

HOROSPHERICAL TWO-ORBIT VARIETIES AS ZERO LOCI

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ABSTRACT. We present geometric realizations of horospherical two-orbit varieties, by showing that their blow-up along the unique closed invariant orbit is the zero locus of a general section of a homogeneous vector bundle over some auxiliary variety. As an application, we compute the cohomology ring of the G_2 -variety, including its quantum version. We also consider the $Spin_7$ -variety, which deserves a different treatment.

1. INTRODUCTION

Homogeneous varieties play an important role in the classification of complex Fano manifolds, one of the main building blocks in the classification of complex projective varieties. Already in dimension three, the Fano-Iskovskih classification of Fano threefolds of Picard number one and index one reveals that many of them (those of genus between 6 and 10, to be precise) can be realized as complete intersections in certain homogeneous spaces [IP99]. In genus 12, one has to consider an equivariant bundle over a Grassmannian in order to realize the Fano threefolds with this genus as zero loci of global sections. If other approaches are also possible, this *vector bundle method* was applied systematically by Mukai and many others for Fano threefolds and K3 surfaces. It has the great advantage of allowing an easy access to the geometry of these varieties [Muk95]. Very recently, the vector bundle method was used in order to cover the whole of Mori-Mukai's classification of Fano threefolds [DBFT22].

In higher dimensions, complete intersections and, more generally, zero loci of sections of homogeneous vector bundles on homogeneous varieties also allow to construct lots of interesting varieties (in dimension four, see [Kü95, Ben18] for a sample of these techniques). Nevertheless, it is certainly important to enlarge the class of manifolds on which one could use the vector bundle method. Close to homogeneous varieties, one can consider quasi-homogeneous varieties (those varieties whose automorphism group acts with a dense orbit), especially those that have been classified by combinatorial data, such as spherical varieties, or even more special ones, such as symmetric varieties or horospherical varieties. Under the hypothesis that the Picard group is cyclic (which implies that these varieties are Fano), symmetric varieties were classified by Ruzzi [Ruz11]. They are in fact homogeneous, or hyperplane sections of homogeneous varieties, up to two exceptions. It is remarkable that these two exceptional varieties can both be realized geometrically by the vector bundle method; this was used in [Man18, Man20] in order to study their geometries and compute their (quantum) intersection rings.

In this paper we consider the case of horospherical varieties. Before stating our main result, let us recall the classification of smooth projective horospherical varieties with Picard group \mathbb{Z} .

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Theorem 1. [Pas09, Th. 0.1] *Let G be a connected reductive algebraic group. Let X be a smooth projective horospherical G -variety with Picard group \mathbb{Z} , which is not homogeneous.*

Then X is horospherical of rank one, and its automorphism group is a connected non-reductive linear algebraic group, strictly containing G , acting with exactly two orbits.

Moreover, X is uniquely determined by its two closed G -orbits Y and Z , isomorphic to G/P_Y and G/P_Z respectively. With the convention that Z is fixed by $\text{Aut}(X)$, the possible triples (G, P_Y, P_Z) are the following:

- (1) $(B_m, P(\varpi_{m-1}), P(\varpi_m))$ with $m \geq 3$
- (2) $(B_3, P(\varpi_1), P(\varpi_3))$
- (3) $(C_m, P(\varpi_{i+1}), P(\varpi_i))$ with $m \geq 2$ and $i \in \{1, \dots, m-1\}$
- (4) $(F_4, P(\varpi_2), P(\varpi_3))$
- (5) $(G_2, P(\varpi_2), P(\varpi_1))$

We denote by $P(\varpi_i)$ the maximal parabolic subgroup of G corresponding to the dominant weight ϖ_i , with the notations of Bourbaki [Bou68]. We will also denote it P_i for simplicity. We refer to [GPPS22] for more geometric information on these varieties (in particular their dimension and their index).

The varieties from case (3) are also known as the odd symplectic Grassmannians, often denoted $IG(i+1, 2m+1)$. They can be defined as the zero locus of a generic section of the vector bundle $\wedge^2 \mathcal{U}_{i+1}^\vee$ over the Grassmannian $\mathbb{G}(i+1, 2m+1)$ of $(i+1)$ -planes in \mathbb{C}^{2m+1} , where \mathcal{U}_{i+1} denotes the tautological bundle. These interesting varieties, whose automorphism groups are the so-called odd-symplectic groups, have been studied by several authors [Mih07, Pec13, GPPS22].

Apart from those varieties, note that for each group G there is at most one variety in the previous list, that we will call the G -variety. The $Spin_7$ -variety is a generic hyperplane section of the spinorial variety $Spin_{10}/P_5 \subset \mathbb{P}(\Delta)$, where Δ denotes any of the half-spin representations of $Spin_{10}$. Since its dimension is 9 and its index is 7, this follows from Mukai's classification of Fano varieties of coindex three [Muk89]).

Our main result provides geometric models of the remaining varieties, cases (1), (4) and (5) of Theorem 1, and another geometric model for the odd symplectic Grassmannian (case (3)).

In case (2) of Theorem 1, the fundamental weights in the triple do not correspond to consecutive simple roots of the Dynkin diagram, contrary to the other cases. This explains why we do not obtain a geometric model for the $Spin_7$ -variety from our construction.

Theorem 2. *Let X be a smooth projective horospherical G -variety with Picard group \mathbb{Z} , which is not homogeneous, and with $G \neq Spin_7$. Let $q: \tilde{X} \rightarrow X$ be the blow-up of X along the closed orbit Z fixed by $\text{Aut}(X)$. Then \tilde{X} can be realized as the zero locus of a general section of a vector bundle over some homogeneous space.*

More precisely, there exist a fundamental G -module V , and a positive integer k , such that \tilde{X} coincides with the zero locus of a general section s of the vector bundle $\mathcal{E} = \mathcal{Q} \boxtimes \mathcal{U}^$ over $G/P_{k+1} \times \mathbb{G}(k+1, V \oplus \mathbb{C})$, where $\mathbb{G}(l, W)$ denotes the Grassmannian of l -planes in W , \mathcal{Q} is the tautological quotient bundle over $G/P_{k+1} \subset \mathbb{G}(k+1, V)$, and \mathcal{U} the tautological bundle over $\mathbb{G}(k+1, V \oplus \mathbb{C})$.*

Finally, the projection p to G/P_{k+1} realizes \tilde{X} as the projective bundle $\mathbb{P}(\mathbb{C} \oplus \mathcal{V}^)$, if \mathcal{V} denotes the tautological rank $k+1$ bundle over G/P_{k+1} .*

We can illustrate the theorem by the following diagram.

$$\begin{array}{ccccc}
 & & \mathcal{Q} \boxtimes \mathcal{U}^* & & \\
 & & \downarrow \uparrow \text{ } s \text{ general} & & \\
 & & G/P_{k+1} \times \mathbb{G}(k+1, V \oplus \mathbb{C}) & & \mathcal{U}^* \\
 & \swarrow & \downarrow & \searrow & \downarrow \\
 \mathbb{G}(k+1, V) & \hookrightarrow & G/P_{k+1} & & \mathbb{G}(k+1, V \oplus \mathbb{C}) \\
 & & \parallel & & \uparrow \\
 & & G/P_{k+1} & & X \\
 & \swarrow p & \downarrow \tilde{X} & \searrow q & \\
 & & & &
 \end{array}$$

In the second part of the paper, we use these simple geometric models to improve our understanding of the cohomology (or Chow) ring of the horospherical varieties. The Chevalley formulas for those varieties have been obtained in [GPPS22], including the quantum version, which was recently used to prove that Galkin's Conjecture \mathcal{O} does hold for these varieties [BFSS20]. We give a complete treatment of the G_2 -variety in Section 3. We also discuss the cohomology ring of the $Spin_7$ -variety in Section 4 and extend these results to quantum cohomology.

