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Differentiable and analytic families of continuous martingales in manifolds with connection

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Summary. We prove that the derivative of a differentiable family $X_t(a)$ of continuous martingales in a manifold M is a martingale in the tangent space for the complete lift of the connection in M, provided that the derivative is bi-continuous in t and a. We consider a filtered probability space $(\Omega, (\mathscr{F}_t)_{0\le t\le 1}, \mathbb{P})$ such that all the real martingales have a continuous version, and a manifold M endowed with an analytic connection and such that the complexification of M has strong convex geometry. We prove that, given an analytic family $a \mapsto L(a)$ of random variable with values in M and such that $L(0) \equiv x_0 \in M$, there exists an analytic family $a \mapsto X(a)$ of continuous martingales such that $X_1(a) = L(a)$. For this, we investigate the convexity of the tangent spaces $T^{(n)}M$, and we prove that any continuous martingale in any manifold can be uniformly approximated by a discrete martingale up to a stopping time T such that $\mathbb{P}(T < 1)$ is arbitrarily small. We use this construction of families of martingales in complex analytic manifolds to prove that every $\mathscr{F}_1$-measurable random variable with values in a compact convex set V with convex geometry in a manifold with a C^1 connection is reachable by a V-valued martingale.

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1. Motivations, preliminaries, main results

Let $(\Omega, (\mathscr{F}_t)_{0\le t\le 1}, \mathbb{P})$ be a filtered probability space.

The main motivation of this paper is, given a manifold M of dimension d with a connection ∇, and a $\mathscr{F}_1$-measurable random variable L with values in a small compact subset of M, to prove the existence of a ∇-martingale X with respect to the filtration $(\mathscr{F}_t)$, such that $X_1 = L$. If such a martingale exists, we will say that L is reachable.

This problem has been solved in convex geometry (see section 2 for the different notions of convexity) in [K1] and [P1] with Brownian filtrations and in [P2] with more general filtrations like filtrations generated by independent Brownian motions and Poisson processes (here the solutions are cadlag martingales). In both cases, the method used is a discretisation of the filtration and the convergence of discrete martingales into solutions of the problem is established. In [D2], a solution is given using backward stochastic differential equations, in the context of convex geometry and Brownian filtrations, and in [D3] the author shows how to deduce existence on a manifold M with convex geometry which is an increasing union of compact sets with convex geometry (and with additional geometric assumptions) from the existence on each compact set.

The solution given here is valid with the assumptions that all the real $(\mathscr{F}_t)_{0\le t\le 1}$-martingales have a continuous version and L takes its values in a compact convex set with convex geometry (with the help of [D3] we can then deduce existence results in the non compact case).

This result is a consequence of a construction which will be done with much stronger assumptions on the manifold. Indeed we assume first that the manifold is endowed with an analytic connection and that its complexification has strong convex geometry (see sections 2 and 4). The terminal value L is replaced by an analytic family $a\mapsto L(a), a\in[0,1]$, of $\mathscr{F}_1$-measurable random variables with values in M, such that $L(1)=L$ and $L(0)\equiv x_0\in M$. An analytic family of martingales $a\mapsto X(a)$ with terminal value $L(a)$ will be constructed (theorem 6.17).

In a first approach, assume that the real analytic family $a\mapsto X(a)$ of M-valued martingales such that $X_1(a)=L(a)$ exists. Then the derivative $W^n=\frac{\partial^n X(a)}{\partial a^n}|_{a=0}$ take its values in the spaces $M_S^n\subset T^{(n)}M$ of n-th order derivatives at time 0 of paths γ such that $\gamma(0)=x_0$. As theorem 3.3 below will show, a consequence is that the process W^n is a martingale with respect to the n-th complete lift $\nabla^{(n)}$ of the connection ∇, with terminal value $L^{(n)}(0)$. If one knows W^n for all n, then one knows $a\mapsto X(a)$. But it will be shown in section 6 that constructing M_S^n-valued $\nabla^{(n)}$-martingales with prescribed terminal value can be performed with an easy induction. To give an idea of this induction, assume that M is an open subset of $\mathbb{R}^d$. Then if the coordinates of the martingales W^n are $(W_1,\dots,W_n)$, one can write

$$X_t(a)=x_0+\sum_{n=1}^{\infty}\frac{a^n}{n!}W_n(t)\ . \tag{0}$$

If Y is a semimartingale in $\mathbb{R}^d$, let $\widetilde{Y}$ denote its finite variation part ($\widetilde{Y}_0=0$). Since $X(a)$ is a martingale we have

$$d\widetilde{X}(a)=-\frac{1}{2}\Gamma_{jk}(X(a))d\langle X^j(a),X^k(a)\rangle \tag{1}$$

where Γ_{jk}^i are the Christoffel symbols. Assume here that expansions in a power series are allowed in both sides of (1) and commute with bracket and finite variation part, one obtains for the first order term

$$d\widetilde{W}_1(t) = 0, \qquad W_1(t) = \mathbb{E}[L'(0)|\mathscr{F}_t] \tag{2}$$

and for the second order term

$$\frac{1}{2}d\widetilde{W}_2(t) = -\frac{1}{2}\Gamma_{jk}(x_0)d\langle W_1^j, W_1^k\rangle_t \ ,$$

$$W_2(t) = \mathbb{E}\big[L''(0) + \Gamma_{jk}(x_0)\big(\langle W_1^j, W_1^k\rangle_1 - \langle W_1^j, W_1^k\rangle_t\big)|\mathscr{F}_t\big] \tag{3}$$

(if one assumes that $\Gamma(x_0) = 0$, like in an exponential chart centered at x_0, then W_2 is a martingale). More generally, if $W_1, W_2, \dots, W_{n-1}$ are known, then it is possible to compute W_n, since $W_n(1)$ is the last coordinate of $L^{(n)}(0)$ and the expression of $d\widetilde{W}_n(t)$ given by (0) and (1) does not involve W_n. Indeed, if W_n appears in one term, then the process $X(0) \equiv x_0$ must appear in the bracket, and hence the bracket is 0.

This formal computation leads to the following approach of the problem. Now $a \mapsto L(a)$ is a given analytic family with $L(0) \equiv x_0$ and one constructs a family of semimartingales W^n with values respectively in M_S^n, with terminal value $L^{(n)}(0)$ and which are $\nabla^{(n)}$-martingales, i.e. such that in local coordinates,

$$\frac{1}{n!}d\widetilde{W}_n = -\frac{1}{2}\sum_{\substack{r\geq 0,\, p,q>0 \\ \alpha_1+\dots+\alpha_r+p+q=n \\ 1\leq i_1,\dots,i_r\leq d}} C_{\alpha,p,q,i} D_i\Gamma_{jk}(x_0)W_{\alpha_1}^{i_1}\dots W_{\alpha_r}^{i_r} d\langle W_p^j, W_q^k\rangle \tag{4}$$

where $\alpha = (\alpha, \dots, \alpha_r), i = (i_1, \dots, i_r)$ and $C_{\alpha,p,q,i}$ are constants defined in (18). The question of the existence of $X(a)$ boils down to establishing the convergence of the series with n-th order derivative W^n at $a = 0$, i.e. the convergence of (0) in local coordinates, and proving that the sum is a martingale. This will be done in section 6.

We give now a brief description of the content of the paper.

In section 2, one defines different notions of convexity in a manifold.

Section 3 is devoted to the differentiability of families of martingales. It is not supposed there that M or ∇ are analytic. One shows that if $a \mapsto X(a)$ is a differentiable family of ∇-martingales such that almost surely $(t, a) \mapsto \frac{\partial X_t(a)}{\partial a}$ is continuous, then $\frac{\partial X(a)}{\partial a}$ is a martingale in TM for the complete lift ∇' of ∇ (theorem 3.3). This is proved via an approximation of ∇' by a family of connections on a neighbourhood of the diagonal in the product manifold (proposition 3.1). Then it is shown that under convexity assumptions, if $X_1(a)$ is differentiable $\frac{\partial X_1(a)}{\partial a}$ is the terminal value of a ∇'-martingale $Y(a)$, then $X_t(a)$ is differentiable and $\frac{\partial X_t(a)}{\partial a} = Y_t(a)$ (proposition 3.7).

The end of section 3 is devoted to a discrete analogue to theorem 3.3. Proposition 3.8 shows that exponential expectation commutes with differentiation if the derivative random variable takes its values in a small subset of the tangent bundle.

Going back to the problem of convergence of (0), one needs to complexify the real manifold and the real connection, and to transform a into a complex parameter. This is made possible in section 4. It is shown that under the hypothesis of proposition 3.7, if $a \mapsto X_1(a)$ is holomorphic, then $a \mapsto X_t(a)$ is holomorphic (corollary 4.5). Under convexity conditions, the discrete equivalent of the problem of convergence of (0) is solved, and one shows that the solution in holomorphic in a (corollary 4.3). This gives upper bounds of $\|((W_1^{discr}, \dots, W_n^{discr}))\|$ independent of the discrete filtration (corollary 4.4).

The aim of section 5 is to prove that the martingales W^n described above can be approximated by discrete martingales $W^{n\,discr}$. This is done in a larger context. Therorem 5.1 shows that any continuous martingale $(X_t)_{0\le t\le 1}$ in any manifold can be uniformly approximated up to a stopping time T such that $\mathbb{P}(T < 1)$ is as small as we want, by a discrete martingale.

The result of theorem 5.1 is used only in theorem 6.4, which shows that if (0) converges absolutely, then the sum is a ∇-martingale. Here the commutativity of affine mappings with exponential barycenters (proposition 2.10), and complexification are used.

In section 6, the convergence of the series defined by the sequence $(W^n)_{n\in\mathbb{N}}$ (the convergence of (0) in local coordinates) is investigated. For this, it is useful to study the geometry of $T^{(n)}M$ and its symmetrized space M_S^n (see propositions 6.3, 6.5, 6.6). It is shown that the martingales W^n live in compact convex subsets of M_S^n with convex geometry (corollary 6.12). For this, one constructs convex functions (proposition 6.6) with a Hessian bounded below by a scalar product and uses the fact that convex geometry is implied by uniqueness of martingales with prescribed terminal value (see [K3] and [K4]). Complexification is used again to give sharper bounds for $\|((W_1, \dots, W_n))\|$ via the construction of a convex function on M_S^n (proposition 6.13). This leads to the main result (theorem 6.17) which asserts that (0) converges a.s. absolutely and uniformly in t, ω and that the sum is a martingale $X(a)$ with terminal value $L(a)$, provided that the complexification of the manifold M has strong convex geometry.

Section 7 is devoted to the existence of martingales with prescribed terminal value in a compact convex set V with convex geometry. It is first proven in lemma 7.1 that it is sufficient to solve the problem replacing V by a neighbourhood V_x of x for every point $x \in V$. This is done with an argument of connexity of the set of reachable random variables for the topology of almost sure uniform convergence. Then local existence of martingales with prescribed terminal value is established by approximating a C^1 connection by analytic connections, and this gives reachability of every V-valued random variable (theorem 7.3).

From now on, we assume that all the real martingales with respect to $(\Omega, (\mathscr{F}_t)_{0\le t\le 1}, \mathbb{P})$ *have a continuous version. All the manifolds considered are smooth.*

By a connection on a manifold, we will mean a smooth torsion-free connection.

2. Convexity on manifolds

Definition 2.1. – *Let M be a manifold endowed with a C^p* connection $\nabla (p \in \mathbb{N}\backslash\{0\}$ *or* $p = \infty)$. *A subset V of M will be called a convex set if for all*

$x, y \in V$, there exists a unique geodesic $[0,1] \ni t \mapsto \gamma_{x,y}(t)$ with values in V, such that $\gamma_{x,y}(0) = x$ and $\gamma_{x,y}(1) = y$, and if $\dot{\gamma}_{x,y}(0)$ depends in a C^p way on x and y. This vector will be denoted by $\overrightarrow{xy}$.

Definition 2.2. – *Let M be a manifold endowed with a connection ∇, and V a compact subset of M. We shall denote by $\mathscr{C}(V)$ the set of functions f on V such that there exists an open neighbourhood U of V such that f is defined and convex on U.*

Recall that a function f is convex on an open subset U of a manifold M with connection, if for every geodesic γ with values in U, the function $f \circ \gamma$ is convex.

Definition 2.3. – *We shall say that a manifold M endowed with a connection ∇ has convex geometry if there exists a convex function $\psi : M \times M \to \mathbb{R}_+$ such that $\psi^{-1}(\{0\}) = \{(x,x),\ x \in M\}$.*

We shall say that a compact subset K of M has convex geometry if there exists an open neighbourhood of K which has convex geometry.

Definition 2.4. – *1) We shall say that a manifold M endowed with a connection ∇ has strong convex geometry if the following conditions are fulfilled*

(*i*) *there exists a convex function $\psi : M \times M \to \mathbb{R}_+$* such that *$\psi^{-1}(\{0\})$ is exactly the diagonal $\Delta = \{(x,x),\ x \in M\}$, the function ψ is smooth outside Δ and its Hessian ∇df is definite positive outside Δ,*

(*ii*) *there exists an open neighbourhood U of the null section in TM such that the application*

$$U \to M \times M, \qquad u \mapsto (\pi(u), \exp_{\pi(u)} u)$$

(*where π stands for the canonical projection $TM \to M$*) *is a smooth diffeomorphism. Its inverse mapping defined on $M \times M$ will be denoted by $(x,y) \mapsto \overrightarrow{xy}$,*

(*iii*) *for any probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and any random variable $\omega \mapsto L(\omega)$ with values in M, there exists a unique point x in M such that $\mathbb{E}\big[\overrightarrow{xL}\big] = 0$. This point will be denoted by $\mathscr{E}(L)$, and will be called the exponential expectation of L,*

(*iv*) *for any probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and any application $(\omega, a) \mapsto L(a)(\omega)$ defined on $\Omega \times I$ where I is an interval of $\mathbb{R}$, such that almost surely the map $a \mapsto L(a)(\omega)$ is differentiable on I, the map*

$$M \times I \to TM, \qquad (x,a) \mapsto \mathbb{E}\left[\overrightarrow{xL(a)}\right]$$

is differentiable and its differential with respect to x is everywhere inversible.

2) We shall say that a compact subset K of M has strong convex geometry if K is convex and has an open neighbourhood with strong convex geometry.

Definition 2.5. – *Let $p \in [1, \infty)$. We shall say that a manifold M endowed with a connection ∇ has p-convex geometry if M has convex geometry and there is a Riemannian distance δ on M and two constants $0 < a < A$ such that the function ψ of definition 2.3 satisfies $a\delta^p \le \psi \le A\delta^p$.*

We shall say that a manifold M endowed with a connection ∇ has strong p-convex geometry if M has p-convex geometry and strong convex geometry.

If $p \le p'$ and M has p-convex geometry, then M has p'-convex geometry. Simply connected Riemannian manifolds with negative sectional curvatures have 1-convex geometry. We know from [E] that any point x of any manifold M with a connection ∇ has a neighbourhood with strong 2-convex geometry, and Kendall has shown in [K2] that regular geodesic balls in Riemannian manifolds have p-convex geometry if p is sufficiently large.

Remark that the function ψ of definition 2.5 need not be the same as the function ψ of definition 2.3.

Among the two notions of convexity of definition 2.1 and definition 2.3, no one implies the other: every open subset of $\mathbb{R}^d$ has convex geometry, even if it is not convex. It is much more difficult to find a compact convex subset of a manifold which has not convex geometry. An example has been given in [K5].

Definition 2.6. – *If $\mathscr{G}$ is a σ-field included in $\mathscr{F}$ and if X is a random variable taking its values in a compact convex subset V (possibly random and $\mathscr{G}$-measurable) of a manifold M, one defines the conditional expectation of X with respect to $\mathscr{G}$ as the set of $\mathscr{G}$-measurable random variables Y taking their values in V and such that $f \circ Y \le \mathbb{E}[f \circ X|\mathscr{G}]$ for every convex function f belonging to $\mathscr{C}(V)$ (possibly depending in a $\mathscr{G}$-measurable way on ω). This set (possibly empty) will be denoted by $\mathbb{E}[X|\mathscr{G}]$.*

By [E,M] and [A2], we know that exponential conditional expectations are a particular case of conditional expectations.

Definition 2.7. – *A conditional expectation Y with respect to $\mathscr{G}$ of a random variable X taking its values in a ($\mathscr{G}$-measurable random) convex set will be called an exponential conditional expectation if it satisfies*

$$\mathbb{E}\left[\overrightarrow{YX}|\mathscr{G}\right] = 0 \ .$$

If it is unique up to a negligible set, it will be denoted by $\mathscr{E}(X|\mathscr{G})$.

By [K1] theorem 7.3, we know that convex geometry implies uniqueness for the exponential conditional expectation.

