A DESCRIPTION OF THE SARKISOV PROGRAM OF HOROSPHERICAL VARIETIES VIA MOMENT POLYTOPES

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Abstract. Let X and Y be horospherical Mori fibre spaces which are equivariantly birational with respect to the group action. Then, there is a horospherical Sarkisov program from X/S to Y/T.

Contents

1. Introduction

In this paper, we work over the field of complex numbers. Horospherical varieties are examples of complex varieties endowed with an action of a linear algebraic group, with finitely many orbits. They are for example rational. An important class of horospherical varieties is given by toric varieties, on which the group $\mathbb{G}_m^n$ acts. Similarly to toric varieties, horospherical varieties admit a combinatorial description. Indeed, to a horospherical variety Z and an ample divisor D one can attach a polytope Q_D, called *the moment polytope*, describing the geometry of the variety and of the action. For instance, some specific facets of Q_D are in bijection with the divisors of the variety which are stable by the group.

In [Pas15] the second-named author described a minimal model program for horospherical varieties, or *horospherical MMP*, completely in terms of moment polytopes by considering a one-parameter family of polytopes of the form $\{Q_{D+\epsilon K_Z}\}_{\epsilon\in\mathbb{Q}}$. For small values of ϵ the polytope $Q_{D+\epsilon K_Z}$ still defines the same variety Z, but, since K_Z is not pseudoeffective, as ϵ grows, facets of the polytope start collapsing. Eventually, the dimension of the polytope drops, defining a fibration to a variety of smaller dimension.

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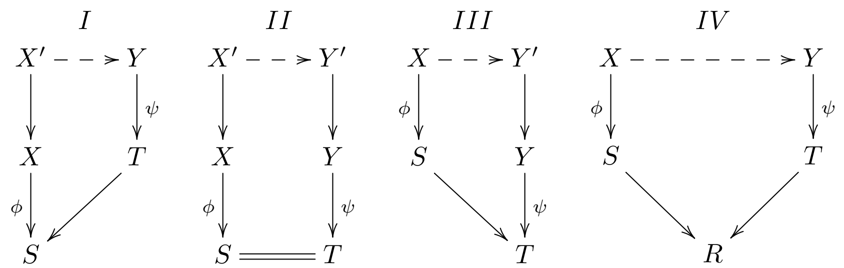
This process, for a suitable choice of D, is a minimal model program and ends with a Mori fibre space $X \to S$.

The natural question arises then of the relation between different horospherical Mori fibre spaces which are outcome of two MMP on the same variety for two different ample divisors.

Two birational Mori fibre spaces, without any further structure, are connected by a *Sarkisov program* by the cornerstone results of [Cor95] and [HM13]. A Sarkisov program is a sequence of diagrams, called *links*, of one of the following forms



where $W' \to W$ are extremal divisorial contractions, horizontal dashed arrows are isomorphisms in codimension 1 and all the other arrows are extremal contractions. In type IV, we have two cases depending on wether $S \to R$ and $T \to R$ are fibrations (type $\mathrm{IV_m}$) or small birational maps (type $\mathrm{IV_s}$).

If X/S and Y/T are two Mori fibre spaces carrying the action of a complex connected group and which are birational equivariantly with respect to the group, then by [Flo20] there is an equivariant Sarkisov program, that is one in which all the arrows in the links are equivariant with respect to the group.

In this work, we aim to produce a Sarkisov program in the spirit of [Pas15] and prove the following.

Theorem 1. *Let G be a complex connected reductive algebraic group. Let X and Y be horospherical G-varieties which are G-equivariantly birational. Assume moreover that there are Mori fibre space structures X/S and Y/T. Then, there is a horospherical Sarkisov program from X/S to Y/T.*

Given two Mori fibre spaces that are horospherical varieties, a *horospherical Sarkisov program* is a Sarkisov program that can be realized by deforming moment polytopes. More precisely, if X/S and Y/T are Mori fibre spaces that are horospherical varieties, Z a horospherical G-variety, D and D' ample divisors on Z such that X/S (resp. Y/T) is the outcome of a horospherical K_Z-MMP with scaling of D (resp. of D') as in [Pas15], we construct a 2-parameter family of pseudo-moment polytopes

$$\{P^{\delta,\epsilon}\}_{(\delta,\epsilon)\in\mathbb{Q}^2}$$

with the following properties: there are ε_0 and ε_1 such that $\{P^{0,\epsilon}\}_{\epsilon\in[0,\epsilon_0]}$ gives the K_Z-MMP with scaling of D and $\{P^{1,\epsilon}\}_{\epsilon\in[0,\epsilon_1]}$ gives the K_Z-MMP with scaling of D'. Moreover, for every $\delta \in [0,1]$ the divisor $\delta D+(1-\delta)D'$ is ample on Z and $P^{\delta,0}$ is a pseudo-moment polytope for Z. The Sarkisov program $X/S \dashrightarrow Y/T$ is obtained by considering the polytopes $P^{\delta,\epsilon}$ for (δ,ϵ) moving along a certain piecewise-linear curve, which we call the Mori Polygonal Chain.

More precisely, we set

$$P^{\delta,\epsilon} := \{x \in \mathbb{Q}^n \mid Ax \geq (1-\delta)B + \delta B' + \epsilon C\},$$

where the matrix A corresponds to primitive elements of edges (in a lattice isomorphic to $\mathbb{Z}^n$) of the colored fan of the horospherical G-variety Z, the column matrices B and B' correspond to the divisors D and D', and the column matrix C correspond to K_Z. See also Definition 7.1 and Theorem 30.

We illustrate the intuitive idea behind the theorem with an example. Let X/S be the toric variety $\mathbb{P}^1 \times \mathbb{P}^1$ with the first projection to $\mathbb{P}^1$. We consider the polytopes P_ϵ defined by $A_0 x \geq B_0 + \epsilon C_0$ where

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 0 \\ 0 \\ -2 \\ -1 \end{pmatrix}, \text{ and } C_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

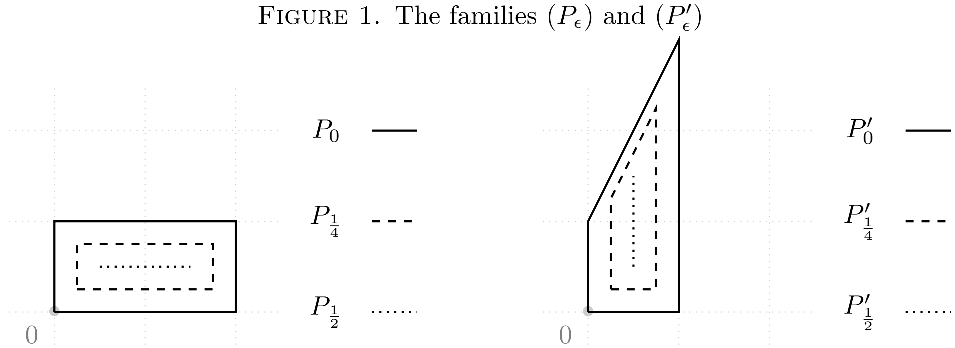
For any $\epsilon \in [0, \frac{1}{2}[$, the polytopes P_ϵ are rectangles and are moment polytopes of $X = \mathbb{P}^1 \times \mathbb{P}^1$, and $P_{\frac{1}{2}}$ is a (horizontal) segment which is a moment polytope of $\mathbb{P}^1$ (see Figure 1).

Now let Y/T be the projective bundle $\mathbb{F}_2 = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2))$. It is a two-dimensional toric Mori fibre space. Consider the polytopes P'_ϵ, defined by $A_1 x \geq B_1 + \epsilon C_1$ where

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 2 & -1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ -1 \end{pmatrix}, \text{ and } C_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

For any $\epsilon \in [0, \frac{1}{2}[$, the polytopes P_ϵ are moment polytopes of Y, and $P_{\frac{1}{2}}$ is a (vertical) segment which is a moment polytope of $\mathbb{P}^1$ (see Figure 1).

FIGURE 1. The families (P_ϵ) and (P'_ϵ)



Thus we may add inequalities to the systems defining P_ϵ and P'_ϵ, or, equivalently, add lines to the matrices in order to get $A_0 = A_1 = A$ and $C_0 = C_1 = C$ (then B_0

becomes B and B_1 becomes B'). For example, we can consider

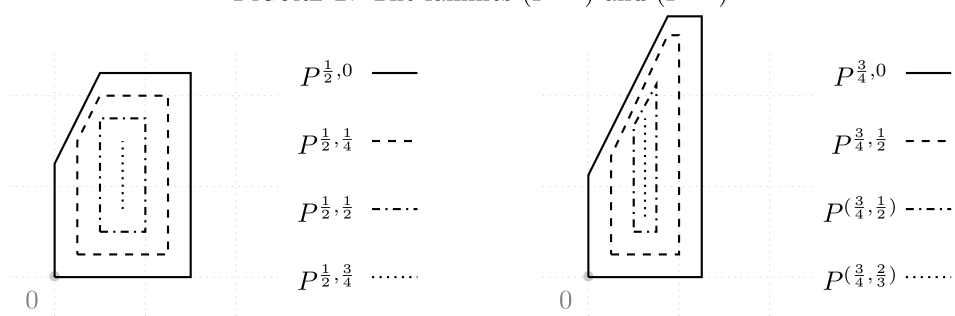
$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 2 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ -2 \\ -1 \\ -1 \end{pmatrix}, \quad B' = \begin{pmatrix} 0 \\ 0 \\ -1 \\ -3 \\ -1 \end{pmatrix}, \text{ and } C_= \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Then the two-parameter family $(P^{\delta,\epsilon})_{(\delta,\epsilon)\in\mathbb{Q}^2}$ defined by

$$P^{\delta,\epsilon} := \{x \in \mathbb{Q}^n \mid Ax \geq (1-\delta)B + \delta B' + \epsilon C\},$$

is such that, for any $\epsilon \geq 0$, $P_\epsilon = P^{0,\epsilon}$ and $P'_\epsilon = P^{1,\epsilon}$.

FIGURE 2. The families $(P^{\frac{1}{2},\epsilon})$ and $(P^{\frac{3}{4},\epsilon})$



Note that the lines of A correspond to the primitive elements of the edges of the fan of a two-dimensional toric variety W resolving the indeterminacies of $X \dashrightarrow Y$. Since we added only one line to A_0 (resp A_1) to get A, we have $\rho(W/X) = 1$ (resp. $\rho(W/Y) = 1$). We get therefore a type II link

$$\begin{array}{ccc} W & = & W \\ \downarrow & & \downarrow \\ \mathbb{P}^1 \times \mathbb{P}^1 & & \mathbb{F}^2 \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & = & \mathbb{P}^1 \end{array}$$

The variety W obtained here does not have terminal singularities, hence this is not a standard Sarkisov link, but terminality can be achieved by considering higher resolutions. We do this in Example 6.1.

Structure of the paper. We recall in section 2 basic definitions and properties of horospherical varieties. In section 3 and 4 we describe the results in [Pas15] and prove that the ample divisors D such that the K_Z-MMP scaled by D ends with X/S form an euclidian open set. Section 5 contains the main technical results allowing to translate the MMP and Sarkisov program into simple operation on polytopes. Those results rely on the study of certain two-dimensional polytopes. All the details of this study are reported in the Appendix. In section 6 we present two examples illustrating the horospherical Sarkisov program, and in section 7 we give the proof of Theorem 1.

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2. Horospherical varieties

We begin by recalling very briefly the Luna-Vust theory of horospherical embeddings and by setting the notation used in the rest of the paper. For more details on horospherical varieties, we refer the reader to [Pas08], and for basic results on Luna-Vust theory of spherical embeddings, we refer to [Kno91].

Let G be connected reductive algebraic group. A closed subgroup H of G is said to be *horospherical* if it contains the unipotent radical U of a Borel subgroup B of G. This is equivalent to say that ([Pas08, Prop. and Rem. 2.2]), there exists a parabolic subgroup P containing H such that the map $G/H \longrightarrow G/P$ is a torus fibration; or to say that there exists a parabolic subgroup P containing H such that H is the kernel of finitely many characters of P.

Note that $B \subset P = N_G(H)$. We also fix a maximal torus T of B. Then we denote by S the set of simple roots of (G, B, T). Also denote by R the subset of S of simple roots of P. In particular, if P is a minimal parabolic subgroup, then R consists of only one element. Let $\mathfrak{X}(T)$ (resp. $\mathfrak{X}(T)^+$) be the lattice of characters of T (resp. the set of dominant characters). Similarly, we define $\mathfrak{X}(P)$ and $\mathfrak{X}(P)^+ = \mathfrak{X}(P) \cap \mathfrak{X}(T)^+$. Note that the lattice $\mathfrak{X}(P)$ and the dominant chamber $\mathfrak{X}(P)^+$ are generated by the fundamental weights ϖ_α with $\alpha \in S\backslash R$ and the weights of the center of G.

We denote by M the sublattice of $\mathfrak{X}(P)$ consisting of characters of P vanishing on H. The rank of M is called the *rank of* G/H and denoted by n. Let $N := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$.

For any free lattice $\mathbb{L}$, we denote by $\mathbb{L}_{\mathbb{Q}}$ the $\mathbb{Q}$-vector space $\mathbb{L} \otimes_{\mathbb{Z}} \mathbb{Q}$.

For any simple root $\alpha \in S\backslash R$, the restriction of the coroot $\alpha^\vee$ to M is a point of N, which we denote by $\alpha^\vee_M$.

Moreover, we define the *walls of the dominant chamber* $\mathfrak{X}(P)^+$ in the following way. For any $\alpha \in S \setminus R$ we set

$$W_{\alpha,P} = \mathfrak{X}(P)^+ \cap \{\alpha^\vee = 0\}.$$

Definition 1. A G/H-embedding is a couple (X, x), where X is a normal algebraic G-variety and x a point of X such that $G \cdot x$ is open in X and isomorphic to G/H. The variety X is called a *horospherical variety*.

By abuse of notation, we often forget the point x, so that we call X a G/H-embedding. But there are several non-isomorphic G/H-embeddings (X, x) for the same horospherical variety X. Two points x_1 and x_2 differ by an element of the torus P/H, which acts on the right on G/H. Similarly to toric varieties (which are $(\mathbb{C}^*)^n$-embeddings with the above defintion), G/H-embeddings are classified by colored fans in $N_{\mathbb{Q}}$. For example, for the toric variety $\mathbb{P}^2$ we have different non-isomorphic $(\mathbb{C}^*)^2$-embeddings, whose fans are the same up to the action of $\mathrm{SL}_2(\mathbb{Z})$. In this paper, since we define horospherical varieties by their colored fans, or equivalently by some of their moment polytopes, we are implicitly fixing a G/H-embedding up to isomorphism.

Definition 2. (1) A colored cone of $N_{\mathbb{Q}}$ is an couple $(\mathcal{C},\mathcal{F})$ where $\mathcal{C}$ is a convex cone of $N_{\mathbb{Q}}$ and $\mathcal{F}$ is a set of colors (called the set of colors of the colored cone), such that

(i) $\mathcal{C}$ is generated by finitely many elements of N and contains $\{\alpha_M^\vee \mid \alpha\in\mathcal{F}\}$,

(ii) $\mathcal{C}$ does not contain any line and $\mathcal{F}$ does not contain any α such that $\alpha_M^\vee$ is zero.

(2) A colored face of a colored cone $(\mathcal{C},\mathcal{F})$ is a couple $(\mathcal{C}',\mathcal{F}')$ such that $\mathcal{C}'$ is a face of $\mathcal{C}$ and $\mathcal{F}'$ is the set of $\alpha\in\mathcal{F}$ satisfying $\alpha_M^\vee\in\mathcal{C}'$. A colored fan is a finite set $\mathbb{F}$ of colored cones such that

(i) any colored face of a colored cone of $\mathbb{F}$ is in $\mathbb{F}$,

(ii) and any element of $N_{\mathbb{Q}}$ is in the interior of at most one colored cone of $\mathbb{F}$.

The main result of Luna-Vust Theory of spherical embeddings is the one-to-one correspondence between colored fans and isomorphic classes of G/H-embeddings (see for example [Kno91]). It generalizes the classification of toric varieties in terms of fans, case where $G=(\mathbb{C}^*)^n$ and $H=\{1\}$. We will rewrite this result in Section 2.1 for projective horospherical varieties in terms of polytopes and describe explicitly the correspondence.
If X is a G/H-embedding, we denote by $\mathbb{F}_X$ the *colored fan* of X in $N_{\mathbb{Q}}$ and we denote by $\mathcal{F}_X$ the subset $\cup_{(\mathcal{C},\mathcal{F})\in\mathbb{F}_X}\mathcal{F}$ of $S\backslash R$, which we call the *set of colors* of X.

We now recall the description of divisors of horospherical varieties.
We denote by $X_1,\dots,X_r$ the G-stable irreducible divisors of a G/H-embedding X. For any $i\in\{1,\dots,r\}$, we denote by x_i the primitive element in N of the colored edge associated to X_i. The B-stable and not G-stable irreducible divisors of a G/H-embedding X are the closures in X of B-stable irreducible divisors of G/H, which are the inverse images by the torus fibration $G/H\longrightarrow G/P$ of the Schubert divisors of the flag variety G/P. The B-stable irreducible divisors of G/H are indexed by simple roots of $S\backslash R$, we write them D_α with $\alpha\in S\backslash R$.

We can now recall the characterization of Cartier, $\mathbb{Q}$-Cartier and ample divisors of horospherical varieties due to M. Brion in the more general case of spherical varieties ([Bri89]). This will permit to define a polytope associated to a divisor of a horospherical variety.

Theorem 2. *(Section 3.3, [Bri89]) Let G/H be a horospherical homogeneous space. Let X be a G/H-embeding. Then every divisor of X is equivalent to a linear combination of $X_1,\dots,X_r$ and D_α with $\alpha\in S\backslash R$. Now, let $D=\sum_{i=1}^r a_iX_i+\sum_{\alpha\in S\backslash R}a_\alpha D_\alpha$ be a $\mathbb{Q}$-divisor of X.*

(1) D is $\mathbb{Q}$-Cartier if and only if there exists a piecewise linear function h_D, linear on each colored cone of $\mathbb{F}_X$, such that for any $i\in\{1,\dots,r\}$, $h_D(x_i)=a_i$ and for any $\alpha\in\mathcal{F}_X$, $h_D(\alpha_M^\vee)=a_\alpha$.

(2) Suppose that D is a divisor (i.e. $a_1,\dots,a_r$ and the a_α with $\alpha\in S\backslash R$ are in $\mathbb{Z}$). Then D is Cartier if moreover, for any colored cone $(\mathcal{C},\mathcal{F})$ of $\mathbb{F}_X$, the linear function $(h_D)_{|\mathcal{C}}$, can be defined as an element of M (instead of $M_{\mathbb{Q}}$ for $\mathbb{Q}$-Cartier divisors).

(3) Suppose that D is $\mathbb{Q}$-Cartier. Then D is ample, resp. nef if and only if the piecewise linear function h_D is strictly convex, resp. convex, and for any $\alpha\in(S\backslash R)\backslash\mathcal{F}_X$, we have $h_D(\alpha_M^\vee)<a_\alpha$, resp. $\leq a_\alpha$.

(4) *Suppose that D is Cartier. Let $\tilde{Q}_D$ be the polytope in $M_{\mathbb{Q}}$ defined by the following inequalities, where $\chi \in M_{\mathbb{Q}}$: for any colored cone $(\mathcal{C},\mathcal{F})$ of $\mathbb{F}_X$, $(h_D)+\chi \geq 0$ on $\mathcal{C}$, and for any $\alpha \in (S\backslash R)\backslash\mathcal{F}_X$, $\chi(\alpha_M^\vee)+a_\alpha \geq 0$. Note that here the weight of the canonical section of D is $v^0 := \sum_{\alpha\in S\backslash R} a_\alpha\varpi_\alpha$. Then the G-module $H^0(X,D)$ is the direct sum, with multiplicity one, of the irreducible G-modules of highest weights $\chi + v^0$ with χ in $\tilde{Q}_D \cap M$.*

In all the paper, a divisor of a horospherical variety is always supposed to be B-stable, i.e. of the form $\sum_{i=1}^r a_iX_i + \sum_{\alpha\in S\backslash R} a_\alpha D_\alpha$.

Corollary 3. *A G/H-embbeding X is $\mathbb{Q}$-factorial if and only if all the cones in $\mathbb{F}_X$ are simplicial and for any $\alpha \in \mathcal{F}_X$, $\alpha_M^\vee$ generates a ray of $\mathbb{F}_X$.*
Moreover, in that case, the Picard number of X is the number of their B-stable prime divisors minus the rank n (Equality (4.1.1) [Pas08]).

In what follows we denote by $WDiv(X)_{\mathbb{Q}}$ (resp. $WDiv_0(X)_{\mathbb{Q}}$) the vector space of B-stable (linearly equivalent to zero) $\mathbb{Q}$-Weil divisors on a horospherical variety X and by $Amp(X)$ (resp. $Nef(X)$) the cones in $WDiv(X)_{\mathbb{Q}}$ of ample (resp. nef) divisors. Both cones are polyhedral and, if X is $\mathbb{Q}$-factorial, full-dimensional.

2.1. **Projective horospherical varieties and polytopes.** In this section we recall how many properties of G/H-embeddings can be formulated in terms of moment polytopes. Theorem 2 gives the following

Proposition 4. *(Corollary 2.8, [Pas15]) Let X be a projective G/H-embedding and $D = \sum_{i=1}^r a_iX_i + \sum_{\alpha\in S\backslash R} a_\alpha D_\alpha$ be a $\mathbb{Q}$-divisor of X. Suppose that D is $\mathbb{Q}$-Cartier and ample.*

(1) *The polytope $\tilde{Q}_D$ defined in Theorem 2 is of maximal dimension in $M_{\mathbb{Q}}$ and we have*

$$\tilde{Q}_D = \{m \in M_{\mathbb{Q}} \mid \langle m, x_i\rangle \geq -a_i,\ \forall i \in \{1,\dots,r\} \text{ and } \langle m, \alpha_M^\vee\rangle \geq -a_\alpha,\ \forall \alpha \in \mathcal{F}_X\}.$$

(2) *Let $v^0 := \sum_{\alpha\in S\backslash R} a_\alpha\varpi_\alpha$. The polytope $Q_D := v^0 + \tilde{Q}_D$ is contained in the dominant chamber $\mathfrak{X}(P)^+$ of $\mathfrak{X}(P)$ and it is not contained in any wall of the dominant chamber.*

(3) *Let $(\mathcal{C},\mathcal{F})$ be a maximal colored cone of $\mathbb{F}_X$, then the element $v^0 - (h_D)_{|\mathcal{C}}$ of $M_{\mathbb{Q}}$ is a vertex of Q_D. In particular, if D is Cartier, then Q_D is a lattice polytope (i.e. has its vertices in $v^0 + M$).*

(4) *Conversely, let v be a vertex of Q_D. We define $\mathcal{C}_v$ to be the cone of $N_{\mathbb{Q}}$ generated by inward-pointing normal vectors of the facets of Q_D containing v. We set $\mathcal{F}_v = \{\alpha \in S\backslash R \mid v$ the corresp. wall of the dominant chamber$\}$. Then $(\mathcal{C}_v,\mathcal{F}_v)$ is a maximal colored cone of $\mathbb{F}_X$.*

The polytope Q_D is called the *moment polytope* of (X,D) (or of D), and the polytope $\tilde{Q}_D$ the *pseudo-moment* polytope of (X,D) (or of D).
The projective G/H-embeddings are classifed in terms of G/H-polytopes (defined below in Definition 2.3), and we can give an explicit construction of a G/H-embedding from a G/H-polytope.

Definition 3. Let Q be a polytope in $\mathfrak{X}(P)^+_{\mathbb{Q}}$ (not necessarily a lattice polytope). We say that Q is a G/H-polytope, if its direction is $M_{\mathbb{Q}}$ and if it is contained in no wall $W_{\alpha,P}$ with $\alpha \in S\backslash R$.

Let Q and Q' be two G/H-polytopes in $\mathfrak{X}(P)^+_{\mathbb{Q}}$. Consider any polytopes $\tilde{Q}$ and $\tilde{Q}'$ in $M_{\mathbb{Q}}$ obtained by translations from Q and Q' respectively. We say that Q and Q' are equivalent G/H-polytopes if the following conditions are satisfied.

