1. $GL_n(D)$ and Hecke algebras

Let F be a local field, o its ring of integers and i the maximal ideal of o. Let q be the cardinal of the residual field o/i. Let D be a central division algebra of dimension d^2 over F. Let O be the ring of integers of D and I the maximal ideal of O. Let π be a uniformizer for D. Set $G = GL_n(D)$. Set $K_0 = GL_n(O)$ and, for all $k \in \mathbb{N}^*$, $K_j = 1 + M_n(I^{dj})$. Let H be the convolution algebra of locally constant functions on G with compact support. For each j, let H_j be the sub-algebra of H formed by the K_j bi-invariant functions. H_j will be called the **Hecke algebra of level** j. Let Z be the center of G. If $g \in G$, g is called **regular semi-simple** if its characteristic polynomial has distinct roots in an algebraic closure of F. It is called *elliptic* if moereover its characteristic polynomial is irreducible (sometimes this is called regular elliptic).

Recall the Cartan decomposition. Let $\mathcal{A}$ be the set of matrices $(a_{i,j})_{1\leq i,j\leq n}$ such that $a_{i,j} = \delta_{i,j}\pi^{a_i}$ where $\delta_{i,j}$ is the Kronecker symbol and $a_1 \leq a_2 \leq ... \leq a_n$. Then we have:

$$G = \coprod_{A\in\mathcal{A}} K_0 A K_0.$$

So, the caracteristic functions of sets K_0AK_0 form a basis of H_0 when A lies in $\mathcal{A}$. If $j \in \mathbb{N}$, then K_j is a normal sub-group or K_0. The kernel of the natural projection from K_0 onto $GL_n(O/I^j)$ is K_j, so there is a canonical isomorphim $K_0/K_j \simeq GL_n(O/I^j)$. So we will identify these two groups. In particular we write:

$$K_0 = \coprod_{B\in GL_n(O/I^j)} K_j B = \coprod_{B\in GL_n(O/I^j)} BK_j.$$

Now set $T_j = GL_n(O/I^j) \times GL_n(O/I^j)$. The Cartan decomposition may then be written:

$$G = \coprod_{A\in\mathcal{A}} \cup_{(B,C)\in T_j} K_j BAC^{-1} K_j.$$

It is not a disjoint union. However, two sets like in the union are either equal, either disjoint. Let X_A be the sub-group of $GL_n(O) \times GL_n(O)$ made of couples (B, C) such that $BAC^{-1} = A$. Let $H_{A,j}$ be the image of X_A in T_j. Then we have $K_jBAC^{-1}K_j = K_jbAc^{-1}K_j$ if and only if $(b^{-1}B, c^{-1}C) \in H_{A,j}$. So, the set $K_jBAC^{-1}K_j$ is well defined for $(B, C) \in T_j/H_{A,j}$, and we have:

$$G = \coprod_{A\in\mathcal{A}} \coprod_{(B,C)\in T_j/H_{A,j}} K_j BAC^{-1} K_j.$$

So, the set of caracteristic functions of sets $K_jBAC^{-1}K_j$ is a basis of H_j when A lies in $\mathcal{A}$ and, for every such A, (B, C) lies in $T_j/H_{A,j}$. For al this, see [Ba2].

2. $GL_n(D)$ AND CLOSE FIELDS

Now suppose L is another local field. All the objects we described before are defined for L too, and will take an index F or L in the following, to specify the field they are attached to. Suppose that there is an isomorphism $\lambda_j : o_F/i_F^j \simeq o_L/i_L^j$ for some positive integer j. We say then that the fields F and L are **j-close**. If D_L is the central division algebra of dimension d^2 over L with the same Hasse invariant as D_F, then λ_j induces an isomorphism $O_F/I_F^{dj} \simeq O_L/I_L^{dj}$, which we still denote by λ_j. Fix an uniformizer π_L of D_L such that the image by λ_j of the class of π_L is the class of π_F. The set $\mathcal{A}_L$ is defined with respect to this choice, and we get a natural bijection still denoted λ_j from $\mathcal{A}_F$ onto $\mathcal{A}_L$. It is clear that the isomorphisme $\lambda_j : O_F/I_F^{dj} \simeq O_L/I_L^{dj}$ induces an isomorphism $\lambda_j : T_{j,F} \simeq T_{j,L}$. One may prove that the restriction of this isomorphism induces, for every $A \in \mathcal{A}_F$ an isomorphisme between the sub-groups $H_{A,j,F}$ and $H_{\lambda_j(A),j,L}$. So we get a natural bijection between the basis of $H_{j,F}$ and $H_{j,L}$ which defines un isomorphisme λ_j between these two vector spaces.

One may show that, if $l \leq j$, λ_j induces an isomorphism between o_F/i_F^l and o_L/i_L^l, so the fields F and L are also l-close. If we use this isomorphism and the same choice of uniformizer for D_F and D_L, then the isomorpism $\lambda_l : H_{l,F} \simeq H_{l,L}$ obtained is induced by the restriction of the isomorphisme $\lambda_j : H_{j,F} \simeq H_{j,L}$. If K is a compact subset of G_F bi-invariant by $K_{j,F}$, its characteristic function is an element of $H_{j,F}$, and the image by λ_j of this function in $H_{j,L}$ is the charactersitic function of an open compact set denoted $\lambda_j(K)$. Fix Haar mesures on G_F (resp. G_L) such that the volume of the sub-group $K_{0,F}$ (resp. $K_{0,L}$) be one. Then the volume of $\lambda_j(K)$ equals the volume of K. All these results are proven in [Ba2].

