

Sketch of proof. In case (d), X is the quadric $g(x)=0$ of [9, Prop.5].

In all other cases, the proof uses elementary linear algebra (as in the proofs of Propositions 1.8 or 1.9) and always follows the same strategy. Indeed, we pick a particular element

x in the flag variety and we prove that $H := \mathrm{Stab}_G\, x$ satisfies the wanted properties. With natural notations, here is a way to choose x in Cases (a), (b), (f), (a'), (b'), (c') and (d') respectively:

$x = [e_1 \wedge e_2 + e_3 \wedge e_4] \in \mathbb{P}(\bigwedge^2 \mathbb{C}^5)$;
$x = [e_1 \wedge \cdots \wedge e_i + e_1 \wedge \cdots \wedge e_i \wedge (e_{i+1} + e_{-(i+1)})] \in \mathbb{P}(\bigwedge^i \mathbb{C}^n \oplus \bigwedge^{i+1} \mathbb{C}^n) \simeq \mathbb{P}(\bigwedge^{i+1} \mathbb{C}^{n+1})$
x is the subspace of $\mathbb{C}^{2n}$ generated by e_1, e_2 and e_{-2};
$x = [e_1 \otimes f_1 + e_2 \otimes f_2] \in \mathbb{P}(\mathbb{C}^{n+1} \otimes \mathbb{C}^2)$;
x is the subspace of $\mathbb{C}^{2n+4}$ generated by $e_1, \ldots, e_{n-1}, e_{n+1}, e_n + e_{n+2}, e_{-n} + e_{-(n+2)}$;
$x = [e_1 \wedge \ldots \wedge e_{n-1} + (e_1 \wedge \ldots \wedge e_n) \otimes e_{n+1} - (e_1 \wedge \ldots \wedge e_{n-1} \wedge e_{-n}) \otimes e_{-(n+1)}]$
$\in \mathbb{P}(V(\omega_{n-1}) \oplus (V(\omega_n) \otimes \mathbb{C}^2)) \simeq \mathbb{P}(V_{\mathrm{Sp(2n+2)}}(\omega_{n+1}))$;
and $x = [e_1 \otimes f_1 + e_2 \otimes f_2] \in \mathbb{P}(\mathbb{C}^{2n} \otimes \mathbb{C}^2)$.

□

Proposition 2.18. *In Cases (c) and (i), the varieties X are respectively the homogeneous varieties $F_4/P(\omega_1)$ and $E_6/P(\omega_2)$.*

Sketch of proof in Case (c) (Case (i) is very similar). Let $G = \mathrm{Spin}(9) \subset F_4$, $P := P(\omega_4)$ and $Q := P(\omega_2)$ in G. We consider the flag variety $F_4/P(\omega_1)$ in the projective space $\mathbb{P}(V_{F_4}(\omega_1)) = \mathbb{P}(V_G(\omega_4) \oplus V_G(\omega_2))$.

First, $F_4/P(\omega_1)$ does not contain the closed orbit G/P of $\mathbb{P}(V_G(\omega_4))$, because a Levi subgroup of P cannot be included in a Levi subgroup of a group conjugated to $P(\omega_1) \subset F_4$.

Moreover $F_4/P(\omega_1)$ contains necessarily G/Q.

Now we consider the rational map ϕ from $F_4/P(\omega_1)$ to $\mathbb{P}(V_G(\omega_4))$ defined by projection. It is G-equivariant and its image contains the closed orbit G/P of $\mathbb{P}(V_G(\omega_4))$. Let us denote by X'_0 the fiber of ϕ over the point $[v_{\omega_4}]$ of G/P. It is stable under the action of P and its dimension is at most 5 because $\dim(F_4/P(\omega_1)) = 15$ and $\dim(G/P) = 10$.

Then one can prove that the closure X' of X'_0 contains $P/(P \cap Q)$, and deduce that X' contains an element of the form $[v_{\omega_4} + v']$, where v' is a non-zero element of the sub-P-module V' generated by $v_{\omega_2} \in V_G(\omega_2)$.

Remark now that V' is the P-module $\bigwedge^2 \mathbb{C}^7$, and that the projective space $\mathbb{P}(\mathbb{C}.v_{\omega_4} \oplus V')$ has 4 types of P-stable subvarieties: the point $[1,0]$, the quadric $P/(P \cap Q)$ in $\mathbb{P}(V')$, one quadric of dimension 5 with the two latter closed orbits and infinitely many quadrics of dimension 5 with only one closed orbit $P/(P \cap Q)$.