(1) There exist an integer j and $2j$ affine half-spaces $\mathcal{H}_1^+,\dots,\mathcal{H}_j^+$ and $\mathcal{H'}_1^+,\dots,\mathcal{H'}_j^+$ of $M_{\mathbb{Q}}$ (respectively delimited by the affine hyperplanes $\mathcal{H}_1,\dots,\mathcal{H}_j$ and $\mathcal{H'}_1,\dots,\mathcal{H'}_j$) such that $\tilde{Q}$ is the intersection of the $\mathcal{H}_i^+$, $\tilde{Q}'$ is the intersection of the $\mathcal{H'}_i^+$, and for all $i\in\{1,\dots,j\}$, $\mathcal{H}_i^+$ is the image of $\mathcal{H'}_i^+$ by a translation.

(2) With the notation of (1), for all subset J of $\{1,\dots,j\}$, the intersections $\cap_{i\in J}\mathcal{H}_i\cap Q$ and $\cap_{i\in J}\mathcal{H'}_i\cap Q'$ have the same dimension.

(3) Q and Q' intersect exactly the same walls of the dominant chamber.

Remark that this definition does not depend on the choice of $\tilde{Q}$ and $\tilde{Q}'$. We now give a classification of projective horospherical varieties in terms of polytopes.

Proposition 5. *(Proposition 2.10, [Pas15]) The correspondence between moment polytopes and colored fans gives a bijection between the set of equivalence classes of G/H-polytopes and (isomorphism classes of) projective G/H-embeddings.*

Proposition 6. *(Proposition 2.11 and Remark 2.12, [Pas15]) Let Q be a G/H-polytope. Up to multiplying Q by an integer, there exists a very ample Cartier divisor D of the corresponding G/H-embedding X such that $Q=Q_D$. More precisely, X is isomorphic to the closure of the G-orbit $G\cdot[\sum_{\chi\in(v^0+M)\cap Q}v_\chi]$ in $\mathbb{P}(\oplus_{\chi\in(v^0+M)\cap Q}V(\chi))$.*

Lemma 7. *Let X be a G/H-embedding and let D be a nef divisor on X. Then there is a horospherical subgroup $H\subsetneq H'\subseteq G$ such that Q_D is the polytope of a G/H'-embedding X' and a suitable multiple of D defines a morphism $X\to X'$.*

Proof. If D is nef, the polytope Q_D can be defined as in Proposition 4. The colored fan that we can then construct is not necessarily $\mathbb{F}_X$, but the colored fan of a horospherical G-variety Y (not necessarily a G/H-embedding), where the G-equivariant map ϕ_D associated to the nef divisor goes from X to Y.

Indeed, up to multiplying D or Q by an integer, the proof of [Pas15, Proposition 2.11] and [Pas15, Remark 2.12] gives the projective G-equivariant map

$$\begin{array}{cccc}\phi_D: & X & \longrightarrow & \mathbb{P}(H^0(X,D)^\vee)\\ & x & \longmapsto & [s\mapsto s(x)]\end{array}$$

whose image is the closure of the G-orbit $G\cdot[\sum_{\chi\in(v^0+M)\cap Q}v_\chi]$ in $\mathbb{P}(\oplus_{\chi\in(v^0+M)\cap Q}V(\chi))$. This closure is the variety Y by applying Proposition 6 to $Q=Q_D$. Thus Q is a G/H'-polytope with $H\subseteq H'$, and Y is the G/H'-embedding corresponding to Q. □

By the duality between colored fans and moment polytopes we easily get that if X is a $\mathbb{Q}$-factorial G/H-embedding then for every ample B-stable divisor D the polytope Q_D is simple. We can go further and give the following result, which is a translation of Corollary 3 in terms of polytopes, by using Proposition 4 (3) and (4). The matrix inequality $Ax\geq B$ comes from Proposition 4 (1).

Lemma 8. *Let X be a G/H-embedding and let D be an ample B-stable divisor. The polytope $\tilde{Q}_D$ can be defined by a matrix inequality $Ax\geq B$, where the lines of A are given by the x_i's and the $\alpha_M^\vee$'s, and the column matrix B is given by minus* *the coefficients of D. For any vertex v of $\tilde{Q}_D$, denote I_v the set of lines of A such that $A_{I_v}x = B_{I_v}$.*

Then X is $\mathbb{Q}$-factorial if and only if A_{I_v} is surjective for any vertex v of $\tilde{Q}_D$.

Similarly, we have the following result.

Lemma 9. *Let X be a G/H-embedding and let D be an ample B-invariant divisor. Let D' be a B-invariant divisor of X and denote by B' the column matrix is given by minus the coefficients of D'.*

Then D' is $\mathbb{Q}$-Cartier if and only if for any vertex v of $\tilde{Q}_D$, B' is in the image of A_{I_v}.

Remark 1. The existence of G-equivariant morphisms between horospherical varieties can be characterized in terms of colored fans [Kno91] or equivalently in terms of moment polytopes [Pas15, section 2.4]. In this text, we will rather use Lemma 7.

3. Minimal Model Program

In this section, we recall the results in [Pas15] where the second-named author describes a minimal model program from a horospherical variety in terms of a one-parameter family of polytopes.
We start by recalling some standard terminology for the minimal model program for projective varieties. We refer to [KM98] for the basic notions on the minimal model program.

Let X be a projective variety with terminal singularities such that K_X is not nef. Then by the cone and contraction theorem there is a morphism $\varphi\colon X \to Y$ such that $\rho(X) = \rho(Y) + 1$ and for every curve C in X which is contracted to a point by φ we have $K_X \cdot C < 0$. Moreover

- if $\dim Y < \dim X$ then φ is called a Mori fibre space;
- if $\dim Y = \dim X$ and $Exc(\varphi)$ has codimension 1 in X then φ is said to be divisorial;
- if $\dim Y = \dim X$ and $Exc(\varphi)$ has codimension at least 2 in X then φ is said to be small.

In the last case, by [HM07] and [HM10], there is $\varphi^+\colon X^+ \to Y$ such that $Exc(\varphi^+)$ has codimension at least 2 in X^+ and for every curve C in X^+ contracted by φ^+ we have $K_{X^+} \cdot C > 0$. The data of φ and φ^+ is called a *flip*.

In the second and third case Y is birational to X. We set $X_1 = Y$ and $\varphi_1 = \varphi$ in the second case and $X_1 = X^+$ and $\varphi_1 = (\varphi^+)^{-1} \circ \varphi$ in the third case. If K_{X_1} is not nef, then by the cone and contraction theorem there is again a morphism $X_1 \to Y_1$. An *MMP*, or minimal model program, is a sequence of birational maps $\varphi_i\colon X_{i-1} \dashrightarrow X_i$ obtained as above. We say that it *terminates* if there is an integer k such that K_{X_k} is nef or there is a Mori fibre space $X_k \to T$. A proper morphism with connected fibres is called a *contraction*. A birational morphism $\varphi\colon X \to Y$ such that $\rho(X) = \rho(Y) + 1$ is called an *extremal contraction*. We say that a morphism φ is *K-negative* (resp. *K-positive*) if for every curve C in X which is contracted to a point by φ we have $K_X \cdot C < 0$ (resp. $K_X \cdot C > 0$).

3.1. **HMMP scaled by an ample divisor.** Let (X, D) be a polarized horospherical variety: X is a G/H-embedding and D is a B-stable ample $\mathbb{Q}$-divisor of X. Write $D = \sum_{i=1}^r b_i X_i + \sum_{\alpha \in S\setminus R} b_\alpha D_\alpha$. An anticanonical divisor of X is

$-K_X=\sum_{i=1}^r X_i+\sum_{\alpha\in S\backslash R}a_\alpha D_\alpha$, where a_α are integers greater or equal than 2, and given by an explicit formula [Bri97, Th. 4.2].

We consider the one-parameter family of polytopes $(\tilde{Q}^\epsilon)_{\epsilon\geq 0}$ defined by $\langle x,x_i\rangle\geq -b_i+\epsilon$ for all $i\in\{1,\dots,r\}$ and $\langle x,\alpha_M^\vee\rangle\geq -b_\alpha+\epsilon a_\alpha$ for any $\alpha\in S\backslash R$. Equivalently, let A be the matrix associated to the linear map $\varphi(m)=(\langle m,x_i\rangle_{i=1\dots r},\ \langle m,\alpha_M^\vee\rangle_{\alpha\in\mathcal{F}})$. Let B be the column matrix whose coordinates are minus the coefficents of D; and C the column matrix whose coordinates are the coefficents of $-K_X$.

We define $(Q^\epsilon)_{\epsilon\geq 0}$ by $Q^\epsilon=\tilde{Q}^\epsilon+\sum_{\alpha\in S\backslash R}b_\alpha-\epsilon a_\alpha\varpi_\alpha$.

Remark 2. For small $\epsilon\in\mathbb{Q}$, $\tilde{Q}^\epsilon=\tilde{Q}_{D+\epsilon K_X}$ and $Q^\epsilon=Q_{D+\epsilon K_X}$ is the moment polytope associated to the ample divisor $D+\epsilon K_X$.

Theorem 10. *(Corollary 3.16 and Section 4, [Pas15]) For every ample divisor D over a $\mathbb{Q}$-Gorenstein horospherical variety X, there exists $\epsilon_{max}>0$, and, there exist non-negative integers $k,j_0,\dots,j_k$, rational numbers $\alpha_{i,j}$ for $i\in\{0,\dots,k\}$ and $j\in\{0,\dots,j_i\}$ and $\alpha_{k,j_k+1}\in\mathbb{Q}_{>0}\cup\{+\infty\}$ ordered as follows with the convention that $\alpha_{i,j_i+1}=\alpha_{i+1,0}$ for any $i\in\{0,\dots,k-1\}$:*

(1) $\alpha_{0,0}=0$;
(2) *for any $i\in\{0,\dots,k\}$, and for any $j<j'$ in $\{0,\dots,j_i+1\}$ we have $\alpha_{i,j}<\alpha_{i,j'}$;*

and such that the different G/H-embedding associated to the polytopes in the family $(Q^\epsilon)_{\epsilon\in\mathbb{Q}_{\geq 0}}$ are given by the following intervals:

(1) *$X_{i,0}$ when $\epsilon\in[\alpha_{i,0},\alpha_{i,1}[$, with $i\in\{0,\dots,k\}$;*
(2) *$X_{i,j}$ when $\epsilon\in]\alpha_{i,j},\alpha_{i,j+1}[$, with $i\in\{0,\dots,k\}$ and $j\in\{1,\dots,j_i\}$;*
(3) *$Y_{i,j}$ when $\epsilon=\alpha_{i,j}$ with $i\in\{0,\dots,k\}$ and $j\in\{1,\dots,j_i\}$;*
(4) *T such that $\dim T<\dim X$ when $\epsilon=\alpha_{k,j_k+1}=\epsilon_{max}$.*

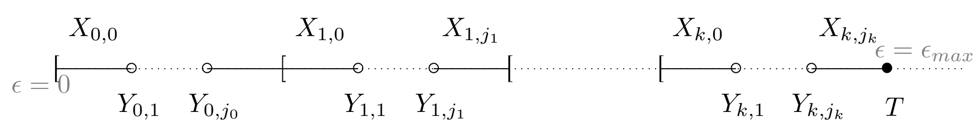
Moreover we get dominant G-equivariant morphisms:

(1) *$\phi_{i,j}:X_{i,j-1}\longrightarrow Y_{i,j}$ for any $i\in\{0,\dots,k\}$ and $j\in\{1,\dots,j_i\}$;*
(2) *$\phi_{i,j}^+:X_{i,j}\longrightarrow Y_{i,j}$ for any $i\in\{0,\dots,k\}$ and $j\in\{1,\dots,j_i\}$;*
(3) *$\phi_i:X_{i,j_i}\longrightarrow X_{i+1,0}$ for any $i\in\{0,\dots,k-1\}$;*
(4) *and $\phi:X_{k,j_k}\longrightarrow T$.*

For every i,j the morphism $\phi_{i,j}$ is K-negative, the morphism $\phi_{i,j}^+$ is K-positive and their exceptional loci have codimension at least 2. For every i the morphism ϕ_i is a K-negative divisorial contraction and ϕ is a fibration.

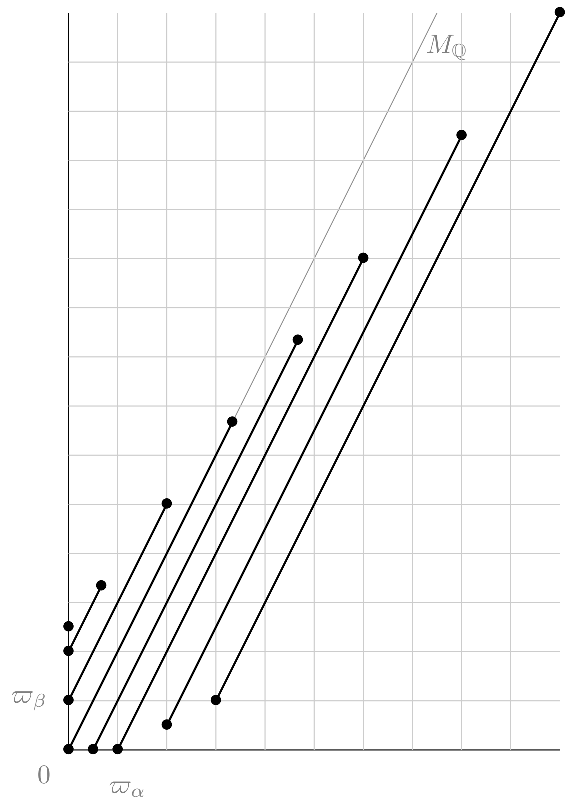
We can illustrate this result by the diagram in Figure 3: we draw the segment $[0,\epsilon_{max}]$ which is partitioned by points, and open or semi-open segments, so that each set of the partition corresponds to a horospherical G-variety.

FIGURE 3.



We now give an example of the implementation of Theorem 10.

FIGURE 4. The family $(Q^\epsilon)_{\epsilon\geq 0}$ for $\epsilon = 0,\ \frac{1}{2},\ 1,\ \frac{4}{3},\ \frac{5}{3},\ 2,\ \frac{7}{3}$ and $\frac{5}{2}$ (from right to left).



Example 1. Consider a rank-one horospherical homogeneous space with two colors D_α and D_β such that $\alpha^\vee_M = 1$ and $\beta^\vee_M = 2$, $a_\alpha = a_\beta = 3$. In particular, $M = \mathbb{Z}(\varpi_\alpha + 2\varpi_\beta)$.

Let X be the G/H-embedding without picked color, and let $D = 2X_1 + 5X_2 + 5D_\alpha + 5D_\beta$, which is an ample divisor of X. Then the family $(\tilde{Q}^\epsilon)_{\epsilon\geq 0}$ is defined by $\tilde{Q}^\epsilon := \{x \in M_{\mathbb{Q}} \mid Ax \geq B + \epsilon C\}$ where

$$A = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 2 \end{pmatrix}, \quad B = \begin{pmatrix} -2 \\ -5 \\ -5 \\ -5 \end{pmatrix}, \text{ and } C = \begin{pmatrix} 1 \\ 1 \\ 3 \\ 3 \end{pmatrix}.$$

And the family $(Q^\epsilon)_{\epsilon\geq 0}$ is defined by $Q^\epsilon := (5-3\epsilon)(\varpi_\alpha + \varpi_\beta) + \tilde{Q}^\epsilon$, and illustrated in Figure 4.

For any $\epsilon \in [0, 1[$, the polytopes are associated to $X = X_{0,0}$.
For any $\epsilon \in [1, \frac{5}{3}[$, the polytopes are associated to the G/H-embedding $X_{1,0}$ with picked color β. For $\epsilon = \frac{5}{3}$, the polytope corresponds to the (non-$\mathbb{Q}$-factorial) G/H-embedding $Y_{1,1}$ with the two picked colors.
For any $\epsilon \in]\frac{5}{3}, \frac{5}{2}[$, the polytopes are associated to the G/H-embedding $X_{1,1}$ with

picked color α.
And for $\epsilon = \frac{5}{2}$, the polytope corresponds to the variety $T = G/P(\varpi_\beta)$.

We first have a divisorial contraction, followed by a flip and we finish with a Mori fibre space.

We will refer to the series of maps of Theorem 10 as *HMMP*, for horospherical minimal model program.

Remark 3. We notice that the algorithm described in Theorem 10 is not necessarily an MMP, as the morphisms involved are not extremal contractions. Nevertheless, if X is $\mathbb{Q}$-factorial, it is an MMP for a general choice of D, in particular ϕ is a Mori fibre space.

Remark 4. In Theorem 10, we can also use the family $(\tilde{Q}^\epsilon)_{\epsilon\in\mathbb{Q}_{\geq 0}}$. By replacing the walls of the dominant chamber by the inequalities in $Ax \geq B + \epsilon C$ coming from an $\alpha \in S\backslash R$, the study of the family and associated horospherical variety is simpler.

4. Resolutions and Horospherical MMP

The main goal of this section is to prove that if X/T is a Mori fibre space and $Z \to X$ a resolution of the singularites of X, then there is a euclidian open set in $WDiv(Z)_\mathbb{Q}$ of ample divisors A such that the HMMP from Z with scaling of A ends with X/T.

Lemma 11. *Let X and Y be G/H-embeddings. Then there is a smooth G/H-embedding Z and $Z \to X \times Y$ a resolution of the indeterminacy of $X \dashrightarrow Y$.*

Proof. The existence of a smooth resolution of the indeterminacy of $X \dashrightarrow Y$ is a consequence of the same result for toric varieties. Indeed, we can first unpick colors both for X and Y to obtain toroidal varieties. Then, we can apply De Concini-Procesi theorem to the corresponding fans, to get a commun smooth resolution of these two toroidal varieties (see for example [Ewa96, Page 252]). □

Lemma 12. *Let X and Z be terminal and $\mathbb{Q}$-factorial G/H-embeddings such that Z is a resolution of the singularities of X.*

Let $E_1, \dots, E_k$ be the exceptional divisors of $\phi\colon Z \longrightarrow X$. Then, for any non-negative rational numbers $d_1, \dots, d_k$, we have $Q_D = Q_{\phi^*(D)+\sum_{i=1}^k d_iE_i}$.

Proof. Since X has terminal singularites, we have $K_Z = \phi^*K_X + \sum_{i=1}^k a_iE_i$ with $a_i > 0$ for every i.

We denote by $X_1, \dots, X_r$ the G-stable irreducible divisors of X. For any $i \in \{1, \dots, r\}$, we denote by x_i the primitive element in N of the colored edge associated to X_i. Then the G-stable irreducible divisors of Z are $X_1, \dots, X_r, E_1, \dots, E_k$. We denote by e_i the primitive element in N of the colored edge associated to E_i. Let $D = \sum_{i=1}^r b_iX_i + \sum_{\alpha\in S\backslash R} b_\alpha D_\alpha$ be an ample B-stable divisor of X. Denote by $c_1, \dots, c_k$ the rational numbers such that $\phi^*(D) = \sum_{i=1}^r b_iXi + \sum_{j=1}^k c_jE_j + \sum_{\alpha\in S\backslash R} b_\alpha D_\alpha$. Then, for any non-negative rational numbers $d_1, \dots, d_k$, the polytope Q_D coincides with

$Q_{\phi^*(D)+\sum_{i=1}^k d_iE_i} :=$
$$\{m \in M_{\mathbb{Q}} \mid \langle m, x_i\rangle \geq -b_i, \langle m, e_j\rangle \geq -c_j - d_j, \langle m, \alpha_M^\vee\rangle \geq -a_\alpha \text{ for all } i, j, \alpha\}.$$

By [Bri89], the divisors D and $\phi^*(D)$ are the divisors associated to the same piecewise linear function h_D defined in Theorem 2. Therefore, the coefficients d_j are the values $h_D(e_j)$. In terms of polytopes, since D is ample, this means that $\mathcal{H}_j := \{m \in M_{\mathbb{Q}} \mid \langle m, e_j\rangle = -c_j\}$ intersects Q_D along a face of Q_D. In particular, intersecting Q_D with the halfspace $\{m \in M_{\mathbb{Q}} \mid \langle m, e_j\rangle \geq -c_j - d_j\}$ does not change the polytope. □

Lemma 13. *Let X/T be a Mori fibre space such that X is a G/H-embedding. The set of ample B-stable $\mathbb{Q}$-divisors D of X such that the HMMP from X scaled by D gives $X \longrightarrow T$ is an open cone C_1 in $WDiv(X)_{\mathbb{Q}}$.*

Proof. Let R be the extremal ray of $NE(X)$ corresponding to the Mori fibration $X \longrightarrow T$. Let F be the dual facet of R in $Nef(X)_{\mathbb{Q}}$. By abuse of notation we denote by F the facet $F + WDiv_0(X)_{\mathbb{Q}}$ of the inverse image of $Nef(X)_{\mathbb{Q}}$ in $WDiv(X)_{\mathbb{Q}}$.

Then the HMMP from X scaled by D gives $X \longrightarrow T$ if and only if the half line $D + \mathbb{Q}_+K_X$ intersects the boundary of $Nef(X)_{\mathbb{Q}} + WDiv_0(X)_{\mathbb{Q}}$ in the relative interior $\mathring{F}$ of F. Equivalently, D is in $C_1' := \mathring{F} - \mathbb{Q}_+K_X = \mathbb{Q}_+(\mathring{F} - K_X)$.

Note that $-K_X \cdot C > 0$ for any $[C] \in R$, so that $-K_X$ is not in F and then C_1' is of maximal dimension. This implies that $D' - \epsilon K_X$ is ample for any $D' \in \mathring{F}$ and $\epsilon > 0$ small enough and then C_1' intersects $Amp(X)_{\mathbb{Q}} + WDiv_0(X)_{\mathbb{Q}}$ non trivially.

Finally, the set $C_1 = C_1' \cap Amp(X)_{\mathbb{Q}} + WDiv_0(X)_{\mathbb{Q}}$ is the sought open cone. □

Proposition 14. *Let X/T be a Mori fibre space such that X is a G/H-embedding. Let $Z \to X$ be a resolution of singularities in the category of G/H-embeddings. Then there is an euclidian open neighborhood U_X of $WDiv(Z)_{\mathbb{Q}}$ such that every divisor in U_X is ample and for every $A \in U_X$ the Mori fibre space X/T is the outcome of the HMMP from Z with scaling of A.*

Proof. Let C_1 be the open cone in $WDiv(X)_{\mathbb{Q}}$ of Lemma 13. Let $D \in C_1$. Let η be small enough such that $D - \eta K_X$ ample.

We have $WDiv(Z)_{\mathbb{Q}} = \phi^*(WDiv(X)_{\mathbb{Q}}) \oplus Vect(E_1, \ldots, E_k)$, and $\phi^*(Nef(X))$ is a face of $Nef(Z) = \overline{Amp(Z)}$. Set

$$A := \phi^*(D - \eta K_X) + \sum_{i=1}^k b_iE_i.$$

There exists an open polyhedron Pol_D of $\mathbb{Q}^k$ containing 0 in its boundary, such that for any $(b_1, \ldots, b_k) \in Pol_D$ the divisor A is ample and $d_j := b_j + \eta a_j > 0$ for all $j \in \{1, \ldots, k\}$.

Since $A + \eta K_Z = \phi^*(D) + \sum_{i=1}^k d_iE_i$, we have $Q_{A+\eta K_Z} = Q_D$ by Lemma 12. This, together with Lemma 13, proves that the HMMP scaled by A ends with the Mori fibre space X/T. Indeed, the inequalities coming from the exceptional divisors E_i are necessary to define the polytope Q_A but not the polytope $Q_{A+\eta K_Z}$ and the polytopes $Q_{A+\epsilon K_Z}$ with $\epsilon \geq \eta$.

Choose U_X to be the set of divisors $A = \phi^*(D - \eta K_X) + \sum_{i=1}^k b_iE_i$, with $D \in C_1$, $\eta > 0$ such that $D - \eta K_X$ is ample and $(b_1, \ldots, b_k) \in Pol_{D,\eta}$. Remark that by construction, since $Amp(X)$ and $Amp(Z)$ are polyhedral, the condition $D-\eta K_X$ ample is given by an inequality depending linearly on the coefficents of D, and the polyhedron Pol_D is given by inequalities depending linearly on the coefficents of D and δ. In particular, U_X is also an open polyhedron.

□

5. Two-parameters families of polytopes

In this section, we study some two-parameters families of polytopes. We present in this section many auxiliary results whose proofs, mainly relying on linear algebra tools, are in the Appendix.

