Introduction to spherical varieties and description of special classes of spherical varieties

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Abstract

The aim of these lectures is to give an overview on spherical varieties, especially recent results. Spherical varieties form a very large class of varieties containing in particular toric and flag varieties. They are very useful to test conjecture or theory, or simply to understand better what can happen in algebraic geometry. And, to my point of view, there is still a lot of results to prove for these varieties.

Contents

Introduction

The idea to get classes of examples of varieties is to study particular varieties with a group action. More precisely, let X be a normal algebraic variety over an algebraically closed field $\mathbb{k}$. And let G be a connected reductive algebraic group over $\mathbb{k}$ acting on X with an open orbit (equivalently with finitely many orbits). For example take $G = \mathrm{GL(n)}$, $\mathrm{SL(n)}$, $\mathrm{SO(n)}$, $\mathrm{Sp(2n)}$ or $(\mathbb{k}^*)^n$.

What we can do now is to:

- define an invariant called the complexity, which is a non-negative integer (it is the minimal codimension of an orbit under the action of a Borel subgroup)
- give a partial combinatorial description of spherical ones (*i.e.* with complexity zero); for varieties of positive complexity very few is known.
- give a precise classification and a general theory for spherical ones when the open orbit is isomorphic to a given homogenous space.
- give a classification of spherical varieties under the action of groups of type A, D (SL_n, $\mathrm{SO}(2n)$) and E.
- describe more or less precisely some classes of spherical ones, for example: toric varieties, flag varieties, horospherical varieties, symmetric varieties, wonderful varieties.

Even if some results are also true over all algebraically closed field, still now we will only work over $\mathbb{C}$.

Notice that spherical varieties have an equivalent in real symplectic geometry, indeed they correspond to real symplectic manifold with a multiplicity-free Hamiltonian action of a compact Lie group [Wo98]. But here, we will only consider the algebraic geometrical point of view of spherical varieties.

1 Two well-known classes of spherical varieties

1.1 Toric varieties

To have more details to what we will say in this section, see for example [Fu93] or [Od78]. Here the group acting will be an algebraic torus $(\mathbb{C}^*)^n$.

Definition 1.1. A toric variety of dimension n is a normal algebraic variety where $(\mathbb{C}^*)^n$ acts with an open orbit isomorphic to itself.

Examples 1.2.

- $\mathbb{C}^n$, $\mathbb{P}^n$, $Bl_Y\mathbb{P}^n$ for all Y stable under the action of $(\mathbb{C}^*)^n$,...

- Products of toric varieties are toric varieties.
- A singular example: the surface $xz = y^2$ in $\mathbb{C}^3$ where $(\mathbb{C}^*)^2$ acts by $(a,b)\cdot(x,y,z) = (ax, by, \frac{b^2}{a}z)$.

A way to construct affine toric varieties

Take a strictly convex cone $\mathcal{C}$ of $\mathbb{R}^n \supset \mathbb{Z}^n$ generated by finitely many elements of $\mathbb{Z}^n$. Define the dual $\mathcal{C}^\vee$ of $\mathcal{C}$ to be the subset of $\mathbb{R}^n$ (identified to its dual) of elements v such that $\langle v, u\rangle \geq 0$. Then define

$$X = \operatorname{Spec}\mathbb{C}[\underline{X}^\chi, \chi \in \mathcal{C}^\vee \cap \mathbb{Z}^n]$$

where $\underline{X}^\chi = X_1^{\chi_1}\cdots X_n^{\chi_n}$ for $\chi = (\chi_1, \dots, \chi_n)$ in a fixed basis of $\mathbb{Z}$ and $X_1, \dots, X_n$ are variables.

Examples 1.3. We obtain a variety isomorphic to $\mathbb{C}^n$ as soon as $\mathcal{C}$ is generated by a basis of $\mathbb{Z}^n$. The cone consisting of the origin 0 in $\mathbb{R}^n$ gives $\mathbb{C}^n\backslash\{0\}$.
The singular example above is obtained choosing $\mathcal{C}$ to be generated by $(2,1)$ and $(0,1)$.

Remark 1.4. Let us fix a basis of weight of $(\mathbb{C}^*)^n$. Then, to an affine toric variety X, we associate a convex cone in $\mathbb{R}^n$ spanned by the weights of $\mathbb{C}[X]$. The dual of this cone is then a strictly convex cone $\mathcal{C}_X$ of $\mathbb{R}^n$. And both construction are inverse from each other.

A way to construct projective toric varieties

We now consider a convex lattice polytope Q of dimension n in $\mathbb{R}^n$ (*i.e.* whose vertices are in $\mathbb{Z}^n$). And we define

$$X = \overline{(\mathbb{C}^*)^n \cdot [1, \dots, 1]} \subset \mathbb{P}(\oplus_{\chi\in Q\cap\mathbb{Z}^n}\mathbb{C}_\chi)$$

where $\mathbb{C}_\chi$ is the affine line where acts by $(t_1, \dots, t_n)\cdot z = t_1^{\chi_1}\cdots t_n^{\chi_n} z$ for $\chi = (\chi_1, \dots, \chi_n)$ in a fixed basis of $\mathbb{Z}^n$.

Examples 1.5. The simplex with vertices the origin and the elements of a basis of $\mathbb{Z}^n$ gives the projective space $\mathbb{P}^n$.
The square with vertices $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$ gives $\mathbb{P}^1\times\mathbb{P}^1$ in $\mathbb{P}^3$ via the Segre embedding.

We construct in fact, by this way, polarized varieties *i.e.* projective varieties together with an ample line bundle. In the examples above, the ample line bundles are respectively $\mathcal{O}(1)$ and $\mathcal{O}(1)\times\mathcal{O}(1)$.

A classification in terms of fans

Definition 1.6. A fan $\mathbb{F}$ in $\mathbb{R}^n \supset \mathbb{Z}^n$ is a finite set of strictly convex cones generated by finitely many lattice points such that:

- every face of a cone of $\mathbb{F}$ in also in $\mathbb{F}$;
- two cones intersect along a common face.

Proposition 1.7. *Let X be a toric variety. Then the set of cones associated to the $(\mathbb{C}^*)^n$-stable affine subvarieties of X is a fan denoted by $\mathbb{F}_X$.*

The map $X \longmapsto \mathbb{F}_X$ from the set of isomorphism classes of toric varieties of dimension n and the set of isomorphism classes of fans in $\mathbb{R}^n$ is a bijection.

A lot of properties of a toric variety can be seen directly on its fan. Here is some examples of such properties.

Proposition 1.8.

- *X is complete if and only if the fan $\mathbb{F}_X$ is complete (i.e.for all point x of $\mathbb{R}^n$, there exists a cone of $\mathbb{F}_X$ containing x).*
- *X is smooth if and only if all cones of $\mathbb{F}_X$ are generated by a part of a basis of $\mathbb{Z}^n$.*
- *X is $\mathbb{Q}$-factorial if and only if all cones of $\mathbb{F}_X$ are generated by linearly independent elements of $\mathbb{Z}^n$ (i.e. all cones are simplicial).*
- *The orbits of $(\mathbb{C}^*)^n$ in X of dimension d are in bijection with the cones of codimension D of $\mathbb{F}^n$. In particular, fixed points are parameterized by cones of dimension n and $(\mathbb{C}^*)^n$-stable divisors by one-dimensional cones.*

Proposition 1.9.

- *The group of Weil divisors of X is presented by generators $X_1, \dots, X_r$ indexed by the one-dimensional cones of $\mathbb{F}_X$ and the following relations*

  $$\text{for all } v \in \mathbb{Z}^n,\ \sum_{i=1}^{r} \langle v, x_i \rangle X_i = 0$$

  where x_i is the primitive element of the i-th one-dimensional cone of $\mathbb{F}_X$.
  In particular, if X is $\mathbb{Q}$-factorial, the Picard number ρ equals $r - n$.
- *An anticanonical bundle of X is $-K_X = X_1 + \cdots + X_r$.*

Proposition 1.10. *Let X be a (complete) toric variety and D be a Weil divisor of the form $\sum_{i=1}^{r} b_i X_i$ where the b_i are integers.*

Then D is Cartier if and only if for all cone $\mathcal{C}$ of $\mathbb{F}_X$, there exists $\chi_{\mathcal{C}}$ in $\mathbb{Z}^n$ such that for all primitive element x_i of an edge of $\mathcal{C}$, $\langle x_i, \chi_{\mathcal{C}} \rangle = b_i$.

When D is Cartier, we can define a piecewise linear function h_D of $\mathbb{R}^n$ as follows. Let $x \in \mathbb{R}^n$. There exists a unique maximal cone $\mathcal{C}$ of $\mathbb{F}_X$ such that $x \in \mathcal{C}$ and then we define $h_D(x) = \langle x, \chi_{\mathcal{C}} \rangle$.

Then a Cartier divisor D is ample if and only if the piecewise linear function h_D is strictly convex (i.e. for all distinct maximal cones $\mathcal{C}$ and $\mathcal{C}'$ of $\mathbb{F}_X$ and for all element x in the interior of $\mathcal{C}$ we have $h_D(x) = \langle x, \chi_{\mathcal{C}} \rangle > \langle x, \chi'_{\mathcal{C}} \rangle$).

The Fano case

Definition 1.11. A normal projective variety is said to be Fano if it is Gorenstein and if its anticanonical divisor is ample.

V. Batyrev classified Fano toric varieties in terms of particular lattice polytopes called reflexive polytopes [Ba94]. Geometric properties of Fano toric varieties can be characterized by properties of its reflexive polytope: for example the anticanonical degree $(-K_X)^n$ of a Fano toric variety X is the volume of the polytope dual to the reflexive polytope associated to X.

V. Batyrev used this classification to obtain a result of mirror symmetry on Calabi-Yau subvarieties of Fano toric varieties. But this classification was also used to have results on the degree [De03], on the Picard number and on the pseudo-index [Ca06] of Fano toric varieties.

1.2 Flag varieties

The references to have more details on algebraic groups (or Lie algebras) and flag varieties are [Hu75], [Sp98] and [Bo75]. In this section, G is a semi-simple (*i.e.* the radical $R(G)$ of G is trivial where $R(G)$ is the maximal closed, connected, normal, solvable subgroup of G) and connected, algebraic group over $\mathbb{C}$. Such groups are almost classified by there Dynkin diagrams (of type A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4 and G_2) or equivalently by their root systems.

Let me give a very short summary of this classification (due to C. Chevalley). One can read this summary together with the example of SL_n below in order to understand what really happens.

Fix a Borel subgroup of G (*i.e.* a maximal solvable closed and connected subgroup of G) and a maximal torus $T \subset B$. Let $\mathfrak{g}$ be the Lie algebra of G, it is a semi-simple Lie algebra, and denote by $[\cdot,\cdot]$ the Lie Bracket. There is a natural representation of $\mathfrak{g}$ in $\mathrm{GL}(\mathfrak{g})$, called the adjoint representation, defined by $\mathrm{ad}(x)(y) = [x, y]$. Let $\mathfrak{t}$ be the Lie algebra of T, it is a maximal torus of $\mathfrak{g}$, then we said that a character α of $\mathfrak{t}$ is a root of $(\mathfrak{g},\mathfrak{t})$ if it is a weight of $\mathfrak{t}$ in $\mathfrak{g}$ via the adjoint representation *i.e.* if there exists a non-zero $x \in \mathfrak{g}$ such that for all $h \in \mathfrak{t}$, $[h, x] = \alpha(h)x$. We denote by R the set of roots, it is a finite subset of characters of $\mathfrak{t}$. Define also $\mathfrak{g}_\alpha$ the set of eigenvector of weight α *i.e.* the element $x \in \mathfrak{g}$ satisfying the latter condition. Then we have

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in R} \mathfrak{g}_\alpha.$$

Now let $\mathfrak{b}$ be the Lie algebra of B. Let R^+ be the set of roots α such that $\mathfrak{g}_\alpha \subset \mathfrak{b}$. Denote by R^- the complementary of R^+ in R. Then we have

$$\mathfrak{b} = \mathfrak{t} \oplus \bigoplus_{\alpha\in R^+} \mathfrak{g}_\alpha \text{ and } R^- = \{-\alpha \mid \alpha \in R^+\}.$$

The elements of R^+ (resp. R^-) are called positive (resp. negative) roots. There exists a unique basis S of R^+ (*i.e.* such that all positive roots are linear combinations of elements of S with positive integer coefficients). The elements of S are called simple roots.