Then we conclude from the latter paragraph that X' is the 5-dimensional quadric of Case 1a of Lemma 2.4 with $n = 6$, so that $F_4/P(\omega_1) \subset \mathbb{P}(V_G(\omega_4) \oplus V_G(\omega_2))$ is the variety X satisfying (*) in Case (c). □

2.3 When $R(P) \not\subset H$

When H contains a Levi subgroup of P, we have the following result.

Lemma 2.19 (Th.2.1 of [6]). *Suppose that H contains a Levi subgroup of the maximal parabolic subgroup P.*

Then (G, P, H) is one of the following:

(i) $(\mathrm{PSL}(\mathrm{m}+1), P(\omega_1), \mathrm{GL}(\mathrm{m}))$.

(ii) $G = \mathrm{SO}(2\mathrm{m}+1)$; $P = P(\omega_m)$ is the stabilizer of an isotropic m-dimensional subspace E of $\mathbb{C}^{2m+1}$; H is the stabilizer of E and a non-isotropic vector orthogonal to E.

(iii) $G = \mathrm{Sp}(2\mathrm{m})/\{\pm 1\}$; P is the stabilizer of a line l in $\mathbb{C}^{2m}$; H is the stabilizer of l and a non-isotropic plane containing l.

(iv) $G = G_2$; P is the stabilizer of a line l in the 7-dimensional simple G-module; H is the stabilizer of a line l' in the 14-dimensional simple G-module such that H contains a maximal torus T of G and T acts with the same weight α in l and l'; α is a short root of (G, T).

In Case (i), $\tilde{X} = \mathbb{P}^n \times (\mathbb{P}^n)^*$, and in Case (iii), it is easy to check that $\tilde{X}$ is the partial flag variety $\mathrm{SL}(2\mathrm{m})/(P(\omega_1) \cap P(\omega_2))$. So these two cases cannot give us a variety satisfying (*). And in the next two propositions we are going to prove that $\tilde{X}$ is homogeneous in Case (ii) and that X is homogeneous in Case (iv).

Proposition 2.20. *Let $\tilde{X}$ be the $\mathrm{SO}(2\mathrm{n}+1)$-variety defined by*

$$\tilde{X} := \{(l, V) \in \mathrm{Gr}(1, 2n+1) \times \mathrm{Gr}_q(n, 2n+1) \mid l \subset V^\perp\}$$

where $V^\perp$ is the q-orthogonal subspace of V.

Then $\tilde{X} \simeq \mathrm{SO}(2\mathrm{n}+2)/(P(\omega_1) \cap P(\omega_{n+1}))$.

Remark 2.21. The variety $\tilde{X}$ is the two-orbit variety with an open orbit isomorphic to the homogeneous space of Case (ii) of Lemma 2.19 and with closed orbit of codimension 1.

Proof. We have to prove that

$$\tilde{X} \simeq \tilde{Y} := \{(l', V') \in \mathrm{Gr}_q(1, 2n+2) \times \mathrm{Gr}_q^+(n+1, 2n+2) \mid l' \subset V'\}.$$

Let us decompose $\mathbb{C}^{2n+2}$ in an orthogonal sum $\mathbb{C}^{2n+1} \oplus L$ such that the restriction of q to $\mathbb{C}^{2n+1}$ is of maximal index. Denote by π the orthogonal projection $\mathbb{C}^{2n+2} \longrightarrow \mathbb{C}^{2n+1}$.

Let $(l', V') \in \tilde{Y}$. Then $\pi(l')$ is a line because L is not isotropic. Also $V' \cap \mathbb{C}^{2n+1}$ is an isotropic subspace of $\mathbb{C}^{2n+1}$ of dimension n. Moreover one can check that $\pi(l') \subset (V' \cap \mathbb{C}^{2n+1})^\perp$.

Then we can define a $\mathrm{SO}(2\mathrm{n}+1)$-equivariant morphism

$$\begin{array}{cccc} \phi: & \tilde{Y} & \longrightarrow & \tilde{X} \\ & (l', V') & \longmapsto & (\pi(l'), V' \cap \mathbb{C}^{2n+1}) \end{array}$$

Now, we only need to prove that ϕ is an isomorphism (this is left to the reader) . □

Proposition 2.22. *The variety* $\mathrm{Gr}_q(2,7)$ *has two orbits under the action of* G_2*. It is the two-orbit variety satisfying (*) in Case (iv) of Lemma 2.19.*

Remark 2.23. The fact that $\mathrm{Gr}_q(2,7)$ has two orbits under the action of G_2 is already known, it is a particular case of [11, Th.4.3]. Moreover the two orbits consist of the sets of orthogonal planes in the space of imaginary octonions where the product is zero and not zero respectively.