5.1. **A two-parameters family of polytopes: definitions and first properties.**

Let n and p be two positive integers. Denote by I_0 the set $\{1,\dots,p\}$. Then two matrices $A\in M_{p\times n}(\mathbb{Q})$ and $d\in M_{p\times 1}(\mathbb{Q})$ define a polyhedron

$$P:=\{x\in\mathbb{Q}^n \mid Ax\geq d\}.$$

Remark 5. Suppose that P is non-empty. Then P is a polytope, that is a bounded polyhedron, if and only if the following condition is satisfied.

Condition 1. There is no non-zero $x\in\mathbb{Q}^n$ satisfying $Ax\geq 0$.

Indeed, assume that there is a non-zero $x\in\mathbb{Q}^n$ satisfying $Ax\geq 0$. Let $y\in P$. Then for every $t\in\mathbb{Q}_{\geq 0}$ we get $A(y+tx)=Ax+tAy\geq d$, thus $y+tx$ belongs to P. Conversely, if P is not bounded, P contains at least an affine half-line, and x can be taken to be a generator of the direction of this half-line.

Note that Condition 1 implies that A is injective.

To define a two-parameters family of polytopes, we fix A satisfying Condition 1 and we define divisors depending on two rational parameters δ and ϵ.
Let B, B' and C in $M_{p\times 1}(\mathbb{Q})$. Set $I_0:=\{1,\dots,p\}$ and define

$$\begin{array}{rccc} D\colon & \mathbb{Q}^2 & \to & \mathbb{Q}^p \\ & (\delta,\epsilon) & \mapsto & (1-\delta)B+\delta B'+\epsilon C. \end{array}$$

Note that D is an affine map.

Definition 4. Given A, B, B' and C as above, we define for any $(\delta,\epsilon)\in\mathbb{Q}^2$:

$$P^{\delta,\epsilon}:=\{x\in\mathbb{Q}^n \mid Ax\geq D(\delta,\epsilon)\}.$$

We do not exclude the case where some lines of A are zero. Notice that $P^{\delta,\epsilon}$ can be empty, even for all $(\delta,\epsilon)\in\mathbb{Q}^2$.

Condition 1 implies that for any $(\delta,\epsilon)\in\mathbb{Q}^2$, the set $P^{\delta,\epsilon}$ is a polytope (possibly empty).

Now, we want to describe some equivalence classes of polytopes in this family, looking at their faces that correspond to lines of A. A face $F^{\delta,\epsilon}$ of $P^{\delta,\epsilon}$ is given by some equalities in $Ax\geq D(\delta,\epsilon)$ and then is associated to some $I\subseteq I_0$. We formalize this below.

For any matrix $\mathcal{M}$ and any $i\in I_0$, we denote by $\mathcal{M}_i$ the matrix consisting of the line i of $\mathcal{M}$. More generally, for any subset I of I_0 we denote by $\mathcal{M}_I$ the matrix consisting of the lines $i \in I$ of $\mathcal{M}$. For any subset I of I_0, we can identify D_I with the affine map

$$\begin{array}{cccc} D_I\colon & \mathbb{Q}^2 & \to & \mathbb{Q}^{|I|} \\ & (\delta,\epsilon) & \mapsto & (1-\delta)B_I + \delta B'_I + \epsilon C_I. \end{array}$$

Let $(\delta,\epsilon) \in \mathbb{Q}^2$. Denote by $\mathcal{H}_i^{\delta,\epsilon}$ the hyperplane $\{x \in \mathbb{Q}^n \mid A_i x = D_i(\delta,\epsilon)\}$. For any $I \subseteq I_0$, denote by $F_I^{\delta,\epsilon}$ the face of $P^{\delta,\epsilon}$ defined by

$$F_I^{\delta,\epsilon} := (\bigcap_{i\in I} \mathcal{H}_i^{\delta,\epsilon}) \cap P^{\delta,\epsilon}.$$

Note that for any face $F^{\delta,\epsilon}$ of $P^{\delta,\epsilon}$ there exists a unique maximal $I \subseteq I_0$ such that $F^{\delta,\epsilon} = F_I^{\delta,\epsilon}$ (we include the empty face and $P^{\delta,\epsilon}$ itself).

Definition 5. Let $I \subseteq I_0$. Define Ω_I to be the set of $(\delta,\epsilon) \in \mathbb{Q}^2$ such that $F_I^{\delta,\epsilon}$ is not empty. Define ω_I to be the subset of Ω_I such that, if $I' \subseteq I_0$ satisfies $F_I^{\delta,\epsilon} = F_{I'}^{\delta,\epsilon}$, then $I' \subseteq I$.

In other words,

$$\Omega_I = \{(\delta,\epsilon)| \text{ there is } x \in \mathbb{Q}^n \text{ such that } A_I x = D_I(\delta,\epsilon) \text{ and } Ax \geq D(\delta,\epsilon)\}$$

and

$$\omega_I = \{(\delta,\epsilon)| \text{ there is } x \in \mathbb{Q}^n \text{ such that } A_I x = D_I(\delta,\epsilon) \text{ and } A_{I^c} x > D_{I^c}(\delta,\epsilon)\}$$

where I^c denotes the complement of I in I_0.

To simplify the notation, we often write i instead of $\{i\}$, for any $i \in I_0$.

Remark 6. If $(\delta,\epsilon) \in \omega_\emptyset$, then the polytope $P^{\delta,\epsilon}$ is of dimension n (i.e. has a non-empty interior). And, for any $i \in I_0$, if $(\delta,\epsilon) \in \omega_i$ and $A_i \neq 0$, then $F_i^{\delta,\epsilon}$ is a facet of $P^{\delta,\epsilon}$.

The following lemma describes the first properties of the sets Ω_I and ω_I. It follows from Lemmas 32 and 33.

Proposition 15. *Let $I \subseteq I_0$.*

(1) *The sets Ω_I and ω_I are convex subsets of $\mathbb{Q}^2$.*

(2) *The set ω_I is open, for the euclidean topology, inside $\{(\delta,\epsilon)|\ D_I(\delta,\epsilon) \in \operatorname{Im} A_I\} = D_I^{-1} \operatorname{Im} A_I$.*

(3) *There are four cases: either ω_I is empty, or it is a point, or it is a convex part of an affine line (a segment, a half-line or a line), or it is a non-empty open set in $\mathbb{Q}^2$.*

(4) *If ω_I is not empty, we have $\omega_I \subset \Omega_I = \overline{\omega_I}$.*

The next result will be useful in the next sections. It follows from Lemma 35.

Lemma 16. *Let $I \subseteq I_0$ be such that $\omega_I = \{(\delta_0,\epsilon_0)\}$. Suppose that the image of A_I is of codimension 2. There is $i \in I$ such that the image of $A_{I\setminus\{i\}}$ is of codimension 1. For any such i, the point (δ_0,ϵ_0) belongs to $\Omega_{I\setminus\{i\}} \setminus \omega_{I\setminus\{i\}}$.*

If B and B' are general, then the sets ω_I and Ω_I intersect "nicely", as proved in the next proposition.

We denote by $\operatorname{Aff}(S)$ the affine space generated by a set S.

Proposition 17. *There is an open Zariski subset V of the ample cone such that for every $(B,B')\in V\times V$ the subsets Ω_I and ω_I of Definition 5.2 satisfy the following.*

(1) *If $\omega_I\neq\emptyset$, the dimension of ω_I equals $2-\operatorname{codim}\operatorname{Im}A_I$ (which is then at least 0).*

(2) *There is a finite union of convex parts of affine lines*

$$\mathcal{L}=\partial\Omega_\emptyset\cup\bigcup_{I\subseteq I_0\ \dim\Omega_I\leq 1}\Omega_I\subseteq\mathbb{Q}^2$$

such that if $(\delta,\epsilon)\notin L$ then $P^{\delta,\epsilon}$ is an n-dimensional simple polytope or it is empty.

(3) *For any $I,J\subseteq I_0$ such that $\dim\omega_I=\dim\omega_J=1$, we have $\operatorname{Aff}(\omega_I)=\operatorname{Aff}(\omega_J)$ if and only if $\omega_{I\cap J}$ is a non-empty open convex part of an affine line. Moreover in this case, we have $\omega_{I\cap J}\supseteq\omega_I\cup\omega_J$.*

(4) *For any $I,J\subseteq I_0$ such that $\dim\omega_I=\dim\omega_J=0$, we have $\omega_I=\omega_J$ if and only if $\omega_{I\cap J}$ is a singleton, equal to $\omega_I=\omega_J$.*

(5) *For any $I,J\subseteq I_0$ such that $\dim\omega_I=0$ and $\dim\omega_J=1$, $\operatorname{Aff}(\omega_J)$ contains ω_I if and only if $\omega_{I\cap J}$ is a non-empty open convex part of $\operatorname{Aff}(\omega_J)$.*

The proof of Proposition 17 is given in Section A.2.

Lemma 18. *Let B,B' be as in Proposition 17. Let $I\subseteq I_0$, such that $\omega_I\neq\emptyset$. There exists $J\subseteq I$ as above and such that $\dim\omega_I=\dim\omega_J$ and J is minimal with the property.*

Proof. Follows from Lemma 34(3). ☐

Remark 7. Let $I\subseteq I_1$ be such that $\operatorname{Im}A_I$ and $\operatorname{Im}A_{I_1}$ have codimension 1 and are defined by the same equation $\sum_{i\in I}\lambda_iX_i=0$. Assume moreover that $I=\{i|\ \lambda_i\neq 0\}$. Then for every $j\in I$ the morphism induced by $A_{I_1\backslash\{j\}}$ is surjective.

Notation 1. Let B,B' be as in Proposition 17.

(1) Let $I\subseteq I_0$ be such that ω_I is one-dimensional. By Lemma 18 without loss of generality we can assume that for any $i\in I$, $A_{I\backslash\{i\}}$ is surjective. Then there are rational numbers λ_i^I with $i\in I$ such that $\sum_{i\in I}\lambda_i^IA_i=0$. We notice that $\lambda_i^I\neq 0$ for every $i\in I$. Indeed, if there is j such that $\lambda_j^I=0$, then $\sum_{i\in I\backslash\{j\}}\lambda_i^IX_i$ is a nontrivial equation for the lines of $A_{I\backslash\{j\}}$. We can assume that $\sum_{i\in I}\lambda_i^IC_i$ is zero or one. We fix such numbers.

(2) Let $L\subseteq I_0$ be such that ω_L is a singleton. By Lemma 18 we can assume that L is minimal with the property. As above, we get two independent relations $\sum_{\ell\in L}\lambda_\ell X_\ell=0$ and $\sum_{\ell\in L}\mu_\ell X_\ell=0$ on the lines of A_L. We set $I=\{\ell\in L|\ \lambda_\ell\neq 0\}$ and $J=\{\ell\in L|\ \mu_\ell\neq 0\}$. We can choose the relations such that I and J are proper in L. Then, as the image of $A_{L\backslash\{\ell\}}$ has codimension 1 for every $\ell\in L$, the images of A_I and A_J have codimension 1 and we have $I\cup J=L$.

5.2. **Polyhedral decomposition and the geography of models.** Let G be a reductive group and $H\subseteq G$ be a horospherical subgroup. We choose a basis of $M_{\mathbb{Q}}$ so that $M_{\mathbb{Q}}\cong\mathbb{Q}^n$. We fix a horospherical embedding Z of G/H and set $p=r+|S\backslash R|$. Recall that, by the notation given in Section 2, r is the number of G-stable prime divisor of Z and $|S\backslash R|$ is the number of B-stable prime divisor of G/H. In particular, p is the number of B-stable prime divisor of Z. Let A be the $p\times n$ matrix associated to the linear map $\varphi(m)=(\langle m,x_i\rangle_{i=1\dots r},\ \langle m,\alpha_M^\vee\rangle_{\alpha\in S\backslash R})$. Denote by $J_0\subseteq I_0$ the set of indices $S\backslash R$.

Let $B = (-d_1, \ldots, -d_r, (-d_\alpha)_{\alpha \in S\setminus R})$ and $B' = (-d'_1, \ldots, -d'_r, (-d'_\alpha)_{\alpha \in S\setminus R})$ be such that $D = \sum_{i=1}^d d_i Z_i + \sum_{\alpha \in S\setminus R} d_\alpha D_\alpha$ and $D' = \sum_{i=1}^d d'_i Z_i + \sum_{\alpha \in S\setminus R} d'_\alpha D_\alpha$ are ample divisors on two varieties X and Y respectively, in the sense that $\tilde{Q}_D$ and $\tilde{Q}_{D'}$ are pseudo-moment polytopes for X and Y respectively. Notice that it may occur that some D_i's are not divisors of X or Y as we observe in Lemma 13. Let $C = (c_1, \ldots, c_r, (c_\alpha)_{\alpha \in S\setminus R})$ be such that $c_i > 0$ for every i and $c_\alpha > 0$ for every α. Since X is projective, $\tilde{Q}_D = \{x \in M_{\mathbb{Q}} \mid Ax \geq B\}$ is a polytope and then Condition 1 is verified by A.

From now on, following remark 4, if a polytope Q is defined by $\{x \in M_{\mathbb{Q}} \mid Ax \geq D\}$, with $D = (-d_1, \ldots, -d_r, (-d_\alpha)_{\alpha \in S\setminus R})$, we will say that it is the pseudo-moment polytope associated to the moment polytope $v^0 + Q$ with $v^0 = \sum_{\alpha \in S\setminus R} d_\alpha \varpi_\alpha$, or similarly the pseudo-moment polytope of the associated G/H-embedding as soon as $v^0 + Q$ is a G/H-polytope (Definition 2.3).

Remark 8. Let $(\delta, \epsilon) \in \omega_\emptyset$, then $P^{\delta,\epsilon}$ is of maximal dimension in $M_{\mathbb{Q}}$ and contains no hyperplane defined as $\mathcal{H}_\alpha^{\delta,\epsilon} := \{A_j x = D_j^{\delta,\epsilon}\}$ with $j \in J_0$ associated to the simple root α. In particular, if $v^{\delta,\epsilon} = \sum_{j \in J_0} D_j^{\delta,\epsilon} \varpi_\alpha$, we can check that the polytope $v^{\delta,\epsilon} + P^{\delta,\epsilon}$ is a G/H-polytope.

Consider now the case where $(\delta, \epsilon) \in \Omega_\emptyset \setminus \omega_\emptyset$. Let $M'_{\mathbb{Q}}$ be the linear subspace generated by $P^{\delta,\epsilon}$ and M' the sublattice $M'_{\mathbb{Q}} \cap M$ of M, let R' be the union of R with the set of simple roots $\alpha \in S\setminus R$ such that $P^{\delta,\epsilon} \subset \mathcal{H}_\alpha^{\delta,\epsilon}$, and define P' to be the parabolic subgroup containing B whose simple roots are R'; and H' to be the kernel of characters of $M' \subset \mathfrak{X}(P')$. Then, we can also check that the polytope $v^{\delta,\epsilon} + P^{\delta,\epsilon}$ is a G/H'-polytope.

We suppose that B and B' belong to the open set V existing by Proposition 17.

Then we define the following locally closed subsets of $\Omega_\emptyset$. Recall that by $\mathrm{Aff}(S)$ we denote the affine space generated by a set S.

Definition 6. We set

$$\begin{aligned}
U_2 = &\quad \{(\delta,\epsilon) \in \Omega_\emptyset |\ (\delta,\epsilon) \in \omega_I \Rightarrow \dim \omega_I = 2\} \\
U_1 = &\quad \{(\delta,\epsilon) \in \Omega_\emptyset |\ (\delta,\epsilon) \notin U_2,\ (\delta,\epsilon) \in \omega_I \cap \omega_J \Rightarrow \dim \mathrm{Aff}\, \omega_I \cap \mathrm{Aff}\, \omega_J \geq 1\} \\
U_0 = &\quad \{(\delta,\epsilon) \in \Omega_\emptyset |\ \exists I, \{(\delta,\epsilon)\} = \omega_I\} \\
U'_0 = &\quad \Omega_\emptyset \setminus (U_2 \cup U_1 \cup U_0).
\end{aligned}$$

Note that U_0, U'_0 are finite sets as B and B' are as in Proposition 17. By construction, $\Omega_\emptyset$ is the disjoint union of these four sets.

Moreover, we have $U_2 \subseteq \omega_\emptyset$. Indeed, by Proposition 15, $\Omega_\emptyset = \overline{\omega_\emptyset}$, therefore $\Omega_\emptyset \setminus \omega_\emptyset$ is a union of zero or one-dimensional ω_I's. We also have $U'_0 \subseteq \omega_\emptyset$, because points of U'_0 are intersections of non-colinear one-dimensional ω_I's by Proposition 17. However, note that points of U_1 and U_0 can be either in $\omega_\emptyset$ or in $\Omega_\emptyset \setminus \omega_\emptyset$.

We can now give the following notation, using Remark 8 and Proposition 5.

Notation 2. Let (δ, ϵ) be such that $P^{\delta,\epsilon}$ is not empty.

(1) If $(\delta, \epsilon) \in U_2$, the polytope $P^{\delta,\epsilon}$ is the pseudo-moment polytope of a G/H-embedding which we denote by $X^{\delta,\epsilon}$.

(2) If $(\delta, \epsilon) \in U_1$, there is $H' \supseteq H$ such that $P^{\delta,\epsilon}$ is the pseudo-moment polytope of a G/H'-embedding which we denote by $Y^{\delta,\epsilon}$.

(3) If $(\delta, \epsilon) \in U_0$, there is $H' \supseteq H$ such that $P^{\delta,\epsilon}$ is the pseudo-moment polytope of a G/H'-embedding which we denote by $Z^{\delta,\epsilon}$.

Remark 9. By Proposition 17(2), the generality assumption on B and B' and Lemma 8 for every $(\delta,\epsilon)\in U_2$ the variety $X^{\delta,\epsilon}$ is $\mathbb{Q}$-factorial. In particular, by Corollary 3, its Picard number is

$$\rho(X^{\delta,\epsilon}) = |I_1^{\delta,\epsilon}\sqcup J_0| - n \tag{1}$$

where $I_1^{\delta,\epsilon} := \{i\in I_0\backslash J_0 \mid (\delta,\epsilon)\in\omega_i\}$. Since $(\delta,\epsilon)\in U_2$, we can replace $I_1^{\delta,\epsilon}$ by $I_2^{\delta,\epsilon}=\{i\in I_0\backslash J_0 \mid (\delta,\epsilon)\in\Omega_i\}$ in the formula.

Let $(\delta_0,\epsilon_0)\in U_2$. Let $\ell\colon\mathbb{Q}\to\mathbb{Q}^2$ be the parametrisation of a rational affine line such that $\ell(0)=(\delta_0,\epsilon_0)$. Let $\bar{t}>0$ be the minimum such that $\ell(\bar{t})\in\Omega_\emptyset\setminus U_2$. There is a G-equivariant morphism from X^{δ_0,ϵ_0} to the variety corresponding to $P^{\ell(\bar{t})}$. Indeed, for every $t\in[0,\bar{t})$, the polytope $P^{\ell(t)}$ is the pseudo-moment polytope of a polarized variety $(X^{\ell(t)},D^{\ell(t)})$, and all the $P^{\ell(t)}$ for $t\in[0,\bar{t})$ are equivalent. Thus the corresponding varieties are isomorphic to X^{δ_0,ϵ_0}. The divisor $D(\ell(\bar{t}))$ is nef, but not ample, on X^{δ_0,ϵ_0}. Hence, by Lemma 7 there is a G equivariant morphism from X^{δ_0,ϵ_0} to the variety corresponding to $P^{\ell(\bar{t})}$.

5.2.1. *If $\ell(\bar{t})\in U_1$.* We assume in this paragraph that $\ell(\bar{t})=(\delta_1,\epsilon_1)\in U_1$. The main result of this paragraph is Proposition 23, which describes the different sorts of G-equivariant morphisms we can get from X^{δ_0,ϵ_0} to Y^{δ_1,ϵ_1}. We start with some preparatory lemmas.

Lemma 19. *Let $(\delta,\epsilon)\in U_1$. Let I be such that $(\delta,\epsilon)\in\omega_I$ and $\dim\omega_I=1$. Assume moreover that I is minimal with the property. Let $\{\lambda_i^I\}$ be the coefficients defined in Notation/Construction 1. Denote by $I_+ := \{i\in I \mid \lambda_i^I>0\}$ and $I_- := \{i\in I \mid \lambda_i^I<0\}$. Note that $I=I_+\sqcup I_-$.*

Then $(\delta,\epsilon)\in\omega_\emptyset$ if and only if both I_+ and I_- are not empty.

Proof. Follows from Lemma 46 in the Appendix. □

Lemma 20. *Let $(\delta,\epsilon)\in U_1\cap\omega_\emptyset$. Let I be such that $(\delta,\epsilon)\in\omega_I$ and $\dim\omega_I=1$. Assume moreover that I is minimal with the property. Let $i\in I_0\backslash J_0$. Then $(\delta,\epsilon)\in\Omega_i\backslash\omega_i$ if and only if I_+ or I_- equals $\{i\}$.*

In particular, there is at most one $i\in I_0\backslash J_0$, such that (δ,ϵ) is in $\Omega_i\backslash\omega_i$.

Proof. Follows from Lemma 47 in the Appendix. □

Lemma 21. *Let $(\delta,\epsilon)\in U_1\cap\Omega_\emptyset\backslash\omega_\emptyset$. Let I be such that $(\delta,\epsilon)\in\omega_I$ and $\dim\omega_I=1$. Assume moreover that I is minimal with the property. Let $i\in I_0\backslash I$. Then either $(\delta,\epsilon)\notin\Omega_{I\cup\{i\}}$ or $(\delta,\epsilon)\in\omega_{I\cup\{i\}}$.*

Proof. Follows from Lemma 48 in the Appendix. □

Corollary 22. *Let $(\delta,\epsilon)\in U_1$.*

(1) *Assume that $(\delta,\epsilon)\in\omega_\emptyset$. If there is $i\in I_0\backslash J_0$ such that I_+ or I_- equals $\{i\}$, then the G-stable prime divisors of $Y^{\delta,\epsilon}$ are in bijection with $I_1^{\delta,\epsilon}=I_2^{\delta,\epsilon}\backslash\{i\}$. Otherwise the G-stable prime divisors of $Y^{\delta,\epsilon}$ are in bijection with $I_1^{\delta,\epsilon}=I_2^{\delta,\epsilon}$.*

(2) *Assume that $(\delta,\epsilon)\in\Omega_\emptyset\backslash\omega_\emptyset$. Then the G-stable prime divisors of $Y^{\delta,\epsilon}$ are in bijection with*

$$\{i\in I_0\backslash I \mid (\delta,\epsilon)\in\omega_{I\cup\{i\}}\} = \{i\in I_0\backslash I \mid (\delta,\epsilon)\in\Omega_{I\cup\{i\}}\} = I_2^{\delta,\epsilon}\backslash(I_2^{\delta,\epsilon}\cap I).$$

Proof. The set of B-stable prime divisors of $Y^{\delta,\epsilon}$ is the union of G-stable divisors and colors of the horospherical homogeneous space. The G-stable prime divisors correspond bijectively to the facets $F_i^{\delta,\epsilon}$ with $i\in I_0\backslash J_0$ that are not equal to some $F_j^{\delta,\epsilon}$ with $j\in J_0$.

Assume that $(\delta,\epsilon)\in\omega_\emptyset$. Thus $Y^{\delta,\epsilon}$ is a G/H-embedding, colors are indexed by $S\backslash I$ or equivalently by J_0. If there are i and $j\in I_0$ with $i\neq j$ and such that $F_i^{\delta_1,\epsilon_1}=F_j^{\delta_1,\epsilon_1}$ is a facet, then A_i and A_j are colinear, one of i or j is in J_0, and $I=i,j$. Lemma 20 implies the claim.

Assume that $(\delta,\epsilon)\in\Omega_\emptyset\backslash\omega_\emptyset$. Then $Y^{\delta,\epsilon}$ is a G/H'-embedding with $H'\subsetneq H$ as explained in Remark 8. In the proof of Lemma 46, we prove that $P^{\delta,\epsilon}$ generates the affine subspace $\{x\in M_\mathbb{Q}\mid A_Ix=D_I(\delta,\epsilon)\}$, so that $M'=\ker(A_I)\cap M$. Moreover, the colors of G/H' are indexed by $J_0\backslash(J_0\cap I)$. If for some $i,j\in I_0$, $F_i^{\delta,\epsilon}=F_j^{\delta,\epsilon}$ is a facet, then $i,j\notin I$, A_i and A_j are colinear modulo the vector space $N''_\mathbb{Q}$ generated by the lines of A_I. If $i\neq j$, it would give another relation to the one given by I and then (δ,ϵ) is in a zero dimensional ω_K with $K\supset I\cup\{i,j\}$, that is a contradiction with $(\delta,\epsilon)\in U_1$. Then $i=j$. Note also that for any $i\in I_0\backslash I$, $F_i^{\delta,\epsilon}=F_{I\cup\{i\}}^{\delta,\epsilon}$, and if it is not empty then $A_i\notin N''_\mathbb{Q}$. Indeed, if $A_i\in N''_\mathbb{Q}$, we have another relation to the one given by I.