3. $SL_n(D)$ AND HECKE ALGEBRAS

We forget L for a moment and we turn back to our F, D and the construction of the beginning. Let G' be the sub-group $SL_n(D)$ of G. For all positive integer j, set $K'_j = K_j \cap G'$. The K'_j make a basis of open compact neighborwood of 1 in G'. Let H'_j be the Hecke algebra of level j of G' made by K'_j-bi-invariant functions on G' which have compact support. Set $\mathcal{A}' = \mathcal{A} \cap G'$. The kernel of the natural projection from K'_0 onto $SL_n(O/I^j)$ is K'_j, so there is a canonical isomorphim $K'_0/K'_j \simeq SL_n(O/I^j)$ and we will identify these two groups. Now let T'_j be the sub-group $SL_n(O/I^j) \times SL_n(O/I^j)$ of T_j. For each $A \in \mathcal{A}'$, set $H'_{A,j} = H_{A,j} \cap T'_j$. Let $Z' = Z \cap G'$ be the center of G'.

Proposition 3.1. *For every $(B,C) \in T'_j/H'_{A,j}$, $K'_jBAC^{-1}K'_j$ is well defined and we have*

$$G' = \coprod_{A\in\mathcal{A}'} \coprod_{(B,C)\in T'_j/H'_{A,j}} K'_jBAC^{-1}K'_j.$$

Proof. We use the Cartan decomposition

$$G' = \coprod_{A \in \mathcal{A}'} K_0' A K_0'.$$

As

$$K_0' = \coprod_{B \in K_0'/K_j'} K_j' B = \coprod_{B \in K_0'/K_j'} B K_j'.$$

and $K_0'/K_j' \simeq SL_n(O/I^j)$, we have

$$G' = \coprod_{A \in \mathcal{A}'} \cup_{(B,C) \in T_j'} K_j' B A C^{-1} K_j'.$$

Now, suppose that $K_j' BAC^{-1} K_j' = K_j' bAc^{-1} K_j'$ for some (B,C) and (b,c) in T_j'. If we consider (B,C) and (b,c) as elements of T_j, then we must have in G: $K_j BAC^{-1} K_j = K_j bAc^{-1} K_j$, because these two sets has non-void intersection. So, we know that $(b^{-1}B, c^{-1}C) \in H_{A,j}$. As $(b^{-1}B, c^{-1}C)$ is an element of T_j', we must then have $(b^{-1}B, c^{-1}C) \in H'_{A,j}$. The converse is also true, if $(b^{-1}B, c^{-1}C) \in H'_{A,j}$, then $K_j' BAC^{-1} K_j' = K_j' bAc^{-1} K_j'$ (it suffices to consider a representant of $(b^{-1}B, c^{-1}C)$ in X_A). The proposition is proven. □

Choose a Haar mesure on G' such that the volum of K_0' is 1.

Lemma 3.2. *If $A \in G$, then, for every $j \in \mathbb{N}$ we have*

$$card(K_j'/(AK_j'A^{-1} \cap K_j')) = card(K_j/(AK_jA^{-1} \cap K_j)).$$

As $G = K_0 \mathcal{A} K_0$ and K_0 normalizes K_j and K_j', it suffices to prove the lemma for $A \in \mathcal{A}$.

Write

$$K_j = \coprod_{i=1}^{l} k_i (AK_jA^{-1} \cap K_j).$$

If $A \in \mathcal{A}$, then the diagonal matrix with 1 on the first $n-1$ positions and $det(k_i)^{-1}$ on the last is always in $AK_jA^{-1} \cap K_j$, so we may and will assume that $k_i \in G'$ for all i. Then

$$K_j' = K_j \cap G' = \coprod_{i=1}^{l} \left(k_i(AK_jA^{-1} \cap K_j) \cap G'\right) = \coprod_{i=1}^{l} \left(k_i(AK_jA^{-1} \cap K_j \cap G')\right)$$

because $k_i \in G'$. But G' is a normal sub-group of G, so

$$AK_jA^{-1} \cap K_j \cap G' = (A(K_j \cap G')A^{-1}) \cap (K_j \cap G') = AK_j'A^{-1} \cap K_j'$$

and we proved that

$$K_j' = \coprod_{i=1}^{l} k_i (AK_j'A^{-1} \cap K_j')$$

hence the equality for cardinals. □

Lemma 3.3. *Let $A \in \mathcal{A}$, and let $a_1 \leq a_2 \leq ... \leq a_n$ be the powers of the uniformizer on the diagonal of A. Then:*

$$vol(K'_j A K'_j) = q^{d \sum_{1 \leq i < i' \leq n} a_{i'} - a_i} vol(K'_j).$$

Proof. Using the last lemma, it follows from the proof of lemma 2.10 in [Ba2]. □

Remark. The volumes of K_0 and K'_0 are one and $K_0/K_j \simeq GL_n(O/I^{dj})$ and $K'_0/K'_j \simeq SL_n(O/I^{dj})$. The determinant is a surjective map $GL_n(O/I^{dj})$ to $GL_1(O/I^{dj})$ with kernel $SL_n(O/I^{dj})$. So we have

$$vol(K_j) = card(GL_1(O/I^{dj}))vol(K'_j) = (q^d - 1)q^{dj-d} vol(K'_j).$$

Proposition 3.4. *For every $a \in G$, the automorphism $f_a : x \mapsto axa^{-1}$ of G' is measure prezerving.*