Unfortunately, although the Chow ring of the F_4 -variety could in principle be determined following the same approach, the computational complexity seems too big for such a task to be feasible. Indeed the cohomology ring of a blow-up is in principle accessible, but its effective determination requires a very precise understanding of the cohomology ring of the initial variety, of the subvariety that is blown-up, of the characteristic classes of its normal bundle. Finally, it is a challenging problem to extend this approach to the quantum cohomology ring.

2. PROOF OF THEOREM 2

Let G be a simple algebraic group over \mathbb{C} . Fix a Borel subgroup B of G containing a maximal torus T .

We will show that to almost any horospherical two-orbit G -variety, we can associate a horospherical G -variety of rank one, which is naturally embedded in a homogeneous $G \times GL_{d+1}$ -variety. Then we will prove that this variety is the zero locus of a general section of a vector bundle over this homogeneous space.

2.1. The general construction. Our main construction will involve a certain fundamental G -module V , whose dimension will be denoted by d . Let k be a positive integer, with $k < d$. Denote e_1, \dots, e_{k+1} T -semi-invariant linearly independent vectors of maximal weights in V . We order them starting from the highest weight, in some compatible way with the partial dominance order. In particular $v_k := e_1 \wedge \dots \wedge e_k$ and $v_{k+1} := e_1 \wedge \dots \wedge e_{k+1}$ are B -semi-invariant vectors of $\wedge^k V$ and $\wedge^{k+1} V$, respectively. Denote by V_k (respectively V_{k+1}) the sub- G -module of $\wedge^k V$ (resp. $\wedge^{k+1} V$) generated by v_k (resp. v_{k+1}).

We will suppose that the weights of v_k and v_{k+1} are fundamental weights ϖ_i and ϖ_j of (G, B, T) , in particular $V_k \simeq V(\varpi_i)$ and $V_{k+1} \simeq V(\varpi_j)$. This is a quite restrictive hypothesis, which is sensible only when V is a fundamental module associated to an end of the Dynkin of diagram of G ; in this situation the closed orbit in $\mathbb{P}(V_k)$ can naturally be realized as a subvariety of $\mathbb{G}(k, V)$, as discussed in [LM03] (see in particular Proposition 4.15).

Also denote by α_i and α_j the simple roots and by α_i^\vee and α_j^\vee the simple coroots corresponding respectively to the fundamental weights ϖ_i and ϖ_j . Note that, with Bourbaki's

notation, we will have either $i = j + 1$ or $j = i + 1$ according to the considering cases (1), (3), (4) or (5) of Theorem 1.

Finally, fix a non-zero element e_0 in the trivial G -module \mathbb{C} .

Definition 3. With the notations above, we define \tilde{X} as the G -orbit closure:

$$\tilde{X} := \overline{G \cdot [v_{k+1} \otimes (v_k \wedge (e_{k+1} + e_0))]} \subset \mathbb{P}(V_{k+1} \otimes \bigwedge^{k+1} (V \oplus \mathbb{C})).$$

Denote by P_k and P_{k+1} the maximal parabolic subgroups containing B associated to ϖ_i and ϖ_j respectively. Note that P_k and P_{k+1} are the stabilizers in G of the lines $\mathbb{C}v_k$ and $\mathbb{C}v_{k+1}$, respectively.

Proposition 4. *The variety \tilde{X} is a horospherical G -variety. Moreover, there exists a horospherical G -variety X of Picard group \mathbb{Z} such that \tilde{X} is obtained by blowing-up X along a closed G -orbit.*

To prove the proposition we will use the general theory of horospherical varieties, in particular, the classification in terms of colored fans, the description of divisors and the ampleness criterion. For a survey on this theory, see for example [Pas08] or [Pas17].

Proof. We start by studying the open G -orbit Ω of \tilde{X} more closely. We have $\Omega \simeq G/H$, where $H = \text{Stab}_G[v_{k+1} \otimes (v_k \wedge (e_{k+1} + e_0))]$ is the kernel of $\varpi_j - \varpi_i$ in the parabolic subgroup $P = P_k \cap P_{k+1}$ that stabilizes both $\mathbb{C}v_k$ and $\mathbb{C}v_{k+1}$. In particular Ω is a horospherical homogeneous space of rank 1 with the following spherical data:

- (1) the weight lattice $M = \mathbb{Z}(\varpi_j - \varpi_i) \simeq \mathbb{Z}$;
- (2) the set of colors $\mathcal{D} = \{D_k, D_{k+1}\}$ corresponding to the set of inverse images by $G/H \rightarrow G/P$ of the two Schubert divisors in G/P ;
- (3) the images of D_k and D_{k+1} in $N := \text{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) \simeq \mathbb{Z}$, given respectively by $\alpha_{i|M}^{\vee}$ (i.e. $-1 \in \mathbb{Z}$) and $\alpha_{j|M}^{\vee}$ (i.e. $1 \in \mathbb{Z}$).

According to the Luna-Vust classification of G/H -embeddings in terms of colored fans, there exist four complete G/H -embeddings, obtained by picking or not picking each of the two colors. Each of these embeddings has three G -orbits: Ω and two closed orbits isomorphic to either G/P_k , G/P_{k+1} or G/P (which is a divisor).

The Picard group and the ample divisors of each of these G/H -embeddings can easily be described. In particular, they are all projective and locally factorial, and their Picard number can be 1, 2 or 3. Also, in order to realize these embeddings, we can choose a (small) ample divisor (which is automatically very ample because G/H has rank one [Pas06, Th. 0.3]), and by computing its global sections, we can describe the corresponding embedding into the projective space of the dual G -module of global sections (here we need to suppose that G is simply connected, so that our line bundle can be G -linearized). In another point of view, the projective G/H -embeddings are classified and can be described by moment polytopes, see [Pas15, Section 2.3].

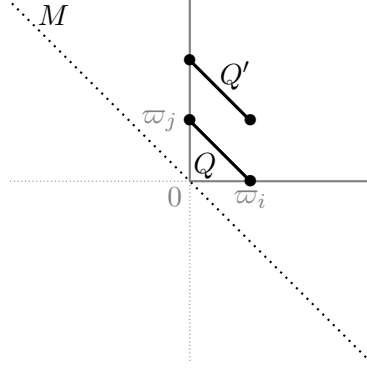
To get the G/H -embedding X obtained by picking the two colors of \mathcal{D} , we choose as (small) ample divisor the divisor D whose corresponding moment polytope is $Q = [\varpi_i, \varpi_j]$, and then

$$X = \overline{G \cdot [v_k + v_{k+1}]} \subset \mathbb{P}(V_k \oplus V_{k+1}),$$

which is of Picard number one; and to get the G/H -embedding X' obtained by picking only the color D_k , we choose D whose corresponding moment polytope is $Q' = [\varpi_i + \varpi_j, 2\varpi_j]$, and then

$$X' = \overline{G \cdot [v_{k+1} \otimes v_k + v_{k+1} \otimes v_{k+1}]} \subset \mathbb{P}((V_{k+1} \otimes V_k) \oplus (V_{k+1} \otimes V_{k+1})),$$

which is of Picard number two. To get the description of X' , we also use that $V(\varpi_i + \varpi_j) \subset V(\varpi_j) \otimes V(\varpi_i) \simeq V_{k+1} \otimes V_k$ and $V(2\varpi_j) \subset V(\varpi_j) \otimes V(\varpi_j) \simeq V_{k+1} \otimes V_{k+1}$. We illustrate the choice of the moment polytope in Figure 1.