Hence Lemma 21 implies the claim. □

Proposition 23. *Let $(\delta_0,\epsilon_0)\in U_2$. Let $\ell\colon\mathbb{Q}\to\mathbb{Q}^2$ be the parametrisation of a rational affine line such that $\ell(0)=(\delta_0,\epsilon_0)$. Let $\bar t>0$ be the minimum such that $\ell(\bar t)\in\Omega_\emptyset\setminus U_2$. Assume that $\ell(\bar t)=(\delta_1,\epsilon_1)\in U_1$.*

Set I minimal such that $(\delta_1,\epsilon_1)\in\omega_I$.

The morphism from X^{δ_0,ϵ_0} to Y^{δ_1,ϵ_1} is an extremal contraction and one of the following occurs.

(1) *If $I=I_+$ or I_- then $\dim Y^{\delta_1,\epsilon_1}<\dim X^{\delta_0,\epsilon_0}$, Y^{δ_1,ϵ_1} is $\mathbb{Q}$-factorial and $\rho(X)=\rho(Y)+1$.*

(2) *If $|I|\geq 2$ and I_+ or I_- is $\{i\}$ with $i\in I_0\backslash J_0$ then Y^{δ_1,ϵ_1} is $\mathbb{Q}$-factorial and $X^{\delta_0,\epsilon_0}\to Y^{\delta_1,\epsilon_1}$ is an extremal divisorial contraction or an isomorphism.*

(3) *In the other cases $X^{\delta_0,\epsilon_0}\to Y^{\delta_1,\epsilon_1}$ is a small extremal contraction.*

Proof. Assume that $I=I_+$ or I_-. By Lemma 19, we have $(\delta_1,\epsilon_1)\in\Omega_\emptyset\setminus\omega_\emptyset$ and by Remark 8 we have $\dim Y^{\delta_1,\epsilon_1}<\dim X^{\delta_0,\epsilon_0}$.

By Lemma 21, the B-stable prime divisors of Y^{δ_1,ϵ_1} are in bijection with $(I_2^{\delta_1,\epsilon_1}\sqcup J_0)\backslash I$. Let $F_J^{\delta_1,\epsilon_1}$ be a face of P^{δ_1,ϵ_1}, choose J such that $(\delta_1,\epsilon_1)\in\omega_J$. Then $\mathrm{Im}(A_I)$ and $\mathrm{Im}(A_J)$ have both codimension 2 in $\mathbb{Q}^{|I|}$ and $\mathbb{Q}^{|J|}$ respectively. We can see $\mathbb{Q}^{|I|}$ and $\mathbb{Q}^{|J\backslash I|}$ as supplementary subsapces of $\mathbb{Q}^{|J|}$, so that $\mathrm{Im}(A_J)\subset\mathrm{Im}(A_I)\oplus\mathbb{Q}^{|J\backslash I|}$. Since both subspaces are two-codimensional subspaces of $\mathbb{Q}^{|J|}$, they are equal. We then deduce that the restriction of $A_{J\backslash I}$ to $\ker(A_I)$ is surjective. This implies, with Lemma 8 applied to the matrix of lines of $A_{I_0\backslash I}$ restricted to $\ker(A_I)$, that Y^{δ_1,ϵ_1} is $\mathbb{Q}$-factorial. Now, the dimension of $\ker(A_I)$ is $n-|I|+1$ and by Corollary 3, the Picard number of Y^{δ_1,ϵ_1} is $|(I_2^{\delta_1,\epsilon_1}\sqcup J_0)\backslash I|-(n-|I|+1)=|I_2^{\delta_1,\epsilon_1}\sqcup J_0|-n-1$. But, since the Ω_i's are closed, $I_2^{\delta_1,\epsilon_1}=I_2^{\delta_0,\epsilon_0}$, so that the relative Picard number of $X^{\delta_0,\epsilon_0}\longrightarrow Y^{\delta_1,\epsilon_1}$ is 1.

Assume that I_+ or I_- is $\{i\}$ with $i\in I_0\backslash J_0$ and $|I|\geq 2$. By Lemma 19 we have $(\delta_1,\epsilon_1)\in\omega_\emptyset$. By Remark 8 we have $\dim Y^{\delta_1,\epsilon_1}=\dim X^{\delta_0,\epsilon_0}$.

By Corollary 22, the B-stable prime divisors of Y^{δ_1,ϵ_1} are in bijection with $(I_2^{\delta_1,\epsilon_1}\backslash\{i\})\sqcup J_0 = (I_2^{\delta_0,\epsilon_0}\backslash\{i\})\sqcup J_0$. For the sake of shortness we denote by $K\setminus\{i\}$ the set $K\cap(I_0\setminus\{i\})$.

Let $F_J^{\delta_1,\epsilon_1}$ be a face of P^{δ_1,ϵ_1}, choose J such that $(\delta_1,\epsilon_1)\in\omega_J$.

Suppose that A_J is not surjective, then ω_J is of dimension 1 (because $(\delta_1,\epsilon_1)\in U_1$). Then $(\delta_1,\epsilon_1)\in U_1\cap\omega_I\cap\omega_J$ implies that ω_I and ω_J are in the same affine line. We conclude by Proposition 17 (3) and the minimality of I that $I\subseteq J$. Then by Remark 7, for any $i\in I$, $A_{J\backslash\{i\}}$ is surjective. This implies, with Lemma 8, that Y^{δ_1,ϵ_1} is $\mathbb{Q}$-factorial. Then its Picard number is $|(I_2^{\delta_0,\epsilon_0}\backslash\{i\})\sqcup J_0|-n$. This is the Picard number of X^{δ_0,ϵ_0} minus one if $i\in I_2^{\delta_0,\epsilon_0}$ and the Picard number of X^{δ_0,ϵ_0} if not. In that latter case, since P^{δ_1,ϵ_1} can be defined as well without the line i, it is an equivalent G/H-polytope to P^{δ_0,ϵ_0}, and then $X^{\delta_0,\epsilon_0}=Y^{\delta_1,\epsilon_1}$.

In the other cases, again by Lemma 19 and Remark 8 we have $\dim Y^{\delta_1,\epsilon_1} = \dim X^{\delta_0,\epsilon_0}$.

Corollary 22 implies that the B-stable prime divisors of Y^{δ_1,ϵ_1} are the same as the ones of X^{δ_0,ϵ_0}, they are therefore in bijection with $I_2^{\delta_0,\epsilon_0}\sqcup J_0$.

Let $F_J^{\delta_1,\epsilon_1}$ be a face of P^{δ_1,ϵ_1}, choose J such that $(\delta_1,\epsilon_1)\in\omega_J$. If A_J is not surjective, then ω_J is one-dimensional and $I\subseteq J$. Then a B-stable $\mathbb{Q}$-divisor of Y^{δ_1,ϵ_1} is $\mathbb{Q}$-Cartier if and only if its coefficients satisfy the equation $\sum_{i\in I}\lambda_i^I X_i$. Also note that for $I=J$, A_J is not surjective.

Then the relative Picard number of $X^{\delta_0,\epsilon_0}\longrightarrow Y^{\delta_1,\epsilon_1}$ is 1. □

Remark 10. If $I^+=I$ or $I^-=I$, for every $i\in I$ the polytope $P^{\delta,\epsilon}$ is contained in $\mathcal{H}_i^{\delta,\epsilon}=\{x\in\mathbb{Q}^n \mid A_ix=D_i^{\delta,\epsilon}\}$. There are two possible cases for I. Either $I=\{i\}$ and $A_i=0$, so that $P^{\delta,\epsilon}$ has dimension n, the associated horospherical variety X has the same rank as G/H, the same lattice M but its open homogeneous space has one color less (α_i); or $A_i\neq 0$ for any $i\in I$ (and $|I|\geq 2$), so that $P^{\delta,\epsilon}$ is of maximal dimension an affine subspace directed by $\ker(A_I)$, then it has dimension $n-|I|+1$, and the associated horospherical variety X has rank $n-|I|+1$, its lattice is $M\cap\ker A_I$ and the colors of its open homogeneous space are the colors of G/H whose index of line in not in I.

If $I^+=\{i\}$ or $I^-=\{i\}$, then the condition $A_ix\geq D_i(\delta,\epsilon)$ is superfluous in the definition of $P^{\delta,\epsilon}$.

5.2.2. *If $\ell(\bar{t})\in U_0$.* We assume in this paragraph that $\ell(\bar{t})=(\delta_2,\epsilon_2)\in U_0$. We want to study the morphisms from X^{δ_0,ϵ_0} to Z^{δ_2,ϵ_2}.

Let L be such that $\{(\delta_2,\epsilon_2)\}=\omega_L$. By Lemma 16, there is I such that ω_I has dimension one and $(\delta_2,\epsilon_2)\in\Omega_I$. Then all but a finite set of points of ω_I are in $\omega_I\cap U_1$. Choose $(\delta_1,\epsilon_1)\in\omega_I\cap U_1$ close to (δ_2,ϵ_2).

By Lemma 7 there are G-equivariant morphisms from X^{δ_0,ϵ_0} to Y^{δ_1,ϵ_1} and Z^{δ_2,ϵ_2}, and from Y^{δ_1,ϵ_1} to Z^{δ_2,ϵ_2}. Moreover the morphism from X^{δ_0,ϵ_0} to Z^{δ_2,ϵ_2} factorizes through Y^{δ_1,ϵ_1}.

We suppose from now on that $\{(\delta_2,\epsilon_2)\}=\omega_L\subset\Omega_\emptyset\backslash\omega_\emptyset$.

Notation 3. Let L be such that $\omega_L=\{(\bar{\delta},\bar{\epsilon})\}\subset\Omega_\emptyset\backslash\omega_\emptyset$ and $\bar{\epsilon}>0$. Then, by Propostion 17, the image of A_L has codimension two. By Lemma 18, we can suppose up to taking a subset of L that for every $\ell \in L$ the image of $A_{L\setminus\{\ell\}}$ has codimension 1 in $\mathbb{Q}^{|L|}$.

We have two possible cases.

(1) If ω_L is a vertex of $\Omega_\emptyset\backslash\omega_\emptyset$, by Notation/Construction 1(2) there exist I and J subsets of L such that, for any $i \in I$ and any $j \in J$, $A_{I\setminus\{i\}}$ and $A_{J\setminus\{j\}}$ are surjective, ω_I, ω_J have dimension 1 and are contained in $\Omega_\emptyset\backslash\omega_\emptyset$. By Lemma 19 we can fix two linearly independent equations for the lines of A_L

$$\begin{array}{lll} (\mathcal{R}_I) & \sum_{i\in I}\lambda_i^I X_i = 0, & \lambda_i^I > 0\ \forall i \in I \\ (\mathcal{R}_J) & \sum_{j\in J}\lambda_j^J X_j = 0, & \lambda_j^J > 0\ \forall j \in J \end{array}$$

We suppose that $\omega_I \subseteq \{\delta < \bar{\delta}\}$ and that $\omega_J \subseteq \{\delta > \bar{\delta}\}$.
We can moreover assume that $\sum_{i\in I}\lambda_i^I C_i = 1$ and $\sum_{j\in J}\lambda_j^J C_j = 1$.

(2) If ω_L is not a vertex of $\Omega_\emptyset\backslash\omega_\emptyset$, then there is $I \subseteq L$ such that, for any $i \in I$ the matrix $A_{I\setminus\{i\}}$ is surjective, and ω_I is a segment of $\Omega_\emptyset\backslash\omega_\emptyset$ containing ω_L. By Lemma 19, we can fix two linearly independent equations for the lines of A_L

$$\begin{array}{lll} (\mathcal{R}_I) & \sum_{i\in I}\lambda_i^I X_i = 0, & \lambda_i^I > 0\ \forall i \in I \\ (\mathcal{R}_L) & \sum_{j\in L}\lambda_j^L X_j = 0, & \lambda_j^L \neq 0\ \forall j \in L\setminus I \end{array}$$

We verify that $\lambda_j^L \neq 0\ \forall j \in L \setminus I$. Indeed if the second equation has one zero coefficient λ_j^L for some $j \in L\backslash I$, then $A_{L\setminus\{j\}}$ would also have an image of codimension 2, contradicting the minimality of L. Moreover, the coefficients with indexes in $L\backslash I$ in the relation $(\mathcal{R}_L)$ do not have all the same sign.

Indeed, if not, we can suppose that that they are positive, and by adding to $(\mathcal{R}_L)$ a positive multiple of $(\mathcal{R}_I)$, we would obtain a second equation, linearly independent with $(\mathcal{R}_I)$, with positive coefficients, associated to some $J \subsetneq I$. By Lemma 19 the set ω_J would be contained in $\Omega_\emptyset \setminus \omega_\emptyset$, contradicting the hypothesis on ω_L, as it would be the vertex between the segments ω_I and ω_J.

With the same argument we can also suppose that $\lambda_j^L \geq 0$ for any $j \in I$ but some are zero.

We set $(L \setminus I)_+ = \{j \in L \setminus I \mid \lambda_j^L > 0\}$ and $(L \setminus I)_- = \{j \in L \setminus I \mid \lambda_j^L < 0\}$.

We can moreover assume that $\sum_{i\in I}\lambda_i^I C_i = 1$ and either $\sum_{j\in L}\lambda_j^L C_j = 1$, or $\sum_{j\in L}\lambda_j^L C_j = 0$ and $\sum_{j\in L}\lambda_j^L(B'_j - B_j) > 0$. Indeed if $\sum_{j\in L}\lambda_j^L C_j < 0$ we take the opposite and add a positive multiple of the first equation and we obtain the wanted equation with $\sum_{j\in L}\lambda_j^L C_j > 0$. And if $\sum_{j\in L}\lambda_j^L C_j = 0$ and $\sum_{j\in L}\lambda_j^L(B'_j - b_j) > 0$, we take the opposite and add a positive multiple of the first equation and we obtain $\sum_{i\in I}\lambda_i^I C_i > 0$.

Lemma 24. *Let $(\delta,\epsilon) \in U_0\cap\Omega_\emptyset\backslash\omega_\emptyset$ and assume that (δ,ϵ) is a vertex. Let $L \subseteq I_0$ be a minimal subset such that $\omega_L = \{(\delta,\epsilon)\}$. Let $i \in I_0\backslash L$. Then either $(\delta,\epsilon) \notin \Omega_{L\cup\{i\}}$ or $(\delta,\epsilon) \in \omega_{L\cup\{i\}}$.*

Proof. Follows from Lemma 49 in the Appendix. □

Lemma 25. *Let $(\delta,\epsilon) \in U_0\cap\Omega_\emptyset\backslash\omega_\emptyset$ and assume that (δ,ϵ) is not a vertex of $\Omega_\emptyset\backslash\omega_\emptyset$. Let $L \subseteq I_0$ be a minimal subset such that $\omega_L = \{(\delta,\epsilon)\}$. Let I be a minimal subset such that $\omega_L \subseteq \omega_I$ and $\dim\omega_I = 1$. Let $i \in I_0\backslash I$. If $(\delta,\epsilon) \in \Omega_{I\cup\{i\}}\backslash\omega_{I\cup\{i\}}$ then $i \in L\backslash I$ and either $L \setminus I_+ = \{i\}$ or $L \setminus I_- = \{i\}$. In particular, there are at most 2 indices i such that $(\delta,\epsilon) \in \Omega_{I\cup\{i\}}\backslash\omega_{I\cup\{i\}}$. If there are 2 such indices, then $|L| = |I| + 2$ and the restriction of $A_{L\setminus I}$ to $\ker(A_I)$ consists of twice the same line if $L\backslash I \subseteq I_0\backslash J_0$.*

Proof. Follows from Lemma 50 in the Appendix. □

Corollary 26. *Let $(\delta,\epsilon)\in U_0\cap\partial\Omega_\emptyset$. Let L be minimal such that $\omega_L=\{(\delta,\epsilon)\}$.*

(1) *If ω_L is a vertex, the set of G-stable prime divisors of $Z^{\delta,\epsilon}$ is in bijection with $I_2^{\delta,\epsilon}\backslash(I_2^{\delta,\epsilon}\cap L)$;*

(2) *if ω_L is not a vertex and there is $i\in I_0\setminus J_0$ such that either $L\backslash I_+$ or $L\backslash I_-$ is $\{i\}$, the set of G-stable prime divisors of $Z^{\delta,\epsilon}$ is in bijection with $I_2^{\delta,\epsilon}\setminus\{i\}$;*

(3) *otherwise, the set of G-stable prime divisors of $Z^{\delta,\epsilon}$ is in bijection with $I_2^{\delta,\epsilon}$.*

Proof. By Remark 8, the variety $Z^{\delta,\epsilon}$ is a G/H'-embedding with $H'\subsetneq H$. We let M' be the lattice of characters vanishing on H and N' its dual. Note that $N'_{\mathbb{Q}}$ is the quotient $N_{\mathbb{Q}}/N''_{\mathbb{Q}}$ where $N''_{\mathbb{Q}}$ is the subspace generated by the lines of A_L.

The G-stable prime divisors of $Z^{\delta,\epsilon}$ correspond bijectively to the facets $F_i^{\delta,\epsilon}$ with $i\in I_0\backslash J_0$ that are not equal to some $F_j^{\delta,\epsilon}$ with $j\in J_0$.

Assume that ω_L is a vertex. The polytope $P^{\delta,\epsilon}$ generates the affine subspace $\{x\in M_{\mathbb{Q}}\mid A_Lx=D_L(\delta,\epsilon)\}$, so that $M'=\ker(A_L)\cap M$. Moreover, the colors of G/H' are indexed by $J_0\backslash(J_0\cap L)$. If for some $i,j\in I_0$, $F_i^{\delta,\epsilon}=F_j^{\delta,\epsilon}$ is a facet, then, $i,j\notin L$, A_i and A_j are colinear modulo $N''_{\mathbb{Q}}$. If $i\neq j$ we would have another relation other than the two given by L. Thus we have $i=j$.

Note also that for any $i\in I_0\backslash L$, we have $F_i^{\delta,\epsilon}=F_{L\cup\{i\}}^{\delta,\epsilon}$. If the latter is non empty and if $A_i\in N''_{\mathbb{Q}}$ we would have another relation other than the two given by L so that $A_i\notin N''_{\mathbb{Q}}$. Hence by Lemma 24 the G-stable prime divisors of the variety $Z^{\delta,\epsilon}$ are in bijection with

$$\{i\in I_0\backslash L\mid(\delta,\epsilon)\in\omega_{L\cup\{i\}}\}=\{i\in I_0\backslash L\mid(\delta,\epsilon)\in\Omega_{L\cup\{i\}}\}=I_2^{\delta,\epsilon}\backslash(I_2^{\delta,\epsilon}\cap L).$$

Assume that ω_L is not a vertex. The polytope $P^{\delta,\epsilon}$ generates the affine subspace $\{x\in M_{\mathbb{Q}}\mid A_Ix=D_I(\delta,\epsilon)\}$, so that $M'=\ker(A_I)\cap M$. Moreover, the colors of G/H' are indexed by $J_0\backslash(J_0\cap I)$. If for some $i\neq j\in I_0$, $F_i^{\delta,\epsilon}=F_j^{\delta,\epsilon}$ is a facet, then A_i and A_j are colinear modulo the vector space $N'_{\mathbb{Q}}$ generated by the lines of A_I, and then $L=I\cup\{i,j\}$. Hence, by Lemma 25, the G-stable prime divisors of the variety $Z^{\delta,\epsilon}$ are in bijection with

$$\{i\in I_0\backslash I\mid(\delta,\epsilon)\in\omega_{I\cup\{i\}}\}=I_2^{\delta,\epsilon}\backslash\{i\}$$

if $L\backslash I_+$ or $L\backslash I_-$ is $\{i\}$ with $i\in I_0\backslash J_0$, and else with

$$\{i\in I_0\backslash I\mid(\delta,\epsilon)\in\omega_{I\cup\{i\}}\}=I_2^{\delta,\epsilon}.$$

□

Proposition 27. *Let $(\delta_0,\epsilon_0)\in U_2$. Let $\ell\colon\mathbb{Q}\to\mathbb{Q}^2$ be the parametrisation of a rational affine line such that $\ell(0)=(\delta_0,\epsilon_0)$. Let $\bar{t}>0$ be the minimum such that $\ell(\bar{t})\in\Omega_\emptyset\setminus U_2$. Assume that $\ell(\bar{t})=(\delta_2,\epsilon_2)\in U_0\cap\Omega_\emptyset\backslash\omega_\emptyset$.*

Let L be a minimal set such that $\{(\delta_2,\epsilon_2)\}=\omega_L$. The morphism from X^{δ_0,ϵ_0} to Z^{δ_2,ϵ_2} has relative Picard number at most 2.

Proof. Suppose first that ω_L is a vertex.

By Corollary 26, the B-stable prime divisors of Z^{δ_2,ϵ_2} are in bijection with $(I_2^{\delta_2,\epsilon_2}\sqcup J_0)\backslash L$. Let $F_J^{\delta_2,\epsilon_2}$ be a face of P^{δ_2,ϵ_2}, choose J such that $(\delta_2,\epsilon_2)\in\omega_J$. Then $L\subseteq J$ and $\omega_J=\omega_L$, so that the restriction of $A_{J\backslash L}$ to $\ker(A_L)$ is surjective. This implies, by Lemma 8 applied to the matrix of lines of $A_{I_0\backslash L}$ restricted to $\ker(A_L)$, that Z^{δ_2,ϵ_2} is $\mathbb{Q}$-factorial. Then its Picard number is

$$|(I_2^{\delta_2,\epsilon_2} \sqcup J_0)\backslash L| - (n - |L| + 2) = |I_2^{\delta_2,\epsilon_2} \sqcup J_0| - n - 2.$$

Since $I_2^{\delta_2,\epsilon_2} \supseteq I_2^{\delta_0,\epsilon_0}$, the relative Picard number of $X^{\delta_0,\epsilon_0} \longrightarrow Z^{\delta_2,\epsilon_2}$ is at most 2.

Suppose now that ω_L is not a vertex. By Corollary 26, the B-stable prime divisors of Z^{δ_2,ϵ_2} are in bijection with a subset of $(I_2^{\delta_2,\epsilon_2} \sqcup J_0)\backslash I$ of cocardinality at most 1. Let $F_J^{\delta_2,\epsilon_2}$ be a face of P^{δ_2,ϵ_2}, choose J such that $(\delta_2, \epsilon_2) \in \omega_J$. Then $I \subseteq J$. Assume that ω_J is one-dimensional. Then the restriction of $A_{J\backslash I}$ to $\ker(A_L)$ is surjective.

Assume that ω_J has dimension zero. Then $\omega_J = \omega_L$, so that the restriction of $A_{J\backslash I}$ to $\ker(A_I)$ implies the same equation for every such J.

If the cocardinality of a subset of $(I_2^{\delta_2,\epsilon_2} \sqcup J_0)\backslash I$ is one, then this equation does not occur for B-stable divisors and then Z^{δ_2,ϵ_2} is $\mathbb{Q}$-factorial. In any case,the Picard number of Z^{δ_2,ϵ_2} is $|(I_2^{\delta_2,\epsilon_2} \sqcup J_0)\backslash I| - (n - |I| + 1) - 1 = |I_2^{\delta_2,\epsilon_2} \sqcup J_0| - n - 2$. But $I_2^{\delta_2,\epsilon_2}$ is either $I_2^{\delta_0,\epsilon_0}$ or the union of $I_2^{\delta_0,\epsilon_0}$ with i as in Lemma 25, so that the relative Picard number of $X^{\delta_0,\epsilon_0} \longrightarrow Z^{\delta_2,\epsilon_2}$ is 1 or 2.

□

Remark 11. If $\omega_L = \{(\delta_2, \epsilon_2)\}$ is a vertex and the relative Picard number of $X^{\delta_0,\epsilon_0} \longrightarrow Z^{\delta_2,\epsilon_2}$ is not 2, then there exists $i \in I_2^{\delta_2,\epsilon_2}\backslash I_2^{\delta_0,\epsilon_0}$. In particular, ω_L is a vertex of Ω_i that does not contain (δ_0, ϵ_0). Hence, there exists $K \subset I_0$, such that $\omega_K \subset \omega_\emptyset$ is one dimensional with one extremity equals to ω_L.
If ω_L is not a vertex, such ω_K exists by definition.