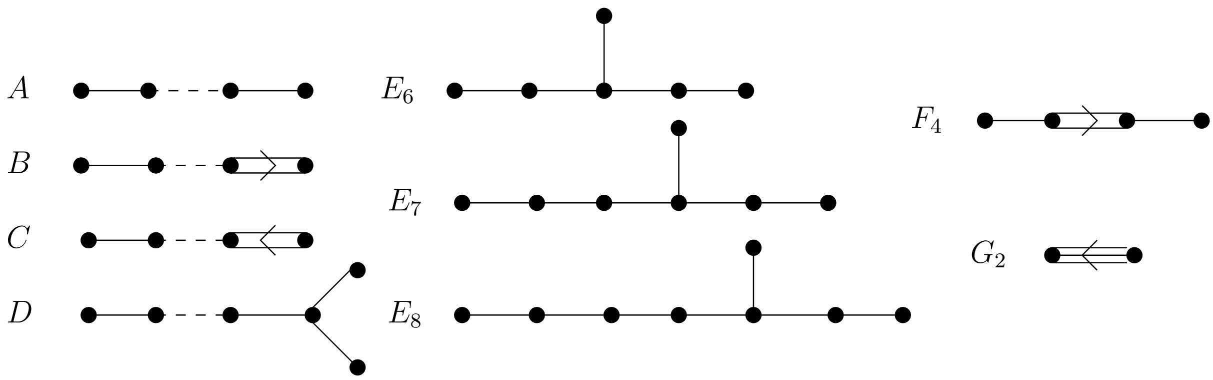
Denote by $\mathfrak{X}$ the set of characters of T, by $\mathfrak{X}^\vee := \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{X}, \mathbb{Z})$ the set of cocharacters of T and by $\langle\cdot,\cdot\rangle$ the pairing between $\mathfrak{X}$ and $\mathfrak{X}^\vee$. Define now the Weyl group of (G, T) (or $(\mathfrak{g},\mathfrak{t})$) to be the quotient $N_G(T)/T$. It is a finite group acting on $\mathfrak{X}$ by: for all $w \in W$, $\chi \in \mathfrak{X}$, $t \in T$, $w\cdot\chi(t) = \chi(w^{-1}tw)$. For all $\alpha \in R$ define G_α to be the subgroup of G with Lie algebra $\mathfrak{g}_{-\alpha}\oplus\mathfrak{t}\oplus\mathfrak{g}_\alpha$. Then there is a unique non trivial element in W which have a representative in $N_{G_\alpha}T$, we denote it by s_α. The s_α for $\alpha \in R$ are reflections and the s_α for $\alpha \in S$ generate W. Remark that $G_{-\alpha} = G_\alpha$ and $s_{-\alpha} = s_\alpha$.

Now, for all $\alpha \in R$, there exists a unique element $\alpha^\vee$ of $\mathfrak{X}^\vee$ with $\langle \alpha, \alpha^\vee \rangle = 2$ such that for all $x \in \mathfrak{X}$, $s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha$. Remark that if $\alpha \in R$ then $(-\alpha)^\vee = -(\alpha^\vee)$. The $\alpha^\vee$ for $\alpha \in R$ are called the coroots of (G, T). The set of coroots is denoted by $R^\vee$, it is a finite subset of $\mathfrak{X}^\vee$.

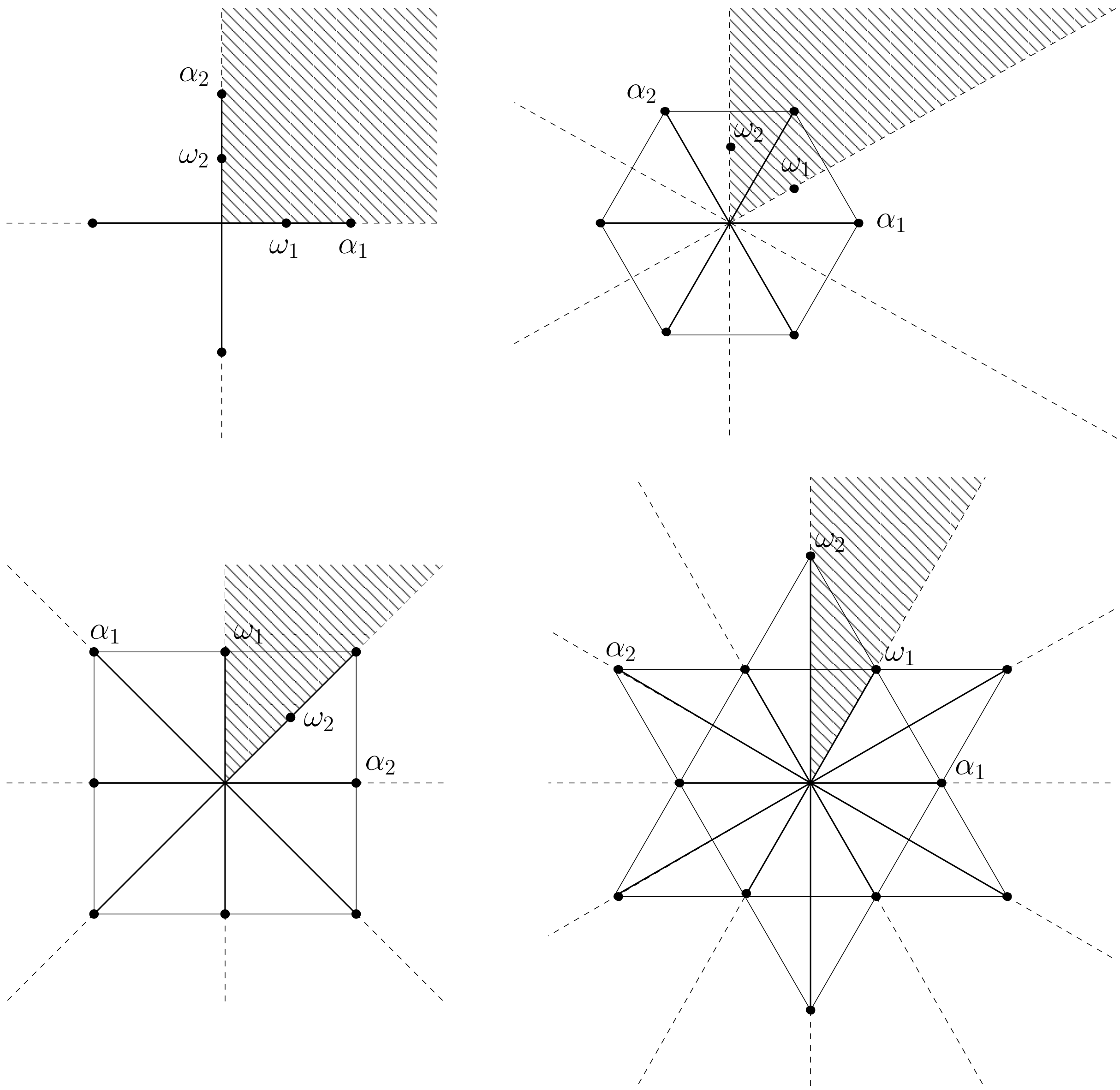
The quadruple $(\mathfrak{X}, R, \mathfrak{X}^\vee, R^\vee)$ is called the root datum of (G, T) and characterize uniquely G.

The root data of semi-simple connected algebraic groups satisfy a lot of nice properties (for example, for all two roots $\alpha \neq \pm\beta$ we have $\langle \alpha, \beta^\vee \rangle = 0, -1, -2$ or -3. And there are classified in terms of Dynkin diagrams. The Dynkin diagram associated to G is the graph Γ defined as follows:
- the vertices of Γ are indexed by the set of simple roots S;
- if, for α and β in S, $\langle \alpha, \beta^\vee \rangle = 0$, there is no arrow between α and β;
- if, for α and β in S, $\langle \alpha, \beta^\vee \rangle = \langle \beta, \alpha^\vee \rangle = -1$, there is a simple not oriented arrow between α and β;
- if, for α and β in S, $\langle \alpha, \beta^\vee \rangle = -2$ and $\langle \beta, \alpha^\vee \rangle = -1$,there is a double arrow from α to β;
- if, for α and β in S, $\langle \alpha, \beta^\vee \rangle = -3$ and $\langle \beta, \alpha^\vee \rangle = -1$,there is a triple arrow from α to β.
(Other cases are not possible.)

Here is all possible Dynkin diagrams.



We can represent the root system of rank n (the cardinality of S or the dimension of T) in an n-dimensional space such that the coroot of a root α is $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$ where $(\cdot, \cdot)$ is the canonical product (with $(\alpha, \alpha) = 2$ for the smallest roots). For example, the root systems $A_1 \times A_1$, A_2, B_2 (that is the same as C_2) and G_2, of rank 2 are represented as follows (respectively).



The points ω_1 and ω_2 are the fundamental weights that will be defined at the end of the section.

Example 1.12. Let $G = \mathrm{SL}_n$, choose T to be the diagonal matrices and B to be the upper triangular matrices. Then $\mathfrak{g} = \mathfrak{sl}_n$ the set of $n \times n$ matrices of zero trace and the Lie bracket is define by $[x, y] = xy - yx$. The Lie subalgebras $\mathfrak{t}$ and $\mathfrak{b}$ of $\mathfrak{g}$ are respectively the diagonal and upper triangular matrices with zero trace. The roots of (G, T) are the characters $\alpha_{ij} : h = \mathfrak{diag}(h_1, \dots, h_n) \longmapsto h_i - h_j$ of $\mathfrak{t}$ for $1 \leq i \neq j \leq n$. We can also see these characters as characters $\alpha_{ij} : t = \mathrm{diag}(t_1, \dots, t_n) \longmapsto t_i t_j^{-1}$ of T. And the Lie subalgebra $\mathfrak{g}_{\alpha_{ij}}$ of $\mathfrak{g}$ is generated by the matrix E_{ij} with 0 everywhere excepted for the coefficient on the ith line and jth column that equals 1. Then $R^+ = \{\alpha_{ij} \mid 1 \leq i < j \leq n\}$ and $S = \{\alpha_i := \alpha_{i(i+1)} \mid 1 \leq i \leq n-1\}$.

Now W is isomorphic to the symmetric group S_n, we can represent it in SL_n by the group of permutation matrices. Then, for $1 \leq i < j \leq n$, $s_{\alpha_{ij}}$ is the transposition $(i\,j)$ and represented in SL_n by the permutation matrix $\sum_{k \neq i,j} E_{kk} + E_{ij} - E_{ji}$.

For all $1 \leq i, j \leq n-1$, the integer $\langle \alpha_i, \alpha_j^\vee \rangle$ equals 2 if $i = j$, 0 if $|i-j| > 2$ and -1 if $|i-j| = 1$. So the Dynkin diagram of G is A_{n-1}, we say that G is of type A_{n-1}.

Remark 1.13. In the latter example, we could replace SL_n by PSL_n and we would have exactly the same sets of roots and coroots, but we would also remark that they have the same Lie algebra. Indeed we have a bijective correspondence between simple Lie algebra and Dynkin diagram, however a simple Lie algebra can be the Lie algebra of several simple algebraic groups (but finitely many). To each Dynkin diagram corresponds a "maximal" and a "minimal" simple algebraic group: a simply connected one and one with trivial center. For A_{n-1}, SL_n is simply connected and PSL_n has trivial center (by definition PSL_n is the quotient of SL_n by its center). We can also remark that the group character of T in PSL_n is smaller than the group character of SL_n.

