An approach of the Minimal Model Program for horospherical varieties via moment polytopes

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Abstract

We describe the Minimal Model Program in the family of $\mathbb{Q}$-Gorenstein projective horospherical varieties, by studying a family of polytopes defined from the moment polytope of a Cartier divisor of the variety we begin with. In particular, we generalize the results on MMP for toric varieties due to M. Reid, and we complete the results on MMP for spherical varieties due to M. Brion in the case of horospherical varieties.

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1 Introduction

In this paper, we work over the complex numbers.

The Minimal Model Program (MMP) takes an important place in birational algebraic geometry in order to get a birational classification of algebraic varieties. A lot of progress has been done in the last three decades. We come back here to an original version of the MMP, summarized by Figure 1 where $\mathcal{H}$ denote a family of $\mathbb{Q}$-factorial varieties (see [Mat02] to have a good overview of this theory). For any $\mathbb{Q}$-Gorenstein variety X, we denote by $NE(X)$ the nef cone of curves on X, by K_X a canonical divisor of X and by $NE(X)_{K_X<0}$ (resp. $NE(X)_{K_X>0}$) the intersection of the nef cone with the open half-space of curves negative (resp. positive) along the divisor K_X.

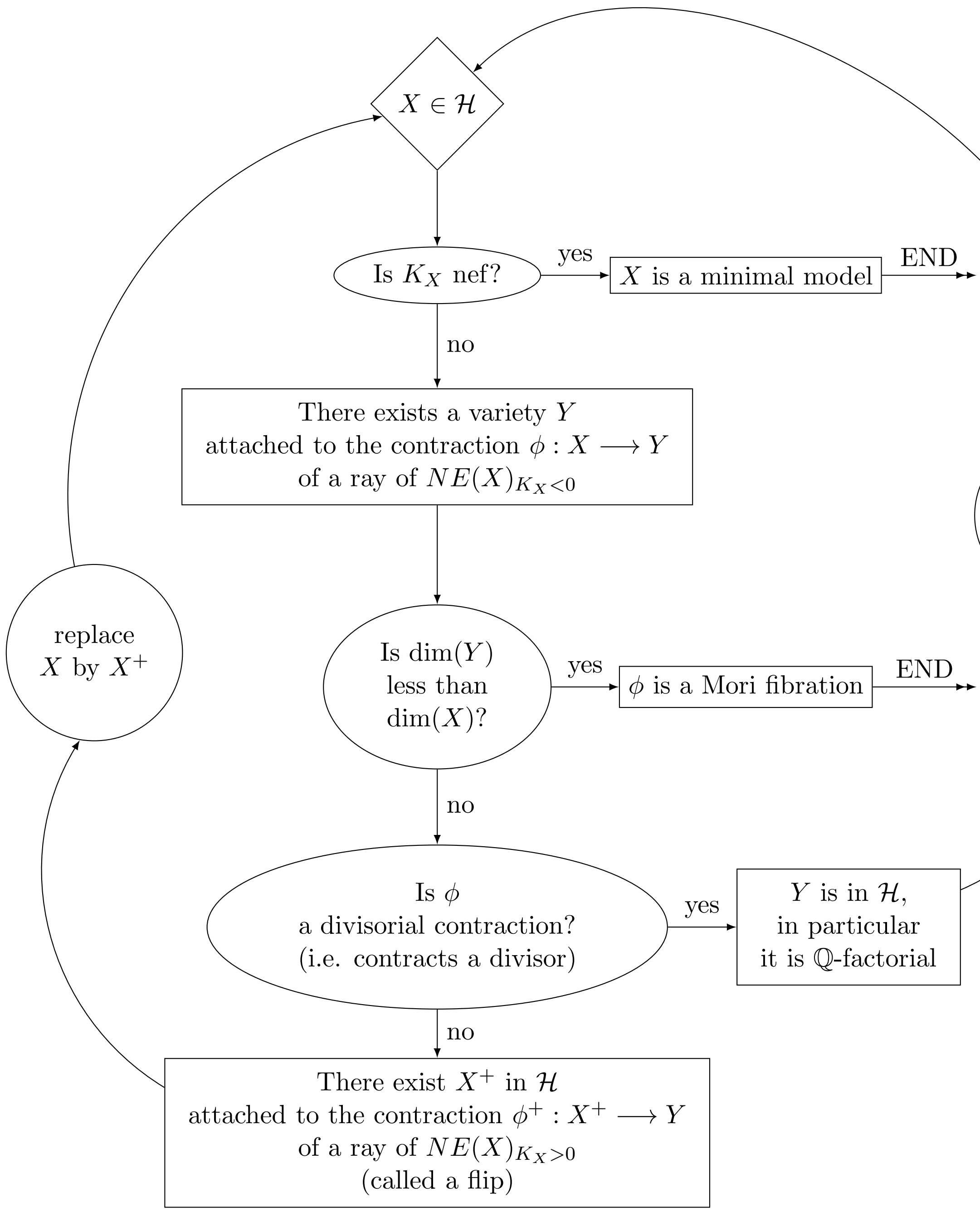
When $\mathcal{H}$ is the family of $\mathbb{Q}$-factorial toric varieties, M. Reid proved in 1983 that the MMP works [Rei83]. In particular, the MMP ends (there is no infinite series of flips). Note that, for this family, the cone $NE(X)$ is polyhedral generated by finitely many rays, and is very well-understood. Note also that, since toric varieties are rational varieties, a minimal model is then a point. Moreover, for $\mathbb{Q}$-factorial toric varieties, the general fibers of Mori fibrations are toric varieties given by a simplex (but not only weighted projective spaces as explained in [Mat02, Remark 14.2.3]).

When $\mathcal{H}$ is the family of $\mathbb{Q}$-factorial spherical G-varieties, for any connected reductive algebraic group G, M. Brion proved in 1993 that the MMP works [Bri93]. For this family, the cone $NE(X)$ is still polyhedral generated by finitely many rays and described in [Bri93]. Note that spherical varieties are also rational varieties, so that minimal models are still points here. Nevertheless, it is very difficult to compute concretely $NE(X)$ and K_X, so that the application of the MMP to explicit examples of this family is laborious. This is why we reduce the study to horospherical varieties, for which a canonical divisor is well-known, and that is also why we present another approach that does not need the computation of $NE(X)$. Unlike the toric case, the general fibers of Mori fibrations for $\mathbb{Q}$-factorial spherical G-varieties are not known.

In this paper, we first consider the case where $\mathcal{H}$ is the family of $\mathbb{Q}$-Gorenstein projective horospherical G-varieties. The family of horospherical varieties is contained in the family of spherical varieties and contains toric varieties. Using a different approach than the one used by M. Reid or M. Brion, we obtain the main result of this paper.

Theorem 1. *The MMP described by Figure 2 works if $\mathcal{H}$ is the family of $\mathbb{Q}$-Gorenstein projective horospherical G-varieties for any connected reductive algebraic group G. Moreover, for any X in*

Figure 1: Original MMP for $\mathbb{Q}$-factorial varieties



$\mathcal{H}$ and for any choice of an ample Cartier divisor of X, we can explicitly describe each step of this MMP until it ends.

Remark that the main difference with the original MMP is that a divisorial contraction can give a flip. This comes from the fact that the varieties are here not necessarily $\mathbb{Q}$-factorial.

We use here convex geometry, and in particular continuous transformations of moment polytopes. This is why we need to restrict to projective varieties. To illustrate Theorem 1, we give two first examples, both from the same toric variety X. Note that the result of the MMP applied to a variety X is not unique: for the original MMP, it depends on the choices of a ray in the effective cone of all varieties appearing in the beginning of each loop of the program, and here it depends on the choice of a $\mathbb{Q}$-Cartier ample divisor of the variety X (only in the beginning of the program). Choosing a $\mathbb{Q}$-Cartier ample divisor of a projective toric variety is equivalent to choosing its moment polytope (see Section 2.3). In the following examples, we assume that the reader is familiar with the classification of toric varieties in terms of fans. We refer to [Ful93] or [Oda88] for more details.

Example 1. In $\mathbb{Q}^3$ consider the simple polytope Q defined by the following inequalities, where (x, y, z) are the coordinates of a point in $\mathbb{Q}^3$.

$$\begin{array}{ccc} z & \geq & -1 \\ -x-y-2z & \geq & -5 \\ 2x-z & \geq & -3 \\ -2x-z & \geq & -3 \\ 2y-z & \geq & -3 \\ -2y-z & \geq & -3. \end{array}$$

It is a pyramid whose top vertex is cut by a plane.

The edges of the fan $\mathbb{F}_X$ of the toric variety X associated to Q are $x_1 := (0,0,1)$, $x_2 := (-1,-1,-2)$, $x_3 := (2,0,-1)$, $x_4 := (-2,0,-1)$, $x_5 := (0,2,-1)$ and $x_6 := (0,-2,-1)$. And the maximal cones of $\mathbb{F}_X$ are the cones respectively generated by (x_1,x_3,x_5), (x_1,x_3,x_6), (x_1,x_4,x_5), (x_1,x_4,x_6), (x_2,x_3,x_5), (x_2,x_3,x_6), (x_2,x_4,x_5), (x_2,x_4,x_6). (See Section 2.3, to have an explanation of the correspondence between moment polytopes and fans.)

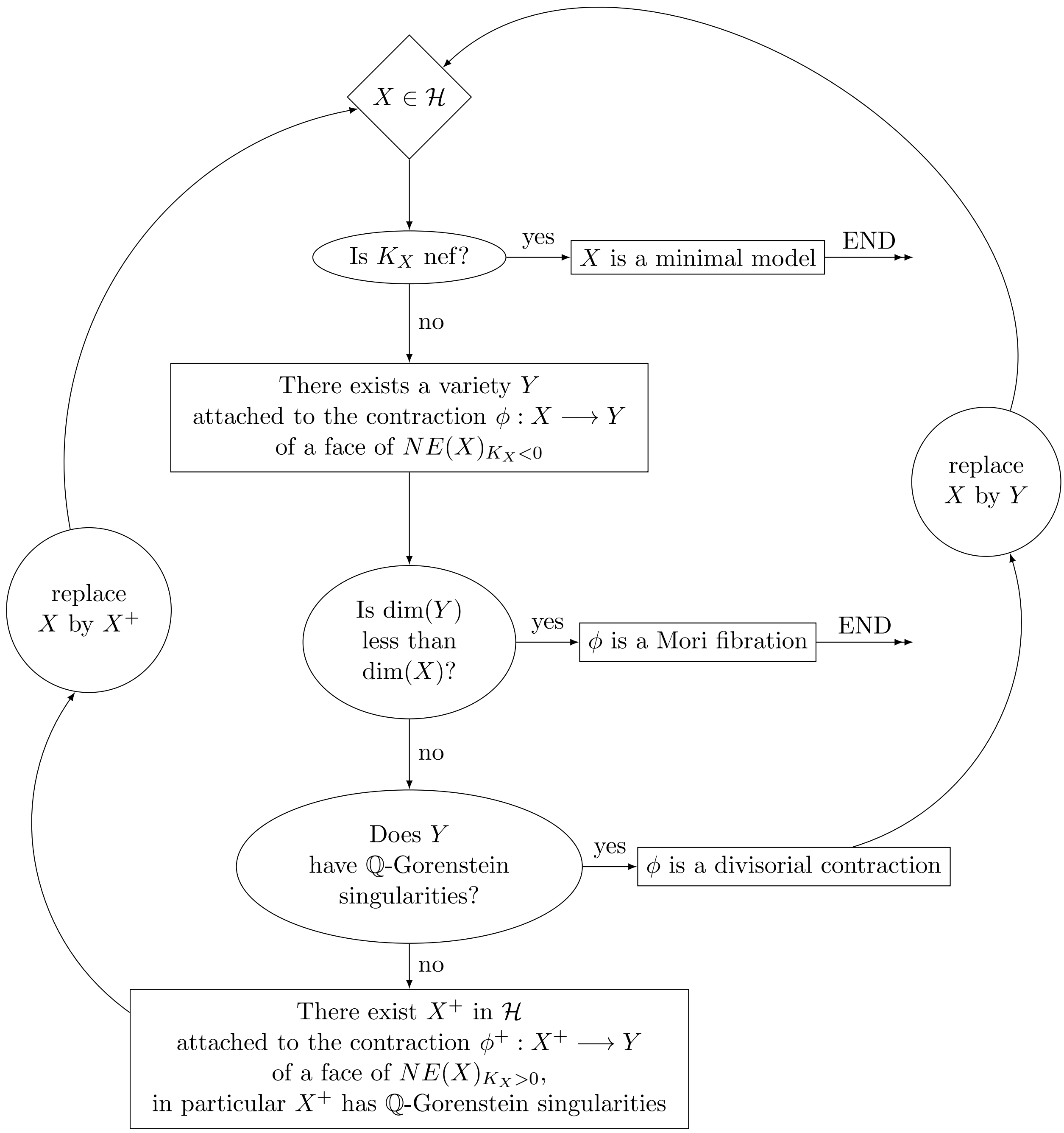
Note that X is $\mathbb{Q}$-factorial, because Q is simple (and $\mathbb{F}_X$ is simplicial).

We consider the family of polytopes Q^ϵ defined by the following inequalities:

$$\begin{array}{ccc} z & \geq & -1+\epsilon \\ -x-y-2z & \geq & -5+\epsilon \\ 2x-z & \geq & -3+\epsilon \\ -2x-z & \geq & -3+\epsilon \\ 2y-z & \geq & -3+\epsilon \\ -2y-z & \geq & -3+\epsilon. \end{array}$$

Note that, for $\epsilon > 0$ and small enough, it is the moment polytope of $D+\epsilon K_X$, where D is the divisor of X whose moment polytope is Q, and K_X is the canonical divisor.

Figure 2: MMP for $\mathbb{Q}$-Gorenstein projective horospherical G-varieties



For $\epsilon \in [0, 1[$, all polytopes Q^ϵ have the same structure so that they all correspond to the toric variety X. For any $\epsilon \in [1, 2[$, the polytopes Q^ϵ are pyramids. They all correspond to the toric variety Y whose fan $\mathbb{F}_Y$ is described by the cones respectively generated by (x_1, x_3, x_5), (x_1, x_3, x_6), (x_1, x_4, x_5), (x_1, x_4, x_6), (x_3, x_4, x_5, x_6). Remark that the last cone is not simplicial, so that Y is not $\mathbb{Q}$-factorial. The fact that Y is $\mathbb{Q}$-Gorenstein comes from the fact that x_3, x_4, x_5 and x_6 are in a common plane. Note also that x_2 is not an edge of $\mathbb{F}_Y$.

And for $\epsilon = 2$, the polytope Q^ϵ is a point. Then the family $(Q^\epsilon)_{\epsilon \in [0,2]}$ reveals a divisorial contraction $\phi : X \longrightarrow Y$ and a Mori fibration from Y to a point.

We illustrate this example in Figure 3.

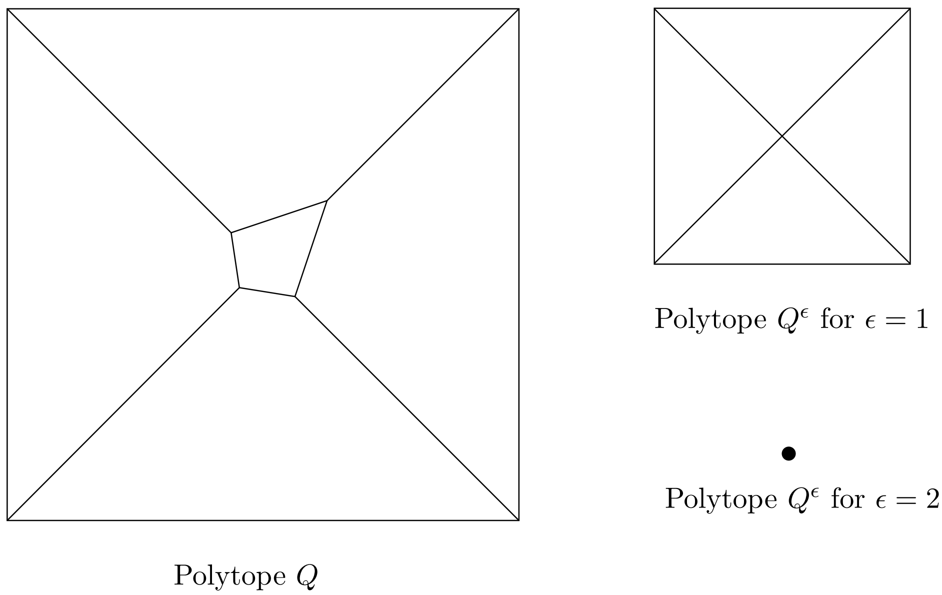


Figure 3: Evolution of Q^ϵ in Example 1 (view from the top of the pyramid)

Before we give the second example, we can notice that, in Example 1, the divisorial contraction ϕ goes from a $\mathbb{Q}$-factorial variety to a not $\mathbb{Q}$-factorial variety. It does not contradict the results of M. Reid, because ϕ is not the contraction of a ray of $NE(X)$ but of a 2-dimensional face of $NE(X)$. This gives a reason to begin with $\mathbb{Q}$-Gorenstein varieties instead of $\mathbb{Q}$-factorial varieties. Fortunately, what occurs in Example 1 can be observed only in very particular cases (see Theorem 2). Now, we consider a second example, with a more general divisor D.

Example 2. Let Q^ϵ be the polytopes defined by the following inequalities,

$$\begin{array}{lcl} z & \geq & -1+\epsilon \\ -x-y-2z & \geq & -5+\epsilon \\ 2x-z & \geq & -4+\epsilon \\ -2x-z & \geq & -4+\epsilon \\ 2y-z & \geq & -3+\epsilon \\ -2y-z & \geq & -3+\epsilon. \end{array}$$

The simple polytope Q^0 is still a pyramid whose top vertex is cut by a plane. It is the moment polytope of a $\mathbb{Q}$-Cartier ample divisor of the toric variety X of Example 1.

For $\epsilon \in [0, \frac{1}{2}[$, all polytopes Q^ϵ have the same structure so that they all correspond to the toric variety X.

Now, for $\epsilon = \frac{1}{2}$, Q^ϵ corresponds to the fan whose maximal cones are the cones respectively generated by (x_1, x_3, x_5), (x_1, x_3, x_6), (x_1, x_4, x_5), (x_1, x_4, x_6), (x_2, x_4, x_5), (x_2, x_4, x_6) and (x_2, x_3, x_5, x_6). The associated toric variety Y is not $\mathbb{Q}$-factorial. In fact, it is even not $\mathbb{Q}$-Gorenstein (because x_2, x_3, x_5 and x_6 are not in a common plane).

For $\epsilon \in]\frac{1}{2}, \frac{3}{2}[$, the polytope Q^ϵ is the moment polytope of a divisor of the toric variety X^+ whose fan has maximal cones respectively generated by (x_1, x_3, x_5), (x_1, x_3, x_6), (x_1, x_4, x_5), (x_1, x_4, x_6), (x_2, x_4, x_5), (x_2, x_4, x_6), (x_2, x_5, x_6) and (x_3, x_5, x_6). The variety X^+ is clearly $\mathbb{Q}$-factorial, and defines a flip $X \longrightarrow Y \longleftarrow X^+$.

Now, for $\epsilon \in [\frac{3}{2}, 2[$, Q^ϵ is a simple polytope with 6 vertices and the moment polytope of a divisor of the toric variety Z whose fan has maximal cones respectively generated by (x_1, x_3, x_5), (x_1, x_3, x_6), (x_1, x_4, x_5), (x_1, x_4, x_6), (x_3, x_5, x_6) and (x_4, x_5, x_6). And we get a divisorial contraction from X^+ to Z.

To finish, for $\epsilon = 2$, Q^ϵ is the interval whose extremities are $(-\frac{1}{2}, 0, 1)$ and $(\frac{1}{2}, 0, 1)$, corresponding to the toric variety $\mathbb{P}^1$. It gives a Mori fibration from Z to the projective line.

We illustrate this example in Figure 4.

Moreover, the general fiber of the Mori fibration is the toric variety associated to the polytope in $\mathbb{Q}^2$ defined by the inequalities $z \geq -1$, $2y - z \geq -3$ and $-2y - z \geq -3$. Its fan is isomorphic to the complete fan in $\mathbb{Q}^2$ whose edges are $(0, 1)$, $(-2, 1)$ and $(-2, -1)$. It is the weight projective plane $\mathbb{P}(1, 1, 2)$.

Example 2 raises a natural question: can we recover the original MMP for $\mathbb{Q}$-factorial projective horospherical varieties, by choosing a good divisor D? The answer is yes and given in the following result.