5.3. **Mori polygonal chain.** We start with the following definition.

Definition 7. The Mori polygonal chain of the family $P^{\delta,\epsilon}$ is

$$MPC := (\Omega_\emptyset\backslash\omega_\emptyset) \cap \{(\delta, \epsilon) \in \mathbb{Q}^2, |\ 0 \le \delta \le 1\}.$$

Throughout this section we suppose that $(0, 0)$ and $(1, 0)$ are in $\omega_\emptyset$, that for every $i \in I_0\backslash J_0$ there exist negative ϵ_0 and ϵ_1 such that $(0, \epsilon_0)$ and $(1, \epsilon_1)$ are in ω_i and that $C \ge 0$. Then $\{(\delta, \epsilon) \in \mathbb{Q}^2 \mid \epsilon \le 0 \text{ and } 0 \le \delta \le 1\}$ is a contained in $\omega_\emptyset$.

Note that, with the above hypothesis, the Mori polygonal chain is contained in the half plane defined by $\epsilon > 0$. It is polygonal because $\Omega_\emptyset\backslash\omega_\emptyset$ is a union of one-dimensional Ω_I.

We say that ω_I is a *segment of the Mori polygonal chain* if $\dim \omega_I = 1$ and the intersection of ω_I with MPC is not empty. If $(\delta, \epsilon) \in MPC$, then either $P^{\delta,\epsilon}$ is not of maximal dimension or there exists i such that $A_i = 0$ and $D_i(\delta, \epsilon) = 0$.

By Lemma 19, if ω_I is a segment of MPC such that $A_{I\backslash\{j\}}$ is surjective for every $j \in I$, then there is an equation $\sum_{i\in I} \lambda_i^I X_i = 0$ for the lines of A_I such that $\lambda_i^I > 0$ for every $i \in I$.

Definition 8. Let $K \subseteq I_0$ be such that $\dim \omega_K = 1$ and the codimension of the image A_K is 1. Let $\sum_{i\in K} \lambda_i^K X_i = 0$ be an equation for the lines of A_K. The slope of ω_K is

$$sl_K = \begin{cases} \frac{\sum_{i\in K} \lambda_i^K (B_i' - B_i)}{\sum_{i\in K} \lambda_i^K C_i} & \text{if } \sum_{i\in K} \lambda_i^K C_i \neq 0, \\ \infty & \text{if } \sum_{i\in K} \lambda_i^K C_i = 0. \end{cases}$$

Remark 12. The segment ω_K is included in the line defined by

$$\epsilon\left(\sum_{i\in K}\lambda_i^K C_i\right)+\delta\left(\sum_{i\in K}\lambda_i^K(B_i'-B_i)\right)+\sum_{i\in K}\lambda_i^K B_i=0$$

and sl_K is the slope of this line.

The next result is the key technical step for the proof of the main theorem. It describes the configuration of sets ω_I around a vertex of the Mori polygonal chain. We show in the main theorem that these configurations correspond to Sarkisov link.

Proposition 28. *Let $(\delta_2,\epsilon_2)\in U_0\cap MPC$ and let L be minimal such that $\omega_L=\{(\delta_2,\epsilon_2)\}$. We refer to the notation of Construction 3. Then there is a partition $L=K^0\sqcup\ldots\sqcup K^{r+1}$, with $r\geq 0$, such that*

(1) *for every $s\in\{0,\ldots,r+1\}$, for every $h,k\in K^s$ we have $-\dfrac{\lambda_h^I}{\lambda_h^J}=-\dfrac{\lambda_k^I}{\lambda_k^J}=:\nu_s\in[-\infty,0]$; for every $s<s'\in\{0,\ldots,r+1\}$, we have $\nu_s>\nu_{s'}$.*

(2) *The set $K_s=L\setminus K^s$ is such that the codimension of the image of A_{K_s} is 1 and for every $k\in K_s$ the map $A_{K_s\setminus\{k\}}$ is surjective. If $K\subseteq L$ is such that the codimension of the image of A_K is 1 and for every $k\in K$ the map $A_{K\setminus\{k\}}$ is surjective, then $K=K_s$ for some $s\in\{0,\ldots,r+1\}$. Moreover, if (δ_2,ϵ_2) is a vertex then $K_0=I$ and $K_{r+1}=J$ and if it is not then $K_0=I$ and $K^0=L\backslash I$.*

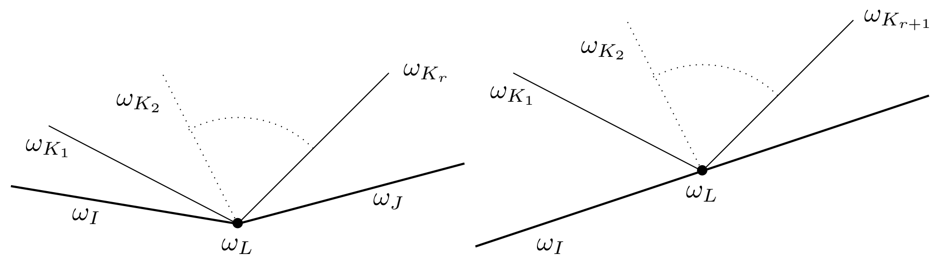
Note that, by Lemma 16, ω_{K_s} is not empty and with an extremity equals to ω_L.

(3) *The slope of ω_{K_s} is*

$$sl_{K_s}=\frac{\sum_{i\in I}\lambda_i^I(B_i'-B_i)+\nu_s\sum_{j\in J}\lambda_j^J(B_j'-B_j)}{1+d\nu_s}$$

with $d=1$ if (δ_2,ϵ_2) is a vertex and $d=\sum_{j\in L}\lambda_j^L C_j$ otherwise.

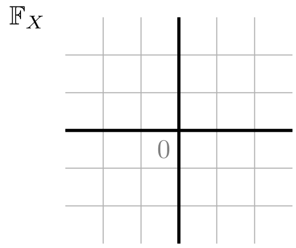
(4) *Up to a rotation, the slopes decrease when s increases (see picture below).*

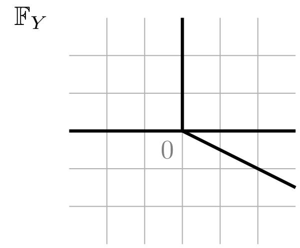


6. Two examples

In this section we present two examples illustrating the horosperical Sarkisov program.

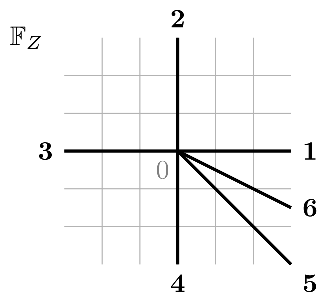
6.1. **A toric example.** Set $X:=\mathbb{P}^1\times\mathbb{P}^1$, $S=\mathbb{P}^1$, Y the projective bundle $\mathbb{P}(\mathcal{O}\oplus\mathcal{O}(2))$ and $T=\mathbb{P}^1$. Fans of X and Y are the following:





Here $G = (\mathbb{C}^*)^2$ and coincides with the Borel subgroup.

A resolution of X and Y is the toric variety Z given by the following fan, where we index the edges as in the picture:



We let $p\colon Z \to X$ and $q\colon Z \to Y$ be the two morphisms resolving the indeterminacy of $X \dashrightarrow Y$.

We will consider the following matrix $A := \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -1 \\ 2 & -1 \end{pmatrix}$.

A G-stable divisor of Z is of the form $D = \sum_{i=1}^{6} d_i D_i$. Note that D_1, D_2, D_3 and D_4 are also prime G-stable divisor of X and D_1, D_2, D_3 and D_6 are also prime G-stable divisor of Y.

Let $D = \sum_{i=1}^{6} d_i D_i$ be a divisor on Z and $B = (-d_1, \ldots, -d_6)$. The divisor D is ample if and only if for any $I \subseteq \{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{5,6\},\{1,6\}\}$ we have $A_{I^c}(A_I^{-1}B_I) > B_I$.

This inequality system reduces to the following system, thus defining the ample cone of Z:

$$d_1 + d_5 > d_6,\ d_2 + d_6 > 2d_1,\ d_3 + d_5 > d_4,\ d_4 + d_6 > 2d_5. \tag{2}$$

The polytope Q_D is the pseudo-moment polytope of (X, p_*D) if and only if for every $I \subseteq \{\{1,2\},\{2,3\},\{3,4\},\{1,4\}\}$ we have $A_{I^c}(A_I^{-1}B_I) > B_I$. Equivalently, if and only if the following inequalities are satisfied

$$d_1 + d_3 > 0,\ d_2 + d_4 > 0,\ d_5 > d_1 + d_4,\ d_6 > 2d_1 + d_4. \tag{3}$$

Note that the first two inequalities correspond to the condition for p_*D to be ample and the last two other correspond to the fact that the lines 5 and 6 are not necessary to define Q_D.

Similarly, Q_D is the pseudo-moment polytope of (Y, q_*D) if and only if for every $I \subseteq \{\{1,2\},\{2,3\},\{3,6\},\{1,6\}\}$ we have $A_{I^c}(A_I^{-1}B_I) > B_I$. Equivalently, if and only if the following inequalities are satisfied

$$d_1 + d_3 > 0,\ d_2 + d_6 > 2d_1,\ d_4 > 2d_3 + d_6,\ d_5 > d_3 + d_6. \tag{4}$$

Note that the first two inequalities correspond to the condition for q_*D to be ample and the last two other correspond to the fact that the lines 4 and 5 are not necessary to define Q_D.
To run the horospherical Sarkisov program, we have to choose D, D' such that

(1) Q_D and $Q_{D'}$ are pseudo-moment polytopes of (X, p_*D) and (Y, q_*D') respectively;
(2) there exist $\epsilon < 0$ and $\epsilon' < 0$ such that $D + \epsilon K_Z$ and $D' + \epsilon' K_Z$ are ample over Z.
(3) the HMMP with scaling of D (resp. D') ends with X/S (resp. Y/T).

Condition (1) is given by (3) and (4).

For condition (2), note that $-K_Z = \sum_{i=1}^6 D_i$ and thus $C = (1, \dots, 1)$. Then $D + \epsilon K_Z$ is ample over Z if and only if

$$d_1 + d_5 > d_6 + \epsilon, \quad d_2 + d_6 > 2d_1, \quad d_3 + d_5 > d_4 + \epsilon \text{ and } d_4 + d_6 > 2d_5.$$

Hence, there exists $\epsilon < 0$ such that $D + \epsilon K_Z$ is ample over Z if and only if $d_2 + d_6 > 2d_1$ and $d_4 + d_6 > 2d_5$. Similarly, there exists $\epsilon' < 0$ such that $D' + \epsilon' K_Z$ is ample over Z if and only if $d_2' + d_6' > 2d_1'$ and $d_4' + d_6' > 2d_5'$.

As for condition (3), note first that the HMMP from any ample divisor of Y ends with $Y \to S$, because only one extremal ray of $NE(Y)$ is K-negative. If $d_1 + d_3 < d_2 + d_4$ (resp. $d_1 + d_3 > d_2 + d_4$) the HMMP from D gives the first (resp. the second) projection $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$

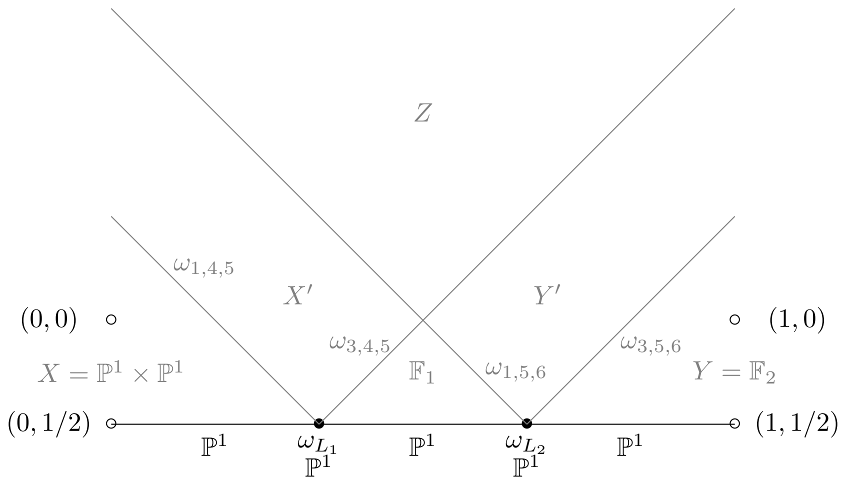
Since D and D' are given up to linearly equivalence, we can choose them such that $d_1 = d_2 = d_1' = d_2' = 0$. In particular, Q_D and $Q_{D'}$ have a "south-west" vertex at 0 (but the same is not true for $Q^{\epsilon,\lambda}$ for $\epsilon \neq 0$). The conditions on D and D' are then

$$d_3 > 0, \quad d_6 > d_4 > 0 \text{ and } d_4 < d_5 < \frac{1}{2}(d_4 + d_6), \text{ with either } d_3 < d_4 \text{ or } d_3 > d_4;$$

$$d_3' > 0, \quad d_6' > 0, \quad d_4' > 2d_3' + d_6' \text{ and } d_3' + d_6' < d_5' < \frac{1}{2}(d_4' + d_6').$$

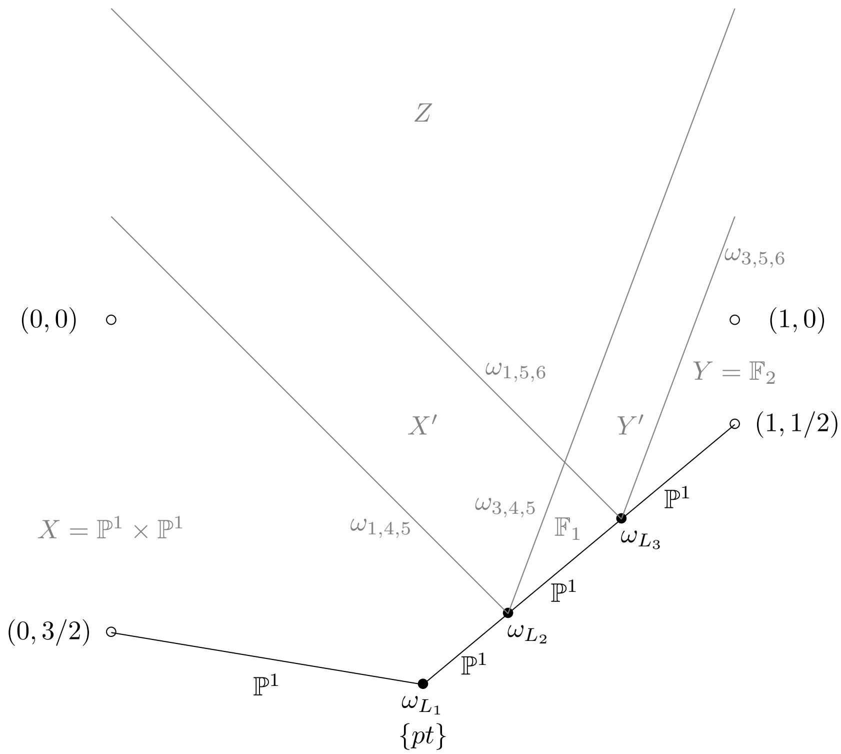
For example we can have $B := \begin{pmatrix} 0 \\ 0 \\ -1 \\ -2 \\ -5/2 \\ -4 \end{pmatrix}$ and $B' := \begin{pmatrix} 0 \\ 0 \\ -1 \\ -6 \\ -7/2 \\ -2 \end{pmatrix}$.

Here is a scheme of $\Omega_\emptyset$ for this choice of B, B'.



Here X' is the blow-up of X at a point and Y' is the blow-up of $\mathbb{F}_1$ at a point in the (-1)-section. Note that $L_1 = \{1, 3, 4, 5\}$ and $L_2 = \{1, 3, 5, 6\}$. Also note that $(1/2, 0)$ is in U'_0.

In the above example, we have $d_3 < d_4$. If we choose $B := \begin{pmatrix} 0 \\ 0 \\ -6 \\ -1 \\ -3/2 \\ -3 \end{pmatrix}$, then $d_3 > d_4$. Thus we obtain the following scheme. Note that the first Sarkisov link is the type IV link from the first to the second projection $\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$.



The fifth line of A does not correspond to a divisor of X or Y, but is added in order to get Z to be smooth. The strategy given in the paper also works by chosing a (not necessarily smooth) resolution of indeterminacies Z', but non-terminal varieties can appear. In this example, if we forget the fifth line of A, we can obtain a unique Sarkisov link of type II with $X' = Y' = Z'$.

6.2. **A rank one horospherical example.** Choose a connected algebraic group G with simple roots. Set

$$S \backslash R = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$$

with the notation of section 2. Let $\chi = \varpi_1 + \varpi_2 - \varpi_3 - \varpi_4$ and $H \subset P$ be the kernel in P of the character χ. In particular, $M = \mathbb{Z}\chi$. We identify χ with 1 in $M \simeq \mathbb{Z}$. Then $N \simeq \mathbb{Z}$, $\alpha_{1M}^\vee = \alpha_{2M}^\vee = 1$, $\alpha_{3M}^\vee = \alpha_{4M}^\vee = -1$ and $\alpha_{5M}^\vee = 0$.

Since G/H is of rank one, projective G/H-embedding are uniquely determined by their colors, that is, by subsets of $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. Denote by X_K the projective G/H-embedding such that $\mathcal{F}_{X_K} = \{\alpha_k \mid k \in K\}$. In particular, $Z := X_\emptyset$ is a common resolution of all X_K.

Denote by P^K the parabolic subgroup of G containing B whose the set of simple roots is $\{\alpha_k \mid k \in K\} \cup R$.

Let $H' \subset P_5$ be the kernel in P_5 of the character χ. It is a horospherical subgroup associated to the same lattice M. Denote by Y_K the projective G/H'-embedding Y such that $\mathcal{F}_{Y_K} = \{\alpha_k \mid k \in K\}$.

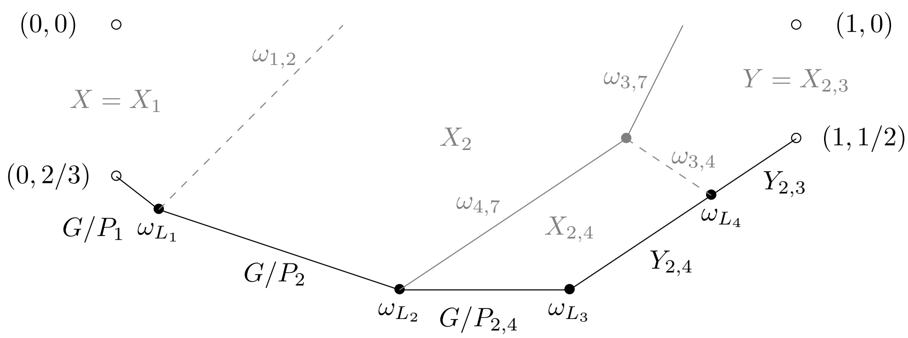
Suppose that the coefficients a_{α_i}'s are respectively 2, 3, 2, 3 and 2 for any $i = 1, \ldots, 5$. We consider A to be the column matrix associated to

$$(\alpha_{1M}^\vee, \alpha_{2M}^\vee, \alpha_{3M}^\vee, \alpha_{4M}^\vee, \alpha_{5M}^\vee, 1, -1) = (1, 1, -1, -1, 0, 1, -1).$$

Then C is the column matrix associated to $(2,3,2,3,2,1,1)$.

Let B and B' be the column matrices associated to $-(0,1,7,6,5,2,2)$ and $-(2,0,6,7,1,3,7)$ respectively. Then the associated horospherical Sarkisov program gives a link of type $\mathrm{IV_m}$, followed by a link of type III, a link of type $\mathrm{IV_m}$ and a last one of type $\mathrm{IV_s}$.

We then obtain the following scheme (only with $\epsilon \geq 0$ and $0 \leq \delta \leq 1$). In black we draw the one dimensional ω_I's giving fibrations and in grey the ones giving birational contractions. We dash the ones corresponding to flips.



7. Proof of the main theorem

Let G be a reductive group and $H \subseteq G$ be a horospherical subgroup. We choose a basis of $M_{\mathbb{Q}}$ so that $M_{\mathbb{Q}} \cong \mathbb{Q}^n$. We fix a horospherical embedding Z of G/H and set $p = r + |S\backslash R|$. Let A be the matrix associated to the linear map $\varphi(m) = (\langle m, x_i\rangle_{i=1\dots r},\ \langle m, \alpha_M^\vee\rangle_{\alpha\in S\backslash R})$. Denote by $J_0 \subseteq I_0$ the set of indices $S\backslash R$.

Let $B = (-d_1,\dots,-d_r,-d_\alpha)$ and $B' = (-d_1',\dots,-d_r',-d_\alpha')$ be such that $\sum d_iZ_i + \sum d_\alpha D_\alpha$ and $\sum d_i'Z_i + \sum d_\alpha' D_\alpha$ are ample divisors. Let $-K_Z = \sum c_iZ_i + \sum c_\alpha D_\alpha$. Let $C = (c_1,\dots,c_r,c_\alpha)$.

Suppose now that for any $i \in I_0\backslash J_0$ there exist negative ϵ_0 and ϵ_1 such that $(0,\epsilon_0)$ and $(1,\epsilon_1)$ are in ω_i. Then for any δ, the intersection $\omega_i \cap \{(\delta,\epsilon) \mid \epsilon \geq 0\}$ is an open segment (possibly empty) with one extremity at $(\delta, 0)$. Then the family $(P^{\delta,\epsilon})_{\epsilon\in\mathbb{Q}_{\geq 0}}$ describes a HMMP.

Proposition 29. *For any $\delta \in [0,1]$ in the complement of a finite set, the HMMP described by the family $(P^{\delta,\epsilon})_{\epsilon\in\mathbb{Q}_{\geq 0}}$ is an MMP.*

Proof. By Theorem 10, the family $(P^{\delta,\epsilon})_{\epsilon\in\mathbb{Q}_{\geq 0}}$ describes an HMMP. Let U'' be the set of δ such that there is a 1-dimensional set ω_I included in $\{\delta\}\times\mathbb{Q}$. Let $p_1 : \mathbb{Q}^2 \to \mathbb{Q}$ be the projection onto the first factor and let $\delta \in [0,1]\backslash p_1(U_0 \cup U_0') \cup U''$. By Proposition 23, the HMMP described by the family $(Q^{\delta,\epsilon})_{\epsilon\in\mathbb{Q}_{\geq 0}}$ consists of extremal contractions.

□

Definition 9. Let G be a connected reductive algebraic group. Let X and Y be horospherical G-varieties which are G-equivariantly birational. Assume moreover that there are Mori fibre space structures X/S and Y/T.

Let Z be a horospherical resolution of the indeterminacy of $X \dashrightarrow Y$, and let A_X and A_Y be ample divisors of Z.

We set

$$P^{\delta,\epsilon} := \{x \in \mathbb{Q}^n \mid Ax \geq (1-\delta)B + \delta B' + \epsilon C\}$$

and $\Omega = \{(\delta,\epsilon) \in \mathbb{Q}^2 \mid P^{\delta,\epsilon} \neq \emptyset\}$. We say that the two-parameter family of polytopes $\{P^{\delta,\epsilon}\}_{\delta,\epsilon}$ is associated to (Z, A_X, A_Y) and describes a horospherical Sarkisov program from X/S to Y/T, if there exist ϵ_0, ϵ_0' in $\mathbb{Q}$, and singletons $\omega_{L_1}, \dots, \omega_{L_\ell} \subseteq \partial\Omega$ such that

(1) every point $(\delta,\epsilon) \in \Omega$ defines the horospherical variety of pseudo-moment polytope $P^{\delta,\epsilon}$;
(2) X/S and Y/T are the outcomes of the horospherical K_Z-MMP with scaling of A_X and A_Y respectively described by the one-parameter families $(P^{0,\epsilon})_{\epsilon\geq\epsilon_0}$ and $(P^{1,\epsilon})_{\epsilon\geq\epsilon_0'}$ respectively;
(3) every ω_{L_i} defines a Sarkisov link involving horospherical varieties of pseudo-moment polytope $P^{\delta,\epsilon}$ with (δ,ϵ) in a neighborhood of ω_{L_i} in Ω, including varieties in the sequences of flips.
(4) the Sarkisov links defined by the ω_{L_i}'s give a Sarkisov program from X/S to Y/T.