 FIGURE 1. Two choices of moment polytopes for G/H

Note that X' can also be obtained by blowing-up X along the closed G -orbit isomorphic to G/P_k . Indeed, blowing-up one of the closed G -orbit in X we obtain another projective G/H -embedding where the closed G -orbit of X has been replaced by a G -stable divisor, which has to be a closed G -orbit isomorphic to G/P . In terms of colored fans, we have deleted a color. Now, observe that

$$(V_{k+1} \otimes V_k) \oplus (V_{k+1} \otimes V_{k+1}) = V_{k+1} \otimes (V_k \oplus V_{k+1}) \subset V_{k+1} \otimes \left(\bigwedge^k V \oplus \bigwedge^{k+1} V \right) \simeq V_{k+1} \otimes \bigwedge^{k+1} (V \oplus \mathbb{C}),$$

and that by this G -equivariant isomorphism,

$$X' \simeq \overline{G \cdot [v_{k+1} \otimes (v_k \wedge (e_{k+1} + e_0))]} \subset \mathbb{P}(V_{k+1} \otimes \bigwedge^{k+1} (V \oplus \mathbb{C})).$$

This exactly means that X' is G -equivariantly isomorphic to \tilde{X} . \square

Remark 5. With the same G -equivariant isomorphism as above, we also have the following G -equivariant embedding

$$X \simeq \overline{G \cdot [v_k \wedge (e_{k+1} + e_0)]} \subset \mathbb{P}(\bigwedge^{k+1} (V \oplus \mathbb{C})).$$

Now, \tilde{X} is by construction a closed subvariety of the homogeneous $G \times \mathrm{GL}_{d+1}$ -space

$$G/P_{k+1} \times \mathbb{G}(k+1, V \oplus \mathbb{C}) \subset \mathbb{P}(V_{k+1}) \times \mathbb{P}(\bigwedge^{k+1} (V \oplus \mathbb{C})) \subset \mathbb{P}(V_{k+1} \otimes \bigwedge^{k+1} (V \oplus \mathbb{C})).$$

Moreover, the immersion $\tilde{X} \rightarrow G/P_{k+1} \times \mathbb{G}(k+1, V \oplus \mathbb{C})$ is G -equivariant so that composing with the projection to the second factor we get a proper G -equivariant morphism $\pi : \tilde{X} \rightarrow \mathbb{G}(k+1, V \oplus \mathbb{C})$. The image of π is the closure of the image of $G \cdot [v_k \wedge (e_{k+1} + e_0)]$ in $\mathbb{P}(\bigwedge^{k+1} (V \oplus \mathbb{C}))$, which is nothing else but X . This means that the projection to $\mathbb{G}(k+1, V \oplus \mathbb{C})$ induces the blow-up $X' \simeq \tilde{X} \rightarrow X$.

Note also that the closed orbit $G \cdot [v_{k+1} \otimes (v_k \wedge e_{k+1})] \simeq G/P_{k+1}$ maps by π to $Y := G \cdot [v_k \wedge e_{k+1}] \simeq G/P_{k+1}$. Moreover $G \cdot [v_{k+1} \otimes (v_k \wedge e_0)] \simeq G/(P_k \cap P_{k+1})$, which is the exceptional divisor of the blow-up, maps by π to $Z := G \cdot [v_k \wedge e_0] \simeq G/P_k$.

2.2. The variety \tilde{X} as the zero locus of a general section. We begin by the following description of \tilde{X} . Recall that G/P_{k+1} is embedded in $\mathbb{G}(k+1, V)$.

Proposition 6. *Let $(A, B) \in G/P_{k+1} \times \mathbb{G}(k+1, V \oplus \mathbb{C})$, in particular A and B are subspaces of dimension $k+1$ of V and $V \oplus \mathbb{C}$, respectively. Then (A, B) is a point of \tilde{X} if and only if the projection of B to V is contained in A , or equivalently $B \subset A \oplus \mathbb{C}$.*

Proof. The points of the G -orbit $G \cdot [v_{k+1} \otimes (v_k \wedge (e_{k+1} + e_0))]$ are of the form $[(x_1 \wedge \cdots \wedge x_{k+1}) \otimes (x_1 \wedge \cdots \wedge (x_{k+1} + e_0))]$, and then correspond to a pair $(A, B) \in G/P_{k+1} \times \mathbb{G}(k+1, V \oplus \mathbb{C})$ where A is generated by x_1, \dots, x_{k+1} , and B is generated by x_1, \dots, x_k and $x_{k+1} + e_0$. In particular, the projection of B in V is contained in A . Since this is a closed condition, it remains true on the closure of $G \cdot [v_{k+1} \otimes (v_k \wedge (e_{k+1} + e_0))]$.

Conversely, if the projection of B in V is contained in A , there exists a basis (x_1, \dots, x_{k+1}) of A such that $(x_1, \dots, x_k, x_{k+1} + e_0)$ or (x_1, \dots, x_{k+1}) or (x_1, \dots, x_k, e_0) is a basis of B . In the first case, (A, B) is a point of the open G -orbit $G \cdot [v_{k+1} \otimes (v_k \wedge (e_{k+1} + e_0))]$. In the two other cases, (A, B) is a point in one of the two closed G -orbits $G \cdot [v_{k+1} \otimes v_{k+1}]$ and $G \cdot [v_{k+1} \otimes (v_k \wedge e_0)]$. \square

We will deduce the following proposition, which completes the proof of Theorem 2. Since G/P_{k+1} is embedded in the Grassmannian $\mathbb{G}(k+1, V)$, it admits a vector bundle \mathcal{Q} obtained by restricting the tautological quotient vector bundle, of rank $d - k - 1$. We also denote by \mathcal{U} the tautological vector bundle, of rank $k + 1$, over $\mathbb{G}(k+1, V \oplus \mathbb{C})$.

Proposition 7. *The variety \tilde{X} is the zero locus of a general section of the vector bundle $\mathcal{E} = \mathcal{Q} \boxtimes \mathcal{U}^*$ over $G/P_{k+1} \times \mathbb{G}(k+1, V \oplus \mathbb{C})$.*

We first prove the following statement.

Lemma 8. *The space of global sections of \mathcal{E} contains $V \otimes (V \oplus \mathbb{C})^*$.*

Proof of the lemma. By the Borel-Weil theorem on the Grassmannian,

$$H^0(\mathbb{G}(k+1, V \oplus \mathbb{C}), \mathcal{U}^*) = (V \oplus \mathbb{C})^*,$$

so it suffices to prove that $H^0(G/P_{k+1}, \mathcal{Q})$ contains V . But by its very definition, \mathcal{Q} is a quotient of the trivial bundle with fiber V , so there is a non trivial equivariant map $V \rightarrow H^0(G/P_{k+1}, \mathcal{Q})$. Since V is irreducible, it must be injective. \square

We can then conclude with the following general statement. Suppose given a globally generated rank r vector bundle \mathcal{F} over some variety Z , and denote by F its space of global sections, of dimension d . Consider a vector space W of dimension $w \geq d$, some positive integer $\ell < w$, and denote by \mathcal{U} the tautological rank ℓ bundle on the Grassmannian $\mathbb{G}(\ell, W)$. The space of global sections of $\mathcal{F} \boxtimes \mathcal{U}^*$ on $Z \times \mathbb{G}(\ell, W)$ is then

$$H^0(Z \times \mathbb{G}(\ell, W), \mathcal{F} \boxtimes \mathcal{U}^*) = F \otimes W^* \simeq \text{Hom}(W, F).$$

Since \mathcal{F} is globally generated, there is an exact sequence

$$0 \longrightarrow \mathcal{M} \longrightarrow F \otimes \mathcal{O}_Z \longrightarrow \mathcal{F} \longrightarrow 0,$$

where \mathcal{M} is a vector bundle of rank $d - r$.