Theorem 2. *Suppose that X is $\mathbb{Q}$-factorial. Then, by taking a general divisor D on X, we have that*

- *at any loops of the MMP described in Figure 2 (until it ends), the morphisms ϕ and ϕ^+ are contractions of rays of $NE(X)$ and $NE(X^+)$;*
- *the MMP described in Figure 2 still works by replacing $\mathbb{Q}$-Gorenstein singularities with $\mathbb{Q}$-factorial ones everywhere;*

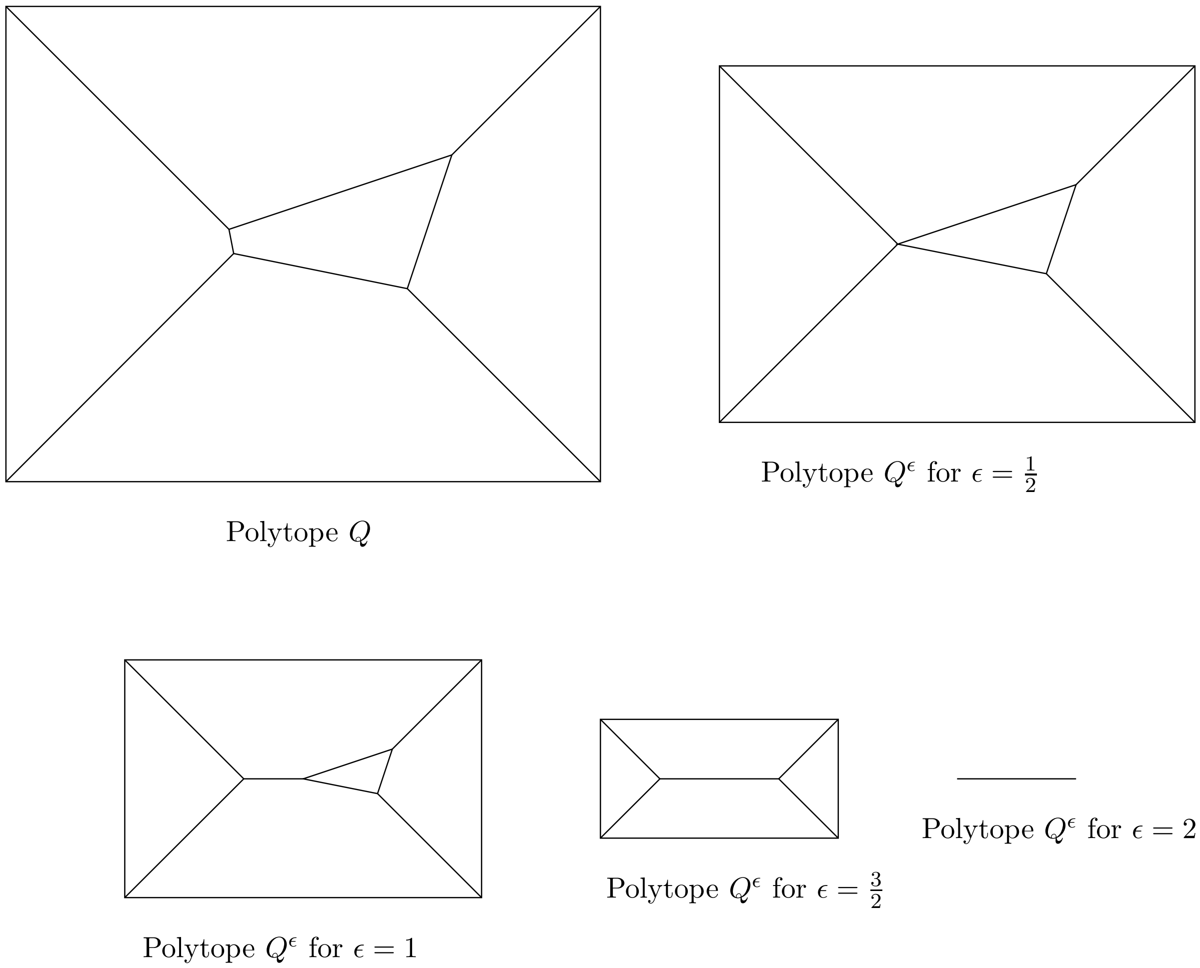


Figure 4: Evolution of Q^ϵ in Example 2 (view from the top of the pyramid)

- *the general fibers of Mori fibrations are projective $\mathbb{Q}$-factorial horospherical varieties with Picard number 1 (whose moment polytopes are simplexes intersecting all walls of a dominant chamber along facets).*

Remark 1. If X is smooth and D general, the general fibers of Mori fibrations are projective smooth horospherical varieties with Picard number 1. These varieties have been classified in [Pas09], in particular we get flag varieties and some two-orbit varieties. Hence, the general fibers of smooth Mori fibrations are not only projective spaces as in the toric case.

The paper is organized as follows.

In Section 2, we recall the theory of horospherical varieties, we state the correspondence between polarized horospherical varieties and moment polytopes, we rewrite the existence criterion of an equivariant morphism between two horospherical varieties in terms of polytopes, and we describe the curves on horospherical varieties using moment polytopes.

In Section 3, we study particular (linear) one-parameter families of polytopes in $\mathbb{Q}^n$ and we define some equivalence relations in these families.

In Section 4, we prove Theorems 1 and 2, by using the previous two sections. In particular, we construct a one-parameter family of moment polytopes whose equivalence classes describe all loops of the MMP given in Figure 2.

In Section 5, we give five examples to illustrate what can happen in the MMP for horospherical (and not toric) varieties. And we describe completely the process for any rank 1 projective horospherical variety.

2 Horospherical varieties and polytopes

2.1 Notation and horospherical embedding theory

We begin by recalling briefly the Luna-Vust theory of horospherical embeddings and by setting the notation used in the rest of the paper. For more details on horospherical varieties, we refer the reader to [Pas08], and for basic results on Luna-Vust theory of spherical embeddings, we refer to [Kno91].

We fix a connected reductive algebraic group G.

Definition 1. A closed subgroup H of G is said to be horospherical if it contains the unipotent radical U of a Borel subgroup B of G.

We fix a maximal torus T of B. Then we denote by S the set of simple roots of (G, B, T). Also denote by R the subset of S of simple roots of P. Let $X(T)$ (respectively $X(T)^+$) be the lattice of characters of T (respectively the set of dominant characters). Similarly, we define $X(P)$ and $X(P)^+ = X(P) \cap X(T)^+$. Note that the lattice $X(P)$ (and the dominant chamber $X(P)^+$) is generated by the fundamental weights ϖ_α with $\alpha \in S\backslash R$ and the weights of the center of G.

Proposition 3. *([Pas08, Prop. and Rem. 2.2]) Let H be a closed subgroup of G. The following assertions are equivalent:*

1. *the subgroup H is horospherical;*
2. *there exists a parabolic subgroup P containing H such that the map $G/H \longrightarrow G/P$ is a torus fibration;*
3. *there exists a parabolic subgroup P containing H such that H is the kernel of finitely many characters of P.*

If the above assertions hold, then the parabolic subgroup P appearing in 2 and 3 is the normalizer $N_G(H)$ of H in G and contains a Borel subgroup B whose unipotent radical U is contained in H.

We denote by M the sublattice of $X(P)$ consisting of characters of P vanishing on H. The rank of M is called the rank of G/H and denoted by n. Let $N := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$.

For any free lattice $\mathbb{L}$, we denote by $\mathbb{L}_{\mathbb{Q}}$ the $\mathbb{Q}$-vector space $\mathbb{L} \otimes_{\mathbb{Z}} \mathbb{Q}$.

For any simple root $\alpha \in S \backslash R$, the restriction of the coroot $\alpha^\vee$ to M is a point of N, which we denote by $\alpha^\vee_M$. For any $\alpha \in S\backslash R$, we define

$$W_{\alpha,P} := \{m \in X(P)_{\mathbb{Q}} \mid \langle m, \alpha^\vee \rangle = 0\}.$$

Note that this hyperplane corresponds to the wall of the dominant chamber of the characters of P.

Definition 2. A G/H-embedding is a couple (X, x), where X is a normal algebraic G-variety and x a point of X such that $G \cdot x$ is open in X and isomorphic to G/H.

By abuse of notation, we will forget the point x and consider that a G/H-embedding is just a normal G-variety X with an open G-orbit isomorphic to G/H.

Similarly to toric varieties, G/H-embeddings are classified by colored fans in $N_{\mathbb{Q}}$.

Definition 3.
1. A colored cone of $N_{\mathbb{Q}}$ is an couple $(\mathcal{C}, \mathcal{F})$ where $\mathcal{C}$ is a convex cone of $N_{\mathbb{Q}}$ and $\mathcal{F}$ is a set of colors (called the set of colors of the colored cone), such that
   - (i) $\mathcal{C}$ is generated by finitely many elements of N and contains $\{\alpha^\vee_M \mid \alpha \in \mathcal{F}\}$,
   - (ii) $\mathcal{C}$ does not contain any line and $\mathcal{F}$ does not contain any α such that $\alpha^\vee_M$ is zero.
2. A colored face of a colored cone $(\mathcal{C}, \mathcal{F})$ is a couple $(\mathcal{C}', \mathcal{F}')$ such that $\mathcal{C}'$ is a face of $\mathcal{C}$ and $\mathcal{F}'$ is the set of $\alpha \in \mathcal{F}$ satisfying $\alpha^\vee_M \in \mathcal{C}'$. A colored fan is a finite set $\mathbb{F}$ of colored cones such that
   - (i) any colored face of a colored cone of $\mathbb{F}$ is in $\mathbb{F}$,
   - (ii) and any element of $N_{\mathbb{Q}}$ is in the interior of at most one colored cone of $\mathbb{F}$.

The main result of Luna-Vust Theory of spherical embeddings is the following classification result (see for example [Kno91]). For horospherical varieties, we will later in Section 2.2 rewrite this result in terms of polytopes and describe explicitly the correspondence.

Theorem 4. *There is an explicit one-to-one correspondence between colored fans and isomorphic classes of G/H-embeddings.*

Complete G/H-embeddings correspond to complete fans i.e. to fans such that $N_{\mathbb{Q}}$ is the union of its colored cones.

If $G = (\mathbb{C}^*)^n$ and $H = \{1\}$, we recover the well-known classification of toric varieties.

If X is a G/H-embedding, we denote by $\mathbb{F}_X$ the colored fan of X in $N_{\mathbb{Q}}$ and we denote by $\mathcal{F}_X$ the subset $\cup_{(\mathcal{C},\mathcal{F})\in\mathbb{F}_X} \mathcal{F}$ of $S\backslash R$, which we call the set of colors of X.

A consequence of Theorem 4 is the following description of G-orbits of G/H-embeddings.

Proposition 5. *The set of G-orbits of a G/H-embedding X is naturally in bijection with the set of colored cones of $\mathbb{F}_X$ (reversing usual orders). In particular, the G-stable irreducible divisors correspond to the colored edges of $\mathbb{F}_X$ of the form $(\mathcal{C}, \varnothing)$.*

We denote by $X_1,\dots,X_r$ the G-stable irreducible divisors of a G/H-embedding X. And for any $i\in\{1,\dots,r\}$, we denote by x_i the primitive element in N of the colored edge associated to X_i.

2.2 Divisors of G/H-embeddings and moment polytopes

In this section, we recall the characterization of Cartier, $\mathbb{Q}$-Cartier and ample divisors of horospherical varieties due to M. Brion in the more general case of spherical varieties ([Bri89]). This permits to define a polytope associated to a divisor of a horospherical variety. And we also give some properties of this polytope.

First, we describe the B-stable irreducible divisors of a G/H-embedding X. The G-stable divisors $X_1,\dots,X_r$ are already defined. And the other ones are the closures in X of B-stable irreducible divisors of G/H, which are the inverse images by the torus fibration $G/H\longrightarrow G/P$ of the Schubert divisors of the flag variety G/P. Recall that the Schubert divisors of G/P are indexed by the subset of simple roots $S\backslash R$ and of the form $\overline{Bw_0s_\alpha P/P}$, where w_0 is the longest element in the Weyl group of (G,T) and s_α is the simple reflection associated to a simple root α. Hence the B-stable irreducible divisors of G/H are of the form $\overline{Bw_0s_\alpha P/H}$, we denote them by D_α for any $\alpha\in S\backslash R$.

Theorem 6. *([Bri89, Section 3.3]) Let G/H be a horospherical homogeneous space. Let X be a G/H-embbeding. Then every divisor of X is equivalent to a linear combinaison of $X_1,\dots,X_r$ and D_α with $\alpha\in S\backslash R$. Now, let $D=\sum_{i=1}^r a_iX_i+\sum_{\alpha\in S\backslash R}a_\alpha D_\alpha$ be a $\mathbb{Q}$-divisor of X.*

1. *D is $\mathbb{Q}$-Cartier if and only if there exists a piecewise linear function h_D, linear on each colored cone of $\mathbb{F}_X$, such that for any $i\in\{1,\dots,r\}$, $h_D(x_i)=a_i$ and for any $\alpha\in\mathcal{F}_X$, $h_D(\alpha^\vee_M)=a_\alpha$.*

2. *Suppose that D is a divisor (i.e. $a_1,\dots,a_r$ and the a_α with $\alpha\in S\backslash R$ are in $\mathbb{Z}$). Then D is Cartier if moreover, for any colored cone $(\mathcal{C},\mathcal{F})$ of $\mathbb{F}_X$, the linear function $(h_D)_{|\mathcal{C}}$, can be define as an element of M (instead of $M_\mathbb{Q}$ for $\mathbb{Q}$-Cartier divisors).*

3. *Suppose that D is $\mathbb{Q}$-Cartier. Then D is ample if and only if the piecewise linear function h_D is strictly convex and for any $\alpha\in(S\backslash R)\backslash\mathcal{F}_X$, we have $h_D(\alpha^\vee_M)<a_\alpha$.*

4. *Suppose that D is Cartier. Let $\tilde{Q}_D$ be the polytope in $M_\mathbb{Q}$ defined by the following inequalities, where $\chi\in M_\mathbb{Q}$: for any colored cone $(\mathcal{C},\mathcal{F})$ of $\mathbb{F}_X$, $(h_D)+\chi\geq 0$ on $\mathcal{C}$, and for any $\alpha\in(S\backslash R)\backslash\mathcal{F}_X$, $\chi(\alpha^\vee_M)+a_\alpha\geq 0$. Note that here the weight of the canonical section of D is $v^0:=\sum_{\alpha\in S\backslash R}a_\alpha\varpi_\alpha$. Then the G-module $H^0(X,D)$ is the direct sum, with multiplicity one, of the irreducible G-modules of highest weights $\chi+v^0$ with χ in $\tilde{Q}_D\cap M$.*

In all the paper, a divisor of a horospherical variety is always supposed to be B-stable, i.e. of the form $\sum_{i=1}^r a_iX_i+\sum_{\alpha\in S\backslash R}a_\alpha D_\alpha$. We can now easily prove the following properties.

Corollary 7. *Let X be a projective G/H-embedding and $D=\sum_{i=1}^r a_iX_i+\sum_{\alpha\in S\backslash R}a_\alpha D_\alpha$ be a $\mathbb{Q}$-divisor of X. Suppose that D is $\mathbb{Q}$-Cartier and ample.*

1. *The polytope $\tilde{Q}_D$ defined in Theorem 6 is of maximal dimension in $M_{\mathbb{Q}}$ and we have*

$$\tilde{Q}_D = \{m \in M_{\mathbb{Q}} \mid \langle m, x_i\rangle \geq -a_i,\, \forall i \in \{1, \dots, r\} \text{ and } \langle m, \alpha_M^\vee\rangle \geq -a_\alpha,\, \forall \alpha \in \mathcal{F}_X\}.$$

2. *Let $v^0 := \sum_{\alpha\in S\backslash R} a_\alpha \varpi_\alpha$. The polytope $Q_D := v^0 + \tilde{Q}_D$ is contained in the dominant chamber $X(P)^+$ of $X(P)$ and it is not contained in any wall $W_{\alpha,P}$, for $\alpha \in S\backslash R$. Moreover, $Q_D \cap W_{\alpha,P} \neq \emptyset$ if and only if $\alpha \in \mathcal{F}_X$.*

3. *Let $(\mathcal{C}, \mathcal{F})$ be a maximal colored cone of $\mathbb{F}_X$, then the element $v^0 - (h_D)_{|\mathcal{C}}$ of $M_{\mathbb{Q}}$ is a vertex of Q_D. In particular, if D is Cartier, then Q_D is a lattice polytope (i.e. has its vertices in $v^0 + M$).*

4. *Conversely, let v be a vertex of Q_D. We define $\mathcal{C}_v$ to be the cone of $N_{\mathbb{Q}}$ generated by inward-pointing normal vectors of the facets of Q_D containing v. And we set $\mathcal{F}_v = \{\alpha \in S\backslash R \mid v \in W_{\alpha,P}\}$. Then $(\mathcal{C}_v, \mathcal{F}_v)$ is a maximal colored cone of $\mathbb{F}_X$.*

5. *With a natural construction, similar to the previous two items, we obtain a bijection between faces of Q_D and colored cones of the colored fan $\mathbb{F}_X$.*

6. *The divisor D can be computed from the pair $(Q, \tilde{Q})$ as follows: the coefficients a_α with $\alpha \in S\backslash R$ are given by the translation vector in $X(P)^+$ that maps $\tilde{Q}$ to Q; and for any $i \in \{1, \dots, r\}$, the coefficient a_i is given by $-\langle v_i, x_i\rangle$ for any element $v_i \in M_{\mathbb{Q}}$ in the facet of $\tilde{Q}$ for which x_i is an inward-pointing normal vector.*

Since we could not find precise references for Corollary 7, we give here a proof for the convenience of the reader.

Proof. By the definition of h_D and 4th item of Theorem 6,

$$\tilde{Q}_D = \{m \in M_{\mathbb{Q}} \mid \langle m, x_i\rangle \geq -a_i,\, \forall i \in \{1, \dots, r\} \text{ and } \langle m, \alpha_M^\vee\rangle \geq -a_\alpha,\, \forall \alpha \in S\backslash R\}.$$

But since D is ample, for any α in $(S\backslash R)\backslash\mathcal{F}_X$, we have $h_D(\alpha_M^\vee) < a_\alpha$. Then if $m \in M_{\mathbb{Q}}$ satisfies $m + h_D \geq 0$, we have $\langle m, \alpha^\vee\rangle \geq -h_D(\alpha_M^\vee) > -a_\alpha$. This means that the inequalities with $\alpha \in (S\backslash R)\backslash\mathcal{F}_X$ are redundant, i.e.

$$\tilde{Q}_D = \{m \in M_{\mathbb{Q}} \mid \langle m, x_i\rangle \geq -a_i,\, \forall i \in \{1, \dots, r\} \text{ and } \langle m, \alpha_M^\vee\rangle \geq -a_\alpha,\, \forall \alpha \in \mathcal{F}_X\} = \{m \in M_{\mathbb{Q}} \mid m + h_D \geq 0\}.$$

Moreover, since D is ample, for some big enough $k \in \mathbb{Z}$, $\mathbb{C}(X)$ is generated by quotients of two sections in $H^0(X, kD)$. But, up to homothetie, the highest weight vectors of $\mathbb{C}(X) = \mathbb{C}(G/H)$ are in bijection with M ([Pas08, Section 2]), and the highest weight vectors of $H^0(X, kD)$ are the points in $kv^0 + (k\tilde{Q}_D \cap M)$. This implies that the lattice points of $k\tilde{Q}_D$ generates M, and then that $\tilde{Q}_D$ is of maximal dimension in $M_{\mathbb{Q}}$.

We have already seen that for any $m \in \tilde{Q}_D$ and any $\alpha \in S\backslash R$, we have $\langle m, \alpha_M^\vee\rangle \geq -a_\alpha$. But by definition, $\langle v^0, \alpha^\vee\rangle = a_\alpha$. We deduce easily that $Q_D \subset X^+(P)$. By the first paragraph of the proof, we also deduce that, if $\alpha \notin \mathcal{F}_X$, $v^0 + m$ cannot be on the wall $W_{\alpha,P}$.

Let $\alpha \in S\backslash R$. By the latter paragraph, if Q_D is contained in the wall $W_{\alpha,P}$, then α is in $\mathcal{F}_X$ and $v^0 + M$ is contained in $W_{\alpha,P}$. This implies that M is contained in $W_{\alpha,P}$ (because both contain 0), i.e. that $\alpha^\vee_M = 0$. This contradicts the fact that $\alpha \in \mathcal{F}_X$, by Definition 3 (ii).

We already proved that $Q_D \cap W_{\alpha,P} \neq \emptyset$ implies that $\alpha \in \mathcal{F}_X$. Inversely, let $\alpha \in \mathcal{F}_X$. Let $(\mathcal{C}, \mathcal{F})$ be the maximal colored cone such that $\alpha^\vee_M \in \mathcal{C}$. Set $m := -(h_D)_{|\mathcal{C}} \in M_\mathbb{Q}$ (m is uniquely defined because $\mathcal{C}$ is of maximal dimension). Then it satisfies $\langle m, \alpha^\vee_M \rangle = -a_\alpha$. And, by convexity of the piecewise linear function h_D, we have $h_D + m \geq 0$. In particular $v^0 + m$ is in $Q_D \cap W_{\alpha,P}$. Note that, since h_D is strictly convex, the set where $h_D + m$ vanishes is exactly $\mathcal{C}$. Hence, since $\mathcal{C}$ is of maximal dimension in $M_\mathbb{Q}$, we have

$$\{m\} = \{m \in M_\mathbb{Q} \mid \langle m, x_i \rangle = -a_i, \forall x_i \in \mathcal{C} \text{ and } \langle m, \alpha^\vee_M \rangle = -a_\alpha, \forall \alpha^\vee_M \in \mathcal{C} \text{ with } \alpha \in \mathcal{F}_X\},$$

and for any $x_i \notin \mathcal{C}$, $\langle m, x_i \rangle > -a_i$, and for any $\alpha \in \mathcal{F}_X$ such that $\alpha^\vee_M \notin \mathcal{C}$, $\langle m, \alpha^\vee_M \rangle > -a_\alpha$. In particular, m is a vertex of $\tilde{Q}_D$. It is easy to check that the colored cone $(\mathcal{C}_m, \mathcal{F}_m)$ equals $(\mathcal{C}, \mathcal{F})$.

Now let v be a vertex of $\tilde{Q}_D$. Then, there exists $A \subset \{x_1, \dots, x_r\}$ and $B \subset \mathcal{F}_X$ such that for all $i \in \{1, \dots, r\}$, $\langle v, x_i \rangle \geq -a_i$ with equality if and only if $x_i \in A$ and for all $\alpha \in \mathcal{F}_X$, $\langle m, \alpha^\vee_M \rangle \geq -a_\alpha$ with equality if and only if $\alpha \in B$. In particular $\mathcal{C}_v$ is generated by the x_i in A and the $\alpha^\vee_M$ with $\alpha \in B$, it contains no line and it is of maximal dimension. Moreover, the restriction of h_D to $\mathcal{C}_v$ is linear (it equals $-v$). Hence, by strictly convexity of h_D, there exists a maximal colored cone $(\mathcal{C}, \mathcal{F})$ in $\mathbb{F}_X$ such that $\mathcal{C}_v \subset \mathcal{C}$. We must have $\mathcal{C}_v = \mathcal{C}$ because v is the same vertex m as the one defined in the latter paragraph from $(\mathcal{C}, \mathcal{F})$. To prove that $(\mathcal{C}_v, \mathcal{F}_v) = (\mathcal{C}, \mathcal{F})$, it is enough to recall that for any $\alpha \in \mathcal{F}_X$, $\alpha \in \mathcal{F}$ if and only if $\alpha^\vee_M \in \mathcal{C}$.

To generalize this to faces of $\tilde{Q}_D$, we just remark that any face of a polytope is the convex hull of the vertices that it contains, and any colored cone of a complete colored fan is the intersection of the maximal colored cones that contain it. We leave the details to the reader.

In particular, the set of facets of $\tilde{Q}_D$ is in bijection with the colored edges of $\mathbb{F}_X$. For the last statement, let $i \in \{1, \dots, r\}$, then x_i corresponds to the facet $\tilde{Q}_D \cap \{m \in M_\mathbb{Q} \mid \langle m, x_i \rangle = -a_i\}$. And then a_i is $-\langle v_i, x_i \rangle$ for any element $v_i \in M_\mathbb{Q}$ in the facet of $\tilde{Q}_D$ □

The polytope Q_D is called the moment polytope of (X, D) (or of D). And let us call the polytope $\tilde{Q}_D$ the pseudo-moment polytope of (X, D) (or of D).

2.3 Correspondence between projective horospherical varieties and polytopes

In this section, we classify projective G/H-embeddings in terms of G/H-polytopes (defined below in Definition 4), and we give an explicit construction of a G/H-embedding from a G/H-polytope.