Theorem 30. *Let G be a connected reductive algebraic group. Let X and Y be horospherical G-varieties which are G-equivariantly birational. Assume moreover that there are Mori fibre space structures X/S and Y/T.*

For any horospherical resolution (smooth or with terminal singularities) of the indeterminacy Z of $X \dashrightarrow Y$, there exist two euclidean open sets $\mathcal{U}_X$ and $\mathcal{U}_X$ of $WDiv(Z)_\mathbb{Q}$, such that for any $A_X \in \mathcal{U}_X$ and $A_Y \in \mathcal{U}_X$, there is a two-parameter family of polytopes associated to (Z, A_X, A_Y) that describes a horospherical Sarkisov program from X/S to Y/T.

Recall that, since the birational map $X \dashrightarrow Y$ is G-equivariant, X and Y are both G/H-embedding with the same horospherical homogeneous space G/H; and by Lemma 11, there exists horospherical resolutions of the indeterminacy of $X \dashrightarrow Y$. Then the second part of Theorem 30 implies Thoerem 1.

Proof of Theorem 30. By Proposition 14 there are euclidean open sets U_X and U_Y of $WDiv(Z)_\mathbb{Q}$ such that every divisor in U_X (resp. U_Y) is ample and for every $A \in U_X$ (resp. in U_Y) the Mori fibre space X/T (resp. Y/S) is the outcome of the HMMP from Z with scaling of A.

Since the open set determined in Proposition 17 is Zariski open, we can now find open subsets $\mathcal{U}_X \subseteq U_X$ and $\mathcal{U}_X \subseteq U_Y$, such that any $(A_X, A_Y) \in \mathcal{U}_X \times \mathcal{U}_Y$ satisfies the generality conditions of Proposition 17.

Let fix such a $(A_X, A_Y) \in \mathcal{U}_X \times \mathcal{U}_Y$.

Let $B = (-d_1, \ldots, -d_r, (-d_\alpha)_{\alpha \in S\setminus R})$ and $B' = (-d'_1, \ldots, -d'_r, (-d'_\alpha)_{\alpha \in S\setminus R})$ and $C = (c_1, \ldots, c_r, (c_\alpha)_{\alpha \in S\setminus R})$ be such that

$$A_X = \sum_{i=1}^{d} d_i Z_i + \sum_{\alpha \in S\setminus R} d_\alpha D_\alpha$$

$$A_Y = \sum_{i=1}^{d} d'_i Z_i + \sum_{\alpha \in S\setminus R} d'_\alpha D_\alpha$$

$$-K_Z = \sum_{i=1}^{d} c_i Z_i + \sum_{\alpha \in S\setminus R} c_\alpha D_\alpha.$$

Let Ω_I and ω_I be the polytopes of Definition 5.2.

Thus there are two segments ω_I and ω_J of the Mori poligonal chain such that every (δ, ϵ) in a euclidean neighborhood of ω_I satisfies $X^{\delta,\epsilon} \cong X$ and if $(\delta_1, \epsilon_1) \in \omega_I$ then $Y^{\delta_1,\epsilon_1} \cong T$. The same holds for J.

The segments ω_I and ω_J disconnect the chain. Let $\mathcal{C} \subseteq MPC$ be such that $\omega_I \cup \mathcal{C} \cup \omega_J$ is connected. Let $\omega_{L_1}, \ldots, \omega_{L_h}$ be the points in $U_0 \cap \mathcal{C}$. We notice that $U'_0 \cap \partial\Omega_\emptyset = \emptyset$. Indeed, if $(\delta, \epsilon) \in \omega_I \cap \omega_J$ and ω_I, ω_J are not aligned, then the convex hull of ω_I and ω_J is non-empty and open (and thus of dimension 2), proving that $(\delta, \epsilon) \in \omega_\emptyset$.

We prove that every ω_{L_i} describes a link. We write ω_L for simplicity.

Let $L = K^0 \sqcup \cdots \sqcup K^{r+1}$ be the partition existing by Proposition 28. We fix a euclidean neighborhood Δ of ω_L and for any $i \in \{0, \ldots, r\}$, let X_i be the horospherical variety corresponding to a point in the open set delimited by ω_{K_i} and $\omega_{K_{i+1}}$. If ω_L is not a vertex, let X_{r+1} be the horospherical variety corresponding to a point in the open set delimited by $\omega_{K_{r+1}}$ and ω_I. Let t be r if ω_L is a vertex and $r+1$ if not. We denote by T_0, respectively T_{t+1}, the horospherical variety corresponding to a point in MPC on the left, respectively on the right, of ω_L.

We prove first that

Claim 31. *the only possible divisorial contractions between two varieties X_i and $X_{i\pm1}$ are $X_1 \to X_0$ and $X_{t-1} \to X_t$.*

Recall that we have Mori fibrations $X_0 \to T_0$ and $X_t \to T_{t+1}$. Let $s \in \{1, \ldots, r\}$. First assume that ω_L is a vertex. Then K_s is such that $K_s^- = K^0 \sqcup \cdots \sqcup K^{s-1}$ and $K_s^+ = K^{s+1} \sqcup \cdots \sqcup K^{r+1}$. By Proposition 23, around ω_{K_s} we have flips except if $s = 1$ and $K^0 = \{i\}$ with $i \in I_0\setminus J_0$ or $s = r$ and $K^{r+1} = \{i\}$ with $i \in I_0\setminus J_0$.

Assume now that ω_L is not a vertex. Let $s \in \{1, \ldots, t\}$, then K_s is such that $K_s^- = K_-^0 \sqcup K^1 \sqcup \cdots \sqcup K^{s-1}$ and $K_s^+ = K_+^0 \sqcup K^{s+1} \sqcup \cdots \sqcup K^{r+1}$. Since K_+^0 and K_-^0 are non-empty (by Notation 3), by Proposition 23 around ω_{K_s} we have flips except if $s = 1$ and $K_-^0 = \{i\}$ with $i \in I_0\setminus J_0$ or $s = r+1$ and $K_+^0 = \{i\}$ with $i \in I_0\setminus J_0$. This finishes the proof of the claim.

Let R be the variety corresponding to ω_L. Notice that we have fibrations from $T_0 \to R$ and $T_{t+1} \to R$. There are three cases:

(1) $R \cong T_0 \cong T_{t+1}$;
(2) $R \cong T_0$ or $R \cong T_{t+1}$, and we are not in case 1;
(3) $R \ncong T_0$ and $R \ncong T_{t+1}$.

In case 1, by Remark 11, we have $t \geq 1$. In particular, $sl_{K_1} < sl_{K_0}$, or $sl_{K_t} > sl_{K_{t+1}}$ (with the convention that $K_{r+2} = I$ if ω_L is not a vertex). A priori, sl_{K_1} and sl_{K_t} could be ∞, but the next paragraph proves that it cannot happen.
If $sl_{K_1} < sl_{K_0}$, then the Mori fibration $X_0 \dashrightarrow R$ factors through $X_0 \dashrightarrow Y \to R$ where Y is the horospherical variety corresponding to a point of ω_{K_1}. Then the map $X_0 \dashrightarrow Y$ is either an isomorphism or a K-negative contraction. It cannot be a K-negative contraction because $sl_{K_1} < sl_{K_0}$. Then $X_0 = Y$, which implies that $X_1 \to X_0$ is a divisorial contraction. Similarly, if $sl_{K_t} > sl_{K_{t+1}}$, $X_{t-1} \to X_t$ has to be a divisorial contraction.
Thus there is $i = 1$ or $i = t-1$ such that $\rho(X_i) = \rho(R)+2$. Assume it is $i = 1$. By Proposition 27 and Claim 31, for every $i \in \{1, \dots, t-1\}$ we have $\rho(X_i) = \rho(X_1)$ so that $\rho(X_{t-1}) = \rho(X_t)+1$ and thus $X_{t-1} \to X_t$ is also a divisorial contraction. We have a type II link.

In case 2, assume that $R \cong T_0$. Again by Proposition 23 the variety R is $\mathbb{Q}$-factorial. Thus $R \ncong T_{t+1}$ implies $\rho(T_{t+1}) = \rho(R)+1$ and $\rho(X_t) = \rho(R)+2$. By Proposition 27 and Claim 31, for every $i \in \{1, \dots, t\}$ we have $\rho(X_i) = \rho(X_t) = \rho(R)+2 = \rho(T_0)+2$. Since $\rho(X_0) = \rho(T_0)+1$, the map $X_1 \to X_0$ is a divisorial contraction and we have a type I link. Similarly if $R \cong T_{r+1}$ we get a type III link.

In case 3 we have $\rho(T_0) = \rho(R)+1$ and $\rho(T_{t+1}) = \rho(R)+1$. Moreover $\rho(X_0) = \rho(T_0)+1$ and $\rho(X_t) = \rho(T_{t+1})+1$. By Proposition 27 and Claim 31, for every $i \in \{0, \dots, t-1\}$ the map $X_i \dashrightarrow X_{i+1}$ is an isomorphism in codimension 1 and we get a type IV link. $\square$

Remark 13. If ω_L is not a vertex then we cannot have a link of type IV with fibrations $T_0 \longrightarrow R$ and $T_{t+1} \longrightarrow R$. Indeed from ω_I (both side) to ω_L we get two birational maps, one of the two can be divisorial (if we are in the hypotheses of Lemma 25 occurs), but not in case 3.

Appendix A.

A.1. **A polyhedral partition of $\mathbb{Q}^2$.**

In this section we collect the proofs of the facts quoted in Section 5.
The following lemma describes the first properties of the sets Ω_I and ω_I defined in Definition 5.2.

Lemma 32. *Let $I \subseteq I_0$.*

(1) *The sets Ω_I and ω_I are convex subsets of $\mathbb{Q}^2$.*

(2) *The set ω_I is open, for the euclidean topology, inside $\{(\delta,\epsilon) |\ D_I(\delta,\epsilon) \in \operatorname{Im} A_I\} = D_I^{-1} \operatorname{Im} A_I$.*

(3) *There are four cases: either ω_I is empty, or it is a point, or it is a convex part of an affine line (a segment, a half-line or a line), or it is a non-empty open set in $\mathbb{Q}^2$.*

(4) *If ω_I is not empty, we have $\omega_I \subset \Omega_I \subset \overline{\omega_I}$.*

Proof. (1) Assume that Ω_I has at least two points, (δ_1, ϵ_1) and (δ_2, ϵ_2). Then there exist x_1 and x_2 in $\mathbb{Q}^n$ such that $A_I x_i = D_I(\delta_i, \epsilon_i)$ and $A_{I^c} x_i \geq D_{I^c}(\delta_i, \epsilon_i)$ for $i = 1, 2$. For any rational number $t \in [0,1]$, we get $A_I(tx_1 + (1-t)x_2) = D_I(t\epsilon_1 + (1-t)\epsilon_2, t\delta_1 + (1-t)\delta_2)$ and $A_{I^c}(tx_1 + (1-t)x_2) \geq D_{I^c}(t\epsilon_1 + (1-t)\epsilon_2, t\delta_1 + (1-t)\delta_2)$, so that $t(\delta_1, \epsilon_1) + (1-t)(\delta_2, \epsilon_2)$ is in Ω_I.

Replacing $\geq$ by $>$, we prove the convexity of ω_I.

(2) The inclusion $\omega_I \subseteq D_I^{-1} \operatorname{Im} A_I$ follows from the definition of ω_I.
If $D_I^{-1} \operatorname{Im} A_I$ is a point or if ω_I is empty, there is nothing to prove.
Let $(\delta_0, \epsilon_0) \in \omega_I$. Then there is x_0 such that $A_I x_0 = D_I(\delta_0, \epsilon_0)$ and $A_{I^c} x_0 > D_{I^c}(\delta_0, \epsilon_0)$.

Assume that $D_I^{-1} \operatorname{Im} A_I$ is two-dimensional. Let (δ_1, ϵ_1) and (δ_2, ϵ_2) in $D_I^{-1} \operatorname{Im} A_I$ such that the three (δ_i, ϵ_i)'s are not in the same line. Let x_1 and x_2 be such that $A_I x_1 = D_I(\delta_1, \epsilon_1)$ and $A_I x_2 = D_I(\delta_2, \epsilon_2)$. Then

$$A_I((1-t_1-t_2)x_0+t_1x_1+t_2x_2) = D_I((1-t_1-t_2)\epsilon_0+t_1\epsilon_1+t_2\epsilon_2, (1-t_1-t_2)\delta_0+t_1\delta_1+t_2\delta_2)$$

and for (t_1, t_2) in a neighborhood of 0 in $\mathbb{Q}^2$, we have

$$A_{I^c}((1-t_1-t_2)x_0+t_1x_1+t_2x_2) > D_{I^c}((1-t_1-t_2)\epsilon_0+t_1\epsilon_1+t_2\epsilon_2, (1-t_1-t_2)\delta_0+t_1\delta_1+t_2\delta_2).$$

The proof for the case $\dim D_I^{-1} \operatorname{Im} A_I = 1$ is analogous. This concludes the proof of the statement.

(3) It follows from the two previous statements.

(4) The first inclusion is obvious.

To prove the second one, remark that for all $(\delta_1, \epsilon_1) \in \omega_I$ and $(\delta_2, \epsilon_2) \in \Omega_I$, the segment $\{(1-t)(\delta_1, \epsilon_1) + t(\delta_2, \epsilon_2) | \, t \in (0, 1]\}$ is contained in ω_I. Indeed, let x_1 and x_2 be such that $A_I x_i = D_I(\delta_i, \epsilon_i)$, $A_{I^c} x_1 > D_{I^c}(\delta_1, \epsilon_1)$ and $A_{I^c} x_2 \geq D_{I^c}(\delta_2, \epsilon_2)$. Then, for any $t \in (0, 1]$, we set $x_t := t x_1 + (1-t) x_2$ and $(\delta_t, \epsilon_t) := (t\epsilon_1 + (1-t)\epsilon_2, t\delta_1 + (1-t)\delta_2)$. Thus $A_I x_t = D_I(\delta_t, \epsilon_t)$ and $A_{I^c} x_t > D_{I^c}(\delta_t, \epsilon_t)$.

This remark implies directly that if $(\delta_2, \epsilon_2) \in \Omega_I$ then $(\delta_2, \epsilon_2) \in \overline{\omega_I}$, as soon as ω_I is not empty.

□

Remark 14. Lemma 32 is still true if we consider polytopes in an $\mathbb{R}$-vector space. The proof is the same after replacing $\mathbb{Q}$ by $\mathbb{R}$ everywhere.

Lemma 33. *Let $I \subseteq I_0$ be such that ω_I is not empty. Then $\Omega_I = \overline{\omega_I}$ and Ω_I is polyhedral in $\mathbb{Q}^2$.*

Proof. If ω_I is reduced to a point, by Lemma 32 we have nothing to prove, so we suppose that ω_I contains at least two points.

For $(\delta, \epsilon) \in \mathbb{R}^2$ we also set

$$P^{\delta,\epsilon} := \{x \in \mathbb{R}^n \mid Ax \geq (1-\delta)B + \delta B' + \epsilon C\}.$$

and we define as before $F_I^{\delta,\epsilon}$.

For $I \subseteq I_0$ we set

$$\Omega_I(\mathbb{R}) = \{(\delta, \epsilon) \in \mathbb{R}^2 \mid F_I^{\delta,\epsilon} \neq \emptyset\}$$

and

$$\omega_I(\mathbb{R}) = \{(\delta, \epsilon) \in \mathbb{R}^2 \mid \text{if } I' \subseteq I_0 \text{ satisfies } F_I^{\delta,\epsilon} = F_{I'}^{\delta,\epsilon}, \text{ then } I' \subseteq I\}.$$

Since the matrices A, B, B' and C have rational coefficients, if $(\delta, \epsilon) \in \mathbb{Q}^2$ and if there exists $x \in \mathbb{R}^n$, such that $A_I x = D_I(\delta, \epsilon)$, then there is $x' \in \mathbb{Q}^n$, such that $A_I x' = D_I(\delta, \epsilon)$; and x' can be chosen arbitrarily close to x. In particular, if $A_{I^c} x > D_{I^c}(\delta, \epsilon)$, then x' can be chosen such that $A_{I^c} x' > D_{I^c}(\delta, \epsilon)$. Hence, $\omega_I = \omega_I(\mathbb{R}) \cap \mathbb{Q}^2$.

Moreover, $\Omega_I = \bigcup_{I \subseteq I' \subseteq I_0} \omega_{I'}$ (over $\mathbb{R}$ and $\mathbb{Q}$), then we also have $\Omega_I(\mathbb{Q}) = \Omega_I(\mathbb{R}) \cap \mathbb{Q}^2$.

We first prove that $\Omega_I(\mathbb{R})$ is closed. This, together with Lemma 32 will imply that $\Omega_I(\mathbb{R}) = \overline{\omega_I(\mathbb{R})}$.

Let $(\delta_k, \epsilon_k)_{k\in\mathbb{N}}$ be a sequence of elements in $\Omega_I(\mathbb{R})$ converging to $(\bar\delta, \bar\epsilon)$ in $\mathbb{R}^2$. The elements (δ_k, ϵ_k) are contained in a compact set K of $\mathbb{R}^2$. Then, for every $k \in \mathbb{N}$, the polytope P^{δ_k,ϵ_k} is contained in the polytope

$$P^K := \{x \in \mathbb{R}^n \mid Ax \geq \mathrm{Min}_{(\delta,\epsilon)\in K} D(\delta, \epsilon)\}$$

where $\mathrm{Min}_{(\delta,\epsilon)\in K} D(\delta, \epsilon)$ is the vector whose i-th coordinate is $\mathrm{Min}_{(\delta,\epsilon)\in K} D(\delta, \epsilon)_i$. The set P^K is compact by Remark 5.

By definition, for any $k \in \mathbb{N}$, there exists $x_k \in \mathbb{R}^n$ such that $A_I x_k = D_I(\delta_k, \epsilon_k)$ and $A_{I^c} x_k \geq D_{I^c}(\delta_k, \epsilon_k)$. Since the $x_k \in P^K$ that is compact, there is $\{k_m\}_{m\in\mathbb{N}}$ such that $k_m \to \infty$ as m tends to infinity, such that $(x_{k_m})_{m\in\mathbb{N}}$ converges to $\bar{x} \in \mathbb{R}$. Then $\bar{x}$ satisfies $A_I\bar{x} = D_I(\bar\delta, \bar\epsilon)$ and $A_{I^c}\bar{x} \geq D_{I^c}(\bar\delta, \bar\epsilon)$. This implies that $(\bar\delta, \bar\epsilon)$ is also in Ω_I.

We now prove that $\Omega_I(\mathbb{R})$ is polyhedral. If $\omega_I(\mathbb{R})$ is either empty, or a point, or a convex part of an affine line, then there is nothing to prove. Suppose that $\omega_I(\mathbb{R})$ is open in $\mathbb{R}^2$. Then the boundary of $\Omega_I(\mathbb{R})$ is

$$\Omega_I(\mathbb{R})\backslash\omega_I(\mathbb{R}) = \bigcup_{I\subsetneq I'\subseteq I_0} \omega_{I'}(\mathbb{R}) = \bigcup_{I\subsetneq I'\subseteq I_0} \Omega_{I'}(\mathbb{R}).$$

But for every I' such that $I \subsetneq I' \subseteq I_0$, the set $\Omega_{I'}(\mathbb{R})$ is either empty, or a point, or a closed convex part of an affine line. We proved that the boundary of $\Omega_I(\mathbb{R})$ is a finite union of closed convex parts of affine lines. Hence, $\Omega_I(\mathbb{R})$ is polyhedral in $\mathbb{R}^2$.

To conclude, we have to prove that the vertices of $\Omega_I(\mathbb{R})$ are rational and that the maximal half lines in the boundary of $\Omega_I(\mathbb{R})$ are rational half-lines, that is, they have a rational extremity and rational direction.

A vertex of $\Omega_I(\mathbb{R})$ is (δ, ϵ) such that there is $I' \neq I$ with $\{(\delta, \epsilon)\} = \omega_{I'}(\mathbb{R})$. By Lemma 32 we have $\{(\delta, \epsilon)\} = D_{I'}^{-1} \operatorname{Im} A_{I'} \otimes \mathbb{R}$. Since $D_{I'}$ and $A_{I'}$ are defined over $\mathbb{Q}$, we have $\{(\delta, \epsilon)\} \in \mathbb{Q}^2$.

In the same way, the boundary is a union of finitely many $\omega_{I'_j}$ for $j = 1 \ldots r$ with $I'_j \neq I$. Let $I' \in \{I'_1, \ldots, I'_r\}$ so that $\omega_{I'}(\mathbb{R})$ has dimension 1. By Lemma 32, we have $\omega_{I'}(\mathbb{R}) \subseteq D_{I'}^{-1} \operatorname{Im} A_{I'} \otimes \mathbb{R}$ open. The latter is an affine subspace defined over $\mathbb{Q}$, thus its direction is rational. The extremities are vertices of $\Omega_I(\mathbb{R})$ and are then rational. □

We now study inclusions between the ω's, generalizing the easy following fact: for any $J \subseteq I \subseteq I_0$, we have $\Omega_I \subseteq \Omega_J$.

Lemma 34. *Let $J \subseteq I \subseteq I_0$, such that $\omega_I \neq \emptyset$.*

(1) *If ω_J is not empty and has the same dimension of ω_I, then $\omega_I \subseteq \omega_J$.*

(2) *If the images of A_I and A_J have the same codimension k, then ω_J is not empty.*

(3) *There exists $J \subseteq I$ as above and such that $\omega_I \subseteq \omega_J$ and for any $j \in J$ the codimension of the image of $A_{J\backslash\{j\}}$ is less than k.*

Proof. (1) If ω_J is not empty, Lemma 33 implies that ω_J is the interior part Ω_J. Since ω_I is not empty, it is as well the interior part of Ω_I. As $\Omega_I \subseteq \Omega_J$, we conclude that $\omega_I \subseteq \omega_J$.

(2) We prove the claim by induction on $h = |I \setminus J|$. If $h = 0$ the claim is true. We assume now that the claim is true for h and let J be such that $|I \setminus J| = h + 1$. Let $i \in I \backslash J$ and $J' = J \cup \{i\}$. By inductive hypothesis $\omega_{J'}$ is not empty. Let $(\delta, \epsilon) \in \omega_{J'}$ and let $x \in \mathbb{Q}^n$ such that $A_{J'}x = D_{J'}(\delta, \epsilon)$ and $A_{(J')^c}x > D_{(J')^c}(\delta, \epsilon)$. Since the codimensions of the image of A_I and of the image of A_J are equal to k, and A_J is the product of A_J with the projection matrix onto the indices $J \subseteq J'$, also the codimension of the image of $A_{J'}$ is k. By the rank theorem, we have $\dim \operatorname{Ker}(A_J) = \dim \operatorname{Ker}(A_{J'}) + 1$. Thus, there is $y \in \operatorname{Ker}(A_J) \backslash \operatorname{Ker}(A_{J'})$. We can choose y such that $A_i y > 0$. Then for $t > 0$ small enough, we have $A_J(x + ty) = D_J(\delta, \epsilon)$, $A_i(x + ty) > D_i(\delta, \epsilon)$ and $A_{(J')^c}(x + ty) > D_{(J')^c}(\delta, \epsilon)$. This proves that $(\delta, \epsilon) \in \omega_J$.

(3) It is enough to take J minimal such that the images of A_I and A_J have the same codimension, and apply the previous statement.