Proof. Let $\mathbb{O}$ be the 8-dimensional (non-associative) $\mathbb{C}$-algebra of octonions. Let $(1, e_1, \ldots, e_7)$ be the basis of $\mathbb{O}$ and the corresponding multiplication table given in [3, Section 2.].

Let us denote by q the norm defined by $q(x_0 + \sum_{i=1}^{7} x_k e_k) = \sum_{k=0}^{7} x_k^2$ and $\mathrm{Im}(\mathbb{O})$ the 7-dimensional subspace generated by $e_1, \ldots, e_7$. Let $SO(7)$ be the special orthogonal group defined by q and acting on $\mathrm{Im}(\mathbb{O})$. Let G be the group of automorphism of $\mathbb{O}$. Remark that, since an automorphism fixes the identity and preserve the norm, G is a subgroup of $SO(7)$. Moreover it is well-known that G is of type G_2.

Let $\mathrm{Gr}_q(2,7)$ the set of isotropic planes in $\mathrm{Im}(\mathbb{O})$. By remark 2.23, $\mathrm{Gr}_q(2,7)$ has two orbits: a closed one isomorphic to $G/P(\omega_2)$ and an open one.

Let us compute the stabilizer of a particular point of the open orbit of $\mathrm{Gr}_q(2,7)$, and check that it is a subgroup of G as in Case (iv) of Lemma 2.19. For this, let us consider a new basis $(z_0, z_1, z_2, z_3, z_{-1}, z_{-2}, z_{-3})$ of $\mathrm{Im}(\mathbb{O})$ having the following multiplication table.

	z_0	z_1	z_2	z_3	z_{-1}	z_{-2}	z_{-3}
z_0	1	z_1	z_2	$-z_3$	$-z_{-1}$	$-z_{-2}$	z_{-3}
z_1	$-z_1$	0	z_3	0	$-1-z_0$	0	$-2z_{-2}$
z_2	$-z_2$	$-z_3$	0	0	0	$-1-z_0$	$2z_{-1}$
z_3	z_3	0	0	0	$2z_2$	$-2z_1$	$-2+2z_0$
z_{-1}	z_{-1}	$-1+z_0$	0	$-2z_2$	0	z_{-3}	0
z_{-2}	z_{-2}	0	$-1+z_0$	$2z_1$	$-z_{-3}$	0	0
z_{-3}	$-z_{-3}$	$2z_{-2}$	$-2z_{-1}$	$-2-2z_0$	0	0	0

Take $z_0 = \imath e_7$, $z_1 = (e_1 + \imath e_3)/\sqrt{2}$, $z_2 = (e_2 + \imath e_6)/\sqrt{2}$, $z_3 = e_4 - \imath e_6$, $z_{-1} = (e_1 - \imath e_3)/\sqrt{2}$, $z_{-2} = (e_2 - \imath e_6)/\sqrt{2}$ and $z_{-3} = e_4 + \imath e_6$.

Let E be the plane of $\mathrm{Im}(\mathbb{O})$ generated by z_1 and z_2 and let $H := \mathrm{Stab}_G E$. Remark that H contains a maximal torus T (the diagonal matrices of G in the new basis). Moreover, H stabilizes the line l generated by z_3 (because $z_1 z_2 = z_3$). In other words, $H \subset P := \mathrm{Stab}_G l$. Moreover T acts on l with weight a short root α of (G,T). Let us note that the adjoint G-module V^{14} is the submodule of the adjoint SO(7)-module $\bigwedge^2 \mathrm{Im}(\mathbb{O})$. Indeed, it is the kernel of the following linear map.

$$\begin{array}{ccc} \bigwedge^2 \mathrm{Im}(\mathbb{O}) & \longrightarrow & \mathrm{Im}(\mathbb{O}) \\ z \wedge z' & \longmapsto & \mathrm{Im}(zz') \end{array}$$

Then, the only line of V^{14} where T acts with weight α is the line l' generated by $z_1 \wedge z_2 + z_0 \wedge z_3$. Now, one can check that H is the stabilizer of l', so that G/H is isomorphic to the homogeneous space of Case (iv) of Lemma 2.19. □

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