Definition 4. Let Q be a polytope in $X(P)^+_\mathbb{Q}$ (not necessarily a lattice polytope). We say that Q is a G/H-polytope, if its direction is $M_\mathbb{Q}$ and if it is contained in no wall $W_{\alpha,P}$ with $\alpha \in S\backslash R$.

Let Q and Q' be two G/H-polytopes in $X(P)^+_\mathbb{Q}$. Consider any polytopes $\tilde{Q}$ and $\tilde{Q}'$ in $M_\mathbb{Q}$ obtained by translations from Q and Q' respectively. We say that Q and Q' are equivalent G/H-polytopes if the following conditions are satisfied.

1. There exist an integer j and $2j$ affine half-spaces $\mathcal{H}^+_1, \dots, \mathcal{H}^+_j$ and $\mathcal{H}'^+_1, \dots, \mathcal{H}'^+_j$ of $M_\mathbb{Q}$ (respectively delimited by the affine hyperplanes $\mathcal{H}_1, \dots, \mathcal{H}_j$ and $\mathcal{H}'_1, \dots, \mathcal{H}'_j$) such that $\tilde{Q}$

is the intersection of the $\mathcal{H}_i^+$, $\tilde{Q}'$ is the intersection of the $\mathcal{H}'^+_i$, and for all $i \in \{1, \dots, j\}$, $\mathcal{H}_i^+$ is the image of $\mathcal{H}'^+_i$ by a translation.

2. With notation of the previous item, for all subset J of $\{1, \dots, j\}$, the intersections $\cap_{i\in J}\mathcal{H}_i \cap Q$ and $\cap_{i\in J}\mathcal{H}'_i \cap Q'$ have the same dimension.

3. Q and Q' intersect exactly the same walls $W_{\alpha,P}$ of $X(P)^+$ (with $\alpha \in S\backslash R$).

Remark that this definition does not depend on the choice of $\tilde{Q}$ and $\tilde{Q}'$. Using Corollary 7, we now rewrite the classification result of Luna-Vust Theory (Theorem 4).

Proposition 8. *The correspondence between moment polytopes and colored fans gives a bijection between the set of equivalence classes of G/H-polytopes and (isomorphism classes of) projective G/H-embeddings.*

Moreover, the set of G-orbits of a projective G/H-embedding is in bijection with the set of faces of one of its moment polytope (preserving the respective orders).

For later use (in Section 4.6), we give an explicit description of the G/H-embedding associated to a G/H-polytope. For any dominant weight χ, we denote by $V(\chi)$ the irreducible G-module of highest weight χ, and we fix a highest weight vector v_χ in $V(\chi)$. The Borel subgroup of G opposite to B is denoted by B^-. In the dual G-module $V(\chi)^\vee$ we pick a highest weight vector $v_\chi^\vee$, under the action of B^-, of weight $w_0\chi$. Recall that w_0 is the longest element of the Weyl group of (G,T). Choosing a basis of T-eigen vectors of $V(\chi)$, the vector $v_\chi^\vee$ is the dual of a highest vector of $V(\chi)$.

Proposition 9. *Let Q be a G/H-polytope. Let D be a divisor of the corresponding G/H-embedding X such that $Q = Q_D$. Suppose that D is Cartier and very ample. Then X is isomorphic to the closure of the G-orbit $G \cdot [\sum_{\chi\in(v^0+M)\cap Q} v_\chi]$ in $\mathbb{P}(\oplus_{\chi\in(v^0+M)\cap Q} V(\chi))$.*

Remark 2. If D is Cartier and only ample, then $(n-1)D$ is very ample ([Pas06, Theorem 0.3]). In particular, the assumptions of Proposition 9 are not really restrictive. Indeed, if D is $\mathbb{Q}$-Cartier and ample, then there exists a non-zero integer p such that pD is Cartier and very ample, and X is isomorphic to the closure of $G \cdot [\sum_{\chi\in(pv^0+M)\cap pQ)} v_\chi]$ in $\mathbb{P}(\oplus_{\chi\in(pv^0+M)\cap pQ} V(\chi))$.

If D is not very ample (but still ample), the closure X' of $G\cdot[\sum_{\chi\in(v^0+M)\cap Q)} v_\chi]$ in $\mathbb{P}(\oplus_{\chi\in(v^0+M)\cap Q} V(\chi))$ is not normal (the proof of this fact is left to the reader: use [Pas06, Lemma 5.1] and affine B^--stable open sets of X').

Proof. Since D is very ample, the morphism

$$\begin{array}{rccc} \phi_D : & X & \longrightarrow & \mathbb{P}(H^0(X,D)^\vee) \\ & x & \longmapsto & [s \mapsto s(x)] \end{array}$$

is a G-equivariant closed immersion.

Let $x_0 := w_0H/H$ in X. Recall that H contains the maximal unipotent radical U, then x_0 is fixed by the unipotent radical U^- of B^- (recall that $B- = w_0Bw_0$ and $U^- = w_0Uw_0$). And then $\phi_D(x_0)$ is also fixed by U^-. But, in a simple G-module, the U-stable line is the line of highest weight vectors. Hence, $\phi_D(x_0)$ is a sum of some highest weight vectors under the action of B^- of weights $w_0\chi$ for $\chi \in (v^0 + M) \cap Q$. But, for any $\chi \in (v^0 + M) \cap Q$, the section v_χ does not vanish on X and, since $B \cdot x_0$ is open in X, we have $v_\chi(x_0) \neq 0$. By definition of ϕ_D, it implies that $\phi_D(x_0) = [\sum_{\chi\in(v^0+M)\cap Q} \lambda_\chi v_\chi^\vee]$, where the $\lambda_\chi \in \mathbb{C}^*$ for any $\chi \in (v^0 + M) \cap Q$. Since the $v_\chi^\vee$ can be chosen up to a non-zero scalar, we can assume that λ_χ equals 1 for any $\chi \in (v^0 + M) \cap Q$.

We complete the proof by the following remark. □

Remark 3. The stabilizer in G of $[\sum_{\chi\in(v^0+M)\cap Q} v_\chi^\vee]$ is the subgroup $H^\vee$ defined as the intersection of the kernels of the characters $-w_0\chi$ of P, for $\chi \in M$. It is not conjugated to H! We can explain this as follows. In all the previous papers on horospherical varieties, in order to simplify the notations, we hid the choice to take $\mathbb{C}(G/H)$ to be the direct sum of the G-modules $V(\chi)$ with χ in M instead of the direct sum of their duals. In other words, we study the $G/H^\vee$-embeddings instead of the G/H-embeddings.

But G/H-embeddings are in bijection with $G/H^\vee$-embeddings. Indeed, we can assume as in [Pas08, Proof of Proposition 3.10] that G is a direct product of a torus T' (isomorphic to P/H) and a semi-simple group G' (that we can also assume to be simply connected). Let ψ be the (outer) automorphism of G, trivial on the torus T', corresponding to the automorphism of the root system $-w_0$ (see [Bou75, Chapter VIII Section 7.5] and [OV90, Section 1.2.10]). Then $\psi(H) = H^\vee$ and the $\psi(G)$-modules $V(\chi)^\vee$ are the G-modules $V(\chi)$.

In particular, the closure of $G \cdot [\sum_{\chi\in(pv^0+M)\cap pQ)} v_\chi]$ in $\mathbb{P}(\oplus_{\chi\in(pv^0+M)\cap pQ} V(\chi))$ is the G/H-embedding corresponding to the embedding defined as the closure of $G \cdot [\sum_{\chi\in(pv^0+M)\cap pQ)} v_\chi^\vee]$ in $\mathbb{P}(\oplus_{\chi\in(pv^0+M)\cap pQ} V(\chi)^\vee)$.

2.4 G-equivariant morphisms and polytopes

The existence of G-equivariant morphisms between horospherical varieties can be characterized in terms of colored fans [Kno91]. In this section, we rewrite this characterization in terms of moment polytopes.

Let (X, D) be a polarized G/H-embedding (i.e. a G/H-embedding X with an ample $\mathbb{Q}$-Cartier divisor D on X), let (X', D') be a polarized G/H'-embedding and denote by Q and Q' the corresponding moment polytopes respectively. (We denote all data corresponding to G/H' and X' with primes: H', P', R', M', N',...). Also we denote by $\tilde{Q}$ and $\tilde{Q}'$ the pseudo-moment polytopes.

A first necessary condition for the existence of a dominant G-equivariant morphism from X to X', is the existence of a projection π from G/H to G/H'. In particular $H' \supset H$, $P' \supset P$ and $R' \supset R$. The projection π induces an injective morphism π^* from M' to M and a surjective morphism π_* from N to N'. We suppose that this necessary condition is satisfied in the rest of the section and we identify M' with $\pi^*(M')$.

Proposition 10. *(F. Knop) There exists a dominant G-equivariant morphism from X to X' if and only if, for any colored cone $(\mathcal{C}, \mathcal{F})$ of X, there exists a colored cone $(\mathcal{C}', \mathcal{F}')$ of X' such that $\pi_*(\mathcal{C}) \subset \mathcal{C}'$ and for any element $\alpha \in \mathcal{F}$, either $\alpha \in R'$ or $\alpha \in \mathcal{F}'$.*

Moreover, if there exists a dominant G-equivariant morphism from X to X', let $\mathcal{O}$ be a G-orbit in X and denote by $(\mathcal{C}, \mathcal{F})$ the associated colored cone of $\mathbb{F}_X$. Then $\phi(\mathcal{O})$ is the G-orbit in X' associated to the minimal colored cone $(\mathcal{C}', \mathcal{F}')$ of $\mathbb{F}_{X'}$ such that $\pi_*(\mathcal{C}) \subset \mathcal{C}'$.

Using Proposition 10, a dominant G-equivariant morphism from X to X' permits to define a map from the set of colored edges in $\mathbb{F}_X$ to $\mathbb{F}_{X'}$ (by taking $(\mathcal{C}', \mathcal{F}')$ minimal such that $\pi_*(\mathcal{C}) \subset \mathcal{C}'$, even $(\{0\}, \emptyset)$ if $\pi_*(\mathcal{C}) = \{0\}$). In order to rewrite Proposition 10 in terms of moment polytopes, we need to define the equivalent of this map in terms of polytope, i.e. a map ψ from the set of facets of $\tilde{Q}$ to the set of faces of $\tilde{Q}'$ (including $\tilde{Q}'$ itself).

First, note a general fact on polytopes: if $\mathcal{P}$ is a polytope in $\mathbb{Q}^r$, then for any affine half-space $\mathcal{H}^+$ delimited by an affine hyperplane $\mathcal{H}$ in $\mathbb{Q}^r$, there exists a unique face F of $\mathcal{P}$ and a point $x \in \mathbb{Q}^r$ such that F is defined by $x + \mathcal{H}$ (i.e. $F = \mathcal{P} \cap (x + \mathcal{H})$ and $\mathcal{P} \subset v + \mathcal{H}^+$). Then, for any facet F of $\tilde{Q}$, let $\mathcal{H}^+$ be the affine half-space in $M_{\mathbb{Q}}$, delimited by an affine hyperplane $\mathcal{H}$, such that $F = \mathcal{H} \cap \tilde{Q}$ and $\tilde{Q} \subset \mathcal{H}^+$. If $\mathcal{H}^+ \cap M'_{\mathbb{Q}} \neq M'_{\mathbb{Q}}$, it is an affine half-space in $M'_{\mathbb{Q}}$ and, applying the above fact to $\mathcal{P} = \tilde{Q}'$, it gives a unique face F' of $\tilde{Q}'$. We set $\psi(F) = F'$. And if $\mathcal{H}^+ \cap M'_{\mathbb{Q}} = M'_{\mathbb{Q}}$, we set $\psi(F) = \tilde{Q}'$.

To understand better what does the map ψ, we give two examples in the family of toric varieties.

Example 3. The first two polytopes correspond respectively to the blow-up of a point in $\mathbb{P}^2$ and $\mathbb{P}^2$. Then $\psi([AB]) = [A'B']$, $\psi([BC]) = [B'C']$, $\psi([CD]) = [C'A']$ and $\psi([AD]) = A'$. The last two polytopes correspond respectively $\mathbb{P}^2$ and $\mathbb{P}^1$. Then $\psi([AB]) = [A'B']$, $\psi([BC]) = B'$ and $\psi([AC]) = A'$.

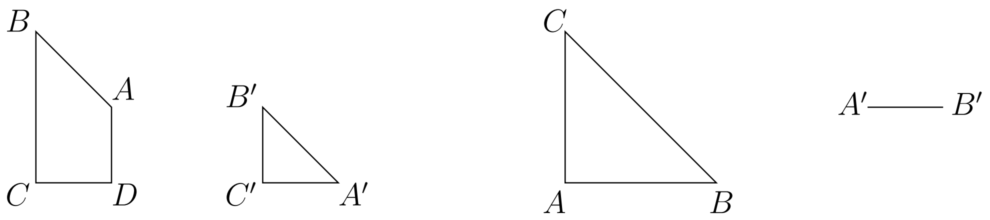


Figure 5: Polytopes in $\mathbb{Q}^2$

We can now characterize the existence of dominant G-equivariant morphisms.

Proposition 11. *Under the above necessary condition, there exists a dominant G-equivariant morphism from X to X', if and only if*

1. *for any set $\mathcal{G}$ of facets of $\tilde{Q}$, $\cap_{F \in \mathcal{G}} F \neq \emptyset$ implies $\cap_{F \in \mathcal{G}} \psi(F) \neq \emptyset$, and*

2. *for any $\alpha \in S \backslash R$ such that $Q \cap W_{\alpha,P} \neq \emptyset$, we have $Q' \cap W_{\alpha,P} \neq \emptyset$.*

Remark that, in Proposition 11, we could replace $\tilde{Q}$ by Q (by extension of the definition of ψ).

In Example 3, we can check that there is a $(\mathbb{C}^*)^2$-equivariant morphism from the blow-up of a point in $\mathbb{P}^2$ to $\mathbb{P}^2$, but there is no morphism from $\mathbb{P}^2$ to $\mathbb{P}^1$ (because $[AC]\cap[BC]=C$ and $\psi([AC])\cap\psi([BC])=\emptyset$).

Proof. For the second condition, remark that if $\alpha\in S\backslash R$ is in R', then Q' is contained in $W_{\alpha,P}$. And, if $\alpha\in S\backslash R'$, $Q'\cap W_{\alpha,P}\neq\emptyset$ is equivalent to $Q'\cap W_{\alpha,P'}\neq\emptyset$ because $Q'\subset X(P')$ and $W_{\alpha,P'}=X(P')\cap W_{\alpha,P}$. Recall also that $Q\cap W_{\alpha,P}\neq\emptyset$ is equivalent to $\alpha\in\mathcal{F}_X$ (and $Q'\cap W_{\alpha,P'}\neq\emptyset$ is equivalent to $\alpha\in\mathcal{F}_{X'}$). In particular the second item in Proposition 11 is equivalent to: for any $\alpha\in S\backslash R$, $\alpha\in\mathcal{F}_X$ implies $\alpha\in\mathcal{F}_{X'}$ or $\alpha\in S\backslash R'$.

Since X and X' are complete, we can rewrite Proposition 10. Denote by y_i with $i\in\{1,\dots,k\}$ (resp. y'_i with $i\in\{1,\dots,k'\}$) the primitive elements of the egdes of the colored fan of X (resp. X'). For all $j\in\{1,\dots,k\}$, let J'_j be the minimal subset of $\{1,\dots,k'\}$ such that $\pi_*(y_j)$ is in the cone $\mathcal{C}'_j$ generated by the y'_i with $i\in J'_j$ (it is never empty because X' is complete). We prove now that there exists a dominant G-equivariant morphism from X to X' if and only if:

- for all (colored) cone $\mathcal{C}$ of X, generated by the y_j with $j\in J_{\mathcal{C}}$, the cone generated by the y'_i with $i\in\cup_{j\in J_{\mathcal{C}}}J'_j$ is contained in a (colored) cone of X';
- and $\mathcal{F}_X$ is contained in $R'\cup\mathcal{F}_{X'}$.

Indeed, let $\mathcal{C}$ be a cone of X, generated by the y_j with $j\in J_{\mathcal{C}}$. If there exists a cone $\mathcal{C}'$ of X' such that $\pi_*\mathcal{C}\subset\mathcal{C}'$, then $\mathcal{C}'$ contains all y_j with $j\in J_{\mathcal{C}}$. Then by minimality of J'_j, it also contains the cones $\mathcal{C}'_j$ with $j\in J_{\mathcal{C}}$ and then the cone generated by the y'_i with $i\in\cup_{j\in J_{\mathcal{C}}}J'_j$. Conversely, if the cone generated by the y'_i with $i\in\cup_{j\in J_{\mathcal{C}}}J'_j$ is contained in a cone $\mathcal{C}'$ of X', then it is obvious that $\pi_*(\mathcal{C})\subset\mathcal{C}'$. And, the condition on colors is the same in both cases, since for any colored cone $(\mathcal{C},\mathcal{F})$ of a colored fan of a G/H-embedding Y, $\mathcal{F}=\{\alpha\in\mathcal{F}_Y\mid\alpha^\vee_M\in\mathcal{C}\}$.

Now, the proposition comes from the bijective correspondence between colored cones of X (resp. X') and faces of Q (resp. Q'), the first paragraph of the proof, and the following fact: the intersection of some facets of Q is not empty if and only if the cone generated by the inward-pointing normal vectors corresponding to these facets is included in a cone of $\mathbb{F}_X$ (and this latter cone corresponds to the face defined as the intersection of these facets). □

Corollary 12. *Suppose there exists a dominant G-equivariant morphism ϕ from X to X'. Let $\mathcal{O}$ be the G-orbit in X associated to a face $\cap_{F\in\mathcal{G}}F$ (where $\mathcal{G}$ is a set of facets). Then $\phi(\mathcal{O})$ is the G-orbit in X' associated to the face $\cap_{F\in\mathcal{G}}\psi(F)$ of Q'.*

Proof. We just rewrite, in terms of polytopes, the last statement of Proposition 10. □

2.5 Curves in horospherical varieties

We begin this section by collecting, in the following theorem, some results on curves in spherical varieties due to M. Brion [Bri93]. We denote by $N_1(X)$ the group of numerical classes of 1-cycles of the variety X. Recall that $NE(X)$ is the convex cone in $N_1(X)$ generated by effective 1-cycles. We denote by $N^1(X)$ the group of numerical classes of Cartier divisors of X, it is the dual of $N_1(X)$.

Theorem 13 (M. Brion). *Let X be a complete spherical variety. Let Y and Z be two distinct closed G-orbits in X. Note that Y and Z are flag varieties, so that they have exactly one point fixed by B.*

1. *There exists a B-stable curve C containing the B-fixed points of Y and Z if and only if the two (maximal) colored cones in the colored fan $\mathbb{F}_X$ corresponding to the two closed G-orbits Y and Z intersect along a one-codimensional colored cone μ. And, in that case, C is unique, isomorphic to $\mathbb{P}^1$, we denote it by C_μ.*

2. *The space $N_1(X)_\mathbb{Q}$ is generated by the classes of the curves C_μ with μ any one-codimensional colored cone in $\mathbb{F}_X$, and some classes $[C_{D,Y}]$ with Y any closed G-orbit of X and D any irreducible B-stable divisor of X not containing Y.*

3. *Any class $[C_{D,Y}]$ that is not in a ray generated by the class of a curve C_μ is represented by a (not unique in general) B-stable curve $C_{D,Y}$ admitting the B-fixed point of Y as unique B-fixed point.*

4. *The cone $NE(X)$ is generated by all the classes $[C_\mu]$ and $[C_{D,Y}]$.*

In the proof of these results, M. Brion also gives some formulas [Bri93, 3.2]. Here, we write them in the case of horospherical varieties, but they are true for spherical varieties. Recall, that any $\mathbb{Q}$-Cartier divisor δ of a G/H-embedding X defines a piecewise linear function h_δ, linear on each cone of $\mathbb{F}_X$.

Lemma 14 (M. Brion). *Let X be a complete G/H-embedding and let $\delta=\sum_{i=1}^r a_iX_i+\sum_{\alpha\in S\backslash R}a_\alpha D_\alpha$ be a Cartier divisor of X.*

Let μ be a one-codimensional colored cone of $\mathbb{F}_X$. Denote by μ_+ and μ_- the two maximal colored cone of $\mathbb{F}_X$ containing μ. Let $\chi_\mu\in M$ be a primitive element of the line orthogonal to μ (in $M_\mathbb{Q}$). Consider χ_μ as a linear function on $N_\mathbb{Q}$ and suppose that χ_μ is positive on μ_+. Then the intersection $\delta\cdot C_\mu$ is the rational number such that

$$(h_\delta)_{|\mu_+}-(h_\delta)_{|\mu_-}=(\delta\cdot C_\mu)\chi_\mu.$$

(Note that $(h_\delta)_{|\mu_+}-(h_\delta)_{|\mu_-}$ and χ_μ both vanish on μ, so they are collinear.)

Let Y be a closed G-orbit of X and D_α be an irreducible B-stable divisor of X that does not contain Y (with $\alpha\in S\backslash R$). Denote by $(\mathcal{C}_Y,\mathcal{F}_Y)$ the maximal cone of $\mathbb{F}_X$ associated to the G-orbit Y. Let C be a curve in the class $C_{D_\alpha,Y}$. Then

$$\delta\cdot C=a_\alpha-(h_\delta)_{|\mathcal{C}_Y}(\alpha_M^\vee).$$

(Note that, $\alpha_M^\vee$ is not in $\mathcal{C}_Y$ by hypothesis, but $(h_\delta)_{|\mathcal{C}_Y}$ is linear so it can be extend to a unique linear function of $N_\mathbb{Q}$, and by abuse of notation, we call them both $(h_\delta)_{|\mathcal{C}_Y}$.)

Now, in the particular case of horospherical varieties, we complete this collection of results by the following proposition that describes an explicit curve in all the classes of the form $[C_{D,Y}]$. Note that a homogeneous G-space has a B-fixed point if and only if it is complete (projective), so that the B-fixed points of a complete G-variety are the (unique) B-fixed points of its closed G-orbits. Recall that s_α is the simple reflection associated to a simple root α, and w_0 is the longest element of the Weyl group of (G,T).

Proposition 15. *1. Let X be a complete horospherical variety. If a B-stable curve C in X contains a unique B-fixed point y, then C is contained in the closed G-orbit $Y := G \cdot y$. In particular, it is a Schubert subvariety of Y.*

2. For any closed G-orbit Y and any divisor D_α that does not contain Y, the class of $[C_{D_\alpha,Y}]$ is represented by the Schubert variety of Y given by the simple root α (i.e. $\overline{Bs_\alpha \cdot y}$, where y is the B-fixed point of Y). We denote it by $C_{\alpha,Y}$.