In lots of situations (for example for flag varieties and horospherical varieties) we will not take care of the choice of the semi-simple algebraic groups of fixed type because it does not matter. But we will see that, for symmetric varieties or wonderful varieties, the choice of the semi-simple algebraic groups of a given type has to be taken into consideration, and in these cases we will always assume (without loss of generality) that G is simply connected.

Now, we can interest us in flag varieties.

Proposition 1.14. *The complete homogeneous spaces are the G/P where P is a closed subgroup containing a Borel subgroup of G. In that case P is called a parabolic subgroup of G and G/P is called a flag variety.*

Flag varieties are smooth projective varieties.

Remark 1.15. All Borel subgroup are conjugated, so we can assume that P contains a fixed Borel subgroup B.

With the notations above, the set of parabolic subgroups of G containing B (and then also the set of isomorphism classes of flag varieties with G fixed) is in bijection with the set of subsets of S.

Example 1.16. Let $G = \mathrm{SL}_n$. Recall that S is in bijection with $\{1, \ldots, n-1\}$ (see example 1.12). Then to a subset $S \backslash I = \{0 = i_0 < i_1 < \cdots < i_s < i_{s+1} = n\}$ ($s \geq 0$) we associate the parabolic subgroup P_I consisting of upper triangular block matrices with diagonal blocks of size $i_1 - i_0, \ldots, i_{s+1} - i_s$. For example, $B = P_\emptyset$ corresponds to the parabolic subgroup associated to $\{1 < 2 < \cdots < n-1\}$. Another example is for maximal parabolic subgroup P_I where $S \backslash I = \{i_1\}$ then G/P is the grassmannian of i_1-dimensional subspaces of $\mathbb{C}^n$, denoted by $\mathrm{Grass}_{i_1,n}$.

Remark 1.17. We have $P_I \subset P_J$ if and only if $I \subset J$.

To understand the geometry of flag varieties, we often use their decomposition into B-orbits.

Proposition 1.18 (Bruhat decomposition)**.**

$$G = \sqcup_{w\in W} B\dot{w}B$$

where $\dot{w}$ is a representative of w in G (in fact, by abuse of notations, we always note w instead of $\dot{w}$). In particular,

$$G/B = \sqcup_{w\in W} B\dot{w}B/B \text{ and } G/P = \sqcup_{w\in W/W_P} B\dot{w}P/P$$

where, if $P = P_I$, W_P is the subgroup of W generated by the simple reflections associated to elements of $S\backslash I$.

Since W is finite we have the remarkable following.

Corollary 1.19. *Flag varieties have an open orbit under the action of a Borel subgroup.*

The closure of the B-orbits in G/P are called the Schubert varieties (denoted by $X(w)$), they play an important role in the study of G/P. The dimension of $X(w)$ equals the length $l(w)$ of w (*i.e.* the minimal number of simple reflections needed to write w as product of simple reflections). In particular, there exists a unique element w_0 of maximal length in W/W_P.

Here is few properties of flag varieties.

Proposition 1.20.

- *The Picard group of G/P_I is the free group generated by the divisors $X(w_0s_\alpha)$ for $\alpha \in S\backslash I$.*
- *Flag varieties are locally rigid, i.e. they admit no local deformation or in other words $H^1(G/P, T_{G/P}) = 0$ (in fact $H^i(G/P, T_{G/P}) = 0$ for any $i > 0$).*
- *Flag varieties are Fano varieties and $-K_X = \sum_{\alpha\in S\backslash I} a_\alpha X(w_0s_\alpha)$ where the a_α are integers ≥ 2 depending on G and $P = P_I$, defined by $a_\alpha = \langle \sum_{\beta\in R^+\backslash R_I^+} \beta, \alpha^\vee\rangle$ where R_I^+ is the set of positive roots generated by the simple roots in I.*

Corollary 1.21. *Flag varieties of Picard number one are the G/P where P is a maximal proper parabolic subgroup of G. In particular, for $G = \mathrm{SL}_n$, flag varieties of Picard number one are the grassmannians.*

Remark 1.22. If G is not semi-simple but reductive (*i.e.* G contains no normal closed subgroup isomorphic to $\mathbb{C}^n$), then G/P is isomorphic to $G'/G'\cap P$ where G' is the semi-simple part (or equivalently, the derived subgroup) of G and then it is still a flag variety.

A canonical way to embed flag varieties in projective spaces

Assume here that G is simply connected. The set $\mathfrak{X}$ of characters of T is generated by characters ω_α called the fundamental weights for $\alpha \in S$. They can be defined by the dual basis of $(\alpha^\vee)_{\alpha\in S}$ relatively to $\langle\cdot,\cdot\rangle$. In fact, ω_α is a generator of the character group of the maximal parabolic subgroup $P(\omega_\alpha) := P_{S\backslash\{\alpha\}}$. A character is said to be dominant if it is the linear combination of the fundamental weights with non-negative coefficient. And for all dominant character χ, there exists a unique simple G-module $V(\chi)$ of highest weight χ, *i.e.* a simple G-module containing a non-zero element v such that for all $b \in B$ we have $b\cdot v = \chi(b)v$. A such element v is called a highest weight vector (of highest weight χ), it is unique up to scalar and denoted by v_χ.

Proposition 1.23. *Let χ be a dominant character of T. Then the G-orbit of $[v_\chi]$ in $\mathbb{P}(V(\chi))$ is isomorphic to G/P_I where $I = \{\alpha \in I \mid \langle \chi, \alpha^\vee \rangle = 0\}$. In fact, P_I is the stabilizer of $[v_\chi]$ in G.*

In particular for all $\alpha \in S$, we have a highest weight module $V(\omega_\alpha)$ and a natural embedding of $G/P(\omega_\alpha) \longrightarrow \mathbb{P}(V(\omega_\alpha))$ defined by $g \longmapsto g \cdot v_{\omega_\alpha}$.

Example 1.24. Let $G = \mathrm{SL}_n$. For all $i \in \{1, \ldots, n-1\}$, $\omega_{\alpha_i}(\mathrm{Diag}(t_1, \ldots, t_n)) = t_1 \cdots t_i$ and the highest weight module $V(\omega_{\alpha_i})$ is the vector space $\bigwedge^i \mathbb{C}^n$ with the natural action of SL_n. Let fix $(e_1, \ldots, e_n)$ a basis of $\mathbb{C}$ (the same basis used to write elements of G as matrices). Then a highest weight vector of $V(\omega_{\alpha_i})$ is $v_{\omega_{\alpha-i}} = e_1 \wedge \cdots \wedge e_i$. The embedding $\mathrm{Grass}_{i,n} \longrightarrow \bigwedge^i \mathbb{C}^n$ is called the Plücker embedding.

Remark that if $G = \mathrm{PSL}_n$, the character ω_{α_1} is not well-defined, as well as the G-action on the corresponding module $\mathbb{C}^n$. Nevertheless the G-action is well-defined on $\mathbb{P}(\mathbb{C}^n)$.

Remark 1.25. We have constructed, by this way, polarized flag varieties.

2 Spherical varieties: definition and theory of Luna-Vust

From now on, G can be any reductive and connected algebraic group over $\mathbb{C}$. Then we can define the root system of G to be the root system of the semi-simple part G' of G. So we refer to section 1.2 for the notations. The only difference between semi-simple and reductive group is that we can have characters of $\mathfrak{X}$ that are not generated by the ω_α with $\alpha \in S$, because there is also characters of the center $C(G)$ of G (that is finite if G is semi-simple).

Examples 2.1. Let $G = (\mathbb{C}^*)^n$. Then $G' = \{1\}$ so that the root system of G is empty (there is no root, no fundamental weight...). But the set of characters of $T = G = C(G)$ is isomorphic to $\mathbb{Z}$.

Let now $G = \mathrm{GL}_n$. Then $G' = \mathrm{SL}_n$, $C(G) = \{\lambda \,\mathrm{Id} \mid \lambda \in \mathbb{C}^*\}$. Then the root system of G is the root system of SL_n and the character group $\mathfrak{X}$ is freely generated by the ω_α with $\alpha \in S$ and the determinant.

In this section, we give the definitions of spherical varieties and a horosphercial varieties. And we explain the theory of Luna-Vust that classify the embeddings of a fixed (horo)spherical homogeneous space.

2.1 Definitions and the general theory in an "easy" case

Definition 2.2. Let X be a normal G-variety. Then X is said to be spherical if X contains an open orbit under the action of a Borel subgroup of G. In particular, it contains an open G-orbit.

At the same way, an homogeneous space G/H is said to be spherical if it contains an open orbit under the action of a Borel subgroup of G. Then a spherical variety is a G/H-embedding for a spherical homogeneous space G/H (*i.e.* a normal G-variety with an open orbit isomorphic to G/H).

Examples 2.3. the first examples of spherical varieties are toric varieties (a Borel subgroup of $(\mathbb{C}^*)^n$ is $(\mathbb{C}^*)^n$ itself) and flag varieties.

The B-stable divisors of spherical varieties will play an important role in their study. We distinguish the G-stable ones (denoted by $X_1, \ldots, X_m$) and the B-stable but not G-stable ones. These latter divisors are indeed the closure in X of the B-stable divisors of the open orbit G/H of X that are called the colors of G/H.

Example 2.4. For toric varieties, there are no colors (because $G = B$) and the G-stable divisors are in correspondence with the one-dimensional cones of the associated fan (see 1.1).

For flag varieties, there is no G-stable divisors and the colors are the codimension one Schubert varieties (see 1.2).

Another more general but not too difficult example: horospherical varieties

Definition 2.5. An homogenous space G/H is said to be horospherical if it is a torus bundle over a flag variety G/P, or equivalently if H contains the unipotent radical of a Borel subgroup, or equivalently if H is the kernel of characters of a parabolic subgroup P of G. The dimension of the torus, fiber of $G/H \longrightarrow G/P$, is called the rank of G/H and denoted by n.

Remark 2.6. The parabolic subgroup P in the definition is uniquely defined as the normalizer $N_G(H)$ of H in G. And we also have $P = TH = BH$ for all maximal torus T of B contained in P and all Borel subgroup B of G contained in P.

Example 2.7. Here is some examples of horospherical homogenous spaces.

	G	H	rank	dimension
1	$(\mathbb{C}^*)^n$	$\{1\}$	n	n
2	G	un sous-groupe parabolique P	0	$\dim G - \dim P$
3	SL_2	$U = \{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\}$	1	2
4	$SL_2 \times \mathbb{C}^*$	$U = \{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\} \times \{1\}$	2	3
5	$SL_2 \times SL_2$	$U = \{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\} \times \{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\}$	2	4
6	SL_3	$U = \{\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}\}$	2	5

Let G/H be a horospherical homogenous space. Then, the colors of G/H are the inverse images of the codimension one Schubert varieties of G/P by the torus fibration $G/H \longrightarrow G/P$. Thus there are indexed by the elements of $S \backslash I$ where I is the subset of S such that $P = P_I$ (see section 1.2). We denote the colors by D_α, for $\alpha \in S \backslash I$.

The Luna-Vust theory for horospherical varieties

Fix a horospherical homogenous space G/H. And keep the notations of the latter paragraph. Choose B and T contained in P.

Denote by M the lattice of characters of P whose restriction to H is trivial. This lattice is of rank n (the rank of G/H). Define now N to be the dual of M and $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$.

We denote by $\mathbb{C}(G/H)^{(B)}$ the set of rational functions f on G/H such that there exists a character χ_f of T (and of B) such that for all $b \in B$ and $g \in G$, $f(bgH) = \chi_f(b)f(gH)$. (We denote by gH the class of g in G/H.)

Lemma 2.8. *The map $f \longmapsto \chi_f$ is a bijection from $(\mathbb{C}(G/H)^{(B)}\backslash\{0\})/\mathbb{C}^*$ to M.*

Proof of the injectivity. Let f_1 and f_2 two non zero elements of $\mathbb{C}(G/H)^{(B)}$ with associated characters $\chi_1 = \chi_2$. Then the quotient $f = f_1/f_2$ is still an element of $\mathbb{C}(G/H)^{(B)}$ with associated character $\chi_1 - \chi_2 = 0$. Then f is constant in the open B-orbit of G/H so it is constant everywhere. $\square$

Remark 2.9. The hypothesis that there exists an orbit under a Borel subgroup is used to prove that there is no multiplicity in the decomposition of $\mathbb{C}(G/H)$ into simple G-module.