Proposition 9. *Let s be a section of $\mathcal{F} \boxtimes \mathcal{U}^*$, defined by $\sigma \in \text{Hom}(W, F)$. Then:*

(1) *the zero locus $Z(s) \subset Z \times \mathbb{G}(\ell, W)$ is the set of pairs (x, U) such that*

$$\sigma(U) \subset \mathcal{M}_x \subset F;$$

(2) *if σ is surjective, $Z(s)$ is smooth and has the structure of a $\mathbb{G}(\ell, d - r)$ -bundle over Z . More precisely, if $K \subset W$ denotes the kernel of σ ,*

$$Z(s) \simeq \mathbb{G}(\ell, K \otimes \mathcal{O}_Z \oplus \mathcal{M}).$$

Proof. The first claim is clear, since the evaluation of s at (x, U) is the vector in $\text{Hom}(U, \mathcal{F}_x)$ obtained by restricting σ to $U \subset W$ and projecting from F to its quotient \mathcal{F}_x .

The condition can be rewritten $U \subset \sigma^{-1}(\mathcal{M}_x)$. When σ is surjective, $\sigma^{-1}(\mathcal{M}_x)$ has constant dimension $e - r$. This implies that the projection to Z is locally trivial, and the second claim follows since $\sigma^{-1}(\mathcal{M}_x)$ can be identified with $K \oplus \mathcal{M}_x$. \square

Conclusion of the proof of Proposition 7. Apply Proposition 9 (1) to the canonical section of \mathcal{E} given by $\sigma \in \text{Hom}(V \oplus \mathbb{C}, V)$, the projection to V . By Proposition 6, its zero locus coincides with \tilde{X} . \square

By Proposition 9 (2), the projection of \tilde{X} to G/P_{k+1} is the fiber-bundle $\mathbb{G}(k+1, \mathbb{C} \oplus \mathcal{V})$, if \mathcal{V} denotes the tautological bundle, restricted from $\mathbb{G}(k+1, V)$. Since $\mathbb{C} \oplus \mathcal{V}$ has rank $k+2$, this is just the hyperplane bundle $\mathbb{P}(\mathbb{C} \oplus \mathcal{V}^*)$, of relative dimension $k+1$. This concludes the proof of the main Theorem.

Remark 10. The stabilizer of σ in $G \times GL_{d+1}$ is $(G \times \mathbb{C}^*) \times V$. Indeed, $(g_1, g_2) \in G \times GL_{d+1}$ acts on $\text{Hom}(V \oplus \mathbb{C}, V) \simeq M_{d,d+1}(\mathbb{C})$ by $(g_1, g_2) \cdot M = g_1 M g_2^{-1}$; then $(g_1, g_2) \cdot (I_d, 0) = (I_d, 0)$ if and only if

$$g_2 = \begin{pmatrix} g_1 & 0 \\ {}^t x & y \end{pmatrix},$$

with $x \in V$ and $y \in \mathbb{C}^*$. As a consequence, the $G \times GL_{d+1}$ -orbit of σ is always dense, and the zero-loci of the corresponding sections are all isomorphic to \tilde{X} .

2.3. More details on the module V . Here discuss what k and the module V are for each case of Theorem 1 (except the $Spin_7$ -variety).

- **Case 1.** $(Spin_{2m+1}, P(\varpi_{m-1}), P(\varpi_m))$ with $m \geq 3$.

The G -module V is the spinorial representation, of dimension $d = 2^m$, and $k = 1$. The T -weights of V are the $\frac{1}{2}(\pm\epsilon_1 + \dots + \pm\epsilon_m)$, all with multiplicity one. Let e_1 be a T -semi-invariant vector of weight $\varpi_m = \frac{1}{2}(\epsilon_1 + \dots + \epsilon_m)$ and let e_2 be a T -semi-invariant vector of weight $\varpi_m - \alpha_m = \frac{1}{2}(\epsilon_1 + \dots + \epsilon_{m-1} - \epsilon_m)$. Then $e_1 \wedge e_2$ is of weight $\varpi_{m-1} = \epsilon_1 + \dots + \epsilon_{m-1}$.

- **Case 3.** $(Sp_{2m}, P(\varpi_{k+1}), P(\varpi_k))$ with $m \geq 2$.

The G -module V is the minimal representation, of dimension $d = 2m$, and k is any positive integer smaller than m . The T -weights of V are the $\pm\epsilon_h$ with $1 \leq h \leq m$, all of multiplicity one. For any $h \in \{1, \dots, k+1\}$, let e_h be a T -semi-invariant vector of weight $\varpi_1 - \alpha_1 - \dots - \alpha_{h-1} = \epsilon_h$. Then $e_1 \wedge \dots \wedge e_k$ and $e_1 \wedge \dots \wedge e_{k+1}$ are of weights $\varpi_k = \epsilon_1 + \dots + \epsilon_k$ and $\varpi_{k+1} = \epsilon_1 + \dots + \epsilon_{k+1}$ respectively.

- **Case 4.** $(F_4, P(\varpi_2), P(\varpi_3))$.

The G -module V is the minimal representation, of dimension $d = 26$, and $k = 2$. The highest weight of V is $\varpi_4 = \epsilon_1$. Consider T -semi-invariant vectors e_1 of weight ϖ_4 , e_2 of weight $\varpi_4 - \alpha_4 = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4)$, and e_3 of weight $\varpi_4 - \alpha_4 - \alpha_3 = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4)$. Then $e_1 \wedge e_2$ is of weight $\varpi_3 = \frac{1}{2}(3\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4)$, while $e_1 \wedge e_2 \wedge e_3$ is of weight $\varpi_2 = 2\epsilon_1 + \epsilon_2 + \epsilon_3$.

- **Case 5.** $(G_2, P(\varpi_2), P(\varpi_1))$.

The G -module V is the minimal representation, of dimension $d = 7$, and $k = 1$. The highest weight of V is ϖ_1 . Consider T -semi-invariant vectors e_1 of weight ϖ_1 , and e_2 of weight $\varpi_1 - \alpha_1$. Then $e_1 \wedge e_2$ is of weight $2\varpi_1 - \alpha_1 = \varpi_2$.