Proof. 1. Let C be a B-stable curve in X. Suppose that C is not contained in Y. By replacing X by the closure of the biggest G-orbit of X that intersects C, we can assume that C intersects the open G-orbit G/H, which is at least of rank one. And then the intersection $C \cap G/H$ is an open set of C. Recall that G/H is a G-equivariant torus fibration over the flag variety G/P. Then $C \cap G/H$ is the fiber P/H of this fibration over P/P. Indeed, the B-orbits of G/H are the inverse image of the Schubert cells of G/P. The (unique) smallest one is P/H, but by hypothesis, P/H has positive dimension. Then, since $C \cap G/H$ is B-stable, it is P/H. Moreover P/H has to be one-dimensional.

We now prove that C has two B-fixed points. By the previous paragraph, P/H is one-dimensional, i.e. X is of rank one. Then, there exists a $\mathbb{P}^1$-bundle $\tilde{X}$ over G/P and a G-equivariant birational morphism $\phi : \tilde{X} \longrightarrow X$ ($\tilde{X}$ is the toroidal variety over X, see [Pas06, Example 1.13 (2)]). The closure of the P-orbit P/H in $\tilde{X}$ is the fiber $\mathbb{P}^1$ over P/P. Then it has exactly two B-fixed points corresponding to the two $\mathbb{C}^*$-fixed points of the toric variety $\mathbb{P}^1$. Moreover, the two closed G-orbits of $\tilde{X}$ are send, by ϕ, to the two closed G-orbits of X respectively (here we use that X is horospherical of rank one). Then $C = \phi(\tilde{C})$ has also two B-fixed points (and is isomorphic to $\mathbb{P}^1$).

2. Let Y be a G-closed orbit of X and let $\alpha \in S\backslash R$ such that D_α does not contain Y. Let $C_{\alpha,Y} := \overline{Bs_\alpha \cdot y}$, where y is the B-fixed point of Y. Let $D = \sum_{i=1}^r b_i X_i + \sum_{\beta\in S\backslash R} b_\beta D_\beta$ be a Cartier divisor of X. We want to compute $D \cdot C_{\alpha,Y}$. Since D is Cartier, there exists an eigenvector $f_Y(D)$ of B in the set of rational functions on X such that the support of $D - \operatorname{div}(f_Y(D))$ is in the union of irreducible B-stable divisors that do not contain Y. Denote by $\chi_Y(D)$ the weight of the eigenvector $f_Y(D)$. For any divisor X_i with $i \in \{1, \dots, r\}$ such that $Y \not\subset X_i$, we clearly have $X_i \cdot C_{\alpha,Y} = 0$ because Y and X_i are disjoint. Hence

$$D \cdot C_{\alpha,Y} = (D - \operatorname{div}(f_Y(D))) \cdot C_{\alpha,Y} = \sum_{\beta,\, Y \not\subset D_\beta} (b_\beta - \langle \chi_Y(D), \beta_M^\vee \rangle) D_\beta \cdot C_{\alpha,Y}.$$

We conclude, by Lemmas 14 and 16. □

Lemma 16. *Let Y be a G-closed orbit of X and let α, $\beta \in S\backslash R$ such that D_α and D_β do not contain Y. Then $D_\beta \cdot C_{\alpha,Y} = \delta_{\alpha\beta}$, where δ is the Kronecker delta.*

Proof. Denote by y the B-fixed point of Y. Remark that y is also fixed by P. By [Pas06, Lemma 2.8], there exists a unique G-equivariant morphism ϕ from the open G-stable set $X_Y := \{x \in X \mid$

$\overline{G \cdot x} \supset Y\}$ to the closed G-orbit Y. Note that $y = \phi(H/H)$ because y is the only point of Y fixed by $U \subset H$ and, similarly, ϕ is the identity on Y.

Recall that D_β is the closure in X of the B-orbit $Bs_\beta w_0 P/H$. Then $D_\beta \cap Y$ is contained in (and then equals) the closure of $\phi(Bs_\beta w_0 P/H)$ in Y, which is the Schubert variety $\overline{Bs_\beta w_0 \cdot y}$ of Y. Moreover, since D_β does not contain Y, $D_\beta \cap Y$ is a divisor of Y. Hence $D_\beta \cdot C_{\alpha,Y}$ equals the intersection number $(D_\beta \cap Y) \cdot C_{\alpha,Y}$, which is $\delta_{\alpha\beta}$ (see for example [Bri05, Section 3.1] for intersections of dual Schubert varieties in a flag variety). ☐

With the correspondence between colored fans and moment polytopes, if X is a projective horospherical variety and D is an ample $\mathbb{Q}$-Cartier divisor of X, we denote by C_μ for any edge μ of Q_D, and by $C_{\alpha,v}$ for any $\alpha \in S\backslash R$ and any vertex v not contained in $W_{\alpha,P}$, the curves defined above in terms of the colored fans (one-codimensional colored cones correspond to edges of moment polytopes, and closed G-orbits correspond to maximal colored cones and then to vertices of moment polytopes).

The following results is a direct consequence of Lemma 14.

Proposition 17. *Let X be a projective horospherical variety and D an ample $\mathbb{Q}$-Cartier divisor of X.*

Then, for any edge μ of the moment polytope Q_D, the intersection number $D.C_\mu$ is the integral length of μ, i.e. the length of μ divided by the length of the primitive element in the direction of μ.

And for any $\alpha \in S\backslash R$ and for any vertex v of Q_D not in the wall $W_{\alpha,P}$, we have $D.C_{\alpha,v} = \langle v, \alpha^\vee \rangle$.

3 One-parameter families of polytopes

In this section, we study some one-parameter families of polytopes. This section can be read independently from the rest of the paper. Note that Corollary 28 is an essential tool in the proofs of Theorems 1 and 2.

3.1 A first one-parameter family of polytopes: definitions and results

Let n and m be two positive integers. Consider three matrices A, B and C respectively in $M_{m\times 1}(\mathbb{Q})$, $M_{m\times 1}(\mathbb{Q})$ and $M_{m\times n}(\mathbb{Q})$. Then we define a first family of polyhedrons indexed by $\epsilon \in \mathbb{Q}$ as follows:

$$P^\epsilon := \{x \in \mathbb{Q}^n \mid Ax \geq B + \epsilon C\}.$$

We do not exclude the case where some lines of A are zero. Note that P^ϵ can be empty (even for all $\epsilon \in \mathbb{Q}$).

If there exists $\epsilon \in \mathbb{Q}$ such that P^ϵ is a (not empty) polytope (i.e. is bounded), then there is no non-zero $x \in \mathbb{Q}^n$ satisfying $Ax \geq 0$. Inversely, if there exists $\epsilon \in \mathbb{Q}$ such that P^ϵ is not bounded then there exists a non-zero $x \in \mathbb{Q}^n$ satisfying $Ax \geq 0$ (indeed, P^ϵ contains at least an affine half-line and x can be taken to be a generator of the direction of this half-line).

From now on, we suppose that there is no non-zero $x \in \mathbb{Q}^n$ satisfying $Ax \geq 0$, so that all P^ϵ are polytopes (maybe empty).

Let $I_0 := \{1, \dots, m\}$. For any matrix $\mathcal{M}$ and any $i \in I_0$, we denote by $\mathcal{M}_i$ the matrix consisting of the line i of $\mathcal{M}$. And more generally, for any subset I of I_0 we denote by $\mathcal{M}_I$ the matrix consisting of the lines $i \in I$ of $\mathcal{M}$.

Let $\epsilon \in \mathbb{Q}$. Denote by $\mathcal{H}_i^\epsilon$ the hyperplane $\{x \in \mathbb{Q}^n \mid A_i x = B_i + \epsilon C_i\}$. For any $I \subset I_0$, denote by F_I^ϵ the face of P^ϵ defined by

$$F_I^\epsilon := (\bigcap_{i \in I} \mathcal{H}_i^\epsilon) \cap P^\epsilon.$$

Note that for any face F^ϵ of P^ϵ there exists a unique maximal $I \subset I_0$ such that $F^\epsilon = F_I^\epsilon$ (we include the empty face and P^ϵ itself).

Let $I \subset I_0$. Define Ω^0_{I,I_0} to be the set of $\epsilon \in \mathbb{Q}$ such that F_I^ϵ is not empty; define Ω^1_{I,I_0} to be the set of $\epsilon \in \mathbb{Q}$ such that, if $I' \subset I_0$ satisfies $F_I^\epsilon = F_{I'}^\epsilon$, then $I' \subset I$. In other words, $\epsilon \in \Omega^0_{I,I_0}$ if and only if there exists $x \in \mathbb{Q}^n$ such that $Ax \geq B + \epsilon C$, and $\epsilon \in \Omega^1_{I,I_0}$ if and only if there exists $x \in \mathbb{Q}^n$ such that $A_I x = B_I + \epsilon C_I$ and $A_{I_0 \backslash I} x > B_{I_0 \backslash I} + \epsilon C_{I_0 \backslash I}$.

To simplify notation, we often write i instead of $\{i\}$, for any $i \in I_0$.

Remark that if $\epsilon \in \Omega^1_{\emptyset,I_0}$, the polytope P^ϵ is of dimension n (i.e. has a non-empty interior). And, for any $i \in I_0$, if $\epsilon \in \Omega^1_{i,I_0}$ and $A_i \neq 0$, F_i^ϵ is a facet of P^ϵ.

Example 4. Consider the following matrices

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix}, \ B = \begin{pmatrix} -4 \\ 1 \\ -1 \\ 0 \\ 0 \\ -5 \end{pmatrix} \text{ and } C = \begin{pmatrix} 3 \\ -1 \\ 1 \\ 0 \\ 0 \\ 3 \end{pmatrix},$$

defining a family of polytopes $(P^\epsilon)_{\epsilon \in \mathbb{Q}}$ in $\mathbb{Q}^2$. In Figure 6, we represent three of these polytopes with the 6 hyperplanes corresponding to the 6 lines of the above matrices.

Then $I_0 = \{1, 2, 3, 4, 5, 6\}$.
For $I = \{2, 3, 5\}$, we have $\Omega^0_{I,I_0} =]-\infty, 1]$ and $\Omega^1_{I,I_0} =]-\infty, 1[$.
For $I = \{2, 3\}$, we have $\Omega^0_{I,I_0} =]-\infty, 1]$ and $\Omega^1_{I,I_0} = \emptyset$.
For $I = \{4, 5\}$, we have $\Omega^0_{I,I_0} = \{1\}$ and $\Omega^1_{I,I_0} = \emptyset$.
For $I = \{2, 3, 4, 5\}$, we have $\Omega^0_{I,I_0} = \{1\}$ and $\Omega^1_{I,I_0} = \{1\}$.
We also have $\Omega^1_{1,I_0} =]-\infty, \frac{5}{4}[$, $\Omega^1_{2,I_0} =]-\infty, 1[$, $\Omega^1_{3,I_0} =]-\infty, \frac{5}{4}[$, $\Omega^1_{4,I_0} =]1, \frac{5}{4}[$, $\Omega^1_{5,I_0} = \emptyset$, $\Omega^1_{6,I_0} =]-\infty, 0[$ and $\Omega^1_{\emptyset,I_0} =]-\infty, \frac{5}{4}[$.

If a subfamily of $(P^\epsilon)_{\epsilon \in \mathbb{Q}}$ is a family of G/H-polytopes (where G/H is a horospherical homogeneous space), then the equivalence of G/H-polytopes given in Definition 6 restricts to an

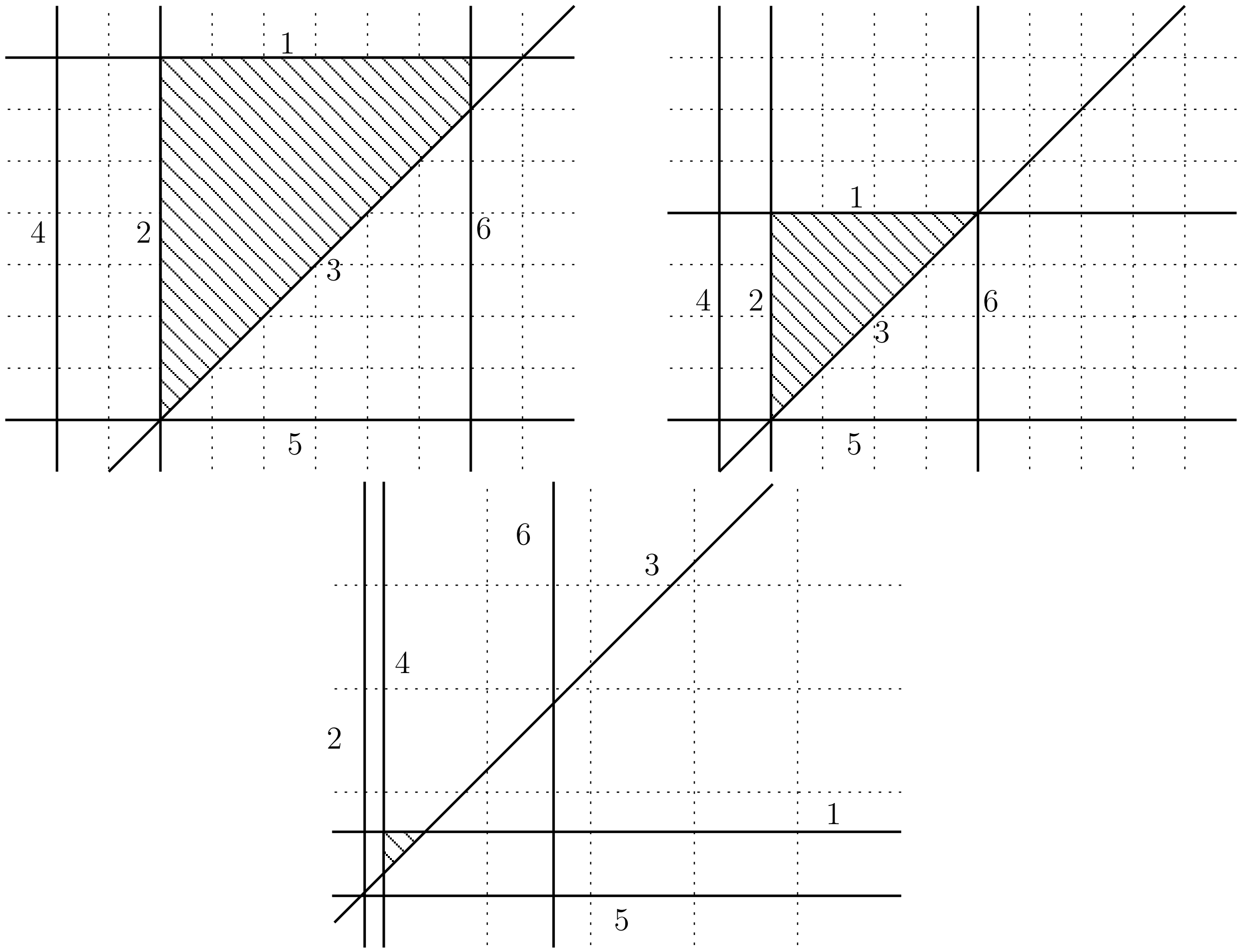


Figure 6: P^{-1}, P^{0} and $P^{\frac{11}{10}}$

equivalence $\mathcal{E}_1$ on the subfamily of $(P^\epsilon)_{\epsilon\in\mathbb{Q}}$. In this section, we independently define an equivalence $\mathcal{E}_2$ on a particular subfamily of $(P^\epsilon)_{\epsilon\in\mathbb{Q}}$. We will prove later that $\mathcal{E}_1 = \mathcal{E}_2$ in the case where the subfamily of $(P^\epsilon)_{\epsilon\in\mathbb{Q}}$ is a family of G/H-polytopes (see Proposition 29).

Let K_0 be the subset of I_0 consisting of indices of zero lines of A. Fix a subset K of I_0 containing K_0 and define $\Omega^{max}_{K,I_0} := \Omega^1_{\emptyset,I_0} \cap \bigcap_{i\in I_0\backslash K} \Omega^1_{i,I_0}$.

Definition 5. Let ϵ and η both in Ω^{max}_{K,I_0}. We say that the polytopes P^ϵ and P^η are equivalent if, for any $I \subset I_0$, ϵ and η are either both in Ω^1_{I,I_0}, either both in $\mathbb{Q}\backslash\Omega^0_{I,I_0}$, or both in $\Omega^0_{I,I_0}\backslash\Omega^1_{I,I_0}$. (In other words, F^ϵ_I and F^η_I are either both not empty faces with I maximal, either both empty, or both not empty with I not maximal.)

Example 5. In Example 4, if $K = \{4,5,6\}$ we have $\Omega^{max}_{K,I_0} =]-\infty, 1[$. And the equivalent classes in the family $(P^\epsilon)_{\epsilon\in\Omega^{max}_{K,I_0}}$ correspond to the two intervals $]-\infty, 0[$ and $]0, 1[$ and the singelton $\{0\}$.

Moreover, $1 \in \Omega^1_{\emptyset,I_0}$ and P^1 can be defined without the second inequality.

The properties of the family $(P^\epsilon)_{\epsilon\in\mathbb{Q}}$ observed in this example can be generalized as follows.

Theorem 18. *With the notation above, we get the following results.*

1. *If Ω^{max}_{K,I_0} is not empty, there exist an integer k and $\alpha_0 < \cdots < \alpha_k$ in $\mathbb{Q}\cup\{\pm\infty\}$, such that the equivalence classes of polytopes in the family $(P^\epsilon)_{\epsilon\in\Omega^{max}_{K,I_0}}$ correspond exactly to the open intervals $]\alpha_i,\alpha_{i+1}[$ for any $i\in\{0,\dots,k-1\}$ and the singletons $\{\alpha_i\}$ for any $i\in\{1,\dots,k-1\}$ with $\alpha_i\in\mathbb{Q}$. In particular, $\Omega^{max}_{K,I_0} =]\alpha_0,\alpha_k[$.*

2. *Suppose that $\alpha_k\in\mathbb{Q}$. If $\alpha_k\in\Omega^1_{\emptyset,I_0}$, there exists $I_1\subset I_0$ containing K such that $P^{\alpha_k} = P^{\alpha_k}_{I_1} := \{x\in\mathbb{Q}^n \mid A_{I_1}x\geq B_{I_1}+\alpha_k C_{I_1}\}$ and $\alpha_k\in\Omega^{max}_{K,I_1}$ (defined for the family $(P^\epsilon_{I_1})_{\epsilon\in\mathbb{Q}}$).*

   If $\alpha_k\notin\Omega^1_{\emptyset,I_0}$, there exist subsets J_1 and I_1 of I_0 such that $J_1\cap I_1=\emptyset$, $P^{\alpha_k} = P^{\alpha_k}_{I_1,J_1} := \{x\in \bigcap_{j\in J_1}\mathcal{H}^{\alpha_k}_j \mid A_{I_1}x\geq B_{I_1}+\alpha_k C_{I_1}\}$ and α_k is in $\Omega^{max}_{I_1\cap K,I_1}$ (defined for the family $(P^\epsilon_{I_1,J_1})_{\epsilon\in\mathbb{Q}}$ of polytopes in $\bigcap_{j\in J_1}\mathcal{H}^{\alpha_k}_j$).

   We have the same result for the other extremity α_0 of Ω^{max}_{K,I_0} (if it is finite).

3.2 Proof of Theorem 18

We study all the sets Ω^0_{I,I_0} and Ω^1_{I,I_0}.

For any $I\subset I_0$, denote by $\bar I$ the complement of I in I_0.

Lemma 19. *For any $I\subset I_0$, the sets Ω^0_{I,I_0} and Ω^1_{I,I_0} are convex subsets (possibly empty) of $\mathbb{Q}$.*

Proof. Let ϵ_1 and ϵ_2 be in Ω^0_{I,I_0}. Then there exist x_1 and x_2 in $\mathbb{Q}^n$ such that $A_Ix_1 = B_I+\epsilon_1C_I$, $A_Ix_2 = B_I+\epsilon_2C_I$, $A_{\bar I}x_1\geq B_{\bar I}+\epsilon_1C_{\bar I}$ and $A_{\bar I}x_2\geq B_{\bar I}+\epsilon_2C_{\bar I}$. Then, for any rational number $\lambda\in[0,1]$, we get easily by adding equalities and inequalities that $A_I(\lambda x_1+(1-\lambda)x_2) = B_I + (\lambda\epsilon_1+(1-\lambda)\epsilon_2)C_I$ and $A_{\bar I}(\lambda x_1+(1-\lambda)x_2)\geq B_{\bar I}+(\lambda\epsilon_1+(1-\lambda)\epsilon_2)C_{\bar I}$, so that $\lambda\epsilon_1+(1-\lambda)\epsilon_2$ is in Ω^0_{I,I_0}.

Replacing $\geq$ by $>$, we similarly prove the convexity of Ω^1_{I,I_0}. □

Lemma 20. *For any $I\subset I_0$, the set Ω^1_{I,I_0} is either reduced to a point, or an open subset (possibly empty) of $\mathbb{Q}$. Moreover, if Ω^1_{I,I_0} is reduced to a point ϵ_0, the system $A_IX = B_I+\epsilon C_I$ has a solution only if $\epsilon=\epsilon_0$; in particular, in that case, Ω^0_{I,I_0} is also reduced to ϵ_0.*

Proof. Suppose that Ω^1_{I,I_0} is neither empty nor reduced to a point. Let ϵ_1 and ϵ_2 two different points of Ω^1_{I,I_0}. Then there exist x_1 and x_2 in $\mathbb{Q}^n$ such that $A_Ix_1 = B_I+\epsilon_1C_I$, $A_Ix_2 = B_I+\epsilon_2C_I$, $A_{\bar I}x_1 > B_{\bar I}+\epsilon_1C_{\bar I}$ and $A_{\bar I}x_2 > B_{\bar I}+\epsilon_2C_{\bar I}$. In particular both linear systems $A_IX = B_I$ and $A_IX = C_I$ have solutions, hence for any $\epsilon\in\mathbb{Q}$, the linear system $A_IX = B_I+\epsilon C_I$ has a set of solutions, depending continuously on ϵ. More precisely, if x_B and x_C are solutions of $A_IX = B_I$ and $A_IX = C_I$ respectively, then the set of solutions of $A_IX = B_I+\epsilon C_I$ is $\ker(A_I)+x_B+\epsilon x_C$. Now, let $\epsilon\in\Omega^1_{I,I_0}$, and let $y := x_0+x_B+\epsilon x_C$ with $x_0\in\ker(A_I)$ such that $A_{\bar I}y > B_{\bar I}+\epsilon C_{\bar I}$. Then there clearly exists a neighborhood $\mathcal{V}$ of ϵ in $\mathbb{Q}$ such that $A_{\bar{I}}(y+(\eta-\epsilon)x_C) > B_{\bar{I}} + \eta C_{\bar{I}}$ for all $\eta \in \mathcal{V}$. We also check easily that $A_I(y+(\eta-\epsilon)x_C) = B_I + \eta C_I$ so that $\mathcal{V} \subset \Omega^1_{I,I_0}$.