We are now able to define special points of the lattice N. Let $\alpha \in S\backslash I$, then recall that the color D_α is a B-stable divisor of G/H. Then, the application from $\mathbb{C}(G/H)^{(B)}$ to $\mathbb{Z}$ the degree of zeros or poles of a rational function on the divisor D_α, is an element of N using the last lemma. It is called the image of the color D_α and denoted by $\alpha_M^\vee$. It is in fact the restriction to M of the coroot $\alpha^\vee : \omega_\beta \longmapsto 1$ if $\alpha = \beta$ and 0 if not.

As the same way, for a G-stable irreducible divisor D of a G/H-embedding we can associate an element $\sigma(D)$ of N.

We are now able to give the classification of G/H-embedding in the particular case of horospherical varieties.

Definition 2.10. (i) A colored cone is a pair $(\mathcal{C}, \mathcal{F})$ with $\mathcal{C} \subset N_{\mathbb{R}}$ and $\mathcal{F} \subset S\backslash I$ having the following properties:

- $\mathcal{C}$ is a convex cone generated by the $\alpha_M^\vee$ with $\alpha \in \mathcal{F}$ and finitely many elements of N;
- $\mathcal{C}$ contains no lines and $\alpha_M^\vee \neq 0$ for all $\alpha \in \mathcal{F}$.

(ii) A colored face of a colored cone $(\mathcal{C}, \mathcal{F})$ is a pair $(\mathcal{C}', \mathcal{F}')$ such that $\mathcal{C}'$ is a face of $\mathcal{C}$ and $\mathcal{F}'$ is the subset of $\mathcal{F}$ of elements α satisfying $\alpha_M^\vee \in \mathcal{C}'$.

(iii) A colored fan is a finite set $\mathbb{F}$ of colored cones with the following properties:

- every colored face of a colored cone of $\mathbb{F}$ is in $\mathbb{F}$;
- for all $u \in N$, there exists at most one $(\mathcal{C}, \mathcal{F}) \in \mathbb{F}$ such that u is in the relative interior of $\mathcal{C}$.

(iv) The support of a colored fan $\mathbb{F}$ is the set of elements of N contained in the cone of a colored cone of $\mathbb{F}$. A color of a colored cone of $\mathbb{F}$ is an element of $\mathcal{D}$ such that there exists $(\mathcal{C}, \mathcal{F}) \in \mathbb{F}$ such that $D \in \mathcal{F}$.

Let X be a G/H-embedding. Let X' be a G-stable open subvariety of X containing a unique closed G-orbit (such a variety is called a simple G/H-embedding). Let $X_1, \dots, X_{m'}$ the G-stable divisors of X. Let $\mathcal{F}'$ be the set of $\alpha \in S\backslash I$ such that the closure of D_α in X contains the closed orbit of X'. Let $\mathcal{C}'$ be the cone of $N_{\mathbb{R}}$ generated by the $\alpha_M^\vee$ with $\alpha \in \mathcal{F}'$ and the $\sigma(X_i)$ with $i = 1 \dots m'$. Then $(\mathcal{C}', \mathcal{F}')$ is a colored cone of $N_{\mathbb{R}}$. Moreover the set of colored cones constructed on this way from X and their colored faces form a colored fan. We denote it by $\mathbb{F}_X$.

Theorem 2.11 (Luna-Vust)**.** *The map $X \longmapsto \mathbb{F}_X$ is a bijection from the isomorphism classes of G/H-embeddings and the set of colored fans.*

Remark 2.12. In the theorem, isomorphisms are isomorphisms of G/H-embeddings (*i.e.* G-equivariant isomorphisms $\phi : (X, x) \longrightarrow (X', x')$ where x and x' are in the open G-orbit of X and X' such that $\phi(x) = x'$).

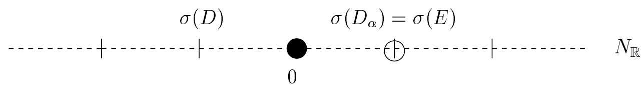
Example 2.13. The horospherical homogeneous space SL_2/U, of rank 1, is isomorphic to $\mathbb{C}^2\backslash\{0\}$. We can choose B (respectively U) equal to the set of upper triangular matrices of SL_2 (respectively upper triangular matrices with ones on the diagonal). Here $P = B$, $S = \{\alpha\}$, $I = \varnothing$ and U is the kernel of $\omega_\alpha : b = (b_{ij})_{1 \leq i,j \leq 2} \in B \longrightarrow b_{11}$ in B. We can remark that the morphism $\mathrm{SL}_2/U \longrightarrow \mathrm{SL}_2/P$ is the usual projection from $\mathbb{C}^2\backslash\{0\}$ to $\mathbb{P}^1$.

The natural action on SL_2 on $\mathbb{C}^2$ induces an action of SL_2 on $\mathbb{P}^2 \simeq \mathbb{P}(\mathbb{C} \oplus \mathbb{C}^2)$. If we denote by x_0, x_1, x_2 the homogenous coordinates of $\mathbb{P}^2$, we remark that $\mathbb{P}^2$ is a SL_2/U-embedding. In fact SL_2/U corresponds to the open set $\{[1, x_1, x_2] \mid (x_1, x_2) \in \mathbb{C}^2\backslash\{0\}\}$ of $\mathbb{P}^2$. Denote by 0 the fixed point $[1, 0, 0]$ of $\mathbb{P}^2$ under the action of SL_2, D the line $\{[x_0, x_1, x_2] \in \mathbb{P}^2 \mid x_0 = 0\}$ (so that $\mathbb{P}^2\backslash D = \mathbb{C}^2$), and E exceptional divisor of the blow-up of 0 in $\mathbb{P}^2$. Then non-trivial SL_2/U-embeddings are the 5 following varieties.

	SL_2/U-embedding X	SL_2-stable divisor(s)	closed SL_2-orbit(s)	color of X
1/	$\mathbb{C}^2$	$\varnothing$	$\{0\}$	D_α
2/	$\mathbb{P}^2\backslash\{0\}$	D	D	$\varnothing$
3/	$\mathbb{P}^2$	D	D and $\{0\}$	D_α
4/	blow-up of 0 in $\mathbb{C}^2$	E	E	$\varnothing$
5/	blow-up of 0 in $\mathbb{P}^2$	D and E	D and E	$\varnothing$

For SL_2/U, the unique color is $\{D_\alpha\} = \{[1, x_1, 0] \mid x_1 \in \mathbb{C}^*\}$. The embeddings 1/, 2/ and 4/ only have one closed SL_2-orbit; they are the simple SL_2/U-embeddings. The embeddings 2/, 4/ and 5/ have no color; there are said to be toroïdal. We also remark that the blow-up of 0 in $\mathbb{P}^2$ is covered by the blow-up of 0 in $\mathbb{C}^2$ and $\mathbb{P}^2\backslash\{0\}$, and also that $\mathbb{P}^2$ is covered by $\mathbb{P}^2\backslash\{0\}$ and $\mathbb{C}^2$.

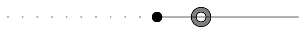
In this example, the lattices M and N are isomorphic to $\mathbb{Z}$. The following represents the line $N_{\mathbb{R}}$ with the images by σ of the B-stable divisors of the blow-up of 0 in $\mathbb{P}^2$:



The non-trivial colored fans of $N_{\mathbb{R}}$ are the following:

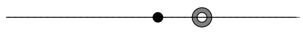
4/

2/



1/

5/



3/

The colored fan i/ corresponds to the SL_2/U-embedding i/ and the trivial colored fan $\{(\{0\},\varnothing)\}$ corresponds to the trivial SL_2/U-embbeding SL_2/U.

Let us now give few properties of horospherical varieties via the classification.

Proposition 2.14.
- *X is complete if and only $\mathbb{F}_X$ is complete i.e. if for all element $x \in N_{\mathbb{R}}$ there exists a cone of $\mathbb{F}_X$ containing x [LV83].*
- *There is a bijective correspondence between G-orbits in X and the colored cones in $\mathbb{F}_X$.*
- *There is a smoothness criterion given independently at the same time in [Pa06a] and [Ti06]. In particular, every G-stable subvariety of a smooth horospherical variety is smooth.*
- *There are also locally factoriality and $\mathbb{Q}$-factoriality criteria.*

Proposition 2.15.
- *If X is $\mathbb{Q}$-factorial, the Picard number of X equals $m + \sharp(S\backslash I) - n = r - n + \sharp(S\backslash I) - \sharp\mathcal{D}_X$ where r is the number of 1-dimensional colored cones in $\mathbb{F}_X$ and $\mathcal{D}_X$ is the set of colors of X (that is a subset of $S\backslash I$).*
- *An anticanonical divisor of X is $-K_X = X_1 + \cdots + X_m + \sum_{\alpha \in S\backslash I} a_\alpha D_\alpha$ where the a_α are the same as in Proposition 1.20 [Br97a].*

In [Br89], M. Brion described the Picard group of all spherical varieties and gave criteria for Weil divisors to be Cartier, globally generated and ample. Here is a summary of these results in the case of complete horospherical varieties. Note that these criteria are very similar to the criteria given for toric varieties in Proposition 1.10.

Theorem 2.16 ([Br89])**.** *Let X be a complete G/H-embedding and D a Weil divisor of the form*

$$\sum_{i=1}^{m} b_i X_i + \sum_{\alpha \in S\backslash I} b_\alpha D_\alpha,$$

where the b_i and b_α are integers.

Then D is Cartier if and only if for all colored cone $(\mathcal{C},\mathcal{F})$ of $\mathbb{F}_X$, there exists $\chi_{\mathcal{C}}$ in M such that for all primitive element x_i of an edge of $\mathcal{C}$, $\langle x_i, \chi_{\mathcal{C}}\rangle = b_i$ and $\forall \alpha \in \mathcal{F}$, $\langle \alpha_M^\vee, \chi_{\mathcal{C}}\rangle = b_\alpha$.

When D is Cartier, we can define a piecewise linear function h_D of $N_{\mathbb{R}}$ as follows. Let $x \in N_{\mathbb{R}}$. There exists a unique maximal colored cone $(\mathcal{C},\mathcal{F})$ of $\mathbb{F}_X$ such that $x \in \mathcal{C}$ and then we define $h_D(x) = \langle x, \chi_{\mathcal{C}}\rangle$.

Then a Cartier divisor D is ample if an only if:

(i) the piecewise linear function h_D is strictly convex (i.e. for all distinct maximal colored cones $(\mathcal{C},\mathcal{F})$ and $(\mathcal{C}',\mathcal{F}')$ of $\mathbb{F}_X$ and for all element x in the interior of $\mathcal{C}$ we have $h_D(x) = \langle x, \chi_{\mathcal{C}}\rangle > \langle x, \chi'_{\mathcal{C}}\rangle$);

(ii) for all colored cone $(\mathcal{C},\mathcal{F})$ of $\mathbb{F}_X$ and for all $\alpha \in (S\backslash I)\backslash\mathcal{F}$, we have $h_D(\alpha_M^\vee) = \langle \alpha_M^\vee, \chi_{\mathcal{C}}\rangle < b_\alpha$.

A way to construct projective horospherical varieties

We consider a convex lattice polytope Q of dimension in $M_{\mathbb{R}}$ whose lattice points are dominant characters. Then we define

$$X = \overline{G\cdot[\sum_{\chi\in Q\cap M} v_\chi]} \subset \mathbb{P}(\oplus_{\chi\in Q\cap M} V(\chi)).$$

It is a projective horospherical (with open orbit G/H if Q is n-dimensional).

Examples 2.17.