Proof. Let us show that $vol(aK'_0a^{-1}) = 1$. Applying the lemma 3.2 to a and a^{-1} we get:

$$card(K'_0/aK'_0a^{-1} \cap K'_0) = card(K_0/aK_0a^{-1} \cap K_0)$$

and

$$card(K'_0/a^{-1}K'_0a \cap K'_0) = card(K_0/a^{-1}K_0a \cap K_0).$$

On the other hand,

$$card(K_0/aK_0a^{-1} \cap K_0) = card(K_0/a^{-1}K_0a \cap K_0)$$

because conjugation with a in G is measure preserving with respect to a Haar measure, and

$$card(K'_0/a^{-1}K'_0a \cap K'_0) = card(aK'_0a^{-1}/aK'_0a^{-1} \cap K'_0)$$

because conjugation with a is an isomorphism between these two groups. The result follows. □

If $g \in G'$, set $h(g) = (vol(K'_j)^{-1})1_{K'_l g K'_j}$.

Lemma 3.5. *a) If $A, A' \in \mathcal{A}'$, then $h(A) * h(A') = h(AA')$.*
b) If $(B, C) \in T'_0$, then $h(B) * h(A) * h(C) = h(BAC)$.

The proof is exactly like for lemma 2.11 in [Ba2].

Let's remark that for every function $f \in H_j$, the restriction of f to G' belongs to H'_j. This restriction commutes with the inclusions $H_j \subset H_i$ and $H'_j \subset H'_i$ for $i \geq j$. Conversely, every function $f' \in H'_j$ can be lifted in a standard way to a function $f \in H_j$, using the natural inclusion of the standard basis of H'_j into the standard basis of H_j. But this operation doesn't commute no longer with the inclusions between Hecke algebras.

4. $SL_n(D)$ and close fields

Let's consider again the situation of the two j-close fields, F and L, and all the other constructions from the section 2. Embody in the situation the groups $G'_F(= SL_n(D_F))$ and $G'_L(= SL_n(D_F))$. The bijection $\lambda_j : \mathcal{A}_F \to \mathcal{A}_L$ induces a bijection $\lambda'_j : \mathcal{A}'_F \to \mathcal{A}'_L$, and the isomorphism $\lambda_j : T_{j,F} \to T_{j,L}$ induces an isomorphism $\lambda'_j : T'_{j,F} \to T'_{j,L}$. As a consequence, the isomorphism $\lambda_j : H_{j,A,F} \to H_{j,\lambda_j(A),L}$ induces an isomorphism $\lambda'_j : H'_{j,A,F} \to H'_{j,\lambda_j(A),L}$. (This last result in the case of GL_n (lemma 2.7 in [Ba2]) needed some painfull calculations in the first part of [Ba2], and to avoid recalling all the notation, we chose to get it here by this imbedding of G' in G). We obtain then an isomorphism λ'_j of vector spaces from $H'_{j,F}$ to $H'_{j,L}$. We recall that, if m is an integer bigger than j, if F and L are m-close, then F and L are also j-close.

Theorem 4.1. *There exist an integer $m \geq j$ such that, if F and L are m-close, then the isomorphism λ'_j is an isomorphism of (Hecke) algebras.*

Proof. A lemma, first:

Lemma 4.2. *Let $\mathcal{C}$ be a finite subset of $\mathcal{A}'_F$, and set*

$$G'_F(\mathcal{C}) = \cap_{A\in\mathcal{C}} K'_{0,F} A K'_{0,F}.$$

Then

a) There exist $m \geq j$ depending on $\mathcal{C}$ such that, for all $g \in G'_F(\mathcal{C})$, we have $gK'_{m,F}g^{-1} \subset K'_{j,F}$.

b) If L is m-close to F, then for all $f_1, f_2 \in H'_{j,F}$ supported on $G'_F(\mathcal{C})$ we have

$$\lambda'_j(f_1 * f_2) = \lambda'_j(f_1) * \lambda'_j(f_2).$$

Proof. This lemma is the analogus for $G' = SL_n$ of the lemma 2.14, proved in [Ba2] for the groupe $G = GL_n$. The point a) here follows obviously "by intersection with G'" from the point a) there. The point b) is then proven exactely like the point b) of the lemma 2.14 in [Ba2]. □

Now, for the proof of the theorem, it goes exactely like the proof of the theorem 2.13 in [Ba2]. □

5. Hecke algebras and representations

We forget the close fields for a moment and turn back to notations in section 3. Let (π, V) be an irreducible smooth representation of G'. If K is a subgroup of G', let V^K be the sub-space of vectors which are fixe under $\pi(k)$ for all $k \in K$. If K is open, V^K has finite dimension. The **level** of π is the lowest integer l such that $V^{K_l} \neq 0$. If $f \in H'_j$, we set

$$\pi(f) = \int_{G'} f(g)\pi(g)dg.$$

The image of $\pi(f)$ is included then in $V^{K'_j}$. In particular, if j is smaller than l, then $\pi(f) = 0$. If j is bigger or equal to l, then $\pi(f)$ induces an endomorphism of $V^{K'_j}$. It is also clear that the trace of $\pi(f)$ equals the trace of this endomorphism. The space $V^{K'_j}$ is a H'_j module with the external low: $f * v = \pi(f)v$ for all $f \in H'_j$ and all $v \in V^{K'_j}$. To any irreducible smooth representation π of level smaller than j we associate this way a H'_j module. This construction is a bijection from the set of equivalence classes of irreducible smooth representations of G' with level lower or equal to j and the set of isomorhism classes of irreducible non-degenerated K'_j-modules (see [Be] for example).