To prove the last statement, we suppose that the system $A_I X = B_I + \epsilon C_I$ has a solution for at least two values of ϵ, and then by the same arguments as above, we deduce that Ω^1_{I,I_0} is open, which gives a contradiction. The result follows immediately. ☐

Lemma 21. *For any $I \subset I_0$ such that Ω^1_{I,I_0} is not empty, we have $\Omega^1_{I,I_0} \subset \Omega^0_{I,I_0} \subset \overline{\Omega^1_{I,I_0}}$.*

Proof. The first inclusion is obvious.

To prove the second one, remark that for all $\epsilon_1 \in \Omega^1_{I,I_0}$ and $\epsilon_2 \in \Omega^0_{I,I_0}$, the interval $[\epsilon_1, \epsilon_2[$ (or $]\epsilon_2, \epsilon_1]$) is contained in Ω^1_{I,I_0}. Indeed, if there exists x_1 and x_2 such that $A_I x_1 = B_I + \epsilon_1 C_I$, $A_I x_2 = B_I + \epsilon_2 C_I$, $A_{\bar{I}} x_1 > B_{\bar{I}} + \epsilon_1 C_{\bar{I}}$ and $A_{\bar{I}} x_2 \geq B_{\bar{I}} + \epsilon_2 C_{\bar{I}}$, then for any $\lambda \in]0,1]$, $x_\lambda := \lambda x_1 + (1-\lambda) x_2$ and $\epsilon_\lambda := \lambda \epsilon_1 + (1-\lambda)\epsilon_2$ clearly satisfy $A_I x_\lambda = B_I + \epsilon_\lambda C_I$, $A_{\bar{I}} x_\lambda > B_{\bar{I}} + \epsilon_\lambda C_{\bar{I}}$, so that $\epsilon_\lambda \in \Omega^1_{I,I_0}$. This remark implies directly that any $\epsilon_2 \in \Omega^0_{I,I_0}$ is in $\overline{\Omega^1_{I,I_0}}$, as soon as Ω^1_{I,I_0} is not empty. ☐

Lemma 22. *Let $I \subset I_0$ be such that Ω^1_{I,I_0} is not empty. Then the supremum and the infimum (well-defined in $\mathbb{R}$) of Ω^1_{I,I_0} are either infinite or rational numbers. Moreover, $\Omega^0_{I,I_0} = \overline{\Omega^1_{I,I_0}}$.*

Proof. If Ω^1_{I,I_0} is reduced to a point, by Lemma 21 we have nothing to prove, so we suppose that Ω^1_{I,I_0} contains at least two points.

Here (and only here), we need to consider the family of polytopes $(P^\epsilon)_{\epsilon \in \mathbb{R}}$. Note that all definitions and all results given until now are still available replacing $\mathbb{Q}$ by $\mathbb{R}$. When it is necessary, we denote by $\Omega(\mathbb{Q})$ and by $\Omega(\mathbb{R})$ the sets Ω defined respectively in $\mathbb{Q}$ and in $\mathbb{R}$. Now, by Lemmas 19 and 21, it is enough to prove that, if they are finite, the supremum and the infimum of $\Omega^1_{I,I_0}(\mathbb{R})$ are rational numbers contained in the set $\Omega^0_{I,I_0}(\mathbb{Q})$.

Suppose that the supremum $\epsilon_0 \in \mathbb{R}$ of $\Omega^1_{I,I_0}(\mathbb{R})$ is finite. Let $\epsilon_1 \in \Omega^1_{I,I_0}(\mathbb{R})$. Then, for any $\epsilon \in [\epsilon_1, \epsilon_0]$, the polytope P^ϵ is contained in the polytope

$$P^{[\epsilon_1,\epsilon_0]} := \{x \in \mathbb{R}^n \mid Ax \geq \mathrm{Min}(B + \epsilon_1 C, B + \epsilon_0 C)\},$$

where the mimimum Min is taken line by line. Now for all $\epsilon \in [\epsilon_1, \epsilon_0[$, let $x^\epsilon \in F^\epsilon_I$ such that $A_{\bar{I}} x^\epsilon > B_{\bar{I}} + \epsilon C_{\bar{I}}$ (and $A_I x^\epsilon = B_I + \epsilon C_I$). Since the points x^ϵ are in the compact set $P^{[\epsilon_1,\epsilon_0]}$ of $\mathbb{R}^n$, there exists $x_0 \in \mathbb{R}^n$ such that $A_I x_0 = B_I + \epsilon_0 C_I$ and $A_{\bar{I}} x_0 \geq B_{\bar{I}} + \epsilon_0 C_{\bar{I}}$, i.e. $x_0 \in F^{\epsilon_0}_I$. It means that ϵ_0 is in $\Omega^0_{I,I_0}(\mathbb{R})$.

Define the maximal subset J of I_0 containing I such that for any $x \in F^{\epsilon_0}_I$ we have $A_J x = B_J + \epsilon_0 C_J$. We may assume that $A_{\bar{J}} x_0 > B_{\bar{J}} + \epsilon_0 C_{\bar{J}}$. We now prove that, for any $\epsilon_1 \in \Omega^1_{I,I_0}$, the face $F^{\epsilon_1}_J$ is empty. Indeed, if it is not empty, there exists $x_1 \in \mathbb{R}^n$ such that $A_J x_1 = B_J + \epsilon_1 C_J$ and $A_{\bar{J}} x_1 \geq B_{\bar{J}} + \epsilon_1 C_{\bar{J}}$. Let $\eta > 0$, let $x_2 := x_0 + \eta(x_0 - x_1)$ and let $\epsilon_2 := \epsilon_0 + \eta(\epsilon_0 - \epsilon_1)$. Then we have $A_J x_2 = B_J + \epsilon_2 C_J$ and, for η small enough, we have $A_{\bar{J}} x_2 > B_{\bar{J}} + \epsilon_2 C_{\bar{J}}$. Hence $F^{\epsilon_2}_I \supset F^{\epsilon_2}_J$ is not empty, which contradicts the fact that $\epsilon_2 > \epsilon_0$ is not in Ω^0_{I,I_0} (see Lemma 21).

We now claim that the intersection of the vector space $\mathrm{Im}(A_J)$ with the affine line $\{B_J + \epsilon C_J \mid \epsilon \in \mathbb{R}\}$ is reduced to the point $B_J + \epsilon_0 C_J$. Indeed, $B_J + \epsilon_0 C_J$ clearly belongs to this intersection, which can be either reduced to a point or the affine line. But, if some $\epsilon \in \Omega^1_{I,I_0}$ is in this intersection, then there exists $x_3 \in \mathbb{R}^n$ such that $A_J x_3 = B_J + \epsilon C_J$. Then, for $\eta > 0$ small enough, we can prove that $\eta x_3 + (1-\eta)x_0$ is in $F_J^{\epsilon_1}$ with $\epsilon_1 = \eta\epsilon + (1-\eta)\epsilon_0$, which is not possible because $F_J^{\epsilon_1}$ is necessarily empty. To conclude that $\epsilon_0 \in \mathbb{Q}$, it is now enough to recall that A, B and C have rational coefficients.

It remains to prove that x_0 can be chosen in $\mathbb{Q}^n$ so that ϵ_0 is in $\Omega^0_{I,I_0}(\mathbb{Q})$. But x_0 can be taken in an open set of the set of solutions of $A_J x = B_J + \epsilon_0 C_J$. We conclude by noticing that $B_J + \epsilon_0 C_J$ has rational coefficients.

For the infimum of Ω^1_{I,I_0}, it is the same proof (for example by replacing ϵ and C by their opposites). □

Corollary 23. *Let $I \subset I_0$. We have the following cases:*

1. *Ω^1_{I,I_0} is an open interval of $\mathbb{Q}$ (with extremities in $\mathbb{Q} \cup \{\pm\infty\}$) and Ω^0_{I,I_0} is the closed interval $\overline{\Omega^1_{I,I_0}}$ (may both be empty);*
2. *Ω^1_{I,I_0} is a reduced to a point ϵ_0, then it equals Ω^0_{I,I_0} and the system $A_I X = B_I + \epsilon C_I$ has a solution only when $\epsilon = \epsilon_0$;*
3. *Ω^1_{I,I_0} is empty and Ω^0_{I,I_0} is the closure of some Ω^1_{J,I_0} with $I \subsetneq J \subset I_0$.*

Moreover, if $I = \emptyset$ or reduced to an index $i \in I_0 \backslash K_0$, then only Cases 1 and 3 are possible, so that Ω^1_{I,I_0} is open (or empty).

Proof. The first two items, and the fact that we cannot have more cases, can be deduced directly from Lemmas 20 and 22.

Suppose now that Ω^1_{I,I_0} is empty, but not Ω^0_{I,I_0}. Let $J \subset I_0$ be the maximal set containing I such that, for all $\epsilon \in \mathbb{Q}$, $F_I^\epsilon = F_J^\epsilon$. Then $J \neq I$ by hypothesis. Indeed, if $I = J$ then there exist subsets $J_1, \ldots, J_k$ of I_0 whose intersection is I, and rational numbers $\epsilon_1, \ldots, \epsilon_k$ in $\Omega^1_{J_1,I_0}, \ldots, \Omega^1_{J_k,I_0}$ respectively. Hence $\frac{\epsilon_1 + \cdots + \epsilon_k}{k}$ is in Ω^1_{I,I_0} that is not possible.

Then $\Omega^0_{I,I_0} = \Omega^0_{J,I_0}$ and by maximality of J, Ω^1_{J,I_0} is not empty. We conclude by using the first two items.

Now, for the last statement, remark that if I is empty or has a unique element $i \in I_0 \backslash K_0$, the system $A_I X = B_I + \epsilon C_I$ has solutions for all ϵ and then, we cannot be in Case 2. □

We now prove three lemmas to get the connectedness of equivalence classes of polytopes.

Lemma 24. *Let $I \subset I_0$ such that Ω^0_{I,I_0} is bounded, but neither empty nor reduced to a point. Denote by ϵ_1 and ϵ_2 the extremities of Ω^0_{I,I_0}. Then, there exist J_1 and J_2 containing I such that $\{\epsilon_1\} = \Omega^1_{J_1,I_0}$ and $\{\epsilon_2\} = \Omega^1_{J_2,I_0}$. In particular, if ϵ_1 and ϵ_2 are in Ω^{max}_{K,I_0}, P^{ϵ_1} is not equivalent to P^{ϵ_2}.*

Proof. By Corollary 23, we can suppose that $\Omega^0_{I,I_0} = \overline{\Omega^1_{I,I_0}} = [\epsilon_1, \epsilon_2]$. Then, there exists $j \in \bar{I}$ such that $x \in P^{\epsilon_1}$ and $A_I x = B_I + \epsilon_1 C_I$ imply that $A_j x = B_j + \epsilon C_j$. Let J_1 be the union of I with the set of all such indices j. In particular, $\epsilon_1 \in \Omega^1_{J_1,I_0}$. Moreover, since $\Omega^1_{J_1,I_0}$ is contained in $\Omega^0_{I,I_0} = [\epsilon_1, \epsilon_2]$, it cannot be open, and then by Corollary 23, $\Omega^1_{J_1,I_0}$ is reduced to ϵ_1.

Similarly, we prove the existence of J_2.

In particular, if ϵ_1 and ϵ_2 are in Ω^{max}_{K,I_0}, by definition, the equivalence classes of P^{ϵ_1} and P^{ϵ_2} are both reduced to one polytope and then distinct. $\square$

Lemma 25. *Let $I \subset I_0$ such that $\Omega^0_{I,I_0} = \{\epsilon_0\}$, with $\epsilon_0 \in \Omega^{max}_{K,I_0}$. There exists $I_1 \subset I_0$ such that $\Omega^0_{I_1,I_0}$ is an interval, not reduced to a point and with upper extremity ϵ_0.*

Proof. First, we can suppose that I is the maximal subset of I_0 such that $\Omega^0_{I,I_0} = \{\epsilon_0\}$, (i.e. such that $\Omega^1_{I,I_0} = \{\epsilon_0\}$). Then there exists $x_0 \in \mathbb{Q}^n$ satisfying $A_I x_0 = B_I + \epsilon_0 C_I$ and $A_{\bar{I}} x_0 > B_{\bar{I}} + \epsilon_0 C_{\bar{I}}$.

Consider a subset I_1 of I such that there exists $\epsilon_1 < \epsilon_0$ in $\Omega^0_{I_1,I_0}$. Choose I_1 maximal in I with this property. It exists (but can be empty) because ϵ_0 is in the open set Ω^{max}_{K,I_0}. Moreover $I_1 \neq I$. Then there exist $\epsilon_1 < \epsilon_0$ and $x_1 \in \mathbb{Q}^n$ such that $A_{I_1} x_1 = B_{I_1} + \epsilon_1 C_{I_1}$ and $A_{\bar{I}_1} x_1 \geq B_{\bar{I}_1} + \epsilon_1 C_{\bar{I}_1}$. Hence, for any $\epsilon \in]\epsilon_1, \epsilon_0[$, the element $x^\epsilon := \frac{(\epsilon_0-\epsilon)x_1 + (\epsilon-\epsilon_1)x_0}{\epsilon_0-\epsilon_1}$ satisfies $A_{I_1} x^\epsilon = B_{I_1} + \epsilon C_{I_1}$ and $A_{\bar{I}_1} x^\epsilon > B_{\bar{I}_1} + \epsilon C_{\bar{I}_1}$ by maximality of I_1. It proves that $]\epsilon_1, \epsilon_0[\subset \Omega^1_{I_1,I_0}$. In particular, with Corollary 23, $\Omega^1_{I_1,I_0}$ is open with closure equal to $\Omega^0_{I_1,I_0}$.

It is now enough to prove that the upper extremity of $\Omega^0_{I_1,I_0}$ is ϵ_0. Suppose the contrary, then there exists $\epsilon_2 > \epsilon_0$ and $x_2 \in \mathbb{Q}^n$ such that $A_{I_1} x_2 = B_{I_1} + \epsilon_2 C_{I_1}$ and $A_{\bar{I}_1} x_2 \geq B_{\bar{I}_1} + \epsilon_2 C_{\bar{I}_1}$. For any $\epsilon \in]\epsilon_0, \epsilon_2[$, the element $y^\epsilon := \frac{(\epsilon_2-\epsilon)x_0 + (\epsilon-\epsilon_0)x_2}{\epsilon_2-\epsilon_0}$ satisfies $A_{I_1} y^\epsilon = B_{I_1} + \epsilon C_{I_1}$ and $A_{\bar{I}_1} y^\epsilon \geq B_{\bar{I}_1} + \epsilon C_{\bar{I}_1}$. Moreover, for $\epsilon = \epsilon_0$ (i.e. $y^\epsilon = x_0$), we have $A_I y^\epsilon = B_I + \epsilon C_I$ and $A_{\bar{I}} y^\epsilon > B_{\bar{I}} + \epsilon C_{\bar{I}}$. Then by continuity, for any $\epsilon < \epsilon_0$ big enough, we have $A_{I_1} y^\epsilon = B_{I_1} + \epsilon C_{I_1}$, $A_{\bar{I}} y^\epsilon > B_{\bar{I}} + \epsilon C_{\bar{I}}$ and $A_{I \backslash I_1} y^\epsilon \geq B_{I\backslash I_1} + \epsilon C_{I \backslash I_1}$. Choose such an ϵ in $]\epsilon_1, \epsilon_0[$. Hence, there exist z^ϵ in the interval $]x^\epsilon, y^\epsilon[$ and some $j \in I \backslash I_1$ such that $A_{I_1} z^\epsilon = B_{I_1} + \epsilon C_{I_1}$, $A_{\bar{I}_1} z^\epsilon \geq B_{\bar{I}_1} + \epsilon C_{\bar{I}_1}$ and $A_j z^\epsilon = B_j + \epsilon C_j$. This contradicts the maximality of I_1, hence we have proved that the upper extremity of $\Omega^0_{I_1,I_0}$ is ϵ_0. $\square$

Lemma 26. *Let $]\eta, \epsilon[\subset \Omega^{max}_{K,I_0}$ such that there exists $I \subset I_0$ satisfying $\emptyset \neq \Omega^0_{I,I_0} \subset]\eta, \epsilon[$. Then P^η is not equivalent to P^ϵ.*

Proof. Among the set of subsets I of I_0 satisfying $\Omega^0_{I,I_0} \subset]\eta, \epsilon[$, choose the one whose lower extremity is minimal.

There are two cases: in the first one, $\Omega^0_{I,I_0} = \{\epsilon_0\}$. By Lemma 25, there exist $I_1 \subset I_0$ and $\epsilon_1 < \epsilon_0$ (may be $-\infty$) such that $\Omega^0_{I_1,I_0} = [\epsilon_1, \epsilon_0]$. By hypothesis of minimality, η is in $\Omega^0_{I_1,I_0}$. But ϵ is not in $\Omega^0_{I_1,I_0}$, so that P^η is not equivalent to P^ϵ.

Consider the second case: $\Omega^0_{I,I_0} = [\epsilon_1, \epsilon_2]$ with $\epsilon_1 < \epsilon_2$. Let J be the maximal subset of I_0 such that: $x \in Q^{\epsilon_1}$ and $A_I x = B_I + \epsilon_1 C_I$ imply that $A_J x = B_J + \epsilon_1 C_J$. In particular, J strictly contains I and $\epsilon_1 \in \Omega^1_{J,I_0}$. Then $\Omega^1_{J,I_0} \subset \Omega^0_{I,I_0}$ cannot be open, hence it is reduced to a point and we conclude by the first case. $\square$

Proof of Theorem 18.

1. By the last statement of Corollary 23, the set Ω^{max}_{K,I_0} is an open interval of $\mathbb{Q}$ (with extremities in $\mathbb{Q} \cup \{\pm\infty\}$, or empty).

   Now, let $\epsilon_0 \in \Omega^{max}_{K,I_0}$. Prove that the equivalence class of P^{ϵ_0} corresponds to a interval of Ω^{max}_{K,I_0} or is reduced to P^{ϵ_0}.

If there exists $I \subset I_0$ such that $\Omega^1_{I,I_0} = \{\epsilon_0\}$, then the equivalence class of P^{ϵ_0} is clearly reduced to P^{ϵ_0}. By Lemma 24, we get the same conclusion if there exists $I \subset I_0$ such that Ω^1_{I,I_0} is not empty and $\epsilon_0 \in \Omega^0_{I,I_0} \backslash \Omega^1_{I,I_0}$ (and also if there exists $I \subset I_0$ such that ϵ_0 is an extremity of Ω^0_{I,I_0}, by using Corollary 23 Case 3).

Suppose now that for all subsets I of I_0, $\Omega^1_{I,I_0} \neq \{\epsilon_0\}$, and ϵ_0 is not an extremity of Ω^0_{I,I_0}. Then the set of $\epsilon \in \Omega^{max}_{K,I_0}$ such that P^{ϵ} is equivalent to P^{ϵ_0} is the intersection of open intervals of $\mathbb{Q}$. Indeed, it is the intersection of some open intervals Ω^1_{I,I_0} and connected components of some $\mathbb{Q} \backslash \Omega^0_{I,I_0}$ (by Lemma 26).

2. Now, let ϵ_1 be an extremity of Ω^{max}_{K,I_0}. Suppose that $\epsilon_1 \in \Omega^1_{\emptyset,I_0}$. Define I_2 to be the set of indices $i \in I_0 \backslash K$ such that ϵ_1 is not in Ω^1_{i,I_0}. Denote by I_1 the complementary of I_2 in I_0. Then by definition, I_1 contains K and ϵ_1 is in the set Ω^{max}_{K,I_1}. Denote by $P^{\epsilon_1}_{I_1}$ the polyhedron $\{x \in \mathbb{Q}^n \mid A_{I_1}x \geq B_{I_1} + \epsilon_1 C_{I_1}\}$, and prove that $P^{\epsilon_1} = P^{\epsilon_1}_{I_1}$. We clearly have $P^{\epsilon_1} \subset P^{\epsilon_1}_{I_1}$, so that we have to prove that $P^{\epsilon_1}_{I_1}$ is contained in all closed half-spaces $\mathcal{H}^{+\epsilon_1}_i := \{x \in \mathbb{Q}^n \mid A_i x \geq B_i + \epsilon_1 C_i\}$ with $i \in I_2$, or equivalently that the interior $\{x \in \mathbb{Q}^n \mid A_{I_1}x > B_{I_1} + \epsilon_1 C_{I_1}\}$ of $P^{\epsilon_1}_{I_1}$ (that is not empty because $\epsilon_1 \in \Omega^1_{\emptyset,I_0}$) is contained in all open half-spaces $\mathcal{H}^{++\epsilon_1}_i := \{x \in \mathbb{Q}^n \mid A_i x > B_i + \epsilon_1 C_i\}$ with $i \in I_2$. Since the interior of $P^{\epsilon_1}_{I_1}$ intersects the connected component $\bigcap_{i \in I_2} \mathcal{H}^{++\epsilon_1}_i$ of $\mathbb{Q}^n \backslash \bigcup_{i \in I_2} \mathcal{H}^{\epsilon_1}_i$, it is enough to prove that the interior of $P^{\epsilon_1}_{I_1}$ does not intersect the union of hyperplanes $\bigcup_{i \in I_2} \mathcal{H}^{\epsilon_1}_i$.

   Let $i \in I_2$. Suppose that the interior of $P^{\epsilon_1}_{I_1}$ intersects the hyperplane $\mathcal{H}^{\epsilon_1}_i$. There exists $x_1 \in \mathbb{Q}^n$ such that $A_{I_1}x_1 > B_{I_1} + \epsilon_1 C_{I_1}$ and $A_i x_1 = B_i + \epsilon_1 C_i$. Since there also exists $x_0 \in \mathbb{Q}^n$ such that $Ax_0 > B + \epsilon_1 C$, any element x_2 of the interval $[x_1, x_0[$ is in the interior of $P^{\epsilon_1}_{I_1}$, in P^{ϵ_1} and in $\mathcal{H}^{\epsilon_1}_j$ for at least one $j \in I_2$. Let $J \subset I_2$ the maximal set such that $A_J x_2 = B_J + \epsilon_1 C_J$. Then the set $\mathcal{C} := \{x \in \mathbb{Q}^n \mid A_J x \geq B_J + \epsilon_1 C_J\}$ is a cone (of apex $x_2 + \ker(A_J)$) with non-empty interior. Let $j \in J$ such that $\mathcal{H}^{\epsilon_1}_j \cap \mathcal{C}$ is a facet of $\mathcal{C}$. By Lemma 27 below, all hyperplanes $\mathcal{H}^{\epsilon_1}_j$ with $j \in J$ are different, then there exists $y \in \mathcal{H}^{\epsilon_1}_j \cap \mathcal{C}$ that is not in the other hyperplanes $\mathcal{H}^{\epsilon_1}_i$ with $j \neq i \in J$. Since x_2 is in the interior of $P^{\epsilon_1}_{I_0 \backslash J}$, for any rational number $\lambda > 0$ small enough, the point $x_2 + \lambda(y - x_2)$ is in the interior of $P^{\epsilon_1}_{I_0 \backslash J}$, also in $\mathcal{H}^{\epsilon_1}_j \cap \mathcal{C}$ and not in $\mathcal{H}^{\epsilon_1}_i$ for all $i \in J$ different from j. This contradicts the fact that $\epsilon_1 \notin \Omega^1_{j,I_0}$.