Definition 2.8. – *Let M be a manifold endowed with a connection ∇, and let $\tau = (T_0 \le T_1 \le \ldots \le T_n)$ be an increasing sequence of stopping times.*

(*i*) *We shall say that a M-valued process $X^{discr(\tau)}$ indexed by $[T_0, T_n]$ is a discrete convex martingale if for all $k \in \{0, \ldots, n-1\}$, the conditional law $\mathscr{L}\left(X^{discr(\tau)}_{T_{k+1}}|\mathscr{F}_{T_k}\right)$ is almost surely included in a (random) compact subset of M with strong convex geometry, $X^{discr(\tau)}$ is constant on $[T_k, T_{k+1})$ and*

$$X_{T_k}^{discr(\tau)} \in \mathbb{E}\left[X_{T_{k+1}}^{discr(\tau)} | \mathscr{F}_{T_k}\right] \ .$$

(*ii*) *We shall say that a M-valued process $X^{discr(\tau)}$ indexed by τ is a discrete exponential martingale if it is a discrete convex martingale and for all $k \in \{0, \dots, n-1\}$,*

$$X_{T_k}^{discr(\tau)} = \mathscr{E}\left(X_{T_{k+1}}^{discr(\tau)} | \mathscr{F}_{T_k}\right)$$

where $\mathscr{E}$ is the exponential conditional exceptation.

The next proposition was proven in [A2].

Proposition 2.9. – *If $(\Omega, \mathscr{F}_1 \mathbb{P})$ is such that conditional laws with respect to any σ-field included in $\mathscr{F}_1$ exist, if V (resp. V') is a compact subset of a manifold M (resp. M') with strong convex geometry, if (L, L') is a random variable with values in $V \times V'$ and $X^{discr(\tau)}$ is a V-valued discrete convex martingale with terminal value L, then there exists a V'-valued discrete convex martingale $X'^{\,discr(\tau)}$ with terminal value L' such that $\left(X^{discr(\tau)}, X'^{\,discr(\tau)}\right)$ is a discrete convex martingale with terminal value (L, L').*

Remark. – If $X^{discr(\tau)}$ is a V-valued discrete exponential martingale, then we can take for $X'^{\,discr(\tau)}$ the V'-valued discrete exponential martingale with terminal value L'.

To end this section, let us prove that exponential expectations commute with affine mappings.

Proposition 2.10. – *Let $\psi : (M, \nabla^M) \to (N, \nabla^N)$ be an affine mapping between two manifolds, and let L be a random variable with values in a compact convex subset V of M such that $\psi(V)$ is included in a compact convex subset of N. Suppose that $x \in V$ is an exponential expectation of L. Then $\psi(x)$ is an exponential expectation $\psi(L)$. If both are unique, then*

$$\mathscr{E}^N(\psi(L)) = \psi(\mathscr{E}^M(L)) \ .$$

Proof. – Let x be an exponential expectation of L. Then $\mathbb{E}[\overrightarrow{xL}] = 0$ and $\psi_*\left(\mathbb{E}[\overrightarrow{xL}]\right) = \mathbb{E}\left[\psi_*(\overrightarrow{xL})\right] = 0$. We have to show that $\mathbb{E}[\overrightarrow{\psi(x)\psi(L)}] = 0$. Because of the equality above, it will be true if for each $y, z \in M$, we have $\psi_*(\overrightarrow{yz}) = \overrightarrow{\psi(y)\psi(z)}$. Let us establish this equality: since ψ is affine, the map $t \mapsto \psi(\exp t\overrightarrow{yz})$ is a geodesic with derivative $\psi_*(\overrightarrow{yz})$ at the origin. Hence for all $t \in [0, 1]$, we have $\exp(t\psi_*(\overrightarrow{yz})) = \psi(\exp t\overrightarrow{yz})$, which for $t = 1$ gives $\exp(\psi_*(\overrightarrow{yz})) = \psi(z)$. Hence $\psi_*(\overrightarrow{yz}) = \overrightarrow{\psi(y)\psi(z)}$. This proves the proposition. □

3. Differentiable families of martingales

Let M be a manifold endowed with a connection ∇. In this section, we will show that under continuity conditions, the derivative of a family of martingales is a martingale in the tangent space TM endowed with the completed lift ∇' of ∇. This complete lift is described in [Y,I]. In local coordinates $(x^1,\dots,x^d,x^{\bar 1},\dots,x^{\bar d})$ (if x has coordinates $(x^1,\dots,x^d)$, a vector u above x has coordinates $(x^1,\dots,x^d,x^{\bar 1},\dots,x^{\bar d})$ with $u=\sum_i x^{\bar i}\frac{\partial}{\partial x^i}$), if the Christoffel symbols of ∇ write Γ^i_{jk}, then the Christoffel symbols of ∇' write

$$\begin{array}{llll} \Gamma'^i_{jk}=\Gamma^i_{jk}, & \Gamma'^i_{j\bar k}=0, & \Gamma'^i_{\bar j k}=0, & \Gamma'^i_{\bar j\bar k}=0, \\ \Gamma'^{\bar i}_{jk}=\langle d\Gamma^i_{jk},u\rangle, & \Gamma'^{\bar i}_{\bar j k}=\Gamma^i_{jk}, & \Gamma'^{\bar i}_{j\bar k}=\Gamma^i_{jk}, & \Gamma'^{\bar i}_{\bar j\bar k}=0\ . \end{array}$$

The geodesics for ∇' are the Jacobi fields for ∇ (see [Y,I] proposition 9.1). Since the ∇-martingales in M are the same as the martingales for the symmetrized connection, we shall assume that ∇ is torsion free, and therefore that ∇' is torsion free. In this case, the geodesics determine the connection.

Let us first give a natural approximation of ∇'.

Define for $a\in(0,1]$

$$\begin{aligned} l_a : U_a &\to M\times M \\ u &\mapsto (\pi(u),\exp_{\pi(u)} au) \end{aligned}$$

where $U_a=\frac{1}{a}U_1$ is a neighbourhood of 0 in TM such that $l_a:U_a\to l_a(U_a)$ is a diffeomorphism. Define $\Delta_1=l_a(U_a)=l_1(U_1)$ and

$$\begin{aligned} \varphi_a : \Delta_1 &\to U_a \\ (x,y) &\mapsto \frac{1}{a}\overrightarrow{xy}=l_a^{-1}((x,y)) \end{aligned}$$

where $\overrightarrow{xy}:=\exp_x^{-1}y$.

The set Δ_1 is a neighbourhood of the diagonal Δ in $M\times M$, and the diffeomorphism φ_a induces a connection ∇^a on U_a, image of the product connection in Δ_1.

Proposition 3.1. – *Let $f\in\mathscr{C}^\infty(TM)$. If V is a relatively compact open subset of TTM, then uniformly in $A\in V$, $\nabla^a df(A,A)$ converges to $\nabla' df(A,A)$ as a tends to 0.*

Proof. – Possibly by replacing V by $\varepsilon V,\varepsilon>0$, one can assume that for all $A\in V$, the ∇'-geodesic $t\mapsto J_A(t)$ in TM such that $J_A'(0)=A$ exists for all $t\in[0,1]$. Let $A\in V$ and J_A be such a geodesic. Define $\gamma_A=\gamma_A^0=\pi(J_A)$ and $t\mapsto\gamma_A^a(t)$ geodesic in M such that $(\dot\gamma_A^0(0),\dot\gamma_A^a(0))=l_{a*}(A)$. This definition is valid if a is small enough. We want first to show that $\frac{\partial}{\partial a}|_{a=0}\gamma_A^a(t)=J_A(t)$ for all $t\in[0,1]$. Since both are Jacobi fields, it is sufficient to show that $\frac{\partial}{\partial a}|_{a=0}\gamma_A^a(0)=J_A(0)$ and $\frac{\nabla}{\partial t}|_{t=0}\frac{\partial}{\partial a}|_{a=0}\gamma_A^a(t)=\frac{\nabla}{\partial t}|_{t=0}J_A(t)$ where $\frac{\nabla}{\partial .}$ denotes the covariant derivative. We have $\gamma_A^a(0)=\exp_{\gamma_A(0)}(aJ_A(0))$ which gives $\frac{\partial}{\partial a}|_{a=0}\gamma_A^a(0)=J_A(0)$; on the other hand,

$$\frac{\nabla}{\partial t}|_{t=0}\frac{\partial}{\partial a}|_{a=0}\gamma_A^a(t)=\frac{\nabla}{\partial a}|_{a=0}\frac{\partial}{\partial t}|_{t=0}\gamma_A^a(t)=\frac{\nabla}{\partial a}|_{a=0}\exp_*(aA)$$
$$=\frac{\nabla}{\partial a}|_{a=0}\frac{\partial}{\partial t}|_{t=0}\exp aJ_A(t)=\frac{\nabla}{\partial t}|_{t=0}\frac{\partial}{\partial a}|_{a=0}\exp aJ_A(t)$$
$$=\frac{\nabla}{\partial t}|_{t=0}J_A(t).$$

The first and the fourth equalities come from the fact that ∇ is torsion-free.

The definition of ∇^a implies that $t\mapsto\psi(a,A)(t)$ defined by $\psi(a,A)(t)=\frac{1}{a}\overrightarrow{\gamma_A(t)\gamma_A^a(t)}=\varphi_a(\gamma_A(t),\gamma_A^a(t))$ is a ∇^a-geodesic in U_a if a is small enough and $a\neq 0$. Let us define $\psi(0,A)=J_A$.

Lemma 3.2. – *There exists $a_0>0$ such that the map $(a,A,t)\mapsto\psi(a,A)(t)$ is smooth on $[0,a_0]\times V\times[0,1]$.*

Let us for the moment assume that the lemma is true. In particular, for all $r\in\mathbb{N}$,

$$\frac{\partial^r J_A}{\partial t^r}(t)=\lim_{a\to 0}\frac{\partial^r}{\partial t^r}\psi(a,A)(t)$$

uniformly in $A\in V$. Hence $(f\circ\psi(a,A))''(t)$ converges to $(f\circ J_A)''(t)$ as a tends to 0, uniformly in $A\in V$ and $t\in[0,1]$. Using the fact that

$$(f\circ\psi(a,A))''(0)=\nabla^a df(\psi(a,A)'(0),\psi(a,A)'(0)),$$
$$(f\circ J_A)''(0)=\nabla' df(A,A),$$

and that $\psi(a,A)'(0)=\varphi_{a*}l_{a*}(A)=A$, this convergence gives

$$\lim_{a\to 0}\nabla^a df(A,A)=\nabla' df(A,A)$$

uniformly in $A\in V$. $\square$

Proof of the lemma. – In local coordinates, $\frac{1}{a}\overrightarrow{xy}$ writes $(x^i,\frac{1}{a}C_j^k(x,y)(y^j-x^j))$ where C_j^k is a smooth function such that $u=\overrightarrow{x\exp_x u}$ has coordinates $((x^i),(C_k^j(x,x)u^k))$ since $T_0\exp_x=\mathrm{Id}$. Hence $\frac{1}{a}\overrightarrow{\gamma_A\gamma_A^a}$ writes $(\gamma_A^i,\frac{1}{a}C_j^k(\gamma_A,\gamma_A^a)(\gamma_A^{aj}-\gamma_A^j))$. But the function which maps (a,A,t) to $\frac{1}{a}(\gamma_A^{aj}(t)-\gamma_A^j(t))$ if $a\neq 0$ and to $J_A^j(t)$ if $a=0$ is equal to $\int_0^1 ds\frac{\partial}{\partial b}|_{b=as}(\gamma_A^b)^j(t)$; it is therefore a smooth function. Hence the coordinates of $\frac{1}{a}\overrightarrow{\gamma_A\gamma_A^a}$ converge to $(\gamma_A^i,C_j^k(\gamma_A,\gamma_A)J_A^j(t))$ as a tends to 0, and the last coordinates are those of $J_A(t)$. This proves the lemma. $\square$

Remark. – In the definition of φ_a, instead of $(x,y)\mapsto\overrightarrow{xy}$, one could have chosen any smooth mapping $(x,y)\mapsto e_x(y)\in T_xM$ defined in a neighbourhood of the diagonal of $M\times M$, such that $e_x(x)=0$ and $T_xe_x=\mathrm{Id}$.

Let us state the main result of this section.

Theorem 3.3. *Let I be an open interval in $\mathbb{R}$ and let $X_t(a))_{a\in I,t\in[0,1]}$ be a family of continuous martingales in M, such that ω as., $\forall t\in[0,1]$, the map $a\mapsto X_t(a)$ is $\mathscr{C}^1$ in $a\in I$, and ω a.s., the map $(t,a)\mapsto\frac{\partial X_t(a)}{\partial(a)}$ is continuous on* $[0,1]\times$ I.

Then $\left(\frac{\partial X(a)}{\partial a}\right)_{a\in I}$ *is a family of* ∇'*- martingales in TM.*

Proof. – One can assume that $0 \in I$. Define $Y = \frac{\partial X(a)}{\partial a}|_{a=0}$. It is sufficient to show that Y is a ∇'-martingale. For this, it is sufficient to show that each point u of TM has a relatively compact neighbourhood V_u such that Y is a martingale during the time it spends in V_u (use [E] lemma (3.5)). But each point u of TM has a neighbourhood V'_u such that every continuous process Z with values in V'_u is a martingale if and only if $\forall f : V'_u \to \mathbb{R}$ smooth, satisfying $\nabla' df > 0, f(Z)$ is a real submartingale (see [D1] [E] [A1]). Let U be such a neighbourhood. Possibly by reducing U, we can assume that there are three relatively compact open subsets U^1, U^2 and U^3 such that $\bar{U} \subset U^1 \subset \bar{U}^1 \subset U^2 \subset \bar{U}^2 \subset U^3$, and that U^3 has the same property. Let g be any Riemannian metric on TM and δ be the associated Riemannian distance, and let $f : U^3 \to \mathbb{R}$ be a smooth convex function such that $\nabla' df > 0$. Then there exists $\varepsilon > 0$ such that $\nabla' df > \varepsilon g$ on $\bar{U}^2$.

Suppose that T and T' are stopping times such that $T \leq T', T'$ is bounded and Y take its values in U on $[T, T']$. If $a > 0$, define $\Delta_a = l_a(U_1)$ with the notations before proposition 3.1. For $a > 0$, define

$$T^a = T' \wedge \inf\left\{t > T, \forall a' \in (0, a], (X_t(0), X_t(a')) \in \bar{\Delta}_{\frac{1}{3}}, \frac{1}{a'}\overrightarrow{X_t(0)X_t(a')} \in \bar{U}^1\right\}$$

and

$$T'^{\,a} = T' \wedge \inf\left\{t > T^a, \exists a' \in (0, a], (X_t(0), X_t(a')) \notin \Delta_{\frac{2}{3}} \text{ or } \frac{1}{a'}\overrightarrow{X_t(0)X_t(a')} \notin U^2\right\}.$$

Then $T \leq T^a \leq T'^{\,a} \leq T'$ and the times T^a and $T'^{\,a}$ are stopping times. Since ω a.s. $\frac{1}{a'}\overrightarrow{X(0)X(a')}$ converges uniformly in t to Y as a' tends to $0\,(\frac{\partial X(a)}{\partial a}$ is jointly continuous) and $\bar{U}^2$ is compact, we have that almost surely T^a decreases stationarily to T as a tends to 0 and for a small enough so that $T^a = T, T'^{\,a}$ increases stationarily to T'.

Hence it is sufficient to show that there exists $a_0 > 0$ such that for all $a \in (0, a_0], f(Y)$ is a submartingale on $[T^a, T'^{\,a}]$.

Choose a_0 such that for all $a \in (o, a_0], U^3 \subset \varphi_a(\Delta_1)$ and f is ∇^a-convex on U^2. This is possible because $\bar{U}^2$ is compact, $\nabla' df > \varepsilon g$ on $\bar{U}^2$, using proposition 3.1 with $V = \{A \in TTM, \pi_2(A) \in U^2, g(A, A) = 1\}$($\pi_2$ is the canonical projection $TTM \to TM$).

For all $a \in (0, a_0]$, for all $a' \in (0, a], f\left(\frac{1}{a'}\overrightarrow{X(0)X(a')}\right)$ is a bounded submartingale on $[T^a, T'^a]$. But a.s. $f\left(\frac{1}{a'}\overrightarrow{X(0)X(a')}\right)$ converges to $f(Y)$ uniformly in t, which implies that $f(Y)$ is a submartingale. This proves the theorem. □

In local coordinates, the martingales $X(a)$ satisfy

$$d\widetilde{X}^i(a) = -\frac{1}{2}\Gamma^i_{jk}(X(a))d\langle X^j(a), X^k(a)\rangle \tag{5}$$

where $\widetilde{X}^i$ stands for the finite variation part of X^i. The formal differentiation with respect to the parameter a at $a = 0$ of the right hand side of this equality gives

$$-\frac{1}{2}\langle d\Gamma^i_{jk}(X(0)), Y\rangle d\langle X^j(0), X^k(0)\rangle$$
$$-\frac{1}{2}\Gamma^i_{jk}(X(0))\big(d\langle Y^j, X^k(0)\rangle + d\langle X^j(0), Y^k\rangle\big)$$

and theorem 3.3 says that these formal derivatives are in fact exactly the coordinates of the drift of Y:

Corollary 3.4. – *Let I be an open interval in* $\mathbb{R}$ *containing 0 and let* $(X(a))_{a\in I}$ *be a family of martingales satisfying the same hypothesis as in theorem 3.3. Define* $Y = \frac{\partial X(a)}{\partial a}|_{a=0}$. *Then in a chart* $(x^1, \ldots, x^d, y^1, \ldots, y^d)$, the finite variation part $\widetilde{Y}^i$ of the i-th component Y^i satisfies the relation

$$d\widetilde{Y}^i = -\frac{1}{2}\langle d\Gamma^i_{jk}(X(0)), Y\rangle d\langle X^j(0), X^k(0)\rangle$$
$$-\frac{1}{2}\Gamma^i_{jk}(X(0))(d\langle Y^j, X^k(0)\rangle + d\langle X^j(0), Y^k\rangle)$$

Proof. – The expression of the Christoffel symbols of the connection ∇' in the chart chosen, the fact that Y is a ∇'-martingale and $\pi(Y) = X(0)$ give the formula. □

The proof of theorem 3.3 was nothing but saying that formal differentiation of equation (1) is rigorous.