6. Close fields and representations

Let F, L and m be like in the theorem 4.1; in view of what has been said in the last section, λ'_j induces a bijection between the set of equivalence classes of irreducible smooth representations of G'_F with level lower or equal to j and the set of equivalence classes of irreducible smooth representations of G'_L with level lower or equal to j. As the applications λ'_i for $i \leq j$ are compatible with the inclusions relations between Hecke algebras, we see that λ'_j is level prezerving. Also, if $f \in H'_{j,F}$ and π is an irreducible smooth representation of level lower or equal to j of G'_F, we have obviously $tr\pi(f) = tr\lambda'_j(\pi)(\lambda'_j(f))$.

Proposition 6.1. *The application λ'_j sends cuspidal representations to cuspidal representations, square integrable representations to square integrable representations and tempered representations to tempered representations.*

Proof. For cuspidal and square integrable representations, the proof is the same as in theorem 2.17 of [Ba2]. Now, the tempered representations of G'_F are its irreducible unitary representations π such that for all $\epsilon > 0$, there exist a non-trivial coefficient of π belonging to $L^{2+\epsilon}(G'_F)$, and the same for G'_L. The same proof as for square integrable representations shows that λ'_j sends tempered representations to tempered representations. □

Corollary 6.2. *If π is a square integrable repesentation of G'_F of level less than or equal to j and f is a pseudo-coefficient of π, then $\lambda'_j(f)$ is a pseudo-coefficient of $\lambda'_j(\pi)$.*

Proof. The corollary is an easy consequence of the above proposition. See [Ba1], lemma 4.2 for details (as well as section 2 of [Ba1] for a definition and a survey of pseudocefficients in all characteristics). □

7. Elliptic orbital integrals

Let F be a local field like in section 1. Here $D = F$. Recall that we fixed Haar meaures dg and dg' on G and G' such that $vol(K_0, dg) = 1$ and $vol(K'_0, dg') = 1$. If γ is an element of G and $Z_G(\gamma)$ is the stabiliser of γ in G we put a Haar measure on $Z_G(\gamma)$ such that the volume of the subgroup of its points over O is one. On $G/Z_G(\gamma)$ we consider the quotient measure. The same if $\gamma \in G'$ and we conider its commutator $Z_{G'}(\gamma) = Z_G(\gamma) \cap G'$ in G'. The orbital integrals $\Phi(f, .)$ of functions $f \in H$ or $f \in H'$ at the point γ will be calculated with respect to this choices of measures.

Let's fix the following notations: if A is a subset of F, $A^{[n]}$ is the set of all power n of elements of A in F. If A is a subset of G, then $det(A)$ is the image in F of A under the determinant map. If A and B are subsets of G, then AB is the set of all products ab with $a \in A$ and $b \in B$.

From now on we suppose that the characteristics of F is either zero either prime with n.

Lemma 7.1. *We have $1 + I^{2n} \subset O^{*[n]}$.*

Proof. If the characteristics of F is zero, the point a) is an obvious consequence of the exercice 2, page 46 in [BS] (which is an easy application of their theorem 3, page 42). In our opinion, there is a mistake in the enouncement of the exercice, and one has to replace $2^\delta + 1$ by $2\delta + 1$, which is stronger and come streight from the standard proof. The δ in the exercice is the greatest power of p dividing n. In particular, we have $\delta < n$, so $2\delta + 1 < 2n$, so the exercice implies our statement. The same proof works in non-zero characteristics if p is prime to n. □

Let S be a set of reprezentatives of $O^*/1 + I^{2n}$ in O^*. Chose a subset S' of S which is a system of reprezentatives of $O^*/O^{*[n]}$ (always possible, thanks to the lemma 7.1).

Let $X_{G'}$ be the set of diagonal matrices in G with 1 in the first $n-1$ places and an element of S' in the last one. Let X be the set of diagonal matrices in G with 1 in the first $n-1$ places and an element of $\{1, \pi, \pi^2, ..., \pi^{n-1}\}$ in the last one.

It is clear that $F^{*[n]} = det(Z)$, $O^{*[n]} = det(Z(O))$ and $F/F^{*[n]} = \coprod_{i=0}^{n-1} \pi^i O^*/O^{*[n]}$.

Using the natural inclusion of G' in G we realise, for all $x \in G$, $x(G'/Z')$ as a subset of G/Z. It is easy to check that $G(O)/Z(O) = \coprod_{x \in X_{G'}} A(G'(O)/Z'(O))$ and $G/Z = \coprod_{A \in X_{G'}X} A(G'/Z')$.

Remark. If $j \geq 2n$, the natural inclusion $K'_j/Z(K'_j) \to K_j/Z(K_j)$ is a bijection (where $Z(K'_j)$ is the center of K'_j and $Z(K_j)$ is the center of K_j).