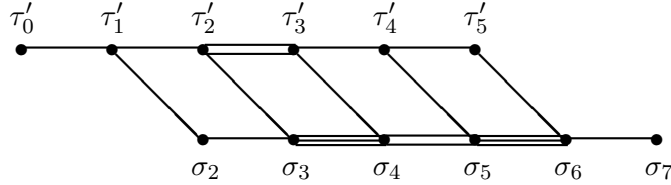
3. COHOMOLOGY OF THE G_2 -VARIETY

3.1. The Hasse diagram. Recall that the Chow ring of the horospherical variety X of type G_2 has two natural basis, made of classes coming from the two closed G_2 -orbits [GPPS22]. The latter are $Z = G_2/P_1$ isomorphic the 5-dimensional quadric \mathbb{Q}^5 , the only closed $\text{Aut}(X)$ -orbit, and the adjoint variety $Y = G_2/P_2$, also of dimension 5. Both are homologically rational projective homogeneous spaces, in the sense that their Hodge

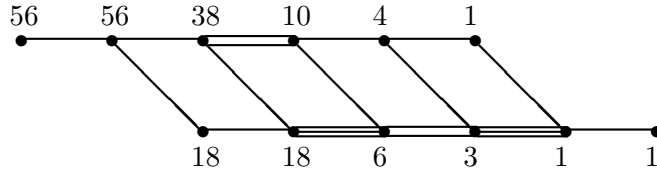
numbers are the same as those of \mathbb{P}^5 . We deduce that the Chow ring of X is free of rank $6 + 6 = 12$.

Let us choose the basis (τ'_i, σ_j) , made of classes indexed by their degrees, where the τ'_i are induced from $Z = G_2/P_1$ in degree 0 to 5, while the σ_j are induced from $Y = G_2/P_2$ in degree 2 to 7. More precisely, the τ'_i are the classes of the closures in X of \mathbb{C}^2 -bundles over Schubert varieties of G_2/P_1 (locally X is a \mathbb{C}^2 -bundle over G_2/P_1), and the σ_j are the classes of the Schubert varieties of G_2/P_2 .

From Proposition 1.14 of [GPPS22] we deduce that the Hasse diagram of X is the following:



Recall that the edges in this diagram encode the multiplication by the hyperplane class h (which is nothing else but τ'_1). For example $h\tau'_2 = 2\tau'_3 + \sigma_3$. In particular we can readily deduce the degrees of the classes (τ'_i, σ_j) , which are given in the following diagram:



3.2. Fundamental class. According to Theorem 2, the blow-up \tilde{X} of X along its closed orbit Z is the zero-locus of a general section of the vector bundle $\mathcal{Q} \boxtimes \mathcal{U}^*$ over the product variety $G_2/P_2 \times \mathbb{G}$, where $\mathbb{G} := \mathbb{G}(2, V_7 \oplus \mathbb{C})$. The Thom-Porteous formula implies that its fundamental class

$$[\tilde{X}] = c_{10}(\mathcal{Q} \boxtimes \mathcal{U}^*) \in A^{10}(G_2/P_2 \times \mathbb{G}).$$

We can easily deduce the fundamental class of $X \subset \mathbb{G}$.

Notation. Recall that the Chow ring of the Grassmannian $G(2, d+2)$ has an additive basis given by the Schubert classes σ_{ij} , for $0 \leq j \leq i \leq d$. In order to get generators of the Chow ring, it is enough to consider the special Schubert classes $\sigma_k = \sigma_{k0}$, which can be defined as the Chern classes of the tautological quotient vector bundle (see e.g. [Man01, Proposition 3.5.5]).

In the sequel, in the hope of minimizing confusions we will denote by $\bar{\sigma}_{ij}$ the Schubert classes on \mathbb{G} , for $0 \leq j \leq i \leq 6$, and by $\bar{\sigma}_i = \bar{\sigma}_{i0}$ the special Schubert classes. We will denote by $\bar{\tau}_{k\ell}$, for $0 \leq \ell \leq k \leq 5$, the Schubert classes on $\mathbb{G}(2, V_7)$, and by $\bar{\tau}_i = \bar{\tau}_{i0}$ the special Schubert classes. In order to keep notations reasonably simple we will use the same notations for the restrictions of the Schubert classes $\bar{\tau}_{k\ell}$ to G_2/P_2 .

Lemma 11. *The fundamental class of $X \subset \mathbb{G}$ is*

$$[X] = 2\bar{\sigma}_{41} + 2\bar{\sigma}_{32} \in A^5(\mathbb{G}).$$

Proof. Decompose $[\tilde{X}] = \sum_k \sum_i \alpha_k^i \otimes \beta_{10-k}^i$, with $\alpha_k^i \in A^k(G_2/P_2)$ and $\beta_\ell^j \in A^\ell(\mathbb{G})$. Since the projection map p from \tilde{X} to \mathbb{G} is birational on its image X , we deduce that

$$[X] = p_*[\tilde{X}] = \sum_i (p_*\alpha_5^i)\beta_5^i.$$

In order to compute this, let us denote by x_1, \dots, x_5 the Chern roots of \mathcal{Q} , and by y_1, y_2 the Chern roots of \mathcal{U}^* . Recall that the m -th elementary symmetric function of x_1, \dots, x_5 is the m -th Chern class of \mathcal{Q} , which is nothing else but the Schubert class $\bar{\tau}_m$ of $\mathbb{G}(2, V_7)$, restricted to G_2/P_2 . We get

$$c_{10}(\mathcal{Q} \boxtimes \mathcal{U}^*) = \prod_{i,j} (y_i + x_j) = \prod_{i=1}^2 (y_i^5 + y_i^4 \bar{\tau}_1 + y_i^3 \bar{\tau}_2 + y_i^2 \bar{\tau}_3 + y_i \bar{\tau}_4 + \bar{\tau}_5).$$

The part of bidegree $(5, 5)$ is

$$[\tilde{X}]_{5,5} = (y_1^5 + y_2^5) \bar{\tau}_5 + (y_1^4 y_2 + y_1 y_2^4) \bar{\tau}_4 \bar{\tau}_1 + (y_1^3 y_2^2 + y_1^2 y_2^3) \bar{\tau}_3 \bar{\tau}_2.$$

In order to project this, we need to evaluate the classes $\bar{\tau}_5$, $\bar{\tau}_4 \bar{\tau}_1$ and $\bar{\tau}_3 \bar{\tau}_2$ on G_2/P_2 . For this we need to recall that $G_2/P_2 \subset \mathbb{G}(2, V_7)$ is the zero locus of a general section of the vector bundle $\mathcal{Q}^*(1)$. In particular, its fundamental class is

$$[G_2/P_2] = c_5(\mathcal{Q}^*(1)) = 2\bar{\tau}_{41} + 2\bar{\tau}_{32} \in A^5(\mathbb{G}(2, V_7)).$$

For any class α restricted from the Grassmannian, it is then straightforward to compute

$$\int_{G_2/P_2} \alpha = \int_{\mathbb{G}(2, V_7)} (2\bar{\tau}_{41} + 2\bar{\tau}_{32}) \alpha.$$

In particular, we get the following evaluations:

$$\int_{G_2/P_2} \bar{\tau}_5 = 0, \quad \int_{G_2/P_2} \bar{\tau}_4 \bar{\tau}_1 = 2, \quad \int_{G_2/P_2} \bar{\tau}_3 \bar{\tau}_2 = 4.$$

Plugging in our formula for the fundamental class of X , we finally get

$$[X] = 2(y_1^4 y_2 + y_1 y_2^4) + 4(y_1^3 y_2^2 + y_1^2 y_2^3) = 2\bar{\sigma}_2(\bar{\sigma}_1^3 - \bar{\sigma}_1 \bar{\sigma}_2) = 2\bar{\sigma}_{41} + 2\bar{\sigma}_{32}. \quad \square$$

3.3. Generators and relations. Our next ingredient in order to compute the intersection product on X is

Lemma 12. *The restriction map $A^*(\mathbb{G})_{\mathbb{Q}} \rightarrow A^*(X)_{\mathbb{Q}}$ is surjective. And τ'_1 and σ_2 are the restrictions of the Schubert classes $\bar{\sigma}_1$ and $\bar{\sigma}_{11}$ of \mathbb{G} , respectively.*

Proof. The ring $A^*(X)_{\mathbb{Q}}$ is generated by τ'_1 and σ_2 . It is therefore enough to prove that these two classes are the restrictions of the Schubert classes $\bar{\sigma}_1$ and $\bar{\sigma}_{11}$ of \mathbb{G} , respectively. Remark that τ'_1 is the hyperplane class in X , while $\bar{\sigma}_1$ is the hyperplane class of \mathbb{G} , so the claim is obvious for τ'_1 .