□

Lemma 35. *Let $I \subseteq I_0$ be such that $\omega_I = \{(\delta_0, \epsilon_0)\}$. Suppose that the image of A_I is of codimension 2. There is $i \in I$ such that the image of $A_{I\backslash\{i\}}$ is of codimension 1. For any such i, the point (δ_0, ϵ_0) belongs to $\Omega_{I\backslash\{i\}} \backslash \omega_{I\backslash\{i\}}$.*

Proof. Let $i \in I$ be such that there exists rational numbers λ_j for $j \in I\backslash\{i\}$ such that $A_i = \sum_{j \in I\backslash\{i\}} \lambda_j A_j$. Then the image of $A_{I\backslash\{i\}}$ is of codimension one. Let $(\delta_1, \epsilon_1) \neq (\delta_0, \epsilon_0)$ such that $D_{I\backslash\{i\}}(\delta_1, \epsilon_1)$ is in $\operatorname{Im}(A_{I\backslash\{i\}})$. Let x and y in $\mathbb{Q}^n$ such that $A_I x = D_I(\delta_0, \epsilon_0)$, $A_{I^c}x > D_{I^c}(\delta_0, \epsilon_0)$ and $A_{I\backslash\{i\}}y = D_{I\backslash\{i\}}(\delta_1, \epsilon_1)$. For any $t \in \mathbb{Q}$, define $z_t := (1 - t)x + ty$ and $(\delta_t, \epsilon_t) := (1 - t)(\delta_0, \epsilon_0) + t(\delta_1, \epsilon_1)$. Then for any t we have $A_{I\backslash\{i\}}z_t = D_{I\backslash\{i\}}(\delta_t, \epsilon_t)$ and for any t small enough we have $A_{I^c}z_t > D_{I^c}(\delta_t, \epsilon_t)$. Now if $A_i y > 0$ we have $A_i z_t > D_i(\delta_t, \epsilon_t)$ for any $t > 0$; and if $A_i y < 0$ we have $A_i z_t > D_i(\delta_t, \epsilon_t)$ for any $t < 0$. Hence, for any t small enough, either positive or negative, we have $A_{(I\backslash\{i\})^c}z_t > D_{(I\backslash\{i\})^c}(\delta_t, \epsilon_t)$ and (δ_t, ϵ_t) is in $\omega_{I\backslash\{i\}}$. We have proved that ω_I is in the closure of $\omega_{I\backslash\{i\}}$.

But $\omega_I = \{(\delta_0, \epsilon_0)\}$ cannot be in $\omega_{I\backslash\{i\}}$, because the relation $A_i = \sum_{j \in I\backslash\{i\}} \lambda_j A_j$ implies that if $A_{I\backslash\{i\}}x = D_{I\backslash\{i\}}(\delta_0, \epsilon_0)$ then $A_I x = D_I(\delta_0, \epsilon_0)$. Hence ω_I is in $\Omega_{I\backslash\{i\}} \backslash \omega_{I\backslash\{i\}}$.

□

Remark 15. A similar result could be proved from a one-dimensional ω_I. Let $I \subset I_0$ such that ω_I is an open convex part of an affine line. Suppose that the image of A_I is of codimension 1. Then there exists $i \in I$ such that ω_I is a subset of $\Omega_{I\backslash\{i\}} \backslash \omega_{I\backslash\{i\}}$ and $A_{I\backslash\{i\}}$ is surjective.

Lemma 36. *Let I and J such that ω_I and ω_J are not empty. Then $\omega_{I\cap J}$ is not empty and contains the strict convex hull of any element of ω_I with any element of ω_J.*

Proof. Let x and y in $\mathbb{Q}^n$ be such that $A_I x = D_I(\delta_1, \epsilon_1)$, $A_{I^c}x > D_{I^c}(\delta_1, \epsilon_1)$, $A_J y = D_J(\delta_2, \epsilon_2)$ and $A_{J^c}y > D_{J^c}(\delta_2, \epsilon_2)$.
For any $t \in]0, 1[$, $z_t := (1 - t)x + ty$ and $(\delta_t, \epsilon_t) := (1 - t)(\delta_1, \epsilon_1) + t(\delta_2, \epsilon_2)$ satisfy $A_{I\cap J}z_t = D_{I\cap J}(\delta_t, \epsilon_t)$, $A_{(I\cap J)^c}z_t > D_{(I\cap J)^c}(\delta_t, \epsilon_t)$. Hence, $\omega_{I\cap J}$ contains the strict convex hull of (δ_1, ϵ_1) and (δ_2, ϵ_2). In particular, $\omega_{I\cap J}$ is not empty. □

Remark 16. If $I = \{i\}$ and $A_i \neq 0$, or more generally if A_I is surjective, Lemma 32 implies that ω_I is open (possibly empty).

If $I=\{i\}$ and $A_i \neq 0$, for any $(\delta,\epsilon) \in \Omega_I \backslash \omega_I$, either the polytope $P^{\delta,\epsilon}$ can be defined without the line i, in other words

$$P^{\delta,\epsilon} = \{x \in \mathbb{Q}^n \mid A_{\{i\}^c} x \geq D_{\{i\}^c}(\delta,\epsilon)\},$$

or the polytope $P^{\delta,\epsilon}$ is not of dimension n. Indeed, if $(\delta,\epsilon) \in \Omega_I \backslash \omega_I$, and $P^{\delta,\epsilon}$ has maximal dimension, then $F_i^{\delta,\epsilon}$ is not a facet of $P^{\delta,\epsilon}$ or equals another $F_j^{\delta,\epsilon}$ and in both cases the inequality $A_i x \geq D_i(\delta,\epsilon)$ is superfluous in the definition of $P^{\delta,\epsilon}$.
If I is such that A_I is invertible, then $F_I^{\delta,\epsilon}$ is either empty or a vertex of $P^{\delta,\epsilon}$. If $(\delta,\epsilon) \in \Omega_I$, then $P^{\delta,\epsilon}$ is simple at the corresponding vertex if $(\delta,\epsilon) \in \omega_I$. But the converse is false: it can happen that $P^{\delta,\epsilon}$ is simple but $(\delta,\epsilon) \notin \omega_I$ if an inequality $A_i x \geq D_i(\delta,\epsilon)$ with $i \notin I$ is superfluous in the definition of $P^{\delta,\epsilon}$ and $F_I^{\delta,\epsilon} = F_{I\cup\{i\}}^{\delta,\epsilon}$.

A.2. **Generality of the polarization.** In this section we prove Proposition 17. Throughout the section by *general* we mean *in a Zariski open set*.

By Lemma 32 (2), ω_I is one-dimensional implies that the codimension of $\mathrm{Im}(A_I)$ is at least 1, and ω_I is zero-dimensional implies that the codimension of $\mathrm{Im}(A_I)$ is at least 2. The lemma tells that, for general B and B', equalities hold for any I in both cases.

Lemma 37. *We can choose B and B' general such that for any $I \subseteq I_0$, we have the following.*

- *If the image of A_I has codimension 1 in $\mathbb{Q}^I$, then ω_I is an open convex part of an affine line (possibly empty).*
- *If the image of A_I has codimension 2 in $\mathbb{Q}^I$, then ω_I is either empty or a point.*
- *If the image of A_I has codimension 3 in $\mathbb{Q}^I$, then ω_I is empty.*

Proof. The set $\{D_I(\delta,\epsilon) \mid (\delta,\epsilon) \in \mathbb{Q}^2\}$ is the affine subspace passing through B_I and directed by $B'_I - B_I$ and C_I. It is a plane for B and B' general (that is, if $B'-B$ is not colinear to C). Now, to have the three conditions above, it is enough to choose B and B' such that:

- if the image of A_I has codimension 1 in $\mathbb{Q}^I$, then B_I and B'_I are not in the image of A_I;
- if the image of A_I has codimension 2 in $\mathbb{Q}^I$, then $B'_I - B_I$ is not in the vector subspace $\mathrm{Im}(A_I) + \mathbb{Q}C_I$ (of codimension at least one in $\mathbb{Q}^I$), and if C_I is in $\mathrm{Im}(A_I)$ then B_I is not in $\mathrm{Im}(A_I)$;
- if the image of A_I has codimension 3 in $\mathbb{Q}^I$, $B'_I - B_I$ is not in $\mathrm{Im}(A_I)$ and B_I is not in the vector subspace $\mathrm{Im}(A_I) + \mathbb{Q}C_I + \mathbb{Q}(B'_I - B_I)$ (of codimension at least one in $\mathbb{Q}^I$).

This, together with Lemma 32, means that it is enough to choose B and B' outside finitely many proper linear subspaces of $\mathbb{Q}^p$, thus B and B' in an open set of $\mathbb{Q}^p$. □

From now, we assume that B and B' general in the sense of Lemma 37.

Corollary 38. *There is a finite union of convex parts of affine lines*

$$\mathcal{L} = \partial\Omega_\emptyset \cup \bigcup_{I \subseteq I_0 \ \dim \Omega_I \leq 1} \Omega_I \subseteq \mathbb{Q}^2$$

such that if $(\delta,\epsilon) \notin \mathcal{L}$ then $P^{\delta,\epsilon}$ is an n-dimensional simple polytope or it is empty.

Proof. Assume that $P^{\delta,\epsilon}$ is not empty. If $P^{\delta,\epsilon}$ has of dimension n and is not simple, then there exists a vertex v of $P^{\delta,\epsilon}$ that is contained in at least $n+1$ facets. Then there is $I \subseteq I_0$ of cardinality $p \geq n+1$ such that the vertex is $F_I^{\delta,\epsilon} = \{v\}$. In particular, if we choose I maximal among the subsets I of I_0 such that $F_I^{\delta,\epsilon} = \{v\}$, we have $(\delta,\epsilon) \in \omega_I$. But A_I is a $p \times n$ matrix with $p \geq n+1$ so it cannot be surjective. By Lemma 37, ω_I is either a segment or a point.

If $P^{\delta,\epsilon}$ is of dimension at most n, then $(\delta,\epsilon) \notin \omega_\emptyset$, which is open and non-empty by hypothesis. Then (δ,ϵ) is in the boundary of $\Omega_\emptyset$, which is a finite union of convex parts of affine lines.

We set then

$$\mathcal{L} = \partial\Omega_\emptyset \cup \bigcup_{I \subseteq I_0 \ \dim \Omega_I \leq 1} \Omega_I.$$

□

Remark 17. Notice that the condition we impose on $\Omega_\emptyset \setminus \mathcal{L}$ is stronger than the polytope $P^{\delta,\epsilon}$ being simple. Indeed, the vertices of $P^{\delta,\epsilon}$ for $(\delta,\epsilon) \in \Omega_\emptyset \setminus \mathcal{L}$ are contained in exactly n affine hyperplanes $\mathcal{H}_i := \{x \in \mathbb{Q}^n \mid A_i x = D_i(\delta,\epsilon)\}$, while a vertex of an affine polytope is contained in exactly n facets.

Notation 4. Let $I \subseteq I_0$ be such that the image of A_I is contained in a hyperplane, and such that for any $i \in I$, $A_{I\setminus\{i\}}$ is surjective. For the sake of clarity we repeat here Notation/Construction 1. Then there are rational numbers λ_i^I with $i \in I$ such that $\sum_{i\in I} \lambda_i^I A_i = 0$. We notice that $\lambda_i^I \neq 0$ for every $i \in I$. Indeed, if there is j such that $\lambda_j^I = 0$, then $\sum_{i\in I\setminus\{j\}} \lambda_i^I X_i$ is a nontrivial equation for the lines of $A_{I\setminus\{j\}}$. We can assume that $\sum_{i\in I} \lambda_i^I C_i$ is zero or one. We fix such numbers.

Let $I \subseteq I_0$ such that ω_I is not empty, the image of A_I has codimension at least two and such that for any $i \in I$, the image of $A_{I\setminus\{i\}}$ has codimension one. In particular, the image of A_I has codimension exactly two. Then there are rational numbers λ_i^I and $\lambda_i'^I$ with $i \in I$ such that $\sum_{i\in I} \lambda_i^I A_i = 0$ and $\sum_{i\in I} \lambda_i'^I A_i = 0$ are two independent relations. For B and B' general as in Lemma 37, the point in ω_I is the only solution of the linear system

$$\begin{cases} (\sum_{i\in I} \lambda_i^I C_i)\epsilon + (\sum_{i\in I} \lambda_i^I (B_i' - B_i))\delta + \sum_{i\in I} \lambda_i^I B_i & = & 0 \\ (\sum_{i\in I} \lambda_i'^I C_i)\epsilon + (\sum_{i\in I} \lambda_i'^I (B_i' - B_i))\delta + \sum_{i\in I} \lambda_i'^I B_i & = & 0 \end{cases}$$

therefore $\sum_{i\in I} \lambda_i^I C_i$ and $\sum_{i\in I} \lambda_i'^I C_i$ cannot be simultaneously zero. After perhaps replacing $\lambda_i'^I$ with $\lambda_i'^I - \lambda_i^I$, we can assume that $\sum_{i\in I} \lambda_i^I C_i = 1$ and $\sum_{i\in I} \lambda_i'^I C_i = 0$.

Moreover, for any $i \in I$, either λ_i^I or $\lambda_i'^I$ is not zero. Indeed, if there is j such that $\lambda_j^I = 0$ and $\lambda_j'^I = 0$, then $\sum_{i\in I\setminus\{j\}} \lambda_i^I X_i$ and $\sum_{i\in I\setminus\{j\}} \lambda_i'^I X_i$ are nontrivial linearly independent equations for the lines of $A_{I\setminus\{j\}}$.

We fix such numbers.

Remark 18. Let $I \neq J$ be such that the images of A_I and A_J have codimension d and for any $i \in I$ and $j \in J$, the images of $A_{I\setminus\{i\}}$ and $A_{J\setminus\{j\}}$ have codimension $d-1$. Then $I \not\subseteq J$ and $J \not\subseteq I$. Indeed if we had $I \subsetneq J$, there would be $j \in J \setminus I$ and $\mathrm{Im}(A_I)$ would have codimension at most the codimension of $\mathrm{Im}(A_{J\setminus\{j\}})$.

Now we prove that if B and B' are general, two sets ω_I and ω_J of dimension 0 or 1 intersect or are aligned only in specific cases.

Lemma 39. *Let $I \neq J$ be such that ω_I and ω_J are non-empty open convex parts of the same affine line, and such that for any $i \in I$ and $j \in J$, $A_{I\setminus\{i\}}$ and $A_{J\setminus\{j\}}$ are surjective. Then B and B' satisfy a quadratic or linear condition.*

Proof. Set $B'' := B' - B$. By remark 18 we have $I \not\subseteq J$ and $J \not\subseteq I$. Since ω_I and ω_J are non-empty open convex parts of affine lines, the images of A_I and A_J have codimension at least one. Since $A_{I\setminus\{i\}}$ and $A_{J\setminus\{j\}}$ are surjective, they have codimension exactly one.

The affine line containing ω_I (resp. ω_J) has as equation $\sum_{i\in I} \lambda_i^I D_i(\delta,\epsilon) = 0$ (resp. $\sum_{i\in I} \lambda_j^J D_j(\delta,\epsilon) = 0$), *i.e.*

$$(\sum_{i\in I} \lambda_i^I C_i)\epsilon + (\sum_{i\in I} \lambda_i^I B_i'')\delta + \sum_{i\in I} \lambda_i^I B_i = 0, \text{ resp. } (\sum_{j\in J} \lambda_j^J C_j)\epsilon + (\sum_{j\in J} \lambda_j^J B_j'')\delta + \sum_{j\in J} \lambda_j^J B_j = 0.$$

Those two equations by hypothesis define the same line. It follows from Construction 4 that the coefficients of ϵ are either both zero or both one.

If $\sum_{i\in I} \lambda_i^I C_i = \sum_{j\in J} \lambda_j^J C_j = 0$, since the two equations define the same line, we have

$$(\sum_{i\in I} \lambda_i^I B_i'')(\sum_{j\in J} \lambda_j^J B_j) = (\sum_{j\in J} \lambda_j^J B_j'')(\sum_{i\in I} \lambda_i^I B_i).$$

This condition is nontrivial: let $i \in I$ and $j \in J$ such that $i \notin J$ and $j \notin I$, then the coefficient in the quadratic condition in $B_i'' B_j$ is $\lambda_i^I \lambda_j^J$ and this is non zero by Construction 4.

If $\sum_{i\in I} \lambda_i^I C_i = \sum_{j\in J} \lambda_j^J C_j = 1$, then $\sum_{i\in I} \lambda_i^I B_i'' = \sum_{j\in J} \lambda_j^J B_j''$ and $\sum_{i\in I} \lambda_i^I B_i = \sum_{j\in J} \lambda_j^J B_j$. □

Corollary 40. *Assume that B and B' are general. Then for any I and J subsets of I_0 such that ω_I and ω_J are nonempty open convex parts of affine lines, ω_I and ω_J are contained in the same affine line if and only if $\omega_{I\cap J}$ is a non-empty open convex part of an affine line. Moreover in this case, we have $\omega_{I\cap J} \supseteq \omega_I \cup \omega_J$.*

Proof. Let I and J be subsets of I_0 such that ω_I and ω_J are non-empty open convex parts of the same affine line L. By Lemma 34, there exist $I' \subseteq I$ and $J' \subseteq J$ such that $\omega_{I'}$ and $\omega_{J'}$ are non-empty open segments contained in L, and such that for every $i \in I'$ and $j \in J'$, $A_{I'\setminus\{i\}}$ and $A_{J'\setminus\{j\}}$ are surjective. By Lemma 39, for B and B' general, we must have $I' = J'$. In particular $I \cap J \supset I'$ and $\omega_I \subset \omega_{I\cap J} \subseteq \omega_{I'}$ so that $\omega_{I\cap J}$ is a non-empty open convex part of an affine line.

Conversely, if ω_I, ω_J and $\omega_{I\cap J}$ are non-empty open convex part of an affine line, then they are contained in the same affine line because $\omega_I \subset \omega_{I\cap J} \supset \omega_J$. □

Lemma 41. *Let $I \neq J$ be such that $\omega_I = \omega_J$ is reduced to a point, and such that for any $i \in I$ and $j \in J$, the images of $A_{I\setminus\{i\}}$ and $A_{J\setminus\{j\}}$ have codimension one. Then B and B' satisfy quadratic conditions.*

Proof. Set $B'' := B' - B$. By remark 18 we have $I \not\subseteq J$ and $J \not\subseteq I$. Since ω_I and ω_J are points, the images of A_I and A_J have codimension at least two. Since $A_{I\setminus\{i\}}$ and $A_{J\setminus\{j\}}$ have codimension exactly one, the images of A_I and A_J have codimension exactly two.

Let (δ_0, ϵ_0) be such that $\omega_I = \omega_J = \{(\delta_0, \epsilon_0)\}$.

By Notation/Construction 4, (δ_0, ϵ_0) is the unique solution of the two following systems:

$$\left\{\begin{array}{lcl} \epsilon + (\sum_{i\in I}\lambda_i^I B_i'')\delta + \sum_{i\in I}\lambda_i^I B_i & = & 0 \\ (\sum_{i\in I}\lambda_i'^I B_i'')\delta + \sum_{i\in I}\lambda_i'^I B_i & = & 0 \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{lcl} \epsilon + (\sum_{j\in J}\lambda_j^J B_j'')\delta + \sum_{j\in J}\lambda_j^J B_j & = & 0 \\ (\sum_{j\in J}\lambda_j'^J B_j'')\delta + \sum_{j\in J}\lambda_j'^J B_j & = & 0 \end{array}\right. .$$

Comparing the first and second equations of the two systems we get

$$(\mathcal{C}_1) : (\sum_{i\in I}\lambda_i'^I B_i'')(\sum_{j\in J}\lambda_j'^J B_j) = (\sum_{j\in J}\lambda_j'^J B_j'')(\sum_{i\in I}\lambda_i'^I B_i).$$

$$(\mathcal{C}_2) : \left((\sum_{i\in I}\lambda_i^I B_i'') - (\sum_{j\in J}\lambda_j^J B_j'')\right)\sum_{i\in I}\lambda_i'^I B_i - \left(\sum_{i\in I}\lambda_i^I B_i - \sum_{j\in J}\lambda_j^J B_j\right)\sum_{i\in I}\lambda_i'^I B_i'' = 0.$$

To conclude, one proves, with an easy but tedious computation that the conditions $(\mathcal{C}_1)$ and $(\mathcal{C}_2)$ cannot be both trivial. □

Corollary 42. *For B and B' general we have the following: for any $I, J \subseteq I_0$ such that ω_I and ω_J are singletons, we have $\omega_I = \omega_J$ if and only if $\omega_{I\cap J}$ is a singleton, equal to $\omega_I = \omega_J$.*

Proof. Let I and J be subsets of I_0 such that $\omega_I = \omega_J = \{(\delta_0, \epsilon_0)\}$. By Lemma 34, there exist $I' \subseteq I$ and $J' \subseteq J$ such that $\omega_{I'} = \omega_{J'}$ and for any $i \in I'$ and $j \in J'$, the images of $A_{I'\setminus\{i\}}$ and $A_{J'\setminus\{j\}}$ have codimension one. By Lemma 41, for general B and B', we must have $I' = J'$. In particular $I \cap J \supset I'$ and $\omega_{I\cap J} \subseteq \omega_{I'} = \{(\delta_0, \epsilon_0)\}$. By Lemma 36 the set $\omega_{I\cap J}$ is not empty, then it must be $\omega_{I\cap J} = \{(\delta_0, \epsilon_0)\}$.

Conversely, if ω_I, ω_J and $\omega_{I\cap J}$ are singletons then $\omega_I = \omega_{I\cap J} = \omega_J$ because $\omega_I \subset \omega_{I\cap J} \supset \omega_J$. □

Lemma 43. *Let I and J be such that $\omega_I = \{(\delta_0, \epsilon_0)\}$, ω_J is an open convex part of an affine line, and ω_I is contained in this affine line. Suppose that for every $i \in I$, the image of $A_{I\setminus\{i\}}$ has codimension one and for every $j \in J$, $A_{J\setminus\{j\}}$ is surjective.*

Then either $J \subseteq I$ or B and B' satisfy a quadratic condition.

Proof. Set $B'' = B' - B$.

The affine line containing both ω_I and ω_J has equation

$$(\sum_{j\in J}\lambda_j^J C_j)\epsilon + (\sum_{j\in J}\lambda_j^J B_j'')\delta + \sum_{j\in J}\lambda_j^J B_j = 0,$$

where $\sum_{j\in J}\lambda_j^J C_j$ is zero or one and (δ_0, ϵ_0) is the unique solution of the system

$$(\mathcal{S}_I) : \quad \left\{\begin{array}{lcl} \epsilon + (\sum_{i\in I}\lambda_i^I B_i'')\delta + \sum_{i\in I}\lambda_i^I B_i & = & 0 \\ (\sum_{i\in I}\lambda_i'^I B_i'')\delta + \sum_{i\in I}\lambda_i'^I B_i & = & 0 \end{array}\right. .$$

If $\sum_{j\in J}\lambda_j^J C_j = 0$ we have

$$(\sum_{i\in I}\lambda_i'^I B_i'')(\sum_{j\in J}\lambda_j^J B_j) = (\sum_{i\in I}\lambda_i'^I B_i)(\sum_{j\in J}\lambda_j^J B_j'').$$

And if $\sum_{j\in J}\lambda_j^J C_j = 1$, we have

$$(\sum_{i\in I}\lambda_i^I B_i'')(\sum_{j\in J}\lambda_j^J B_j) = (\sum_{i\in I}\lambda_i^I B_i)(\sum_{j\in J}\lambda_j^J B_j'').$$

In both cases, if there exists $j \in J$ that is not in I, then there exists $i \in I$ such that the coefficent of $B_j B''_i$ is non zero (it is $\lambda'^I_i \lambda^J_j$ in the first case and $\lambda^I_i \lambda^J_j$ is the second case) so that the condition is not trivial. □

Corollary 44. *For B and B' general we have the following: for any I and J subsets of I_0 such that ω_I is a singleton and ω_J is an open convex part of an affine line ℓ, the affine line ℓ contains ω_I if and only if $\omega_{I\cap J}$ is a non-empty open convex part of ℓ.*

Proof. By Lemma 34, there exist $I' \subseteq I$ and $J' \subseteq J$ such that $\omega_{I'} = \omega_I$, $\omega_J \subset \omega_{J'}$ and for any $i \in I'$ and $j \in J'$, the image of $A_{I'\setminus\{i\}}$ has codimension one and $A_{J'\setminus\{j\}}$ is surjective. Then, by Lemma 43, , for general B and B', we must have $J' \subseteq I'$.

In particular, $J' \subseteq I \cap J$ and by Lemma 34 we have $\omega_{I\cap J} \subset \omega_{J'}$, so that $\omega_{I\cap J}$ is either empty or an open convex part of an affine line (the same affine line as for ω_J).

By with Lemma 36, the set $\omega_{I\cap J}$ is not empty. □

We end this subsection with the following lemma in order to avoid the case where three affine lines intersect into a point except in only one situation.

Lemma 45. *Let I, J and K be distinct subsets of I_0 such that the images of A_I, A_J and A_K have codimension one, and such that for any $i \in I$, $j \in J$ and $k \in K$, $A_{I\setminus\{i\}}$, $A_{J\setminus\{j\}}$ and $A_{K\setminus\{k\}}$ are surjective. If the three affine lines generated by ω_I, ω_J and ω_K intersect, then either B and B' satisfy some linear or quadratic condition or $I\cup J = I\cup K = J\cup K = I\cup J\cup K$.*

Proof. Set $B'' := B' - B$.