Let $\gamma \in G'$. As $Z \subset Z_G(\gamma)$, $det(Z) \subset det(Z_G(\gamma))$. So we may (and will) choose a subset S'_γ of S' which form a system of representatives for $O^*/\det(Z_{G'}(\gamma)(O))$

in O^*. We denote X_γ the corresponding subset of $X_{G'}$. The valuation map send the set $det(Z_{G'}(\gamma))$ into a sub-group W of $\mathbb{Z}$ containing $n\mathbb{Z}$. Consider a system of representatives J_γ of $\mathbb{Z}/W$ in the set $\{0, 1, 2, ..., n\}$. We have

$$F^*/det(Z_{G'}(\gamma)) = \coprod_{j \in J_\gamma} \pi^j O^*/det(Z_{G'}(\gamma)(O)).$$

So, $S'_\gamma J_\gamma$ is a system of representatives of $F^*/det(Z_{G'}(\gamma))$ in F^*. Let Y_γ be the set of diagonal matrices in G with 1 in the first $n-1$ places and π^j, with $j \in J$, in the last one.

One may show that

$$G(O)/Z_G(\gamma)(O) = \coprod_{x \in X_\gamma} x(G'(O)/Z_{G'}(\gamma)(O)).$$

and

$$G/Z_G(\gamma) = \coprod_{x \in X_\gamma Y_\gamma} x(G'/Z_{G'}(\gamma)).$$

Let x_γ be the cardinal of X_γ. The first relation shows that, with our choice of measures, the measure we put on $G'/Z_{G'}(\gamma)$ is x_γ times the restricted measure from $G/Z_G(\gamma)$.

One may also verify that, if δ is conjugated to γ in G, there exist exactly one element $x \in X_\gamma Y_\gamma$ such that δ is conjugated to $x\gamma x^{-1}$ in G'.

Let's look to this construction form another point of view. We say that U is a **system adapted to** γ if for all δ conjugated to γ in G, there exist exactly one element $x \in U$ such that δ is conjugated to $x\gamma x^{-1}$ in G'. Then we have

$$G/Z_G(\gamma) = \coprod_{x \in U} x(G'/Z_{G'}(\gamma)).$$

We just proved that $X_\gamma Y_\gamma$ is a system adapted to γ. But what is remarquable from our discussion is that knowing just the set $det(Z_{G'}(\gamma))$, we may construct a system adapted to γ and we know x_γ (which is the quotient of two cardinals: those of $F^*/det(Z_{G'}(\gamma))$ and of $\mathbb{Z}$ by its sub-group of valuations of elements in $det(Z_{G'}(\gamma))$. Our previous construction allows us to construct a particular such system depending only on $O^*/1 + I^{2n}$ and on the first n powers of π.

Let now U be a system adapted to γ. We suppose that U contains the identity matrix. If we donote $\mathcal{O}_G(\gamma)$ (resp. $\mathcal{O}_{G'}(\gamma)$) the orbit in G (resp. in G') of γ, then:

$$\mathcal{O}_G(\gamma) = \coprod_{x \in U} \mathcal{O}_{G'}(x\gamma x^{-1}).$$

If $d\bar{g}$ (resp. $d\bar{g}'$) is the measure fixed on $G/Z_G(\gamma)$ (resp. $G'/Z_{G'}(\gamma)$), then we have that for every $f \in H(G)$,

$$\Phi(f, \gamma) = \int_{G/Z_G(\gamma)} f(g\gamma g^{-1}) d\bar{g} = \sum_{x \in U} \int_{G'/Z_{G'}(\gamma)} f(xg\gamma g^{-1}x^{-1}) d\bar{g} =$$

$$\sum_{x\in U}\int_{G'/Z_{G'}(\gamma)} f((xgx^{-1})(x\gamma x^{-1})(xg^{-1}x^{-1}))d\bar{g} =$$

$$\sum_{x\in U}\int_{G'/Z_{G'}(\gamma)} f(g(x\gamma x^{-1})g^{-1})d\bar{g},$$

the last equality coming from the proposition 3.4. So, if $f' \in H'$ is the restriction of f to G', we obtained, as $d\bar{g} = \frac{1}{x_\gamma}d\bar{g}'$:

$$\Phi(f,\gamma) = \frac{1}{x_\gamma}\sum_{x\in U}\Phi(f', x\gamma x^{-1}).$$

Suppose now γ is regular semi-simple. Let V_γ be an open and compact neigbourhoud of γ in G' containing only elements of G' which are conjugated under G' to a regular element in the torus $Z_{G'}(\gamma)$. Such a neighbourhood always exists by the submersion tehorem of Harish-Chandra. Then, for all $t \in V_\gamma$, $Z_G(t)$ is conjugated to $Z_G(\gamma)$ in G, which shows that the system U is adapted to t too and $x_t = x_\gamma$ and the formula

$$\Phi(f,t) = \frac{1}{x_\gamma}\sum_{x\in U}\Phi(f', xtx^{-1})$$

is true in the whole neighbourhood V_γ.