Recall that σ_2 is the class of the closed G_2 -orbit Y of X , which is isomorphic to G_2/P_2 . In a neighborhood of Y , more precisely on $X \setminus Z$, X is a vector bundle of rank two over Y . More precisely, the map $X \setminus Z \rightarrow Y$ is given by the restriction of the projection map $\mathbb{G} = \mathbb{G}(2, V_7 \oplus \mathbb{C}) \dashrightarrow \mathbb{G}(2, V_7)$. This projection is defined on $\mathbb{G} \setminus \{W \in \mathbb{G} \mid \mathbb{C} \subset W\}$, which is a neighborhood of the Schubert variety $\mathbb{G}_{11} := \{W \in \mathbb{G} \mid W \subset V_7\}$; and it defines a vector bundle of rank two over $\mathbb{G}(2, V_7)$. It is now clear that the restriction of $\bar{\sigma}_{11}$ (the class of \mathbb{G}_{11}) is σ_2 . \square

Remark 13. The ring $A^*(X)_{\mathbb{Q}}$ is also generated by τ'_1 and τ'_2 . We could also prove directly that τ'_2 is the restriction of the Schubert class $\bar{\sigma}_2$, the class of $\{W \in \mathbb{G} \mid W \cap V_5 \neq \{0\}\}$, where V_5 is a 5-dimensional subspace of V_7 . Indeed, one can check that this Schubert variety intersects X transversely at general points, and that the intersection is the subvariety of X that defines τ'_2 .

In order to simplify the notations we will denote by h, σ our two generators τ'_1, σ_2 of the Chow ring of X . Given the Betti numbers of X , we deduce that the rational Chow ring

$$A^*(X)_{\mathbb{Q}} = \mathbb{Q}[h, \sigma] / \langle R_4, R_6 \rangle$$

for two relations R_4 of degree four and R_6 of degree six.

Proposition 14. *We can choose the relations to be*

$$R_4 = 3\sigma^2 - h^2\sigma \quad \text{and} \quad R_6 = 28h^4\sigma - 9h^6.$$

Proof. The fact that $R_4 = 0$ in $A^*(X)$ follows from the observation that in the Chow ring of $\mathbb{G}(2, V_7)$, we have $[X]\bar{\sigma}_{22} = 2\bar{\sigma}_{54}$, while $[X]\bar{\sigma}_{31} = 4\bar{\sigma}_{54}$. Therefore the class $2\bar{\sigma}_{22} - \bar{\sigma}_{31} = 3\bar{\sigma}_{11}^2 - \bar{\sigma}_1^2\bar{\sigma}_{11}$ restricts to zero on X , and this restriction is $3\sigma^2 - h^2\sigma$.

The fact that $R_6 = 0$ is even easier. Indeed $A^6(X)$ has rank one, so the classes h^6 and $h^4\sigma$ must be proportional. Since the degree of the latter is 18, while the degree of the former is 56, the claim follows immediately. \square

3.4. The multiplication table. Note that the information encoded in the Hasse diagram is already sufficient to express all the classes of X in terms of h and σ . We get:

$$\begin{aligned} \tau'_2 &= h^2 - \sigma, & \tau'_3 &= \frac{h^3}{2} - h\sigma, & \tau'_4 &= \frac{h^4}{2} - \frac{4}{3}h^2\sigma, & \tau'_5 &= \frac{h^5}{2} - \frac{3}{2}h^3\sigma, \\ \sigma_3 &= h\sigma, & \sigma_4 &= \frac{h^2\sigma}{3}, & \sigma_5 &= \frac{h^3\sigma}{6}, & \sigma_6 &= \frac{h^6}{56}, & \sigma_7 &= \frac{h^7}{56}. \end{aligned}$$

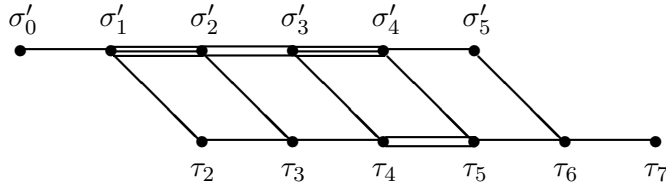
Using the relations of Proposition 14, the multiplication table is then easily obtained:

$$\begin{aligned} (\tau'_2)^2 &= 2\tau'_4 + 3\sigma_4, & \tau'_2\sigma_2 &= 2\sigma_4, & \sigma_2^2 &= \sigma_4, \\ \tau'_3\sigma_2 &= \sigma_5, & \sigma_3\sigma_2 &= 2\sigma_5, & \tau'_3\tau'_2 &= \tau'_5 + 2\sigma_5, & \sigma_3\tau'_2 &= 4\sigma_5, \\ \tau'_4\sigma_2 &= \sigma_6, & \sigma_4\sigma_2 &= 2\sigma_6, & \tau'_4\tau'_2 &= 3\sigma_6, & \sigma_4\tau'_2 &= 4\sigma_6, \\ \tau'_5\sigma_2 &= 0, & \sigma_5\sigma_2 &= \sigma_7, & \tau'_5\tau'_2 &= \sigma_7, & \sigma_5\tau'_2 &= 2\sigma_7, \\ (\tau'_3)^2 &= 2\tau'_6, & \tau'_3\sigma_3 &= 3\sigma_6, & \sigma_3^2 &= 6\sigma_6, \\ \tau'_4\sigma_3 &= \sigma_7, & \sigma_4\sigma_3 &= 2\sigma_7, & \tau'_4\tau'_3 &= \sigma_7, & \sigma_4\tau'_3 &= \sigma_7. \end{aligned}$$

For completeness, we can also compute the Poincaré dual basis (which is not, as in classical Schubert calculus, a permutation of the original basis). Following the notations of [GPPS22, Proposition 1.10], we let $\sigma'_i = \sigma_{7-i}^\vee$ and $\tau_j = (\tau'_{7-j})^\vee$. Then:

$$\begin{aligned} \tau_6 &= \sigma_6, & \tau_5 &= \tau'_5, & \sigma'_5 &= \sigma_5 - 2\tau'_5, & \tau_4 &= 2\tau'_4 - \sigma_4, & \sigma'_4 &= \sigma_4 - \tau'_4, \\ \tau_3 &= 2\tau'_3 - \sigma_3, & \sigma'_3 &= \sigma_3 - \tau'_3, & \tau_2 &= \tau'_2 - 2\sigma_2, & \sigma'_2 &= \sigma_2, & \sigma'_1 &= \tau'_1. \end{aligned}$$

This is the other natural basis of the Chow ring of X , in terms of which the Hasse diagram becomes, in agreement with [GPPS22, Proposition 4.6]:



In this basis, the multiplication table is the following:

$$\begin{aligned} (\sigma'_2)^2 &= 2\sigma'_4, & \sigma'_2\tau_2 &= 0, & \tau_2^2 &= \tau_4, \\ \sigma'_3\tau_2 &= -\tau_5, & \tau_3\tau_2 &= 2\tau_5, & \sigma'_3\sigma'_2 &= \sigma'_5 + 2\tau_5, & \tau_3\sigma'_2 &= 0, \\ \sigma'_4\tau_2 &= -\tau_6, & \tau_4\tau_2 &= 2\tau_6, & \sigma'_4\sigma'_2 &= \tau_6, & \tau_4\sigma'_2 &= 0, \\ \sigma'_5\tau_2 &= -2\tau_7, & \tau_5\tau_2 &= \tau_7, & \sigma'_5\sigma'_2 &= \tau_7, & \tau_5\sigma'_2 &= 0, \\ (\sigma'_3)^2 &= 2\tau_6, & \sigma'_3\tau_3 &= -\tau_6, & \tau_3^2 &= 2\tau_6, \\ \sigma'_4\tau_3 &= -\tau_7, & \tau_4\tau_3 &= 2\tau_7, & \sigma'_4\sigma'_3 &= \tau_7, & \tau_4\sigma'_3 &= -\tau_7. \end{aligned}$$

Remarks.