Lemma 3.5. – *Let* $p \in 2\mathbb{N}^*$ *and* (M, ∇) *be a manifold with strong p-convex geometry, such that the functions* ψ *of definition 2.3 and definition 2.5 coincide and* ψ *is smooth on* $M \times M$. *Let us define the function* $T^{\otimes p}\psi : TM \to \mathbb{R}_+$ *by*

$$T^{\otimes p}\psi(u) = T_2^{(p)}\psi(\pi(u), \pi(u))(u \otimes \cdots \otimes u)$$

($T^{\otimes p}\psi$ *is the p-th derivative of* ψ *with respect to the second variable, on the diagonal; it is p-linear since the derivatives of order less than p vanish, and it is positive if* $u \neq 0$ *since the pth derivative does not vanish*).

Then $T^{\otimes p}\psi$ *is convex for the connection* ∇'.

Proof. – Let J be a Jacobi field on M, let I be an open interval of $\mathbb{R}$ containing 0 and $(\gamma^a)_{a\in I}$ a family of geodesics such that $J = \frac{\partial \gamma^a}{\partial a}|_{a=0}$. For all $a \in I\backslash\{0\}$, the function $t \mapsto \frac{p!}{a^p}\psi(\gamma^0(t), \gamma^a(t))$ is convex, and converges to $t \mapsto T^{\otimes p}\psi(J(t))$ as a tends to 0. It implies that the latter function is convex and $T^{\otimes p}\psi$ is convex. □

Corollary 3.6. – *Let V be a compact subset of TM, where M is a manifold which satisfies the same hypothesis as in lemma 3.5. Then if P is a* $\mathscr{F}_1$*-mesurable random variable with values in V, then there is at most one V-valued* ∇'*-martingale with terminal value P.*

Proof. – If Y and Y' are two V-valued ∇'-martingales with terminal value P, then $\pi(Y)$ and $\pi(Y')$ are two M-valued ∇-martingale with terminal value $\pi(P)$, hence $\pi(Y) = \pi(Y')$ since M has strong convex geometry. It implies that the process $Y - Y'$ is well defined and is a ∇'-martingale with values in a compact of TM, and with terminal value 0. Hence $T^{\otimes p}\psi(Y - Y')$ is a bounded non-negative submartingale with terminal value 0. It implies that $T^{\otimes p}\psi(Y - Y') \equiv 0$ and $Y = Y'$. □

Now we investigate the case where we only know that the terminal value of a family of martingales is differentiable.

Proposition 3.7. – *Let $p \geq 1$ and let M be a manifold with a connection ∇. Let I be an open interval containing 0, and for all $a \in I$, let $X(a)$ be a M-valued ∇-martingale such that almost surely $a \mapsto X_1(a)$ is differentiable; define $L(a) = X_1(a)$ and let Y be a TM-valued* ∇'-martingale with $Y_1 = L'(0)$.

Assume that

(i) for all $a \in I$, almost surely, $X(a)$ takes its values in a compact subset K of M with p-convex geometry

(ii) for all $a \in I$, almost surely, $L'(a)$ and Y take their values in a compact subset V of TM,

(iii) every compact subset of TM has an open neighbourhood with strong convex geometry.

Then almost surely, for all $t \in [0, 1]$, the map $a \mapsto X_t(a)$ is differentiable at $a = 0$, with derivative Y_t.

Remark. – By [K4] corollary 3.4, uniqueness of martingales with prescribed terminal value on a compact subset W of a manifold implie convex geometry for W. From this and corollary 3.6, one can think that assumption (iii) on TM is not too restrictive.

Proof of proposition 3.7. – Let φ be a function defining p-convex geometry on M', with $c\delta^p \leq \varphi \leq C\delta^p$ where δ is a Riemannian distance on M. Then $\varphi(X(0), X(a))$ is a bounded non-negative submartingale, and therefore, for all $t \in [0, 1]$,

$$\varphi(X_t(0), X_t(a)) \leq \mathbb{E}[\varphi(X_1(0), X_1(a)) | \mathscr{F}_t] \ .$$

It gives

$$\delta^p(X_t(0), X_t(a)) \leq \frac{C}{c} \mathbb{E}[\delta^p(X_1(0), X_1(a)) | \mathscr{F}_t]$$

and since $\frac{\partial X_1(a)}{\partial a}$ is bounded, there exists a non-negative constant C' such that $\delta^p(X_t(0), X_t(a)) \leq \frac{C}{c} C'^p a^p$. This gives $\delta(X_t(0), X_t(a)) \leq \left(\frac{C}{c}\right)^{\frac{1}{p}} C' a$, and it implies that there exists a relatively compact open subset V' of TM and $a_0 > 0$ such that for all $a \in (0, a_0)$, $\overrightarrow{X(0)X(a)}$ exists, $\frac{1}{a}\overrightarrow{X(0)X(a)} \in V'$, $Y \in V'$ and $V' \subset U_a$ (see definition of U_a before proposition 3.1). Let U be an open neighbourhood of V' with strong convex geometry. Since the function ψ defining strong convex geometry on U has a strictly positive Hessian outside the diagonal, for any metric g on $U \times U$, for any $\varepsilon > 0$, there exists $a_1 \in (0, a_0)$ such that if $0 < a' < a_1$, then ψ is convex for the connection $\nabla^{a'} \otimes \nabla'$ on $V' \times V' \backslash \psi^{-1}([0, \varepsilon))$ (this is a consequence of proposition 3.1). It implies that for this connection, $\psi_\varepsilon = \sup(\psi, \varepsilon)$ is convex on $V' \times V'$, and that $\psi_\varepsilon\left(\frac{1}{a'}\overrightarrow{X(0)X(a')}, Y\right)$ is a submartingale. But $\psi_\varepsilon\left(\frac{1}{a'}\overrightarrow{X_1(0)X_1(a')}, Y_1\right)$ is bounded and tends almost surely stationarily to ε as a' tends to 0. Hence almost surely, for all $t \in [0,1], \psi_\varepsilon\left(\frac{1}{a'}\overrightarrow{X_t(0)X_t(a')}, Y_t\right)$ tends to ε as a' tends to 0. Since this is true for all $\varepsilon > 0$, we have that almost surely, for all $t \in [0,1], \frac{1}{a}\overrightarrow{X_t(0)X_t(a)}$ tends to Y_t as a tends to 0 (see [K1] lemma 4.5). Therefore almost surely, for all $t \in [0,1]$, the map $a \mapsto X_t(a)$ is differentiable at $a = 0$ with derivative Y.

To end this section, let us prove that exponential expectations commute with differentiation. This can be seen as a discrete analogue to theorem 3.3.

Proposition 3.8. – *Let (M, ∇) be a manifold endowed with a torsion-free connection, and let W be a random variable with values in an open subset V of TM such that (V, ∇') has strong convex geometry. Denote by L the projection $\pi(W)$. Assume that the differential of the exponential expectation $\mathscr{E}$ at point L in the direction W, denoted by $\langle T\mathscr{E}(L), W\rangle$, belongs to V.*

Then $\langle T\mathscr{E}(L), W\rangle$ is the exponential expectation $\mathscr{E}'(W)$ of W with respect to the connection ∇'.

Remark. – If W takes its values in a small set, then $\langle T\mathscr{E}(L), W\rangle$ is very close to this set and it is possible to find an open set V with strong convex geometry such that the assumptions are fulfilled.

Proof of proposition 3.8. – Let $a \mapsto L(a)$ be a smooth family of random variables with values in M, such that almost surely $L'(0) = W$. Define $x(a) = \mathscr{E}(L(a))$ and $U(a) = \overrightarrow{x(a)L(a)}$. Then $x'(0)$ is equal to $\langle T\mathscr{E}(L), W\rangle$.

If $a \mapsto u(a)$ is a differentiable path in TM with projection $\pi_1(u(a)) = x(a)$, one gives the two following equivalent definitions of the complete lip $u^c(a)$ of u:

(i) $u^c(a)$ is the vector in TTM with projection $\pi_2(u^c(a)) = x'(a)$ in TM, and with coordinates $(u^i(a), (u^i)'(a))$;
(ii) $u^c(a)$ is the vector in TTM with projection $\pi_2(u^c(a)) = x'(a)$ in TM, with horizontal part the horizontal lift of $u(a)$ in $T_{x'(a)}TM$ and with vertical part the vector $\frac{\nabla u(a)}{da}$, where $\frac{\nabla}{d}$ denotes the covariant derivative.

Since the first definition does not require the connection and the second one does not require a chart, $u^c(a)$ depends on none of them.

We have $\mathbb{E}[U(a)] = 0$ by definition of the exponential expectation (definition 2.4), and this implies by derivation and by the definition (i) of $U^c(a)$, that $\mathbb{E}[U^c(0)] = 0$. To show that $x'(0)$ is the exponential expectation of $L'(0)$, it remains to verify that if $\exp'$ denotes the exponential map with respect to ∇', then $\exp'_{x'(0)} U^c(0) = L'(0)$. This is exactly what the following lemma says. □

Lemma 3.9. – *Let $a \mapsto x(a)$* and *$a \mapsto l(a)$ be two differentiable curves defined on an interval I of $\mathbb{R}$ containing 0 and with values in V. Denote by $u(a)$ the vector $\overrightarrow{x(a)l(a)} \in T_{x(a)}M$.*

Then $\exp_*(u'(0)) = \exp' u^c(0)$, *where* $\exp'$ *denotes the exponential map defined with* ∇' and $\exp_*$ *denotes the tangent map to* exp.

Proof. – Let $(t,a) \mapsto \gamma(t,a)$ be the V-valued map defined on $[0,1] \times I$, such that for each $a \in I$, the path $t \mapsto \gamma(t,a)$ is a geodesic with $\gamma(0,a) = x(a)$ and $\gamma(1,a) = l(a)$. Then we have for all $a \in I, u(a) = \frac{\partial \gamma(t,a)}{\partial t}|_{t=0}$. Denote by J the Jacobi field $\frac{\partial \gamma}{\partial a}|_{a=0}$. It is sufficient to show that $J'(0) = u^c(0)$. For this, we are going to use the definition (ii) of $u^c(0)$ and show that the projections, and the horizontal and vertical parts of the two vectors coincide.

It is clear that they have the same projection $x'(0) \in V$. Since $\pi_1 \circ J(t) = \gamma(t,0)$, we have $\pi_{1*}(J'(0) = u(0)$, and we deduce that they have the same horizontal part. As for the vertical part, we have

$$\frac{\nabla}{\partial t}|_{t=0}\, J(t) = \frac{\nabla}{\partial t}|_{t=0} \frac{\partial}{\partial a}|_{a=0}\gamma(t,a) \frac{\nabla}{\partial a}|_{a=0} \frac{\partial}{\partial t}|_{t=0}\gamma(t,a) = \frac{\nabla}{\partial a}|_{a=0} u(a) \ .$$

This proves the lemma. □

4. Complexification of a manifold with connection

In this section, the manifold $M^{\mathbb{R}}$ is real analytic, and endowed with a real analytic torsion-free connection $\nabla^{\mathbb{R}}$. According to [W,B], there exists a complexification M of $M^{\mathbb{R}}$ which is a complex analytic manifold. For every point x in $M^{\mathbb{R}}$ and every analytic function $\phi^{\mathbb{R}}$ defined on a neighbourhood $V^{\mathbb{R}}$ of x in $M^{\mathbb{R}}$, there exists a neighbourhood V of x in M and a unique holomorphic extension ϕ of $\phi^{\mathbb{R}}$ to V. We shall show, possibly by reducing the neighbourhood M of $M^{\mathbb{R}}$, that this fact allows us to extend $\nabla^{\mathbb{R}}$ to an holomorphic connection in M such that $M^{\mathbb{R}}$ is a totally geodesic submanifold.

Proposition 4.1. – *Possibly by reducing the complexification M of $M^{\mathbb{R}}$, there exists a connection ∇ on M such that the equation of geodesics in an holomorphic chart is*

$$\ddot{\gamma}^l = -\Gamma^l_{jk}(\gamma)\dot{\gamma}^j\dot{\gamma}^k \tag{6}$$

where the coordinates are taken in $\mathbb{C}$ and the Γ^l_{jk} are holomorphic functions. For this connection, the set $M^{\mathbb{R}}$, is a totally geodesic submanifold and the inclusion $(M^{\mathbb{R}}, \nabla^{\mathbb{R}}) \to (M, \nabla)$ is affine. The equation of continuous martingales in an holomorphic chart is

$$d\widetilde{Z} = -\frac{1}{2}\Gamma_{jk}(Z)d\langle Z^j, Z^k\rangle \ , \tag{7}$$

where the coordinates are taken in $\mathbb{C}$ and $\widetilde{Z}$ denotes the finite variation part in these coordinates.

Proof. – Consider an analytic exponential chart in $M^{\mathbb{R}}$ with image an open ball $B^{\mathbb{R}}(0,R)$ in $\mathbb{R}^d$, such that the Christoffel symbols

$$\Gamma^l_{jk}(x) = \sum_\delta c^l_{jk\delta}x^\delta$$

converge on this domain, where summation is taken over multiindexes $\delta \in \mathbb{N}^d$. We have $\Gamma(0) = 0$. Considering $\mathbb{R}^d$ as a subset of $\mathbb{C}^d$ and denoting by B(0, R) the open ball of center 0 and radius R in $\mathbb{C}^d$, we can assume that $B(0,R)$ is the image of the complexification of the real chart in $M^{\mathbb{R}}$. We can define holomorphic functions on $B(0,R)$ by

$$\Gamma^l_{jk}(z) = \sum_\delta c^l_{jk\delta} z^\delta \quad . \tag{8}$$

The uniqueness of the holomorphic extension of the Christoffel symbols together with the existence of a locally finite covering of $M^{\mathbb{R}}$ with charts as above allow us to say that it is sufficient to consider only one chart and to show that the functions given by (8) define a connection ∇ on $B(0,R)$ which extends the connection $\nabla^{\mathbb{R}}$ on $B^{\mathbb{R}}(0,R)$, and for which this subset is a totally geodesic submanifold of $B(0,R)$.

Denote by $z^a = x^a + ix^\alpha$(and $z^b = x^b + ix^\beta, z^c = x^c + ix^\gamma$) the coordinates in $\mathbb{C}^d$, and by $\Gamma^a_{bc}(z) = \Gamma^{ar}_{bc}(z) + i\Gamma^{ai}_{bc}(z)$ the decomposition into real and imaginary part of the function Γ^a_{bc}. Considering $B(0,R)$ as a real manifold with the system of coordinates (x^a, x^α), one verifies that equation (6) is equivalent to the equation of geodesics for the real connection whose Christoffel symbols Δ are

$$\begin{array}{llll} \Delta^a_{bc} = \Gamma^{ar}_{bc}, & \Delta^a_{b\gamma} = -\Gamma^{ai}_{bc}, & \Delta^a_{\beta c} = -\Gamma^{ai}_{bc}, & \Delta^a_{\beta\gamma} = -\Gamma^{ar}_{bc} \ , \\ \Delta^\alpha_{bc} = \Gamma^{ai}_{bc}, & \Delta^\alpha_{b\gamma} = \Gamma^{ar}_{bc}, & \Delta^\alpha_{\beta c} = \Gamma^{ar}_{bc}, & \Delta^\alpha_{\beta\gamma} = -\Gamma^{ai}_{bc} \ . \end{array}$$

In the same way, one verifies that equation (8) is the same as the equation for martingales in the real manifold $B(0,R)$ endowed with the connection with Christoffel symbols Δ in the canonical coordinates.

The fact that the coefficients $c^c_{ab\delta}$ are real implies that the equation of geodesics in $(B^{\mathbb{R}}(0,R), \nabla^{\mathbb{R}})$ is the same as the equation in $B(0,R)$ with real initial conditions. Hence the inclusion $(B^{\mathbb{R}}(0,R), \nabla^{\mathbb{R}}) \to (B(0,R), \nabla)$ is affine and $B^{\mathbb{R}}(0,R)$ is totally geodesic in $B(0,R)$. □

Let M be as in proposition 4.1. Since the connection in M is holomorphic, we have the following results:

Lemma 4.2. – *Let $\mathscr{I} : TM \to TM$ be the complex structure on TM, i.e. the multiplication by i of the complex coordinates of vectors in TM.*

Then $\mathscr{I}$ is an affine diffeomorphism.