For each $x \in U$, set $V_{x\gamma x^{-1}} = xV_\gamma x^{-1}$ (it is an open and compact neigbourhood of $x\gamma x^{-1}$ in G'). If $A \subset G$ let $Ad_{G'}(A)$ stand for the set of all conjugates of elements in A by elements of G'. The sets $Ad_{G'}(V_{x\gamma x^{-1}})$, $x \in U$ are disjoint (because for every $g \in Ad_{G'}(V_{x\gamma x^{-1}})$, $Z_G(g)$ is conjugated under G' with $xZ_G(\gamma)x^{-1}$ and never with $Z_G(\gamma)$). They are all open and close also. The fact that they are open is obvious (union of open sets). Then the fact that they are close would be a consequence of their union being closed. But their union is $Ad_G(V_\gamma)$, and this is closed: if P is the (continuous!) map *characteristic polynomial* from G to F^n, then $P(V_\gamma)$ is compact beacuse V_γ is, hence the reciprocal image $P^{-1}(P(V_\gamma)) = Ad_G(V_\gamma)$ is closed.

Let now $f' \in H'$. We may write $f' = f'_0 + \sum_{x\in U} f'_x$, where the support of f'_0 does not intersect any $Ad_{G'}(V_{x\gamma x^{-1}})$, and the support of each f'_x is included in $Ad_{G'}(V_{x\gamma x^{-1}})$. The orbital integral of f'_0 vanish on all $xV_\gamma x^{-1}$. The orbital integral of f'_{x_0} vanish on all $xV_\gamma x^{-1}$ with $x \in U\backslash\{x_0\}$. If $f'_1 \in H'_j$, we just lift it to a function $f_1 \in H_j$ and we get by the formula about orbital integrals:

$$\Phi(f', t) = x_\gamma \Phi(f_1, t)$$

for all $t \in V_\gamma$. In particular, if $\Phi(f_1, .)$ is constant in a neighbourhood V of γ, then $\Phi(f', .)$ is constant on $V_\gamma \cap V$.

8. Orbital integral and local fields

We'll deal again with two different fields F and L, and the subscript F or L will indicate the one which the object is attached to. The field F is fixed. If γ is an elliptic element of G'_F, then we fix X_γ like in the previous section, and, if L is a field m close to F, with $m \geq 2n$, we define $\lambda_m(X_\gamma)$ in the following way: we take the image of X_γ in $O_F^*/1 + I_F^m = (O_F/I_F^m)^*$ defined by its last coefficient on the diagonal; then we take the image of this set under the ring isomorphisme $\lambda_m : O_F/I_F^m \to O_L/I_L^m$; we then consider a system S_L of representatns of this set in O_L^* and finaly we let $\lambda_m(X_\gamma)$ be the set of diagonal matrices in G_L with 1 in the first $n-1$ places of the diagonal and an element of S_L in the last. The set Y_γ is defined only in terms of powers of the uniformizer π_F of F, so there is a canonical way of defining the corresponding set $\lambda_m(Y_\gamma)$ using the uniformizer π_L of L. It is also clear how we define $\lambda_m(x)$ for each x in $X_\gamma Y_\gamma$. Actaully, $X_\gamma \subset K_{0,F}$, and $Y_\gamma \subset \mathcal{A}_F$, so every $x \in X_\gamma Y_\gamma$ is an element of type BAC^{-1} (with $C = 1$) like those used in the standard decomposition of G_F. Hence, for all $m \geq 2n$, if L is m-close to F we automaticly have $\lambda_m(x) \in \lambda_m(K_m x K_m)$, so for this particular system we defined a pointwise lifting always compatible with the general lifting of open compact sets.

Theorem 8.1. (Lemaire) *Let γ be an elliptic element of G_F. Let j be a positive integer. Then there exist l and m such that:*

a) for every $f \in H_{j,F}$, $\Phi(f,.)$ is constant on $K_{l,F}\gamma K_{l,F}$, equal to $\Phi(f,\gamma)$,

b) m is bigger than j and l and for every field L which is m close to F, for every $f \in H_{j,F}$, $\Phi(\lambda_j(f),.)$ is constant on $\lambda_l(K_{l,F}\gamma K_{l,F})$, equal to $\Phi(f,\gamma)$,

Proof. [Le1], page 1054.

Lemma 8.2. *Let $\gamma \in G_F$ be an elliptic element and let j be a positive integer. There exist l et m such that*

a) For all $\gamma' \in K_{l,F}\gamma K_{l,F}$, $K_{j,F}Z_G(\gamma')K_{j,F} = K_{j,F}Z_G(\gamma)K_{j,F}$

b) If L and F are m close, then for all $\gamma, \gamma' \in \lambda_l(K_{l,F}\gamma K_{l,F})$, $K_{j,L}Z_{G(L)}(\gamma')K_{j,L}) = K_{j,L}Z_{G(L)}(\gamma)K_{j,L} = \lambda_l(K_{j,F}Z_{G_F}(\gamma)K_{j,F})$.

Proof. It is the proof of (i), page 1043, [Le1]. □

Let $\gamma \in G'_F$ be an elliptic element. Apply the last lemma for a $j \geq 2n$. Then

Proposition 8.3. *a) For all $\gamma' \in K'_{l,F}\gamma K'_{l,F}$, the system $X_\gamma Y_\gamma$ is adapted to γ' and $x_{\gamma'} = x_\gamma$.*

b) If L and F are m close, then for all $\gamma' \in \lambda_l(K'_{l,F}\gamma K'_{l,F})$, the system $\lambda_l(X_\gamma)\lambda_l(Y_\gamma)$ is adapted to γ' and $x_{\gamma'} = x_\gamma$.