- Let us consider $G = \mathrm{SL}_n$ and the segment $Q = [\omega_{\alpha_i}, \omega_{\alpha_{i+1}}]$ in $\mathfrak{X}$ with $1 \leq i \leq n-2$. Define

$$X = \overline{G\cdot e_1\wedge\cdots\wedge e_i + e_1\wedge\cdots\wedge e_{i+1}} \subset \mathbb{P}(\bigwedge^i \mathbb{C}^n \oplus \bigwedge^{i+1}\mathbb{C}^n) = \mathbb{P}(V(\omega_{\alpha_i})\oplus V(\omega_{\alpha_{i+1}})).$$

  Then X is horospherical of rank one with open orbit G/H where $H = \mathrm{Ker}(\omega_{\alpha_i} - \omega_{\alpha_{i+1}}) \subset P_{S\backslash\{\alpha_i,\alpha_{i+1}\}}$.

  In fact, in that case, X is isomorphic to $\mathrm{Grass}_{i+1,n+1}$. Indeed, let e_0 such that $e_0,\dots,e_n$ is a basis of $\mathbb{C}^{n+1}$. Then the morphism

$$\begin{array}{ccc} \bigwedge^i \mathbb{C}^m \oplus \bigwedge^{i+1}\mathbb{C}^m & \longrightarrow & \bigwedge^{i+1}\mathbb{C}^{m+1} \\ x+y & \longmapsto & x\wedge e_0 + y \end{array}$$

  is an isomorphism. Moreover $e_1\wedge\cdots\wedge e_i + e_1\wedge\cdots\wedge e_{i+1}$ is send to $e_1\wedge\cdots\wedge e_i\wedge(e_{i+1}\pm e_0)$ so that X is a subvariety of $\mathrm{Grass}_{i+1,n+1}$. We conclude comparing the dimension.

- Let us now consider $G/H = \mathrm{SL}_3/U$ and the triangle Q with vertices 0, ω_α, ω_β in $M = \mathfrak{X}$. Then, using the same trick as above, we can prove that X is a cone of basis the grassmannian $\mathrm{Grass}_{2,4}$. In particular, it is not smooth (it is only locally factorial).

2.2 The Luna-Vust theory in the general case

Good references for this theory in the general case are the original paper of D. Luna and T. Vust [LV83], the paper of F. Knop [Kn91] where he proves the result in any characteristic and notes of M. Brion [Br97b].

Here G/H is a spherical homogeneous space, *i.e.* containing an open B-orbit for a Borel subgroup B of G. Then we fix this Borel subgroup B and a maximal torus T of B.

Define M as the set of characters χ such that there exists a non-zero $f\in\mathbb{C}(G/H)^{(B)}$ of weight χ. Then, as before, M is isomorphic to $(\mathbb{C}(G/H)^{(B)}\backslash\{0\})/\mathbb{C}^*$.

We denote by $\mathcal{D}$ the set of irreducible B-stable divisor of G/H (*i.e.* the set of colors). Then we have an application σ from $\mathcal{D}$ to N "defining by the degree of zeros and poles of rational functions".

Denote by $\mathcal{V}$ the cone of $N_{\mathbb{R}}$ generated by the image in N of the set of G-stable valuations of G/H. It is a convex polyhedral cone. We have $\mathcal{V} = N_{\mathbb{R}}$ if and only if G/H is horospherical.

Definition 2.18. (i) A colored cone is a pair $(\mathcal{C}, \mathcal{F})$ with $\mathcal{C} \subset N_{\mathbb{R}}$ and $\mathcal{F} \subset \mathcal{D}$ having the following properties:

- $\mathcal{C}$ is a convex cone generated by $\sigma(\mathcal{F})$ and finitely many elements of $\mathcal{V}$;
- the relative interior of $\mathcal{C}$ intersects $\mathcal{V}$ non trivially;
- $\mathcal{C}$ contains no lines and $0 \notin \sigma(\mathcal{F})$.

(ii) A colored face of a colored cone $(\mathcal{C}, \mathcal{F})$ is a pair $(\mathcal{C}', \mathcal{F}')$ such that $\mathcal{C}'$ is a face of $\mathcal{C}$, the relative interior of $\mathcal{C}'$ intersects non trivially $\mathcal{V}$ and $\mathcal{F}'$ is the subset of $\mathcal{F}$ of elements D satisfying $\sigma(D) \in \mathcal{C}'$.

(iii) A colored fan is a finite set $\mathbb{F}$ of colored cones with the following properties:

- every colored face of a colored cone of $\mathbb{F}$ is in $\mathbb{F}$;
- for all $v \in \mathcal{V}$, there exists at most one $(\mathcal{C}, \mathcal{F}) \in \mathbb{F}$ such that v is in the relative interior of $\mathcal{C}$.

(iv) The support of a colored fan $\mathbb{F}$ is the set of elements of $\mathcal{V}$ contained in the cone of a colored cone of $\mathbb{F}$. A color of a colored cone of $\mathbb{F}$ is an element of $\mathcal{D}$ such that there exists $(\mathcal{C}, \mathcal{F}) \in \mathbb{F}$ such that $D \in \mathcal{F}$.

We can now define the colored fan associated to a G/H-embedding as in the horospherical case and we still have the following.

Theorem 2.19 (Luna-Vust)**.** *The map $X \longmapsto \mathbb{F}_X$ is a bijection from the isomorphism classes of G/H-embeddings and the set of colored fans.*

Theorem 2.20.

- *X is complete if and only $\mathbb{F}_X$ is complete i.e. if for all element $x \in \mathcal{V}$ there exists a cone of $\mathbb{F}_X$ containing x.*
- *There is a bijective correspondence between G-orbits in X and the colored cones in $\mathbb{F}_X$.*
- *If X is $\mathbb{Q}$-factorial, the Picard number of X equals $m + \sharp(S\backslash I) - n = r - n + \sharp(S\backslash I) - \sharp\mathcal{D}_X$.*
- *There is no general smoothness criterion but a description of the local structure of spherical varieties [Br89, Chap.1].*
- *An anticanonical divisor of X is $-K_X = X_1 + \cdots + X_m + \sum_{D\in\mathcal{D}} a_D D$ where the a_D are positive integers [Br97a].*
- *There is an ampleness criterion for divisor on spherical varieties [Br89] very similar to the case of horospherical varieties (Theorem 2.16.*

2.3 Some examples

Let $G = \mathrm{SO}_3$ (it is of type A_1, in fact isomorphic to PSL_2). Let us consider the G-module $\mathbb{C}^3$ with the natural action. Let q be a G-invariant non-degenerate quadratic form on $\mathbb{C}^3$. Let X_0 and X_1 be the fibers of 0 and 1, respectively, of $q : \mathbb{C}^3 \longrightarrow \mathbb{C}$. There are affine spherical varieties (left to the reader) which are non G-equivariantly isomorphic because X_0 has two G-orbits and X_1 has only one G-orbit.

Denote by $\langle \cdot, \cdot \rangle$ the scalar product associated to q. Fix a basis (e_1, e_0, e_{-1}) of $\mathbb{C}^3$ such that $\langle e_i, e_j \rangle = 1$ if $i = -j$ and 0 if not. Denote by (f_1, f_0, f_{-1}) the dual basis. Then $\mathbb{C}[X_0] = \mathbb{C}[f_1, f_0, f_{-1}]/(f_0^2 + 2f_1 f_{-1})$ and $\mathbb{C}[X_1] = \mathbb{C}[f_1, f_0, f_{-1}]/(f_0^2 + 2f_1 f_{-1} - 1)$.

Let B be the Borel subgroup of G define by $B = \mathrm{Stab}_G\, e_1$. Then f_{-1} is a highest weight vector of weight $2\omega_\alpha : (b_{ij}) \in B \longmapsto b_{11} = b_{33}^{-1}$. Then we can prove that for $i = 0$ and 1, the lattice M of weights of $\mathbb{C}[X_i]$ is the sublattice of characters of T generated by $2\omega_\alpha$.

For X_0, we can observe that the decomposition of $\mathbb{C}[X_0]$ into simple G-module is the decomposition into polynomials of same degree ($\mathbb{C}[f_1, f_0, f_{-1}]_d/(f_0^2 + 2f_1 f_{-1}) = V(2d\omega_\alpha)^*$). We have one color D whose closure in X_0 is the divisor defining by $f_{-1}(x) = 0$ (it is in fact the line generated by e_1). Then $\sigma(D) = \alpha^\vee/2$. The divisor D defines a G-invariant valuation ν which to a polynomial associates its minimal degree. Seen as an element of N, it has an inverse $-\nu$, which also comes from a G-invariant valuation, which to a polynomial associates minus its maximal degree. Then the valuation cone of G/H in this case is $N_\mathbb{Q}$, in fact X_0 is an horospherical variety ($\mathrm{Stab}_G\, e_1$ is the kernel in B of $2\omega_\alpha$).

For X_1, we have two colors that are the irreducible components of the divisor defined by $f_{-1} = 0$ (if $f_{-1} = 0$ then $f_0 = 1$ or $f_0 = -1$). They have the same image $\alpha^\vee/2$ by σ. Now let ν be a G-invariant valuation of G/H. Then $\nu(f_1) = \nu(f_0) = \nu(f_{-1})$ and $0 = \nu(1) = \nu(f_0^2 + 2f_1 f_{-1}) \geq \min(\nu(f_0^2), \nu(f_1 f_{-1})) = 2\nu(f_{-1})$ so that the image of ν in $N_\mathbb{Q}$ is in the half line generated by $-\alpha^\vee/2$. In fact, the valuation cone of G/H in that case is the half line of $N_\mathbb{Q}$ not containing the color.

In fact X_1 is a symmetric homogeneous space G/H (see Definition 3.5) where $H = G^\theta := \{g \in G \mid \theta(g) = g\}$ and $\theta : G \longrightarrow G$, $g \longmapsto JgJ$ with $J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ (here we chose to take $H = \mathrm{Stab}_G(e_1 + e_2)$ so that BH/H is open). And the only non-trivial G/H-embedding is the quadric Q^2 in $\mathbb{P}[\mathbb{C} \oplus \mathbb{C}^3]$ where G acts trivially on $\mathbb{C}$ and by the usual action on $\mathbb{C}^3$. We could also consider the following symmetric homogenous space $G/N_G(H)$. Note that $N_G(H)/H$ is finite (in that particular case, H is a maximal torus and $G/N_G(H)$ is isomorphic to the Weyl group $W = S_2$). The unique non-trivial $G/N_G(H)$-embedding is $\mathbb{P}(\mathbb{C}^3)$. For this symmetric homogeneous space, we have the same invariants as for G/H except that we have only one color (still with image $\alpha^\vee/2$ by σ).

Note that X_1 degenerates into X_0, it is a general fact: spherical varieties degenerate into horospherical varieties and also into toric varieties.

These varieties belong to a class of spherical varieties classified by D. Akhiezer: smooth complete G-varieties with an open orbit and a boundary consisting of homogeneous divisors [Ak83].

3 A general classification?

For the moment, the spherical homogeneous spaces are only classified in very special cases. Nevertheless, I.V. Losev recently proved a uniqueness property for spherical homogeneous spaces.

3.1 A theorem of uniqueness

We have already seen combinatorial invariants of spherical homogeneous spaces: the lattice M of weights of $\mathbb{C}(G/H)$, the valuation cone and the set $\mathcal{D}$ of colors of G/H together with a map σ from $\mathcal{D}$ to the dual N of M. We just have to add one natural natural family of invariants (the stabilizers in G of the colors) in order to have the uniqueness of spherical homogenous spaces, more precisely:

Theorem 3.1 (I.V. Losev). *Let G/H_1 and G/H_2 be two spherical homogeneous spaces with the same weight lattice M, the same valuation cone in $N := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ and set of colors $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively (together with maps σ_1 and σ_2 from $\mathcal{D}_1$ and $\mathcal{D}_2$ to N respectively) such that there exists a bijection $\iota : \mathcal{D}_1 \longrightarrow \mathcal{D}_2$ satisfying, for all $D \in \mathcal{D}_1$, $\sigma_1(D) = \sigma_2(\iota(D))$ and $\mathrm{Stab}_G D = \mathrm{Stab}_G \iota(D)$.*

Then G/H_1 and G/H_2 are G-equivariantly isomorphic.