Proof. – It is sufficient to prove that the image by $\mathscr{I}$ of a geodesic is a geodesic. One verifies that the equation of a geodesic in TM is in complex coordinates

$$\ddot{J}^l = -\langle T\Gamma^l_{jk}(\gamma), J\rangle \dot{\gamma}^j\dot{\gamma}^k - \Gamma^l_{jk}(\gamma)(\dot{J}^j\dot{\gamma}^k + \dot{\gamma}^j\dot{J}^k) \ .$$

Since the Γ^l_{jk} are holomorphic, we have $\langle T\Gamma^l_{jk}(\gamma), \mathscr{I}J\rangle = i(\langle T\Gamma^l_{jk}(\gamma), J\rangle)$. If we replace $J, \dot{J}^j, \dot{J}^k, \ddot{J}^l$ by $\mathscr{I}J, i\dot{J}^j, i\dot{J}^k, i\ddot{J}^l$, the above equation is still satisfied. Hence $\mathscr{I}J$ is a geodesic and $\mathscr{I}$ is affine. □

Corollary 4.3. – *Assume that M has strong convex geometry. Then the following assertions are true:*

1) *Denote by U an open neighbourhood of the null section in TM such that the application $U \to M \times M$, $u \mapsto (\pi(u), \exp_{\pi(u)} u)$ is a diffeomorphism. Then* $\exp : U \to M$ *is holomorphic.*
2) *If $a \mapsto L(a)$ is a holomorphic family of M-valued random variables, then the application $a \mapsto \mathscr{E}(L(a))$* is holomorphic.
3) *Let $a \mapsto L(a)$ be a holomorphic family of M-valued random variables, and for each a, let $X^{discr}(a)$ be a discrete exponential martingale with terminal value $L(a)$. Then almost surely, for each $t \in$* $[0,1]$, the map $a \mapsto X_t^{discr}(a)$ *is holomorphic.*

Corollary 4.4. – *With the hypothesis of 3) of corollary 4.3, if the map $a \mapsto L(a)$ is defined on some closed disc $\bar{D}(0, A)$, if $M = B(0, R)$ and if we write*

$$X^{discr}(a) = X^{discr}(0) + \sum_{n=1}^{\infty} \frac{a^n}{n!} W_n^{discr} \quad ,$$

then for each $n \in \mathbb{N}$, we have

$$\|W_n^{discr}\| \le n! \frac{R}{A^n} \quad .$$

Remark. – The upper bounds obtained for the discrete process W_n^{discr} are independent of the filtration. The question arises whether this result generalizes for non-discrete filtrations, and a positive answer will be given at the end of section 6.

Proof of corollary 4.3. – 1) Consider two holomorphic maps $a \mapsto K(a) \in M$ and $a \mapsto L(a) \in M$ defined on the same open subset of $\mathbb{C}$, and consider for each a the geodesic $t \mapsto \gamma(a, t)$ which satisfies $\gamma(a, 0) = K(a)$ and $\gamma(a, 1) = L(a)$. It is sufficient to show that for each t, the map $a \mapsto \gamma(a, t)$ is holomorphic and then to differentiate at time $t = 0$.

For each t, the map $a \mapsto \gamma(a, t)$ is differentiable since M has strong convex geometry, and if $T_1\gamma(a, t)$ denotes the differential with respect to a and $u \in \mathbb{C}$, we have that $t \mapsto \langle T_1\gamma(a, t), u\rangle$ is a ∇'-geodesic in TM, $t \mapsto \langle T_1\gamma(a, t), iu\rangle$ is a ∇'-geodesic in TM. But since $\mathscr{I}$ is affine, $t \mapsto \mathscr{I}(\langle T_1\gamma(a, t), u\rangle)$ is a ∇'-geodesic in TM with the same end points as those of $t \mapsto \langle T_1\gamma(a, t), iu\rangle$. The consequence is that $\langle T\gamma(a, t), iu\rangle = \mathscr{I}(\langle T\gamma(a, t), u\rangle)$ and 1) is proved.

2) is a consequence of 1) and of the implicit function theorem applied to $(x, a) \mapsto \mathbb{E}\left[\overrightarrow{xL(a)}\right]$, using (iv) of definition 2.4.

3) is a direct consequence of 2). □

Proof of corollary 4.4. – It is a consequence of the formula

$$W_n^{discr}(t) = n! \frac{1}{2\pi i} \int_{\mathscr{C}(0,A)} \frac{X_t^{discr}(a)}{a^{n+1}} da$$

where $\mathscr{C}(0,A)$ denotes the circle in $\mathbb{C}$ of center 0 and of radius A.

Corollary 4.5. – *Let $A > 0$ and for each $a \in D(0,A)$, let $X(a)$ be an M-valued continuous martingale with terminal value $L(a)$, such that a.s. for all $t \in [0,1], a \mapsto X_t(a)$ is differentiable on $D(0,A)$, and $a \mapsto L(a)$ is holomorphic on $D(0,A)$.*

Assume that

a) the manifold M has strong p-convex geometry with $p \in 2\mathbb{N}^*$, the function ψ of definition 2.3 is the same as the function ψ of definition 2.5 and is smooth on $M \times M$,

b) almost surely, the map $(t,a) \mapsto T_2X_t(a)$ is continuous and bounded on $[0,1] \times D(0,A)(T_2X_t(a)$ denotes the differential with respect to a).

Then almost surely, for all $t \in [0,1], a \mapsto X_t(a)$ is holomorphic on $D(0,A)$.

Proof. – It is sufficient to check that the Cauchy-Riemann equations are satisfied at $a = 0$. For $u \in \mathbb{C}$, set $Y^u = \langle T_2X(a), u\rangle|_{a=0}$. The processes Y^u and Y^{iu} are ∇'-martingales in TM by theorem 3.3. The process $\mathscr{I}Y^u$ is a ∇'-martingale by lemma 4.2. Hence the process $Y^{iu} - \mathscr{I}Y^u$ is a bounded martingale above $X(0)$, with terminal value 0. It implies by lemma 3.5 that $T^{\otimes p}\psi(Y^{iu} - \mathscr{I}Y^u)$ is a bounded non-negative real submartingale with terminal value 0. Hence $T^{\otimes p}\psi(Y^{iu} - \mathscr{I}Y^u) \equiv 0$ and $Y^{iu} \equiv \mathscr{I}Y^u$. □

5. Approximations of martingales in a manifold

In this section, we give general results of approximations of a continuous martingale in a manifold by discrete exponential martingales. They extend the results obtained in [A2] in convex geometry.

Theorem 5.1. – *Let $(X_t)_{0\le t\le 1}$ be a continuous martingale in a manifold M endowed with a connection ∇.*

Then for all $\varepsilon > 0$, for all stopping times S, T such that $S \le T$, for any Riemannian distance on M, there exists a stopping time T_ε such that $S \le T_\varepsilon \le T, \mathbb{P}(T_\varepsilon \ne T) < \varepsilon$ and such that X^{T_ε} can be a.s. uniformly approximated between S and T by a discrete exponential martingale at a distance less than ε.

Proof of theorem 5.1. – For the sake of simplicity, one can take $S = 0, T = 1$ and assume that X_0 is a constant. Since we are interested in the martingale X up to a stopping time equal to 1 with probability less than 1, we can assume that X lives in a compact subset K of M. It makes sense to define square integrable martingales living in a compact set: take any Riemannian metric on the manifold and say that X is square integrable if its Riemannian quadratic variation with respect to this metric has finite expectation. By using a stopping time equal to 1 with probability as close to 1 as we want, we can suppose that X is square integrable.

Let g be a Riemannian metric on M, and δ the associated Riemannian distance. In the following, the Levi-Civita connection of g will not be used. Let $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$; define for $m \in \mathbb{N}^* T_0^m = 0$ and for $i \in \{0, \dots, m-1\}$

$$T_{i+1}^m = \inf\left\{t > T_i^m, \delta(X_{T_i^m}, X_t) = \frac{1}{m^\alpha}\right\}$$

(by convention, these stopping times are equal to 1 when the sets are empty).

Lemma 5.2. – *The probability* $\mathbb{P}(T_m^m \neq 1)$ *tends to* 0 *as* m *tends to* ∞.

Remark. – The proof will only use $\alpha < \frac{1}{2}$.

Proof of lemma 5.2. – Let $\eta : M \times M \to \mathbb{R}$ be a smooth function with compact support, which coincides with δ^2 in a neighbourhood of the diagonal of the compact set K. There exists a constant $c > 0$ such that for all $x \in M, \nabla d\eta(x, \cdot) \leq cg$. In the following, $\nabla d\eta(x, \cdot)$ will be denoted by $\mathrm{Hess}_2\eta(x, \cdot)$, and $d\eta(x, \cdot)$ by $d_2\eta(x, \cdot)$. If S, T are two stopping times such that $S \leq T$, then the Itô formula gives

$$\eta(X_S, X_T) = \int_S^T \langle d_2\eta(X_S, X_t), dX_t\rangle + \frac{1}{2}\int_S^T \mathrm{Hess}_2\,\eta(X_S, X_t)(dX \otimes dX)_t$$

where the first term in the right hand side is an Itô integral. This yields

$$\begin{aligned}\mathbb{E}[\eta(X_S, X_T)] &= \frac{1}{2}\mathbb{E}\left[\int_S^T \mathrm{Hess}_2\eta\,(X_S, X_t)(dX \otimes dX)_t\right] \\ &\leq \frac{c}{2}\mathbb{E}[\langle X|X\rangle_T - \langle X|X\rangle_S]\end{aligned}$$

where $\langle X|X\rangle$ stands for the Riemannian quadratic variation of X. Applying this inequality to $S = T_i^m$ and $T = T_{i+1}^m$, summing over i, and taking m sufficiently large, we can replace η by δ^2 and write

$$\mathbb{E}\left[\sum_{i-0}^{m-1} \delta^2\left(X_{T_i^m}, X_{T_{i+1}^m}\right)\right] \leq \frac{c}{2}\mathbb{E}[\langle X|X\rangle_1] \quad .$$

On $\{T_m^m < 1\}$, we have for all $i, \delta\left(X_{T_i^m}, X_{T_{i+1}^m}\right) = \frac{1}{m^\alpha}$. Hence we have

$$\frac{m}{m^{2\alpha}}\mathbb{P}(\{T_m^m < 1\}) \leq \frac{c}{2}\mathbb{E}[\langle X|X\rangle_1] \quad ,$$

which yields

$$\mathbb{P}(\{T_m^m < 1\}) \leq m^{2\alpha-1}\frac{c}{2}\mathbb{E}[\langle X|X\rangle_1]$$

and the right hand side tends to 0 as m tends to ∞ since $2\alpha - 1 < 0$. This proves the lemma. □

Fix m big enough for any ball of radius $\frac{2}{m^\alpha}$ centered in K to be included in a manifold with strong 2-convex geometry, and for T_m^m to be different of 1 with probability less than $\frac{\varepsilon}{2}$. This is possible thanks to lemma 5.2. Then X takes its values in a manifold with strong 2-convex geometry between times T_i^m and T_{i+1}^m.

Let $n \in \mathbb{N}$ (n will be much greater than m). The idea of the proof of theorem 5.1 is to approximate X uniformly on intervals of the form $[T^{m,n}_{i,0}, T^{m,n}_{i,n}]$(with $T^{m,n}_{i,n} = T^{m,n}_{i+1,0}$), $\mathbb{P}(T^{m,n}_{i,0} \neq T^m_i)$ and $\mathbb{P}(T^{m,n}_{i,n} \neq T^m_{i+1})$ being very small. This will be performed using subdivisions of the form $T^{m,n}_{i,0} \leq T^{m,n}_{i,1} \leq \ldots \leq T^{m,n}_{i,n}$.

First define by induction the stopping times $S^{m,n}_{i,j}, 0 \leq i \leq m-1$, $0 \leq j \leq n-1$ by

$$S^{m,n}_{0,0} = 0, \qquad S^{m,n}_{0,j+1} = T^m_1 \wedge \inf\left\{ t > S^{m,n}_{0,j}, \delta(X_{S^{m,n}_{0,j}}, X_t) = \frac{1}{n^\alpha} \right\} .$$

The induction relation is: for $0 \leq i \leq m-2$, define

$$S^{m,n}_{i+1,0} = S^{m,n}_{i,n}$$

and for $0 \leq j \leq n-1$,

$$S^{m,n}_{i+1,j+1} = T^m_{i+2} \wedge \inf\left\{ t > S^{m,n}_{i+1,j}, \delta(X_{S^{m,n}_{i+1,j}}, X_t) = \frac{1}{n^\alpha} \right\} .$$

Then define the stopping time $R^{m,n}$ by

$$R^{m,n} = T^m_m \wedge \inf_{0 \leq i \leq m-1} \{ S^{m,n}_{i,n} \text{ such that } S^{m,n}_{i,n} < T^m_{i+1} \}$$

where by convention, the infimum is equal to 1 if the set is empty (in fact we expect this set to be empty with a large probability).

Now define for $0 \leq i \leq m-1$ and $0 \leq j \leq n-1$ the stopping times

$$T^{m,n}_{i,j} = S^{m,n}_{i,j} \wedge R^{m,n}$$

(the discrete martingale is stopped at the first time $S^{m,n}_{i,n}$ which is strictly less than T^m_{i+1}). A consequence of lemma 5.2 is that there exists a constant $C > 0$ such that almost surely, for all $i \in \{0, \ldots, m-1\}$,

$$\mathbb{P}\left(\bigcap_{0 \leq i' \leq i-1} \{ S^{m,n}_{i',n} = T^m_{i'+1} \} \cap \{ S^{m,n}_{i,n} < T^m_{i+1} \} | \mathscr{F}_{T^m_i} \right) \leq \frac{C}{n^{1-2\alpha}} ,$$

which yields

$$\mathbb{P}(R^{m,n} < T^m_m) = \sum_{i=0}^{m-1} \mathbb{P}\left(\bigcap_{0 \leq i' \leq i-1} \{ S^{m,n}_{i',n} = T^m_{i'+1} \} \cap \{ S^{m,n}_{i,n} < T^m_{i+1} \} \right) \leq \frac{Cm}{n^{1-2\alpha}} .$$

We shall show that the stopping time $R^{m,n}$ is, for n large enough, the answer to our question.

We have constructed an increasing sequence of $m(n+1)$ stopping times

$$0 = T^{m,n}_{0,0} \leq T^{m,n}_{0,1} \leq \ldots \leq T^{m,n}_{0,n} = T^{m,n}_{1,0} \leq T^{m,n}_{1,1} \leq \ldots \leq T^{m,n}_{m-1,n} = R^{m,n}$$

such that if S and T belong to this sequence and are consecutive, then $\delta(X_S, X_T) \leq \frac{1}{n^\alpha}$. Moreover, between $T^{m,n}_{i,0}$ and $T^{m,n}_{i+1,0}$, the law of X conditionned by $\mathscr{F}_{T^{m,n}_{i,0}}$ is carried by a (random) manifold with strong 2-convex geometry. This increasing sequence of stopping times will be denoted by $\tau^{m,n}$.