By Lemma 39, if the three lines are not distinct, we are done. Suppose then that the three lines are not distinct and meet at the point (δ_0, ϵ_0).

Let $L = I, J, K$. The affine line containing ω_L has equation $\sum_{\ell\in L} \lambda^L_\ell D_\ell(\delta,\epsilon) = 0$. If we set

$$\begin{cases} a_L = \sum_{\ell\in L} \lambda^L_\ell C_\ell \\ b_L = \sum_{\ell\in L} \lambda^L_\ell B''_\ell \\ c_L = \sum_{\ell\in L} \lambda^L_\ell B_\ell \end{cases}$$

then the equation becomes $a_L\epsilon + b_L\delta + c_L = 0$. By hypothesis the point (a_L, b_L, c_L) belongs to the plane $a\epsilon_0 + b\delta_0 + c = 0$ for $L = I, J, K$.

By Construction 4 we have $a_L \in \{0, 1\}$.

Assume that there is L such that $a_L = 0$. Without loss of generality we can assume that $L = I$. Notice that then only I is such that $a_I = 0$, because two affine lines distinct and parallel do not meet.

Then $b_I(c_J - c_K) = c_I(b_J - b_K)$, that is,

$$(\sum_{i\in I}\lambda^I_i B''_i)(\sum_{j\in J}\lambda^J_j B_j - \sum_{k\in K}\lambda^K_k B_k) = (\sum_{i\in I}\lambda^I_i B_i)(\sum_{j\in J}\lambda^J_j B''_j - \sum_{k\in K}\lambda^K_k B''_k).$$

Assuming that we do not have $I\cup J = I\cup K = J\cup K = I\cup J\cup K$ we prove that this condition is not trivial. There are three cases, either $J\cup K \neq I\cup J\cup K$, or $I\cup J \neq I\cup J\cup K$, or $I\cup K \neq I\cup J\cup K$. The last two cases are proved in the same way.

If $J\cup K \neq I\cup J\cup K$ let $i \in I$ be such that $i \notin J\cup K$. As $J \neq K$ we can assume without loss of generality that there is $j \in J, j \notin K$. Then the coefficient of $B''_i B_j$ in the quadratic condition is $\lambda^I_i \lambda^J_j$ which is non zero by Construction 4.

If $I \cup J \neq I \cup J \cup K$ let $k \in K$ such that $k \notin I \cup J$. Then for every $i \in I$ the coefficient of $B_i'' B_k$ in the quadratic condition is $\lambda_i^I \lambda_k^k$ which is non zero by Construction 4.

Assume that $a_L = 1$ for every L. Without loss of generality we can assume that $I \cup J \neq I \cup J \cup K$. With similar computations as above, we prove that the condition

$$(b_I - b_J)(c_I - c_K) = (b_I - b_K)(c_I - c_J)$$

verified by the coefficients of the two linear equations is not trivial.

□

A.3. **Polyhedral decomposition and the geography of models.**

Lemma 46. *Let $(\delta, \epsilon) \in U_1$. Let I be such that $(\delta, \epsilon) \in \omega_I$ and $\dim \omega_I = 1$. Assume moreover that I is minimal with the property. Denote by $I_+ := \{i \in I \mid \lambda_i^I > 0\}$ and $I_- := \{i \in I \mid \lambda_i^I < 0\}$. Note that $I = I_+ \sqcup I_-$.*

Then $(\delta, \epsilon) \in \omega_\emptyset$ if and only if both I_+ and I_- are not empty.

Proof. Let $\sum_{i \in I} \lambda_i^I X_i = 0$ be a relation satified by the lines of A_I as in Construction 4. Notice that $\lambda_i^I \neq 0$ for any $i \in I$ by minimality of I. Indeed, if $\lambda_i^I = 0$ then the image of $A_{I \setminus \{i\}}$ still have codimension 1 and $\omega_I \subseteq \omega_{I \setminus \{i\}}$ by Lemma 34(3).

Fix $x \in \mathbb{Q}^n$ such that $A_I x = D_I(\delta, \epsilon)$ and $A_{I^c} x > D_{I^c}(\delta, \epsilon)$.

Assume first that I_+ and I_- are not empty. Let $i \in I_+$, then

$$A_i = -\sum_{j \in I_+, j \neq i} \frac{\lambda_j^I}{\lambda_i^I} A_j + \sum_{j \in I_-} \frac{-\lambda_j^I}{\lambda_i^I} A_j.$$

By the minimality of I, the matrix $A_{I \setminus \{i\}}$ is surjective. Therefore, there exists $y \in \mathbb{Q}^n$ such that $A_j y > 0$ for any $j \in I \setminus \{i\}$ and $A_j y$ big enough for $j \in I_-$ so that $A_i y$ is also positive. Then for any $t > 0$ small enough, we have $A(x + ty) > D(\delta, \epsilon)$. This means that $(\delta, \epsilon) \in \omega_\emptyset$.

Conversely, suppose that $I^+ = I$ and $I_- = \emptyset$. First, we prove that $\sum_{i \in I} \lambda_i^I D_i(\delta, \epsilon) = 0$. Since $(\delta, \epsilon) \in \omega_I$, there is $x \in \mathbb{Q}^n$ such that $Ax \geq D(\delta, \epsilon)$ and $A_I x = D_I(\delta, \epsilon)$. Thus $0 = \sum_{i \in I} \lambda_i^I A_i x = \sum_{i \in I} \lambda_i^I D_i(\delta, \epsilon)$.

Now, let $y \in \mathbb{Q}^n$ such that $Ay \geq D(\delta, \epsilon)$. Then $0 = \sum_{i \in I} \lambda_i^I A_i y \geq \sum_{i \in I} \lambda_i^I D_i(\delta, \epsilon) = 0$, so that $A_I y = D_I(\delta, \epsilon)$. In particular, $(\delta, \epsilon) \notin \omega_\emptyset$.

The same occurs if $I^- = I$. □

Remark 19. The proof implies in particular that, if both I_+ and I_- are not empty, then there is $y \in \mathbb{Q}^n$ such that $A_I y > 0$.

Lemma 47. *Let $(\delta, \epsilon) \in U_1 \cap \omega_\emptyset$. Let I be such that $(\delta, \epsilon) \in \omega_I$ and $\dim \omega_I = 1$. Assume moreover that I is minimal with the property. Let $i \in I_0 \setminus J_0$. Then $(\delta, \epsilon) \in \Omega_i \setminus \omega_i$ if and only if I_+ or I_- equals $\{i\}$.*

In particular, there is at most one $i \in I_0 \setminus J_0$, such that (δ, ϵ) is in $\Omega_i \setminus \omega_i$.

Proof. Let $\sum_{i \in I} \lambda_i^I X_i = 0$ be a relation satified by the lines of A_I as in Construction 4. Notice that $\lambda_i^I \neq 0$ for any $i \in I$ by minimality of I.

Suppose that there exists $i \in I_0 \setminus J_0$ such that $(\delta, \epsilon) \in \Omega_i \setminus \omega_i$. Then there exists $J \subseteq I_0$ containing i such that $(\delta, \epsilon) \in \omega_J$. Pick such J minimal. By Lemma 34(1), since $(\delta, \epsilon) \notin \omega_i$, the image of A_J has codimension either one or two. By assumption on (δ, ϵ), ω_J is not a point, so that ω_J is one-dimensional and then by Corollary 40 and the generality assumption, $I \subset J$. By minimality of J, we have $J = I \cup \{i\}$.

Assume by contradiction that $i \notin I$. With Notation 4, we have $\lambda_i^J = 0$, that is, J and I give the same relation.

Since $(\delta, \epsilon) \in \omega_\emptyset$, by Lemma 46 and Remark 19 and 7, there exists $y \in \mathbb{Q}^n$ such that $A_I y > 0$ and such that $A_i y = 0$. Let $x \in \mathbb{Q}^n$ such that $A_J x = D_J(\delta, \epsilon)$ and $A_{J^c} x > D_{J^c}(\delta, \epsilon)$. Then for any $t > 0$ small enough $A_i(x + ty) = D_i(\delta, \epsilon)$ and $A_{\{i\}^c}(x+ty) > D_{\{i\}^c}(\delta, \epsilon)$. This is a contradiction with $(\delta, \epsilon) \notin \omega_i$. Then $i \in I$ and $I = J$.

By assumption the matrix $A_{I\backslash\{i\}}$ is surjective. If $I\backslash\{i\}$ is not contained in either I_+ or I_-, we can find $y \in \mathbb{Q}^n$ such that $A_j y > 0$ for any $j \in I\backslash\{i\}$ and $\sum_{j\in I\backslash\{i\}} \lambda_j^I A_j y = 0$, which implies that $A_i y = 0$. Let $x \in \mathbb{Q}^n$ be such that $A_I x = D_I(\delta, \epsilon)$ and $A_{I^c} x > D_{I^c}(\delta, \epsilon)$. Then, for any $t > 0$ small enough we have $A_i(x + ty) = D_i(\delta, \epsilon)$ and $A_{\{i\}^c}(x + ty) > D_{\{i\}^c}(\delta, \epsilon)$. This is a contradiction with $(\delta, \epsilon) \notin \omega_i$. We conclude with Lemma 46 and the hypothesis $(\delta, \epsilon) \in \omega_\emptyset$, that $\{i\} = I_+$ or I_-.

Conversely, suppose that $I_+ = \{i\}$. Note that $(\delta, \epsilon) \in \Omega_i$ is obvious. Since there exists $x \in \mathbb{Q}^n$ such that $A_I x = D_I(\delta, \epsilon)$, we have $D_i(\delta, \epsilon) = \sum_{j\in I\backslash\{i\}} \frac{-\lambda_j^I}{\lambda_i^I} D_j(\delta, \epsilon)$.

For any $y \in \mathbb{Q}^n$, such that $A_{\{i\}^c} y \geq D_{\{i\}^c}(\delta, \epsilon)$ we have

$$A_i y = \sum_{j\in I\backslash\{i\}} \frac{-\lambda_j^I}{\lambda_i^I} A_j y \geq \sum_{j\in I\backslash\{i\}} \frac{-\lambda_j^I}{\lambda_i^I} D_j(\delta, \epsilon) = D_i(\delta, \epsilon).$$

And we have equality if and only if $A_{I\backslash\{i\}} y = D_{I\backslash\{i\}}(\delta, \epsilon)$. Then $(\delta, \epsilon) \notin \omega_i$.

For the last statment, note that if $|I| = 1$ then A_I is a zero line and $i \in J_0$; and if $|I| = 2$ with $|I_+| = 1$ then the two lines of A_I generate the same ray of $N_\mathbb{Q}$ and at most one could be the primitive element of the ray, so that the other line has index in J_0. □

Lemma 48. *Let $(\delta, \epsilon) \in U_1 \cap \Omega_\emptyset \backslash \omega_\emptyset$. Let I be such that $(\delta, \epsilon) \in \omega_I$ and $\dim \omega_I = 1$. Let $i \in I_0 \backslash I$. Then either $(\delta, \epsilon) \notin \Omega_{I\cup\{i\}}$ or $(\delta, \epsilon) \in \omega_{I\cup\{i\}}$.*

Proof. Let $i \in I_0\backslash I$ such that $(\delta, \epsilon) \in \Omega_{I\cup\{i\}}$. There exists $J \subseteq I_0$ containing i and I such that $(\delta, \epsilon) \in \omega_J$. By hypothesis on (δ, ϵ) the set ω_J is one-dimensional and Lemma 34 implies that $\omega_J \subseteq \omega_{I\cup\{i\}}$. In particular, $(\delta, \epsilon) \in \omega_{I\cup\{i\}}$. □

Recall that by Notation/Construction 3 if L is such that $\omega_L = \{(\bar{\delta}, \bar{\epsilon})\} \subset \Omega_\emptyset \backslash \omega_\emptyset$ and $\bar{\epsilon} > 0$ and, for B and B' are general, then there are either I and J subsets of L and two linearly independent equations for the lines of A_L

$$\begin{array}{lll} (\mathcal{R}_I) & \sum_{i\in I} \lambda_i^I X_i = 0, & \lambda_i^I > 0\ \forall i \in I \\ (\mathcal{R}_J) & \sum_{j\in J} \lambda_j^J X_j = 0, & \lambda_j^J > 0\ \forall j \in J \end{array}$$

or there is $I \subseteq L$ and two linearly independent equations for the lines of A_L

$$\begin{array}{lll} (\mathcal{R}_I) & \sum_{i\in I} \lambda_i^I X_i = 0, & \lambda_i^I > 0\ \forall i \in I \\ (\mathcal{R}_L) & \sum_{j\in L} \lambda_j^L X_j = 0, & \lambda_j^L \neq 0\ \forall j \in L \setminus I \end{array}$$

Lemma 49. *Let $(\delta, \epsilon) \in U_0 \cap \Omega_\emptyset \backslash \omega_\emptyset$ and assume that (δ, ϵ) is a vertex. Let $L \subseteq I_0$ be a minimal subset such that $\omega_L = \{(\delta, \epsilon)\}$. Let $i \in I_0 \backslash L$. Then either $(\delta, \epsilon) \notin \Omega_{L\cup\{i\}}$ or $(\delta, \epsilon) \in \omega_{L\cup\{i\}}$.*

Proof. If $(\delta,\epsilon)\in\Omega_{L\cup\{i\}}\backslash\omega_{L\cup\{i\}}$, there exists $J\subseteq I_0$ containing i and L such that $(\delta,\epsilon)\in\omega_J$. By Lemma 34, we have $\omega_{L\cup\{i\}}=\omega_L=\omega_J$. In particular $(\delta,\epsilon)\in\omega_{L\cup\{i\}}$. □

Lemma 50. *Let $(\delta,\epsilon)\in U_0\cap\Omega_\emptyset\backslash\omega_\emptyset$ and assume that (δ,ϵ) is not a vertex of $\Omega_\emptyset\backslash\omega_\emptyset$. Let $L\subseteq I_0$ be a minimal subset such that $\omega_L=\{(\delta,\epsilon)\}$. Let I be a minimal subset such that $\omega_L\subseteq\omega_I$ and $\dim\omega_I=1$. Let $i\in I_0\backslash I$. If $(\delta,\epsilon)\in\Omega_{I\cup\{i\}}\backslash\omega_{I\cup\{i\}}$ then $i\in L\backslash I$ and either $L\setminus I_+=\{i\}$ or $L\setminus I_-=\{i\}$. In particular, there are at most 2 indices i such that $(\delta,\epsilon)\in\Omega_{I\cup\{i\}}\backslash\omega_{I\cup\{i\}}$. If there are 2 such indices, then $|L|=|I|+2$ and the restriction of $A_{L\backslash I}$ to $\ker(A_I)$ consists of twice the same line if $L\backslash I\subseteq I_0\backslash J_0$.*

Proof. The proof is similar to the proof of Lemma 47.

If $(\delta,\epsilon)\in\Omega_{I\cup\{i\}}\backslash\omega_{I\cup\{i\}}$, then there exists $J\subseteq I_0$ containing i and I such that $(\delta,\epsilon)\in\omega_J$ and ω_J is one or zero-dimensional. The set ω_J cannot have dimension one, otherwise Lemma 34 would imply that $(\delta,\epsilon)\in\omega_J\subseteq\omega_{I\cup\{i\}}$. Hence, $\omega_J=\omega_L$ and $J=L\cup\{i\}$.

We prove that $i\in L$. Assume that it is not the case. Since the coefficients λ_j^L do not have the same sign for $j\in L\backslash I$, there exist positive μ_j^L such that $\sum_{j\in L\backslash I}\lambda_j^L\mu_j^L=0$. By Remark 7 there exists $y\in\ker(A_I)$ such that $A_jy=\mu_j^L>0$ for any $j\in L\backslash I$ and so that $A_iy=0$. Since $(\delta,\epsilon)\in\omega_{L\cup\{i\}}$, there is $x\in M_{\mathbb{Q}}$ is such that $A_{L\cup\{i\}}x=D_{L\cup\{i\}}(\delta,\epsilon)$ and $A_{(L\cup\{i\})^c}x>D_{(L\cup\{i\})^c}(\delta,\epsilon)$. Thus for t small, we have $A_{I\cup\{i\}}(x+ty)=D_{I\cup\{i\}}(\delta,\epsilon)$ and $A_{L\backslash I}(x+ty)>D_{L\backslash I}(\delta,\epsilon)$ proving that $(\delta,\epsilon)\in\omega_{I\cup\{i\}}$, which is a contradiction. Thus $i\in L$.

We prove now that all the λ_j^L for $j\in L\backslash(I\cup\{i\})$ have the same sign. Again, by contradiction, if the λ_j^L for $j\in L\backslash(I\cup\{i\})$ do not have the same sign, by Remark 7 we can find $y\in\ker(A_I)$ such that $A_jy>0$ for any $j\in L\backslash(I\cup\{i\})$ and $A_iy=0$. Let $x\in\mathbb{Q}^n$ such that $A_Lx=D_L(\delta,\epsilon)$ and $A_{L^c}x>D_{L^c}(\delta,\epsilon)$. Then for any $t>0$ small enough $A_{I\cup\{i\}}(x+ty)=D_{I\cup\{i\}}(\delta,\epsilon)$ and $A_{(I\cup\{i\})^c}(x+tx')>D_{(I\cup\{i\})^c}(\delta,\epsilon)$. This is a contradiction, as $(\delta,\epsilon)\notin\omega_{I\cup\{i\}}$.

Finally, as the coefficients of $(\mathcal{R}_L)$ do not have all the same sign, λ_i^L has opposite sign than λ_j^L for $j\in L\backslash(I\cup\{i\})$.

The last statment is not difficult.

□

Proposition 51. *Let $(\delta_2,\epsilon_2)\in U_0\cap MPC$ and let L be minimal such that $\omega_L=\{(\delta_2,\epsilon_2)\}$. We refer to the notation of Construction 3 and to Definition 5.5. Then there is a partition $L=K^0\sqcup\ldots\sqcup K^{r+1}$, with $r\geq 0$, such that*

(1) for every $s\in\{0,\ldots,r+1\}$, for every $h,k\in K^s$ we have $-\dfrac{\lambda_h^I}{\lambda_h^J}=-\dfrac{\lambda_k^I}{\lambda_k^J}=:\nu_s\in[-\infty,0]$; for every $s<s'\in\{0,\ldots,r+1\}$, we have $\nu_s>\nu_{s'}$.

(2) The set $K_s=L\setminus K^s$ is such that the codimension of the image of A_{K_s} is 1 and for every $k\in K_s$ the map $A_{K_s\backslash\{k\}}$ is surjective. If $K\subseteq L$ is such that the codimension of the image of A_K is 1 and for every $k\in K$ the map $A_{K\backslash\{k\}}$ is surjective, then $K=K_s$ for some $s\in\{0,\ldots,r+1\}$. Moreover, if (δ_2,ϵ_2) is a vertex then $K_0=I$ and $K_{r+1}=J$ and if it is not then $K_0=I$ and $K^0=L\backslash I$.

Note that, by Lemma 35, ω_{K_s} is not empty and with an extremity equals to ω_L.

(3) *The slope of* ω_{K_s} *is*

$$sl_{K_s} = \frac{\sum_{i\in I}\lambda_i^I(B'_i - B_i) + \nu_s \sum_{j\in J}\lambda_j^J(B'_j - B_j)}{1 + d\nu_s}$$

with $d = 1$ *if* (δ_2, ϵ_2) *is a vertex and* $d = \sum_{j\in L}\lambda_j^L C_j$ *otherwise.*

(4) *Up to a rotation, the slopes decrease when s increases (see picture below).*

Proof. Let $K \subseteq L$ be such that the codimension of the image of A_K is 1 and for every $k \in K$ the map $A_{K\setminus\{k\}}$ is surjective. Then there is an equation

$$(\mathcal{R}_K)\ \ \sum \lambda_k^K X_k = 0$$

satisfied by the lines of A_K.

Assume that (δ_2, ϵ_2) is a vertex and let I, J be as in Construction 3. The relation $(\mathcal{R}_K)$ is a linear combination of $(\mathcal{R}_I)$ and $(\mathcal{R}_J)$. Thus there are μ_I, μ_J such that $(\mathcal{R}_K) = \mu_I(\mathcal{R}_I) + \mu_J(\mathcal{R}_J)$. We notice that if $\mu_I = 0$, then by the minimality of K we have $K = J$ and if $\mu_J = 0$ then by the minimality of K we have $K = I$.

Assume that (δ_2, ϵ_2) is not a vertex and let I be as in Construction 3. The relation $(\mathcal{R}_K)$ is a linear combination of $(\mathcal{R}_I)$ and $(\mathcal{R}_L)$. Thus there are μ_I, μ_L such that $(\mathcal{R}_K) = \mu_I(\mathcal{R}_I) + \mu_L(\mathcal{R}_L)$. We notice that if $\mu_L = 0$ then by minimality of K we have $K = I$ and if $\mu_L \neq 0$ then $L\backslash I \subseteq K$.
If (δ_2, ϵ_2) is not a vertex, we set $J = \{j \in L \mid \lambda_j^L \neq 0\}$. Notice that J gives the relation $(\mathcal{R}_L)$.

If $\mu_I\lambda_k^I + \mu_J\lambda_k^J \neq 0$ for every k, then the image of $A_{L\setminus\{\ell\}}$ has the same codimension as the image of A_L, and this is a contradiction. Therefore the set $\{k|\ \mu_I\lambda_k^I + \mu_J\lambda_k^J = 0\}$ is non-empty and K is the complement of this set in L.
We set $N = \{-\lambda_k^I/\lambda_k^J\} = \{\nu_0 > \ldots > \nu_{r+1}\}$ with the convention that $\nu_{r+1} = -\infty$.
We set $K_s = \{k|\ -\lambda_k^I/\lambda_k^J = \nu_s\}$ and $K_s = L \setminus K^s$.

This proves (1) and (2).

As for (3), the slope sl_{K_s} is

$$\frac{\sum_{k\in K_s}\lambda_k^{K_s}(B'_k - B_k)}{\sum_{k\in K_s}\lambda_k^{K_s}C_k}.$$

Since up to a multiple $\lambda_k^{K_s} = \lambda_k^I + \nu_s\lambda_k^{\tilde{I}}$, we get the third part of the statement, with the assumption that, if (δ_2, ϵ_2) is a vertex, we have $\sum_{i\in I}\lambda_i^I C_i = 1$ and $\sum_{j\in J}\lambda_j^J C_j = 1$, and otherwise $\sum_{i\in I}\lambda_i^I C_i = 1$ and $\sum_{j\in L}\lambda_j^L C_j = 1$ or 0.

Set

$$a = \sum_{i\in I}\lambda_i^I(B'_i - B_i) = sl_I$$

$$b = \sum_{j\in J}\lambda_j^J(B'_j - B_j) = sl_J \text{ and}$$

$$d = \begin{cases} 1 \text{ if } (\delta_2, \epsilon_2) \text{ is a vertex and} \\ \sum_{j\in L}\lambda_j^L C_j \text{ (which is 1 or 0) otherwise.}\end{cases}$$

To prove (4), set $f(x) = \dfrac{a + bx}{1 + dx}$. Note that $sl_{K_s} = f(\nu_s)$. And apply a rotation $\frac{1}{1+a^2}\begin{pmatrix} a & -1 \\ 1 & a \end{pmatrix}$ in order to replace $sl_I = a$ by $+\infty$ (vertical direction). This way, the slopes sl_K are replaced by relative slopes $relsl_K := \frac{a\,sl_K + 1}{a - sl_K}$. Set $g(x) = \frac{a^2+1+x(ab+d)}{x(ad-b)}$ so that $relsl_{K_s} = g(\nu_s)$.

We now prove that $ad - b < 0$. By Construction 3, if (δ_2, ϵ_2) is a vertex we have $\omega_I \subseteq \{\delta < \delta_2\}$ and $\omega_J \subseteq \{\delta > \delta_2\}$. By the convexity of $\Omega_\emptyset$ the slope sl_I is smaller that sl_J, so that $ad - b < 0$. If (δ_2, ϵ_2) is not a vertex and $d = 1$, similarily we have $sl_I < sl_J = \frac{b}{d}$, so that $ad - b < 0$. And if $d = 0$, we have $b > 0$ (still by Construction 3).

In all cases, since g is increasing, $relsl_{K_s}$ is decreasing from $+\infty$ ($s = 0$ and $\nu_s = 0$) to $\frac{ab+d}{ad-b}$ ($\nu_s = -\infty$).

□

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