Proof. a) We have seen that $1 + I_F^{2n} \subset \det(K_{2n}) \subset \det(Z_{G_F}(\gamma')(O_F))$ and $1 + I_F^{2n} \subset \det(K_{2n}) \subset \det(Z_{G_F}(\gamma)(O_F))$. So

$$\det\,(Z_{G_F}(\gamma')(O_F)) = \det\,(K_{j,F}Z_{G_F}(\gamma')(O_F)K_{j,F}) =$$

$$\det\,(K_{j,F}Z_{G_F}(\gamma)(O_F)K_{j,F}) = \det\,(Z_{G_F}(\gamma)(O)).$$

b) Using the point b) of the previous lemma, we get

$$K_{j,L}Z_{G_L}(\gamma)K_{j,L} = \lambda_l(K_{j,F}Z_{G_F}(\gamma)K_{j,F}).$$

Now, if V is a $K_{j,F}$ bi-invariant set, then $det(V)$ is invariant by $1 + I_F^j$, and the image of $det(V)$ in $O_F^*/1 + I_F^j$ correspond to the image of $det\lambda_j(V)$ in $O_L^*/1 + I_L^j$ under the isomorphism $\lambda_j : O_F/I_F^j \to O_L/I_L^j$ (it suffices to verify this on basic type sets $K_jBAC^{-1}K_j$, and this is obvious). □

Let γ be an elliptic element of G'_F.

Theorem 8.4. *Let $f' \in H$. There exists p and m such that*
a) $\Phi(f', .)$ is constant on $K'_p\gamma K'_p$, equal to $\Phi(f', \gamma)$ and
b) for every field L m-close to F, $\Phi(\lambda_m(f'), .)$ is constant on $\lambda_m(K'_p\gamma K'_p)$, equal to $\Phi(f', \gamma)$.

Proof. We start with a lemma studying the behaviour of the lifting under conjugation. It imply for exemple that if two open compact sets are obtained one from another by conjugation with an element, the same is true for thier lifting to a field close enough.

Lemma 8.5. *Let H_1, H_2 be open compact subsets of G_F and $g \in G_F$ such that $gH_1g^{-1} \subset H_2$. If H_1 and H_2 are bi-invariant under some $K_{j,F}$, then $K_{j,F}gK_{j,F}H_1K_{j,F}g^{-1}K_{j,F} \subset H_2$. Moreover, there exist $m > j$ such that, if L is m-close to F, then $\lambda_m(K_{j,F}gK_{j,F})\lambda_m(H_1)\lambda_m(K_{j,F}g^{-1}K_{j,F}) \subset \lambda_m(H_2)$.*

Proof. As $gH_1g^{-1} \subset H_2$ and H_1 and H_2 are bi-invariant under $K_{j,F}$, we obviously have $K_{j,F}gK_{j,F}H_1K_{j,F}g^{-1}K_{j,F} \subset H_2$. For the second assertion, it suffices to show that $\lambda_m(K_{j,F}xK_{j,F}yK_{j,F}) = \lambda_m(K_{j,F}xK_{j,F})\lambda_m(K_{j,F}yK_{j,F})$ for all $x, y \in G_F$. But $K_{j,F}xK_{j,F}yK_{j,F}$ is the support of the function obtained by the convolution product of characteristic functions $1_{K_{j,F}xK_{j,F}}$ and $1_{K_{j,F}yK_{j,F}}$. So, when m is big enough such that the linear isomorphism between $H_{j,F}$ and $H_{j,L}$ is an algebra isomorphism (theorem 4.1), we also have our relation. □

The proof is now clear: thaks to the proposition 8.3 and the lemme 8.5, if L is m-close to F, m big enough, then the construction for L at the end of the last section is paralel to that for F (just pick a γ_L in $\lambda_m(V_\gamma)$ and use the lemma 8.5 to show (for m big enough) that, for all $x \in X_\gamma Y_\gamma$, $\lambda_m(V_{x\gamma x^{-1}}) = V_{\lambda_m(x)\gamma_L\lambda_m(x)^{-1}}$).

To conclude for the point b) of our theorem, just use the point b) of the thorem of lemaire 8.1. □

9. The orthogonality relations for characters

If $^-$ denote the complex conjugation, we have the following:

Theorem 9.1. *Let F be a local field of non-zero characteristic p. Let n be a positive integer such that p doesn't divide n. Then, if π is a square integrable representation of $G'_F = SL_n(F)$, if f'_π is a pseudocoefficient of π, if g is a regular elliptic element of G'_F, then*

$$\chi_\pi(g) = \overline{\Phi(f'_\pi, g)}.$$

Proof. The proof is then the same as for the theorem 4.3 in [Ba1]. □

Corollary 9.2. *The orthogonality relations for characters hold on G'_F.*

Proof. The proof is the same as for the theorem 4.4. in [Ba1], as Lemaire showed the local integrability of characters on SL_n in non-zero characteristics ([Le2]). □