We can refer to [Lo09a] for the complete proof or to [Lo09b] for a good summary of the proof.

Remark 3.2. We can also replace the valuation cone by the so called spherical roots defined by the set Σ of primitive elements of M such that $\mathcal{V} = \{n \in N_{\mathbb{Q}} \mid \langle n, m\rangle \leq 0, \forall m \in \Sigma\}$.

Example 3.3. The addition condition on the stabilizers of the colors in G is necessary as the following example shows us.

Let $G = \mathrm{SL}_n$ with $n \geq 5$, B a Borel subgroup of G and T a maximal torus. Let H_1 be the kernel in $P(\omega_{\alpha_1}) \cap P(\omega_{\alpha_2})$ of the fundamental weight ω_{α_1}. Let H_2 be the the kernel of the same weight ω_{α_1} but in the $P(\omega_{\alpha_1}) \cap P(\omega_{\alpha_3})$. Then, both G/H_1 and G/H_2 are horospherical, with the same weight lattice $M = \mathbb{Z}$ generated by ω_{α_1} (and so with the same valuation cone $\mathbb{R}$ because they are horospherical) and each with 2 colors whose images in N are the same ($\alpha_{1M}^\vee$ and 0). But the stabilizers of the colors in G are not the same (for the color with image 0). In fact we could have already noticed that G/H_1 and G/H_2 are not isomorphic because they do not have the same dimension!

In fact a more simple example can also be found just in flag varieties.

For other examples, see previous section.

3.2 Classification in very special cases

3.2.1 Horospherical varieties

Proposition 3.4. *The horospherical G-varieties are classified by couple (I, M) where I is a subset of the set S of simple roots and M is a subgroup of the group $\mathfrak{X}(P_I) \subset \mathfrak{X}$ of characters of P_I.*

Proof. We fix a Borel subroup B and we consider the horospherical homogeneous spaces G/H such that H contains the unipotent radical U of B. Since all Borel subgroup are conjugated, we obtain in this way all other horospherical G/H.

Recall that, for each horospherical homogenous space G/H, we define I to be subset of S such that $N_G(H) = P_I$ and M to be the subgroup of $\mathfrak{X}$ consisting of characters of P_I whose restriction to H is trivial.

Inversely, for all couple (M, I), then the subgroup H equal the intersection of kernels of characters of M in P_I. $\square$

3.2.2 Symmetric varieties

Definition 3.5. Let G be connected semi-simple algebraic group over $\mathbb{C}$ and θ be an involution of G. Let H be a closed subgroup of G such that $G^\theta \subset H \subset N_G(G^\theta)$. Then G/H is said to be a symmetric homogeneous space. And G/H-embeddings are called symmetric varieties. (We will always assume that θ is not the identity.)

Examples 3.6. The homogenous spaces $X_1 = \mathrm{SO}(3)/H$ and $\mathrm{SO}(3)/N_{\mathrm{SO}_3}H$ of section 2.3 are symmetric.

For all connected semi-simple algebraic group G, $G \times G/G$ where G is diagonally embedded in $G \times G$, is a symmetric homogeneous space.

The classification of involutions for each G was established by E. Cartan in the 1920ies (see in Wikipedia website or in appendix A of [BC08]). And then T. Vust proved that symmetric varieties are spherical and described the valuation cone and the colors associated to a symmetric space [Vu90]. Let us describe quickly the combinatorics of symmetric homogeneous spaces.

First, let T^1 be a torus in G such that $\forall t \in T^1$, $\theta(t) = t^{-1}$ and maximal for this property. And let T be a maximal torus of G containing T^1. Then T is θ-stable and the set R of roots of (G, T) inherits an involution θ. Define the restricted root system associated to θ to be

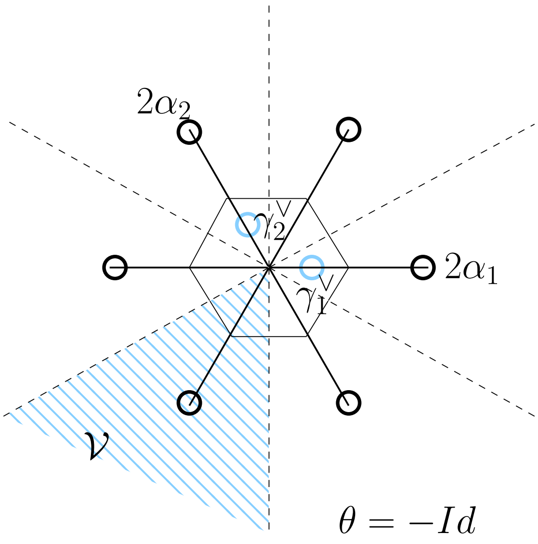
$$R_{G,\theta} := \{\alpha - \theta(\alpha) \mid \alpha \in R\}.$$

Moreover we may choose a Borel subgroup B of G containing T such that for all positive root relative to B we have: $\theta(\alpha)$ is either α or a negative root. Then BH is open in G and the set $S_{G,\theta} := \{\alpha - \theta(\alpha) \mid \alpha \in S\}$ is a basis of $R_{G,\theta}$. We denote by $\gamma_1, \ldots, \gamma_s$ the elements of $S_{G,\theta}$.

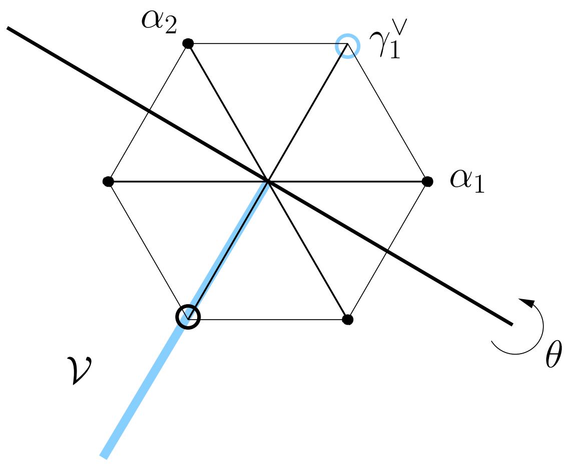
Then the lattice M can be identified to the character group of $T^1/(T^1 \cap H)$ and the valuation cone $\mathcal{V}$ is the antidominant Weyl chamber of the dual root system $R^\vee_{G,\theta}$, *i.e.* $\mathcal{V} = \{x \in N_\mathbb{Q} \mid \langle \alpha_i, x\rangle \leq 0, \forall i = 1, \ldots, s\}$.

The image of the colors by σ are the simple coroots $\gamma_1^\vee, \ldots, \gamma_s^\vee$ in $R^\vee_{G,\theta}$. Then, for all $i = 1, \ldots, s$, there exist at most two colors (and at least one) with image $\gamma_i^\vee$. The set of images of colors is fixed by G and θ, but the set of colors depends on the subgroup H such that $G^\theta \subset H \subset N_G(G^\theta)$ (see Examples 2.3).

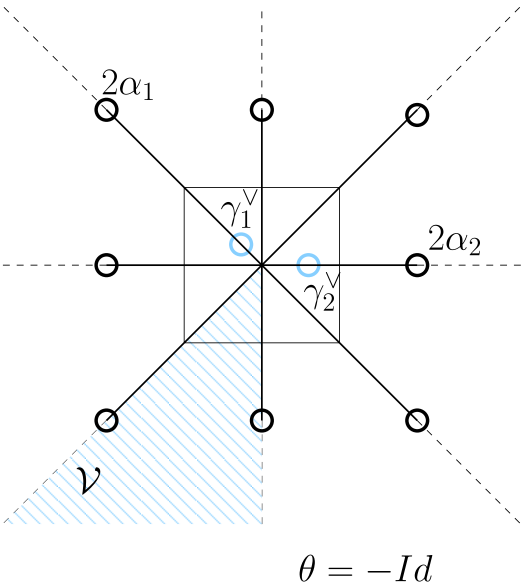
Examples 3.7. Here is the combinatorics for the symmetric spaces G/H where G is simple of rank 2 (*i.e.* S of cardinality 2).



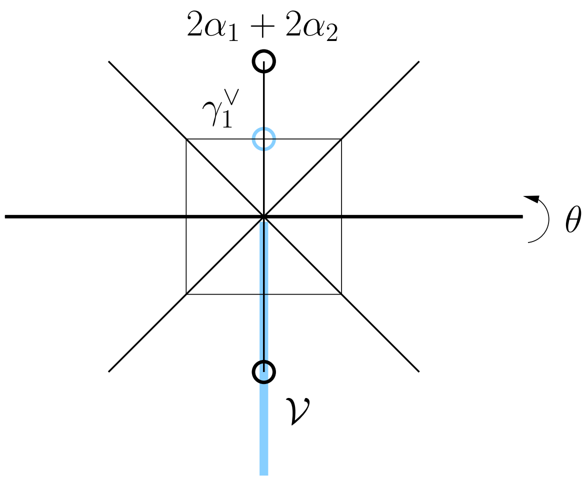
A I $G = \mathrm{SL}_3$ and $H = \mathrm{SO}_3$, $\mathrm{rk}(G/H) = 2$.



A IV $G = \mathrm{SL}_3$ and $H = \mathrm{GL}_2$, $\mathrm{rk}(G/H) = 1$.



B I $G = \mathrm{SO}_5$ and $H = \mathrm{SO}_2 \times \mathrm{SO}_3$, $\mathrm{rk}(G/H) = 2$.



B II $G = \mathrm{SO}_5$ and $H = \mathrm{SO}_4$, $\mathrm{rk}(G/H) = 1$.

Drawing the corresponding picture in the following case is left to the reader: G $G = G_2$ and $H = \mathrm{SL}_2 \times \mathrm{SL}_2$, $\mathrm{rk}(G/H) = 2$, θ acts on R by $-Id$.

3.2.3 Wonderful varieties

Wonderful varieties play a central role inside the theory of classification of spherical varieties. Indeed, to every spherical G-varieties X, we can associate in a functorial way a wonderful variety

$\mathbb{X}$ such that X is determined by the spherical system of $\mathbb{X}$ together with some additional data [Lu01, Th. 3]. The wonderful variety associated to a spherical variety is often used to obtain results on spherical varieties.

Definition 3.8. An algebraic G-variety X is said to be wonderful of rank n if

1. it is smooth and complete,
2. it has an open G-orbit whose complement is the union of smooth irreducible G-divisors D_i for $i = 1, \ldots, n$ with normal crossings and such that $\cap_{i=1}^{n} D_i \neq \varnothing$,
3. if x, x' are such that $\{i \mid x \in D_i\} = \{i \mid x' \in D_i\}$ then the orbits of x and x' are the same.

A wonderful variety X is called strict if each of its points has a selfnormalizing stabilizer. The type of a wonderful variety is the type of G (or the type of the Dynkin diagram of G).

Theorem 3.9 ([Lu96]). *Wonderful varieties are projective and spherical.*

In particular, a wonderful variety with open orbit G/H is the unique G/H-embedding which is wonderful.

Examples 3.10.

- Flag varieties are wonderful varieties of rank 0.
- The only toric and wonderful variety is the point. Indeed, a complete toric variety does not satisfy $\cap_{i=1}^{n} D_i \neq \varnothing$ except in zero dimension.
- The unique non-trivial G/H-embeddings Q^2 and $\mathbb{P}^2$ seen in Section 2.3 are wonderful varieties of rank 1.
- For all symmetric homogenous space G/H, C. De Concini and C. Procesi constructed the wonderful G/H-embedding in [DP83].

D. Luna introduced several invariants attached to any wonderful variety X, called together the spherical system of X. Let us define these invariants and some of there properties. We will give give the complete definitions only for strict wonderful varieties, so see [Lu01] or [BL09] for the complete definitions and details in the general setting).

Let X be a wonderful G-variety. Let Y be the (unique) closed G-orbit of X and $z \in Y$ be the (unique) point fixed by the opposite Borel subgroup B^- of B (the subgroup B^- is the unique Borel subgroup of G such that $B \cap B^- = T$. The spherical roots of X are the weights of T of the quotient T_zX/T_zY where T_zX (resp. T_zY) denotes the tangent space at z of X (resp. of Y). Denote by Σ_X the set of spherical roots.