- (1) One important difference between the two multiplication tables is that the second one has some negative signs, while the first one has none. This is due to the fact that τ_2 is the class of $G_2/P_1 \subset X$, which is the closed $\text{Aut}(X)$ -orbit Z in X and is therefore not movable. On the contrary, σ is the class of a restricted Schubert cycle, and is therefore movable in X .
- (2) The degree four relation R_4 can be expressed as $\sigma_2\tau_2 = 0$, and obviously follows from the fact that the two closed G -orbits of X do not meet.
- (3) More generally, and for any horospherical variety X with Picard number one, we have inside $A^*(X)$ two subalgebras A_1 (here generated by the σ_i 's) and A_2 (here generated by the τ_j 's) such that

$$hA_1 \subset A_1, \quad hA_2 \subset A_2, \quad A_1A_2 = 0.$$

Formally we can even decompose $A^*(X) = A_1 \oplus A_2^\vee = A_2 \oplus A_1^\vee$ (where A_i^\vee is the submodule generated by the Poincaré duals of the Schubert classes in A_i). Then

$$A_1^\vee A_2 \subset A_2 \quad \text{and} \quad A_2^\vee A_1 \subset A_1.$$

If we add Poincaré duality and the Chevalley formula, do we get enough information to determine $A^*(X)$?

3.5. Quantum cohomology. Recall that X has index four, so that the quantum parameter in its quantum cohomology ring $QA^*(X) = A^*(X)_\mathbb{Q}[q]$ has degree four. By the general results of Siebert and Tian [ST97], this quantum cohomology ring admits a presentation of the form

$$QA^*(X)_\mathbb{Q} = \mathbb{Q}[h, \sigma, q] / \langle R_4(q), R_6(q) \rangle$$

for two relations $R_4(q)$ of degree four and $R_6(q)$ of degree six, which are q -deformations of R_4 and R_6 . In particular we can write

$$R_4(q) = R_4 + r_4q \quad \text{and} \quad R_6(q) = R_6 + r_6q,$$

where r_4 has degree zero (a rational number) and r_6 is a class of degree two. Note that since there is no term of degree bigger than one in q , these relations are determined by degree one Gromov-Witten invariants only.

The quantum Chevalley formula has been computed in Proposition 4.6 of [GPPS22], in terms of our Poincaré dual basis. In the basis (σ_i, τ'_j) , this reads

$$\begin{aligned} h * h &= \sigma_2 + \tau'_2, & \tau'_2 * h &= 2\tau'_3 + \sigma_3, & \sigma_2 * h &= \sigma_3, \\ \tau'_3 * h &= \tau'_4 + \sigma_4 + q, & \sigma_3 * h &= 3\sigma_4 + q, \\ \tau'_4 * h &= \tau'_5 + \sigma_5 + qh, & \sigma_4 * h &= 2\sigma_5 + qh, \\ \tau'_5 * h &= \sigma_6 + q\sigma_2, & \sigma_5 * h &= 3\sigma_6 + q\tau'_2, \\ \sigma_6 * h &= \sigma_7 + q\tau'_3, & \sigma_7 * h &= q\tau'_4 + 2q^2. \end{aligned}$$

One immediately deduces the quantum Giambelli formulas (that is, how to determine each Schubert class in terms of the generators), and also the relation $R_6(q)$. Indeed, we get in quantum cohomology

$$\tau'_2 * h^4 = 18\tau'_6 + q(10\sigma_2 + 4\tau'_2), \quad h^6 = 56\tau'_6 + 16q(2\sigma_2 + \tau'_2),$$

and therefore $R_6(q) = R_6 + 8q(h^2 + 3\sigma)$. So the only ingredient missing is the computation of σ^2 in quantum cohomology, which is given by

$$\sigma^2 = \sigma_4 + I_1(\sigma, \sigma, \sigma_7)q.$$

Lemma 15. *The Gromov-Witten invariant $I_1(\sigma, \sigma, \sigma_7) = 0$.*

Proof. Since the class τ is movable, the Gromov-Witten invariant $I_1(\sigma, \sigma, \sigma_7)$ is enumerative [GPPS22, Section 3.2]: it counts the number of lines ℓ in X that pass through a general point x (representing a plane P_x in $V_7 \oplus \mathbb{C}$), and meet general Schubert cycles of class σ_{11} , that is, two general sub-Grassmannians $\mathbb{G}(2, A_7)$ and $\mathbb{G}(2, B_7)$, for A_7, B_7 two general hyperplanes of $V_7 \oplus \mathbb{C}$. But a projective line ℓ in $\mathbb{G}(2, V_7 \oplus \mathbb{C})$ is made of planes containing a common line L_1 (and contained in a common three dimensional space L_3). We would thus get the inclusion $L_1 \subset P_x \cap A_7 \cap B_7 = 0$, a contradiction. \square

Since there is no other quantum correction, we deduce:

Proposition 16. *The quantum cohomology ring of X is*

$$QA^*(X) = \mathbb{Q}[h, \sigma, q] / \langle 3\sigma^2 - h^2\sigma + q, 28h^4\sigma - 9h^6 + 8q(h^2 + 3\sigma) \rangle.$$

\square

Using the quantum Giambelli formulas

$$\begin{aligned} \tau'_2 &= h^2 - \sigma, & \tau'_3 &= \frac{h^3}{2} - h\sigma, & \tau'_4 &= \frac{h^4}{2} - \frac{4}{3}h^2\sigma - \frac{2}{3}q, & \tau'_5 &= \frac{h^5}{2} - \frac{3}{2}h^3\sigma - qh, \\ \sigma_3 &= h\sigma, & \sigma_4 &= \frac{h^2\sigma}{3} - \frac{1}{3}q, & \sigma_5 &= \frac{h^3\sigma}{6} - \frac{2}{3}qh, \\ \sigma_6 &= \frac{h^6}{56} + \frac{2}{7}q\sigma - \frac{4}{7}qh^2, & \sigma_7 &= \frac{h^7}{56} + \frac{9}{7}qh\sigma - \frac{15}{14}qh^3, \end{aligned}$$