   Suppose now that $\epsilon_1 \notin \Omega^1_{\emptyset,I_0}$. Let J_1 the maximal subset of I_0 such that P^{ϵ_1} is contained in the subspace $\mathcal{H}_{J_1} := \bigcap_{j \in J_1} \mathcal{H}^{\epsilon_1}_j$ of $\mathbb{Q}^n$. Then P^{ϵ_1} is clearly the polytope $\{x \in \bigcap_{j \in J_1} \mathcal{H}^{\epsilon_1}_j \mid A_{I_0 \backslash J_1}x \geq B_{I_0 \backslash J_1} + \epsilon_1 C_{I_0 \backslash J_1}\}$. Then $\epsilon_1 \in \Omega^1_{\emptyset, I_0 \backslash J_1}$ (defined in for the family of polytopes in $\mathcal{H}_{J_1}$). If $\epsilon_1 \in \Omega^{max}_{K \cap (I_0 \backslash J_1), I_0 \backslash J_1}$, then $I_1 = I_0 \backslash J_1$ gives the result. If not, ϵ_1 is an extremity of $\Omega^{max}_{K \cap (I_0 \backslash J_1), I_0 \backslash J_1}$ and $\epsilon_1 \in \Omega^1_{\emptyset, I_0 \backslash J_1}$. Then we apply what we proved in the previous paragraph in order to find $I_1 \subset I_0 \backslash J_1$ that gives the result.

□

Lemma 27. *If Ω^{max}_{K,I_0} is not empty, then for any i and j in $I_0 \backslash K$, the existence of $\lambda \in \mathbb{Q}_{>0}$ such that $A_i = \lambda A_j$, implies that $i = j$.*

Proof. Let $\epsilon \in \Omega^{max}_{K,I_0}$. Suppose that there exists $i \neq j$ in $I_0 \backslash K$ such that $A_i = \lambda A_j$ for some $\lambda \in \mathbb{Q}_{>0}$. Then, since $\epsilon \in \Omega^1_{i,I_0} \cap \Omega^1_{j,I_0}$, there exist x and y in $\mathbb{Q}^n$, such that $A_i x = B_i + \epsilon C_i$, $A_j x > B_j + \epsilon C_j$, $A_j y = B_j + \epsilon C_j$ and $A_i y > B_i + \epsilon C_i$. And we have $\lambda(B_j + \epsilon C_j) < \lambda A_j x = A_i x = B_i + \epsilon C_i < A_i y = \lambda A_j y = \lambda(B_j + \epsilon C_j)$, which gives a contradiction. □

3.3 A second one-parameter family of polytopes

In Section 4, we apply Theorem 18 to a family $(\tilde{Q}^\epsilon)_{\epsilon \in \mathbb{Q}_{\geq 0}}$ constructed by iteration from the family $(P^\epsilon)_{\epsilon \in \mathbb{Q}}$, using the last statement of Theorem 18 in the following way.

Definition 6. Let A, B and C be as in Theorem 18. Let $K \subset I_0$ containing K_0. Suppose that $0 \in \Omega^{max}_{K,I_0}$.

For any $\epsilon \in \mathbb{Q}_{\geq 0} \cap \Omega^{max}_{K,I_0}$, define $\tilde{Q}^\epsilon$ to be P^ϵ. If Ω^{max}_{K,I_0} has a finite supremum ϵ_1 and if $\epsilon_1 \in \Omega_{\emptyset,I_0}$ consider I_1 as in Theorem 18. Then we define, for any $\epsilon \in [\epsilon_1, +\infty[\cap \Omega^{max}_{K,I_1}$, $\tilde{Q}^\epsilon$ to be the polytope $P^\epsilon_{I_1}$. If $\epsilon_1 \notin \Omega_{\emptyset,I_0}$, we stop the construction and we define $\tilde{Q}^\epsilon = \emptyset$ for any $\epsilon > \epsilon_1$ by convention.

Iterating the construction, we obtain a family of polytopes $(\tilde{Q}^\epsilon)_{\epsilon \in \mathbb{Q}_{\geq 0}}$. Remark that the construction depends on K. Also note that the process is finite because the sequence $(I_k)_k$ of subsets constructed this way satisfies $I_0 \supsetneq I_1 \supsetneq \cdots \supsetneq K$.

We say that $\tilde{Q}^\epsilon$ is equivalent to $\tilde{Q}^\eta$ in the family $(\tilde{Q}^\epsilon)_{\epsilon \in \mathbb{Q}_{\geq 0}}$ if they are both defined at the same step of the construction above (in particular they correspond to P^ϵ_I and P^η_I, with ϵ and η both in $\Omega^{max}_{K,I}$ for some $I \subset I_0$ containing K) and if P^ϵ_I and P^η_I are equivalent according to Definition 5.

Example 6. We still consider the same example as in Examples 4 and 5. We have already seen that for $K = \{4,5,6\}$ we have $\Omega^{max}_{K,I_0} =]-\infty, 1[$ and $1 \in \Omega^1_{\emptyset,I_0}$. Moreover, $1 \in \Omega^{max}_{K,I_1} =]-infty, \frac{5}{4}[$ where $I_1 = \{1,3,4,5,6\}$. Then for any $\epsilon \in [1, \frac{5}{4}[$, $\tilde{Q}^\epsilon$ is in the same equivalent class. And $\frac{5}{4} \notin \Omega^1_{\emptyset,I_1}$, indeed $\tilde{Q}^{\frac{5}{4}}$ is a point.

To summarize this example, the equivalent classes in the family of polytopes $(\tilde{Q}^\epsilon)_{\epsilon \in \mathbb{Q}_{\geq 0}}$ are given by the following intervals: $\{0\}$, $]0,1[$, $\{1\}$, $]1, \frac{5}{4}[$ and $\{\frac{5}{4}\}$.

Applying Theorem 18 to each subfamilies $(P^\epsilon_{I_i})$ appearing in the construction of the family $(\tilde{Q}^\epsilon)_{\epsilon \in \mathbb{Q}_{\geq 0}}$, we obtain immediately the following result.

Corollary 28 (of Theorem 18)**.** *Suppose that $\tilde{Q}^0$ and $\tilde{Q}^\epsilon$ are equivalent for $\epsilon > 0$ small enough (to avoid what occurs in Example 6).*

There exist non-negative integers $k, j_0, \ldots, j_k$, rational numbers $\alpha_{i,j}$ for $i \in \{0, \ldots, k\}$ and $j \in \{0, \ldots, j_i\}$ and $\alpha_{k,j_k+1} \in \mathbb{Q}_{>0} \cup \{+\infty\}$ ordered as follows with the convention that $\alpha_{i,j_i+1} = \alpha_{i+1,0}$ for any $i \in \{0, \ldots, k-1\}$:

1. *$\alpha_{0,0} = 0$;*

2. *for any $i \in \{0, \ldots, k\}$, and for any $j < j'$ in $\{0, \ldots, j_i + 1\}$ we have $\alpha_{i,j} < \alpha_{i,j'}$;*

and such that the equivalence classes in the family of polytopes $(\tilde{Q}^\epsilon)_{\epsilon \in \mathbb{Q}_{\geq 0}}$ are given by the following intervals:

1. $[\alpha_{i,0}, \alpha_{i,1}[$, *with* $i \in \{0, \dots, k\}$;

2. $]\alpha_{i,j}, \alpha_{i,j+1}[$, *with* $i \in \{0, \dots, k\}$ *and* $j \in \{1, \dots, j_i\}$;

3. $\{\alpha_{i,j}\}$ *with* $i \in \{0, \dots, k\}$ *and* $j \in \{1, \dots, j_i\}$;

4. *if* $\alpha_{k,j_k+1} \neq +\infty$, $\{\alpha_{k,j_k+1}\}$ *and* $]\alpha_{k,j_k}, +\infty[$.

Remark 4. For any $i \in \{0, \dots, k\}$, the rational numbers $\alpha_{i,0}, \dots, \alpha_{i,j_i}$ can come from several consecutive steps of the construction of $(\tilde{Q}^\epsilon)_{\epsilon\in\mathbb{Q}_{\geq 0}}$. Indeed, for example, α_{0,j_0} is not necessarily the supremum ϵ_1 of Ω^{max}_{K,I_0} because $\{\tilde{Q}^{\epsilon_1}\}$ may be an equivalence class (see Example 6).

Remark 5. We have $\alpha_{k,j_k+1} = +\infty$ if, in the construction of $(\tilde{Q}^\epsilon)_{\epsilon\in\mathbb{Q}_{\geq 0}}$, some set $\Omega^{max}_{K,I}$ has no upper bound. And if $\alpha_{k,j_k+1} \neq +\infty$, it is the upper extremity of some $\Omega^{max}_{K,I}$ that is not an element of $\Omega^1_{\emptyset,I}$. In this latter case, $]\alpha_{k,j_k+1}, +\infty[$ is the class of empty polytopes and α_{k,j_k+1} corresponds to the last not empty polytope. If $C > 0$, we are always in that case.

4 MMP via a one-parameter family of polytopes

Let X be a projective horospherical G-variety, with open G-orbit isomorphic to G/H. Let D be an ample $\mathbb{Q}$-Cartier divisor on X. Suppose that X is $\mathbb{Q}$-Gorenstein (i.e. the canonical divisor K_X of X is $\mathbb{Q}$-Cartier). We keep the notation given in Section 2.

4.1 The one-parameter family of polytopes associated to X

To construct the one-parameter family of polytopes that permits to run the MMP from X, we start in the same way as in the classical approach of the MMP. Indeed, for any rational number $\epsilon > 0$ small enough, the divisor $D + \epsilon K_X$ is still ample (and $\mathbb{Q}$-Cartier by hypothesis), so that it defines a moment polytope Q^ϵ and a pseudo-moment polytope $\tilde{Q}^\epsilon$. More precisely, using Corollary 7, (for ϵ small enough) $Q^\epsilon := \{x \in v^\epsilon + M_\mathbb{Q} \mid Ax \geq B + \epsilon C\}$ and $\tilde{Q}^\epsilon := \{x \in M_\mathbb{Q} \mid Ax \geq \tilde{B} + \epsilon\tilde{C}\}$ where the matrices A, B, C, $\tilde{B}$, $\tilde{C}$ and the vector v^ϵ are defined below.

Recall that $x_1, \dots, x_r$ denote the primitive elements of the rays of the colored fan of X that are not generated by a vector $\alpha^\vee_M$ with $\alpha \in \mathcal{F}_X$. We choose an order in $S\backslash R$ and we then denote by $\alpha_1, \dots, \alpha_s$ its elements. We fix a basis $\mathcal{B}$ of M and we denote by $\mathcal{B}^\vee$ the dual basis in N. Now define $A \in \mathcal{M}_{r+s,n}(\mathbb{Q})$ whose first r lines are the coordinates of the vectors x_i in the basis $\mathcal{B}^\vee$ with $i \in \{1, \dots, r\}$ and whose last s lines are the coordinates of the vectors $\alpha^\vee_{jM}$ in $\mathcal{B}^\vee$ with $j \in \{1, \dots, s\}$. Let $\tilde{B}$ be the column matrix such that the pseudo-moment polytope of D is defined by $\{x \in M_\mathbb{Q} \mid Ax \geq \tilde{B}\}$. In fact, if $D = \sum_{i=1}^r b_i X_i + \sum_{\alpha\in S\backslash R} b_\alpha D_\alpha$, then $\tilde{B}$ is the column matrix associated to the vector $(-b_1, \dots, -b_r, -b_{\alpha_1}, \dots, -b_{\alpha_s})$. Similarly, the column matrix $\tilde{C}$ corresponds to the vector $(1, \dots, 1, a_{\alpha_1}, \dots, a_{\alpha_s})$, where $-K_X = \sum_{i=1}^r X_i + \sum_{\alpha\in S\backslash R} a_\alpha D_\alpha$ (the coefficients are explicitly defined with $a_\alpha = \langle 2\rho_P, \alpha^\vee\rangle$, where ρ_P is the sum of positive roots of G that are not roots of P). Now, define $v^\epsilon := \sum_{\alpha\in S\backslash R}(b_\alpha - \epsilon a_\alpha)\varpi_\alpha$. Since $Q^\epsilon = v^\epsilon + \tilde{Q}^\epsilon$, we compute that B and C are respectively the column matrices associated to the vectors $(-b_1 +$

$\langle x_1, \sum_{\alpha\in S\backslash R} b_\alpha\varpi_\alpha\rangle, \dots, -b_r + \langle x_r, \sum_{\alpha\in S\backslash R} b_\alpha\varpi_\alpha\rangle, 0, \dots, 0)$ and $(1 - \langle x_1, \sum_{\alpha\in S\backslash R} a_\alpha\varpi_\alpha\rangle, \dots, 1 - \langle x_r, \sum_{\alpha\in S\backslash R} a_\alpha\varpi_\alpha\rangle, 0, \dots, 0)$.

From the matrices A, $\tilde{B}$ and $\tilde{C}$ we construct the family $(\tilde{Q}^\epsilon)_{\epsilon\in\mathbb{Q}_{\geq 0}}$ of polytopes in $M_\mathbb{Q}$ (isomorphic to $\mathbb{Q}^n$) as in Section 3. And for any $\epsilon \geq 0$ we define $Q^\epsilon = v^\epsilon + \tilde{Q}^\epsilon$. Since $\tilde{C} > 0$, by Remark 5, there exists $\epsilon_{max} \in \mathbb{Q}_{>0}$ (it is the number α_{k,j_k+1} given in Corollary 28) such that for all $\epsilon \in [0, \epsilon_{max}[$, Q^ϵ is a G/H-polytope and for all $\epsilon > \epsilon_{max}$, $Q^\epsilon = \emptyset$. The polytope $Q^{\epsilon_{max}}$ is neither empty nor a G/H-polytope, but it is a G/H^1-polytope for some subgroup H^1 of G containing H (see Section 4.2).

Proposition 29. *The two partitions of $[0, \epsilon_{max}[$ given by equivalence classes of G/H-polytopes (Definition 4) in the family $(Q^\epsilon)_{\epsilon\in[0,\epsilon_{max}[}$ and by equivalence classes in the family $(\tilde{Q}^\epsilon)_{\epsilon\in[0,\epsilon_{max}[}$ according to Definition 6 with $K = \{r+1, \dots, r+s\}$ are the same.*

Proof. Let ϵ and η be two rational numbers.

Suppose that Q^ϵ and Q^η are equivalent G/H-polytopes. In Definition 4, take j to be minimal, so that the hyperplanes $\mathcal{H}_1, \dots, \mathcal{H}_j$ and $\mathcal{H'}_1, \dots, \mathcal{H'}_j$ correspond respectively to $\mathcal{H}^\epsilon_{i_1}, \dots, \mathcal{H}^\epsilon_{i_j}$ and $\mathcal{H}^\eta_{i_1}, \dots, \mathcal{H}^\eta_{i_j}$ for $J := \{i_1, \dots, i_j\} \subset \{1, \dots, r+s\}$. Then, if we denote by I the union $J \cup K$, the polytopes $\tilde{Q}^\epsilon$ and $\tilde{Q}^\eta$ respectively equal P^ϵ_I and P^η_I such that ϵ and η are in $\Omega^{max}_{K,I}$. Then we have to prove that P^ϵ_I and P^η_I are equivalent according to Definition 5, which comes directly from the definition of equivalence of G/H-polytopes.

Suppose now that $\tilde{Q}^\epsilon$ and $\tilde{Q}^\eta$ are equivalent according to Definition 6. They are constructed at the same step so that there exists $I \subset \{1, \dots, r+s\}$ containing K such that $\tilde{Q}^\epsilon$ and $\tilde{Q}^\eta$ respectively equal P^ϵ_I and P^η_I, with ϵ and η in $\Omega^{max}_{K,I}$. Then $\tilde{Q}^\epsilon = \{x \in M_\mathbb{Q} \mid A_I \geq \tilde{B}_I + \epsilon\tilde{C}_I\}$ and $\tilde{Q}^\eta = \{x \in M_\mathbb{Q} \mid A_I \geq \tilde{B}_I + \eta\tilde{C}_I\}$). This directly gives the first item of Definition 4. The second item is also clear from Definition 5. And the third item comes from the fact that $\Omega^0_{i,I}$ contains both ϵ and η or none of the two, for any $i \in K$. ☐

4.2 Construction of varieties and morphisms

We apply Corollary 28 to the family $(\tilde{Q}^\epsilon)_{\epsilon\in\mathbb{Q}_{\geq 0}}$. Then the family $(Q^\epsilon)_{\epsilon\in\mathbb{Q}_{\geq 0}}$ gives a list of G/H-embeddings:

1. $X_{i,j}$ for any $i \in \{0, \dots, k\}$ and $j \in \{0, \dots, j_i\}$, respectively associated to moment polytopes Q^ϵ with $\epsilon \in]\alpha_{i,j}, \alpha_{i,j+1}[$;

2. $Y_{i,j}$ for any $i \in \{0, \dots, k\}$ and $j \in \{1, \dots, j_i\}$, respectively associated to moment polytopes $Q^{\alpha_{i,j}}$;

It also gives a projective horospherical G-variety Z associated to the moment polytope $Q^{\alpha_{max}}$. Indeed, let $M^1_\mathbb{Q}$ be the minimal vector subspace containing $\tilde{Q}^{\alpha_{max}}$ and let $M^1 := M^1_\mathbb{Q} \cap M$. Let R^1 be the union of R with the set of $\alpha \in S\backslash R$ such that $Q^{\alpha_{max}}$ is contained in the wall $W_{\alpha,P}$. Then we define the subgroup H^1 of $P^1 := P_{R^1}$ to be the intersection of kernels of characters of P^1 in M^1. Then $Q^{\alpha_{max}}$ is a G/H^1-polytope and corresponds to a G/H^1-embedding Z.

Remark that, by definition, $M^1 \subset M$ and $R \subset R^1$ so that we have a projection $\pi : G/H \longrightarrow G/H^1$.

Now, by Proposition 11, we get dominant G-equivariant morphisms:

1. $\phi_{i,j} : X_{i,j-1} \longrightarrow Y_{i,j}$ for any $i \in \{0, \dots, k\}$ and $j \in \{1, \dots, j_i\}$;
2. $\phi^+_{i,j} : X_{i,j} \longrightarrow Y_{i,j}$ for any $i \in \{0, \dots, k\}$ and $j \in \{1, \dots, j_i\}$;
3. $\phi_i : X_{i,j_i} \longrightarrow X_{i+1,0}$ for any $i \in \{0, \dots, k-1\}$;
4. and $\phi : X_{k,j_k} \longrightarrow Z$.

In the next section, we prove that the morphisms $\phi_{i,j}$ and $\phi^+_{i,j}$ give flips (may be divisorial, see Remark 5 and Example 12), that the morphisms ϕ_i are divisorial contractions and that the morphism ϕ is a Mori fibration.

4.3 Description of contracted curves

Proposition 30. *1. For any $i \in \{0, \dots, k\}$ and $j \in \{1, \dots, j_i\}$, the curves C that are contracted by the morphism $\phi_{i,j}$ satisfy $K_{X_{i,j-1}} \cdot C < 0$; for any $i \in \{0, \dots, k-1\}$, the curves C that are contracted by the morphism ϕ_i satisfy $K_{X_{i,j_i}} \cdot C < 0$; and the curves C that are contracted by the morphism ϕ satisfy $K_{X_{k,j_k}} \cdot C < 0$.*

2. For any $i \in \{0, \dots, k\}$ and $j \in \{1, \dots, j_i\}$, the curves C that are contracted by the morphism $\phi^+_{i,j}$ satisfy $K_{X_{i,j}} \cdot C > 0$.

3. For any $i \in \{0, \dots, k-1\}$, the morphism ϕ_i contracts at least a G-stable divisor of X_{i,j_i}.

Proof. 1. Let $]a, b[$ or $[a, b[$ corresponding to an equivalence class in the family of polytopes $(Q^\epsilon)_{\epsilon \in \mathbb{Q}_{\geq 0}}$. In particular, there exists $I \subset \{1, \dots, r+s\}$ containing K such that for any $\epsilon \in]a, b[$, $\tilde{Q}^\epsilon = P^\epsilon_I$ and $\epsilon \in \Omega^{max}_{K,I}$.

Denote by X the variety given by the G/H-polytopes Q^ϵ with $\epsilon \in]a, b[$, denote by Y^b the variety given by the G/H-polytope Q^b or to the G/H^1-polytope Q^b if $b = \epsilon^{\alpha_{max}}$. And denote by ψ the projective G-equivariant morphism from X to Y^b.

Fix $c \in]a, b[$. Recall that a G-orbit of X corresponding to a face F^c_J of Q^c with $J \subset I$, is sent to the G-orbit of Y^b corresponding to the face F^b_J of Q^b (see Corollary 12).

Recall also that we described the curves on horospherical varieties in Section 2.5, we saw in particular that a curve C_μ intersects exactly two closed G-orbits (in one point for each closed G-orbit). Hence, a curve C_μ of X is contracted by ψ if and only if the G-orbit of X intersecting C_μ in an open set (i.e. the G-orbit corresponding to the edge μ of Q^c) is sent to a closed G-orbit of Y^b; and a curve $C_{\alpha,v}$ of X is contracted by ψ if the closed G-orbit corresponding to v is sent to a closed G-orbit isomorphic to G/P' where α is a root of the parabolic subgroup P'. In other words, C_μ is contracted by ψ if and only if for any $J \subset I$ such that $\mu = F^c_J$, the face F^b_J of Q^b is in fact a vertex of Q^b. And $C_{\alpha,v}$ is contracted by ψ

if and only if for any $J \subset I$ such that $v = F_J^c$ (which is not in $W_{\alpha,P}$), the vertex F_J^b of Q^b is in $W_{\alpha,P}$.

Let D^c be the $\mathbb{Q}$-Cartier divisor on X defined by the moment polytope Q^c and the pseudo-moment polytope $\tilde{Q}^c$. Then, for any $\epsilon \in [c, b[$ the $\mathbb{Q}$-Cartier (and ample) divisor of X defined by $(Q^\epsilon, \tilde{Q}^\epsilon)$ is $D^c + (\epsilon - c)K_X$. But, by Proposition 17, $(D + (\epsilon - c)K_X) \cdot C_\mu$ is the integral length of the edge μ in Q^ϵ for any $\epsilon \in [c, b[$, we get by continuity that C_μ is contracted by ψ if and only if $(D + (b - c)K_X) \cdot C_\mu = 0$. Similarly, since $(D + (\epsilon - c)K_X) \cdot C_{\alpha,v} = \langle v, \alpha^\vee \rangle$ for any $\epsilon \in [c, b[$, we get that $C_{v,\alpha}$ is contracted by ψ if and only if $(D + (b - c)K_X) \cdot C_{\alpha,v} = 0$. In particular, for any curve C that is contracted by ψ, we have $K_X \cdot C < 0$.