Then Σ_X is a finite subset of the monoïd $\mathbb{N}S$, the rank of X is the cardinality of ΣX and Σ is a basis of the lattice M associated to the open orbit G/H of X.

We say that an element of $\mathbb{N}S$ is a spherical root of G if it is the spherical root of a wonderful variety of rank 1. Note that the spherical roots of all reductive connected algebraic group are known because of the classification of wonderful varieties of rank 1 (or equivalently, equivariant completions of homogeneous algebraic varieties by one homogeneous divisor), due to D. Akhiezer [Ak83].

Let P_X be the stabilizer of the point $z \in Y$, it is a parabolic subgroup of G. Let S^p_X be the subset of S such that $P_X = P_{S^p_X}$. Then P_X coincides with the stabilizer of the colors of X.

In the case of strict wonderful varieties, the couple (S^p_X, Σ_X) shares nice properties: it is a strict spherical system for G. Spherical systems were introduced by D. Luna in [Lu01] as triples which fulfill certain axiomatic conditions. Let us give the definition of strict spherical systems.

Definition 3.11. A couple (S^p, Σ) is a strict spherical system for G if it consists of a subset S^p of simple roots and a set Σ of spherical roots of G that satisfy the following properties:
(Σ_0) $\Sigma \cap S = \varnothing$;
(Σ_1) If $2\alpha \in \Sigma \cap 2S$ then $\frac{1}{2}\langle \gamma, \alpha^\vee \rangle$ is a non-positive integer for every $\gamma \in \Sigma \backslash \{2a\}$;
(Σ_2) If α, $\beta \in S$ are orthogonal (*i.e.* $\langle \alpha, \beta^\vee \rangle = 0$) and $\alpha + \beta \in \Sigma$ or $\frac{1}{2}(\alpha+\beta) \in \Sigma$ then $\langle \gamma, \alpha^\vee \rangle = \langle \gamma, \beta^\vee \rangle$ for every $\gamma \in \Sigma$;
(S) For every $\gamma \in \Sigma$, there exists a wonderful G-variety X of rank 1 such that $\{\gamma\} = \Sigma_X$ and $S^p = S^p_X$;
(R) For every $\gamma \in \Sigma$, there exists no rank 1 wonderful G-variety X such that $\{2\gamma\} = \Sigma_X$ and $S^p = S^p_X$;

In full generality, a spherical system is defined by D. Luna as a triple $(S^p, \Sigma, \mathbf{A})$ with (S^p, Σ) as above satisfying conditions (Σ_0), (Σ_1), (Σ_2) and (S). The datum $\mathbf{A}$ is a multiset of functionals on $\mathbb{Z}\Sigma = M$ related to the simple roots in Σ with some extra conditions.

Definition 3.12. The set of colors $\mathcal{D}$ of a given strict spherical system (S^p, Σ) is defined as $\mathcal{D} = (S \backslash S^p)/\sim$ with $\alpha \sim \beta$ whenever α and β are orthogonal and $\alpha + \beta \in \Sigma$ or $2(\alpha+\beta) \in \Sigma$. We denote by D_α the color associated to $\alpha \in S \backslash S^p$.

We define a map $\sigma : \mathcal{D} \longrightarrow (\mathbb{Z}\Sigma)^* = N$ as follows: if $2\alpha \in \Sigma$ then $\sigma(D_\alpha)$ is the restriction to M of $\frac{1}{2}\alpha^\vee$ and otherwise $\sigma(D_\alpha)$ is the restriction to M of $\alpha^\vee$. In particular σ is injective.

The set of colors of a spherical system can be also defined, it depends on $\mathbf{A}$ and σ is then not necessarily injective.

In the case where the spherical system is given by a wonderful G-variety X, the set $\mathcal{D}$ coincides with the set of colors of the open orbit G/H of X.

Conjecture 3.13 ([Lu01])**.** *Wonderful varieties are classified by their spherical systems.*

Theorem 3.14. *This conjecture has been answered positively for wonderful varieties:*

- *of rank 2, by B. Wasserman in [Wa96];*
- *of type A, by D. Luna in [Lu01];*
- *of type D, by P. Bravi and G. Pezzini in [BP05];*
- *of type E, by P. Bravi in [Br07];*
- *and strict wonderful varieties, first by P. Bravi and S. Cupit-Foutou in [BC08] and also, with a smaller proof, by S. Cupit-Foutou in [Cu08].*

4 Fano horospherical varieties

We have already defined Fano varieties in Definition 1.11. Fano varieties play an important role in order to understand algebraic varieties, indeed Mori's program predicts that every uniruled variety is birational to a fiberspace whose general fiber is a Fano variety with terminal singularities. We know all deformation families of smooth Fano varieties in dimension 1, 2 and 3, but in higher dimension we only know that the number of these families is finite. That is why we are interested in having classes of examples of Fano varieties as Fano toric varieties or more generally Fano horospherical varieties.

4.1 Their classification

Let X be a complete G/H-embedding. Recall that $-K_X = \sum_{i=1}^m X_i + \sum_{\alpha\in S\backslash I} a_\alpha D_\alpha$, where the X_i are the irreducible G-stable divisors of X and the D_α are the colors and denote by h_{-K_X} the piecewise linear function associated to $-K_X$ (see Theorem 2.16). And define the set

$$Q(X) := \{u \in N_{\mathbb{R}} \mid h_{-K_X}(u) \leq 1\}.$$

When X is Fano, $Q(X)$ is a G/H-reflexive polytope in the following sense.

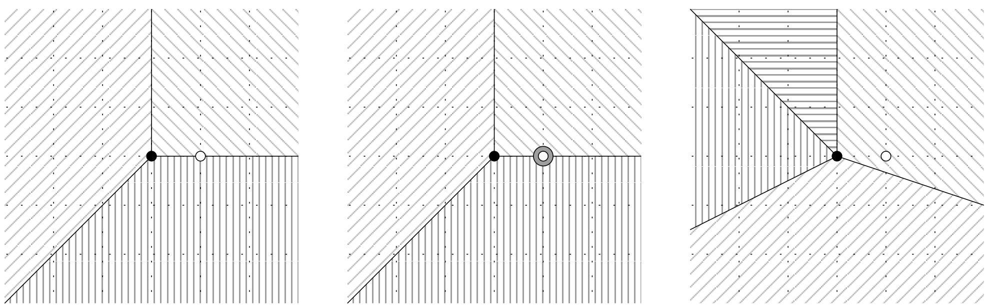
Definition 4.1. Let G/H be a homogeneous horospherical space. A convex polytope Q in $N_{\mathbb{R}}$ is said to be G/H-reflexive if the following three conditions are satisfied:

(1) The vertices of Q are in $N \cup \{\frac{\alpha_M^\vee}{a_\alpha} \mid \alpha \in S\backslash I\}$, and the interior $\overset{o}{Q}$ of Q contains 0.

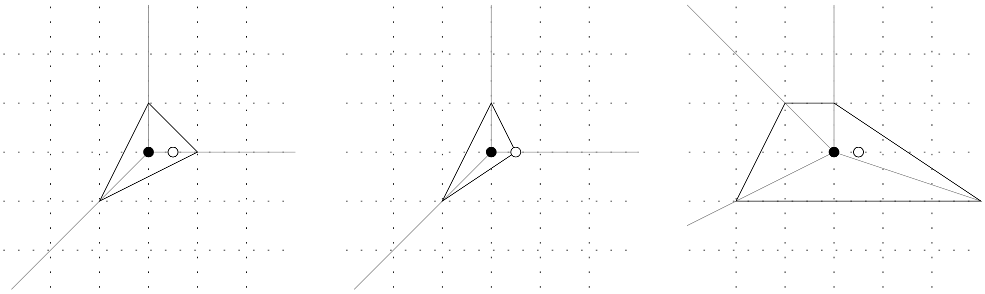
(2) $Q^* := \{v \in M_{\mathbb{R}} \mid \forall u \in Q,\ \langle v, u\rangle \geq -1\}$ is a lattice polytope (*i.e.* its vertices are in M).

(3) For all $\alpha \in S\backslash I$, $\frac{\alpha_M^\vee}{a_\alpha} \in Q$.

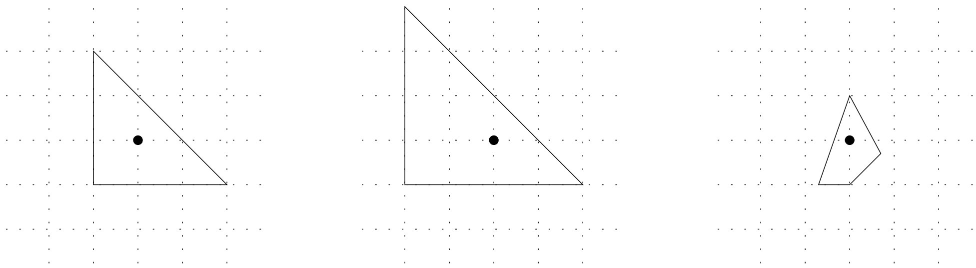
Examples 4.2. In the case where $G/H = (SL_2 \times \mathbb{C}^*)/U$, $N_{\mathbb{R}} = \mathbb{R}^2$ and there is one color which is a primitive element. Consider the following colored fans:



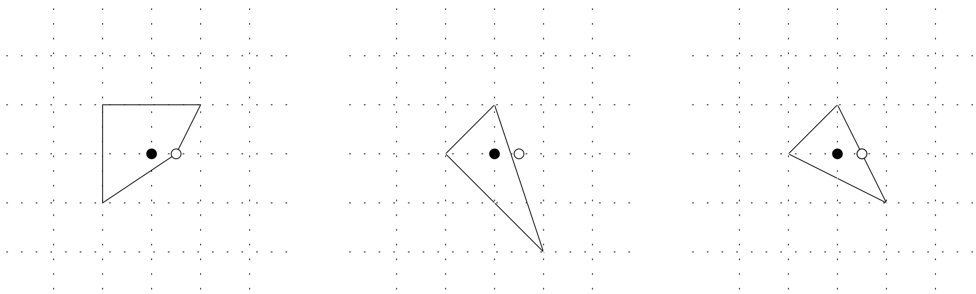
Then the corresponding $Q(X)$ are the following polytopes (in general it is not necessarily a polytope). The first two polytopes are G/H-reflexive. The third polytope is not G/H-reflexive because its dual is not a lattice polytope.



Their dual polytopes are respectively:



Here is other examples of polytopes in the case where $G/H = (SL_2 \times \mathbb{C}^*)/U$.



The first and the third polytopes are G/H-reflexive. But, the second one is not, because $\frac{\alpha_M^\vee}{a_\alpha}$ is not in the polytope.

Then we have the following.

Proposition 4.3. *Let G/H be a horospherical homogeneous space. Then the application $X \longmapsto Q(X)$ is a bijection between the set of isomorphism classes of Fano G/H-embeddings and the set of G/H-reflexive polytopes.*

For all Fano horospherical variety X, the dual polytope $Q(X)^*$ translated to $2\rho^P := \sum_{\alpha \in S \setminus I} a_\alpha \omega_\alpha$ lies in the cone of dominant weights of (G, B, T), and it is the moment polytope of the polarized G-variety $(X, -K_X)$. In other words, X can be embedded in

$$\mathbb{P}(\bigoplus_{\chi \in Q(X)^* \cap M} V(2\rho^P + \chi))$$

as in the construction used in Example 2.17.

Remark 4.4. A. Ruzzi also give the list of all Fano symmetric varieties [Ru06].

4.2 Some results on their geometry

Theorem 4.5 ([Pa08a]). *Let X be a locally factorial Fano horospherical variety, of dimension d, rank n and Picard number ρ.*
If $\rho > 1$ then

$$(-K_X)^d \leq d!\, d^{d\rho+n}.$$

And if $\rho = 1$, we have

$$(-K_X)^d \leq d!\, (d+1)^{d+n}.$$

O. Debarre proved similar inequalities for smooth Fano toric varieties [De03]. Remark that a locally factorial toric variety is always smooth, but there is locally factorial horospherical varieties that are not smooth.