Lemma 5.3. – *Let $p \geq 1$ and let N be a manifold with strong p-convex geometry.*

Then there exists a constant $C' > 0$, such that for any martingale Y with values in N and for any sequence of stopping times

$$\tau = \tau(q) = \{S_0 = 0 \leq S_1 \leq \ldots \leq S_q\}$$

which satisfies

$$\sup_{S_i \leq t \leq S_{i+1}} \delta(Y_{S_i}, Y_t) \leq \frac{1}{q^\alpha} \quad ,$$

if $Y^{discr(\tau)}$ is the exponential discrete martingale with terminal value Y_{S_q}, then almost surely,

$$\sup_{t \in [0, S_q]} \delta(Y_t, Y_t^{discr(\tau)}) \leq \frac{C'}{q^{3\alpha - 1}} \quad .$$

Proof. – The proof goes as in [A2] proposition 3.3, following the construction of Picard ([P2], proof of théorème 6.3). Since $\delta(Y_{S_{q-1}}, Y_{S_q}) \leq \frac{1}{q^\alpha}$, we have by [A2] proposition 2.15, $\delta(Y_{S_{q-1}}, Y_{S_{q-1}}^{discr(\tau)}) \leq \frac{C}{q^{3\alpha}}$ where C is a constant depending only on the manifold. Define $\tau(q-1) = \{S_0 = 0 \leq S_1 \leq \ldots \leq S_{q-1}\}$ and $Y^{discr(\tau(q-1))}$ the discrete exponential martingale with terminal value $Y_{S_{q-1}}$. Then $(Y^{discr(\tau(q))}, Y^{discr(\tau(q-1))})$ is a discrete exponential martingale up to time S_{q-1}, and since N has strong p-convex geometry, for any Riemannian distance δ, there exists a constant C'' depending only on N and δ, such that

$$\begin{aligned} \delta^p\left(Y_0^{discr(\tau(q-1))}, Y_0^{discr(\tau(q))}\right) &\leq C'' \mathbb{E}\left[\delta^p\left(Y_{S_{q-1}}^{discr(\tau(q-1))}, Y_{S_{q-1}}^{discr(\tau(q))}\right)\right] \\ &\leq C''\left(C\frac{1}{q^{3\alpha}}\right)^p, \end{aligned}$$

which yields

$$\delta\left(Y_0^{discr(\tau(q-1))}, Y_0^{discr(\tau(q))}\right) \leq C''^{\frac{1}{p}}\left(C\frac{1}{q^{3\alpha}}\right) \quad .$$

Replacing S_q by S_{q-1}, and then S_{q-1} by S_{q-2} and so on, we obtain q similar inequalities, and we add them, It yields

$$\delta\left(Y_0, Y_0^{discr(\tau(1))}\right) + \cdots + \delta\left(Y_0^{discr(\tau(q-1))}, Y_0^{discr(\tau(q))}\right) \leq qC''^{\frac{1}{p}}\left(C\frac{1}{q^{3\alpha}}\right)$$

and hence

$$\delta\left(Y_0, Y_0^{discr(\tau)}\right) \leq C''^{\frac{1}{p}}\left(C\frac{1}{q^{3\alpha-1}}\right)$$

This proves the inequality of the lemma at time 0 and similarly, at time $S_j, 0 \leq j \leq q$. It is not difficult to extend this inequality for all times since $\frac{q^{3\alpha-1}}{q^\alpha}$ tends to 0 as q tends to ∞.

Remarks. – 1) Since $\alpha > \frac{1}{3}$ we have $3\alpha - 1 > 0$ and the upper bound tends to 0 as q tends to ∞.

2) In general, a subdivision $\tau(q)$ does not exist up to time 1. It exists up to a time which is strictly less than 1 with a probability tending to 0 as q tends to ∞.

We are now able to finish the proof of theorem 5.1.

Let C'' be a constant such that for each $x \in K$, for each ball of radius $\frac{2}{m^\alpha}$ and of center x, the function ψ defining the 2-convexity of this ball satisfies $a\delta^2 \leq \psi \leq A\delta^2$ with $\frac{A}{a} \leq C''$. Define $b(n) = C' \frac{1}{n^{3\alpha-1}}$ where C' is as in lemma 5.3 (here C' depends on all the balls of radius $\frac{2}{m^\alpha}$ centered in K). Suppose that n is big enough so that the inequality

$$m\sqrt{C''}^m b(n) < \frac{1}{m^\alpha} \tag{9}$$

is satisfied. Then it will be shown that the discrete exponential $\tau^{m,n}$-martingale $Z^{m,n}$ with terminal value $X_{R^{m,n}}$ exists (where $\tau^{m,n}$ is the subdivision constructed before lemma 5.3). It exists clearly from time $T^{m,n}_{m-1,0}$ to time $R^{m,n}$, and by lemma 5.3, we have

$$\delta\left(X_{T^{m,n}_{m-1,0}}, Z^{m,n}_{T^{m,n}_{m-1,0}}\right) \leq C' \frac{1}{n^{3\alpha-1}} = b(n) \ . \qquad (10)(m)$$

Using condition (9), inequality (10)(m) and the fact that balls of radius $\frac{2}{m^\alpha}$ have strong 2-convex geometry, the construction of $Z^{m,n}$ can be performed on $[T^{m,n}_{m-2,0}, T^{m,n}_{m-1,0}]$ since the conditional law $\mathscr{L}\left(Z^{m,n}_{T^{m,n}_{m-1,0}} | \mathscr{F}_{T^{m,n}_{m-2,0}}\right)$ is almost surely included in a ball of radius $\frac{2}{m^\alpha}$ and centered at $X_{T^{m,n}_{m-2,0}}$.

Denote by $\tau^{m-1,n}$ the subdivision $\tau^{m,n}$ stopped at time $T^{m,n}_{m-1,0} (= T^{m,n}_{m-2,n})$. In the same way, the discrete exponential $\tau^{m-1,n}$-martingale $Z^{m-1,n}$ with terminal value $X_{T^{m,n}_{m-1,0}}$ exists from time $T^{m,n}_{m-2,0}$ to time $T^{m,n}_{m-1,0}$ and from the strong 2-convex geometry of balls centered in $X_{T^{m,n}_{m-2,0}}$ and of radius $\frac{2}{m^\alpha}$ we obtain that $(Z^{m-1,n}, Z^{m,n})$ is an exponential discrete martingale on $[T^{m,n}_{m-2,n}, T^{m,n}_{m-1,n}]$. It yields

$$\delta\left(Z^{m-1,n}_{T^{m,n}_{m-2,0}}, Z^{m,n}_{T^{m,n}_{m-2,0}}\right) \leq \sqrt{C''} b(n) \ . \tag{11}$$

Like (10)(m), the following inequality (10)($m-1$) is valid:

$$\delta\left(X_{T^{m,n}_{m-2,0}}, Z^{m-1,n}_{T^{m,n}_{m-2,0}}\right) \leq C' \frac{1}{n^{3\alpha-1}} \ , \qquad (10)(m-1)$$

and inequalities (10)($m-1$) and (11) yield

$$\delta\left(X^{m,n}_{T^{m,n}_{m-2,0}}, Z^{m,n}_{T^{m,n}_{m-2,0}}\right) \leq b(n)(1+\sqrt{C''}) \leq m\sqrt{C''}^m b(n) < \frac{1}{m^\alpha} \ .$$

It implies that the conditional law $\mathscr{L}\left(Z^{m,n}_{T^{m,n}_{m-2,0}} | \mathscr{F}_{T^{m,n}_{m-3,0}}\right)$ is almost surely included in a ball of radius $\frac{2}{m^\alpha}$ and centered in $X_{T^{m,n}_{m-3,0}}$.

By the same method, defining $Z^{m-2,n}$, discrete martingales can be constructed from time $T^{m,n}_{m-3,0}$ to time $T^{m,n}_{m-2,0}$ and one obtains

$$\delta\left(X_{T^{m,n}_{m-3,0}}, Z^{m,n}_{T^{m,n}_{m-3,0}}\right)$$
$$\leq \delta\left(X_{T^{m,n}_{m-3,0}}, Z^{m-2,n}_{T^{m,n}_{m-3,0}}\right) + \delta\left(Z^{m-2,n}_{T^{m,n}_{m-3,0}}, Z^{m-1,n}_{T^{m,n}_{m-3,0}}\right) + \delta\left(Z^{m-1,n}_{T^{m,n}_{m-3,0}}, Z^{m,n}_{T^{m,n}_{m-3,0}}\right)$$
$$\leq b(n)\left(1+\sqrt{C''}+\sqrt{C''}^2\right).$$

By iteration, one can construct $Z^{m,n}$ on the whole subdivision $\tau^{m,n}$ and one has

$$\delta(X_0, Z_0^{m,n}) \leq b(n)\left(1+\sqrt{C''}+\sqrt{C''}^2+\ldots+\sqrt{C''}^m\right) \leq m\sqrt{C''}^m b(n) < \frac{1}{m^\alpha} \tag{12}$$

using (9). It is obvious that in (12) the time 0 can be replaced by any element of $\tau^{m,n}$.

Let $\varepsilon > 0$. Choose m such that every ball of radius $\frac{2}{m^\alpha}$ is included in a manifold with strong 2-convex geometry, $\mathbb{P}(T_m^m \neq 1) < \frac{\varepsilon}{2}$ and $\frac{1}{m^\alpha} < \frac{\varepsilon}{2}$, choose n_0 such that $m\sqrt{C''}^m b(n_0) < \frac{\varepsilon}{2}$, $n > n_0$ such that $\frac{1}{n^\alpha} < \frac{\varepsilon}{2}$ and $\mathbb{P}(R^{m,n} < T_m^m) < \frac{\varepsilon}{2}$. Define $T_\varepsilon = R^{m,n}$. We have

$$\|\sup_{t\leq T_\varepsilon}\delta(Z_t^{m,n}, X_t)\|_\infty \leq$$
$$\left\|\sup_{T\in\tau^{m,n}}\delta(Z_T^{m,n}, X_T)\right\|_\infty + \left\|\sup_{T_i\in\tau^{m,n},\ t\in[T_i,T_{i+1}]}\delta(X_{T_i}, X_t)\right\|_\infty < \varepsilon\ . \tag{13}$$

This proves the theorem. □

Remarks. – 1) Although T_ε is strictly less than 1 with probability less than ε, we can not expect to obtain a uniform convergence of the discrete martingales almost surely for $0 \leq t \leq 1$. We can not even define discrete martingales up to time 1.
2) If $(\Omega, \mathscr{F}_1, \mathbb{P})$ is such that conditional laws with respect to any σ-field included in $\mathscr{F}_1$ exist, then the conclusion of theorem 5.1 and inequality (13) are still valid if we replace $Z^{m,n}$ by a convex discrete martingale with terminal value X_{T_ε}. Proposition 2.9 is then required at different steps of the proof.

6. Construction of holomorphic families of continuous martingales with prescribed terminal value

Let M be a manifold. Set $T^{(0)}M = M$ and for $n \geq 1$, define by induction $T^{(n)}M = TT^{(n-1)}M$. If ∇ is a connection on M, set $\nabla^{(0)} = \nabla$, and for $n \geq 1$, denote by $\nabla^{(n)}$ the complete lift in $T^{(n)}M$ of the connection $\nabla^{(n-1)}$ in $T^{(n-1)}M$.

For $n \in \mathbb{N}^*$, define $\pi_n : T^{(n)}M \to T^{(n-1)}M$ the canonical projection. Define for $x_0 \in M$,

$$M^n = M^n(x_0) = \{w \in T^{(n)}M, \pi_1 \circ \pi_2 \circ \ldots \circ \pi_n(w) = x_0\}$$

and by induction $T_S^{(0)}M = M, T_S^{(1)}M = TM$ and for $n \geq 2$

$$T_S^{(n)}M = \{w \in TT_S^{(n-1)}M, \pi_n(w) = \pi_{n-1_*}(w)\} \ .$$

Define for $n \geq 1$,

$$M_S^n = M^n \cap T_S^{(n)}M \ .$$

Note that the dimension of M_S^n is nd and that $T_S^{(n)}M$ is the set of n-th derivatives of smooth paths with values in M. A point of $T_S^{(n)}M$ can also be seen as an equivalence class of smooth curves in M, for the equivalence relation given by n-th order tangency at one point of M.

Lemma 6.1. – *If $\psi : (V, \nabla^V) \to (W, \nabla^W)$ is an affine mapping between two manifolds endowed with torsion-free connections, then $\psi_* : (TV, \nabla^{V'}) \to (TW, \nabla^{W'})$ is affine.*

Proof. – It is sufficient to show that the images by ψ_* of geodesics in TV are geodesics in TW.

Let $t \mapsto J(t)$ be a geodesic in TV defined on $[0, \varepsilon)$. If $\varepsilon > 0$ is small enough, then there exists a smooth map $(a, t) \mapsto \gamma(a, t)$ defined on $[0, \varepsilon') \times [0, \varepsilon)$ such that for every $a \in [0, \varepsilon'), t \mapsto \gamma(a, t)$ is a geodesic in V and $\frac{\partial}{\partial a}|_{a=0}\gamma(a, t) = J(t)$.

Since ψ is affine, for every $a \in [0, \varepsilon'), t \mapsto \psi(\gamma(a, t))$ is a geodesic in W and therefore $t \mapsto \frac{\partial}{\partial a}|_{a=0}\psi(\gamma(a, t)) = \psi_*(J(t))$ is a $\nabla^{W'}$-geodesic in TW. □

Lemma 6.2. – *If V and W are two manifolds with a connection, $\psi : V \to W$ is an affine submersion and if W' is a totally geodesic submanifold of W, then $\psi^{-1}(W')$ is a totally geodesic submanifold of V.*

Proof. – The set $V' = \psi^{-1}(W')$ is a submanifold of V since ψ is a submersion. Let $x \in V', u \in T_xV'$ and denote by γ the geodesic in V which satisfies $\dot{\gamma}(0) = u$. Then since ψ is affine, $\psi \circ \gamma$ is a geodesic in W with initial condition $(\psi \circ \gamma)'(0) = T_x\psi(u)$ and this vector belongs to $T_{\psi(x)}W'$. It implies that $\psi \circ \gamma$ is a geodesic in W' and γ is a geodesic in V'. This proves that V' is totally geodesic. □

Lemma 6.1 and lemma 6.2 yield:

Proposition 6.3. – *Assume that the manifold M is endowed with a connection ∇. Then the submanifolds $(M^n, \nabla^{(n)}), (T_S^{(n)}M, \nabla^{(n)})$ $(M_S^n, \nabla^{(n)})$ are totally geodesic in $T^{(n)}M$.*

The manifolds M_S^n are the ones we are interested in, because the n-th derivatives at $a = 0$ of families of martingales defined by an equation like (0) live in M_S^n. Assume that M is the domain of a chart centered in x_0. An element $w^n \in M_S^n$ (resp. $w^n \in T_S^{(n)}M$) will be denoted by $(w_1, \ldots, w_n)$ (resp. $(x, w_1, \ldots, w_n)$) in canonical coordinates, forgetting the repetitions of coordinates.

So as to show that equations (0) and (1) in section 1 define a martingale $X(a)$, we have to show that the series (0) converges, and that if it converges, the sum is a ∇-martingale. First we give an answer to the second point.

Assume $M = B(0,R)$ is an holomorphic manifold endowed with an holomorphic connection and that we are given a holomorphic family

$$X(a) = X(0) + \sum_{n=1}^{\infty} \frac{a^n}{n!} W_n(\cdot) \tag{14}$$

of processes such that the series converges, $X(0)$ is a ∇-martingale, and for each $n \geq 1$, the process $W^n \in T_S^{(n)}M$ with coordinates $(X(0), W_1, \ldots, W_n)$ is a $\nabla^{(n)}$-martingale. The question arises whether $X(a)$ is a ∇-martingale, and with the help of corollary 4.4 and theorem 5.1, it is possible to give the following answer:

Theorem 6.4. – *Assume that $M = B(0,R) \subset \mathbb{C}^d$ is a complex manifold endowed with an holomorphic connection ∇, such that the holomorphic Christoffel symbols are defined on M (see section 4 for details). Assume that almost surely the series* (14) *converges absolutely and uniformly in t, ω on the closed disc $\bar{D}(0,A) \subset \mathbb{C}$ of center 0 and radius A and that there exists a compact subset V of M with strong convex geometry such that almost surely, for all $(t,a) \in [0,1] \times \bar{D}(0,A), X_t(a)$ takes its values in V.*

Then for every a belonging to the closed disc $\bar{D}(0,A), X(a)$ is a ∇-martingale.

Proof. – It is sufficient to obtain the result for every a belonging to the open disc $D(0,A)$, since we can then obtain it in the closed disc using the uniform convergence. According to [A2] proposition 2.12 (one verifies that the strong convex geometry assumption of definition 2.4 is sufficient to apply this result) and [A1] proposition 3.4, it is sufficient to show that for any function $f \in \mathscr{C}(V)$, for all $a \in D(0,A), f(X(a))$ is a real submartingale. Let f be such a function and $s,t \in [0,1]$ such that $s \leq t$. We are going to show that for all $a \in D(0,A), f(X_s(a)) \leq \mathbb{E}[f(X_t(a))|\mathscr{F}_s]$.

Let $\varepsilon > 0, A' \in (0,A)$ and $N \in \mathbb{N}$ such that almost surely, for all $a \in \bar{D}(0,A)$, for all u,

$$\left\| X_u(a) - X_u(0) - \sum_{n=1}^{N} \frac{a^n}{n!} W_n(u) \right\| < \varepsilon \tag{15}$$

and such that for every increasing sequence of stopping times $\tau = \{T_0 \leq T_1 \leq \ldots \leq T_k\}$ and every holomorphic family $L(a)$ of random variables defined on $\bar{D}(0,A)$, with values in V, such that $L(a) = Y_{T_k}^{discr(\tau)}(a)$ where $Y^{discr(\tau)}(a) = Y^{discr(\tau)}(0) + \sum_{n=1}^{\infty} \frac{a^n}{n!} W_n^{discr(\tau)}(\cdot)$ is an exponential discrete martingale, we have for $a \in \bar{D}(0,A')$

$$\left\| Y^{discr(\tau)}(a) - Y^{discr(\tau)}(0) - \sum_{n=1}^{N} \frac{a^n}{n!} W_n^{discr(\tau)}(\cdot) \right\| < \varepsilon \ . \tag{16}$$

Note that this is possible because of the majoration of corollary 4.4, which gives

$$\left\| \sum_{n=N+1}^{\infty} \frac{a^n}{n!} W_n^{discr(\tau)} \right\| \leq R \frac{\left(\frac{A'}{A}\right)^{N+1}}{1 - \frac{A'}{A}}$$

as soon as $|a| \leq A'$.