2. By very similar arguments, we prove that if $]a, b[$ is an equivalence class in the family of polytopes $(Q^\epsilon)_{\epsilon \in \mathbb{Q}_{\geq 0}}$, if Y^a is the G/H-embedding associated to the G/H-polytope Q^a, and if $\psi^+ : X \longrightarrow Y^a$ denotes the G-equivariant morphism, then for any curve C that is contracted by ψ^+, we have $K_X \cdot C > 0$.

3. Now consider the case where $]a, b[$ (or $[a, b[$) and $[b, b'[$ correspond to two successive equivalence classes in the family of polytopes $(Q^\epsilon)_{\epsilon \in \mathbb{Q}_{\geq 0}}$. The subset I of $\{1, \dots, r+s\}$, the varieties X and Y^b, and the morphism ψ are defined as above. Here, by hypothesis, $b \notin \Omega_{K,I}^{max}$ and there exists a proper subset I' of I containing K such that $\tilde{Q}^b = P_{I'}^b$. Then for any $c \in]a, b[$ and for any $i \in I \backslash I' \subset \{1, \dots, r\}$, F_i^c is a facet of Q^c (and corresponds to a G-stable divisor of X), but F_i^b is not a facet of Q^b (and corresponds to a G-stable divisor of Y^b of codimension at least 2). Hence, ψ contracts these G-stable divisors.

□

4.4 $\mathbb{Q}$-Gorenstein singularities

In this section, we prove in particular that all the varieties $X_{i,j}$ defined in Section 4.2 are $\mathbb{Q}$-Gorenstein. We begin by giving a $\mathbb{Q}$-Gorenstein criterion in terms of moment polytopes.

Proposition 31. *Let X be a projective G/H-embedding and let D be an ample $\mathbb{Q}$-Cartier divisor on X. Denote by $\tilde{Q}$ the pseudo-moment polytope of (X, D). Let A and $\tilde{C}$ be the matrices defined in Section 4.1. For any vertex V of $\tilde{Q}$, we denote by I_V the maximal subset of $\{1, \dots, r+s\}$ such that $V = F_{I_V}$.*

Then X is $\mathbb{Q}$-Gorenstein if and only if, for any vertex V of $\tilde{Q}$, the linear system $A_{I_V} X = \tilde{C}_{I_V}$ has a solution.

Proof. The proposition is just a translation, in terms of moment polytopes, of Theorem 6 applied to the divisor K_X. (Recall that X is $\mathbb{Q}$-Gorenstein if and only if K_X is $\mathbb{Q}$-Cartier.) □

By applying this criterion to the family $(\tilde{Q}^\epsilon)_{\epsilon \in \mathbb{Q}_{\geq 0}}$, we easily get the following result.

Corollary 32. *Let $\epsilon \geq 0$ such that Q^ϵ is a G/H-polytope. Let X^ϵ be the G/H-embedding corresponding to the G/H-polytope Q^ϵ. Denote by I the subset of I_0 containing K, such that $\tilde{Q}^\epsilon = P_I^\epsilon$ and $\epsilon \in \Omega_{K,I}^{max}$. For any vertex V^ϵ of $\tilde{Q}^\epsilon$, we denote by I_{V^ϵ} the maximal subset of I such that $V^\epsilon = F_{I_{V^\epsilon}}$.*

Then X^ϵ is $\mathbb{Q}$-Gorenstein if and only if for any vertex V^ϵ of $\tilde{Q}^\epsilon$, the linear system $A_{I_{V^\epsilon}}X = \tilde{C}_{I_{V^\epsilon}}$ has a solution.

Now, using Section 3, we can exactly know which X^ϵ are $\mathbb{Q}$-Gorenstein (except for Z).

Proposition 33. *The varieties $X_{i,j}$ with $i \in \{0,\dots,k\}$ and $j \in \{0,\dots,j_i\}$ are $\mathbb{Q}$-Gorenstein. And the varieties $Y_{i,j}$ with $i \in \{0,\dots,k\}$ and $j \in \{0,\dots,j_i\}$ are not $\mathbb{Q}$-Gorenstein.*

Proof. Let $i \in \{0,\dots,k\}$ and $j \in \{0,\dots,j_i\}$. The variety $X_{i,j}$ is defined by G/H-poltyopes Q^ϵ with ϵ in an open interval of $\mathbb{Q}_{\geq 0}$. In particular, for all these rational numbers ϵ and for any vertex V^ϵ of $\tilde{Q}^\epsilon$, the linear system $A_{I_{V^\epsilon}}X = \tilde{B}_{I_{V^\epsilon}} + \epsilon\tilde{C}_{I_{V^\epsilon}}$ has a solution. Hence, $A_{I_{V^\epsilon}}X = \tilde{C}_{I_{V^\epsilon}}$ has also a solution. It proves that $X_{i,j}$ is $\mathbb{Q}$-Gorenstein.

Now, let $i \in \{0,\dots,k\}$ and $j \in \{0,\dots,j_i\}$. The variety $Y_{i,j}$ is defined by the G/H-poltyope $Q^{\alpha_{i,j}}$. By Corollary 23, there exist J and I subsets of $\{1,\dots,r+s\}$ such that $J \subset I$ and $\alpha_{i,j}$ is the extremity of the interval $\Omega^0_{J,I}$ or $\{\alpha_{i,j}\} = \Omega^1_{J,I}$. But by Lemma 24, we can always choose J and I such that $\{\alpha_{i,j}\} = \Omega^1_{J,I}$. Moreover, by taking J maximal with this property, we get a vertex V of $Q^{\alpha_{i,j}}$ satisfying $\{\alpha_{i,j}\} = \Omega^1_{I_V,I}$. In particular, by Corollary 23, the system $A_{I_V}X = \tilde{C}_{I_V}$ has no solution. It proves that $Y_{i,j}$ is not $\mathbb{Q}$-Gorenstein. $\square$

4.5 $\mathbb{Q}$-factorial singularities

In this section, we prove that, for D general, the MMP works for the family of projective $\mathbb{Q}$-factorial horospherical varieties.

Proposition 34. *Suppose that X is $\mathbb{Q}$-factorial. Choose D such that the vector $\tilde{B}$ is in the open set*

$$\bigcup_{I\subset\{1,\dots,r+s\},\,|I|>n} \pi_I^{-1}(\mathbb{Q}^{|I|}\setminus \mathrm{Im}(A_I)),$$

where π_I is the canonical projection of $\mathbb{Q}^{r+s}$ to its vector subspace corresponding to the coordinates in I.

Then, for any $i \in \{0,\dots,k\}$ and any $j \in \{0,\dots,j_i\}$, the variety $X_{i,j}$ is $\mathbb{Q}$-factorial.

Proof. Applying Theorem 6, it is not difficult to check that a horospherical variety X is $\mathbb{Q}$-factorial if and only if all colored cones of $\mathbb{F}_X$ are simplicial and any color of X is the unique color of a colored edge of $\mathbb{F}_X$. In terms of moment polytopes, if Q is a moment polytope of X, then X is $\mathbb{Q}$-factorial if and only if Q is simple (i.e. each vertex belongs exactly to n facets, where n is the dimension of Q), Q intersects walls $W_{\alpha,P}$ with $\alpha \in S\backslash R$, only along facets and a facet of Q is never in 2 different walls ($W_{\alpha,P}$ and $W_{\beta,P}$ with $\alpha \neq \beta \in S\backslash R$). Hence, with notation of Section 4.1, X is $\mathbb{Q}$-factorial if and only if, for any $I \subset \{1,\dots,r+s\}$ of cardinality greater than n, the face F_I of Q is empty. (And we have the same criterion for any variety associated to a polytope of the family $(Q^\epsilon)_{\epsilon\in[0,\epsilon_{max}[}$). Let $i \in \{0,\dots,k-1\}$ and $j \in \{0,\dots,j_i\}$. Let $\epsilon \in]\alpha_{i,j},\alpha_{i,j+1}[$ so that Q^ϵ is a moment polytope of $X_{i,j}$. Let $I \subset I_0$ containing K such that $\tilde{Q}^\epsilon = P^\epsilon_I$ and $\epsilon \in \Omega^{max}_{K,I}$.

Let V be a vertex of Q^ϵ. Denote by I_V the maximal subset of I such that $V = F^\epsilon_{I_V}$. Note that $|I_V| \geq n$. We want to prove that $|I_V| = n$. The rational number ϵ is a point of $\Omega^1_{I_V}$, which is open (because it contains $]\alpha_{i,j},\alpha_{i,j+1}[$). It implies that, for any $\eta\in\mathbb{Q}$, $\tilde{B}_{I_V}+\eta\tilde{C}_{I_V}$ is in the image of A_{I_V}. In particular, $\tilde{B}_{I_V}$ is in the image of A_{I_V}. By hypothesis on D, the cardinality of I_V has to be n. This proves that $X_{i,j}$ is $\mathbb{Q}$-factorial. $\square$

Remark 6. The open set where $\tilde{B}$ is chosen, is clearly not empty and dense in $\mathbb{Q}^{r+s}$, because for any I of cardinality greater than n, the image of A_I is of codimension at least one. And, since X is $\mathbb{Q}$-factorial, any vector $\tilde{B}\in\mathbb{Q}^{r+s}$ gives a $\mathbb{Q}$-Cartier divisor.

Taking D general, we also get the following result.

Proposition 35. *Suppose that X is $\mathbb{Q}$-factorial. If D is general in the set of ample $\mathbb{Q}$-Cartier divisors, all morphisms $\phi_{i,j}$, $\phi_{i,j}^+$, ϕ_i and ϕ defined in Section 4.2 are contractions of rays of the corresponding effective cone $NE(X_{i,j})$.*

Proof. Let Y be a projective $\mathbb{Q}$-factorial G/H-embedding whose colored fan is made from some edges generated by x_i with $i\in\{1,\dots,r\}$ and $\alpha_M^\vee$ with $\alpha\in S\backslash R$. In particular, the vector space $\mathrm{Cartier}(X)_\mathbb{Q}$ of $\mathbb{Q}$-Cartier divisors of X projects onto the vector space $\mathrm{Cartier}(Y)_\mathbb{Q}$ of $\mathbb{Q}$-Cartier divisors of Y. Denote by p_Y this projection. Let V be a vector subspace of $N_1(Y)_\mathbb{Q}$ of dimension at least 2. Denote by V^* the dual of V in $N^1(Y)_\mathbb{Q}$. Then $V^*+\mathbb{Q}K_Y$ is a proper vector subspace of $N^1(Y)_\mathbb{Q}$. Hence, the intersection $\mathcal{I}_Y$, over all faces of $NE(Y)$ of dimension at least 2 and of direction a vector subspace V, of the sets $N^1(Y)\backslash(V^*+\mathbb{Q}K_Y)$ is a dense open set of $N^1(Y)_\mathbb{Q}$. If we denote by q_Y the projection of $\mathrm{Cartier}(Y)_\mathbb{Q}$ in $N^1(Y)_\mathbb{Q}$, then $(q_Y\circ p_Y)^{-1}(\mathcal{I}_Y)$ is open and dense in $\mathrm{Cartier}(X)_\mathbb{Q}$.

Consider now, an ample $\mathbb{Q}$-Cartier divisor D in the dense open intersection of the set given in Proposition 34 with the sets $(q_Y\circ p_Y)^{-1}(\mathcal{I}_Y)$ for all projective $\mathbb{Q}$-factorial G/H-embeddings Y whose colored fan is made from some edges generated by x_i with $i\in\{1,\dots,r\}$ and $\alpha_M^\vee$ with $\alpha\in S\backslash R$. Then, for any $i\in\{0,\dots,k\}$ and $j\in\{1,\dots,j_i\}$, the divisor $p_{X_{i,j-1}}(D)+\alpha_{i,j}K_{X_{i,j-1}}$ vanishes at most on a ray of $NE(X_{i,j-1})$, so that $\phi_{i,j}$ is the contraction of a ray of $NE(X_{i,j-1})$. Similarly, for any $i\in\{0,\dots,k\}$ and $j\in\{1,\dots,j_i\}$, the divisor $p_{X_{i,j}}(D)+\alpha_{i,j}K_{X_{i,j}}$ vanishes at most on a ray of $NE(X_{i,j})$, so that $\phi_{i,j}^+$ is the contraction of a ray of $NE(X_{i,j})$. And for any $i\in\{0,\dots,k\}$, the divisor $p_{X_{i,j_i}}(D)+\alpha_{i,j_i+1}K_{X_{i,j_i}}$ vanishes at most on a ray of $NE(X_{i,j_i})$, so that ϕ_i (ϕ if $i=k$) is the contraction of a ray of $NE(X_{i,j_i})$. $\square$

Remark 7. Proposition 35 is not true without the hypothesis of $\mathbb{Q}$-factoriality (at least for flips): see Example 11 (such an example also exists for toric varieties of dimension 3).

4.6 General fibers of Mori fibrations

In this section, we study the general fibers of the morphism $\phi: X_{k,j_k}\longrightarrow Z$ defined in Section 4.2 and we prove the last statement of Theorem 2. We may assume that $k=1$ and $j_0=0$, in particular $X_{k-1,j_{k-1}}=X$. And then, it is enough to prove the following more precise result.

Theorem 36. *Let X be a $\mathbb{Q}$-Gorenstein G/H-embedding. Let D be an ample $\mathbb{Q}$-divisor on X. Let ϵ_1 be the maximal positive rational number such that for any $\epsilon\in[0,\epsilon_1[$ the G/H-polytope Q^ϵ is equivalent to $Q=Q^0$.*

Suppose that Q^{ϵ_1} is not a G/H-polytope. And let H^1, P^1, R^1 and M^1 defined as in Section 4.2. In particular Q^{ϵ_1} is a G/H^1-polytope. Denote by Z the associated G/H^1-embedding and by ϕ the G-equivariant morphism from X to Z.

Then the general fibers of ϕ are either the flag variety P^1/P or a projective $\mathbb{Q}$-Gorenstein horospherical variety F_ϕ. Moreover in the second case, F_ϕ is a L^1/H^2-embedding, where $L^1 := H^1/R_u(H^1)$ and $H^2 := H/R_u(H^1)$ ($R_u(H^1)$ denoting the unipotent radical of H^1 in G). And a moment polytope of F_ϕ is the projection of Q in $\mathcal{X}(P)_{\mathbb{Q}}/M^1_{\mathbb{Q}}$.

Assume now that X is $\mathbb{Q}$-factorial. Then, for general D, the general fibers P^1/P or F_ϕ have Picard number one.

Theorem 36 is proved by the following 5 lemmas.

We begin by the easiest case.

Lemma 37. *Suppose that Q^{ϵ_1} has the same dimension as Q. Then the general fibers of the morphism ϕ are isomorphic to the flag variety P^1/P. Moreover, if X is $\mathbb{Q}$-factorial, P^1/P is of Picard number one for general D.*

Proof. In that case, Q^{ϵ_1} is in some wall $W_{\alpha,P}$ with $\alpha \in S\backslash R$, i.e. $R^1 \neq R$. Moreover H^1/H is isomorphic to P^1/P that proves the first claim.

By Proposition 35, if X is $\mathbb{Q}$-factorial, then for D general, we have $|R^1\backslash R| = 1$ and then P^1/P has Picard number 1. Indeed, if $|R^1\backslash R| > 1$, ϕ is not a contraction of an extremal ray, because $G/P \longrightarrow G/P^1$ is clearly not a contraction of an extremal ray. □

Now we consider the case where the dimension of Q^{ϵ_1} is less than the dimension of Q. Let x_0 (resp. x_0^1) be the unique point of the open G-orbit of X (resp. Z) fixed by H (resp. H^1).

Lemma 38. *A general fiber F_ϕ of ϕ is isomorphic to the closure in X of the H^1-orbit $H^1 \cdot x_0$.*

Proof. Since $G \cdot x_0^1$ is open in Z, the fiber $\phi^{-1}(x_0^1)$ is a general fiber of ϕ. This fiber intersected by the open G-orbit $G \cdot x_0$ of X is $H^1 \cdot x_0$. Moreover, since ϕ is G-equivariant, the open set $\phi^{-1}(G/H)$ of X is isomorphic to the bundle $G \times^{H^1} F_\phi$. Then F_ϕ is irreducible and we get the result. □

Note that, since an open set of X is isomorphic to the bundle $G \times^{H^1} F_\phi$, the fiber F_ϕ has the same singularities as X (normal and $\mathbb{Q}$-Gorenstein, $\mathbb{Q}$-factorial or smooth according to the hypothesis done on X).

Lemma 39. *The variety F_ϕ is a L^1/H^2-embedding.*

Note that, by definition, H^2 contains the unipotent radical $U/R_u(H^1)$ of the Borel subgroup $(B \cap H^1)/R_u(H^1)$ of L^1. Moreover $L^1 = H^1/R_u(H^1)$ is clearly reductive so that L^1/H^2 is a horospherical homogenous space.

Proof. By the previous lemma, F_ϕ is a H^1-variety. The unipotent radical $R_u(H^1)$ of H^1 (which is also the unipotent radical of P^1) acts trivially on x_0, then it also acts trivially on $H^1 \cdot x_0$ and F_ϕ. Hence, F_ϕ is a L^1-variety. Also the open L^1-orbit of F_ϕ is clearly isomorphic to L^1/H^2. We already saw that F_ϕ is normal because X is, hence F_ϕ is a L^1/H^2-embedding. □

Lemma 40. *A moment polytope of F_ϕ is the projection Q^2 of Q in $X(P)_{\mathbb{Q}}/M^1_{\mathbb{Q}}$.*

Proof. First describe the combinatorial data associated to the horospherical homogeneous space L^1/H^2. The simple roots of L^1 are those of P^1, i.e. the simple roots in R^1. Then the normalizer P^2 of H^2 in L^1 is the parabolic subgroup of L^1 whose set of simple roots is R. Moreover, the set of characters of P^2 trivial on H^2 is isomorphic to the quotient of the set of characters of P trivial on H with M^1. Then we set $M^2 := M/M^1$. The set of colors of the horospherical homogeneous space L^1/H^2 is $R^1\backslash R$.

Let I be the maximal subset of $\{1,\dots,r+s\}$ such that $\tilde{Q}^{\epsilon_1}\subset\{x\in M \mid A_Ix=\tilde{B}_I+\epsilon_1\tilde{C}_I\}$. In particular, $M^1_{\mathbb{Q}}$ equals the kernel of A_I. Now, we prove that the projection of $\tilde{Q}$ in $M^2_{\mathbb{Q}}:=M_{\mathbb{Q}}/M^1_{\mathbb{Q}}$ is the polytope $\tilde{Q}^2:=\{x\in M_{\mathbb{Q}}/\operatorname{Ker}(A_I)\mid A_Ix\geq\tilde{B}_I\}$. One inclusion is obvious. To prove the second one, let $x+\operatorname{Ker}(A_I)\in\tilde{Q}^2$ be a vertex F^2_J of $\tilde{Q}^2$ where $J\subset I$. Then $A_Jx=\tilde{B}_J$ and $A_Ix\geq\tilde{B}_I$. Moreover, by maximality of I, there exists $x'\in M_{\mathbb{Q}}$ such that $A_Ix'=\tilde{B}_I+\epsilon_1\tilde{C}_I$ and $A_{\bar{I}}x'>\tilde{B}_{\bar{I}}+\epsilon_1\tilde{C}_{\bar{I}}$ (where $\bar{I}=\{1,\dots,r+s\}\backslash I$). Hence, for $\lambda>0$ small enough, we have $x'':=\lambda x+(1-\lambda)x'$ satisfies $A_Jx''=\tilde{B}_J+\epsilon''\tilde{C}_J$, $A_Ix''\geq\tilde{B}_I+\epsilon''\tilde{C}_I$ and $A_{\bar{I}}x''>\tilde{B}_{\bar{I}}+\epsilon''\tilde{C}_{\bar{I}}$, with $\epsilon''=(1-\lambda)\epsilon_1$, so that x'' is a point of the face $F^{\epsilon''}_J$ of $\tilde{Q}^{\epsilon''}$. Since $0\leq\epsilon''<\epsilon_1$, the face F_J of $\tilde{Q}$ is not empty. But every point of F_J projects in F^2_J, which is a point. Then, every vertex of $\tilde{Q}^2$ is the image of a point of $\tilde{Q}$ by the projection in $M^2_{\mathbb{Q}}$, which proves the second inclusion.

By translation, we get that the projection of Q in $X(P)_{\mathbb{Q}}/M^1_{\mathbb{Q}}$ is the polytope $Q^2:=\{x\in\bar{v^0}+M_{\mathbb{Q}}/\operatorname{Ker}(A_I)\mid A_Ix\geq B_I\}$, where $\bar{v^0}$ is the image of v^0 in $X(P)_{\mathbb{Q}}/M^1_{\mathbb{Q}}$.

Suppose now that D is Cartier and very ample (or replace D by a multiple of D, see Remark 2). Then, by Proposition 9, X is isomorphic to the closure of $G\cdot[\sum_{\chi\in(v^0+M)\cap Q}v_\chi]$ in $\mathbb{P}(\oplus_{\chi\in(v^0+M)\cap Q}V(\chi))$. Then F_ϕ is isomorphic to $L^1\cdot[\sum_{\chi\in(v^0+M)\cap Q}v_\chi]$ in $\mathbb{P}(\oplus_{\chi\in(v^0+M)\cap Q}V_{L^1}(\chi))$. But, for any $\chi\in X(P)$ and any $\chi'\in M^1$, the L^1-modules $V_{L^1}(\chi)$ and $V_{L^1}(\chi+\chi')$ are isomorphic, so that Y is isomorphic to $L^1\cdot[\sum_{\chi\in(\bar{v^0}+M^2)\cap Q^2}v_\chi]$ in $\mathbb{P}(\oplus_{\chi\in(\bar{v^0}+M^2)\cap Q^2}V_{L^1}(\chi))$. It proves that F_ϕ is the L^1/H^2-embedding associated to the polytope Q^2. Remark that $\tilde{Q}^2$ is of maximal dimension in $M^2_{\mathbb{Q}}$ because $\tilde{Q}$ is of maximal dimension in $M_{\mathbb{Q}}$. □

Lemma 41. *If X is $\mathbb{Q}$-factorial, then, for general D, the variety F_ϕ has Picard number one.*

Proof. We prove that, for D general, the set I defined in the latter proof is of cardinality $\operatorname{codim}(Q^{\epsilon_1})+1=\dim(Q^2)+1$, and that Q^2 is a simplex of $M^2_{\mathbb{Q}}=M_{\mathbb{Q}}/M^1_{\mathbb{Q}}$ such that any wall W_{α,P^2} in $X(P^2)_{\mathbb{Q}}$ with $\alpha\in R^1\backslash R$ gives a facet of Q^2. Suppose that D (and then $\tilde{B}$) satisfies the following condition: for any subset J of $\{1,\dots,r+s\}$ with $|J|\geq\dim(\operatorname{Im}(A_J))+2$, $\tilde{B}_J$ is not in the (proper) vector subspace of $\mathbb{Q}^{r+s}$ generated by $\operatorname{Im}(A_J)$ and $\tilde{C}_J$. Such a D is general, since X is $\mathbb{Q}$-factorial. Note that $\operatorname{codim}(Q^{\epsilon_1})=\operatorname{codim}(\operatorname{Ker}(A_I))=\dim(\operatorname{Im}(A_I))$. Now, if $|I|>\operatorname{codim}(Q^{\epsilon_1})+1$ then by hypothesis on D, $\tilde{B}_I+\epsilon\tilde{C}_I$ is never in $\operatorname{Im}(A_I)$, which contradicts that $F^{\epsilon_1}_I=Q^{\epsilon_1}$ is not empty. Moreover, Q^2 is a polytope (and then bounded) of maximal dimension in $M^2_{\mathbb{Q}}$, so the number $|I|$ of inequalities defining Q^2 must be at least $\dim(Q^2)+1$. Hence, $|I|=\dim(Q^2)+1$.