The proof of this result uses the fact that the degree of a Fano horospherical variety X can be expressed as the integral of an explicit function on the dual polytope $Q(X)^*$ of the G/H-reflexive $Q(X)$.

Theorem 4.6 ([Pa08a]). *Let X be a $\mathbb{Q}$-factorial Fano horospherical variety, of dimension d, rank n and Picard number ρ.*

Then

$$\rho \leq n + d \leq 2d$$

with equality $\rho = 2d$ if and only if d is even and $X = (S_3)^{d/2}$ where S_3 is the blowing-up of three general points in $\mathbb{P}^2$.

C. Casagrande proved exactly the same result for $\mathbb{Q}$-factorial Fano toric varieties in [Ca06].

The next result answer positively a conjecture of L. Bonavero, C. Casagrande, O. Debarre and S. Druel [BCDD], on the pseudo-index of smooth Fano varieties, in the special case of horospherical varieties. This conjecture had been already proved in the case of toric varieties by C. Casagrande [Ca06] and in dimension less or equal than 4 [BCDD].

Definition 4.7. The pseudo-index ι_X of a Fano variety is the positive integer defined by

$$\iota_X := \min\{-K_X \cdot C \mid C \text{ rational curve in } X\}.$$

Theorem 4.8 ([Pa08b]). *Let X be a $\mathbb{Q}$-factorial horospherical Fano variety of dimension d, Picard number ρ_X and pseudo-index ι_X. Then*

$$(\iota_X - 1)\rho_X \leq d.$$

Moreover, equality holds if and only if X is isomorphic to $(\mathbb{P}^{\iota_X - 1})^{\rho_X}$.

To describe the pseudo-index combinatorially through G/H-reflexive polytopes, we used special curves of spherical varieties and theirs intersections with the irreducible B-stable divisors, described in [Br93].

5 Smooth projective varieties with Picard number one

In this section we consider a special class of spherical varieties, those with Picard number one, that can be seen as minimal varieties in the following sense.

Proposition 5.1. *Let X be a projective spherical variety with Picard number one. Then every G-equivariant birational morphism $f : X \longrightarrow Y$ is an isomorphism.*

5.1 Toric varieties and flag varieties

Proposition 5.2. *The unique n-dimensional smooth projective toric variety with Picard number one is the projective space $\mathbb{P}^1$.*

Proof. By Propositions 1.8 and 1.9, the fan of such a variety is complete, with $n+1$ edges such that the primitive elements of each subset of n edges form a basis of $\mathbb{Z}^n$. Then it is easy to prove that these $(n+1)$ edges are the cones generated respectively by elements of a basis $(e_1, \dots, e_n)$ of $\mathbb{Z}^n$ and $-e_1 - \cdots - e_n$. And the unique fan with such edges is the fan of $\mathbb{P}^n$. □

We have already seen that flag varieties are smooth and projective, and that the flag varieties with Picard number one are the G/P with P maximal (Corollary 1.21).

5.2 Horospherical varieties

In the case of horospherical varieties, there exist non-homogeneous smooth projective varieties with Picard number one. Most of the results of this section are picked up from [Pa09].

Let us begin with an infinite family of examples.

Example 5.3. Let ω be a skew-form of maximal rank on $\mathbb{C}^{2m+1}$. For $i \in \{1, \dots, m\}$, define the odd symplectic grassmannian $\mathrm{Gr}_\omega(i, 2m+1)$ as the variety of i-dimensional ω-isotropic subspaces of $\mathbb{C}^{2m+1}$. Odd symplectic grassmannians are horospherical varieties and, for $i \neq 1, m$ they have two orbits under the action of their automorphism group (the odd symplectic group) which is a connected non-reductive linear algebraic group (see [Mi07] for more details). In fact we can realize $\mathrm{Gr}_\omega(i, 2m+1)$ in

$$\mathbb{P}(\bigwedge^{i} \mathbb{C}^{2m} \oplus \bigwedge^{i+1} \mathbb{C}^{2m}) \simeq \mathbb{P}(\bigwedge^{i+1} \mathbb{C}^{2m+1}).$$

Odd symplectic grassmannians have nice properties. For example, they can be realized as linear sections of (even) symplectic grassmannians. And they are locally rigid (*i.e.* they admit no local deformation or equivalently $H^1(X, T_X) = 0$), like flag varieties.

Theorem 5.4. *Let X be a smooth projective horospherical G-variety with Picard number 1.*
Then we have the following alternative:

(i) X is homogeneous (under its automorphism group), or

(ii) X is horospherical of rank 1. Its automorphism group is a connected non-reductive linear algebraic group, acting with exactly two orbits.

Moreover in the second case, X is uniquely determined by its two closed G-orbits Y and Z, isomorphic to G/P_Y and G/P_Z respectively; and (G, P_Y, P_Z) is one of the triples of the following list.

1. $(B_m, P(\omega_{m-1}), P(\omega_m))$ *with* $m \geq 3$
2. $(B_3, P(\omega_1), P(\omega_3))$
3. $(C_m, P(\omega_i), P(\omega_{i+1}))$ *with* $m \geq 2$ *and* $i \in \{1, \dots, m-1\}$
4. $(F_4, P(\omega_2), P(\omega_3))$
5. $(G_2, P(\omega_2), P(\omega_1))$

Recall that $P(\omega_i)$ is the maximal parabolic subgroup of G corresponding to the dominant weight ω_i (with the numbering of Bourbaki [Bo75]).

Note that Case 3 of Theorem 5.4 corresponds to odd symplectic grassmannians.

Steps of the proof.

- It is not difficult to prove, using that $\rho = 1$ and that X is locally factorial, that:
  the maximal cones of $\mathbb{F}_X$ are generated by n elements among $e_1, \dots, e_n, -e_1 - \cdots - e_n$ with $(e_1, \dots, e_n)$ a basis of N;
  and for all $\alpha \in S \backslash I$, α is a color of $\mathbb{F}_X$ and $\alpha_M^\vee \in \{e_1, \dots, e_n, -e_1 - \cdots - e_n\}$.
- Then, using that X is smooth, we prove that X is a projective space if:
  $n \geq 2$;
  and if $n = 1$ and there is at most one color.
- We give a list of 8 possible cases when X is of rank 1 and has two colors. In every case, we can embedded X in the projectivization of the sum of two fundamental representation (as in Example 2.17). Note also that X has 3 G-orbits (one open and two closed).
- In 3 cases, we prove directly that X is homogenous, exhibiting the corresponding flag variety. And in the other 5 cases, we compute the global sections of the normal sheaf of the closed G-orbits in X. For one of the two closed G-orbit, there is no global sections, so that this orbit is stabilized by the automorphism group of X.
- Then we study more in details the automorphism group of X.

□

We can also complete this theorem by the following, remarking that the non-homogeneous varieties in Theorem 5.4 have two orbits even when they are blown up at their closed orbit.

Theorem 5.5. *Let X be a smooth projective variety with Picard number 1 and put $G := \mathrm{Aut}^0(X)$.*

Assume that X has two orbits under the action of G and denote by Z the closed orbit. Then the codimension of Z is at least 2.

Assume furthermore that the blow-up of Z in X still has two orbits under the action of G. Then, there exists two varieties $\mathbf{X_1}$ and $\mathbf{X_2}$ such that one of the following happens:

- *G is not semi-simple and X is one of the two-orbit varieties classified in Theorem 5.4;*
- *$G = F_4$ and $X = \mathbf{X_1}$;*
- *$G = G_2 \times \mathrm{PSL}(2)$ and $X = \mathbf{X_2}$.*

Note that two-orbit varieties are classified and are known to be spherical varieties [Cu03]. But in fact, to prove Theorem 5.5, we only use the classification of equivariant completions of homogeneous algebraic varieties by homogeneous divisors [Ak83].

Two natural questions on these two-orbit varieties (that are already answered in the case of odd symplectic grassmannians):

- Can we realize them as (linear) sections of flag varieties?
- Are they locally rigid?

The first question is still open except for odd symplectic grassmannians and the variety in Case 3 of Theorem 5.4 that is an hyperplane section of the spinorial variety $\mathbb{S}_{10}$.

The answer of the second one follows from.

Theorem 5.6. *Let X be one of the two-orbit varieties of Theorem 5.5, then we have the alternative:*

- *if X is the variety of Case 5 of Theorem 5.4, then $H^1(X, T_X) = \mathbb{C}$ and $H^i(X, T_X) = 0$ for any $i \geq 2$, moreover X deforms into the orthogonal grassmannian $\mathrm{Gr}_q(2, 7)$;*
- *else $H^i(X, T_X) = 0$ for any $i \geq 1$.*

This result is a quasi-immediate corollary of a more general vanishing theorem of the higher cohomology of a subsheaf of the tangent bundle of some spherical varieties, called quasi-regular varieties [PP08], including all smooth horospherical varieties and smooth spherical varieties of rank one.

5.3 Symmetric varieties

We summarize here results of A. Ruzzi on smooth complete symmetric varieties with Picard number one [Ru07] and [Ru08].

The idea to classify smooth complete symmetric varieties with Picard number one is the following.

First, there is a smooth criterion for symmetric varieties. In particular, X smooth implies that σ is injective, so that the cardinality of the set $\mathcal{D}$ of colors is n.

A second step is to use the Picard number expression to prove that there are exactly two cases: either X is simple (*i.e.* $\mathbb{F}_X$ has one maximal cone) and has $n-1$ colors, or $\mathbb{F}_X$ has two maximal cones (of dimension n) and n colors.

A third step is to use in particular the smooth criterion to prove that G is either a torus or a semi-simple group, and that in the second case the restricted root system is either irreducible (*i.e.* is a root system of a simple group) or of type $A_1 \times A_1$.

Then a case by case study permits to conclude (see Theorem 3.1 of [Ru07] to have the complete list of smooth complete symmetric varieties with Picard number one). We can at lest mention that for a given symmetric homogeneous space G/H there is at most one smooth complete G/H-embedding with Picard number one and it is then projective.

Theorem 5.7 ([Ru08]). *Let X be a smooth projective symmetric varieties with Picard number one which are not homogeneous under the action of their automorphism group. Then its open G-orbit G/H is one of the 6 following. And X is a linear section of a flag variety $\mathbb{X}$.*

	G/H	$\mathbb{X}$	*linear section*	$[\mathrm{Aut}^0(X) : \mathrm{Aut}(X)]$
1/	$G_2/(\mathrm{SL}_2 \times \mathrm{SL}_2)$	$\mathrm{Grass}_{3,7} \subset \mathbb{P}^{34}$	*section of dimension 27*	*1*
2/	$G_2 = G_2 \times G_2/G_2$	$\mathbb{S}_{14} \subset \mathbb{P}^{63}$	*section of dimension 49*	*2*
3/	$\mathrm{SL}_3 / \mathrm{SO}_3$	$\mathrm{Gr}_\omega(3,6) \subset \mathbb{P}^{13}$	*hyperplane section*	*2*
4/	$\mathrm{SL}_3 = \mathrm{SL}_3 \times \mathrm{SL}_3 / \mathrm{SL}_3$	$\mathrm{Grass}_{3,6} \subset \mathbb{P}^{19}$	*hyperplane section*	*4*
5/	$\mathrm{SL}_6 / \mathrm{Sp}_6$	$\mathbb{S}_{12} \subset \mathbb{P}^{31}$	*hyperplane section*	*2*
6/	E_6/F_4	$E_7/P_7 \subset \mathbb{P}^{55}$	*hyperplane section*	*2*

Remark that all these varieties are of rank 2.

Note also that the flag varieties $\mathbb{X}$ in Cases 3/, 4/, 5/ and 6/ are the Legendrian varieties in the third row of the Freudenthal magic square (see for example [LM01] to know more about the flag varieties of the Freudenthal magic square).

The proof of Theorem 5.7 is essentially a case by case study of smooth projective symmetric varieties with Picard number one classified in [Ru07].

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