10. Removing the condition $p \not| n$

What happen if F is of non zero characteristics p, and p divides n? First of all, the theorem 8.1 is absolutly independent of that. Otherwise, the decomposition of G/Z as cosets of G'/Z' is no longer finite, because $F^{*[n]}$ does no longer contain an open neighbourhood of 1. But, if a field E is an extension of F, then the norm map from E^* to F^* contain an open neighbourhood of 1, say $1 + I^{p_{E^*}}$ ([We], proposition 5, page 143). So, if γ is an elliptic element of G'_F, then we may still consider a system of reprezentatives of $O^*/1 + I^{p_{Z_{G_F}(\gamma)}}$ in O^*, and it will be a finite set. The diagonal matrix with 1 on the firs $n-1$ positions and an element of this system of representatives on the last will be our X_γ, adapted to γ. All the other fields involved when applying the close fields theory to G_F and G'_F will be of zero characteristics, so, for them, the valuation of the entries of the matrix in X_δ will be uniformly bounded by $2n$, like before, idependently of the field or of the element δ. So, the place of $2n$ as a bound for valuation has just to be replaced by $\max(2n, p_{Z_{G_F}(\gamma)})$. All the proofs go then the same. Let's remark that the proposition 8.3 b) imply in the end that, even in this case of bad characteristics, we have $x_\gamma \leq 2qn^2$, where q is the cardinal of the residual field.

11. Removing the condition $D = F$

Let d^2 be the dimension of D over F. If γ is a regular semi-simple element of $GL_n(D)$, if δ is an element of $GL_{dn}(F)$, we say that δ **corresponds** to γ if the caracteristic polynomial of δ is equal to that of γ. Such δ always exist and are regular semi-simple. If γ is elliptic, then such δ are always elliptic. If

$f \in H(GL_n(D_F))$ with compact support, one may function $e \in H(GL_{nd}(F))$ such that the orbital integral of f in any regular semi-simple element γ is equal to the orbital integral of e in any element of $GL_{nd}(F)$ corresponding to γ, and such that the orbital integral of e vanishes in every regular semi-simple element of $GL_{nd}(F)$ which doesn't correspond to any regular semi-simple element of $GL_n(D)$. This result is proven in [DKV] for F of characteristics zero and in [Ba3] for F of non-zero characteristics. We will call it **orbital integrals transfer** over F.

Now, if $\gamma \in GL_n(D)$ is elliptic and $\delta \in GL_{nd}(F)$ corresponds to γ, then $Z_{GL_n(D)}(\gamma)$ is canonicaly isomorphic to $Z_{GL_{nd}(F)}(\delta)$. This isomorhism commutes with the determinant map, so $S'_\gamma = S'_\delta$, and the theory of the set X_γ and adapted systems is exactly the same as before. In particular, as $S'_\gamma = S'_\delta$, and any adapted system to δ is also an adapted system for γ. Let's prove an analogus of the theorem 8.1 for $GL_n(D)$. This version of Lemaire's thorem that we prove below is weaker, but we need Lemaire's result only for any *fixed* function, as we used it only for a finite number of functions in the proof of our main theorem.

Theorem 11.1. *Let γ be an elliptic element of $G = GL_n(D_F)$. Let $f \in H(GL_r(D_F))$. Suppose that the support of f is included in the regular elliptic set. Then there exist l and m such that:*
a) $\Phi(f, .)$ is constant on $K_{l,F}\gamma K_{l,F}$, equal to $\Phi(f, \gamma)$,
b) m is bigger than l and for every field L which is m close to F, $\Phi(\lambda_j(f), .)$ is constant on $\lambda_l(K_{l,F}\gamma K_{l,F})$, equal to $\Phi(f, \gamma)$.

Proof. As f is fixed, the real problem is b). We get it by transfering integral orbitals to $GL_{dn}(F)$, an using the theorem 8.1. So we will deal with four groups: $GL_n(D_F)$, $GL_{nd}(F)$, $GL_n(D_L)$ and $GL_{nd}(L)$, where L is a local field of zero characteristics m-close to F for some m. Let $M \in GL_{nd}(F)$ be the companion matrix of the caracteristic polynomial of γ. Then M corresponds to γ. We prove the

Lemma 11.2. *Let U_1 and U_2 be neighbourhoods of γ and M respectively. Then there exist open compact neighbourhoods V_1 of γ and V_2 of M and an integer m such that,*
i) $V_1 \subset U_1$ and $V_2 \subset U_2$.
ii) for all field L m-proche to F, $\lambda_m(V_1)$ ($\subset GL_n(D_L)$) and $\lambda_m(V_2)$ ($\subset GL_{nd}(L)$) are well defined and for all $g \in \lambda_m(V_2)$ there exists $h \in \lambda_m(V_1)$ corresponding to g.

Proof. It is a direct consequence of the propositions 4.5 and 4.10 in [Ba2]. The reader may verify it by formal logic, without knowing what "polynômes proches" means. □

Now, in [Ba3], we proved that, if m is big enough, then the orbital integrals transfer over F and over L commute with the map λ_m for functions. So our proposition follows from the last lemma and the theorem 8.1 applied after transering f. □

The analogus of the proposition 8.3 in $D \neq F$ case is also true. If the V_2 of the lemma 11.2 is included in the $K_{l,F}\gamma K_{l,F}$ of the proposition 8.3, if we apply the proposition and the lemma we find that the proposition is true for $GL_n(D)$. One has just to replace the neighbourhood $K_{l,F}\gamma K_{l,F}$ of γ with the V_1 of the lemma.

Last but not least is the fact that the characters of irreducible smooth representations of $SL_n(D)$ are locally integrable in non zero characteristics. This result may be find in Lemaire. The proof of the orthogonality relations for $SL_n(D)$ is now exactly the same as the proof for $GL_n(F)$.

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