By theorem 5.1, there exists a stopping time T with $s \leq T \leq t$ and $\mathbb{P}(T \neq t) < \varepsilon$ and a subdivision τ of $[s, T]$ such that the discrete exponential martingale $W^{N,discr(\tau)}$ with values in $T_S^{(n)}M$ and with terminal value $W_T^{N,discr(\tau)} = W_T^N$, with coordinates $(X^{discr(\tau)}(0), W_1^{discr(\tau)}, \ldots, W_N^{discr(\tau)})$ exists and satisfies:

$$\forall n \in \{0, 1, \ldots, N\}, \qquad \left\| W_n^{discr(\tau)} - W_n \right\| \leq n! \frac{\varepsilon}{(N+1)A^n} \tag{17}$$

where one defines $W_0^{discr(\tau)} = (W^0)^{discr(\tau)} = X^{discr(\tau)}(0)$. Since the canonical projections $T_S^{(n)}M \to T_S^{(n-1)}M$ are affine, a consequence of proposition 2.10 is that $\pi_n\big((W^n)^{discr(\tau)}\big) = (W^{n-1})^{discr(\tau)}$ for all $n \in \{1, \ldots, N\}$. We can assume that for all $n \in \{1, \ldots, N\}$, if T_k and T_{k+1} are two times of the discretisation, then the random variable $W_{T_{k+1}}^{n,discr(\tau)}$ takes its values in a random $\mathscr{F}_{T_k}$-measurable set $V(n, k, \omega)$ with strong convex geometry and that $\langle T\mathscr{E}\big(W_{T_{k+1}}^{n-1,discr(\tau)} | \mathscr{F}_{T_k}\big), W_{T_{k+1}}^{n,discr(\tau)} \rangle$ belongs to $V(n, k, \omega)$. The assumptions of proposition 3.8 are fulfilled and we deduced that

$$W_{T_k}^{n,discr(\tau)} = \left\langle T\mathscr{E}\left(W_{T_{k+1}}^{n-1,discr(\tau)} | \mathscr{F}_{T_k}\right), W_{T_{k+1}}^{n,discr(\tau)} \right\rangle .$$

By induction on the indexes of the subdivision τ (going from the upper index to the lower index) and by induction on the derivatives, we deduce that for $n \in \{1, \ldots, N\}$, the process $(W^n)^{discr(\tau)}$ is the n-th derivative of $X^{discr(\tau)}(a)$ at time $a = 0$. Hence, since the family of discrete exponential martingales $X^{discr(\tau)}(a)$ with terminal values $X_T(a)$ is holomorphic in a, there exist $\mathbb{C}^d$-valued discrete processes $W_n'^{discr(\tau)}(\cdot), n > N$ such that the processes $X^{discr(\tau)}(a)$ write

$$X^{discr(\tau)}(a) = X^{discr(\tau)}(0) + \sum_{n=1}^{N} \frac{a^n}{n!} W_n^{discr(\tau)}(\cdot) + \sum_{n=N+1}^{\infty} \frac{a^n}{n!} W_n'^{discr(\tau)}(\cdot) .$$

Putting together (15), (16) and (17), we obtain that on $[s, T]$, for all $a \in D(0, A')$, the distance between $X^T(a)$ and $X^{discr(\tau)}(a)$ is less than 3ε. But $X^{discr(\tau)}(a)$ is a discrete martingale with values in K and hence $f\big(X^{discr(\tau)}(a)\big)$ is a bounded discrete submartingale on $[s, T]$, which yields $f\big(X_s^{discr(\tau)}(a)\big) \leq \mathbb{E}\big[f\big(X_T^{discr(\tau)}(a)\big) | \mathscr{F}_s\big]$. By letting ε tend to 0, and using the fact that f is bounded on V, we deduce that $f(X_s(a)) \leq \mathbb{E}[f(X_t(a)) | \mathscr{F}_s]$. □

We are going to investigate the convergence of the series (0) of section 1. It will be shown that the martingales W^n live in compacts of manifolds with strong convex geometry, and this will give us the appropriate bounds.

The following result is immediate:

Proposition 6.5. – *Let M be a manifold with connection and $x_0 \in M$. Then for each $n \geq 1$, given two points in $M_S^n = M_S^n(x_0)$, there exists one and only one geodesic living in M_S^n and joining these two points.*

Proof. – The equation of geodesics in a coordinates system of M_S^n gives the same induction relation as in section 1 with martingales and it is possible to give explicit solutions. One obtains

$$w_n(t) = w_n(0) + t\left(w_n(1) - w_n(0) - \int_0^1 ds \int_0^s du \ddot{w}_n(u)\right) + \int_0^t ds \int_0^s du \ddot{w}_n(u) \ ,$$

and as equation (18) below shows it, $\ddot{w}_n$ is a polynomial in t if for all $i \leq n-1, w_i$ and $\dot{w}_i$ are polynomials in t.

Proposition 6.6. – *Let M be a manifold with connection, $x_0 \in M$, and for $n \geq 1$, denote by $(w_1, \ldots, w_n)$ the coordinates in M_S^n inherited from a chart in a neighbourhood of x_0. Then for each $n \geq 1$, there exists a convex function $h_n : M_S^n \to \mathbb{R}_+$ such that*

(*i*) *h_n^2 is a polynomial in $\|w_i\|, i \in \{1, \ldots, n\}$, without any monomial of order 0 or 1.*
(*ii*) *$h_n \geq \|w_n\|$,*
(*iii*) *there exists a polynomial $P_n(X_1, \ldots, X_n)$ such that for any geodesic $w^n(t)$ in M_S^n,*

$$\left(((h_n \circ w^n)^2)'_+(t)\right)^2 \leq P_n(\|w_1(t)\|, \ldots, \|w_n(t)\|)(\|\dot{w}_1(t)\|^2 + \ldots + \|\dot{w}_n(t)\|^2) \ ,$$

(iv) *for any geodesic $w^n(t)$* in M_S^n, for all $p \in \mathbb{N}, p \geq 2$, *the measure $(h_n^p \circ w^n)''$ is greater than the measure in $\mathbb{R}$ with density $p\|w_n\|^{p-2}\|\dot{w}_n\|^2$.*

Remark. – Let $n, p \in \mathbb{N}^*$. If $h_n : M_S^n \to \mathbb{R}_+$ is defined, then it extends to a convex function on M_S^{n+p} by composition by the affine mapping $\pi_{n+1} \circ \ldots \pi_{n+p} : M_S^{n+p} \to M_S^n$. It will be still denoted by h_n.

Proof of proposition 6.6. – Let us prove this proposition by induction.

For $n = 1$, take $h_1(w^1) = \|w_1\|$.

Let $n \in \mathbb{N}^*$, and suppose that $h_1, \ldots, h_{n-1}$ exist. Let $w^n(t)$ be a geodesic in M_S^n. A dimension argument together with the definition of $\nabla^{(n)}$ show that $w_n(t)$ is the n-th order derivative at $a = 0$ of a family (indexed by a) of geodesics $\gamma(t, a)$ such that for all $t, \gamma(t, 0) = x_0$. Differentiating n times the equation $\ddot{\gamma} = -\Gamma_{jk}(\gamma)\dot{\gamma}^j\dot{\gamma}^k$ yields the equation for $w_n(t)$

$$\frac{1}{n!}\ddot{W}_n = -\sum_{\substack{r \geq 0, p,q > 0 \\ \alpha_1 + \ldots + \alpha_r + p + q = n \\ 1 \leq i_1, \ldots, i_r \leq d}} C_{\alpha,p,q,i} D_i \Gamma_{jk}(x_0) w_{\alpha_1}^{i_1} \ldots w_{\alpha_r}^{i_r} \dot{w}_p^j \dot{w}_q^k \tag{18}$$

where $\alpha = (\alpha_1, \ldots, \alpha_r), i = (i_1, \ldots, i_r)$ and $C_{\alpha,p,q,i}$ are constants uniquely determined. Note that the right hand side involves only indexes less than $n-1$.

From the inequality

$$|a_1 a_2 \dots a_{n-1} a_n| \leq 1 + a_1^4 + a_2^8 + \dots + a_{n-1}^{2n} + a_n^{2^n}$$

which comes from the iteration of $|ab| \leq a^2 + b^2$ (and using $a_0 = 1$), we obtain that each term $|C_{\alpha,p,q,i} D_i \Gamma_{jk} w_{\alpha_1}^{i_1} \dots w_{\alpha_r}^{i_r} \dot{w}_p^j \dot{w}_q^k|$ is bounded above by a sum of terms of the form $C\|w_i\|^m \|\dot{w}_j\|^2$ with $m \in 2\mathbb{N}, i, j, \in \{1, \dots, n-1\}$. But $\|w_i\|^m \|\dot{w}_j\|^2$ is bounded above by the second derivative of $\frac{1}{2}(h_i^m + h_j^2)^2 \circ w^n(t)$. Indeed, this second derivative is equal to

$$\left(\left((h_i^m \circ w^i)'_+(t)\right) + \left((h_j^2 \circ w^j)'_+(t)\right)\right)^2$$
$$+ \left(h_i^m(w^i) + h_j^2(w^j)\right)\left((h_j^m \circ w^i)'' + (h_j^2 \circ w^j)''\right)$$

(where $(\cdot)'_+$ denotes a right derivative) and the term in the right is greater than $2\|w_i\|^m \|\dot{w}_j\|^2$ by the induction relation. The second derivative to $t \mapsto \|w_n(t)\|$ is

$$\frac{1}{\|w_n\|^3}\left(\|w_n\|^2 \|\dot{w}_n\|^2 - \langle w_n, \dot{w}_n \rangle^2\right) + \left\langle \frac{w_n}{\|w_n\|}, \ddot{w}_n \right\rangle$$

added to a non-negative Dirac mass at $\|w_n\| = 0$; the first term is non-negative and $\left\langle \frac{w_n}{\|w_n\|}, \ddot{w}_n \right\rangle$ is greater than

$$- \sum_{\substack{r \geq 0,\, p,q > 0 \\ \alpha_1 + \dots + \alpha_r + p + q = n \\ 1 \leq i_1, \dots, i_r \leq d}} |C_{\alpha,p,q,i} D_i \Gamma_{jk} w_{\alpha_1}^{i_1} \dots w_{\alpha_r}^{i_r} \dot{w}_p^j \dot{w}_q^k| \ .$$

This means that there exists a sum $f_n(w^{n-1})$ of terms $C(h_i^m + h_j^2)^2$, $m \in 2\mathbb{N},\ i, j \in \{1, \dots, n-1\}, C > 0$, such that

$$f_n(w^{n-1})'' \geq \sum_{\substack{r \geq 0,\, p,q > 0 \\ \alpha_1 + \dots + \alpha_r + p + q = n \\ 1 \leq i_1, \dots, i_r \leq d}} |C_{\alpha,p,q,i} D_i \Gamma_{jk} w_{\alpha_1}^{i_1} \dots w_{\alpha_r}^{i_r} \dot{w}_p^j \dot{w}_q^k|$$

and $w^n \mapsto \|w_n\| + f_n(w^{n-1})$ is convex. This is not exactly the function we are looking for, because the second derivative of its square is not big enough: let us denote by $\dot{w}_n^t$ the tangential part $\left\langle \frac{w_n}{\|w_n\|}, \dot{w}_n \right\rangle$ of $\dot{w}_n$ and by $\dot{w}_n^\circ = \dot{w}_n - \dot{w}_n^t \frac{w_n}{\|w_n\|}$ its normal part. The right derivative of $t \mapsto \left(\|w_n(t)\| + f_n(w^{n-1}(t))\right)^2$ is

$$2\left(\|w_n(t)\| + f_n(w^{n-1}(t))\right)\left(\left\langle \frac{w_n}{\|w_n\|}, \dot{w}_n \right\rangle + (f_n \circ w^{n-1})'_+\right) \tag{19}$$

if $\|w_n(t)\| \neq 0$. The second derivative of $t \mapsto \frac{1}{2}\left(\|w_n(t)\| + f_n(w^{n-1}(t))\right)^2$ is

$$\left(\dot{w}_n^t + (f_n \circ w^{n-1})'_+\right)^2 + \left(\|w_n\| + f_n(w^{n-1})\right)\left(\frac{\|\dot{w}_n^\circ\|^2}{\|w_n\|} + \left\langle \frac{w_n}{\|w_n\|}, \ddot{w}_n \right\rangle + (f_n \circ w^{n-1})''\right)$$

added to a non-negative Dirac mass at $\|w_n\| = 0$, and it is greater than

$$\left(\dot{w}_n^t + (f_n \circ w^{n-1})'\right)^2 + \|\dot{w}_n^\circ\|^2 \ .$$

Applying the inequality $(a+b)^2 \geq \frac{3}{4}a^2 - 3b^2$ to the first term, one obtains that the second derivative of $t \mapsto \left(\|w_n(t)\| + f_n(w^{n-1}(t))\right)^2$ is greater than

$$\|\dot{w}_n\|^2 - 6(f_n \circ w^{n-1})'^2_+ \ .$$

The right derivative of $(h_i^m + h_j^2)^2 \circ w^{n-1}$ is

$$2(h_i^m + h_j^2)\left(\frac{m}{2}h_i^{m-2}\left((h_i \circ w^i)^2\right)'_+ + \left((h_j \circ w^j)^2\right)'_+\right) \ . \tag{20}$$

Hence using (iii) and (i) of the induction relation at the order $n-1$, it is possible to bound $(f_n \circ w^{n-1})'^2$ by a sum of terms $C\|w_i\|^m\|\dot{w}_j\|^2$ with $m \in 2\mathbb{N}, i,j \in \{1,\ldots,n-1\}$. Adding to $\left(\|w_n\| + f_n(w^{n-1})\right)^2$ a sum $S_n(w^{n-1})$ of terms of the form $C(h_i^m + h_j^2)^2, m \in 2\mathbb{N}, i,j \in \{1,\ldots,n-1\}, C > 0$, one obtains a convex function with second derivative greater than $\|\dot{w}_n\|^2$. Define

$$h_n(w^n) = \sqrt{(\|w_n\| + f_n(w^{n-1}))^2 + S_n(w^{n-1})}$$

and let us show that it is convex and satisfies (i) to (iv):

Lemma 6.7. – *Let $f, g : [0,1] \to \mathbb{R}$ be two convex non-negative functions. Then $\sqrt{f^2+g^2}$ is convex.*

Proof. – Let $h = \sqrt{f^2+g^2}$. Since f and g are convex, we have for $t \in [0,1],\ a,b \in [0,1]$,

$$h((1-t)a+tb) \leq \sqrt{((1-t)f(a)+tf(b))^2 + ((1-t)g(a)+tg(b))^2} \ ,$$

the right hand side term is

$$\sqrt{(1-t)^2h^2(a) + t^2h^2(b) + 2t(1-t)(f(a)f(b)+g(a)g(b))}$$

and this is less than

$$\sqrt{(1-t)^2h^2(a) + t^2h^2(b) + 2t(1-t)h(a)h(b)} = (1-t)h(a) + th(b). \ \square$$

From lemma 6.7 and the shape of $S_n(w^{n-1})$, we deduce that h_n is convex.

Conditions (i), (ii) and (iv) are fulfilled by construction (remark that if (iv) is fulfilled at order 2, it is fulfilled at order $p \geq 2$). As for condition (iii), using the induction relation and (20), it is sufficient to bound the square of the derivative (19). But one can bound

$$\left(\left\langle \frac{w_n}{\|w_n\|}, \dot{w}_n \right\rangle + (f_n \circ w^{n-1})'_+\right)^2$$

by $2\left(\|\dot{w}_n\|^2 + \left((f_n \circ w^{n-1})'_+\right)^2\right)$ and it yields (iii). $\square$

Definition 6.8. – *Let $n \in \mathbb{N}^*$. On M_S^n, define*

$$f_n = h_1^2 + h_2^2 + \ldots + h_n^2 \ .$$

Then f_n is non-negative, convex, greater than $\|w^n\|^2$ and its Hessian is greater than the scalar product on TM_S^n.