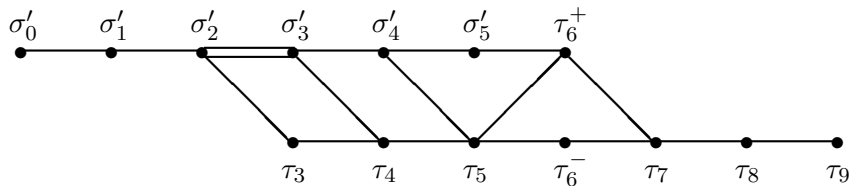
it would then be easy to deduce the quantum multiplication table in our basis (or the dual one). Let us just mention that the quantum multiplication by σ is given by the following formulas:

$$\begin{aligned} \tau'_2\sigma &= 2\sigma_4 + q, & \sigma_2\sigma &= \sigma_4, & \tau'_3\sigma &= \sigma_5 + qh, & \sigma_3\sigma &= 2\sigma_5, \\ \tau'_4\sigma &= \sigma_6 + q\tau'_2, & \sigma_4\sigma &= 2\sigma_6 + q\tau'_2, & \tau'_5\sigma &= q\tau'_3, & \sigma_5\sigma &= \sigma_7 + 2q\tau'_3, \\ \sigma_6\sigma &= q\tau'_4 + q^2, & \sigma_7\sigma &= q\tau'_5 + q^2h. \end{aligned}$$

We could also check the generic semi-simplicity of $QA^*(X)$, which in [GPPS22] was directly deduced from the quantum Chevalley formula.

4. COHOMOLOGY OF THE $Spin_7$ -VARIETY

Since Theorem 2 does not give a geometric model, we have to use another method to compute the cohomology of the $Spin_7$ -variety X , by using the fact that X is a generic hyperplane section of the spinor variety \mathbb{S} for $Spin_{10}$, its cohomology is easily described. First observe that the two closed orbits in X are quadrics \mathbb{Q}^5 and \mathbb{Q}^6 of dimensions 5 and 6, so that the topological Euler number is $6 + 8 = 14$. If we use the Schubert basis (σ'_i, τ_j) , where the classes τ_j are induced from the closed orbit \mathbb{Q}^6 , we get the following Hasse diagram:



We deduce that the Chow ring is generated by the hyperplane class $h = \sigma'_1$ and class $\tau = \tau_3$. Moreover, from the Chevalley formula we get

$$\begin{aligned} \sigma'_2 = h^2, \quad \sigma'_3 = \frac{h^3 - \tau}{2}, \quad \tau_4 = h\tau, \quad \sigma'_4 = \frac{h^4 - 3h\tau}{2}, \quad \tau_5 = h^2\tau, \quad \sigma'_5 = \frac{h^5 - 5h^2\tau}{2}, \\ \tau_6^+ = \frac{h^6 - 5h^3\tau}{2}, \quad \tau_6^- = \frac{7h^3\tau - h^6}{2}, \quad \tau_7 = \frac{h^7}{12}, \quad \tau_8 = \frac{h^8}{12}, \quad \tau_9 = \frac{h^9}{12}. \end{aligned}$$

There must be two relations between the generators, in degrees six and seven. For the latter (which is not uniquely defined) we can choose $R_7 = 6h^4\tau - h^7$. In order to find the former we use the fact that the restriction map from the Chow ring of \mathbb{S} to the Chow ring of X is surjective in degree three, by the Lefschetz hyperplane theorem. The Schubert classes in $A^*(\mathbb{S})$ will be denoted γ_λ , for λ a strict partition with parts smaller than five. The Schubert classes γ_3 and γ_{21} have projective degree seven and five, respectively, and we deduce that τ is just the restriction of the difference $\gamma_3 - \gamma_{21}$. Since $A^6(\mathbb{S})$ has rank two there must exist a linear relation between γ_3^2 , $\gamma_{21}\gamma_3$ and γ_{21}^2 . By applying the Pieri formulas for the spinor variety we get $\gamma_3^2 = 2\gamma_{21}^2$. By restricting to X we deduce the relation we are looking for, namely

$$R_6 = \tau^2 - 6h^3\tau + h^6.$$

(Note that it follows that $h\tau^2 = 0$.)

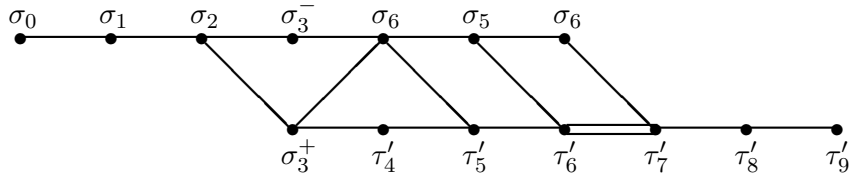
This provides enough information to write down the multiplication table. The multiplication by τ is given by

$$\begin{aligned} \sigma'_1\tau = \tau_4, \quad \sigma'_2\tau = \tau_5, \quad \sigma'_3\tau = \tau_6^+, \quad \tau_3\tau = \tau_6^- - \tau_6^+, \quad \tau_4\tau = 0, \\ \sigma'_4\tau = \tau_7, \quad \tau_5\tau = 0, \quad \sigma'_5\tau = \tau_8, \quad \tau_6^+\tau = \tau_9, \quad \tau_6^-\tau = -\tau_9. \end{aligned}$$

And the missing products are the following:

$$\begin{aligned} (\sigma'_3)^2 = \tau_6^+ + \tau_6^-, \quad \sigma'_3\sigma'_4 = \tau_7, \quad \sigma'_3\tau_4 = \tau_7, \quad \sigma'_3\sigma'_5 = 0, \quad \sigma'_3\tau_5 = \tau_8, \\ \sigma'_3\tau_6^+ = 0, \quad \sigma'_3\tau_6^- = \tau_9, \quad (\sigma'_4)^2 = 0, \quad \sigma'_4\tau_4 = \tau_8, \quad (\tau_4)^2 = \tau_8, \\ \sigma'_4\sigma'_5 = -\tau_9, \quad \sigma'_4\tau_5 = \tau_9, \quad \tau_4\sigma'_5 = \tau_9, \quad \tau_4\tau_5 = 0. \end{aligned}$$

In terms of the Poincaré dual basis we get the reversed Hasse diagram



Since the index of X is seven, the quantum cohomology is very easy to deduce from the quantum Chevalley formula, computed in [GPPS22], Proposition 4.4: quantum corrections do appear only for $h\tau_6^- = \tau_7 + q$, $h\tau_7 = \tau_8 + qh$, $h\tau_8 = \tau_9 + q\sigma'_2$ and $h\tau_9 = q\sigma'_3$. In particular we get $h(\tau_6^- - \tau_6^+) = q$, and since the Giambelli type formulas above are valid in quantum cohomology up to degree six, we deduce that

$$R_6(q) = R_6, \quad R_7(q) = 6h^4\tau - h^7 - q.$$

Moreover the Giambelli type formulas in degree bigger than six must be corrected as

$$\tau_7 = \frac{h^7}{12} - \frac{5}{12}q, \quad \tau_8 = \frac{h^8}{12} - \frac{17}{12}qh, \quad \tau_9 = \frac{h^9}{12} - \frac{29}{12}qh^2.$$

Finally, the quantum multiplication by the generator τ is given, in degree bigger than six, by the following formulas:

$$\begin{aligned} \tau_4\tau &= q, & \sigma'_4\tau &= \tau_7 - q, & \tau_5\tau &= qh, & \sigma'_5\tau &= \tau_8 - qh, & \tau_6^+\tau &= \tau_9, \\ \tau_6^-\tau &= -\tau_9 + qh^2, & \tau_7\tau &= q\sigma'_3, & \tau_8\tau &= q\sigma'_4, & \tau_9\tau &= q\sigma'_5. \end{aligned}$$

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