In particular, Q^2 is a simplex of $M^2_{\mathbb{Q}}$. Moreover, the set $R^1\backslash R$ of colors of L^1/H^2 is clearly contained in I by definition of I and R^1, so that, for any $\alpha\in R^1\backslash R$, $W_{\alpha,P^2}\cap Q^2$ is a face of Q^2. In particular $R^1\backslash R=\mathcal{F}_{F_\phi}$. Moreover, since each $i\in I$ corresponds necessarily to a facet of Q^2

because $|I| = \dim(Q^2) + 1$, this latter face is necessarily a facet. And, with the same argument, $W_{\alpha,P^2} \cap Q^2 \neq W_{\beta,P^2} \cap Q^2$ for any $\alpha \neq \beta \in R^1 \backslash R$.

But, by [Pas06, Eq. (4.5.1)] (or as corollary of Theorem 6), the Picard number of the $\mathbb{Q}$-factorial horospherical variety F_ϕ of moment polytope Q^2 is

$$|\{\text{facets of } Q^2\}| - \dim(Q^2) + |R^1 \backslash R| - |\mathcal{F}_{F_\phi}|.$$

Then F_ϕ is a projective $\mathbb{Q}$-factorial L^1/H^2-embedding of Picard number 1. □

5 Examples

In this section we give 5 examples with $G = \mathrm{SL}_3$. The first three ones give the MMP for the same horospherical smooth variety but with a different ample Cartier divisor. The fourth one gives a flip consisting of exchanging colors (see [Bri93, Section 4.5]). And the last one give the MMP for a $\mathbb{Q}$-Gorenstein (not $\mathbb{Q}$-factorial) variety, where we observe a flip from a divisorial contraction.

Fix a Borel subgroup B of G. Denote by α and β the two simple roots of G.

Example 7. Consider the horospherical subgroup H defined as the kernel of the character $\varpi_\alpha + \varpi_\beta$ of B. In that case we have N and M isomorphic to $\mathbb{Z}$. The horospherical homogeneous space has two colors α and β whose image in N are respectively $\alpha^\vee_M = \beta^\vee_M = 1$.

If χ is a character of B, we denote by $\mathbb{C}_\chi$ the line $\mathbb{C}$ where B acts by $b \cdot z = \chi(b)z$ for any $b \in B$ and $z \in \mathbb{C}$.

Let X be the $\mathbb{P}^1$-bundle $X = G \times^B \mathbb{P}(\mathbb{C}_0 \oplus \mathbb{C}_{\varpi_\alpha + \varpi_\beta})$ over G/B, it is a smooth G/H-embedding. Its colored fan is the unique complete fan (of dimension 1) without color. Denote by X_1 and X_2 the two irreducible G-stable divisors of X, respectively corresponding to the primitive elements $x_1 = 1$ and $x_2 = -1$ of N. Here $-K_X = X_1 + X_2 + 2D_\alpha + 2D_\beta$.

Choose $D = X_1 + 2X_2 + 2D_\alpha + 2D_\beta$. Then the moment polytope Q is the interval $[\varpi_\alpha + \varpi_\beta, 4(\varpi_\alpha + \varpi_\beta)]$ in the dominant chamber of (G, B).

The family $(Q^\epsilon)_{\epsilon \geq 0}$ is given by:

- for any $\epsilon \in [0, 1[$, Q^ϵ is the interval $[(1-\epsilon)(\varpi_\alpha + \varpi_\beta), (4-3\epsilon)(\varpi_\alpha + \varpi_\beta)]$;
- for any $\epsilon \in [1, \frac{4}{3}[$, Q^ϵ is the interval $[0, (4-3\epsilon)(\varpi_\alpha + \varpi_\beta)]$;
- $Q^{\frac{4}{3}}$ is the point 0.

Hence, the MMP from (X, D) gives a divisorial contraction from X to the projective G/H-embedding with the two colors α and β, which is not $\mathbb{Q}$-factorial but $\mathbb{Q}$-Gorenstein, and Fano. It finishes by a Mori fibration from this Fano variety to a point.

Note that, here, the divisorial contraction contracts 2 divisors: the zero and infinite section of the $\mathbb{P}^1$-bundle X. In particular, it is not the contraction of a ray of $NE(X)$.

Example 8. We keep the same G/H-embedding X but we choose another ample divisor $D = X_1 + 2X_2 + 3D_\alpha + 2D_\beta$. Then the moment polytope Q is the interval $[2\varpi_\alpha + \varpi_\beta, 5\varpi_\alpha + 4\varpi_\beta]$.

The family $(Q^\epsilon)_{\epsilon \geq 0}$ is given by:

- for any $\epsilon \in [0,1[$, Q^ϵ is the interval $[2\varpi_\alpha + \varpi_\beta - \epsilon(\varpi_\alpha + \varpi_\beta), 5\varpi_\alpha + 4\varpi_\beta - 3\epsilon(\varpi_\alpha + \varpi_\beta)]$;
- for any $\epsilon \in [1, \frac{4}{3}[$, Q^ϵ is the interval $[\varpi_\alpha, 5\varpi_\alpha + 4\varpi_\beta - 3\epsilon(\varpi_\alpha + \varpi_\beta)]$;
- $Q^{\frac{4}{3}}$ is the point ϖ_α.

Hence, the MMP from (X, D) gives a divisorial contraction from X to the projective G/H-embedding with the color β. This is a contraction of the ray of $NE(X)$ generated by C_{β,X_1}. It finishes by a Mori fibration from this $\mathbb{Q}$-factorial variety to the flag variety G/P_α.

Example 9. We still keep the same G/H-embedding X and we now choose the ample divisor $D = X_2 + 2D_\alpha + 2D_\beta$. Then the moment polytope Q is the interval $[2(\varpi_\alpha + \varpi_\beta), 3(\varpi_\alpha + \varpi_\beta)]$.

The family $(Q^\epsilon)_{\epsilon \geq 0}$ is given by:

- for any $\epsilon \in [0, \frac{1}{2}[$, Q^ϵ is the interval $[(2-\epsilon)(\varpi_\alpha + \varpi_\beta), (3-3\epsilon)(\varpi_\alpha + \varpi_\beta)]$;
- $Q^{\frac{1}{2}}$ is the point $\frac{3}{2}(\varpi_\alpha + \varpi_\beta)$.

Hence, the MMP from (X, D) gives the Mori fibration $X \longrightarrow G/B$, whose fibers are projective lines.

In the following example, we illustrate a flip consisting of exchanging colors.

Example 10. Consider the horospherical subgroup H defined as the kernel of the character $\varpi_\alpha + 2\varpi_\beta$ of B. In that case we have N and M isomorphic to $\mathbb{Z}$. The horospherical homogeneous space has two colors α and β whose image in N are respectively $\alpha_M^\vee = 1$ and $\beta_M^\vee = 2$.

Let X be the $\mathbb{Q}$-factorial G/H-embedding whose colored fan is the complete fan with color β. Denote by X_1 the irreducible G-stable divisor of X, corresponding to the primitive elements $x_1 = -1$. Here $-K_X = X_1 + 2D_\alpha + 2D_\beta$.

Consider $D = 3X_1 + 2D_\alpha + 2D_\beta$. Then the moment polytope Q is the interval $[\varpi_\alpha, 5\varpi_\alpha + 8\varpi_\beta]$.

The family $(Q^\epsilon)_{\epsilon \geq 0}$ is given by:

- for any $\epsilon \in [0,1[$, Q^ϵ is the interval $[(1-\epsilon)\varpi_\alpha, 5\varpi_\alpha + 8\varpi_\beta - \epsilon(3\varpi_\alpha + 4\varpi_\beta)]$;
- Q^1 is the interval $[0, 2\varpi_\alpha + 4\varpi_\beta]$;
- for any $\epsilon \in]1, \frac{5}{3}[$, Q^ϵ is the interval $[2(\epsilon - 1)\varpi_\beta, 5\varpi_\alpha + 8\varpi_\beta - \epsilon(3\varpi_\alpha + 4\varpi_\beta)]$;
- $Q^{\frac{5}{3}}$ is the point $\frac{4}{3}\varpi_\beta$.

Hence, the MMP from (X, D) first gives a flip $X \longrightarrow Y \longleftarrow X^+$, where Y is the G/H-embedding corresponding to the complete colored fan with the two colors α and β and X^+ is the G/H-embedding corresponding to the complete colored fan with the color α. It finishes by a Mori fibration from X^+ to the flag variety G/P_β.

In fact, in rank 1 we can describe the MMP for projective $\mathbb{Q}$-Gorenstein horospherical varieties, without using moment polytopes (by simply using the classifications of embeddings, equivariant morphisms and divisors in terms of colored fans).

Proposition 42. *Let X be a projective horospherical variety of rank 1. Then the lattice M is isomorphic to $\mathbb{Z}$ (and $M_\mathbb{Q} \simeq \mathbb{Q}$). Its colored fan $\mathbb{F}_X$ is a complete colored fan whose maximal cones are the two half-lines of $\mathbb{Q}$ from 0. Recall that an anticanonical divisor of X is $\sum_{i=1}^r X_i + \sum_{\alpha\in S\backslash R} a_\alpha D_\alpha$, for some positive interger a_α (here $r \leq 2$). The colors of the open G-orbit G/H of X can be partitioned as follows*

$$S\backslash R = \bigsqcup_{i=1}^{k} S_i^- \sqcup S^0 \sqcup \bigsqcup_{i=1}^{l} S_i^+$$

such that for any $i \in \{1,\ldots,k\}$ the rational numbers $\frac{\alpha_M^\vee}{a_\alpha}$ with $\alpha \in S_i^-$ are all equal and negative, for any $\alpha \in S^0$ the integer $\alpha_M^\vee$ is zero and, for any $i \in \{1,\ldots,l\}$ the rational numbers $\frac{\alpha_M^\vee}{a_\alpha}$ with $\alpha \in S_i^+$ are all equal and positive. We order the sets S_i^- and S_i^+ such that for any $i < j$ in $\{1,\ldots,k\}$ (respectively $\{1,\ldots,l\}$), for any $\alpha \in S_i^-$ and $\beta \in S_j^-$ (respectively $\alpha \in S_i^+$ and $\beta \in S_j^+$), we have $|\frac{\alpha_M^\vee}{a_\alpha}| < |\frac{\beta_M^\vee}{a_\beta}|$.

Then X is $\mathbb{Q}$-Gorenstein if and only if the set $\mathcal{F}_X$ of colors of X intersects at most one S_i^- and one S_i^+. (And recall that $\mathcal{F}_X$ never intersects S^0).

Suppose now that X is $\mathbb{Q}$-Gorenstein.

A birational extremal contraction ϕ from X to Y consists in adding one color α. We distinguish 3 cases:

1. *the color α is in a half-line with no color of X, then ϕ is a divisorial contraction and Y is $\mathbb{Q}$-Gorenstein;*

2. *the color α is in the same subset S_i^- or S_i^+ as some color of X, then Y is still $\mathbb{Q}$-Gorenstein but ϕ is not a divisorial contraction;*

3. *there exists some $i \in \{1,\ldots,k\}$ (respectively $\{1,\ldots,l\}$), some not empty $A \subset S_i^-$ (respectively S_i^+), such that α is in some S_j^- (respectively S_j^+) with $j \neq i$, then Y is not $\mathbb{Q}$-Gorenstein and we have a flip map $\phi^+ : X^+ \longrightarrow Y$ where X^+ is the $\mathbb{Q}$-Gorenstein G/H-embedding obtained from Y by erasing all colors in A;*

A not birational extremal contraction ϕ is

4. *either a map from X to G/P', where P' is the parabolic subgroup of G containing B whose set of simple roots is $R \sqcup \mathcal{F}_X$;*

5. *or a map from X to the projective G/H'-embedding X' such that H' is the rank one horospherical subgroup $H \cap P'$, P' is the parabolic subgroup of G containing B whose set of simple roots is $R \sqcup \alpha$ with $\alpha \in S^0$, and $\mathcal{F}_{X'} = \mathcal{F}_X$.*

If C is a curve of X contracted by ϕ, then, in the above 5 cases we have:

1. *the sign of $K_X.C$ is the sign of $|\frac{\alpha_M^\vee}{a_\alpha}| - 1$;*

2. *$K_X.C = 0$;*

3. *the sign of $K_X.C$ is the sign of $j-i$, and if C^+ is a curve of X^+ contracted by ϕ^+, the sign of $K_{X^+}.C^+$ is opposite to the sign of $K_X.C$;*

4. $K_X.C<0$*;*

5. $K_X.C<0$*.*

The proof of this proposition is left to the reader.

Example 11. Consider the case where $S\backslash R=S_1^-\sqcup S_2^-\sqcup S_1^+$ with $S_1^-=\{\alpha\}$, $S_2^-=\{\beta,\gamma\}$ and $S_1^+=\{\delta\}$. Consider the projective $\mathbb{Q}$-Gorenstein G/H-embedding X such that $\mathcal{F}_X=\{\beta,\gamma\}$. Then we can add the color δ and get a divisorial contraction. Or we can also add the color α and get a flip $X\stackrel{\phi}{\longrightarrow}Y\stackrel{\phi^+}{\longleftarrow}X^+$, where Y and X^+ are respectively the G/H-embedding such that $\mathcal{F}_Y=\{\alpha,\beta,\gamma\}$ and $\mathcal{F}_{X^+}=\{\alpha\}$. In that case, the morphism ϕ^+ can factor in two ways as follows $X^+\longrightarrow Y'\longrightarrow Y$, where $\mathcal{F}_{Y'}=\{\alpha,\beta\}$ or $\mathcal{F}_{Y'}=\{\alpha,\gamma\}$. In particular ϕ^+ is not an extremal contraction. Moreover in both cases Y' is not $\mathbb{Q}$-Gorenstein, so that we cannot have a flip from the contraction ϕ such that ϕ^+ is an extremal contraction and X^+ is $\mathbb{Q}$-Gorenstein.

We now give a last example in rank 2.

Example 12. Consider the case where the horospherical subgroup H is the maximal unipotent subgroup U of B. Then the lattice M is the lattice of characters of B with basis $(\varpi_\alpha,\varpi_\beta)$, and N is the coroot lattice with basis $(\alpha^\vee,\beta^\vee)$. Here $\alpha^\vee=\alpha_M^\vee$ and $\beta^\vee=\beta_M^\vee$.

Let X be the G/H-embedding whose colored fan is the complete colored fan with color β, and edges generated by $x_1:=-\beta^\vee$, $x_2:=\alpha^\vee$ and $x_3:=\beta^\vee-\alpha^\vee$. Note that X is not $\mathbb{Q}$-factorial because $\beta^\vee$ is not in an edge of $\mathbb{F}_X$. An anticanonical divisor of X is $X_1+X_2+X_3+2D_\alpha+2D_\beta$, and it is Cartier so that X is $\mathbb{Q}$-Gorenstein.

Consider $D=3X_1+X_3+D_\alpha+D_\beta$. Then D is an ample Cartier divisor whose moment polytope Q is the triangle with vertices ϖ_α, $\varpi_\alpha+4\varpi_\beta$ and $5\varpi_\alpha+4\varpi_\beta$. The matrices defining the family $(Q^\epsilon)_{\epsilon\geq 0}$ are

$$A=\begin{pmatrix}0&-1\\1&0\\-1&1\\1&0\\0&1\end{pmatrix},\ B=\begin{pmatrix}-4\\1\\-1\\0\\0\end{pmatrix}\text{ and }C=\begin{pmatrix}3\\-1\\1\\0\\0\end{pmatrix}.$$

Note that these matrices are the same as in Example 4 without the sixth line.

Then the family $(Q^\epsilon)_{\epsilon\geq 0}$ is given by:

- for any $\epsilon\in[0,1[$, Q^ϵ is the triangle with vertices $(1-\epsilon)\varpi_\alpha$, $\varpi_\alpha+4\varpi_\beta-\epsilon(\varpi_\alpha+3\varpi_\beta)$ and $5\varpi_\alpha+4\varpi_\beta-\epsilon(4\varpi_\alpha+3\varpi_\beta)$;
- Q^1 is the triangle with vertices 0, ϖ_β and $\varpi_\alpha+\varpi_\beta$;
- for any $\epsilon\in]1,\frac{5}{4}[$, Q^ϵ is the triangle with vertices $(\epsilon-1)\varpi_\beta$, $(4-3\epsilon)\varpi_\beta$ and $5\varpi_\alpha+4\varpi_\beta-\epsilon(4\varpi_\alpha+3\varpi_\beta)$;

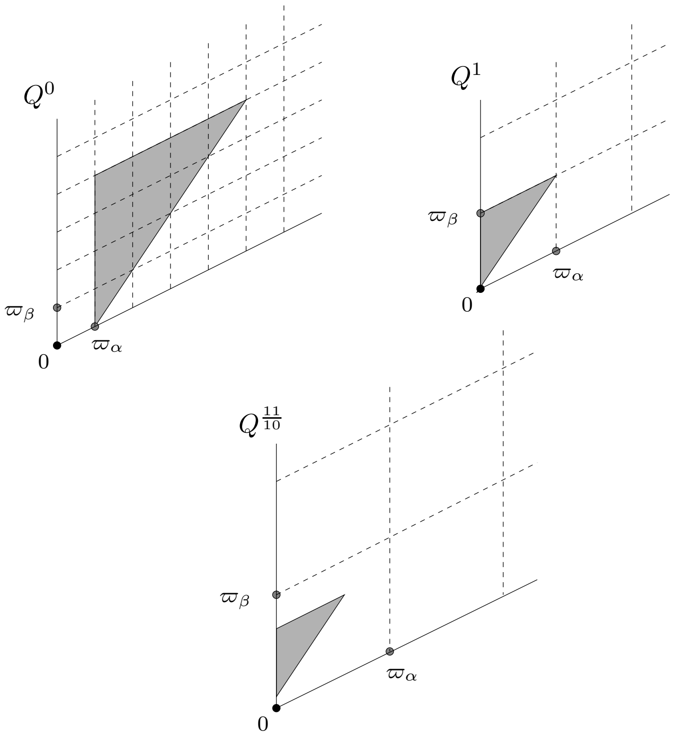


Figure 7: Evolution of Q^ϵ in Example 12

- $Q^{\frac{5}{4}}$ is the point $\frac{1}{4}\varpi_\beta$.

We illustrate the first three classes of the family $(Q^\epsilon)_{\epsilon\geq 0}$ in Figure 7.

Hence, the MMP from (X, D) first gives a flip $X \longrightarrow Y \longleftarrow X^+$, where Y is the G/H-embedding corresponding to the same complete colored fan as X but with the two colors α and β and X^+ is the G/H-embedding corresponding to the same complete colored fan as X but with the color α (instead of β). Note that the map $X \longrightarrow Y$ contracts the divisor X_2 and that the map $X^+ \longrightarrow Y$ does not contract a divisor. It finishes by a Mori fibration from X^+ to the flag variety G/P_β.

References

[Bou75] N. Bourbaki, *Éléments de mathématique*, Hermann, Paris, 1975, Fasc. XXXVIII: Groupes et algèbres de Lie. Chapitre VII: Sous-algèbres de Cartan, éléments réguliers. Chapitre VIII: Algèbres de Lie semi-simples déployées, Actualités Scientifiques et Industrielles, No. 1364.

[Bri89] Michel Brion, *Groupe de picard et nombres caractéristiques des variétés sphériques*, Duke Math. J. **58** (1989), no. 2, 397–424.

[Bri93] ———, *Variétés sphériques et théorie de Mori*, Duke Math. J. **72** (1993), no. 2, 369–404.

[Bri05] ———, *Lectures on the geometry of flag varieties*, Topics in cohomological studies of algebraic varieties, Trends Math., Birkhäuser, Basel, 2005, pp. 33–85.

[Ful93] William Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993, The William H. Roever Lectures in Geometry.

[Kno91] Friedrich Knop, *The Luna-Vust theory of spherical embeddings*, Proceedings of the Hyderabad Conference on Algebraic Groups (Hyderabad, 1989) (Madras), Manoj Prakashan, 1991, pp. 225–249.

[Mat02] Kenji Matsuki, *Introduction to the Mori program*, Universitext, Springer-Verlag, New York, 2002.

[Oda88] Tadao Oda, *Convex bodies and algebraic geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 15, Springer-Verlag, Berlin, 1988, An introduction to the theory of toric varieties, Translated from the Japanese.

[OV90] A. L. Onishchik and È. B. Vinberg, *Lie groups and algebraic groups*, Springer Series in Soviet Mathematics, Springer-Verlag, Berlin, 1990, Translated from the Russian and with a preface by D. A. Leites.

[Pas06] Boris Pasquier, *Variétés horosphériques de Fano*, Ph.D. thesis, Université Joseph Fourier, Grenoble 1, Available at http://tel.archives-ouvertes.fr/tel-00111912, 2006.

[Pas08] ———, *Variétés horosphériques de Fano*, Bull. Soc. Math. France **136** (2008), no. 2, 195–225.

[Pas09] ———, *On some smooth projective two-orbit varieties with Picard number 1*, Math. Ann. **344** (2009), no. 4, 963–987.

[Rei83] Miles Reid, *Decomposition of toric morphisms*, Arithmetic and geometry, Vol. II, Progr. Math., vol. 36, Birkhäuser Boston, Boston, MA, 1983, pp. 395–418.