As a consequence, one can give a stochastic version of proposition 6.5:

Proposition 6.9. – *Let M be a manifold endowed with a connection and $x_0 \in M$. Then for each $n \geq 1$, given a bounded random variable L^n with values in $M_S^n = M_S^n(x_0)$, there exists one and only one bounded martingale W^n with terminal value L^n. Furthermore, if in a chart $(W_1, \ldots, W_n)$ are the coordinates of W^n, and the decomposition of W_k for $1 \leq k \leq n$ into martingale and finite variation part is $M_k + \widetilde{W}_k$, then for all $p \geq 1$, the random variables $M_k(1)$ and $\int_0^1 |d\widetilde{W}_k(t)|$ belong to L^p.*

Proof. – Let $n \geq 1$ and L^n be a bounded random variable with values in M_S^n. For $1 \leq m \leq n$, denote by L^m the random variable $\pi_{m+1} \circ \ldots \pi_n(L^n)$, with coordinates $(L_1, \ldots, L_m)$. One can construct $W_k (1 \leq k \leq n)$ by induction. At rank 1, W_1 is only the bounded martingale with terminal value L_1. One assumes that up to rank m, the processes W_k are bounded and the random variables $M_k(1)$ and $\int_0^1 |d\widetilde{W}_k(t)|$ belong to L^p for all $p \geq 1$. With equation (4) at rank $m+1$, using the fact that all the coefficients in the right are less than m, using the induction assumptions and the Burkholder-Davis-Gundy inequalities, one obtains that $\int_0^1 |d\widetilde{W}_{m+1}(t)|$ belongs to L^p for all $p \geq 1$. Then using the relation $W_{m+1}(1) = L_{m+1}$ and the fact that the latter is bounded, we obtain that $M_{m+1}(1)$ belongs to L^p for all $p \geq 1$. Finally, using the fact that $f_{m+1}(W^{m+1})$ is a bounded submartingale, one obtains that W^{m+1} is bounded. □

Proposition 6.6 gives a sequence of convex functions h_n such that if $t \mapsto w^n(t)$ is a geodesic, the second derivative of $t \mapsto h_n^2 \circ w^n(t)$ is greater than $\|\dot{w}_n\|^2$. It will be useful to have a stochastic application of this result.

Proposition 6.10. – *Let M be a manifold endowed with a connection ∇, let b be a continuous non-negative section of the symmetric tensor product $T^*M \odot T^*M$ and let $f : M \to \mathbb{R}$ be a function such that for any geodesic γ in M, the second derivative $(f \circ \gamma)''$ is a measure and is greater than $b(\dot{\gamma}, \dot{\gamma})dt$.*

Then for any ∇-martingale X, for any stopping times S, T with $S \leq T$, we have

$$\mathbb{E}[f(X_T)|\mathscr{F}_S] - f(X_S) \geq \frac{1}{2}\mathbb{E}\left[\int_S^T b(dX_t, dX_t)|\mathscr{F}_S\right] .$$

Proof. – We will follow the proof of [E,Z] theorem 2 where the case $b = 0$ is considered.

Let g be a Riemannian metric on M and δ the associated Riemannian metric. We can assume that $S = 0$, T is bounded, X lives in a compact set and the Riemannian quadratic variation $\langle X|X\rangle_T$ is bounded. The fact that b is non-negative implies that f is convex. We can assume that on $[0, T]$, the martingale X lives in a small (and relatively compact) ball V of center X_0 and with strong convex geometry. Following [E,Z], we can assume that every point of V is the center of a normal chart $(e^i(a,b))_{1 \leq i \leq d}$, where $e^i(a,b)$ is the

i-th coordinate of $\exp_a^{-1}(b)$. We will denote by $\Gamma^i_{jk}(a,\cdot)$ the Christoffel symbols in the chart $e^i(a,\cdot)$. One can write

$$\begin{aligned} f(X_T)-f(X_0) \geq & \left\langle df, \overrightarrow{X_0X_T}\right\rangle \\ & + \int_0^1 dt \int_0^t ds \Big(b\Big(\exp_*\left(s\overrightarrow{X_0X_T}\right)\left(\overrightarrow{X_0X_T}\right), \exp_*\left(s\overrightarrow{X_0X_T}\right)\left(\overrightarrow{X_0X_T}\right)\Big)\Big) \end{aligned} \tag{21}$$

where $\left\langle df, \overrightarrow{X_0X_T}\right\rangle$ denotes a Gâteaux differential. Let $\varepsilon > 0$ and replace 0 and T by R_n and R_{n+1} where $(R_n)_{n\in\mathbb{N}}$ in such that $R_0 = 0$, and R_{n+1} is equal to

$$T \wedge (R_n + \varepsilon) \wedge \inf\left\{ t > R_n, \sup_{i,j,k} |\Gamma^i_{jk}(X_{R_n}, X_t)| \geq \varepsilon \right\} .$$

Then following the proof of [E,Z] theorem 2, one can write

$$\mathbb{E}[f(X_T) - f(X_0)] = \sum_{n\in\mathbb{N}} \mathbb{E}[f(X_{R_{n+1}}) - f(X_{R_n})] .$$

Using (21) and summing over n, we know from [E,Z] that the first term in the right hand side is greater than $-\varepsilon C\mathbb{E}[\langle X|X\rangle_T]$ where C is a constant depending only on the manifold. As for the second term, we are going to show that its expectation converges to $\frac{1}{2}\mathbb{E}\left[\int_0^T b(dX_t, dX_t)\right]$. Indeed, the absolute value of the difference between

$$\int_0^1 dt \int_0^t ds\, b\Big(\exp_*\left(s\overrightarrow{X_{R_n}X_{R_{n+1}}}\right)\left(\overrightarrow{X_{R_n}X_{R_{n+1}}}\right), \exp_*\left(s\overrightarrow{X_{R_n}X_{R_{n+1}}}\right)\left(\overrightarrow{X_{R_n}X_{R_{n+1}}}\right)\Big)$$

and $\frac{1}{2}b\left(\overrightarrow{X_{R_n}X_{R_{n+1}}}, \overrightarrow{X_{R_n}X_{R_{n+1}}}\right)$ is less than $h_1(\varepsilon)\delta^2(X_{R_n}, X_{R_{n+1}})$ where

$$h_1(\varepsilon) = \sup_{\|v\|\leq\varepsilon, \pi(v)\in V, \|u\|=1} |b(\exp_*(v)(u), \exp_*(v)(u)) - b(u,u)|$$

which tends to 0 as ε tends to 0. There exists a constant $C' > 0$ (see the proof of lemma 5.2) such that

$$\mathbb{E}[\delta^2(X_{R_n}, X_{R_{n+1}})] \leq C'\mathbb{E}\left[\langle X|X\rangle_{R_{n+1}} - \langle X|X\rangle_{R_n}\right] .$$

Summing over n, one can bound the term obtained by $C'h_1(\varepsilon)\mathbb{E}[\langle X|X\rangle_T]$. Hence it suffices to show that the expectation of

$$\frac{1}{2}\sum_{n\in\mathbb{N}} b\left(\overrightarrow{X_{R_n}X_{R_{n+1}}}, \overrightarrow{X_{R_n}X_{R_{n+1}}}\right)$$

converges to $\frac{1}{2}\mathbb{E}\left[\int_0^T b(dX_t, dX_t)\right]$ as ε tends to 0.

Let b_{ij} denote the coordinates of b inherited from the chart $e(X_{R_n}, \cdot)$. The difference

$$\left|\mathbb{E}\left[\int_{R_n}^{R_{n+1}} b(dX_t, dX_t)\right] - \mathbb{E}\left[b\left(\overrightarrow{X_{R_n}X_{R_{n+1}}}, \overrightarrow{X_{R_n}X_{R_{n+1}}}\right)\right]\right|$$

is bounded by

$$\left|\mathbb{E}\left[\int_{R_n}^{R_{n+1}} |b_{ij}(X_s) - b_{ij}(X_{R_n})| d\langle e^i(X_{R_n}, X), e^j(X_{R_n}, X)\rangle_s\right]\right|$$
$$+\left|\mathbb{E}\left[\int_{R_n}^{R_{n+1}} b_{ij}(X_{R_n}) d\langle e^i(X_{R_n}, X), e^j(X_{R_n}, X)\rangle\right.\right.$$
$$\left.\left.-b_{ij}(X_{R_n}) e^i(X_{R_n}, X_{R_{n+1}}) e^j(X_{R_n}, X_{R_{n+1}})\right]\right|$$

and since b is uniformly continuous (V is relatively compact), there exists a function $h_2(\varepsilon)$ tending to 0 as ε goes to 0 such that the first term is less than $h_2(\varepsilon)\mathbb{E}\left[\langle X|X\rangle_{R_{n+1}} - \langle X|X\rangle_{R_n}\right]$. As for the second term, since X is a martingale, it is bounded by

$$\left|\mathbb{E}\left[\int_{R_n}^{R_{n+1}} b_{ij}(X_{R_n})\left(\Gamma^i_{kl}(X_{R_n}, X) e^j(X_{R_n}, X)\right.\right.\right.$$
$$\left.\left.\left.+\Gamma^j_{kl}(X_{R_n}, X) e^i(X_{R_n}, X)\right) d\langle e^k(X_{R_n}, X), e^l(X_{R_n}, X)\rangle\right]\right|$$

and since the Christoffel symbols are bounded by ε in absolute value, $e^i(X_{R_n}, \cdot), e^j(X_{R_n}, \cdot), b_{ij}$ are bounded on V, there is a constant C'' such that this is bounded by $C''\varepsilon\mathbb{E}\left[\langle X|X\rangle_{R_{n+1}} - \langle X|X\rangle_{R_n}\right]$. Adding the majorations and summing over n, one obtains that there is a non-negative function $h_3(\varepsilon)$ tending to 0 as ε goes to 0, such that

$$\left|\mathbb{E}\left[\sum_{n\in\mathbb{N}} b\left(\overrightarrow{X_{R_n}X_{R_{n+1}}}, \overrightarrow{X_{R_n}X_{R_{n+1}}}\right)\right] - \mathbb{E}\left[\int_0^T b(dX_t, dX_t)\right]\right| \le h_3(\varepsilon)\mathbb{E}[< X|X >_T] \ .$$

This proves the proposition. □

From proposition 6.10 and proposition 6.9 and [K3], one obtains

Corollary 6.11. – *For every $n \ge 1$, if $M^{n\prime}$ is a compact subset of M^n_S, then $M^{n\prime}$ has convex geometry.*

Proof. – The hypothesis are not exactly the same as those of Kendall [K3] theorem 4.2, because the convex function on $M^{n\prime}$ we are going to use is not smooth. But Kendall's proof works in this case, because we only need that the expectation of the quadratic variation of $M^{n\prime}$-valued martingales is bounded by a constant depending only on $M^{n\prime}$, and here by proposition 6.10, we know that $2\sup_{M^{n\prime}} f_n$ is a bound for these quadratic variations. In fact, with [K4] corollary 3.4 which says that in a compact set, uniqueness of martingale with prescribed value implies convex geometry, one can avoid using proposition 6.10 and the strict convexity of $M^{n\prime}$. □

For the rest of this section, the manifold M is a ball $B(0,R) \subset \mathbb{C}^d$ endowed with an holomorphic connection such that the complex Christoffel symbols converge on M and vanish at 0 (see section 4). We furthermore assume that M has strong convex geometry. We are given an almost surely holomorphic map $a \mapsto L(a)(\omega)$ with values in M and defined on $\bar{D}(0,1)$, and such that $L(0)(\omega) \equiv 0 \in M$.

As a consequence of proposition 6.9, we have

Corollary 6.12. – *With the assumptions above, for every $n \geq 1$, there exists one and only one process $W^n(t)$ with values in M_S^n defined by the formal equations* (0) *and* (1) (*i.e.* W^n *is a* $\nabla^{(n)}$*-martingale*) *and by* $W^n(1) = L^{(n)}(0)$, *living in a compact set.*

Proof. – The random variable $L^{(n)}(0)$ is bounded because $a \mapsto L(a)$ is almost surely holomorphic and bounded. Hence we can apply proposition 6.9. □

It is not sufficient to show that $W_n(t)$ is a bounded process. We have to show that the series (0) converges. Convex functions inherited from the complex structure will be useful.

Assume that there exists a convex function $f{:}M = B(0,R) \to \mathbb{R}_+$ of the form $z \mapsto h(\|z\|)$ with $h : [0,R) \to \mathbb{R}_+$ continuous, satisfying $h(0) = 0$, strictly increasing. Note that since $\Gamma(0) = 0$, this condition is not too restrictive.

For each $n \geq 1$ and for each $w^n \in M_S^n$ with coordinates $(w_1, \ldots, w_n)$, define the set $E_n(w^n)$ of sequences $(w_{n+1}, \ldots)$ of elements of $\mathbb{C}^d$ such that for all $a \in D(0,1)$, $\sum_{k=1}^{\infty} \frac{a^k}{k!} w_k$ converges and the sum belongs to M; if $E_n(w^n)$ is not empty, define $g^n(w^n)$ by

$$g^n(w^n) = \inf_{(w_{n+1},\ldots)\in E_n} \sup_{a\in D(0,1)} f\left(\sum_{k=1}^{\infty} \frac{a^k}{k!} w_k\right) .$$

Proposition 6.13. – *For every $n \geq 1$, the function g^n is convex on*

$$M_S^{n\prime} = \{w^n \in M_S^n,\ E_n(w^n) \text{ is not empty}\} .$$

Proof. – Let v^n and w^n be two points of $M_S^{n\prime}$ with coordinates respectively $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_n)$, and $v^\infty = (v_1, \ldots, v_n,\ v_{n+1}, \ldots)$, $w^\infty = (w_1, \ldots, w_n, w_{n+1}, \ldots)$ two sequences such that $\forall a \in D(0,1), f\left(\sum_{k=1}^{\infty} \frac{a^k}{k!} v_k\right) < h(R)$ and $f\left(\sum_{k=1}^{\infty} \frac{a^k}{k!} w_k\right) < h(R)$. Define

$$x(a) = \sum_{k=1}^{\infty} \frac{a^k}{k!} v_k, \qquad y(a) = \sum_{k=1}^{\infty} \frac{a^k}{k!} w_k, \qquad x(t,a) = \exp t\overrightarrow{x(a)y(a)} .$$

Since $t \mapsto x(t,a)$ is a geodesic and $x(t,0) = 0$, we have by corollary 4.31)

$$\exp t\overrightarrow{x(a)y(a)} = \sum_{k=1}^{\infty} \frac{a^k}{k!} w_k(t)$$

where $w^k(t)$ is the geodesic in M_S^k such that $w^k(0) = v^k$ and $w^k(1) = w^k$.

We have to show that for all $t \in [0,1]$,

$$g^n(w^n(t)) \leq (1-t)g^n(v^n) + tg^n(w^n) .$$

But since f is convex, we have

$$f(x(t,a)) \leq (1-t)f(x(a)) + tf(y(a))$$

and hence

$$f(x(t,a)) \leq (1-t) \sup_{a\in D(0,1)} f(x(a)) + t \sup_{a\in D(0,1)} f(y(a))$$

and

$$\sup_{a\in D(0,1)} f(x(t,a)) \leq (1-t) \sup_{a\in D(0,1)} f(x(a)) + t \sup_{a\in D(0,1)} f(y(a))$$

which gives

$$g^n(w^n(t)) \leq (1-t) \sup_{a\in D(0,1)} f(x(a)) + t \sup_{a\in D(0,1)} f(y(a)) \ .$$

This inequality is true for all sequences v^∞, w^∞ which extend v^n, w^n and therefore

$$g^n(w^n(t)) \leq (1-t) g^n(v^n) + t g^n(w^n) \ .$$

The function g^n is convex. □

Proposition 6.14. – *For $r_0 \in [0,R)$, denote by $M_S^{n\prime}(r_0)$ the set of elements $w_n \in M_S^{n\prime}$ such that $g^n(w^n) \leq h(r_0)$. Let $w^\infty = (w_1, w_2, \ldots)$ be a sequence such that for all $n \geq 1$, the element w^n* in *M_S^n with* $= (w_1, \ldots, w_n)$ belongs to $M_S^{n\prime}(r_0)$.

Then $\sum_{k=1}^{\infty} \frac{a^k}{k!} w_k$ converges on $D(0,1)$ and

$$f\left(\sum_{k=1}^{\infty} \frac{a^k}{k!} w_k\right) \leq h(r_0) \ .$$

Proof. – Let $n \geq 1$. Since w^n belongs to $M_S^{n\prime}(r_0)$, for all $\alpha > 0$, there exists a sequence $w^{\infty\prime} = (w_1, \ldots, w_n, w'_{n+1}, \ldots)$ such that for all $a \in D(0,1)$,

$$\left\| \sum_{k=1}^{n} \frac{a^k}{k!} w_k + \sum_{k=n+1}^{\infty} \frac{a^k}{k!} w'_k \right\| \leq r_0 + \alpha \ .$$

It implies that for all $n \geq 1$, $\|w_n\| \leq n! r_0$ and for all $k \geq 1$, $\|w'_k\| \leq k!(r_0 + \alpha)$. Hence for all $n \geq 1$, for all $a \in D(0, 1-\varepsilon)$, we have

$$\left\| \sum_{k=n+1}^{\infty} \frac{a^k}{k!} w_k \right\| \leq r_0 \frac{(1-\varepsilon)^{n+1}}{\varepsilon}, \qquad \left\| \sum_{k=n+1}^{\infty} \frac{a^k}{k!} w'_k \right\| \leq (r_0 + \alpha) \frac{(1-\varepsilon)^{n+1}}{\varepsilon}$$

and

$$\begin{aligned}\left\| \sum_{k=1}^{\infty} \frac{a^k}{k!} w_k \right\| &\leq \left\| \sum_{k=1}^{n} \frac{a^k}{k!} w_k + \sum_{k=n+1}^{\infty} \frac{a^k}{k!} w'_k \right\| + \left\| \sum_{k=n+1}^{\infty} \frac{a^k}{k!} w_k \right\| + \left\| \sum_{k=n+1}^{\infty} \frac{a^k}{k!} w'_k \right\| \\ &\leq r_0 \left(1 + \left(2 + \frac{\alpha}{r_0} \right) \frac{(1-\varepsilon)^{n+1}}{\varepsilon} \right) + \alpha \ .\end{aligned}$$

This is valid for all $n \geq 1$ and all $\alpha > 0$, hence for all $a \in D(0, 1-\varepsilon)$,

$$\left\|\sum_{k=1}^{\infty}\frac{a^k}{k!}w_k\right\| \le r_0 \ .$$

This is true for all $\varepsilon > 0$, hence this is true for all $a \in D(0,1)$. □

Convex geometry is, as next lemma shows it, a useful tool to construct, given a convex set, convex non-negative functions which vanish only on this set:

Lemma 6.15. *Let N be a manifold with convex geometry and let A be a relatively compact convex subset of N.*

Then there exists a convex non-negative function ψ_A on N such that $\psi_A^{-1}(\{0\}) = \bar{A}$.

Proof. – Let $\psi : N \times N \to \mathbb{R}$ be a non-negative convex function which vanishes exactly on the diagonal of $N \times N$. Define

$$\psi_A : N \to \mathbb{R}, \qquad x \mapsto \inf_{y \in A} \psi(y,x) \ .$$

Since $\bar{A}$ is compact in N, it is clear that $\psi_A^{-1}(\{0\}) = \bar{A}$. Let us show that ψ_A is convex:

Let $t \mapsto \gamma(t)$ be a geodesic in N defined on [0,1]. We have to show that for all $t \in [0,1]$,

$$\psi_A(\gamma(t)) \le (1-t)\psi_A(\gamma(0)) + t\psi_A(\gamma(1)) \ . \tag{22}$$

Let $\varepsilon > 0$ and $y, z, \in A$ such that $\psi_A(\gamma(0)) > \psi(y, \gamma(0)) - \varepsilon$ and $\psi_A(\gamma(1)) > \psi(z, \gamma(1)) - \varepsilon$. Since A is convex, there exists a geodesic φ which takes its values in A and such that $\varphi(0) = y$ and $\varphi(1) = z$. Since ψ is convex, we have that for all $t \in [0,1]$,

$$\psi(\varphi(t), \gamma(t)) \le (1-t)\psi(\varphi(0), \gamma(0)) + t\psi(\varphi(1), \gamma(1)) \ .$$

The left hand side term is greater than $\psi_A(\gamma(t))$ since $\varphi(t) \in A$ and the right hand side term is less than $(1-t)\psi_A(\gamma(0)) + t\psi_A(\gamma(1)) + \varepsilon$ by definition of y and z. It yields

$$\psi_A(\gamma(t)) \le (1-t)\psi_A(\gamma(0)) + t\psi_A(\gamma(1)) + \varepsilon$$

for all $\varepsilon > 0$. This establishes inequality (22). □

Corollary 6.16. – *Let N be a manifold with convex geometry and let K be a compact convex subset of N.*

If W is a N-valued martingale such that almost surely $W_1 \in K$ and W lives in a compact subset N' of N, then almost surely, for all $t \in [0,1]$, W_t belongs to K.

Proof. – By lemma 6.15, there exists a convex non-negative function ψ_K on N which vanishes exactly on K. Furthermore, $\psi_K(W)$ is a bounded non-negative submartingale since $\psi_K(N')$, is compact, and it satisfies $\psi_K(W_1) = 0$. It implies that $\psi_K(W) \equiv 0$ and W lives in K. □

We are now able to prove the convergence of (0).

Theorem 6.17. – *Let $M = B(0,R)$ be a complex manifold with a holomorphic connection ∇ such that the holomorphic Christoffel symbols Γ are defined on M and satisfy $\Gamma(0) = 0$ (see section 4 for details). Assume furthermore that M has strong convex geometry and there exists a convex function $f : M \to \mathbb{R}_+$ of the form $z \mapsto h(\|z\|)$ with $h : [0,R) \to \mathbb{R}_+$ continuous, satisfying $h(0) = 0$, strictly increasing.*

Let $a \mapsto L(a)(\omega)$ be an almost surely holomorphic map with values in a closed ball $\bar{B}(0,r) \subset M (r < R)$ and defined for $a \in D(0,1)$, and such that a.s. $L(0)(\omega) \equiv 0 \in M$.

Then the bounded M_S^n-valued $\nabla^{(n)}$-martingales W^n with terminal values $L^{(n)}(0)$ define a family $a \mapsto X(a)$ of ∇-martingales for $a \in D(0,1)$, such that for all a $X_1(a) = L(a)$, almost surely, for all $t \in [0,1]$, the map $a \mapsto X_t(a)(\omega)$ is holomorphic on $D(0,1)$ and $\frac{\partial^n a}{\partial a^n} X(a)|_{a=0} = W^n$. In coordinates, this writes

$$X(a) = \sum_{n=1}^{\infty} \frac{a^n}{n!} W_n(\cdot) \qquad (0')$$

and the series converges absolutely in $D(0,1)$.

Proof. – By proposition 6.14, to show that the series converges, it is sufficient to show that for all $n \in \mathbb{N}^*, g(W^n) \leq h(r)$.

By corollary 6.12, W^n lives in a convex compact set K^n since $W^n(1)$ is bounded.

By corollary 6.11, the compact set K^n has convex geometry.

By proposition 6.13, the set $M_S^{n\prime}(r) = (g^n)^{-1}([0, h(r)])$ is a convex compact set.

By corollary 6.16, since $M_S^{n\prime}(r)$ is convex and K^n has convex geometry and is compact, W^n lives in $M_S^{n\prime}(r)$, and hence $g(W^n) \leq h(r)$. Using proposition 6.14, this proves the convergence of $(0')$.

By theorem 6.4, using the fact that the series converges absolutely on $D(0, 1-\varepsilon)$ for all $\varepsilon > 0$ and takes its values in $\bar{B}(0,r)$, the sum of this series is a martingale. □

Corollary 6.18. – *The assumptions on M are the same as in theorem 6.17. Let $r \in [0,R)$. Given a random variable L with values in $B(0,r) \subset M$, there exists a ∇-martingale X with terminal value L.*

If M is the complexification of the real ball $M^{\mathbb{R}} = B^{\mathbb{R}}(0,R)$ (see section 4) and if L takes it values in $M^{\mathbb{R}}$, then X takes it values in $M^{\mathbb{R}}$, *and is a $\nabla^{\mathbb{R}}$-martingale.*

Proof. – Define for $a \in \bar{D}(0,1), L(a) = aL$ and apply theorem 6.17. The solution to our problem is the martingale $X(1)$.

If L takes its value in $M^{\mathbb{R}}$, then since $K = M^{\mathbb{R}} \cap \bar{B}(0,r)$ is a compact convex subset of M, by corollary 6.16, the martingale $X(1)$ takes its values in K and hence in $M^{\mathbb{R}}$. □

One knows that any point of any real analytic manifold has an open neighbourhood $M^{\mathbb{R}}$ such that there is a complexification M of $M^{\mathbb{R}}$ which satisfies the assumptions of theorem 6.17. This together with corollary 6.18 yield the following result.

Corollary 6.19. – *Let N be a real analytic manifold endowed with a real analytic connection $\nabla^{\mathbb{R}}$. Every point x of N has a neighbourhood V_x such that if L is a $\mathscr{F}_1$-measurable random variable with values in V_x, then there exists a unique V_x-valued $\nabla^{\mathbb{R}}$-martingale X such that $X_1 = L$.*

It would be interesting to know in which domain of $\mathbb{C}$ the holomorphic family of martingales $a \mapsto X(a)$ of theorem 6.17 extends, and on which holomorphic extension of M the extension of X takes its values. In the case when $M^{\mathbb{R}}$ is an open hemisphere, if $L(a) = aL$ and L takes its values in a compact subset of $M^{\mathbb{R}}$, then for all $x \in \mathbb{R}, L(ix)$ takes its values in a hyperbolic space and one would expect that the family of martingales $x \mapsto X(ix)$ with terminal values $x \mapsto L(ix)$ is analytic.

7. Existence of martingales with prescribed terminal value in compact convex sets with convex geometry

This section is devoted to quantitative results on existence of martingales with prescribed terminal value, and to the case where it is not supposed any more that M and ∇ are analytic.

Lemma 7.1. – *Let N be a manifold endowed with a C^1 connection ∇, and let V be a compact convex subset of N with convex geometry. Assume that every point x of V has a neighbourhood V_x such that for every $\mathscr{F}_1$-measurable random variable L with values in V_x, there exists a V_x-valued ∇-martingale X such that $X_1 = L$. Then for every $\mathscr{F}_1$-measurable random variable L with values in V, there exists a V-valued ∇-martingale X such that $X_1 = L$.*

Proof. – Let ψ be a function on $U \times U$ defining the convex geometry of V, where U is an open neighbourhood of V. Since ψ is convex on $U \times U$ and equal to 0 on the diagonal and V is compact, one can find a Riemannian distance δ on V which satisfies $\psi \le \delta$. Conversely, with the same arguments as in [K1] lemma 4.3, one shows that there exists a nonnegative increasing function $h : \mathbb{R}_+ \to \mathbb{R}_+$ continuous at 0, which satisfies $h(0) = 0$ and such that $\delta \le h \circ \psi$ on $V \times V$. Let $\mathscr{R}(V)$ be the set of reachable random variables, i.e. the set of V-valued $\mathscr{F}_1$-measurable random variables L such that there exists a ∇-martingale X which satisfies $X_1 = L$. Since $\mathscr{R}(V)$ is not empty and the set of V-valued $\mathscr{F}_1$-measurable random variables is connected for the topology of a.s. uniform convergence (with respect to δ), it is sufficient to show that $\mathscr{R}(V)$ is both open and closed for this topology.

Let us show that $\mathscr{R}(V)$ is closed. Let $(L^n)_{n\in\mathbb{N}}$ be a sequence of elements of $\mathscr{R}(V)$ converging to L, and let $(X^n)_{n\in\mathbb{N}}$ be the sequence of martingales such that $X_1^n = L^n$ for all $n \in \mathbb{N}$. Let $n, m \in \mathbb{N}$. Since (X^n, X^m) is a martingale, we have

$$\psi(X_t^n, X_t^m) \le \mathbb{E}[\psi(L^n, L^m)|\mathscr{F}_t] \le \|\psi(L^n, L^m)\|_\infty \le \|\delta(L^n, L^m)\|_\infty$$

for all $t \in [0,1]$. But this implies that as n, m tends to infinity, $\| \sup_{0 \le t \le 1} \delta(X_t^n, X_t^m)\|_\infty$ converges to 0 (here we use the fact that $\delta \le h \circ \psi$). Hence $(X^n)_{n \in \mathbb{N}}$ is a Cauchy sequence for the topology of a.s. uniform convergence. It converges to a ∇-martingale X with terminal value L, since the set of ∇-martingales is closed for the topology of uniform convergence in probability ([E 4.43]).

Let us now show that $\mathscr{R}(V)$ is open. For $x \in V$ and $\alpha > 0$, let $B(x, \alpha) = \{y \in V, \psi(x, y) \le \alpha\}$. Let $\alpha > 0$ be small enough so that for every $x \in V$,

$$V_x = B(B(x, \alpha), \alpha) = \{z \in V, \exists y \in B(x, \alpha),\ z \in B(y, \alpha)\}$$

satisfies the assumption of the lemma (in particular one easily verifies that V_x is convex). Let L be an element of $\mathscr{R}(V)$. Since $\psi \le \delta$, it is sufficient to show that every random variable L' satisfying almost surely $\psi(L, L') \le \alpha$ is in $\mathscr{R}(V)$. There exists an increasing sequence of stopping times

$$0 = T^0 \le T^1 \le \ldots \le T^n$$

such that $(T^n)_{n \in N}$ converges stationarily to 1 and between two consecutive stopping times S and T of this discretization, the martingale X with terminal value L lives in the set $B(X_S, \alpha)$. For $n \in \mathbb{N}$, let L^n be the random variable defined by $L^n = L'$ on $\{T^n = 1\}$ and $L^n = X_{T^n}$ on $\{T^n < 1\}$. We have almost surely $\psi(X_{T_n}, L^n) \le \alpha$ and $L^n \in V_{X_{T^{n-1}}}$. Hence, conditioning by $\mathscr{F}_{T^{n-1}}$, we have that between the times T^{n-1} and T^n there exists a martingale X^n with terminal value L^n. Since (X, X^n) is a martingale, it satisfies

$$\psi(X_{T^{n-1}}, X^n_{T^{n-1}}) \le \mathbb{E}[\psi(X_{T^n}, L^n)|\mathscr{F}_{T^{n-1}}] \le \alpha \ .$$

Let $k \le n-1$ and assume that we have constructed the martingale X^n between the times T^{k+1} and T^n and that almost surely $\psi(X_{T^{k+1}}, X^n_{T^{k+1}}) \le \alpha$. By the same method, conditionning by $\mathscr{F}_{T^k}$ gives the construction of X^n between the times T^k and T^{k+1}. Hence we have a martingale X^n with terminal value L^n. Using [D3 proposition 4.4] (note that convex geometry is sufficient to apply this result), we deduce that as n tends to infinity, X^n converges uniformly in probability to a V-valued martingale X' with terminal value L'. □

A direct consequence of lemma 7.1 and lemma 6.19 is the following result.

Corollary 7.2. – *Let M be a real analytic manifold with an analytic connection ∇. Let V be a compact convex subset of M with convex geometry. Then for every random variable L with values in V, there exists a V-valued ∇-martingale X with terminal value L.*

We can now state the main result of this section.

Theorem 7.3. – *Let M be a manifold with a C^1 connection ∇. Let V be a compact convex subset of M with convex geometry. Then for every random variable L with values in V, there exists a V-valued ∇-martingale X with terminal value L.*

Proof. – By lemma 7.1, it is sufficient to prove that every point of V has a neighbourhood V_x with the desired property. Hence one can assume that V is contained in the domain of an exponential chart. We will identify this domain with a ball in $\mathbb{R}^d$ containing 0, such that the Christoffel symbols satisfie $\Gamma^k_{i,j}(0) = 0$. Possibly by reducing V, we can assume that the functions $\Gamma^k_{i,j}$ are the restrictions to V of C^1 functions $\mathbb{R}^d \to \mathbb{R}$ with compact support. We can also assume that the map $\psi : V \times V \to \mathbb{R}_+$ defined by $\psi(x,x') = \frac{1}{2}(\varepsilon^2 + \|x + x'\|^2)\|x' - x\|^2$ is convex for the product connection (see [E4 4.59]). By convolution with the analytic functions $\phi_n(z) = \frac{n^d}{(\sqrt{2\pi})^d} e^{-\frac{n^2\|z\|^2}{2}}$, one can approximate the functions Γ^k_{ij} uniformly in V by analytic functions ${}^n\Gamma^{k\,\prime}_{ij}$, with the further property that the derivatives of ${}^n\Gamma^{k\,\prime}_{ij}$ converge uniformly to those of Γ^k_{ij}. Setting ${}^n\Gamma^k_{ij}(x) = {}^n\Gamma^{k\,\prime}_{ij}(x) - {}^n\Gamma^{k\,\prime}_{ij}(0)$, we have the same properties with ${}^n\Gamma^k_{ij}(0) = 0$ as additional one. These functions define analytic connections ∇^n on V which converge uniformly to ∇. The assumptions on the derivatives of the Christoffel symbols together with the proof of [E4 4.59] allow us to say that for n sufficiently large, ψ is convex for the product connection $\nabla^n \times \nabla^n$. Let L be a random variable with values in V. By corollary 7.2, there exists a ∇^n-martingale X^n with terminal value L. We are left to show that $(X^n)_{n\in\mathbb{N}}$ converges to a ∇-martingale X as n tends to infinity. But since $(\nabla \times \nabla)d\psi$ is strictly positive outside the diagonal, for every $\varepsilon > 0$, there exists $n(\varepsilon)$ such that if $m, n \geq n(\varepsilon)$, the function $\psi_\varepsilon = \sup(\psi, \varepsilon)$ is $(\nabla^n \times \nabla^m)$-convex. This implies that $\psi_\varepsilon(X^n, X^m)$ is the constant submartingale equal to ε. Hence X^n converges a.s. uniformly to a continuous adapted process X. Now let f be a function on V such that $\nabla df > 0$. Then $\nabla^n df > 0$ for n sufficiently large, and this implies that $f(X)$ is a submartingale. This is true for all f with $\nabla df > 0$, hence by [A1] proposition 3.4 and [A2] proposition 2.12, X is a ∇-martingale and $X_1 = L$. □

Remark. – From Theorem 7.3 together with theorem 5.2 of [D3] we deduce the existence of martingales with prescribed terminal value in a convex manifold M with convex geometry (with some additional assumptions on the function which defines the convex geometry and an integrability condition on the terminal value) if M is an increasing union of compact sets with convex geometry.

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